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Third Order Degree Regular Graphs

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The title of the paper is:

"Third Order Degree Regular Graphs"

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Mathematics and Statistics

Dr. Allen Schwenk
Mathematics and Statistics

Dr. Arthur Falk
Philosophy
Third Order Degree Regular Graphs

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Abstract
A graph $G$ is regular of degree $d$ if for every vertex $v$ in $G$ there exist exactly $d$ vertices at distance 1 from $v$. A graph $G$ is $k$th order regular of degree $d$ if for every vertex $v$ in $G$, there exist exactly $d$ vertices at distance $k$ from $v$. In this paper, third order regular graphs of degree 1 with small order are characterized.

1 Definitions and Examples
We denote the distance between two vertices $u$ and $v$ in a connected graph $G$ of order $p$ by $d(u, v)$. For an integer $k$ with $1 \leq k \leq p - 1$, the $k$-neighborhood $N_k(v)$ is defined by

$$N_k(v) = \{ u | u \in V(G) \text{and} d(u, v) = k \}$$

and the closed $k$-neighborhood by

$$N_k[v] = N_k(v) \cup \{v\}.$$ 

Definition The $k$th order degree of $v$ is $\deg_k v = |N_k(v)|$. 

1
Definition A graph $G$ is $k$th order regular of degree $d$ if $\deg_k(v) = d$ for every vertex $v$ of $G$. Consequently, first order regularity of degree $d$ is synonymous with regularity of degree $d$.

For example, the graph $G_1$ of Figure 1 is second order regular of degree 4, while $G_2$ is first order regular of degree 4, second order regular of degree 5, and third order regular of degree 2.

![Figure 1](image)

2 $k$th Order Regular Graphs of Degree 1

Alavi, Lick and Zou [1] conjectured that for $k \geq 2$, every connected, $k$th order regular graph of degree 1 is either a path of length $2k - 1$ or has diameter $k$. This conjecture is verified in our first result.

**Theorem 1** For $k \geq 2$, every connected, $k$th order regular graph of degree 1 is either a path of length $2k - 1$ or has diameter $k$.

**Proof.** I still have to write this one up! □
3 Third Order Regular Graphs of Degree 1

Let $nK_2$ denote the complete $n$-partite graph in which each partite set contains exactly 2 vertices. In [1] it is shown that a connected graph $G$ is second order regular of degree 1 if and only if $G$ is either a path of length 3 or $G$ is $nK_2$ for some $n \geq 2$. We present some similar results for graphs that are third order regular of degree 1. We first introduce the following definitions.

**Definition** A $k$th order regular graph $G$ of degree $d$ is maximal if for every pair $u, v$ of nonadjacent vertices the graph $G + uv$ is not $k$th order regular of degree $d$.

For example, for $k \geq 2$, the graphs $P_{2k}$ and $C_{2k}$ are $k$th order regular of degree 1 and $C_{2k}$ is maximal, while $P_{2k}$ is not.

**Definition** Two vertices $u$ and $v$ are defined as antipodal in a graph $G$ if $d(u, v) = \text{diam } G$. Since paths are not maximal, recall from theorem 1 that maximal $k$th order regular graphs of degree 1 have diameter $k$. Hence, if $x$ is a vertex in a maximal $k$th order regular graph of degree 1, the antipodal vertex of the vertex $x$, denoted $x'$, is the unique vertex at distance $k$ from $x$. We refer to a vertex and its antipode as an antipodal pair.

**Proposition 1** Let $x$ and $y$ be distinct vertices in a connected, maximal third order regular graph $G$ of degree 1. Then $xy \in E(G)$ if and only if $x'y' \in E(G)$.

**Proof.** Assume that $xy \in E(G)$, and suppose, to the contrary, that $x'y' \notin E(G)$. Then $d(x, x') = d(y, y') = 3$. Since $P_6$ is not a maximal third order regular graph of degree 1, it follows from Theorem 1 that $G$ has diameter 3. Since $x'$ is the unique vertex at distance 3 from $x$, we have that $d(x', y) \leq 2$. If $d(x', y) = 1$, then, since $d(y, x) = 1$, it follows that $d(x', x) \leq 2$, producing a contradiction. Thus $d(x', y) = 2$. Similarly, $d(y', x) = 2$.

Since $x'y' \notin E(G)$, it follows that $G + x'y'$ is not third order regular of degree 1. Hence, for some pair $u, v$ of antipodal vertices in $G$, there exists a $w - w'$ path of length at most 2 in $G + x'y'$. Since $d(x, x') = d(y, y') = 3$ in $G + x'y'$, there must exist $z, z' \in V(G)$, with $\{z, z'\} \cap \{x, x', y, y'\} = \emptyset$ such that there exist $z - x'$ and $z' - y'$ paths or $z - y'$ and $z' - x'$ paths. Then

$$d(z, x') + 1 + d(y', z') < 3$$

or

$$d(z', x') + 1 + d(y', y) < 3,$$

producing a contradiction. \qed
Proposition 1 implies that there is an automorphism of a maximal third order regular graph of degree 1 that interchanges each vertex with its antipode.

**Corollary 1** Let $G$ be a connected, maximal third order regular graph of degree 1. Let $x, x'$ be a pair of antipodal vertices in $G$. Then for $v \in V(G)$ with $v \neq x, x'$, either $vx \in E(G)$ or $vx' \in E(G)$.

**Proof.** Let $v$ be a vertex of $G$ such that $v \neq x, x'$. Assume, to the contrary, that $v$ is neither adjacent to $x$ nor to $x'$. Since diam $G = 3$ and $x'$ is the unique vertex at distance 3 from $x$, it follows that $d(v, x) = d(v, x') = 2$. By the previous proposition, $d(v', x) = d(v', x') = 2$.

Since $G$ is maximal, $G + vx$ is not third order regular of degree 1. Since in $G + vx$, we know that $d(x, x') = d(v, v') = 3$, there must exist some pair $z, z'$ for which $d(z, z') \leq 2$ in $G + vx$. Thus

$$d(z, v) + 1 + d(x, z') \leq 2$$

or

$$d(z', v) + 1 + d(x, z) \leq 2,$$

producing a contradiction.

Thus $vx \in E(G)$ and, by the symmetry shown in Proposition 1, we have that $v'x \in E(G)$. □

**Corollary 2** follows immediately.

**Corollary 2** Let $G$ be a connected, maximal third order regular graph of degree 1 of order $p$. Then $G$ is regular of degree $(p - 2)/2$.

These results provide us with the following theorems that characterize third order regular graphs of degree 1 of small order.

**Theorem 2** The only connected, maximal third order regular graph of degree 1 and of order 6 is $C_6$.

**Proof.** From Corollary 2, a maximal third order regular graph of degree 1 of order $p$ is regular of degree $(p - 2)/2$. Let $G$ be a connected, maximal third order regular graph of degree 1 and of order 6. Then $G$ is regular of degree 2. Since $G$ is connected, $G$ must be a cycle with six vertices. □

**Theorem 3** The only connected and maximal third order regular graph of degree 1 and of order 8 is the cube.
Proof. Let $G$ be a connected, maximal third order regular graph of degree 1 and order 8. From Corollary 2, $G$ is regular of degree 3. If we delete two antipodal vertices $x$ and $x'$ from $G$, the resulting graph of order 6 is regular of degree 2. Only two such graphs are possible, namely $G_3$ and $G_4$, shown in Figure 2.

We now construct $G$ from these graphs. Since, in adding two vertices to $G_3$, the resulting graph must be connected, we always have $G_4$ as a subgraph of $G$. Thus we need only examine the possible graphs that can be constructed from $G_4$. Since no pair of antipodal vertices are adjacent to the same vertex, there are two such graphs possible, namely $G_5$ and $G_6$ shown in Figure 3.
The graph $G_8$ is not third order regular of degree 1. Hence $G_6$, which is isomorphic to the cube, is the only maximal third order regular graph on eight vertices. \hfill\Box

**Theorem 4**  The only connected, maximal third order regular graphs of degree 1 and of order 10 are $K_2 \times K_5$, the antiprism on 10 vertices, and the cube with two pyramids. The latter two are shown in Figure 4.

![Figure 4](image)

The antiprism on 10 vertices  

the cube with two pyramids

**Proof.** Let $G$ be a connected, maximal third order regular graph of degree 1 and order 10. From Corollary 2, the graph $G$ is regular of degree 4. If we delete a pair of antipodal vertices, $u$ and $u'$, from $G$, the resulting graph on eight vertices is regular of degree 3. The three such graphs possible are $G_5, G_6,$
and $G_7$ (see Figure 5).

We now construct $G$ from $G - uv$. We begin with $G_7$. Since $G$ is connected, in adding two vertices to $G_7$ we create $G_5$ as a subgraph. Hence, we need only examine the possible graphs that can be constructed from $G_5$ and $G_6$.

Consider $G_5$. Without loss of generality, let $ud \in E(G)$. This forces $u'd' \in E(G)$. Since $d(u, b) = d(b, b') = 3$, we must have $u'b \in E(G)$, and thus $ub' \in E(G)$. Since antipodal vertices cannot both be adjacent to $u$ or to $u'$, we now
have two possible graphs, namely the graphs $G_8$ and $G_9$ shown in Figure 6.

Both $G_8$ and $G_9$ are maximal third order regular of degree 1. The graph $G_8$ is isomorphic to the antiprism on ten vertices, and $G_9$ is isomorphic to the cube with two pyramids.

We now examine $G_6$. There are four ways in which two vertices $u$ and $u'$ can be added to $G_6$ such that antipodal vertices are not both adjacent to $u$ or
to \( u' \), as represented by \( G_{10}, G_{11}, G_{12}, \) and \( G_{13} \) of Figure 7.

\[ G_{10} \quad G_{11} \]
\[ G_{12} \quad G_{13} \]

However, observe that \( G_{10} \cong G_{12} \cong G_9 \). Since \( N[u] = N[v] \) in \( G_{11} \), it is not
third order regular of degree 1. Finally, $G_{13} \cong K_2 \times K_5$ shown in Figure 8.

Figure 8. $K_2 \times K_5$. 

\[ \square \]

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