Hamiltonicity and Connectivity
in Distance-Colored Graphs

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For a connected graph G and a positive integer k, the kth power G^k of G is the graph with $V(G^k) = V(G)$ where $uv \in E(G^k)$ if the distance $d_G(u,v)$ between u and v is at most k. The edge coloring of G^k defined by assigning each edge uv of G^k the color $d_G(u,v)$ produces an edge-colored graph G^k called a distance-colored graph.

A distance-colored graph G^k is Hamiltonian-colored if G^k contains a properly colored Hamiltonian cycle. The minimum k for which G^k is Hamiltonian-colored is the Hamiltonian coloring exponent hce(G). It is shown that for each pair k, d of integers with $4 \le k \le d$, there exists a tree T with hce$(T) = k$ and diam$(T) = d$. Hamiltonian coloring exponents are determined for several well-known classes of graphs. It is also shown that for each integer $k \ge 2$, there exists a tree T_k with hce$(T_k) = k$ such that every properly colored Hamiltonian cycle in the kth power of T_k must use all colors $1, 2, \ldots, k$. For a grid $G = P_n \,\square\, P_m$ with $n, m \ge 2$, it is shown that G^2 is Hamiltonian-colored if and only if $nm \equiv 0 \pmod 4$ and a complete solution is presented for restricted Hamiltonian-colored cycles in G^3.

A distance-colored graph is properly p-connected if two distinct vertices u and v in the graph are connected by p internally disjoint properly colored $u - v$ paths. For a connected graph G and an integer $k \ge 2$, the color-connectivity of G^k is

the maximum positive integer p for which G^k is properly p-connected. The color-connectivities are determined for some well-known classes of graphs. It is shown that G^2 is properly 2-connected for every 2-connected graph that is not complete, a double star is the only tree T for which T^2 is properly 2-connected and G^3 is properly 2-connected for every connected graph G of diameter at least 3. All pairs k, n of positive integers for which P_n^k is properly k-connected are determined and other related results are presented.

Various color-related distance parameters and rainbow concepts in distance-colored graphs are introduced and studied. Results and open problems are also presented in this area of research.