

Modular and Graceful Edge Colorings of Graphs

Ryan Jones, Ph.D.

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A proper vertex coloring of a graph G is an assignment of colors to the vertices of G such that adjacent vertices are assigned distinct colors and the minimum number of colors in a proper vertex coloring of G is the chromatic number $\chi(G)$ of G . For a connected graph G of order 3 or more and an edge coloring $c : E(G) \rightarrow \mathbb{Z}_k$ ($k \geq 2$) where adjacent edges may be colored the same, the color sum $s(v)$ of a vertex v of G is the sum in \mathbb{Z}_k of the colors of the edges incident with v . The edge coloring c is a modular k -edge coloring of G if $s(u) \neq s(v)$ in \mathbb{Z}_k for all pairs u, v of adjacent vertices in G . The modular chromatic index $\chi'_m(G)$ of G is the minimum k for which G has a modular k -edge coloring. The modular chromatic indices of several well-known classes of graphs are determined and the relationship between $\chi'_m(G)$ and $\chi'_m(H)$, when H is a subgraph of G , has been investigated. It is shown that $\chi(G) \leq \chi'_m(G) \leq \chi(G) + 1$ for every connected graph G of order at least 3 and $\chi'_m(G) = \chi(G) + 1$ if and only if $\chi(G) \equiv 2 \pmod{4}$ and every proper $\chi(G)$ -coloring of G results in color classes of odd size. Furthermore, every graph G has a modular k -edge coloring for each $k \geq \chi'_m(G)$.

Let G be a connected graph of order $n \geq 3$ and size m and let $f : E(G) \rightarrow \mathbb{Z}_n$ be an edge labeling of G . Define an induced vertex labeling $f' : V(G) \rightarrow \mathbb{Z}_n$ in terms of f by $f'(v) = \sum_{u \in N(v)} f(uv)$ where the sum is computed in \mathbb{Z}_n . If f' is

one-to-one, then f is called a modular edge-graceful labeling and G is a modular edge-graceful graph. A 1991 conjecture states that every tree of order n where $n \not\equiv 2 \pmod{4}$ is modular edge-graceful. We show that this conjecture is true and extend it to all nontrivial connected graphs. It is shown that the modular edge-gracefulness of a connected graph G of order $n \geq 3$ is $n + 1$. We also study nowhere-zero modular edge-graceful labelings in which the label 0 is not permitted and establish several results on nowhere-zero graceful graphs.