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Modular and Graceful Edge Colorings of Graphs

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Western Michigan University

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Modular and Graceful Edge Colorings of Graphs

by
Ryan Jones

A Dissertation
Submitted to the
Faculty of The Graduate College
in partial fulfillment for the requirement for the
Degree of Doctor of Philosophy
Department of Mathematics
Advisor: Ping Zhang, Ph.D.

Western Michigan University
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Modular and Graceful Edge Colorings of Graphs

Ryan Jones, Ph.D.
Western Michigan University, 2011

A proper vertex coloring of a graph $G$ is an assignment of colors to the vertices of $G$ such that adjacent vertices are assigned distinct colors and the minimum number of colors in a proper vertex coloring of $G$ is the chromatic number $\chi(G)$ of $G$. For a connected graph $G$ of order 3 or more and an edge coloring $c : E(G) \to \mathbb{Z}_k$ ($k \geq 2$) where adjacent edges may be colored the same, the color sum $s(v)$ of a vertex $v$ of $G$ is the sum in $\mathbb{Z}_k$ of the colors of the edges incident with $v$. The edge coloring $c$ is a modular $k$-edge coloring of $G$ if $s(u) \neq s(v)$ in $\mathbb{Z}_k$ for all pairs $u, v$ of adjacent vertices in $G$. The modular chromatic index $\chi'_m(G)$ of $G$ is the minimum $k$ for which $G$ has a modular $k$-edge coloring. The modular chromatic indices of several well-known classes of graphs are determined and the relationship between $\chi'_m(G)$ and $\chi'_m(H)$, when $H$ is a subgraph of $G$, has been investigated. It is shown that $\chi(G) \leq \chi'_m(G) \leq \chi(G) + 1$ for every connected graph $G$ of order at least 3 and $\chi'_m(G) = \chi(G) + 1$ if and only if $\chi(G) \equiv 2 \pmod{4}$ and every proper $\chi(G)$-coloring of $G$ results in color classes of odd size. Furthermore, every graph $G$ has a modular $k$-edge coloring for each $k \geq \chi'_m(G)$.

Let $G$ be a connected graph of order $n \geq 3$ and size $m$ and let $f : E(G) \to \mathbb{Z}_n$ be an edge labeling of $G$. Define an induced vertex labeling $f' : V(G) \to \mathbb{Z}_n$ in terms of $f$ by $f'(v) = \sum_{u \in N(v)} f(uv)$ where the sum is computed in $\mathbb{Z}_n$. If $f'$ is
one-to-one, then $f$ is called a modular edge-graceful labeling and $G$ is a modular edge-graceful graph. A 1991 conjecture states that every tree of order $n$ where $n \not\equiv 2 \pmod{4}$ is modular edge-graceful. We show that this conjecture is true and extend it to all nontrivial connected graphs. It is shown that the modular edge-gracefulness of a connected graph $G$ of order $n \geq 3$ is $n + 1$. We also study nowhere-zero modular edge-graceful labelings in which the label 0 is not permitted and establish several results on nowhere-zero graceful graphs.
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Chapter 1

Introduction

1.1 Invitation to Graph Colorings

Graph coloring is one of the most popular research areas in graph theory. The subject of graph colorings goes back to 1852 when the young British mathematician Francis Guthrie observed that the counties in a map of England could be colored with four colors so that every two adjacent counties are colored differently. This led to the Four Color Problem of determining whether the regions of every plane map could be colored with four or fewer colors in such a way that every two adjacent regions are colored differently. Of course, the Four Color Problem has an affirmative solution, as was announced in 1976 by Kenneth Appel and Wolfgang Haken. The popularity of the Four Color Problem and the mathematics that was developed in the process of trying to solve the problem led to graph theory becoming a major area of mathematics and coloring becoming a major topic within graph theory. In fact, many of the theorems in graph theory can be traced back to problems involving coloring and thus back to the Four Color Problem.
Graph Theory is an area of mathematics whose past is always present.

There are many problems that can be represented by a graph and whose solution involves finding a specific coloring of this graph. In 1951, the Danish mathematician Gabriel Andrew Dirac stated the following:

*The colouring of abstract graphs is a generalization of the colouring of maps, and the study of the colouring of abstract graphs opens a new chapter in the combinatorial part of mathematics.*

Among the most studied colorings are proper vertex colorings and proper edge colorings. A **proper vertex coloring** of a graph $G$ is an assignment of colors to the vertices of $G$, one color to each vertex, so that adjacent vertices are colored differently. When it is understood that we are dealing with a proper vertex coloring, we ordinarily refer to this more simply as a **coloring** of $G$. While the colors used can be elements of any set or actual colors (such as red, blue, green, and yellow), positive integers (typically 1, 2, \ldots, $k$ for some positive integer $k$) are commonly used for the colors. Thus, a (proper) coloring can be considered as a function $c : V(G) \rightarrow \mathbb{N}$ (where $\mathbb{N}$ is the set of positive integers) such that $c(u) \neq c(v)$ if $u$ and $v$ are adjacent in $G$. If each color used is one of $k$ given colors, then we refer to the coloring as a $k$-**coloring**. In a $k$-coloring, we may then assume that it is the colors 1, 2, \ldots, $k$ that are being used. While all $k$ colors are typically used in a $k$-coloring of a graph, there are occasions when only some of the $k$ colors are used. The minimum number of colors in a proper vertex coloring of $G$ is the **chromatic number** $\chi(G)$ of $G$. 
A proper edge coloring (or simply an edge coloring) of a graph $G$ is an assignment of colors to the edges of $G$, one color to each edge, such that adjacent edges are assigned distinct colors. In a $k$-edge coloring, we assume that it is the colors $1, 2, \ldots, k$ that are being used (or on some occasions, only some of these $k$ colors are used). The minimum number of colors in a proper edge coloring of $G$ is the chromatic index $\chi'(G)$ of $G$.

As a consequence of the Four Color Theorem, it is possible to distinguish every two adjacent regions of every plane map $M$ by coloring the regions of $M$ with at most four colors. A coloring that provides a method of distinguishing every two adjacent vertices is said to be neighbor-distinguishing. Thus a proper vertex coloring of a graph is neighbor-distinguishing. A number of neighbor-distinguishing vertex colorings other than the standard proper colorings have been introduced (see [6, 7, 8, 9], for example). Furthermore, edge colorings (proper or nonproper) have also been introduced to distinguish every pair of adjacent vertices in a graph (see [1, 3, 14, 23] or [10, p 385], for example). Another neighbor-distinguishing vertex coloring was introduced in [19] for the purpose of finding solutions to a checkerboard problem.

1.2 A Checkerboard Problem

Suppose that the squares of an $m \times n$ checkerboard ($m$ rows and $n$ columns), where $1 \leq m \leq n$ and $n \geq 2$, are alternately colored black and red. Figure 1 1(a) shows a $5 \times 7$ checkerboard where a shaded square represents a black square. Two squares
are said to be *neighboring* if they belong to the same row or to the same column and there is no square between them. Thus every two neighboring squares are of different colors. A combinatorial problem was introduced in [19] and the following conjecture was stated.

**The Checkerboard Conjecture**  *It is possible to place coins on some of the squares of an $m \times n$ checkerboard (at most one coin per square) such that for every two squares of the same color the numbers of coins on neighboring squares are of the same parity, while for every two squares of different colors the numbers of coins on neighboring squares are of opposite parity.*

![Checkerboard Conjecture Diagram](image)

(a) (b)

Figure 1.1: A $5 \times 7$ checkerboard and a coin placement on the checkerboard

Figure 1.1(b) shows a placement of 6 coins on the $5 \times 7$ checkerboard such that the number of coins on neighboring squares of every red square is even and the number of coins on neighboring squares of every black square is odd. Thus for every two squares of different colors, the numbers of coins on neighboring squares are of opposite parity. Consequently, the Checkerboard Conjecture is true for a $5 \times 7$ checkerboard. Observe that all 6 coins on the $5 \times 7$ checkerboard of Figure 1.1(b)
are placed only on red squares. Thus the number of coins on neighboring squares of every red square is 0 and is therefore even, while the number of coins on neighboring squares of each black square is 1 and this is shown in Figure 11(b) as well. Indeed, for any \( m \times n \) checkerboard for which the Checkerboard Conjecture is true, there is always a solution in which all coins are placed only on squares of the same color. In fact, it was shown in [20] that the Checkerboard Conjecture is true for a checkerboard of any size.

**The Checkerboard Theorem**  
For every pair \( m, n \) of positive integers, it is possible to place coins on some of the squares of an \( m \times n \) checkerboard (at most one coin per square) such that for every two squares of the same color the numbers of coins on neighboring squares are of the same parity, while for every two squares of different colors the numbers of coins on neighboring squares are of opposite parity.

### 1.3 The Modular Chromatic Number of a Graph

The Checkerboard Problem described in Section 1.2 can be stated in terms of graphs. We refer to the book [5, 10] for graph theory notation and terminology not described in this work. For an \( m \times n \) checkerboard, we can associate a graph \( G \) whose vertices are the squares of the checkerboard and where two vertices of \( G \) are adjacent if the corresponding squares are neighboring. Thus \( G \) is a bipartite graph of order \( mn \), the partite sets of which are the set of black vertices and the set of red vertices. In fact, this graph is the Cartesian product \( P_m \times P_n \) of paths of orders \( m \) and \( n \).
and \( n \), which is commonly called a grid. Determining whether the Checkerboard Conjecture is true for an \( m \times n \) checkerboard is equivalent to determining whether it is possible to color each vertex of \( P_m \times P_n \) with one of the colors 0 or 1 from \( \mathbb{Z}_2 \) such that the sum of the colors (in \( \mathbb{Z}_2 \)) of the neighboring vertices of each red vertex, say, is 0 and the sum of the colors (in \( \mathbb{Z}_2 \)) of the neighboring vertices of each black vertex is 1. Of course, each black vertex can be colored 0 and so it is only a matter of determining whether there is an appropriate coloring of the red vertices of the grid with the colors 0 and 1. The placement of coins on the 5 \( \times \) 7 checkerboard shown in Figure 1.1(b) is then equivalent to the vertex coloring of the grid \( P_5 \times P_7 \) shown in Figure 1.2 using the element of \( \mathbb{Z}_2 \) as colors. This observation suggests new vertex colorings for graphs in general, also introduced in [19].

![Figure 1.2: A coloring of \( P_5 \times P_7 \) with colors in \( \mathbb{Z}_2 \)](image)

For a vertex \( v \) of a graph \( G \), let \( N(v) \) denote the neighborhood of \( v \) (the set of vertices adjacent to \( v \)). For a graph \( G \) without isolated vertices, let \( c : V(G) \to \mathbb{Z}_k \) \((k \geq 2)\) be a vertex coloring of \( G \) where adjacent vertices may be colored the same. The color sum \( \sigma(v) \) of a vertex \( v \) of \( G \) is defined as the sum in \( \mathbb{Z}_k \) of the colors of
the vertices in $N(v)$, that is,

$$\sigma(v) = \sum_{u \in N(v)} c(u).$$

The coloring $c$ is called a modular sum $k$-coloring or simply a modular $k$-coloring of $G$ if $\sigma(x) \neq \sigma(y)$ in $\mathbb{Z}_k$ for all pairs $x, y$ of adjacent vertices of $G$. A coloring $c$ is a modular coloring if $c$ is a modular $k$-coloring for some integer $k \geq 2$. The modular chromatic number $mc(G)$ of $G$ is the minimum $k$ for which $G$ has a modular $k$-coloring. For every graph $G$, $mc(G)$ is bounded below by $\chi(G)$, the chromatic number of $G$. The Checkerboard Theorem can consequently be stated in terms of graphs and modular colorings as follows.

**Theorem 1.3.1**  For every two positive integers $m$ and $n$ with $mn \geq 2$,

$$mc(P_m \times P_n) = 2.$$

To illustrate the concepts introduced above, consider the bipartite graph $G$ of Figure 1.3, which also shows a modular 3-coloring of $G$ (where the color of a vertex is placed within the vertex) together with the color sum $\sigma(v)$ for each vertex $v$ of $G$ (where the color sum of a vertex is placed next to the vertex). Thus $mc(G) \leq 3$.

Next, we show that $mc(G) \geq 3$. Assume, to the contrary, that there exists a modular 2-coloring $c$ of $G$. By the symmetry of the graph $G$, we may assume that $\sigma(u_i) = 0$ and $\sigma(v_i) = 1$ for $1 \leq i \leq 4$. Since $\sigma(v_1) = 1$, it follows that $\{c(u_1), c(u_2)\} = \{0, 1\}$, which in turn implies that $c(u_3) = 0$ since $\sigma(v_2) = 1$. 
Figure 1.3: A bipartite graph $G$ with $mc(G) = 3$

Because $\sigma(v_3) = 1$, it follows that $c(u_4) \neq c(u_2)$ and so $c(u_4) = c(u_1)$. However then, $\sigma(v_4) = 0$, which is impossible. Therefore, $mc(G) \geq 3$, which implies that $mc(G) = 3$. Among the results appearing in [19, 21] are the following.

**Theorem 1.3.2** For every nontrivial connected graph $G$, there exists a modular $k$-coloring of $G$ for some integer $k \geq 2$.

**Theorem 1.3.3** If $G$ is a complete multipartite graph, then $mc(G) = \chi(G)$.

**Theorem 1.3.4** If $T$ is a nontrivial tree, then $mc(T) = 2$ or $mc(T) = 3$.

A nontrivial tree $T$ is of type one if $mc(T) = 2$ and is of type two if $mc(T) = 3$. It is shown in [21] that all nontrivial trees of diameter at most 6 are of type one. A *caterpillar* is a tree of order 3 or more, the removal of whose end-vertices produces a path. A characterization of caterpillars that are of type two was established.
Chapter 2

Modular Edge Colorings

2.1 Notation and Terminology

We now introduce a neighbor-distinguishing edge coloring that is closely related to modular vertex colorings. For a graph $G$ without isolated vertices, let $c : E(G) \to \mathbb{Z}_k$ ($k \geq 2$) be an edge coloring of $G$ where adjacent edges may be colored the same. The color sum $s_c(v)$ of a vertex $v$ of $G$ is defined as the sum in $\mathbb{Z}_k$ of the colors of the edges incident with $v$, that is, if $E_v$ is the set of edges incident with $v$ in $G$, then

$$s_c(v) = \sum_{e \in E_v} c(e),$$

that is, the sum (in $\mathbb{Z}_k$) is taken over all edges incident with $v$. (We also denote $s_c(v)$ by $s(v)$ if the coloring $c$ is clear.) An edge coloring $c$ is a modular neighbor-distinguishing $k$-edge coloring of $G$ if $s(u) \neq s(v)$ in $\mathbb{Z}_k$ for all pairs $u, v$ of adjacent vertices of $G$. We refer to such edge colorings more simply as modular $k$-edge colorings. An edge coloring $c$ is a modular edge coloring if $c$ is a modular $k$-edge coloring.
coloring for some integer $k \geq 2$. The modular chromatic index $\chi'_m(G)$ of $G$ is the minimum $k$ for which $G$ has a modular $k$-edge coloring. If $G$ contains a component isomorphic to $K_2$, say $V(K_2) = \{u, v\}$, then $s(u) = s(v)$ for any edge coloring of $G$, which implies that $G$ does not have a modular edge coloring. On the other hand, every graph containing neither isolated vertices nor components isomorphic to $K_2$ has a modular edge coloring.

**Proposition 2.1.1** If a graph contains neither isolated vertices nor components isomorphic to $K_2$, then its modular chromatic index exists.

**Proof.** Let $G$ be such a graph and $E(G) = \{e_1, e_2, \ldots, e_m\}$, where $m \geq 2$. Define an edge coloring $c$ of $G$ by $c(e_i) = 2^{i-1}$ for $1 \leq i \leq m$ and let $k = \sum_{i=1}^{m} 2^{i-1} = 2^m - 1$. Since $1 \leq s(v) \leq k$ for every $v \in V(G)$ and $s(u) \neq s(v)$ in $\mathbb{Z}_k$ for every two distinct vertices $u$ and $v$ in $G$, it follows that $c$ is a modular $k$-edge coloring of $G$ and so $\chi'_m(G)$ exists.

In view of Proposition 2.1.1, we consider connected graphs of order 3 or more in this work. If $c$ is a modular $k$-edge coloring of a graph $G$, then $s(u) \neq s(v)$ in $\mathbb{Z}_k$ for every pair $u, v$ of adjacent vertices of $G$. Thus the coloring $c^*$ of $G$ defined by $c^*(v) = s(v), v \in V(G)$, is a proper vertex coloring of $G$ with at most $k$ colors. This observation shows that $\chi(G)$ is a lower bound for $\chi'_m(G)$.

**Proposition 2.1.2** For every connected graph $G$ of order at least 3,

$$\chi'_m(G) \geq \chi(G).$$
To illustrate the concepts introduced above, consider the tree $T$ of order 10 in Figure 2.1(a). An edge coloring of $T$ is shown in Figure 2.1(b), where each edge is colored with an element in $\mathbb{Z}_3 = \{0, 1, 2\}$ and each vertex is labeled with its color sum. Observe that $s(u) \neq s(v)$ in $\mathbb{Z}_3$ for every pair $u, v$ of adjacent vertices of $T$. Thus this edge coloring is a modular 3-edge coloring of $T$ and so $\chi'_m(T) \leq 3$. Since $\chi'_m(T) \geq 2$ by Proposition 2.1.2, it follows that $\chi'_m(T)$ is either 2 or 3. To show that $\chi'_m(T) = 3$, assume, to the contrary, that there exists a modular 2-edge coloring $c$ of $T$. Thus $s(v) = 0$ or $s(v) = 1$ for each $v \in V(T)$. By the symmetry of the tree, we may assume that $s(u_1) = 0$ and $s(u_i) = 1$ for $1 \leq i \leq 5$. Hence, $c(u_1w_5) = 0$ and $c(u_5w_i) = 1$ for $1 \leq i \leq 4$. However, this implies that $s(u_5) = c(u_5w_5) = s(w_5)$, which is not possible. Therefore, $\chi'_m(T) \geq 3$, that is, $\chi'_m(T) = 3 > \chi(T)$.

![Figure 2.1: A modular 3-edge coloring of a graph](image)

There are also graphs $G$ for which $\chi'_m(G) = \chi(G)$. For example, consider the Petersen graph $P$ in Figure 2.2. Since $\chi(P) = 3$ and there exists a modular 3-edge coloring of $P$ (also shown in the figure), $\chi'_m(P) = 3 = \chi(P)$. 
2.2 Complete Graphs, Cycles and Bipartite Graphs

In this section we determine the modular chromatic indices of several classes of graphs. We first consider complete graphs and begin with an observation.

Observation 2.2.1 Let $G$ be a connected graph of order at least 3. If $c : E(G) \rightarrow \mathbb{N}$ is an edge coloring of $G$, then

$$\sum_{v \in V(G)} s(v) = 2 \sum_{e \in E(G)} c(e). \quad (2.1)$$

Thus if $c$ is a modular $k$-edge coloring of $G$, then

$$\sum_{v \in V(G)} s(v) \equiv 2 \sum_{e \in E(G)} c(e) \pmod{k}.$$

In Observation 2.2.1, if $c : E(G) \rightarrow \mathbb{N}$ is defined by $c(e) = 1$ for each $e \in E(G)$, then $s(v) = \text{deg}v$ for each $v \in V(G)$. Thus if the size of $G$ is $m$, then the First Theorem of Graph Theory and equation (2.1) give rise to the following

$$\sum_{v \in V(G)} s(v) = 2m.$$
2.2.1 Complete Graphs

We now determine $\chi'_m(K_n)$ for each integer $n \geq 3$.

**Theorem 2.2.2** For each integer $n \geq 3$,

$$\chi'_m(K_n) = \begin{cases} n + 1 & \text{if } n \equiv 2 \pmod{4} \\ n & \text{otherwise.} \end{cases}$$

**Proof.** Let $G = K_n$, where $V(G) = \{v_1, v_2, \ldots, v_n\}$. If $n$ is odd, then let $c_1 : E(G) \rightarrow \mathbb{Z}_n$ be an edge coloring given by

$$c_1(e) = \begin{cases} i & \text{if } e = v_i v_n \ (1 \leq i \leq n - 1) \\ 0 & \text{otherwise.} \end{cases}$$

Then $s(v_i) = i$ for $1 \leq i \leq n$, implying that $c_1$ is a modular $n$-edge coloring of $G$. It then follows by Proposition 2.1.2 that $\chi'_m(G) = n$ if $n$ is odd. If $n$ is even, then we consider two cases.

Case 1. $n \equiv 0 \pmod{4}$. Let $n = 4p$ for some positive integer $p$. Define an edge coloring $c_2 : E(G) \rightarrow \mathbb{Z}_{4p}$ by

$$c_2(e) = \begin{cases} p & \text{if } e \in \{v_i v_{i+1} : 1 \leq i \leq 4p - 2\} \cup \{v_1 v_{4p-1}\} \\ i & \text{if } e = v_i v_{4p} \text{ and } 1 \leq i \leq 4p - 1 \text{ and } i \neq 2p \\ 0 & \text{otherwise.} \end{cases}$$

Thus for $1 \leq i \leq 4p$

$$s(v_i) = \begin{cases} 2p & \text{if } i = 2p \\ 0 & \text{if } i = 4p \\ 2p + i & \text{otherwise} \end{cases}$$

in $\mathbb{Z}_{4p}$. Hence $c_2$ is a modular $4p$-edge coloring of $G$. The result now follows by Proposition 2.1.2.
Case 2. $n \equiv 2 \pmod{4}$. Let $n = 4p + 2$ for some positive integer $p$. Define an edge coloring $c_3 : E(G) \to \mathbb{Z}_{4p+3}$ by

$$c_3(e) = \begin{cases} 
  i - 1 & \text{if } e = v_iv_{4p+2} \text{ and } 2 \leq i \leq 2p + 1 \\
  i + 1 & \text{if } e = v_iv_{4p+2} \text{ and } 2p + 2 \leq i \leq 4p + 1 \\
  1 & \text{if } e \in \{v_iv_{i+1} : 1 \leq i \leq 2p\} \cup \{v_1v_{2p+1}\} \\
  0 & \text{otherwise.}
\end{cases}$$

Thus for $1 \leq i \leq 4p + 2$

$$s(v_i) = \begin{cases} 
  0 & \text{if } i = 4p + 2 \\
  i + 1 & \text{otherwise}
\end{cases}$$

in $\mathbb{Z}_{4p+3}$ and so $c_3$ is a modular $(4p + 3)$-edge coloring of $G$. Thus, $\chi'_m(G) \leq n + 1$ if $n \equiv 2 \pmod{4}$. On the other hand, assume, to the contrary, that there exists a modular $(4p + 2)$-edge coloring $c'$ of $G$. Then by Observation 2.2.1

$$2 \sum_{e \in E(G)} c'(e) = \sum_{i=1}^{4p+2} s(v_i) = 0 + 1 + \cdots + (4p + 1) = 2p + 1$$

in $\mathbb{Z}_{4p+2}$, which is impossible. Therefore, $\chi'_m(G) \geq n + 1$, which in turn implies that $\chi'_m(G) = n + 1$ if $n \equiv 2 \pmod{4}$. 

It is well known that if $v$ is a vertex in a nontrivial graph $G$, then either

$$\chi(G - v) = \chi(G) \text{ or } \chi(G - v) = \chi(G) - 1.$$ 

Also, if an edge $e$ is deleted from an nonempty graph $G$, then

$$\chi(G - e) = \chi(G) \text{ or } \chi(G - e) = \chi(G) - 1.$$
This, however, is not the case for the modular chromatic index of a graph. For example, let $G = K_n$ with $n \equiv 2 \pmod{4}$. By Theorem 2.2.2, $\chi'_m(G) = n + 1$, while $\chi'_m(G - v) = \chi'_m(K_{n-1}) = n - 1$ as $n - 1 \not\equiv 2 \pmod{4}$, implying that $\chi'_m(G - v) = \chi'_m(G) - 2$ for each $v \in V(G)$. Furthermore, $\chi'_m(G - e) = \chi'_m(G) - 2$ for each $e \in E(G)$, as we show next. It is known that $\chi(K_n - e) = n - 1$ for each integer $n \geq 3$.

**Theorem 2.2.3** For each integer $n \geq 3$, $\chi'_m(K_n - e) = n - 1$.

**Proof.** Let $G = K_n - e$ and $V(G) = \{v_1, v_2, \ldots, v_n\}$. We consider two cases.

**Case 1.** $n$ is odd. Without loss of generality, assume that $v_{[n/2] + 1} \in V(G)$. Let $H$ be the connected spanning subgraph of $G$ such that

\[
\deg_H v_i = \begin{cases} 
  i & \text{if } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\
  i - 1 & \text{if } \left\lceil \frac{n}{2} \right\rceil \leq i \leq n.
\end{cases}
\]

Thus $\deg_H v_{[n/2]} = \deg_H v_{[n/2]}$ while $\deg_H x \neq \deg_H y$ for all pairs $x, y$ of distinct vertices of $H$ with $\{x, y\} \neq \{v_{[n/2]}, v_{[n/2]}\}$. Define an edge coloring $c_1$ of $G$ by $c_1(e) = 1$ if $e \in E(H)$ and $c_1(e) = 0$ otherwise. Then $s(v) = \deg_H v$ for each $v \in V(G)$ and this is a modular $(n - 1)$-edge coloring of $G$.

**Case 2.** $n$ is even. Suppose that $v_{n/2 - 1}v_{n/2} \notin E(G)$. Construct the subgraph $H'$ with the vertex set $\{v_1, v_2, \ldots, v_{n-1}\}$ as described in Case 1. Let $H$ be the spanning subgraph of $G$ obtained from $H'$ by adding the isolated vertex $v_n$. Then

\[
\deg_H v_i = \begin{cases} 
  i & \text{if } 1 \leq i \leq \frac{n}{2} - 1 \\
  i - 1 & \text{if } \frac{n}{2} \leq i \leq n - 1 \\
  0 & \text{if } i = n.
\end{cases}
\]
In this case, \(\deg_H v_n = 0, \deg_H v_{n/2-1} = \deg_H v_{n/2}\) and \(\deg_H x \neq \deg_H y\) for all pairs \(x, y\) of distinct vertices of \(H\) with \(\{x, y\} \neq \{v_{n/2-1}, v_{n/2}\}\). Define an edge coloring \(c_2\) of \(G\) by \(c_2(e) = 1\) if \(e \in E(H)\) and \(c_2(e) = 0\) otherwise. Then \(s(v) = \deg_H v\) for each \(v \in V(G)\) and so \(c_2\) is a modular \((n-1)\)-edge coloring of \(G\).

Since \(\chi'_m(G) \geq n - 1\) by Proposition 2.1.2, the result now follows.

A fundamental property of the chromatic number is that if \(H\) is a subgraph of a graph \(G\), then \(\chi(H) \leq \chi(G)\). For the modular chromatic index, the situation is different. If \(H\) is a subgraph of \(G\) for which \(\chi'_m(H) = \chi(H)\), then \(\chi'_m(H) = \chi(H) \leq \chi(G) \leq \chi'_m(G)\). On the other hand, if \(H = K_n\) with \(n \equiv 2 \pmod{4}\) is a subgraph of a graph \(G\), then it is possible that \(\chi'_m(H) > \chi'_m(G)\). In fact, if \(H \subset G \subseteq \text{cor}(H)\), where \(\text{cor}(H)\) is the corona of \(H\) (the graph obtained from \(H\) by adding a pendant edge at each vertex of \(H\)), then \(\chi'_m(G) = n < \chi'_m(H)\). Furthermore, if \(G\) is the Cartesian product \(H \times K_2\), then \(\chi'_m(G) = n < \chi'_m(H)\) as well. We now verify both statements.

Let \(n \geq 6\) be an integer where \(n \equiv 2 \pmod{4}\). Then \(n = 4p + 2\) for some positive integer \(p\). Let \(H = K_n\) where \(V(H) = \{u_0, u_1, \ldots, u_{4p+1}\}\) and let

\[
G = (u_0, u_1, \ldots, u_{4p+1}, u_{4p+2} = u_0)
\]

be a Hamiltonian cycle of \(H\). We define edge colorings

\[
c_1 : E(H) \to \mathbb{Z}_{4p+2}
\]
of $H$ ($i = 1, 2$) by

$$c_1(e) = \begin{cases} 
i & \text{if } e \in E(C) \text{ and } e \text{ is incident with } u_{2i} \text{ for } 0 \leq i \leq p \\ -i & \text{if } e \in E(C) \text{ and } e \text{ is incident with } v_{4p+2-2i} \text{ for } 1 \leq i \leq p \\ 0 & \text{otherwise} \end{cases}$$

and

$$c_2(e) = \begin{cases} 2p + 1 & \text{if } e = u_0u_{2p+1} \\ c_1(e) & \text{otherwise}. \end{cases}$$

The color sums $s_1(u_i)$ in $\mathbb{Z}_{4p+2}$, $0 \leq i \leq 4p + 1$, obtained from $c_1$ are

$$s_1(u_i) = \begin{cases} 0 & \text{if } i = 2p + 1 \\ i & \text{otherwise}; \end{cases}$$

while the color sums $s_2(u_i)$ in $\mathbb{Z}_{4p+2}$, $0 \leq i \leq 4p + 1$, obtained from $c_2$ are

$$s_2(u_i) = \begin{cases} 2p + 1 & \text{if } i = 0, 2p + 1 \\ s_1(u_i) & \text{otherwise}. \end{cases}$$

The colorings $c_1, c_2, s_1$ and $s_2$ are illustrated for $H = K_{14}$ in Figure 2.3 where only the edges not colored 0 are shown.
The edge colorings $c_1$ and $c_2$ do not result in proper vertex colorings $s_1$ and $s_2$, respectively. This, of course, is not surprising however since $\chi_m(K_{4p+2}) = 4p + 3$ and there is no modular edge coloring of $K_{4p+2}$ using the elements of $\mathbb{Z}_{4p+2}$. In the colorings $s_1$ and $s_2$, only $u_0$ and $u_{2p+1}$ are colored the same, namely $s_1(u_0) = s_1(u_{2p+1}) = 0$ and $s_2(u_0) = s_2(u_{2p+1}) = 2p + 1$.

Now let $G$ be a graph such that $H \subseteq G \subseteq \text{cor}(H)$ obtained from $H$ by adding $j$ new vertices $w_0, w_1, \ldots, w_{j-1}$ (for some integer $j$ with $1 < j < 4p + 2$) and joining $w_i$ to $u_i$ for $0 < i < j - 1$. Then the edge coloring $c'_2 : E(G) \to \mathbb{Z}_{4p+2}$ defined by

$$c'_2(e) = \begin{cases} 
2p + 1 & \text{if } e = u_0w_0 \\
0 & \text{if } e = u_iw_i \text{ and } 1 \leq i \leq j - 1 \\
c_2(e) & \text{otherwise}
\end{cases}$$

is a modular edge coloring of $G$ and so $\chi'_m(G) = 4p + 2 < \chi'_m(H)$.

If $G = H \times K_2$, then let $H_1$ be a copy of $H$ whose edges are colored according to $c_1$ and $H_2$ a copy of $H$ whose edges are colored according to $c_2$. Join the vertex
$u_0$ in $H_1$ to the vertex $u_0$ in $H_2$ by an edge colored $2p + 1$. For $1 \leq i \leq 4p + 1$, join the vertex $u_i$ in $H_1$ to the vertex $u_{4p+2-i}$ in $H_2$ by an edge colored 0. This produces a modular edge coloring of $G$ using the elements in $\mathbb{Z}_{4p+2}$ and so $\chi'(G) = 4p + 2 < \chi'(H)$.

2.2.2 Cycles

It is known that $\chi(C_n) = 2$ if $n$ is even, while $\chi(C_n) = 3$ if $n$ is odd. Next we determine $\chi'_m(C_n)$ for every integer $n \geq 3$. For an ordering $v_1, v_2, \ldots, v_n$ of the vertices of a connected graph $G$ of order $n \geq 3$ and a modular edge coloring $c$ of $G$, define the color sum sequence of $G$ with respect to $c$ by

$$s_c: s(v_1), s(v_2), \ldots, s(v_n).$$

Theorem 2.2.4 For every integer $n \geq 3$,

$$\chi'_m(C_n) = \begin{cases} 2 & \text{if } n \equiv 0 \pmod{4} \\ 3 & \text{if } n \equiv 1, 2, 3 \pmod{4}. \end{cases}$$

Proof. Since $\chi'_m(C_3) = \chi'_m(K_3) = 3$ by Theorem 2.2.2, suppose that $n \geq 4$. Let $C_n = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$. If $n \equiv 0 \pmod{4}$, then let $c_1 : E(C_n) \to \mathbb{Z}_2$ be a 2-edge coloring of $C_n$ such that $c_1(v_i, v_{i+1}) = 1$ if and only if $i \equiv 0, 3 \pmod{4}$. Then the color sum sequence of $c_1$ is 1, 0, 1, 0, \ldots, 1, 0. Thus $c_1$ is a modular 2-edge coloring and so $\chi'_m(C_n) = 2$ by Proposition 2.1.2.
If $n \not\equiv 0 \pmod{4}$, write $n = 4p + q$, where $p$ is a positive integer and $q \in \{1, 2, 3\}$.

Consider an edge coloring $c_2 : E(C_n) \to \mathbb{Z}_3$ such that

$$c_2(v_i, v_{i+1}) = \begin{cases} 0 & \text{if } 1 \leq i \leq 4p \text{ and } i \equiv 1, 2 \pmod{4} \\ 1 & \text{if (i) } 1 \leq i \leq 4p, i \equiv 0, 3 \pmod{4}, q \in \{1, 2\} \text{ or} \\ & \text{(ii) } q = 3 \text{ and } i = n \\ 2 & \text{otherwise.} \end{cases}$$

The color sum sequence of $c_2$ is

$$s_{c_2} : \begin{cases} 2, 0, 1, 2, 1, 0, 1, 2, 1, \ldots, 0, 1, 2, 1, 0, 1, 2, 0 & \text{if } q = 1 \\ 2, 0, 1, 2, 1, 0, 1, 2, 1, \ldots, 0, 1, 2, 1, 0, 1, 2, 0, 1 & \text{if } q = 2 \\ 1, 0, 1, 2, 1, 0, 1, 2, 1, \ldots, 0, 1, 2, 1, 0, 1, 2, 0, 1, 0 & \text{if } q = 3. \end{cases}$$

Thus $c_2$ is a modular 3-edge coloring of $C_n$. Therefore, $\chi'_m(C_n) \leq 3$ if $n \not\equiv 0 \pmod{4}$. Furthermore, $\chi'_m(C_n) \geq \chi(C_n) = 3$ if $n$ is odd by Proposition 2.1.2.

It remains to show that $\chi'_m(C_n) \geq 3$ for $n \equiv 2 \pmod{4}$. Let $n = 4p + 2$ where $p$ is a positive integer and assume, to the contrary, that there exists a modular 2-edge coloring $c'$ of $C_n$. Then there are $2p + 1$ vertices having color sum 0 and there are $2p + 1$ vertices having color sum 1. However, by Observation 2.2.1

$$0 = 2 \sum_{i=1}^{n} c'(v_i, v_{i+1}) = \sum_{i=1}^{n} s(v_i) = 2p + 1 = 1$$

in $\mathbb{Z}_2$, which is impossible. Hence, $\chi'_m(C_n) \geq 3$ if $n \equiv 2 \pmod{4}$, completing the proof.

We have now presented two classes of graphs $G$ for which $\chi'_m(G) > \chi(G)$, namely the complete graphs $K_n$ and the cycles $C_n$ where $n \equiv 2 \pmod{4}$. That
\( \chi'_m(G) > \chi(G) \) in both instances is a special case of the following more general result.

**Theorem 2.2.5** Let \( G \) be a graph such that \( \chi(G) \equiv 2 \pmod{4} \). If each color class in every proper \( \chi(G) \)-coloring of \( G \) consists of an odd number of vertices, then \( \chi'_m(G) > \chi(G) \).

**Proof.** Suppose that \( \chi(G) = 4p + 2 \) for some nonnegative integer \( p \). If \( \chi'_m(G) = \chi(G) \), then there exists a modular \((4p + 2)\)-edge coloring \( c : E(G) \to \mathbb{Z}_{4p+2} \). Let \( V_0, V_1, \ldots, V_{4p+1} \) be the resulting color sum classes from the coloring \( c \), where \( s(v) = i \) if \( v \in V_i \) \((0 \leq i \leq 4p + 1)\). By Observation 2.2.1,

\[
\sum_{i=0}^{4p+1} i \cdot |V_i| \equiv 2r \pmod{4p + 2}
\]

for some integer \( r \) with \( 0 \leq r \leq 2p \). However, this is impossible since each \( |V_i| \) is odd. \( \blacksquare \)

By Theorem 2.2.5, if \( G = K_{n_1,n_2,\ldots,n_k} \) is a complete \( k \)-partite graph where \( k \equiv 2 \pmod{4} \) and each \( n_i, 1 \leq i \leq k \), is odd, then \( \chi'_m(G) > \chi(G) \). In particular, the complete regular \( k \)-partite graph \( G = K_{r,r,\ldots,r} \) where \( k \equiv 2 \pmod{4} \) and \( r \) is odd has the property that \( \chi'_m(G) > \chi(G) \). In fact, \( \chi'_m(G) = \chi(G) + 1 \). To see this, it suffices to show that \( G \) has a modular \((k + 1)\)-edge coloring. Let \( V_0, V_1, \ldots, V_{k-1} \) be the partite sets of \( G \). There are \( r \) pairwise vertex-disjoint copies \( G_1, G_2, \ldots, G_r \) of \( K_k \) in \( G \), where \( V(G_i) = \{ v_0, v_1, \ldots, v_{k-1} \} \) with \( v_j \in V_j \) for \( 1 \leq i \leq r \) and \( 0 \leq j \leq k-1 \). Coloring the edges of each \( G_i \) the same as the edges of \( K_k \) described
earlier and assigning 0 to all other edges of $G$ produces a modular $(k + 1)$-edge coloring of $G$ in which $s(v) = j$ if $v \in V_j$ for $0 \leq j \leq k - 1$. Thus $\chi'_m(G) = \chi(G) + 1$.

We make another observation here. Note that $H = K_k$ is a subgraph of the complete regular $k$-partite graph $G = K_{r^k}$. If $k \equiv 2 \pmod{4}$ and $r$ is odd, then $\chi'_m(H) > \chi(H)$ and $\chi'_m(G) > \chi(G)$ by Theorems 2.2.2 and 2.2.5, while $\chi'_m(H) = \chi'_m(G)$.

2.2.3 Bipartite Graphs

For an arbitrary bipartite graph $G$, the possible values of the modular chromatic number of $G$ are not known. In fact, it is not even known whether there is a constant $C$ such that the modular chromatic number of every connected bipartite graph is bounded above by $C$. This, however, is not the case for the modular chromatic index of a bipartite graph, as we show in this section. We first determine the modular chromatic index of a path.

**Theorem 2.2.6** For each integer $n \geq 3$,

$$\chi'_m(P_n) = \begin{cases} 2 & \text{if } n \equiv 0, 1, 3 \pmod{4} \\ 3 & \text{if } n \equiv 2 \pmod{4} \end{cases}.$$ 

**Proof.** Let $P_n = (v_1, v_2, \ldots, v_n)$. For $n \equiv 0 \pmod{4}$ or $n \equiv 3 \pmod{4}$, define the 2-edge coloring $c_1 : E(P_n) \to \mathbb{Z}_2$ such that $c_1(v_i, v_{i+1}) = 1$ if and only if $i \equiv 1, 2 \pmod{4}$. Then the color sum sequence of $c_1$ is

$$s_{c_1} : \begin{cases} 1, 0, 1, 0, \ldots, 1, 0 & \text{if } n \equiv 0 \pmod{4} \\ 1, 0, 1, 0, \ldots, 1, 0, 1 & \text{if } n \equiv 3 \pmod{4} \end{cases}.$$
For $n \equiv 1 \pmod{4}$, define the 2-edge coloring $c_2 : E(P_n) \to \mathbb{Z}_2$ such that $c_2(v_iv_{i+1}) = 1$ if and only if $i \equiv 2, 3 \pmod{4}$. Then the color sum sequence of $c_2$ is $0, 1, 0, 1, 0, \ldots, 1, 0$. Hence $c_1$ and $c_2$ are modular 2-edge colorings and so $\chi_m'(P_n) = 2$ if $n \equiv 0, 1, 3 \pmod{4}$.

For $n \equiv 2 \pmod{4}$, define the 3-edge coloring $c_3 : E(P_n) \to \mathbb{Z}_3$ by

\[
c_3(v_iv_{i+1}) = \begin{cases} 
0 & \text{if } i = n - 1 \\
1 & \text{if } 1 \leq i \leq n - 2 \text{ and } i \equiv 1, 2 \pmod{4} \\
2 & \text{if } 1 \leq i \leq n - 2 \text{ and } i \equiv 0, 3 \pmod{4}.
\end{cases}
\]

Then the color sum sequence of $c_3$ is

$$1, 2, 0, 1, 0, 2, 0, 1, 0, \ldots, 2, 0, 1, 0, 2, 0, 1, 2, 0.$$  

Thus $c_3$ is a modular 3-edge coloring and so $\chi_m'(P_n) \leq 3$. It then follows by Theorem 2.2.5 that $\chi_m'(P_n) = 3$.

Suppose that $G$ is a connected bipartite graph of order $n \geq 3$ and let $U$ and $W$ be the partite sets of $G$ with $|U| = r$ and $|W| = s$. If $\chi_m'(G) = 2$, then at least one of $r$ and $s$ must be even by Theorem 2.2.5. Let us next determine the modular chromatic indices of complete bipartite graphs.

**Proposition 2.2.7** For positive integers $r$ and $s$ where $r + s \geq 3$,

\[
\chi_m'(K_{r,s}) = \begin{cases} 
3 & \text{if } r \text{ and } s \text{ are odd} \\
2 & \text{otherwise}.
\end{cases}
\]

**Proof.** We may assume that $1 \leq r \leq s$. First suppose that $r = 1$ and $s \geq 2$. If $s$ is even, then the coloring assigning the color 1 to every edge is a modular 2-edge
coloring of $K_{1,s}$. Hence, $\chi_m'(K_{1,s}) = 2$ in this case. Suppose next that $s$ is odd. By Theorem 2.2.5, $\chi_m'(K_{1,s}) \geq 3$. On the other hand, the coloring assigning the color 1 to two edges and the color 0 to the remaining $n - 3$ edges is a modular 3-edge coloring of $K_{1,s}$. Thus the result holds for $r = 1$.

Next suppose that $r, s \geq 2$. By Proposition 2.1.2, $\chi_m'(K_{r,s}) \geq \chi(K_{r,s}) = 2$. Let $U$ and $W$ be the partite sets of $K_{r,s}$ with $|U| = r$ and $|W| = s$. If at least one of $r$ and $s$, say $r$, is even, then let $w \in W$ and consider a 2-edge coloring assigning the color 1 to an edge $e$ if and only if $e$ is incident with $w$. Then this is a modular 2-edge coloring of $K_{r,s}$ and so $\chi_m'(K_{r,s}) = 2$.

If both $r$ and $s$ are odd, then $\chi_m'(K_{r,s}) \geq 3$. Write $r = 6p + q \geq 3$, where $p$ is a nonnegative integer and $q \in \{1, 3, 5\}$, and let $w \in W$. If $q \neq 1$, then the edge coloring $c_1$ given by $c_1(e) = 1$ if $e$ is incident with $w$ and $c_1(e) = 0$ otherwise is a modular 3-edge coloring of $K_{r,s}$. If $q = 1$, then $r \geq 7$. Let $U = \{u_1, u_2, \ldots, u_r\}$ and observe that the edge coloring $c_2$ given by

$$c_2(e) = \begin{cases} 
2 & \text{if } e \in \{u_1w, u_2w\} \\
1 & \text{if } e = u_iw \ (3 \leq i \leq r) \\
0 & \text{otherwise}
\end{cases}$$

is a modular 3-edge coloring of $K_{r,s}$.

We now turn our attention to trees and show that the modular chromatic index of every tree of order 3 or more is either 2 or 3. Moreover, we characterize all trees whose modular chromatic index is 2 (or is 3).
Theorem 2.2.8 Let $T$ be a tree of order $r + s \geq 3$ whose partite sets have orders $r$ and $s$, respectively. Then

$$
\chi'_m(T) = \begin{cases} 
3 & \text{if } r \text{ and } s \text{ are odd} \\
2 & \text{otherwise}
\end{cases}
$$

Proof. We first show that every nontrivial tree of odd order is modular 2-edge colorable. Assume, to the contrary, that there exists a tree of odd order whose modular chromatic index is greater than 2. Let $T$ be such a tree of the minimum order $r + s$ and suppose that $U$ and $W$ are the partite sets of $T$ with $|U| = r$ and $|W| = s$. It follows by Proposition 2.2.7 that $T$ is not a star and so we may assume that $r + s \geq 5$ where $r \geq 2$ is even and $s \geq 3$ is odd. Also, since $T$ is not a path by Theorem 2.2.6, there are at least three end-vertices, implying that there are two end-vertices $r$ and $y$ belonging to the same partite set. Let $T'$ be the tree obtained from $T$ by deleting $r$ and $y$. Therefore, $\chi'_m(T') = 2$ by assumption and so let $c'$ be a modular 2-edge coloring of $T'$. Furthermore, let $U' \subseteq U$ and $W' \subseteq W$ be the partite sets of $T'$ and observe that $|U'|$ is even while $|W'|$ is odd. Hence, $c'$ assigns colors to the edges of $T'$ so that $s_{c'}(v) = 1$ if and only if $v \in U'$ by Observation 2.2.1. If $r, y \in W$, then the edge coloring $c$ of $T$ given by $c(e) = c'(e)$ if $e \in E(T')$ and $c(e) = 0$ otherwise is a modular 2-edge coloring of $T$, which contradicts our assumption. Thus, we may assume that $r, y \in U$. Let $w_1 \in N(r)$ and $w_2 \in N(y)$ and consider the $w_1 - w_2$ path $P$ in $T'$. (If $d(r, y) = 2$, then $w_1 = w_2$ and so $E(P) = \emptyset$.) We define an edge coloring $c$ of $T$ as follows

$$
c(e) = \begin{cases} 
c'(e) + 1 & \text{if } e \in E(P) \\
1 & \text{if } e \in \{rw_1, yw_2\} \\
c'(e) & \text{otherwise}
\end{cases}\]
We verify that $c$ is a modular 2-edge coloring of $T$. If $v \in V(T') - V(P)$, then $s_c(v) = s_{c'}(v)$; while if $v \in V(P)$, then $s_c(v) = s_{c'}(v) + 2 = s_{c'}(v)$. Hence, $s_c(v) = s_{c'}(v)$ for every $v \in V(T')$, that is, $s_c(v) = 1$ if $v \in U'$ and $s_c(v) = 0$ if $v \in W'$. Since $s_c(x) = s_c(y) = 1$, this is indeed a modular 2-edge coloring of $T$, which is again impossible. Hence, such a tree $T$ does not exist and so $\chi'_m(T) = 2$ if $r + s$ is odd.

Next assume that $r + s \geq 4$ is even. If both $r$ and $s$ are even, then it can be verified that $T$ is modular 2-edge colorable by an argument similar to the one used in the case when $r + s$ is odd. Thus we may assume that both $r$ and $s$ are odd. Let $r + s = 2k$ where $k \geq 2$. We need only verify that $\chi'_m(T) \leq 3$ by Theorem 2.2.5. We proceed by induction on $k$. For $k = 2$, $T = K_{1,3}$ and the result immediately follows by Proposition 2.2.7. Suppose that for some $k \geq 2$ every tree of order $2k$ that is a spanning subgraph of $K_{r,2k-r}$ for some odd integer $r$ (1 $\leq r \leq 2k - 1$) is modular 3-edge colorable. Let $T$ be a tree of order $2(k+1)$ with $T \subseteq K_{r,2(k+1)-r}$ for some odd integer $r$ with $1 \leq r \leq 2(k+1) - 1$. Since $T$ is not a star, let $U$ and $W$ be the partite sets of $T$ such that $|U| = r \geq 3$ and $|W| = 2(k+1) - r \geq 3$. Also, since $T$ is not a path, there exist at least three end-vertices in $T$, two of which belong to the same partite set. We may assume that $x$ and $y$ are end-vertices both belonging to $U$. Also, let $w_1$ and $w_2$ be the vertices in $W$ such that $xw_1, yw_2 \in E(T)$. Consider the tree $T'$ of order $2k$ obtained from $T$ by deleting $x$ and $y$. Then the sets $U' = U - \{x, y\}$ and $W' = W$ are the partite sets of $T'$ and, furthermore, both $|U'|$ and $|W'|$ are odd. Hence, $\chi'_m(T') = 3$ and so let $c' : E(T') \to \mathbb{Z}_3$ be a modular 3-edge coloring of $T'$. We consider the following three cases.
Case 1. \(0 \in \{s_{c'}(v) : v \in U'\}\). Then the edge coloring \(c\) given by \(c(xw_1) = c(yw_2) = 0\) and \(c(e) = c'(e)\) for every \(e \in E(T')\) is a modular 3-edge coloring of \(T\).

Case 2. \(\{s_{c'}(v) : v \in W'\} = \{0\}\). Note that \(d(x, y) = d\) is a positive even integer. Let \(P = (w_1 = v_1, v_2, \ldots, v_{d-1} = w_2)\) be the \(w_1 - w_2\) path in \(T'\). (If \(d = 2\), then \(w_1 = w_2\) and \(E(P) = \emptyset\).) Therefore, \(v_i \in W'\) if \(i\) is odd and \(v_i \in U'\) if \(i\) is even. Define an edge coloring \(c\) of \(T\) by

\[
c(e) = \begin{cases} 
  c'(e) + 1 & \text{if } e = v_iv_{i+1} \in E(P) \text{ and } i \text{ is odd} \\
  c'(e) + 2 & \text{if } e = v_iv_{i+1} \in E(P) \text{ and } i \text{ is even} \\
  2 & \text{if } e = xw_1 \\
  1 & \text{if } e = yw_2 \\
  c'(e) & \text{otherwise.}
\end{cases}
\]

To verify that \(c\) is a modular 3-edge coloring of \(T\), first observe that \(s_{c'}(v) = s_{c'}(v)\) for every \(v \in V(T') - V(P)\). Also, \(s_{c'}(v) = s_{c'}(v) + 3 = s_{c'}(v)\) for every \(v \in V(P)\). In particular, \(s_{c}(w_1) = s_{c}(w_2) = 0\). Thus, \(s_{c}(x) = 1 \neq s_{c}(w_1)\) and \(s_{c}(y) = 2 \neq s_{c}(w_2)\), implying that \(c\) is indeed a modular 3-edge coloring of \(T\).

Case 3. \(\{s_{c'}(v) : v \in U'\} = \{A\}\) and \(B \in \{s_{c'}(v) : v \in W'\}\) where \(\{A, B\} = \{1, 2\}\). We consider three subcases.

Subcase 3.1. \(d(x, y) = d \geq 4\). If \(s_{c'}(w_1) = s_{c'}(w_2) = B\), then let \(c\) be an edge coloring of \(T\) such that \(c(xw_1) = c(yw_2) = A\) and \(c(e) = c'(e)\) for every \(e \in E(T')\) and observe that \(c\) is a modular 3-edge coloring of \(T'\).
If $s_{c'}(w_1) = 0$ or $s_{c'}(w_2) = 0$, say the former, then let $P = (w_1 = v_1, v_2, \ldots, v_{d-1} = w_2)$ be the $w_1 - w_2$ path in $T'$ and define an edge coloring $c$ of $T$ by

$$c(e) = \begin{cases} 
\frac{d'(e) + A}{2} & \text{if } e = v_iv_{i+1} \in E(P) \text{ and } i \text{ is odd} \\
\frac{d'(e) + B}{2} & \text{if } e = v_iv_{i+1} \in E(P) \text{ and } i \text{ is even} \\
A & \text{if } e \in \{xw_1, yw_2\} \\
c'(e) & \text{otherwise.}
\end{cases}$$

Then

$$s_{c'}(v) = \begin{cases} 
A & \text{if } v \in \{x, y\} \\
2A - B & \text{if } v = w_1 \\
s_{c'}(v) & \text{otherwise}
\end{cases}$$

and it is straightforward to verify that this is a modular 3-edge coloring of $T$.

**Subcase 3.2.** $d(x, y) = 2$. Let $w_1 = w_2 = w$. If $s_{c'}(w) = 0$, then let $c$ be an edge coloring such that $c(xw) = c(yw) = A$ and $c(e) = c'(e)$ for every $e \in E(T')$ and observe that this is a modular 3-edge coloring of $T$.

Hence, suppose finally that $s_{c'}(w) = B$. Since $T$ is not a star, there exists an end-vertex $z$ in $T$ such that $d(x, z) \geq 3$. Let $P = (w = v_1, v_2, \ldots, v_d = z)$ be the $w - z$ path in $T'$, where $d = d(x, z)$.

**Subcase 3.2.1.** $d$ is odd. Then $z \in W$ and so $s_{c'}(z) \in \{0, B\}$. Then the edge coloring $c$ defined by

$$c(e) = \begin{cases} 
\frac{c'(e) - s_{c'}(z) + A}{2} & \text{if } e = v_iv_{i+1} \in E(P) \text{ and } i \text{ is odd} \\
\frac{c'(e) + s_{c'}(z) + B}{2} & \text{if } e = v_iv_{i+1} \in E(P) \text{ and } i \text{ is even} \\
A & \text{if } e \in \{xw, yw\} \\
c'(e) & \text{otherwise}
\end{cases}$$
is a modular 3-edge coloring of $T$, since $s_c(z) \in \{0, B\}$ and

$$s_c(v) = \begin{cases} 
  A & \text{if } v \in \{x, y\} \\
  2s_c(z) + B & \text{if } v = z \\
  B - s_c(z) \in \{0, B\} & \text{if } v = w \\
  s_c(v) & \text{otherwise.}
\end{cases}$$

Subcase 3.2.2. $d$ is even. Then $z \in U$ and so $s_c(z) = A$. Let $w_3$ be the neighbor of $z$ in $T$, that is, $w_3 = v_{d-1}$. Then consider the edge coloring $c$ defined by

$$c(e) = \begin{cases} 
  d'(e) - s_c(w_3) + A & \text{if } e = v_i v_{i+1} \in E(P) \text{ and } i \text{ is odd and } i \neq d - 1 \\
  d'(e) + s_c(w_3) + B & \text{if } e = v_i v_{i+1} \in E(P) \text{ and } i \text{ is even} \\
  A & \text{if } e \in \{xw, yw, zw_3\} \\
  d'(e) & \text{otherwise}
\end{cases}$$

and one can verify that $c$ is a modular 3-edge coloring of $T$. 

With the aid of Theorem 2.2.8, we are now able to classify all connected bipartite graphs according to their modular chromatic indices.

**Theorem 2.2.9** If $G$ is a connected bipartite graph of order $r + s \geq 3$ such that $G \subseteq K_{r,s}$, then

$$\chi'_m(G) = \begin{cases} 
  3 & \text{if } r \text{ and } s \text{ are odd} \\
  2 & \text{otherwise.}
\end{cases}$$

**Proof.** If $G$ is a tree, then the result clearly holds by Theorem 2.2.8. If $G$ is not a tree, then let $T$ be a spanning tree of $G$ and observe that $T \subseteq K_{r,s}$. Let $c_T$ be a modular edge coloring of $T$ and define an edge coloring $c$ of $G$ by $c(e) = c_T(e)$ if $e \in E(T)$ and $c(e) = 0$ otherwise. Then $s_c(v) = s_{c_T}(v)$ for every vertex $v$ in $G.$
Therefore, every modular edge coloring of $T$ induces a modular edge coloring of $G$ using the same number of colors, which implies that $\chi'_m(G) \leq \chi'_m(T)$. The result now follows by Theorems 2.2.5 and 2.2.8.

If $H$ is any connected bipartite graph each of whose partite sets contains an odd number of vertices, then $\chi'_m(H) = 3 > \chi(H)$ by Theorem 2.2.9. If $G$ is a graph such that $H \subseteq G \subseteq \text{cor}(H)$ with an odd number of pendant edges or $G = H \times K_2$, then $G$ is also bipartite and $\chi'_m(G) = 2$ by Theorem 2.2.9. This provides us with another well-known class of graphs $G$ containing a subgraph $H$ such that $\chi'_m(H) > \chi'_m(G)$.

### 2.3 Chromatic Number and Modular Chromatic Index

For every graph $G$ we have encountered so far, either $\chi'_m(G) = \chi(G)$ or $\chi'_m(G) = \chi(G) + 1$. This gives rise to the following question.

**Question 2.3.1** For a connected graph $G$ of order at least 3, is it true that $\chi'_m(G) \leq \chi(G) + 1$?

Furthermore, for every graph $G$ we have encountered for which $\chi'_m(G) = \chi(G) + 1$, the order of $G$ is even, $\chi(G) \equiv 2 \pmod{4}$ and every proper $\chi(G)$-coloring of $G$ results in color classes of odd size. Thus we have another question.

**Question 2.3.2** Let $G$ be a connected graph of order $n \geq 3$ with $\chi'_m(G) = \chi(G) + 1$. Is it true that $n$ is even, $\chi(G) \equiv 2 \pmod{4}$ and every proper $\chi(G)$-coloring of
We have seen that each of Questions 2.3.1 and 2.3.2 has an affirmative answer when $G$ is a complete graph, a cycle or a bipartite graph. This is, in fact, true for all connected graphs of order at least 3. In this section, we present an affirmative answer to Question 2.3.1 and we will present an affirmative answer to Question 2.3.2 in the next section. By Theorem 2.2.9, if $G$ is a connected bipartite graph of order at least 3, then $\chi'_m(G) \leq \chi(G) + 1$. Thus we need only consider connected graphs that are not bipartite. For an integer $k \geq 2$, a graph $G$ is modular $k$-edge colorable if there is a modular $k$-edge coloring of $G$. It is clear that if $G$ is a $k$ chromatic graph of order $n$, then a proper $k$-coloring of $G$ can induce a proper $k'$-coloring of $G$ for each integer $k'$ with $k \leq k' \leq n$ by introducing a new color to a vertex of $G$.

**Theorem 2.3.3** Let $G$ be a connected graph that is not bipartite. For a positive integer $k$, if $G$ is $(2k + 1)$ colorable, then $G$ is modular $(2k + 1)$-edge colorable. Furthermore, for a given proper $(2k+1)$-vertex coloring $c : V(G) \to \{1, 2, \ldots, 2k+1\}$, there is a modular $(2k + 1)$-edge coloring $c : E(G) \to \mathbb{Z}_{2k+1}$ such that $s_c(v) = c'(v)$ for every $v \in V(G)$.

**Proof.** Let $V(G) = \{v_1, v_2, \ldots, v_n\}$, where $n \geq 3$ and let $c : V(G) \to \{1, 2, \ldots, 2k+1\}$ be a proper $(2k + 1)$-vertex coloring of $G$. We define recursively a sequence of $n + 1$ edge colorings $c_0, c_1, \ldots, c_n$, where $c_i : E(G) \to \mathbb{Z}_{2k+1}$ for $0 \leq i \leq n$ such that (i) if $i = 0$, then $s_{c_0}(v) = 0$ for $v \in V(G)$ and (ii) if $1 \leq i \leq n$, then $s_{c_i}(v_j) = c'(v_j)$.
for $1 \leq j \leq i$ and $s_{c_i}(v) = 0$ for $v \in V(G) - \{v_1, v_2, \ldots, v_i\}$. This will implies that $c = c_n$ is a modular $(2k + 1)$-edge coloring with $s_c(v) = c'(v)$ for every $v \in V(G)$.

First, we define the edge coloring $c_0 : E(G) \to \mathbb{Z}_{2k+1}$ by $c_0(e) = 0$ for all $e \in E(G)$. Thus $s_{c_0}(v) = 0$ for $v \in V(G)$. Next, we define the edge coloring $c_1 : E(G) \to \mathbb{Z}_{2k+1}$ of $G$ from $c_0$ such that $s_{c_1}(v_1) = c'(v_1)$ and $s_{c_1}(v) = 0$ if $v \in V(G) - \{v_1\}$. Suppose that $c'(v_1) = a$. Since $\gcd(2, 2k + 1) = 1$, it follows that $2 \mid a$ in $\mathbb{Z}_{2k+1}$ and so $a = 2b$ for some $b \in \mathbb{Z}_{2k+1}$. We consider two cases.

Case 1. $v_1$ lies on an odd cycle $C$ of $G$. Let $C = (v_1, u_1, u_2, \ldots, u_p, u_{p+1} = u_1)$ where $p \geq 3$ is an odd integer. The coloring $c_1 : E(G) \to \mathbb{Z}_{2k+1}$ is defined by

$$c_1(e) = \begin{cases} c_0(e) & \text{if } e \notin E(C) \\ c_0(e) + b & \text{if } e = u_iu_{i+1}, \text{ } i \text{ is odd and } 1 \leq i \leq p \\ c_0(e) - b & \text{if } e = u_iu_{i+1}, \text{ } i \text{ is even and } 2 \leq i \leq p - 1. \end{cases} \quad (2.2)$$

Thus $s_{c_1}(v_1) = 2b = a = c'(v_1)$ and $s_{c_1}(v_i) = s_{c_0}(v_i) = 0$ for $2 \leq i \leq n$.

Case 2. $v_1$ lies on no odd cycle of $G$. Since $G$ is connected, there is a path $P$ joining $v_1$ and a vertex on an odd cycle $C$ of $G$. Suppose that $C = (u_1, u_2, \ldots, u_p, u_{p+1} = u_1)$ in $G$. We may assume, without loss of generality that $P = (v_1 = w_1, w_2, \ldots, w_t = u_1)$ such that $u_1$ is the only vertex on $P$ that belongs to $C$. We consider two subcases, according to whether $t$ is even or $t$ is odd.
Subcase 2.1. $t$ is even. The coloring $c_1 : E(G) \to \mathbb{Z}_{2k+1}$ is defined by

$$c_1(e) = \begin{cases} c_0(e) & \text{if } e \notin E(C) \cup E(P) \\ c_0(e) + a & \text{if } e = w_jw_{j+1}, j \text{ is odd and } 1 \leq j \leq t - 1 \\ c_0(e) - a & \text{if } e = w_jw_{j+1}, j \text{ is even and } 2 \leq j \leq t - 2 \\ c_0(e) - b & \text{if } e = u_iu_{i+1}, i \text{ is odd and } 1 \leq i \leq p \\ c_0(e) + b & \text{if } e = u_iu_{i+1}, i \text{ is even and } 2 \leq i \leq p - 1 \end{cases} \quad (2.3)$$

Thus $s_{c_1}(v_1) = 2b = a = c'(v_1)$ and $s_{c_1}(v_i) = s_{c_0}(v_i) = 0$ for $2 \leq i \leq n$. Note that $s_{c_1}(u_1) = a - 2b = 0$.

Subcase 2.2. $t$ is odd. The coloring $c_1 : E(G) \to \mathbb{Z}_{2k+1}$ is defined by

$$c_1(e) = \begin{cases} c_0(e) & \text{if } e \notin E(C) \cup E(P) \\ c_0(e) + a & \text{if } e = w_jw_{j+1}, j \text{ is odd and } 1 \leq j \leq t - 1 \\ c_0(e) - a & \text{if } e = w_jw_{j+1}, j \text{ is even and } 2 \leq j \leq t - 2 \\ c_0(e) + b & \text{if } e = u_iu_{i+1}, i \text{ is odd and } 1 \leq i \leq p \\ c_0(e) - b & \text{if } e = u_iu_{i+1}, i \text{ is even and } 2 \leq i \leq p - 1 \end{cases} \quad (2.4)$$

Thus $s_{c_1}(v_1) = 2b = a = c'(v_1)$ and $s_{c_1}(v_i) = s_{c_0}(v_i) = 0$ for $2 \leq i \leq n$. Note that $s_{c_1}(u_1) = -a + 2b = 0$.

In each case, $s_{c_1}(v_1) = c'(v_1)$ and $s_{c_1}(v) = s_{c_0}(v) = 0$ for all $v \in V(G) - \{v_1\}$. (The coloring $c_1$ is neither a proper edge coloring nor a modular edge coloring of $G$.) In general, for an integer $i$ with $1 \leq i \leq n - 1$, suppose that the coloring $c_i : E(G) \to \mathbb{Z}_{2k+1}$ is defined such that $s_{c_i}(v_j) = c'(v_j)$ for $1 \leq j \leq i$ and $s_{c_i}(v) = 0$ for all $v \in V(G) - \{v_1, v_2, \ldots, v_i\}$. Then the coloring $c_{i+1} : E(G) \to \mathbb{Z}_{2k+1}$ is defined from $c_i$ in the same fashion as described in (2.2) – (2.3), namely by replacing $c_0$ and $c_1$ in (2.2) – (2.3) by $c_i$ and $c_{i+1}$, respectively. An argument similar to the one
used in the case dealing with $c_1$ and $c_0$ shows that $s_{c_{i+1}}(v_j) = s_c(v_j) = c'(v_j)$ for $1 \leq j \leq i$, $s_{c_{i+1}}(v_{i+1}) = c'(v_{i+1})$ and $s_{c_{i+1}}(v) = 0$ for $v \in V(G) - \{v_1, v_2, \ldots, v_{i+1}\}$. In particular, $c_n : E(G) \to \mathbb{Z}_{2k+1}$ has the property that $s_{c_n}(v_i) = s_{c_{n-1}}(v_i) = c'(v_i)$ for $1 \leq i \leq n - 1$ and $s_{c_n}(v_n) = c'(v_n)$. Therefore, $c_n$ is a modular $(2k + 1)$-edge coloring of $G$ and so $G$ is modular $(2k + 1)$-edge colorable.

The following corollaries are consequences of Theorems 2.1.2 and 2.3.3.

**Corollary 2.3.4** If $G$ is a connected graph of order at least 3, then

$$\chi(G) \leq \chi_m'(G) \leq \chi(G) + 1.$$  

Furthermore, if $\chi(G)$ is odd, then $\chi_m'(G) = \chi(G)$.

**Proof.** We have seen that $\chi_m'(G) \geq \chi(G)$ in Theorem 2.1.2. For the upper bound, let $G$ be a connected graph of order $n \geq 3$. If $G = K_n$, then the result follows by Theorem 2.2.2. Thus, we may assume that $G \neq K_n$ and $\chi(G) = \chi \leq n - 1$. If $\chi$ is even, then $G$ is $(\chi + 1)$-colorable and so $G$ is modular $(\chi + 1)$-edge colorable by Theorem 2.3.3. Thus $\chi_m'(G) \leq \chi(G) + 1$. If $\chi$ is even, then $G$ is $\chi$-colorable and so $G$ is modular $\chi$-edge colorable by Theorem 2.3.3. Therefore, $\chi_m'(G) \leq \chi(G)$ and so $\chi_m'(G) = \chi(G)$.

By Corollary 2.3.4, if $G$ is a connected graph of order at least 3 such that $\chi_m'(G) = \chi(G) + 1$, then $\chi(G)$ is even and so $\chi(G) \equiv 0 \pmod{4}$ or $\chi(G) \equiv 2 \pmod{4}$. By Theorem 2.1.2 and Corollary 2.3.4, we have the following result for connected graphs $G$ with $\chi(G) \equiv 2 \pmod{4}$. 


Corollary 2.3.5  Suppose that $G$ is a connected graph of order at least 3 such that $\chi(G) \equiv 2 \pmod{4}$. If each color class in every proper $\chi(G)$-coloring of $G$ consists of an odd number of vertices, then $\chi'_m(G) = \chi(G) + 1$.

2.4  A Characterization of Type 1 Graphs

As a consequence of Corollary 2.3.4, the modular chromatic index $\chi'_m(G)$ of a graph $G$ is either $\chi(G)$ or $\chi(G) + 1$. Graphs $G$ for which $\chi'_m(G) = \chi(G)$ are called type 0 graphs and graphs $G$ for which $\chi'_m(G) = \chi(G) + 1$ are called type 1 graphs. So every connected graph of order at least 3 is either type 0 or type 1. This gives rise to a natural question: Which graphs are type 0 and which graphs are type 1? By Theorems 2.2.2, 2.2.4 and 2.2.9 and Corollary 2.3.4, if $G = K_n$ or $G = C_n$, then $G$ is type 1 if and only if $n \equiv 2 \pmod{4}$, while if $G$ is bipartite, then $G$ is type 1 if and only if each partite set of $G$ has an odd number of vertices. Furthermore, if $G$ is type 1, then $\chi(G)$ must be even. In this section, we characterize all connected type 1 graphs. In order to do this, we first determine all type 1 complete multipartite graphs.

2.4.1  Complete Multipartite Graphs

For positive integers $n_1, n_2, \ldots, n_\ell$ ($\ell \geq 2$), let $G = K_{n_1, n_2, \ldots, n_\ell}$ be a complete $\ell$-partite graph of order $n_1 + n_2 + \cdots + n_\ell$ whose partite sets are $V_1, V_2, \ldots, V_\ell$ where $|V_i| = n_i$ for $1 \leq i \leq \ell$. If $\ell = 2$ or $\ell$ is odd, then $\chi'_m(G)$ is determined by Theorem 2.2.9 and Corollary 2.3.4. Thus, we may assume that $\ell \geq 4$ is an even integer. For even integers $\ell \geq 4$, we first determine a class of complete $\ell$-partite
graphs $G$ for which $\chi'_m(G) = \chi(G)$.

**Theorem 2.4.1** Let $G = K_{n_1, n_2, \ldots, n_\ell}$ be a complete $\ell$-partite graph where $\ell \geq 4$ is even. If there exists a set $S \subseteq \{n_1, n_2, \ldots, n_\ell\}$ such that $|S| = \ell/2$ and the sum of the integers in $S$ is even, then $\chi'_m(G) = \chi(G)$.

**Proof.** Let $V_1, V_2, \ldots, V_\ell$ be the partite sets of $G$ where $|V_i| = n_i$ for $1 \leq i \leq \ell$. We may assume that $S = \{n_1, n_2, \ldots, n_{\ell/2}\}$ and $n_1 + n_2 + \cdots + n_{\ell/2}$ is even. Furthermore, let $V = \bigcup_{i=1}^{\ell/2} V_i$. Let $u \in V_\ell$ and define an edge coloring $c_0 : E(G) \to \mathbb{Z}_\ell$ by

$$c_0(e) = \begin{cases} 1 & \text{if } e = uv \text{ where } v \in V \\ 0 & \text{otherwise.} \end{cases}$$

Then $s_{c_0}(v) = 1$ if $v \in V$, $s_{c_0}(v) = 0$ if $v \in V(G) - (V \cup \{u\})$ and $s_{c_0}(u) \equiv 0 \pmod{2}$.

Let $v_1, v_2, \ldots, v_n = u$ be an ordering of the vertices of $G$. We define a sequence $c_1, c_2, \ldots, c_n$ of edge colorings of $G$ recursively such that for each $i$ with $1 \leq i \leq n$ the edge coloring $c_i : E(G) \to \mathbb{Z}_\ell$ induces a vertex coloring $s_{c_i} : V(G) \to \mathbb{Z}_\ell$ for which

$$s_{c_i}(v_i) = \begin{cases} 2j - 1 & \text{if } v_i \in V_j \subseteq V \\ 2j - 2 & \text{if } v_i \in V_j \subseteq V(G) - V \end{cases} \quad (2.5)$$

$$s_{c_i}(v) = s_{c_{i-1}}(v) \quad \text{if } v \in V(G) - \{v_i\}. \quad (2.6)$$

We begin with the coloring $c_1$. Suppose that $v_1 \in V_j$ for some $j$ with $1 \leq j \leq \ell$. Since $\ell \geq 4$, the vertex $v_1$ lies on a triangle $C$ in $G$, say $C = (v_1, u_1, w_1, v_1)$. Define $c_1 : E(G) \to \mathbb{Z}_\ell$ by

$$c_1(e) = \begin{cases} c_0(e) & \text{if } e \notin E(C) \\ c_0(e) + (j - 1) & \text{if } e \in \{v_1u_1, v_1w_1\} \\ c_0(e) - (j - 1) & \text{if } e = u_1w_1. \end{cases}$$
If \( v_i \in V \), then \( s_{c_0}(v_1) = 1 \); while if \( v_i \in V(G) - V \), then \( s_{c_0}(v_1) = 0 \). This implies that

\[
s_{c_1}(v_1) = \begin{cases} 
1 + 2(j - 1) = 2j - 1 & \text{if } v_1 \in V_j \subseteq V \\
0 + 2(j - 1) = 2j - 2 & \text{if } v_1 \in V_j \subseteq V(G) - V 
\end{cases}
\]

\[
s_{c_1}(v) = s_{c_0}(v) \quad \text{if } v \in V(G) - \{v_1\}.
\]

Thus \( s_{c_1} \) satisfies (2.5) and (2.6).

For an integer \( i \) with \( 1 \leq i \leq n - 1 \), suppose that the edge colorings \( c_1, c_2, \ldots, c_i \) have been defined, all of which satisfy (2.5) and (2.6). We now define the coloring \( c_{i+1} \) from \( c_i \) in the same fashion as we defined \( c_1 \) from \( c_0 \). More precisely, assume that \( v_{i+1} \in V_j \) for some \( j \) with \( 1 \leq j \leq \ell \) and that \( v_{i+1} \) lies on a triangle \( C = (v_{i+1}, u_{i+1}, w_{i+1}, v_{i+1}) \). Then define \( c_{i+1} : E(G) \rightarrow \mathbb{Z}_\ell \) by

\[
c_{i+1}(e) = \begin{cases} 
c_i(e) & \text{if } e \notin E(C) \\
c_i(e) + (j - 1) & \text{if } e \in \{v_{i+1}u_{i+1}, v_{i+1}w_{i+1}\} \\
c_i(e) - (j - 1) & \text{if } e = u_{i+1}w_{i+1}.
\end{cases}
\]

We now consider the induced vertex coloring \( s_{c_i} : V(G) \rightarrow \mathbb{Z}_\ell \). Since \( s_{c_i}(v_{i+1}) = s_{c_{i-1}}(v_{i+1}) = \cdots = s_{c_0}(v_{i+1}) \), it follows that \( s_{c_i}(v_{i+1}) = 1 \) if \( v_{i+1} \in V \); while \( s_{c_i}(v_{i+1}) = 0 \) if \( v_{i+1} \in V(G) - V \). Therefore,

\[
s_{c_{i+1}}(v_{i+1}) = \begin{cases} 
1 + 2(j - 1) = 2j - 1 & \text{if } v_{i+1} \in V_j \subseteq V \\
0 + 2(j - 1) = 2j - 2 & \text{if } v_{i+1} \in V_j \subseteq V(G) - V 
\end{cases}
\]

\[
s_{c_{i+1}}(v) = s_{c_0}(v) \quad \text{if } v \in V(G) - \{v_{i+1}\}.
\]

Thus \( s_{c_{i+1}} \) satisfies (2.5) and (2.6). Continuing in this manner, we obtain the edge coloring \( c_n : E(G) \rightarrow \mathbb{Z}_\ell \) that induces a vertex coloring \( s_{c_n} : V(G) \rightarrow \mathbb{Z}_\ell \) such that

\[
s_{c_n}(v_i) = \begin{cases} 
2j - 1 & \text{if } v_{i+1} \in V_j \subseteq V \\
2j - 2 & \text{if } v_{i+1} \in V_j \subseteq V(G) - V.
\end{cases}
\]
This implies that $s_{c_n}$ is a proper vertex $\ell$-coloring of $G$ using the $\ell$ colors in $\mathbb{Z}_\ell$. Thus $c_n$ is a modular $\ell$-edge coloring of $G$. Therefore, $\chi'_m(G) \leq \ell$ and so $\chi'_m(G) = \ell = \chi(G)$. 

We are now prepared to determine all complete multipartite graphs that are type 1.

**Theorem 2.4.2** Let $G = K_{n_1, n_2, \ldots, n_\ell}$ be a complete $\ell$-partite graph where $\ell \geq 2$. Then $\chi'_m(G) = \chi(G) + 1$ if and only if $\ell \equiv 2 \pmod{4}$ and each $n_i$ is odd for $1 \leq i \leq \ell$.

**Proof.** If $\ell \equiv 2 \pmod{4}$ and each $n_i$ is odd for $1 \leq i \leq \ell$, then $\chi'_m(G) = \chi(G) + 1$ by Corollary 2.3.5. It remains to verify the converse. If $\ell$ is odd, then $\chi'_m(G) = \chi(G)$ by Corollary 2.3.4. Thus, we may assume that $\ell$ is even and so either $\ell \equiv 0 \pmod{4}$ or $\ell \equiv 2 \pmod{4}$. Consider the set $N = \{n_1, n_2, \ldots, n_\ell\}$. If $N$ contains an even integer or $\ell \equiv 0 \pmod{4}$, then there is a subset $S \subseteq N$ with $|S| = \ell/2$ such that the sum of the integers in $S$ is even. It then follows by Theorem 2.4.1 that $\chi'_m(G) = \chi(G)$. 

**2.4.2 Type 1 Graphs**

If $G$ is a connected graph with $\chi(G) = \ell$, then the vertex set of $G$ can be partitioned into $\ell$ independent sets $V_1, V_2, \ldots, V_\ell$, where say $|V_i| = n_i$ for $1 \leq i \leq \ell$. Thus, $G$ is a subgraph of $K_{n_1, n_2, \ldots, n_\ell}$. 

Lemma 2.4.3 For an integer \( \ell \geq 4 \), let \( K_{n_1, n_2, \ldots, n_\ell} \) be a complete \( \ell \)-partite graph whose partite sets are \( V_1, V_2, \ldots, V_\ell \) with \( |V_i| = n_i \) for \( 1 \leq i \leq \ell \). If \( G \) is a connected graph with \( \chi(G) = \ell \geq 4 \) such that (i) \( G \subseteq K_{n_1, n_2, \ldots, n_\ell} \) and (ii) for each pair \( u, v \) of vertices of \( G \) where \( u \in V_i, v \in V_j \) and \( i \neq j \), we have \( uv \in E(G) \) whenever \( G \) contains a \( u-v \) path of odd length, then \( G = K_{n_1, n_2, \ldots, n_\ell} \).

Proof. For each vertex \( v \) of \( G \), where say \( v \in V_i \) for some \( i \) with \( 1 \leq i \leq \ell \), we show that \( v \) is adjacent to every vertex in \( V(G) - V_i \). Without loss of generality, suppose that \( v \in V_1 \).

We first claim that \( d_G(u, v) \leq 2 \) for every \( u \in V(G) - V_1 \). If this is not the case, then suppose that \( u \in V_2 \) and \( d_G(v, u) = d \geq 3 \). Let \((u = u_0, v_1, v_2, \ldots, v_d = v)\) be a \( u-v \) geodesic in \( G \). If \( v_3 \notin V_2 \), then there is a \( u-v_3 \) path of length 3 and so \( uv_3 \in E(G) \), creating a \( u-v \) path of length \( d - 2 \). Hence, \( d \geq 4 \) and \( u, v_3 \in V_2 \) while \( v_1, v_2 \notin V_2 \). If \( d \geq 5 \), then it can be similarly shown that \( v_5 \in V_2 \) since \( uv_5 \notin E(G) \). However then, \( v_2v_5 \in E(G) \), which cannot occur. Therefore, \( d = 4 \) and \( v_1, v \in V_1 \). Assume further that \( v_2 \in V_3 \), say. Now consider the partite set \( V_4 \). If \( w \in V_4 \) and is adjacent to \( v_2 \), then \( w \) must be adjacent to both \( u \) and \( v \) since there is a path of length 3 from \( w \) to each of \( u \) and \( v \). However, this creates a \( u-v \) path of length 2, which is impossible. Therefore, we may assume that there is a vertex \( w' \in V_3 - \{v_2\} \) such that \( uw' \in E(G) \) or otherwise \( V_3 \cup V_4 \) is independent. Now, since \( G \) is connected, let \( P \) be a \( u-w \) path of length \( t \). If \( t \) is odd, then \( uw \in E(G) \) and so there is a \( v-w \) path of length 5. On the other hand, if \( t \) is even, then there exists a \( u-w' \) path of length either \( t - 1 \) or \( t + 1 \) and so
$uw' \in E(G)$, which in turn implies that there is a $v - w'$ path of length 5. However then, either $w$ or $w'$ is adjacent to both $u$ and $v$, contradicting the assumption that $d_G(u, v) = 4$. Thus $d_G(u, v) \leq 2$, as claimed.

Let

$$X = \{x \in V(G) - V_1 : d_G(v, x) = 1\} = N(v)$$

$$Y = \{y \in V(G) - V_1 : d_G(v, y) = 2\}.$$  

We show that $Y = \emptyset$. Assume, to the contrary, that $Y \neq \emptyset$. If $Y$ is not independent, say $y, y' \in Y$ and $yy' \in E(G)$, then there is either a $v - y$ path or a $v - y'$ path of length 3, which cannot occur since $vy, vy' \notin E(G)$. Therefore, $Y$ is an independent set. Then $X$ cannot be independent since $\{V_1, X, Y\}$ is a partition of $V(G)$ and $\chi(G) \geq 4$. Let $x, x' \in X$ and $xx' \in E(G)$. Without loss of generality, we may assume that $x \in V_2$ and $x' \in V_3$. Let $y \in Y$. If $y$ is adjacent to either $x$ or $x'$, then there is a $v - y$ path of length 3, which is again a contradiction. Therefore, $yx, yx' \notin E(G)$. Then there exists a vertex $x'' \in X - \{x, x'\}$ so that $(v, x'', y)$ is a $v - y$ geodesic. However, this implies that $xy \in E(G)$ if $y \notin V_2$ and $x'y \in E(G)$ otherwise, neither of which can occur. Thus $Y = \emptyset$ and $v$ is adjacent to every vertex in $V(G) - V_1$. This completes the proof. 

Recall that for an integer $k \geq 2$, a graph $G$ is modular $k$-edge colorable if $G$ has a modular $k$-edge coloring.

**Lemma 2.4.4** Let $G$ be a connected graph of order at least 3 containing two nonadjacent vertices $u$ and $v$ that are connected by a path of odd length. Let $k \geq 2$
be an integer. Then $G + uv$ is modular $k$-edge colorable if and only if there is a modular $k$-edge coloring of $G$ with respect to which $s(u) \neq s(v)$

**Proof.** If there is a modular $k$-edge coloring $c : E(G) \to \mathbb{Z}_k$ such that $s_c(u) \neq s_c(v)$, then the coloring $c' : E(G + uv) \to \mathbb{Z}_k$ defined by $c'(uv) = 0$ and $c'(e) = c(e)$ for $e \in E(G)$ is a modular $k$-edge coloring of $G + uv$. Thus $G + uv$ is modular $k$-edge colorable.

For the converse, assume that $G + uv$ is modular $k$-edge colorable and let $c$ be a modular $k$-edge coloring of $G + uv$. Suppose that $P$ is a $u - v$ path of odd length in $G$, say $P = (u = v_1, v_2, \ldots, v_p = v)$ where $p \geq 4$ is even. Then the $k$-edge coloring $c'$ of $G$ defined by

$$c'(e) = \begin{cases} c(e) & \text{if } e \notin E(P) \\ c(e) + c(uv) & \text{if } e = v_i v_{i+1}, 1 \leq i \leq p - 1 \text{ and } i \text{ is odd} \\ c(e) - c(uv) & \text{if } e = v_i v_{i+1}, 2 \leq i \leq p - 2 \text{ and } i \text{ is even} \end{cases}$$

is a modular $k$-edge coloring of $G$ with the property that $s_{c'}(u) \neq s_{c'}(v)$.

The following is an immediate consequence of Lemma 2.4.4.

**Corollary 2.4.5** Let $G$ be a connected graph of order at least 3 containing two nonadjacent vertices $u$ and $v$ that are connected by a path of odd length. If $G + uv$ is modular $k$-edge colorable for some integer $k \geq 2$, then $G$ is also modular $k$-edge colorable.

By Corollary 2.4.5, if $G$ is a connected graph of order at least 3 containing two nonadjacent vertices $u$ and $v$ that are connected by a path of odd length,
then the fact that $G + uv$ is modular $k$-edge colorable implies that $G$ is modular $k$-edge colorable. This motivates the next definition. Let $G$ be a connected graph with $\chi(G) = \ell \geq 4$. Suppose that the vertex set of $G$ can be partitioned into $\ell$ independent sets $V_1, V_2, \ldots, V_\ell$ with $|V_i| = n_i$ for $1 \leq i \leq \ell$. Then $G$ is a subgraph of a complete $\ell$-partite graph $K = K_{n_1,n_2,\ldots,n_\ell}$. Define the odd path closure of $G$ with respect to $K$, denoted by $C(G; K)$ (or $C_o(G)$), to be the graph obtained from $G$ by recursively joining pairs of nonadjacent vertices that belong to different partite sets in $K$ and are connected by a path of odd length in $G$. Thus, we have the following result as a consequence of Lemma 2.4.3 and Corollary 2.4.5.

**Corollary 2.4.6** Let $G$ be a connected graph of order at least 3 with $\chi(G) = \ell \geq 4$. Then $G \subseteq K$ for some complete $\ell$-partite graph $K$ and $\chi'_m(G) \leq \chi'_m(C(G; K)) = \chi'_m(K)$.

We are now prepared to present a characterization of type 1 graphs, which provides an affirmative answer to Question 2.3.2.

**Theorem 2.4.7** Let $G$ be a connected graph of order at least 3. Then $\chi'_m(G) = \chi(G) + 1$ if and only if $\chi(G) \equiv 2 \pmod{4}$ and every proper $\chi(G)$-coloring of $G$ results in color classes of odd size.

**Proof.** By Corollary 2.3.5, if $\chi(G) \equiv 2 \pmod{4}$ and every proper $\chi(G)$-coloring of $G$ results in color classes of odd size, then $\chi'_m(G) = \chi(G) + 1$. Thus, it remains to verify the converse. Let $\chi(G) = \ell$ and suppose that $\chi'_m(G) = \ell + 1$. Consider a proper $\ell$-coloring of $G$ and let $V_1, V_2, \ldots, V_\ell$ be the resulting color classes, where
say $|V_i| = n_i$ for $1 \leq i \leq \ell$. Then $G$ is a subgraph of $K_{n_1,n_2, \ldots, n_\ell}$ and so $\ell + 1 = \chi'_m(G) \leq \chi'_m(K_{n_1,n_2, \ldots, n_\ell}) \leq \ell + 1$ by Corollaries 2.3.4 and 2.4.6. The result now follows by Theorem 2.4.2.

### 2.5 Modular $k$-Colorable Graphs

For a positive integer $k$, a graph $G$ is *$k$-colorable* if there is a proper coloring of $G$ using $k$ colors. Recall that for an integer $k \geq 2$, a graph $G$ is *modular $k$-edge colorable* if there is a modular $k$-edge coloring of $G$. It is clear that if $G$ is a $k$-chromatic graph of order $n$, then a proper $k$-coloring of $G$ can induce a proper $k'$-coloring of $G$ for each integer $k'$ with $k \leq k' \leq n$ by introducing a new color to a vertex of $G$. Therefore, it is easy to see every graph $G$ of order $n$ is $k$-colorable for all $k$ with $\chi(G) \leq k \leq n$. In the case of modular edge colorings, the situation is quite different, that is, for positive integers $k$ and $k'$ where $k' > k$, a modular $k$-edge coloring of a graph $G$ may not induce a modular $k'$-edge coloring of $G$ by introducing a new color to the edge of $G$. For example, consider the stars $K_{1,4}$ and $K_{1,6}$ of orders 5 and 7, respectively. The modular chromatic index is 2 for both graphs by Theorem 2.2.9 and furthermore, the edge coloring assigning the color 1 to every edge is the only modular 2-edge coloring of $G \in \{K_{1,4}, K_{1,6}\}$. This same coloring is also a modular 3-edge coloring of $K_{1,6}$, while it is not the case for $K_{1,4}$. In fact, no edge coloring assigning to every edge the same color in $\mathbb{Z}_3$ is a modular 3-edge coloring of $K_{1,4}$. On the other hand, there are three modular 3-edge colorings of $K_{1,4}$ and observe that each such coloring uses two colors, say
a, b ∈ ℤ₄, and assigns the color a to two edges and the color b to the remaining two edges. For G ∈ \{K₁,₄, K₁,₅\}, therefore, we see that χₘ(G) = 2 and there also exists a modular 3-edge coloring of G, but a modular 2-edge coloring does not necessarily induce a modular 3-edge coloring of G. Therefore, it is much more challenging to show that every connected graph G of order at least 3 is modular k-colorable for all integers k with k ≥ χₘ(G). By an argument similar to the proof of Theorem 2.3.3, we have the following

**Theorem 2.5.1** If G is a nontrivial connected graph that is not bipartite, then G is modular k-edge colorable for each odd integer k ≥ χₘ(G).

In this section, we show that every connected graph G of order at least 3 is k-colorable for all k with k ≥ χₘ(G). We begin with complete multipartite graphs.

### 2.5.1 Complete Multipartite Graphs

We have seen that if G = Kₜ₁,…,ₜᵢ is a complete ℓ-partite graph where ℓ ≥ 2, then χₘ(G) = ℓ or χₘ(G) = ℓ + 1. Furthermore, χₘ(G) = ℓ + 1 if and only if ℓ ≡ 2 (mod 4) and each nᵢ is odd for 1 ≤ i ≤ ℓ. We show that if G is a complete ℓ-partite graph of order at least 3, then G is modular k-edge colorable for each integer k ≥ ℓ + 1. We begin with complete bipartite graphs. By Theorem 2.5.1, it remains to show (1) every complete bipartite graph of order at least 3 is modular k-edge colorable for each integer k ≥ 3 and (2) every complete ℓ-partite graph where ℓ ≥ 3 is modular k-edge colorable for each even integer k ≥ ℓ + 1.
Theorem 2.5.2  Every complete bipartite graph of order at least 3 is modular $k$-edge colorable for each integer $k \geq 3$

Proof. Let $G = K_{n_1, n_2}$ and let $V_1$ and $V_2$ be the partite sets of $G$ where $|V_i| = n_i$ for $i = 1, 2$. For a fixed integer $k \geq 3$, we show that $G$ is modular $k$-edge colorable by considering two cases.

Case 1 $n_1 \not\equiv 1 \pmod{k}$ or $n_2 \not\equiv 1 \pmod{k}$, say the former. Let $u \in V_2$ and define the coloring $c : E(G) \to \mathbb{Z}_k$ by

$$c(e) = \begin{cases} 1 & \text{if } e = uv \text{ where } v \in V_1 \\ 0 & \text{otherwise} \end{cases}$$

Since $s_c(v) = 1 \in V_1$, $s_c(v) = 0$ for $v \in V_2$ and $v \neq u$ and $s_c(u) = n_1 \not\equiv 1 \pmod{2}$, it follows that $c$ is a modular $k$-edge coloring of $G$.

Case 2 $n_1 \equiv 1 \pmod{k}$ and $n_2 \equiv 1 \pmod{k}$ Since the order of $G$ is at least 3, it follows that $n_1 \geq 2$ or $n_2 \geq 2$, say the former. Let $v_1, v_2 \in V_1$ be two distinct vertices of $G$. Define the coloring $c : E(G) \to \mathbb{Z}_k$ by

$$c(e) = \begin{cases} 1 & \text{if } e = v_i v, i = 1, 2 \text{ and } v \in V_2 \\ 0 & \text{otherwise} \end{cases}$$

Since $s_c(v) = 2 \in V_2$, $s_c(v) = 0$ for $v \in V_1 - \{v_1, v_2\}$ and $s_c(v_i) = n_i \equiv 1 \pmod{2}$ for $i = 1, 2$, it follows that $c$ is a modular $k$-edge coloring of $G$.

We consider the complete $\ell$-partite graphs where $\ell \geq 3$. First, we present a lemma.

Lemma 2.5.3  Let $A$ be a set of $\ell \geq 3$ positive integers. Then for each even integer $k > \ell$, there exists a proper nonempty subset $B$ of $A$ such that (i) $|B| \leq k/2$, (ii) the sum of the integers in $B$ is even and (iii) $1 \leq |A - B| \leq k/2$.
Proof. We proceed by induction on \( |A| = \ell \). For \( \ell = 3 \), let \( k \geq 4 \) be any even integer. Let \( n \) and \( n' \) be two integers in \( A \) that are of the same parity. Then \( B = \{n, n'\} \) has the desired property. Assume that if \( A \) is a set of positive integers with \( |A| = \ell \) for some integer \( \ell \geq 3 \) and \( k > \ell \) is any even integer, then there is a proper nonempty subset \( B \) of \( A \) such that \( B \) satisfies (i) – (iii).

Let \( A = \{n_1, n_2, \ldots, n_{\ell+1}\} \) be a set of \( \ell+1 \geq 4 \) positive integers and let \( k > \ell+1 \) be any even integer. Let \( A' = A - \{n_{\ell+1}\} \). Since \( k > \ell + 1 > \ell \), it follows by the induction hypothesis that there is a proper nonempty subset \( B' \) of \( A' \) such that \( |B'| \leq k/2 \), the sum \( \sum_{n \in B'} n \) of the integers in \( B' \) is even and \( 1 \leq |A' - B'| \leq k/2 \). Hence \( |A' - B'| = |A'| - |B'| = \ell - |B'| \leq k/2 \). If \( |A - B'| \leq k/2 \), then \( B = B' \) has the desired property. Thus we may assume that \( |A - B'| = \ell + 1 - |B'| > k/2 \) and so \( \ell - |B'| > k/2 - 1 \). Thus \( \ell - |B'| \geq k/2 \). On the other hand, \( \ell - |B'| \leq k/2 \). Hence \( \ell - |B'| = k/2 \) and so \( |B'| = \ell - k/2 \). Since \( k \geq \ell + 2 \), it follows that \( \ell \leq k - 2 \) and so

\[
|B'| = \ell - \frac{k}{2} \leq (k-2) - \frac{k}{2} = \frac{k}{2} - 2 \tag{2.7}
\]

Next, consider \( A - B' \). Since \( \ell + 1 \geq 4 \) and \( k \geq \ell + 2 \) is even, it follows that \( k \geq 6 \). Thus \( |A - B'| \geq k/2 \geq 3 \). Let \( n_i \) and \( n_j \) be two distinct integers of same parity in \( A - B' \) and let \( B = B' \cup \{n_i, n_j\} \). Observe that (i) \( |B| = |B'| + 2 \leq \frac{k}{2} - 2 + 2 = \frac{k}{2} \) by (2.7), (ii) \( \sum_{n_i \in B} n_i = n_1 + n_2 + \sum_{n \in B'} n \) is even and (iii)

\[
|A - B| = (\ell + 1) - (|B'| + 2) = \ell - 1 - |B'|
\]

\[
= (\ell - 1) + \left( \ell - \frac{k}{2} \right) = \frac{k}{2} - 1 < \frac{k}{2}
\]

Therefore, \( B \) has the desired property. ■
We are now prepared to present the following result.

**Theorem 2.5.4**  For each pair \( \ell, k \) of integers with \( k \geq \ell + 1 \geq 3 \), every complete \( \ell \)-partite graph is modular \( k \)-edge colorable.

**Proof.** By Theorems 2 5 2 and 2 5 1, we may assume that \( \ell \geq 3 \) and \( k \) is even.

Let \( G = K_{n_1, n_2, \ldots, n_\ell} \) be a complete \( \ell \)-partite graph where \( \ell \geq 3 \) and let \( V_1, V_2, \ldots, V_\ell \) be the partite sets of \( G \) where \( |V_i| = n_i \) for \( 1 \leq i \leq \ell \). By Lemma 2 5 3, there is a proper nonempty subset \( S \) of the set \( \{n_1, n_2, \ldots, n_\ell\} \) such that (i) \( |S| \leq k/2 \), (ii) the sum of the integers of \( S \) is even, and (iii) \( 1 \leq \ell - |S| \leq k/2 \). We may assume that \( S = \{n_1, n_2, \ldots, n_s\} \) where \( 1 \leq s \leq k/2 \) and \( n_1 + n_2 + \ldots + n_s \) is even.

Furthermore, let \( V = V_1 \cup V_2 \cup \ldots \cup V_\ell \).

Let \( u \in V_\ell \) and define the coloring \( c_0 \) of \( E(G) \to \mathbb{Z}_k \) by

\[
c_0(e) = \begin{cases} 1 & \text{if } e = uv \text{ where } v \in V_i \text{ and } 1 \leq i \leq s \\ 0 & \text{otherwise} \end{cases}
\]

Observe that \( s_{c_0}(v) = 1 \in V_i \) for \( 1 \leq i \leq s \), \( s_{c_0}(v) = 0 \) for \( s \leq i \leq \ell \) and \( v \neq u \) and \( s_{c_0}(u) \equiv 0 \pmod{2} \). Note that \( c_0 \) is not a modular coloring of \( G \).

Let \( v_1, v_2, \ldots, v_n = u \) be an ordering of the vertices of \( G \). We define a sequence \( c_1, c_2, \ldots, c_n \) of edge colorings of \( G \) recursively such that for each \( i \) with \( 1 \leq i \leq n \), the edge coloring \( c_i : E(G) \to \mathbb{Z}_k \) induces a vertex coloring \( s_{c_i} : V(G) \to \mathbb{Z}_k \) for which

\[
s_{c_i}(v_i) = \begin{cases} 2j - 1 & \text{if } v_i \in V_j \subseteq V \\ 2j - 2 & \text{if } v_i \in V_j \subseteq V(G) - V \\ \end{cases} \quad (2.8)
\]

\[
s_{c_i}(v) = s_{c_{i-1}}(v) \quad \text{if } v \in V(G) - \{v_i\} \quad (2.9)
\]
We begin with the coloring $c_1$. Suppose that $v_1 \in V_j$ for some $j$ with $1 \leq j \leq \ell$. Since $\ell \geq 3$, the vertex $v_1$ lies on a triangle $C$ in $G$, say $C = (v_1, u_1, w_1, v_1)$. Define $c_1 : E(G) \to \mathbb{Z}_k$ by

$$c_1(e) = \begin{cases} 
    c_0(e) & \text{if } e \notin E(C) \\
    c_0(e) + (j - 1) & \text{if } e \in \{v_1u_1, v_1w_1\} \\
    c_0(e) - (j - 1) & \text{if } e = u_1w_1.
\end{cases}$$

If $v_1 \in V$, then $s_{c_0}(v_1) = 1$; while if $v_1 \in V(G) - V$, then $s_{c_0}(v_1) = 0$. This implies that

$$s_{c_1}(v_1) = \begin{cases} 
1 + 2(j - 1) = 2j - 1 & \text{if } v_1 \in V_j \subseteq V \\
0 + 2(j - 1) = 2j - 2 & \text{if } v_1 \in V_j \subseteq V(G) - V.
\end{cases}$$

Thus $s_{c_1}$ satisfies (2.8) and (2.9).

For an integer $i$ with $1 \leq i \leq n - 1$, suppose that the edge colorings $c_1, c_2, \ldots, c_i$ have been defined, all of which satisfy (2.8) and (2.9). We now define the coloring $c_{i+1}$ from $c_i$ in the same fashion as we defined $c_1$ from $c_0$. More precisely, assume that $v_{i+1} \in V_j$ for some $j$ with $1 \leq j \leq \ell$ and that $v_{i+1}$ lies on a triangle $C = (v_{i+1}, u_{i+1}, w_{i+1}, v_{i+1})$. Then define $c_{i+1} : E(G) \to \mathbb{Z}_k$ by

$$c_{i+1}(e) = \begin{cases} 
c_i(e) & \text{if } e \notin E(C) \\
c_i(e) + (j - 1) & \text{if } e \in \{v_1u_1, v_1w_1\} \\
c_i(e) - (j - 1) & \text{if } e = u_1w_1.
\end{cases}$$

We now consider the induced vertex coloring $s_{c_i} : V(G) \to \mathbb{Z}_k$. Since $s_{c_i}(v_{i+1}) = s_{c_{i-1}}(v_{i+1}) = \cdots = s_{c_0}(v_{i+1})$, it follows that $s_{c_i}(v_{i+1}) = 1$ if $v_{i+1} \in V$; while
Thus $s_{c_{i+1}}$ satisfies (2.8) and (2.9). Continuing in this manner, we obtain the edge coloring $c_n : E(G) \to \mathbb{Z}_k$ that induces a vertex coloring $s_{c_n} : V(G) \to \mathbb{Z}_k$ such that

$$s_{c_n}(v) = s_{c_n}(v) \quad \text{if } v \in V(G) - \{v_{i+1}\}.$$ 

This implies that $s_{c_n}$ is a proper vertex $k$-coloring of $G$ using the $k$ colors in $\mathbb{Z}_k$. Thus $c_n$ is a modular $k$-edge coloring of $G$ and so $G$ is modular $k$-edge colorable.

### 2.5.2 General Graphs

Let $G$ be a connected graph with $\chi(G) = \ell$. Suppose that the vertex set of $G$ can be partitioned into $\ell$ independent sets $V_1, V_2, \ldots, V_\ell$ with $|V_i| = n_i$ for $1 \leq i \leq \ell$. Then $G$ is a subgraph of a complete $\ell$-partite graph $K_{n_1,n_2,\ldots,n_{\ell}}$. In order to show that $G$ is modular $k$-edge colorable for all $k \geq \ell + 1$, we first present a lemma.

**Lemma 2.5.5** Let $G$ be a connected graph with $\chi(G) = 3$. Among all partitions of $V(G)$ into three independent sets, let $\{V_1, V_2, V_3\}$ be one such that $\min\{|V_i| : i = 1, 2, 3\}$ is minimum. Let $|V_i| = n_i$ for $1 \leq i \leq 3$ and let $K_{n_1,n_2,n_3}$ be the complete 3-partite graph whose partite sets are $V_1, V_2, V_3$. If for each pair $u, v$ of vertices of $G$ where $u \in V_i, v \in V_j$ and $i \neq j$, we have $uv \in E(G)$ whenever $G$ contains a $u - v$ path of odd length, then $G = K_{n_1,n_2,n_3}$.
Proof. We have seen that $G$ is a subgraph of $K_{n_1,n_2,n_3}$. For each vertex $v$ of $G$, where say $v \in V_i$ for some $i$ with $1 \leq i \leq \ell$, we show that $v$ is adjacent to every vertex in $V(G) - V_i$. Assume, without loss of generality, that $n_1 = \min\{n_1, n_2, n_3\}$. First suppose that $v \in V_1$. We show that $v$ is adjacent to every vertex in $V(G) - V_1 = V_2 \cup V_3$. We first claim that $d_G(u, v) \leq 2$ for every $u \in V_2 \cup V_3$. If this is not the case, then suppose that $u \in V_2 \cup V_3$ and $d_G(v, u) = d \geq 3$. We may assume without loss of generality, that $u \in V_2$. Let $(u = v_0, v_1, v_2, \ldots, v_d = v)$ be a $u - v$ geodesic in $G$. If $v_3 \notin V_2$, then there is a $u - v_3$ path of length 3 and so $uv_3 \in E(G)$, creating a $u - v$ path of length $d - 2$ and a contradiction. Hence, $d \geq 4$ and $u, v_3 \in V_2$ while $v_1, v_2 \notin V_2$. If $d \geq 5$, then it can be similarly shown that $v_5 \in V_2$ since $uv_5 \notin E(G)$. However then, $(v_2, v_3, v_4, v_5)$ is a $v_2 - v_5$ path of odd length and so $v_2v_5 \in E(G)$, which cannot occur. Therefore, $d = 4$ and $v_1, v \in V_1$ (for otherwise $v_1v \in E(G)$, which is impossible). Since $v_2$ is adjacent to a vertex in $V_1$ and a vertex in $V_2$, it follows that $v_2 \in V_3$. Since $v_1, v \in V_1$, it follows that $n_3 \geq n_1 \geq 2$. Note that there is $w \in V_3 - \{v_2\}$ such that $uw \in E(G)$. For otherwise, $V_1 - \{v\}$, $V_2$, $V_3 \cup \{v\}$ is a partition of $V(G)$ into three independent sets with $|V_1 - \{v\}| = n_1 - 1$, which contradicts the defining property of $\{V_1, V_2, V_3\}$. Thus $(u, v_1, v_2, v_3, v, w)$ is a $u - w$ path of length 5 and so $uw \in E(G)$. However then, $(u, w, v)$ is a $u - v$ path of length 2, which is a contradiction. Therefore, as claimed, $d_G(u, v) \leq 2$ for every $u \in V_2 \cup V_3$. 
Now partition the set \( V_2 \cup V_3 \) into two sets \( X \) and \( Y \) defined by

\[
X = \{ x \in V_2 \cup V_3 : d_G(v, x) = 1 \} = N(v)
\]

\[
Y = \{ y \in V_2 \cup V_3 : d_G(v, y) = 2 \}.
\]

We show that \( Y = \emptyset \). Assume, to the contrary, that \( Y \neq \emptyset \). If \( Y \) is not independent, say \( y, y' \in Y \) and \( yy' \in E(G) \), then there is either a \( v - y \) path or a \( v - y' \) path of length 3, which cannot occur since \( vy, vy' \notin E(G) \). Therefore, \( Y \) is an independent set. On the other hand, we claim that \( X \) cannot be independent. Assume, to the contrary, that \( X \) is independent. If \( V_1 = \{ v \} \), then \( X, Y \cup \{ v \} \) is a partition of \( V(G) \) into two independent sets and so \( \chi(G) \leq 2 \), which contradicts the fact that \( \chi(G) = 3 \). Thus we may assume that \( V_1 - \{ v \} \neq \emptyset \). First, observe that every vertex \( x \in X \) is adjacent to every vertex in \( y \in Y \). Since \( y \) must be adjacent to some vertex \( x_y \in X \). If \( x \neq x_y \), then \((x, v, x_y, y)\) is an \( x - y \) path of odd length and so \( xy \in E(G) \). Hence the subgraph \( G[X \cup Y] \) induced by \( X \cup Y \) is a complete bipartite graph with partite set \( X \) and \( Y \). Since (i) \( X \cup Y = V_2 \cup V_3 \), (ii) \( G[X \cup Y] \) is a complete bipartite graph and (iii) \( V_2 \) and \( V_3 \) are independent, it follows that \( \{X, Y\} = \{V_2, V_3\} \), say \( X = V_2 \) and \( Y = V_3 \). However then, \( V_1 - \{ v \} \), \( X, Y \cup \{ v \} \) are three independent sets of \( G \), which contradicts to the defining property of \( \{V_1, V_2, V_3\} \). Thus, as claimed, \( X \) cannot be independent.

Let \( x, x' \in X \) and \( xx' \in E(G) \). Without loss of generality, we may assume that \( x \in V_2 \) and \( x' \in V_3 \). Let \( y \in Y \). If \( y \) is adjacent to either \( x \) or \( x' \), then there is a \( v - y \) path of length 3, which is again a contradiction. Therefore, \( yx, yx' \notin E(G) \). Then there exists a vertex \( x'' \in X - \{x, x'\} \) so that \((v, x'', y)\) is a \( v - y \) geodesic.
However, this implies that $xy \in E(G)$ if $y \notin V_2$ and $x'y \in E(G)$ otherwise, neither of which can occur. Thus $Y = \emptyset$ and $v$ is adjacent to every vertex in $V_2 \cup V_3$.

Next we show that every vertex in $V_2$ is adjacent to every vertex in $V_3$ in $G$. Since there is at least one edge between $V_2$ and $V_3$, we can assume that $v_2 \in V_2$ and $v_3 \in V_3$ such that $v_2v_3 \in E(G)$. If $V_2 - \{v_2\} \neq \emptyset$, then for each $v'_2 \in V_2 - \{v_2\}$, the path $(v'_2, v, v_2, v_3)$ is a path of odd length and so $v'_2v_3$ is an edge in $G$. Thus every vertex in $V_2$ is adjacent to the vertex $v_3 \in V_3$. Now if $V_3 - \{v_3\} \neq \emptyset$, then for each $v'_3 \in V_3 - \{v_3\}$, the path $(v'_3, v, v_3, v_2)$ is a path of odd length and so $v'_3v_2$ is an edge in $G$. This implies that every vertex in $V_2$ is adjacent to every vertex in $V_3$ in $G$. Therefore, $G = K_{n_1,n_2,n_3}$.

Suppose that the vertex set of $G$ can be partitioned into $\ell$ independent sets $V_1, V_2, \ldots, V_\ell$ with $|V_i| = n_i$ for $1 \leq i \leq \ell$. Then $G$ is a subgraph of a complete $\ell$-partite graph $K = K_{n_1,n_2,\ldots,n_\ell}$. Recall that the odd path closure of $G$ with respect to the partition $\{V_1, V_2, \ldots, V_\ell\}$, denoted by $C_0(G)$, is the graph obtained from $G$ by recursively joining pairs of nonadjacent vertices that belong to different partite sets in $K$ and are connected by a path of odd length in $G$. We first show that the odd path closure of a connected bipartite graph of order at least 3 with respect to the two given partite sets is a complete bipartite graph.

**Proposition 2.5.6** Let $G$ be a connected bipartite graph with partite sets $U$ and $W$ where $|U| = r$ and $|W| = s$ and $r + s \geq 3$. Then the odd path closure $C_0(G)$ of $G$ with respect to the partition $\{U, W\}$ is $K_{r,s}$. 
Proof. First, observe that $C_o(G)$ is a bipartite graph with partite sets $U$ and $W$. If $C_o(G) \neq K_{r,s}$, then there are vertices $u \in U$ and $w \in W$ such that $uw \notin E(C_o(G))$. Since $C_o(G)$ is bipartite,

$$
U = \{v \in V(C_o(G)) : d_{C_o(G)}(u,v) \text{ is even}\}
$$

$$
W = \{v \in V(C_o(G)) : d_{C_o(G)}(u,v) \text{ is odd}\}.
$$

Since $w \in W$, it follows that $d_{C_o(G)}(u,w)$ is odd. Thus $uw \in E(C_o(G))$, which is a contradiction. 

The following is a consequence of Lemmas 2.4.3 and 2.5.5 and Proposition 2.5.6.

**Corollary 2.5.7** Let $G$ be a connected graph with $\chi(G) = \ell \geq 2$. Then there is a partition $\mathcal{P} = \{V_1, V_2, \ldots, V_\ell\}$ of $V(G)$ into $\ell$ independent sets, where $|V_i| = n_i$ for $1 \leq i \leq \ell$, such that the odd path closure $C_o(G)$ of $G$ with respect to the partition $\mathcal{P}$ is $K_{n_1, n_2, \ldots, n_\ell}$.

**Proposition 2.5.8** Let $G$ be a connected graph with $\chi(G) = \ell \geq 2$, let $\mathcal{P} = \{V_1, V_2, \ldots, V_\ell\}$ be a partition of $V(G)$ into $\ell$ independent sets and let $C_o(G)$ be the odd path closure with respect to $\mathcal{P}$. For each integer $k \geq 2$, if $C_o(G)$ is modular $k$-edge colorable, then so is $G$.

Proof. Let $u \in V_i$ and $v \in V_j$ where $i \neq j$ and $u$ and $v$ are connected by a path of odd length. It suffices to show that if $G + uv$ is modular $k$-edge colorable, then $G$ is modular $k$-edge colorable. Let $c$ be a modular $k$-edge coloring of $G + uv$. 


Suppose that $P = (u = v_1, v_2, \ldots, v_p = v)$ is a $u - v$ path of odd length in $G$, where then $p \geq 4$ is even. Then the $k$-edge coloring $c'$ of $G$ defined by

$$
c'(e) = \begin{cases} 
c(e) & \text{if } e \notin E(P) \\
c(e) + c(uv) & \text{if } e = v_i v_{i+1}, 1 \leq i \leq p - 1 \text{ and } i \text{ is odd} \\
c(e) - c(uv) & \text{if } e = v_i v_{i+1}, 2 \leq i \leq p - 2 \text{ and } i \text{ is even}
\end{cases}
$$

is a modular $k$-edge coloring of $G$. ■

By Theorem 2.5.4, Corollary 2.5.7 and Proposition 2.5.8, we establish the main result of this section.

**Theorem 2.5.9** If $G$ is a connected graph of order at least 3, then $G$ is modular $k$-edge colorable for each $k \geq \chi'_m(G)$.

Recall that the modular chromatic index $\chi'_m(G)$ of a graph $G$ exists only when $G$ contains no components isomorphic to $K_2$ and so only connected graphs of order at least 3 have been studied. For disconnected graphs, the following is a consequence of Theorem 2.5.9.

**Corollary 2.5.10** If $G$ is a disconnected graph consisting of components $G_1, G_2, \ldots, G_t$ each of which contains at least three vertices, then

$$
\chi'_m(G) = \max\{\chi'_m(G_i) : 1 \leq i \leq t\}.
$$
Chapter 3

Modular Edge-Graceful Graphs

Over the past few decades the subject of graph labelings has been growing in popularity. Indeed, Gallian [11] has compiled a periodically updated survey of many kinds of labelings and numerous results, obtained from well over a thousand referenced research articles. When one speaks of a labeling of a graph $G$, we typically refer to a function $f : V(G) \to S$ or a function $g : E(G) \to S$, where $S$ is often, but not always, a set of integers and where $f$ and $g$ are often, but not always, one-to-one. In many instances, the labelings that have attracted the most interest have given rise to other labelings that satisfy some prescribed property.

3.1 A History of Graceful Labelings

One of the major figures in 19th century combinatorics was Thomas Penyngton Kirkman. To many, Kirkman was thought to be an amateur mathematician whose contribution to mathematics consisted of a single problem he invented that dealt with fifteen schoolgirls. Kirkman was no amateur mathematician, however. In-
deed, he authored some 60 papers that made significant contributions to mathematics.

Thomas Kirkman was born in 1806 in a small town near Manchester, England. He was by far the best student in the grammar school he attended. Kirkman’s father was a cotton dealer of modest wealth and insisted that his son should work in the family business. Kirkman was forced to leave school at age 14 and for the next nine years worked in his father’s office. At the age of 23 he broke away from his father and entered Trinity College in Dublin, Ireland. After four years, in 1833, he had earned a B.A. degree, studying mathematics, philosophy, science and the classics. He returned to England that year and entered the Church of England.

Kirkman spent five years as a parish priest. In 1839 he became pastor of a parish in Lancashire, a position he held for the next 52 years. By the time he was approaching 40 years old, his wife had inherited some property and, although not a wealthy man, Reverend Kirkman now held a respected position and was financially secure. There was no evidence, however, that he had any special interest in mathematics. Nevertheless, he had mathematical ability. All that was needed was something that would draw him into mathematics. The catalyst turned out to be a problem he encountered in a magazine.

A popular journal during the 1840s was the *Lady’s and Gentleman’s Diary*. In 1844 the editor Wesley Woolhouse stated what was called Prize Question No. 1733 in the magazine:

*Determine the number of combinations that can be made of n symbols.*
\( p \) symbols in each; with this limitation, that no combination of \( q \) symbols which may appear in any one of them shall be repeated in any other.

The 1845 journal contained many attempted but unsuccessful solutions. After a year, the problem was replaced by the special case where \( p = 3 \) and \( q = 2 \). The editor drew attention to the difficulties of the problem, pointing out that when \( n = 10 \), it is impossible to find a system of triples \( (p = 3) \) in which each pair \( (q = 2) \) occurs exactly once.

On 15 December 1846 Kirkman presented a paper dealing with this Prize Question to the Literary and Philosophical Society of Manchester. Shortly afterwards an article by him was published in the *Cambridge and Dublin Mathematical Journal*. In it, he addressed the following problem.

*How many triples can be formed with \( x \) symbols in such a way that no pair of symbols occurs more than once in the triple?*

While it is easy to show that a system \( S_n \) with \( n \) symbols in which each pair occurs exactly once is possible only if \( n \equiv 1 \pmod{6} \) or \( n \equiv 3 \pmod{6} \), the main result in Kirkman’s paper was a proof of the converse – that such a system \( S_n \) of triples exists for all such values of \( n \).

The history of this problem did not end with this paper and the solution it contained. Six years later, in 1853, the famous geometer Jakob Steiner wrote a short note in which the existence of such triples is questioned, obviously being unaware of Kirkman’s paper. Six years after Steiner’s note was published, Steiner’s
question was answered by M. Reiss. That is, Reiss answered Steiner’s question 12 years after Kirkman had posed and solved this problem. This caused Kirkman to write:

... how did the Cambridge and Dublin Mathematical Journal Vol. II, p. 191, contrive to steal so much from a later paper in Crelle’s Journal Vol. LVI, p. 326, on exactly the same problem on combinations?

Even though the triple system described by Kirkman occurred in print much earlier, they were eventually named after Jacob Steiner, giving rise to the concept of Steiner triple systems. Among the observations made by Kirkman in his paper [15] was the following.

If there is a Steiner triple system $S_{2k+1}$, there is also a Steiner triple system $S_{4k+3}$.

If $2k + 1 \equiv 1 \pmod{6}$, then $4k + 3 \equiv 3 \pmod{6}$; while if $2k + 1 \equiv 3 \pmod{6}$, then $4k + 3 \equiv 1 \pmod{6}$. In graph theory terminology, this says the following.

If $K_{2k+1}$ is $K_3$-decomposable, then so is $K_{4k+3}$.

This can be verified as follows. Consider the complete graph $K_{4k+3}$ as the join of $K_{2k+1}$ and $K_{2k+2}$, as shown in Figure 3.1. By assumption, $K_{2k+1}$ is $K_3$-decomposable. There are $(2k + 2)(2k + 1)$ edges joining $K_{2k+1}$ and $K_{2k+2}$. The graph $K_{2k+2}$ is 1-factorable and has size $\binom{2k+2}{2}$. Observe that

$$(2k + 2)(2k + 1) = 2\binom{2k + 2}{2}.$$
There are $2k+1$ 1-factors in a 1-factorization of $K_{2k+2}$. Thus there is a one-to-one correspondence between the vertices $i$ ($1 \leq i \leq 2k+1$) of $K_{2k+1}$ and the 1-factors $F_i$ in a 1-factorization of $K_{2k+2}$. By joining $i$ to the incident vertices of each edge in $F_i$, a total of $k+1$ triangles are produced having only the vertex $i$ in common.

For example, consider $k = 1$.

Although Kirkman’s work on triple systems did not involve graphs, finding a triple system of $n$ symbols, where $n \equiv 1 \pmod{6})$ or $n \equiv 3 \pmod{6}$, is equivalent to finding a $K_3$-decomposition of $K_n$. For example, when $n = 7$, the following is a Steiner triple system.
From the technique that Kirkman used in his paper, it is also possible to construct a Steiner triple system for \( n = 15 \) from one constructed for \( n = 7 \). From a graph theory point of view, Kirkman would have considered \( K_{15} \) as follows.

We already know that there is \( K_3 \)-decomposition of \( K_7 \). Also \( K_8 \) is 1-factorable. Suppose that the vertex set of \( K_8 \) is \( \{a, b, c, d, e, f, g, h\} \). Then each column below describes the edges in seven 1-factors in a 1-factorization of \( K_8 \).

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<td>( dh )</td>
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<td>( df )</td>
</tr>
</tbody>
</table>

There are 56 edges joining the vertices of \( K_7 \) and the vertices of \( K_8 \) and 28 edges in \( K_8 \). From these 84 = 28 \cdot 3 \) edges, 28 pairwise edge-disjoint triangles can
be constructed. For example, the first column gives rise to four triangles and thus four triples.

![Four triangles](image)

**Figure 3.5: Four triples**

These 28 triples together with the 7 triangles in the $K_3$-decomposition of $K_7$ produce 35 pairwise edge-disjoint triangles in $K_{15}$, that is, a Steiner triple system for $n = 15$.

After the publication of Kirkman’s 1847 paper [15], he noticed that there was a Steiner triple system $S_{15}$ that had an additional property. After this original paper, Kirkman observed that from a set of 15 symbols, there are 35 triples that can be divided into seven sets of five triples in a such a way that each of the 15 symbols occurs once in each set of five triples.

<table>
<thead>
<tr>
<th>123</th>
<th>145</th>
<th>167</th>
<th>357</th>
<th>346</th>
<th>256</th>
<th>247</th>
</tr>
</thead>
<tbody>
<tr>
<td>4ae</td>
<td>2bd</td>
<td>2ac</td>
<td>1eb</td>
<td>1cd</td>
<td>1ef</td>
<td>1gh</td>
</tr>
<tr>
<td>5cg</td>
<td>3fh</td>
<td>3eg</td>
<td>2fg</td>
<td>2eh</td>
<td>3bc</td>
<td>3ad</td>
</tr>
<tr>
<td>6bh</td>
<td>6aq</td>
<td>4bf</td>
<td>4ch</td>
<td>5af</td>
<td>4dg</td>
<td>5be</td>
</tr>
<tr>
<td>7df</td>
<td>7ce</td>
<td>5dh</td>
<td>6de</td>
<td>7bg</td>
<td>7ah</td>
<td>6cf</td>
</tr>
</tbody>
</table>

In the *Lady’s and Gentleman’s Diary* for 1850 Kirkman challenged readers in a unique way to discover such an arrangement for themselves.

*Fifteen young ladies in a school walk out three abreast for seven days in succession; it is required to arrange them daily, so that no two shall*
walk twice abreast.

Norman Biggs wrote

*It is unfortunate that such a trifle should overshadow the many more significant contributions which its author was to make to mathematics*

*Nevertheless, it is his most lasting memorial*

In any case, Steiner triple systems with this additional property gave rise to a concept that was named after Kirkman. A *Kirkman triple system* of order $n$ is a Steiner triple system $S_n$ with the added property that the triples in $S_n$ can be partitioned into subsets so that each of the $n$ symbols appears exactly once in each subset. Thus a solution to Kirkman's Schoolgirl Problem constitutes a Kirkman triple system.

We know that there is a Steiner triple system of order $n$ if and only if $n \equiv 1 \pmod{6}$ or $n \equiv 3 \pmod{6}$. It is not difficult to show that if a Steiner triple system is also a Kirkman triple system, then $n \equiv 3 \pmod{6}$ and so $n = 6k + 3$ for some nonnegative integer $k$. Hence $n = 3(2k + 1)$, that is, $n$ is an odd integer multiple of 3. Over a hundred years later, in 1971, Dijen Ray-Chaudhuri and Richard Wilson showed that there exists a Kirkman triple system of order $n$ if and only if $n \equiv 3 \pmod{6}$. If $n \equiv 3 \pmod{6}$, then $n = 3r$ for some odd integer $r$. Showing that there is a Kirkman triple system of order $n$ is then equivalent to showing that $K_n$ is $rK_3$-decomposable.

After Kirkman, many graph theorists have shown an interest in decompositions of complete graphs. One of these mathematicians was Anton Kotzig, who
was born in Czechoslovakia in 1919. He obtained his doctorate in 1947 with a thesis on statistics. In 1961, he was granted the degree of Doctor of Science, the highest possible scientific degree in Czechoslovakia. This time he wrote his thesis on connectivity on graphs.

Kotzig did a great deal of research on several areas of combinatorics, including graph labelings, Latin squares and decompositions of complete graphs. In 1963 Gerhard Ringel conjectured that for every tree $T$ of size $m$, the complete graph $K_{2m+1}$ is $T$-decomposable. Kotzig conjectured that $K_{2m+1}$ is cyclically $T$-decomposable. Kotzig directed the research of many graduate students. One of these students was Alexander Rosa.

Rosa received his Candidate of Science degree (roughly equivalent to the Ph.D.) in 1966 and Doctoral of Natural Science degree in 1968. Rosa's research was greatly influenced by Kotzig's interests. Rosa became interested in Kirkman's Schoolgirl Problem, which led to his interest in Steiner triple systems, which led to graph decompositions, which then led to an interest in graph labelings.

During July 5-8, 1966, Rosa attended, with his advisor Kotzig, the Theory of Graphs International Symposium in Rome and spoke on "On certain valuations of the vertices of a graph". Rosa [22] defined a valuation of a graph $G$ of order $n$ and size $m$ as a one-to-one function $f : V(G) \rightarrow \mathbb{N} \cup \{0\}$ from the vertex set of $G$ to the set of nonnegative integers that gives rise to values of its edges, where the value of an edge $e = uv$ is defined as $|f(u) - f(v)|$. By placing various conditions required of $f$ and the values of the edges of $G$, four different valuations were introduced: $\alpha$-valuations, $\beta$-valuations, $\gamma$-valuations, $\rho$-valuations. In particular, the $\beta$-valuation
requires that

\[ f : V(G) \to \{0, 1, 2, \ldots, m\} \]

and

\[ \{|f(u) - f(v)| \mid uv \in E(G)\} = \{1, 2, \ldots, m\} \]

For example, \( \beta \)-valuations of \( C_3 \) and \( C_4 \) are shown below

![Figure 3.6 \( \beta \)-valuations of \( C_3 \) and \( C_4 \)](image)

The following is a consequence of Rosa’s work [22]

**Theorem 3.1.1** If \( T \) is a tree of size \( m \) having a \( \beta \) valuation, then \( K_{2m+1} \) is cyclically \( T \)-decomposable.

It is known that the cycle \( C_n \) of order \( n \) has a \( \beta \) valuation if and only if \( n \equiv 0 \pmod{4} \) or \( n \equiv 3 \pmod{4} \). Thus by Theorem 3.1.1, if \( n \equiv 0 \pmod{4} \) or \( n \equiv 3 \pmod{4} \), then \( K_{2n+1} \) is cyclically \( C_n \)-decomposable. Also, it is well known that \( K_{2n+1} \) is \( C_{2n+1} \)-decomposable for each positive integer \( n \). Another well-known example of this is the result that \( K_{2n+1} \) is \( K_3 \) decomposable whenever \( 3 \mid \binom{2n+1}{2} \), that is, whenever \( 2n + 1 \equiv 1 \pmod{6} \) or \( 2n + 1 \equiv 3 \pmod{6} \). This result was verified by Thomas Kirkman, although such a \( K_3 \)-decomposition is known as a Steiner triple system (for Jacob Steiner), as we mentioned earlier. These cycle
decompositions are special cases of a more general result. A special case of a conjecture due to Brian Alspach states the following. Let \( m \) and \( n \) be integers such that \( 3 < m < n \) where \( n \) is odd. If \( K_n \) is \( C_m \)-decomposable, then \( \binom{n}{2} \) is a multiple of \( m \) (or \( m | \binom{n}{2} \)). Alspach conjectured that the converse is true, that is, if \( 3 \leq m \leq n \), where \( n \) is odd and \( m | \binom{n}{2} \), then \( K_n \) is \( C_m \)-decomposable. Alspach and Heather Jordon [2] verified this conjecture when \( m \) is odd in 2001 and Mateja Šajna [18] verified this conjecture when \( m \) is even in 2002.

### 3.2 Graceful and Edge-Graceful Labelings

In 1972 Solomon Golomb [13] referred to a \( \beta \)-valuation of a graph as a graceful labeling and a graph admitting a graceful labeling as a graceful graph. It is this terminology that became standard. In fact, many of graph labelings that have been studied since then have been patterned after graceful labelings. The three graphs shown below are the only connected graphs of order 5 that are not graceful.

![Three graphs that are not graceful](image)

Figure 3.7: Three graphs that are not graceful

Among the results obtained on graceful graphs are the following:

1. The cycle \( C_n \) is graceful if and only if \( n \equiv 0 \pmod{4} \) or \( n \equiv 3 \pmod{4} \).

2. The complete graph \( K_n \) is graceful if and only if \( n \leq 4 \).
3. The graph $K_{s,t}$ is graceful for all positive integers $s$ and $t$.

4. The $n$-cube $Q_n$ is graceful for all positive integers $n$.

5. The grid $P_s \times P_t$ is graceful for all positive integers $s$ and $t$.

6. The path $P_n$ is graceful for all positive integers $n$.

7. Every caterpillar is graceful.

8. Every tree with at most four end-vertices is graceful.

9. Every tree of order at most 27 is graceful.

10. Almost all graphs are not graceful.

The following conjecture is due to Anton Kotzig and Gerhard Ringel.

**The Graceful Tree Conjecture** Every tree is graceful.

In 1985 Sheng-Ping Lo [17] introduced an edge labeling version of graceful labeling. A graph $G$ of order $n$ and size $m$ is *edge-graceful* if there exists a bijective function

$$f : E(G) \to \{1, 2, \ldots, m\}$$

such that the induced vertex labeling

$$f' : V(G) \to \{0, 1, 2, \ldots, n - 1\}$$
defined by
\[ f'(u) = \sum_{v \in N(u)} f(uv), \]
where the sum is computed in \( \mathbb{Z}_n \), is a bijective function. Figure 3.8 shows two edge-graceful graphs.

An elementary observation is the following (see [17]).

**Proposition 3.2.1** (Lo’s Condition) **If** \( G \) **is an edge-graceful graph of order** \( n \) **and size** \( m \), **then**
\[ \binom{n}{2} \equiv 2 \binom{m+1}{2} \pmod{n}. \]

**Proof.** Since \( G \) is an edge-graceful graph of order \( n \) and size \( m \),
\[ \sum_{u \in V(G)} f'(u) = \sum_{i=0}^{n-1} i = \binom{n}{2}, \]
while
\[ \sum_{u \in V(G)} f'(u) = 2 \sum_{e \in E(G)} f(e) = 2 \sum_{j=1}^{m} j = 2 \binom{m+1}{2}. \]
Therefore, \( \binom{n}{2} = 2 \binom{m+1}{2} \) and the result follows. \( \blacksquare \)
Since \( \binom{n}{2} = 2^{(m+1)} \) if \( G \) is a tree of order \( n \) in Proposition 3.2.1, a tree satisfies Lo's Condition if and only if its order is odd. It is known that the path \( P_n \) of odd order \( n \) is edge-graceful. Sin-Min Lee [16] made the following conjecture:

**Conjecture 3.2.2** [16] Every nontrivial tree of odd order is edge-graceful.

In fact, it was conjectured by Lee that every connected graph of order \( n \) with \( n \neq 2 \pmod{4} \) is edge-graceful. Among the results obtained on edge-graceful graphs are the following:

1. The complete graph \( K_n \) is edge-graceful if and only if \( n \neq 2 \pmod{4} \).
2. Every odd cycle is edge-graceful.
3. The Cartesian product \( C_m \times C_n \) is edge-graceful if and only if \( m \) and \( n \) are both odd.

In the definition of an edge-graceful labeling of a connected graph \( G \) of order \( n \geq 2 \) and size \( m \), the edge labeling \( f : E(G) \to \{1, 2, \ldots, m\} \) is required to be one-to-one. However, since the induced vertex labels \( f'(u) \) are obtained by summation in \( \mathbb{Z}_n \), the function \( f \) is actually a function from \( E(G) \) to \( \mathbb{Z}_n \) and is in general not one-to-one. Let

\[
m = nq + r, \text{ where } q = \lfloor m/n \rfloor \text{ and } 0 \leq r \leq n - 1
\]

Hence in an edge-graceful labeling of \( G \), \( q + 1 \) edges are labeled \( r \) for each \( r \) with \( 1 \leq r \leq r \) and \( q \) edges are labeled \( r \) for each \( r \) with \( r + 1 \leq r \leq n \). Thus \( f : E(G) \to \mathbb{Z}_n \) is one-to-one only when \( m = n - 1 \) or \( m = n \).
3.3 Modular Edge-Graceful Colorings

We saw in Chapter 1 that in 2008 a vertex coloring of a graph was introduced in [19] in connection with finding a solution to a coin placement problem on a checkerboard. Let’s review this vertex coloring. Recall that for a graph $G$ without isolated vertices, let $c: V(G) \to \mathbb{Z}_k (k \geq 2)$ be a vertex coloring of $G$ where adjacent vertices may be colored the same. Then a vertex coloring $c'$ of $G$ is defined such that $c'(v)$ (denoted by $\sigma(v)$ in Chapter 1) is the sum in $\mathbb{Z}_k$ of the colors of the vertices in the neighborhood of $v$. The coloring $c$ is called a modular $k$-coloring of $G$ if $c'(u) \neq c'(v)$ in $\mathbb{Z}_k$ for every pair $u, v$ of adjacent vertices of $G$. The modular chromatic number of $G$ is the minimum $k$ for which $G$ has a modular $k$-coloring. This coloring was studied further in [20], which led to a complete solution of the checkerboard problem under investigation.

Also, we saw in Chapter 2 that the modular coloring described above led to an edge version of this coloring. We review this as well. For a graph $G$ without isolated vertices, let $c: E(G) \to \mathbb{Z}_k (k \geq 2)$ be an edge coloring of $G$ where adjacent edges may be colored the same. Then a vertex coloring $c'$ is defined such that $c'(v)$ (denoted by $s_c(v)$ in Chapter 2) is the sum in $\mathbb{Z}_k$ of the colors of the edges incident with $v$. An edge coloring $c$ is a modular $k$-edge coloring of $G$ if $c'(u) \neq c'(v)$ in $\mathbb{Z}_k$ for all pairs $u, v$ of adjacent vertices of $G$. The modular chromatic index of $G$ is the minimum $k$ for which $G$ has a modular $k$-edge coloring.

Combining the concepts of graceful labeling and modular edge coloring gives rise to a modular edge-graceful labeling. Henceforth, we assume all graphs under...
consideration are connected graphs of order at least 3.

In what follows, we use the notation $f$ and $f'$ (instead of $c$ and $\sigma$ or $c$ and $s_c$) to be consistent with the notation commonly used in graph labelings.

Let $G$ be a connected graph of order $n \geq 3$ and size $m$ and let $f : E(G) \to \mathbb{Z}_n$, where $f$ need not be one-to-one. Define $f' : V(G) \to \mathbb{Z}_n$ by

$$f'(v) = \sum_{u \in N(v)} f(uv),$$

where the sum is computed in $\mathbb{Z}_n$. If $f'$ is one-to-one, then $f$ is called a modular edge-graceful labeling and $G$ is a modular edge-graceful graph. Consequently, every edge-graceful graph is a modular edge-graceful graph. This concept was introduced independently in the 1991 Ph.D. dissertation of R. B. Gnana Jothi [12] under the terminology of line-graceful graphs. Jothi gave a necessary condition for a graph to be modular edge-graceful. For completeness, we include an independent proof of this result.

First, we give an example of a graph that is not modular edge-graceful by showing that $G = C_6$ (see Figure 3.9) has no modular edge-graceful labeling. Suppose that $G$ has such a labeling, say $f : E(G) \to \mathbb{Z}_6$, where $f(e_i) \in \mathbb{Z}_6$. Then $f'(v_i) = f(e_i) + f(e_{i-1})$ (where $e_0 = e_6$). Since $\{f'(v_i) : 1 \leq i \leq 6\} = \mathbb{Z}_6$, it follows that $\sum_{i=1}^{6} f'(v_i) \equiv 3 \pmod{6}$. However, $\sum_{i=1}^{6} f'(v_i) = 2 \sum_{i=1}^{6} f(e_i)$ and so $\sum_{i=1}^{6} f'(v_i) \equiv 0, 2 \text{ or } 4 \pmod{6}$, which is a contradiction.

The fact that $C_6$ is not modular edge-graceful illustrates a more general observation, which provides a necessary condition not only for a graph to be modular
edge-graceful but also for a graph to be edge-graceful (see [11]).

**Proposition 3.3.1** If $G$ is a modular edge-graceful connected graph of order $n \geq 3$, then $n \not\equiv 2 \pmod{4}$.

**Proof.** Assume, to the contrary, that there exists a modular edge-graceful graph of order $n \geq 3$ with $n \equiv 2 \pmod{4}$. Let $f : E(G) \to \mathbb{Z}_n$ be a modular edge-graceful labeling of $G$ and let $f' : V(G) \to \mathbb{Z}_n$ be the induced vertex labeling. Hence

$$\{f'(v) : v \in V(G)\} = \mathbb{Z}_n$$

and so

$$\sum_{v \in V(G)} f'(v) \equiv \frac{n}{2} \pmod{n},$$

where $n/2$ is odd since $n \equiv 2 \pmod{4}$. On the other hand, observe that

$$\sum_{v \in V(G)} f'(v) = 2 \sum_{uv \in E(G)} f(uv),$$

implying that $\sum_{v \in V(G)} f'(v)$ is even, a contradiction. \hfill \blacksquare
The graphs $G_1 = C_4$ and $G_2$ in Figure 3.10 are both modular edge-graceful. Modular edge-graceful labelings are shown in Figure 3.10 as well. We saw that the graph $G_2$ (also shown in Figure 3.7) is not graceful. Moreover, $G_2$ is not edge-graceful by Proposition 3.2.1.

![Figure 3.10. Two modular edge-graceful graphs](image)

There is a useful observation concerning the two modular edge-graceful graphs in Figure 3.10. One edge of $G_1$ is labeled 0 and two edges of $G_2$ are labeled 0. Of course, these edges contribute 0 to the label of any incident vertex. Thus, if we were to delete these edges labeled 0 from $G_1$ and $G_2$, then we arrive at two graphs, trees in both cases (see Figure 3.11), that also possess modular edge-graceful labelings. As we will see, the examples presented above illustrate more general observations.

![Figure 3.11. Two modular edge-graceful trees](image)

**Proposition 3.3.2** If $G$ is a modular edge-graceful connected graph, then every graph containing $G$ as a spanning subgraph is also modular edge-graceful.
Proof. Let $H$ be a graph of order $n \geq 3$ containing a spanning subgraph $G$ that is modular edge-graceful. Thus there exists a modular edge-graceful labeling $f_G : E(G) \to \mathbb{Z}_n$ of $G$ such that the induced vertex labeling $f'_G$ is one-to-one. This edge labeling $f_G$ of $G$ can then be extended to an edge labeling $f : E(H) \to \mathbb{Z}_n$ of $H$ defined by $f(uv) = f_G(uv)$ if $uv \in E(G)$ and $f(uv) = 0$ if $uv \in E(H) - E(G)$. Since $f'(v) = f'_G(v)$ for each vertex $v$ of $H$, it follows that $f'$ is also one-to-one and so $H$ is modular edge-graceful.

By Proposition 3.3.2, a connected graph is modular edge-graceful if it contains a spanning tree that is modular edge-graceful. This places additional importance on determining those trees that are modular edge-graceful. By Proposition 3.3.1, no tree of order $n \geq 3$ with $n \equiv 2 \pmod{4}$ is modular edge-graceful. As described in [11], Jothi determined a number of classes of graphs that are modular edge-graceful. In order to state these results, we present additional definitions. A vertex $v$ in a graph is odd if $\deg v$ is odd while $v$ is even if $\deg v$ is even. For each integer $k \geq 2$, a (complete) $k$-ary tree is a rooted tree in which every vertex has either $k$ children or no children. The corona $\text{cor}(G)$ of a graph $G$ is that graph obtained from $G$ by adding a new vertex $v'$ to $G$ for each vertex $v$ of $G$ and joining $v'$ to $v$.

Theorem 3.3.3 [12] The following graphs of order at least 3 are modular edge-graceful.

(a) All stars $K_{1,n-1}$ for which $n \not\equiv 2 \pmod{4}$.
(b) All paths $P_n$ for which $n \neq 2 \pmod{4}$

(c) All cycles $C_n$ for which $n \neq 2 \pmod{4}$

(d) All trees of order $n$ with $n \neq 2 \pmod{4}$ containing exactly one even vertex

(e) All $k$-ary trees of order $n$ for which $n \neq 2 \pmod{4}$ and $k$ is even

(f) All trees of order $n \leq 9$ and $n \neq 6$

(g) All coronas $\text{cor}(P_n)$ of paths $P_n$ for which $n$ is even

(h) All coronas $\text{cor}(C_n)$ of cycles $C_n$ for which $n$ is even

Of course, by Proposition 3.3.2, (c) follows from (b), (e) follows from (d) and (h) follows from (g). Furthermore, it is a consequence of (b) that every traceable graph (a graph containing a Hamiltonian path) of order $n$ where $n \neq 2 \pmod{4}$ is modular edge-graceful. In particular, every Hamiltonian graph of order $n$ where $n \neq 2 \pmod{4}$ is modular edge-graceful. In fact, as we will see soon, each known result stated in Theorem 3.3.3, except for (d), is a consequence of Proposition 3.3.2 and the more general results on trees obtained in this work. Jothi also made the following conjecture (see [11])

**Conjecture 3.3.4** [12] If $T$ is a tree of order $n \geq 3$ for which $n \neq 2 \pmod{4}$, then $T$ is modular edge-graceful
3.4 On Modular Edge-Graceful Trees

We saw that if a graph $G$ contains a modular edge-graceful spanning tree, then $G$ is also modular edge-graceful. Thus it is important to study trees, as we mentioned in the previous section. In this section, we present several classes of trees that are modular edge-graceful. We first show that all trees of order $n \geq 3$ with $n \neq 2 \pmod{4}$ having at most two even vertices are modular edge-graceful, beginning with those trees without even vertices.

**Theorem 3.4.1** A tree of order $n \geq 3$ in which every vertex has odd degree is modular edge-graceful if and only if $n \equiv 0 \pmod{4}$.

**Proof.** Let $T$ be a tree of order $n$ each of whose vertices is odd. Since the order has to be even, $n = 4k$ for some positive integer $k$. If $T$ is a star, then let $f : E(T) \to \mathbb{Z}_n$ such that $\{f(uv) : uv \in E(T)\} = \mathbb{Z}_n - \{k\}$ and this is a modular edge-graceful labeling. Therefore, assume that $T$ is not a star. Let $u$ be a cut-vertex of $T$ and

$$N_i = \{v \in V(T) : d(v, u) = i\}$$

for $0 \leq i \leq e$, where $e = e(u)$ is the eccentricity of $u$. Therefore, $N_0 = \{u\}$ and $N_1 = N(u)$. Let $X$ be the set of cut-vertices of $T$. For each $i$ with $0 \leq i \leq e - 1$, let

$$N_i^* = N_i \cap X = \{v_{i,1}, v_{i,2}, \ldots, v_{i,s_i}\},$$

where $s_i$ is the number of cut-vertices distance $i$ from $u$. In particular, $N_0^* = \{u\}$ and $s_0 = 1$. Since $T$ is not a star, the number of cut-vertices
of \( T \) is at least 2.

For each cut-vertex \( v_{i,j} \in N_i^* \), where \( 0 \leq i \leq e - 1 \) and \( 1 \leq j \leq s_i \), let

\[
 r_{i,j} = |N(v_{i,j}) \cap N_{i+1}|.
\]

Thus \( r_{0,1} = \deg u \geq 3 \) is odd and \( r_{i,j} = \deg v_{i,j} - 1 \geq 2 \) is even for \( 1 \leq i \leq e - 1 \) and \( 1 \leq j \leq s_i \). Consider \( s \) pairwise disjoint subsets

\[
 A_{0,1}, A_{1,1}, A_{1,2}, \ldots, A_{1,s_1}, A_{2,1}, A_{2,2}, \ldots, A_{2,s_2}, \ldots, A_{e-1,1}, A_{e-1,2}, \ldots, A_{e-1,s_{e-1}}
\]

of \( \mathbb{Z}_n \) for which

(i) \( \cup_{i=0}^{e-1} (\cup_{j=1}^{s_i} A_{i,j}) = \mathbb{Z}_n - \{0, \pm k, 2k\} \),

(ii) \( |A_{0,1}| = r_{0,1} - 3 \) and \( |A_{i,j}| = r_{i,j} \) for \( 1 \leq i \leq e - 1 \) and \( 1 \leq j \leq s_i \) (and so each \( A_{i,j} \) has an even cardinality) and

(iii) \( a \in A_{i,j} \) if and only if \( -a \in A_{i,j} \) for each \( A_{i,j} \).

Define an edge labeling \( f : E(T) \to \mathbb{Z}_n \) of \( T \) such that for each pair \( i, j \) of integers with \( 0 \leq i \leq e - 1 \) and \( 1 \leq j \leq s_i \)

\[
 \{ f(vv_{i,j}) : v \in N(v_{i,j}) \cap N_{i+1} \} = \begin{cases} 
 A_{0,1} \cup \{0, k, 2k\} & \text{if } (i,j) = (0,1) \\
 A_{i,j} & \text{otherwise.}
\end{cases}
\]

Observe that

\[
 \{ f'(v) : v \in N_i \} = \begin{cases} 
 \{-k\} & \text{if } i = 0 \\
 A_{0,1} \cup \{0, k, 2k\} & \text{if } i = 1 \\
 A_{i-1,1} \cup A_{i-1,2} \cup \cdots \cup A_{i-1,s_{i-1}} & \text{if } 2 \leq i \leq e.
\end{cases}
\]
Therefore, $f$ is a modular edge-graceful labeling of $T$. 

As we mentioned earlier, each result stated in Theorem 3.3.3, except for (d), is a consequence of Proposition 3.3.2 and results obtained in this section. For completeness, we present an independent proof of Theorem 3.3.3(d), which is stated in the following result.

**Theorem 3.4.2** A tree of order $n \geq 3$ containing exactly one even vertex is modular edge-graceful if and only if $n$ is odd.

**Proof.** Let $T$ be a tree of order $n$ with exactly one even vertex. Then $n$ must be odd. It remains to show that $T$ is modular edge-graceful. If $T$ is a star, then let $f : E(T) \to \mathbb{Z}_n$ such that $\{f(uv) : uv \in E(T)\} = \mathbb{Z}_n - \{0\}$ and this is a modular edge-graceful labeling. We therefore assume that $T$ is not a star. Let $u$ be the only even vertex of $T$ and $N_i = \{v \in V(T) : d(v, u) = i\}$ for $0 \leq i \leq e$, where $e = e(u)$ is the eccentricity of $u$. Let $X$ be the set of cut-vertices of $T$. For each $i$ with $0 \leq i \leq e - 1$, let $N^*_i = N_i \cap X = \{v_{i,1}, v_{i,2}, \ldots, v_{i,s_i}\}$, where $s_i$ is the number of cut-vertices distance $i$ from $u$. Since $T$ is not a star, the number of cut-vertices $|X| = s_0 + s_1 + \cdots + s_{e-1} = s$ of $T$ is at least 2. For each cut-vertex $v_{i,j} \in N^*_i$, where $0 \leq i \leq e - 1$ and $1 \leq j \leq s_i$, let $r_{i,j} = |N(v_{i,j}) \cap N_{i+1}|$. Thus $r_{0,1} = \deg u \geq 2$ is even and $r_{i,j} = \deg v_{i,j} - 1 \geq 2$ is even for $1 \leq i \leq e - 1$ and $1 \leq j \leq s_i$. Consider $s$ pairwise disjoint subsets $A_{i,j}$ ($0 \leq i \leq e - 1$ and $1 \leq j \leq s_i$) of $\mathbb{Z}_n$ for which

(i) $\bigcup_{i=0}^{e-1} \left( \bigcup_{j=1}^{s_i} A_{i,j} \right) = \mathbb{Z}_n - \{0\}$,

(ii) $|A_{i,j}| = r_{i,j}$ for each $A_{i,j}$ and
(iii) \( a \in A_{i,j} \) if and only if \(-a \in A_{i,j}\) for each \( A_{i,j} \).

Define an edge labeling \( f : E(T) \to \mathbb{Z}_n \) of \( T \) such that

\[
\{f(vv_{i,j}) : v \in N(v_{i,j}) \cap N_{i+1}\} = A_{i,j}
\]

for each pair \( i, j \) of integers with \( 0 \leq i \leq e - 1 \) and \( 1 \leq j \leq s_i \). Since

\[
\{f'(v) : v \in N_i\} = \begin{cases} \{0\} & \text{if } i = 0 \\ A_{i-1,1} \cup A_{i-1,2} \cup \ldots \cup A_{i-1,s_{i-1}} & \text{if } 1 \leq i \leq e, \end{cases}
\]

it follows that \( f \) is a modular edge-graceful labeling of \( T \).

As a consequence of Theorems 3.4.1 and 3.4.2, a star is modular edge-graceful if and only if \( n \equiv 2 \pmod{4} \), which is Theorem 3.3.3(a).

**Theorem 3.4.3** A tree of order \( n \geq 3 \) in which exactly two vertices have even degree is modular edge-graceful if and only if \( n \equiv 0 \pmod{4} \).

**Proof.** Let \( T \) be a tree of order \( n \) where \( u \) and \( w \) are the only even vertices. In this case, \( n \) must be even and so \( n = 4k \) for some positive integer \( k \). Suppose that the eccentricity of \( u \) is \( e(u) = e \). Let \( N_i = \{v \in V(T) : d(v, u) = i\} \) for \( 0 \leq i \leq e \).

Therefore, \( N_0 = \{u\} \) and \( N_1 = N(u) \). Let \( X \) be the set of cut-vertices of \( T \). For each \( i \) with \( 0 \leq i \leq e - 1 \), let

\[
N_i^* = N_i \cap X = \{v_{i,1}, v_{i,2}, \ldots, v_{i,s_i}\},
\]

where \( s_i \) is the number of cut-vertices distance \( i \) from \( x \). The number of cut-vertices

\[
|X| = s_0 + s_1 + \cdots + s_{e-1} = s \text{ of } T \text{ is at least } 2.
\]
Suppose that \( w = v_{w,1} \). For each cut-vertex \( v_{i,j} \in N_i^* \) (\( 0 \leq i \leq e - 1 \) and \( 1 \leq j \leq s_i \)), let \( r_{i,j} = |N(v_{i,j}) \cap N_{i+1}| \). Thus \( r_{0,1} = \deg u \geq 2 \) is even, \( r_{m,1} \geq 1 \) is odd and \( r_{s,1} = \deg v_{s,1} - 1 \geq 2 \) is even if \( 1 \leq i \leq e - 1 \), \( 1 \leq j \leq s_i \) and \((i,j) \neq (0,1)\).

Consider \( s \) pairwise disjoint subsets \( A_{i,j} \) (\( 0 \leq i \leq e - 1 \) and \( 1 \leq j \leq s_i \)) of \( \mathbb{Z}_n \) for which

(i) \( \bigcup_{i=0}^{e-1} \left( \bigcup_{j=1}^{s_i} A_{i,j} \right) = \mathbb{Z}_n - \{0, \pm k, 2k\} \),

(ii) \( |A_{0,1}| = r_{0,1} - 2 \), \( |A_{0,1}| = r_{w,1} - 1 \) and \( |A_{i,j}| = r_{i,j} \) for \( 1 \leq i \leq e - 1 \), \( 1 \leq j \leq s_i \) and \((i,j) \neq (i_0,1)\) and

(iii) \( a \in A_{i,j} \) if and only if \( -a \in A_{i,j} \) for each \( A_{i,j} \).

Define an edge labeling \( f: E(T) \to \mathbb{Z}_n \) of \( T \) such that for each pair \( i, j \) of integers with \( 0 \leq i \leq e - 1 \) and \( 1 \leq j \leq s_i \)

\[
\{f(v_{i,j}) : v \in N(v_{i,j}) \cap N_{i+1}\} = \begin{cases} 
A_{0,1} \cup \{k, 2k\} & \text{if } (i, j) = (0, 1) \\
A_{w,1} \cup \{0\} & \text{if } (i, j) = (i_0, 1) \\
A_{i,j} & \text{otherwise.}
\end{cases}
\]

For each \( i \) with \( 0 \leq i \leq e - 1 \), let \( A_i = A_{i,1} \cup A_{i,2} \cup \cdots \cup A_{i,s_i} \), where then \( A_0 = A_{0,1} \).

Observe that

\[
\{f'(v) : v \in N_i\} = \begin{cases}
\{-k\} & \text{if } i = 0 \\
A_0 \cup \{k, 2k\} & \text{if } i = 1 \\
A_{i_0} \cup \{0\} & \text{if } i = i_0 + 1 \\
A_{i-1} & \text{if } 2 \leq i \leq e \text{ and } i \neq i_0 + 1.
\end{cases}
\]

Thus \( f \) is a modular edge-graceful labeling of \( T \).

By Theorems 3.4.1 - 3.4.3, we have the following corollary.
Corollary 3.4.4  If $T$ is a tree of order $n \geq 3$ with $n \not\equiv 2 \pmod{4}$ having at most two even vertices, then $T$ is modular edge-graceful.

A double star is a tree whose diameter is 3. Since a double star has at most two even vertices, the following is a consequence of Corollary 3.4.4.

Corollary 3.4.5  A double star of order $n \geq 4$ is modular edge-graceful if and only if $n \not\equiv 2 \pmod{4}$

Next, we show that if $T$ is a tree of order $n$ with $n \not\equiv 2 \pmod{4}$ such that the set of even vertices in $T$ induces a path, then $T$ is modular edge-graceful.

Theorem 3.4.6  A tree of order $n \geq 3$ in which the set of even vertices induces a path is modular edge-graceful if and only if $n \not\equiv 2 \pmod{4}$

Proof.  Let $T$ be a tree of order $n$ containing $\ell$ even vertices, where the set
\[
\{v_0, v_1, \ldots, v_{\ell-1}\}
\]
of even vertices induces a path $P = (v_0, v_1, v_2, \ldots, v_{\ell-1})$. By Corollary 3.4.4 we may assume that $\ell \geq 3$. Let $N_i = \{v \in V(T) \mid d(v, v_0) = i\}$ for $0 \leq i \leq e$, where $e = e(v_0)$ is the eccentricity of $v_0$. (Note that $e \geq \ell$.) Let $X$ be the set of cut vertices of $T$. For each $i$ with $0 \leq i \leq e - 1$, let $N_i^* = N_i \cap X = \{v_{i1}, v_{i2}, \ldots, v_{is_i}\}$, where $s_i$ is the number of cut-vertices distance $i$ from $v_0$. Furthermore, assume that $v_{i1} = v_i$ for $0 \leq i \leq \ell - 1$. Observe that $|X| = s_0 + s_1 + \ldots + s_{e-1} = s \geq \ell \geq 3$. For each cut-vertex $v_{ij} \in N_i^*$, where $0 \leq i \leq e - 1$ and $1 \leq j \leq s_i$, let $r_{i,j} = |N(v_{ij}) \cap N_{i+1}|$. We consider two cases according to the parity of $\ell$. 

Case 1. $\ell$ is odd. Then $n$ is odd. We first define an edge labeling $f_1 : E(P) \to \mathbb{Z}_n$. Let $f_1(v_0v_1) = 1$. If $\ell = 3$, then let $f_1(v_1v_2) = -1$. Otherwise, we define $f_1$ recursively as follows. Suppose that $f_1(v_{i-1}v_i)$ has been defined for $1 \leq i \leq \lfloor \ell/2 \rfloor - 1$. Then $f_1(v_i v_{i+1}) = i + 2 - f_1(v_{i-1}v_i)$. For $\lfloor \ell/2 \rfloor \leq i \leq \ell - 1$, we let 

\[ f_1(v_i v_{i+1}) = f_1(v_{\ell-2-i} v_{\ell-1-i}) \]

We also define an edge labeling $f_2 : E(T) - E(P) \to \mathbb{Z}_n$ as follows. First consider $s$ pairwise disjoint subsets $A_{i,j}$ ($0 \leq i \leq \ell - 1$ and $1 \leq j \leq s_i$) of $\mathbb{Z}_n$ for which

(i) $\bigcup_{i=0}^{\ell-1} \left( \bigcup_{j=1}^{s_i} A_{i,j} \right) = \mathbb{Z}_n - \{0, \pm 1, \pm 2, \ldots, \pm \lfloor \ell/2 \rfloor \},$

(ii) $|A_{0,1}| = r_{0,1} - 2$, $|A_{i,1}| = r_{i,1} - 1$ for $1 \leq i \leq \ell - 1$ and $|A_{i,j}| = r_{i,j}$ otherwise and

(iii) $a \in A_{i,j}$ if and only if $-a \in A_{i,j}$ for each $A_{i,j}$.

For $0 \leq i \leq \ell - 1$ and $1 \leq j \leq s_i$, define

\[ E_{i,j} = \{ vv_{i,j} : v \in N(v_{i,j}) \cap N_{i+1} \} - E(P). \]

Let $f_2 : E(T) - E(P) \to \mathbb{Z}_n$ be given by

\[ \{ f_2(vv_{i,j}) : vv_{i,j} \in E_{i,j} \} = \begin{cases} A_{0,1} \cup \{1\} & \text{if } (i,j) = (0,1) \\ A_{\ell-1,1} \cup \{-1\} & \text{if } (i,j) = (\ell - 1,1) \\ A_{i,j} & \text{otherwise.} \end{cases} \]

Now define the edge labeling $f : E(T) \to \mathbb{Z}_n$ by

\[ f(uv) = \begin{cases} f_1(uv) & \text{if } uv \in E(P) \\ f_2(uv) & \text{if } uv \in E(T) - E(P). \end{cases} \]
Then $f$ is a modular edge-graceful labeling of $T$.

**Case 2. $\ell$ is even.** Then $\ell \geq 4$ and we may assume that $n = 4k$ for some integer $k \geq 2$ since $n$ must be even. Again we first define an edge labeling $f_1 : E(P) \to \mathbb{Z}_n$. If $\ell = 4$, then let $(f_1(v_0v_1), f_1(v_1v_2), f_1(v_2v_3)) = (-1, 0, k)$. If $\ell = 6$, then

$$
(f_1(v_0v_1), f_1(v_1v_2), \ldots, f_1(v_4v_5)) = \begin{cases} 
(2, -2, -1, 0, 2) & \text{if } k = 2, \text{ i.e., if } T = P_8 \\
(1, -1, -1, 0, k) & \text{if } k \geq 3.
\end{cases}
$$

If $\ell \geq 8$, then let $f_1(v_0v_1) = 1$. We define $f_1$ recursively as follows. Suppose that $f_1(v_{i-1}v_i)$ has been defined for $1 \leq i \leq \ell/2 - 3$. Then

$$
f_1(v_iv_{i+1}) = \begin{cases} 
 i + 2 - f_1(v_{i-1}v_i) & \text{if } i \leq \min\{k, \ell/2\} - 3 \\
 i + 3 - f_1(v_{i-1}v_i) & \text{if } \min\{k, \ell/2\} - 2 \leq i \leq \ell/2 - 2.
\end{cases}
$$

Now for $\ell/2 - 2 \leq i \leq \ell - 2$ let

$$
f_1(v_iv_{i+1}) = \begin{cases} 
 -f_1(v_{i-5}v_{i-4}) & \text{if } \ell/2 - 2 \leq i \leq \ell - 5 \\
 -1 & \text{if } i = \ell - 4 \\
 0 & \text{if } i = \ell - 3 \\
k & \text{if } i = \ell - 2.
\end{cases}
$$

We next define an edge labeling $f_2 : E(T) - E(P) \to \mathbb{Z}_n$ as follows. First consider $s$ pairwise disjoint subsets $A_{i,j}$ ($0 \leq i \leq s - 1$ and $1 \leq j \leq s_i$) of $\mathbb{Z}_n$ for which

(i) $\bigcup_{i=0}^{s-1} \left( \bigcup_{j=1}^{s_i} A_{i,j} \right) = \mathbb{Z}_n - \{0, \pm 1, \pm 2, \ldots, \pm \alpha, \pm k, 2k\}$, where $\alpha = \ell/2 - 1$ if $\ell \leq 2k$ and $\alpha = \ell/2$ if $\ell \geq 2k + 2$,

(ii) $|A_{0,1}| = r_{0,1} - 2$, $|A_{i,1}| = r_{i,1} - 1$ for $1 \leq i \leq \ell - 1$ and $|A_{i,j}| = r_{i,j}$ otherwise and

(iii) $a \in A_{i,j}$ if and only if $-a \in A_{i,j}$ for each $A_{i,j}$. 
For $0 < r < \ell - 1$ and $1 < j < s$, define $E_{i,j} = \{vv \in N(v_i) \cap N_{i+1} \} - E(P)$

Let $f_2: E(T) - E(P) \rightarrow \mathbb{Z}_n$ be given by

$$
\{f_2(vv_{i,j}) \in E_{i,j} \} = \begin{cases} 
A_{0,1} \cup \{1\} & \text{if } (r,j) = (0,1) \\
A_{\ell-1,1} \cup \{2k\} & \text{if } (r,j) = (\ell - 1,1) \\
A_{i,j} & \text{otherwise}
\end{cases}
$$

Define the edge labeling $f: E(T) \rightarrow \mathbb{Z}_n$ as described in (3.2). One can now verify that $f$ is a modular edge-graceful labeling of $T$.

As a consequence of Theorem 3.4.6, a path of order $n \geq 3$ is modular edge-graceful if and only if $n \equiv 2 \pmod{4}$, which is Theorem 3.3.3(b).

We have seen in Theorem 3.3.3(a) and Corollary 3.4.5 that if $T$ is a star or a double star of order $n$ with $n \not\equiv 2 \pmod{4}$, then $T$ is modular edge-graceful. Thus every tree of order $n \geq 3$ with diameter 2 or 3 is modular edge-graceful if and only if $n \not\equiv 2 \pmod{4}$. Next we determine all trees of diameter 4 that are modular edge-graceful.

**Theorem 3.4.7** A tree of order $n \geq 5$ having diameter 4 is modular edge-graceful if and only if $n \equiv 2 \pmod{4}$

**Proof.** Let $T$ be a tree of order $n \geq 5$ and diam$(T) = 4$. Let $v_0$ be the central vertex and $N(v_0) = N_0 \cup N_1$, where $v \in N_0$ if $v \in N(v_0)$ and $v$ is even while $v \in N_1$ if $v \in N(v_0)$ and $v$ is odd. We may assume that $N(v_0) = \{v_1, v_2, \ldots, v_d\}$, where $d = \deg v_0$, and $v_i \in N_0$ if and only if $1 \leq i \leq |N_0|$. We consider two cases, according to the parity of $d$. 

Case 1. \( d \) is even. By Theorem 3.4.6, we may assume that \(|N_0| \geq 3\). Furthermore, \(|N_0|\) and \(|N_1|\) are of the same parity and so there are two subcases.

Subcase 1.1. Both \(|N_0|\) and \(|N_1|\) are even. Then \(|N_0| \geq 4\). Since there are exactly \( n - (|N_0| + 1) \) odd vertices, \( n \) is odd. Consider \( d + 1 \) pairwise disjoint sets

\[ A_0, A_1, A_2, \ldots, A_d \subseteq \mathbb{Z}_n \]

such that

(i) \( \bigcup_{i=0}^{d} A_i = \mathbb{Z}_n - \{0, \pm 1, \pm 2, \ldots, \pm |N_0|\} \),

(ii) \(|A_0| = |N_1|, |A_i| = \deg v_i - 2\) if \( 1 \leq i \leq |N_0| \) and \(|A_i| = \deg v_i - 1\) if \(|N_0| + 1 \leq i \leq d\) and

(iii) \( a \in A_i\) if and only if \(-a \in A_i\) for \( 0 \leq i \leq d \).

Let \( f : E(T) \rightarrow \mathbb{Z}_n \) be an edge labeling of \( T \) such that for \( 1 \leq i \leq d \),

\[ \{f(v_0v_i) : |N_0| + 1 \leq i \leq d\} = A_0 \]

and

\[ f(v_0v_i) = \begin{cases} 1 & \text{if } 1 \leq i \leq |N_0|/2 \\ -1 & \text{if } |N_0|/2 + 1 \leq i \leq |N_0| \end{cases} \]

\[ \{f(vv_i) : v \in N(v_i) - \{v_0\}\} = \begin{cases} A_i \cup \{2i - 1\} & \text{if } 1 \leq i \leq |N_0|/2 \\ A_i \cup \{|N_0| + 1 - 2i\} & \text{if } |N_0|/2 + 1 \leq i \leq |N_0| \\ A_i & \text{if } |N_0| + 1 \leq i \leq d. \end{cases} \]

Then \( f \) is a modular edge-graceful labeling of \( T \).
Subcase 1.2. Both $|N_0|$ and $|N_1|$ are odd. Then $n = 4k$ for some integer $k \geq 2$.

We first assume that $|N_0| \leq k$. Consider $d + 1$ pairwise disjoint sets

$$A_0, A_1, A_2, \ldots, A_d \subseteq \mathbb{Z}_n$$

such that

(i) $\bigcup_{i=0}^{d} A_i = \mathbb{Z}_n - \{0, \pm 1, \pm 2, \ldots, \pm (|N_0| - 1), \pm k, 2k\}$,

(ii) $|A_0| = |N_1| - 1$, $|A_i| = \deg v_i - 2$ if $1 \leq i \leq |N_0|$ and $|A_i| = \deg v_i - 1$ if $|N_0| + 1 \leq i \leq d$ and

(iii) $a \in A_i$ if and only if $-a \in A_i$ for $0 \leq i \leq d$.

Let $f : E(T) \to \mathbb{Z}_n$ be an edge labeling of $T$ such that for $1 \leq i \leq d$,

$$\{f(v_0v_i) : |N_0| + 1 \leq i \leq d\} = A_0 \cup \{0\}$$

and

$$f(v_0v_i) = \begin{cases} 
1 & \text{if } 1 \leq i \leq \lfloor |N_0|/2 \rfloor \\
-1 & \text{if } \lceil |N_0|/2 \rceil \leq i \leq |N_0| - 1 \\
k & \text{if } i = |N_0|
\end{cases}$$

$$\{f(vv_i) : v \in N(v_i) - \{v_0\}\} = \begin{cases} 
A_i \cup \{2i - 1\} & \text{if } 1 \leq i \leq \lfloor |N_0|/2 \rfloor \\
A_i \cup \{|N_0| - 2i\} & \text{if } \lfloor |N_0|/2 \rfloor \leq i \leq |N_0| - 1 \\
A_{|N_0|} \cup \{2k\} & \text{if } i = |N_0| \\
A_i & \text{if } |N_0| + 1 \leq i \leq d.
\end{cases}$$

Then $f$ is a modular edge-graceful labeling of $T$. Such a labeling is shown in Figure 3.12 for a tree of order $n = 20$, where $|N_0| = k = 5$. (The solid vertices are the ones belonging to $N_0$.)
When $|N_0| \geq k + 1$, a modular edge-graceful labeling of $T$ can be constructed by a similar argument except that for each $E_i = \{vv_i \in N(v_i) - \{v_0\}\}$ with $1 \leq i \leq |N_0| - 1$, no edges in $E_i$ are assigned labels in $\{\pm(k-1), \pm k\}$. A labeling is shown in Figure 3.13 for a tree of order $n = 16$, where $|N_0| = 5 > 4 = k$. In this case, no label in $\{\pm 3, \pm 4\}$ is used to label any edge in $E_i$ for $1 \leq i \leq 4$.

Figure 3.13 A modular edge-graceful labeling in Subcase 1.2 where $|N_0| \geq k + 1$

**Case 2**  

**$d$ is odd** By Corollary 3.4.4, we may assume that $|N_0| \geq 3$. In this case, $|N_0|$ and $|N_1|$ are of the opposite parity.

**Subcase 2.1**  

$|N_0|$ is odd Then $n$ is odd since there are $n - |N_0|$ odd vertices.

Consider $d + 1$ pairwise disjoint sets $A_0, A_1, A_2, \ldots, A_d \subseteq \mathbb{Z}_n$ such that
(i) $\bigcup_{i=0}^{d} A_i = \mathbb{Z}_n - \{0, \pm 1, \pm 2, \ldots, \pm (|N_0| - 1), \pm [n/2]\},$

(ii) $|A_0| = |N_1|$, $|A_i| = \deg v_i - 2$ if $1 \leq i \leq |N_0|$ and $|A_i| = \deg v_i - 1$ if $|N_0| + 1 \leq i \leq d$ and

(iii) $a \in A_i$ if and only if $-a \in A_i$ for $0 \leq i \leq d$.

Let $f : E(T) \to \mathbb{Z}_n$ be an edge labeling of $T$ such that for $1 \leq i \leq d$,

$$\{f(v_0v_i) : |N_0| + 1 \leq i \leq d\} = A_0$$

and

$$f(v_0v_i) = \begin{cases} 
1 & \text{if } 1 \leq i \leq [|N_0|/2] \\
-1 & \text{if } [|N_0|/2] \leq i \leq |N_0| - 1 \\
[n/2] & \text{if } i = |N_0|
\end{cases}$$

$$\{f(vv_i) : v \in N(v_i) - \{v_0\}\} = \begin{cases} 
A_i \cup \{2i - 1\} & \text{if } 1 \leq i \leq [|N_0|/2] \\
A_i \cup \{|N_0| - 2i\} & \text{if } [|N_0|/2] \leq i \leq |N_0| - 1 \\
A_{|N_0|} \cup \{[n/2]\} & \text{if } i = |N_0| \\
A_i & \text{if } |N_0| + 1 \leq i \leq d.
\end{cases}$$

Then $f$ is a modular edge-graceful labeling of $T$.

**Subcase 2.2.** $|N_0|$ is even. Then $|N_0| \geq 4$ and $|N_1| \geq 1$. Also, $n = 4k$ for some integer $k \geq 3$. Suppose first that $|N_0| \leq k + 1$. Consider $d + 1$ pairwise disjoint
sets $A_0, A_1, A_2, \ldots, A_d \subseteq \mathbb{Z}_n$ such that

(i) $\bigcup_{i=0}^{d} A_i = \mathbb{Z}_n - \{0, \pm 1, \pm 2, \ldots, \pm (|N_0| - 2), \pm k, \pm (2k - 1), 2k\},$

(ii) $|A_0| = |N_1| - 1$, $|A_i| = \deg v_i - 2$ if $1 \leq i \leq |N_0|$ and $|A_i| = \deg v_i - 1$ if $|N_0| + 1 \leq i \leq d$ and
(iii) $a \in A_i$ if and only if $-a \in A_i$ for $0 \leq i \leq d$

Then let $f : E(T) \rightarrow \mathbb{Z}_n$ be an edge labeling of $T$ such that for $1 \leq i \leq d$,

$$\{f(v_0v_i) \mid N_0 + 1 \leq i \leq d\} = A_0 \cup \{2k\}$$

and

$$f(v_0v_i) = \begin{cases} 1 & \text{if } 1 \leq i \leq |N_0|/2 - 1 \\ -1 & \text{if } |N_0|/2 \leq i \leq |N_0| - 2 \\ k + 1 & \text{if } i = |N_0| - 1 \\ 2k - 1 & \text{if } i = |N_0| \end{cases}$$

$$\{f(v)v \in N(v_i) - \{v_0\}\} = \begin{cases} A_i \cup \{2i - 1\} & \text{if } 1 \leq i \leq |N_0|/2 - 1 \\ A_i \cup \{|N_0| - 1 - 2i\} & \text{if } |N_0|/2 \leq i \leq |N_0| - 2 \\ A_{|N_0|-1} \cup \{2k - 1\} & \text{if } i = |N_0| - 1 \\ A_{|N_0|} \cup \{2k + 1\} & \text{if } i = |N_0| \\ A_i \cup \{|N_0| + 1 - i\} & \text{if } |N_0| + 1 \leq i \leq d \end{cases}$$

Then $f$ is a modular edge-graceful labeling of $T$. The case with $|N_0| \geq k + 2$ can be treated by a similar manner described in Subcase 12.

By an argument similar to the one used in the proof of Theorem 3.4.7 and a case-by-case analysis, we are also able to determine all trees of diameter 5 that are modular edge-graceful. Hence, we have the following

**Theorem 3.4.8** A tree of order $n \geq 3$ having diameter at most 5 is modular edge-graceful if and only if $n \not\equiv 2 \pmod{4}$

Figure 3.14 shows six trees of order 9 having diameter 6 along with modular edge-graceful labelings.
Furthermore, if $T$ is a tree of order 8 having diameter 6 or if $T$ is a tree of order 9 having diameter 7, then $T$ is modular edge-graceful. To see this, let $T_1$ be a tree of order 9 having diameter 7 that is constructed from the path $P_8 = (v_1, v_2, \ldots, v_8)$ of order 8 by adding the edge $v_7v_9$ for some $r$ with $2 \leq r \leq 7$. Then the coloring $f_1 : E(T_1) \to \mathbb{Z}_9$ defined by $f_1(v_jv_{j+1}) = j$ for $1 \leq j \leq 4$, $f_1(v_9v_0) = f_1(v_1v_0) = 0$, $f_1(v_7v_9) = 2$ and $f_1(v_{7}v_{8}) = 6$ is a modular edge-graceful labeling of $T_1$. Next, let $T_2$ be a tree of order 8 having diameter 6 that is constructed from the path $P_7 = (v_1, v_2, \ldots, v_7)$ of order 7 by adding the edge $v_7v_8$ for some $r$ with $2 \leq r \leq 6$. Then the coloring $f_2 : E(T_2) \to \mathbb{Z}_8$ defined by $f_2(v_jv_{j+1}) = j$ for $1 \leq j \leq 4$, $f_2(v_7v_8) = f_2(v_6v_7) = 2$ and $f_2(v_1v_8) = 0$ is a modular edge-graceful labeling of $T_2$. Thus $T_1$ and $T_2$ are modular edge-graceful. It then follows from Corollary 3.4.4 and Theorems 3.4.6 and 3.4.8 that all trees of order $3 \leq n \leq 9$ and $n \neq 6$ are modular edge-graceful, which is Theorem 3.3.3(f).
3.5 A Characterization of Modular Edge-Graceful Trees

In the preceding section, we not only showed that Conjecture 3.3.4 holds for many classes of trees, we described a modular edge-graceful labeling for the modular edge-graceful trees in those classes. In this section, we verify Conjecture 3.3.4 in its entirety, that is, if \( T \) is a tree of order \( n \geq 3 \) for which \( n \not\equiv 2 \pmod{4} \), then \( T \) is modular edge-graceful. First, we present a result dealing with modular edge-graceful graphs that has the same flavor as the Bondy and Chvátal theorem on Hamiltonian graphs and closures (see [4]). First, we present a lemma.

**Lemma 3.5.1** Let \( G \) be a connected graph of order at least 3 containing two nonadjacent vertices \( u \) and \( v \) that are connected by a path of odd length. Then the graph \( G + uv \) is modular edge-graceful if and only if \( G \) is modular edge-graceful.

**Proof.** Since \( G \) is a connected spanning subgraph of \( G + uv \), it then follows by Proposition 3.3.2 that if \( G \) is modular edge-graceful, then so is \( G + uv \). For the converse, assume that \( G + uv \) is modular edge-graceful and let \( f \colon V(G+uv) \rightarrow \mathbb{Z}_n \) be a modular edge-graceful labeling of \( G + uv \). Suppose that \( P \) is a \( u - v \) path of odd length in \( G \), say \( P = (u = v_1, v_2, \ldots, v_p = v) \) where \( p \geq 4 \) is even. Now define the edge labeling \( g \colon V(G) \rightarrow \mathbb{Z}_n \) of \( G \) by

\[
g(e) = \begin{cases} 
  f(e) & \text{if } e \notin E(P) \\
  f(e) + f(uv) & \text{if } e = v_i v_{i+1}, 1 \leq i \leq p - 1 \text{ and } i \text{ is odd} \\
  f(e) - f(uv) & \text{if } e = v_i v_{i+1}, 2 \leq i \leq p - 2 \text{ and } i \text{ is even}
\end{cases}
\]
Since \( g'(r) = f'(r) \) in \( \mathbb{Z}_n \) for all \( r \in V(G) \), it follows that \( g \) is a modular edge graceful labeling of \( G \). Thus \( G \) is modular edge-graceful.

Let \( G \) be a connected graph of order at least 3 and let \( \mathcal{P} \) be a partition of \( V(G) \) into two or more independent sets. Recall that the odd path closure of \( G \) with respect to \( \mathcal{P} \), denoted by \( C_0(G, \mathcal{P}) \) (or simply by \( C_0(G) \) if the partition \( \mathcal{P} \) under consideration is clear), to be the graph obtained from \( G \) by recursively joining pairs of nonadjacent vertices that belong to different independent sets in \( \mathcal{P} \) and that are connected by a path of odd length in \( G \). Repeated applications of Lemma 3.5.1 give us the following result on modular edge-graceful graphs and odd path closures.

**Corollary 3.5.2** Let \( G \) be a connected graph of order at least 3, let \( \mathcal{P} \) be a partition of \( V(G) \) into two or more independent sets, and let \( C_0(G) \) be the odd path closure of \( G \) with respect to \( \mathcal{P} \). Then \( C_0(G) \) is modular edge-graceful if and only if \( G \) is modular edge-graceful.

Of course, every nontrivial tree is a connected bipartite graph. We saw in Chapter 2 (Proposition 2.5.6) that the odd path closure of a connected bipartite graph of order at least 3 with respect to given partite sets is a complete bipartite graph. We restate this result and its proof here for completion.

**Lemma 3.5.3** Let \( G \) be a connected bipartite graph with partite sets \( U \) and \( W \) where \( |U| = r \) and \( |W| = s \) and \( r + s \geq 3 \). Then the odd path closure \( C_0(G) \) of \( G \) with respect to the partition \( \{U, W\} \) is \( K_{r,s} \).
Proof. First, observe that $C_0(G)$ is a bipartite graph with partite sets $U$ and $W$. If $C_0(G) \neq K_{r,s}$, then there are vertices $u \in U$ and $w \in W$ such that $uw \notin E(C_0(G))$. Since $C_0(G)$ is bipartite,

\[ U = \{ v \in V(C_0(G)) \mid d_{C_0(G)}(u, v) \text{ is even} \} \]
\[ W = \{ v \in V(C_0(G)) \mid d_{C_0(G)}(u, v) \text{ is odd} \} \]

Since $w \in W$, it follows that $d_{C_0(G)}(u, w)$ is odd. Thus $uw \in E(C_0(G))$, which is a contradiction.

For positive integers $a$ and $b$, let $S_{a,b}$ be the double star of order $a + b$ whose central vertices have degrees $a$ and $b$, respectively. By Theorem 3.4.8, every double star $S_{a,b}$ is modular edge-graceful if $a + b \neq 2 \pmod{4}$. We are now prepared to present the following modular edge-graceful trees theorem.

**Theorem 3.5.4** A tree $T$ of order $n \geq 3$ is modular edge-graceful if and only if $n \neq 2 \pmod{4}$

Proof. We have seen that if $n \equiv 2 \pmod{4}$, then $T$ is not modular edge-graceful. For the converse, assume that $n \neq 2 \pmod{4}$. Let $U$ and $W$ be the partite sets of $T$ with $|U| = r$ and $|W| = s$. By Lemma 3.5.3, the odd path closure $C_0(G)$ of $G$ with respect to the partition $\{U, W\}$ is $K_{r,s}$. By Corollary 3.5.2, it suffices to show that $G = K_{r,s}$ is modular edge-graceful. If $r = 1$ or $s = 1$, then $K_{r,s}$ is a star and so it is modular edge-graceful by Theorem 3.4.8. If $r \geq 2$ and $s \geq 2$, then the double star $S_{r,s}$ of order $r + s$ is a modular edge-graceful spanning subgraph.
of $K_{r,s}$. It then follows by Proposition 3.3.2 that $K_{r,s}$ is modular edge-graceful. Therefore, $T$ is modular edge-graceful by Corollary 3.5.2.

Unlike the results presented in the preceding section where modular edge-graceful labelings were described for the classes of trees presented there, the argument used to verify Theorem 3.5.4 only shows the existence of modular edge-graceful labelings of trees. The following is a consequence of Proposition 3.3.2 and Theorem 3.5.4.

**Corollary 3.5.5** Let $G$ be a connected graph of order $n \geq 3$. Then $G$ is a modular edge-graceful graph if and only if $n \equiv 2 \pmod{4}$.

### 3.6 Modular Edge-Gracefulness of Graphs

If $G$ is a connected graph of order $n \geq 3$ for which $n \equiv 2 \pmod{4}$, then, of course, $G$ is not modular edge-graceful. Then there is a smallest integer $k > n$ for which there exists an edge labeling $f : E(G) \to \mathbb{Z}_k$ such that the induced vertex labeling $f' : V(G) \to \mathbb{Z}_k$ defined by

$$f'(v) = \sum_{u \in N(v)} f(uv),$$

where the sum is computed in $\mathbb{Z}_n$, is one-to-one. This number $k$ is referred to as the *modular edge-gracefulness* $\text{meg}(G)$ of $G$. Thus $\text{meg}(G) \geq n$ and $\text{meg}(G) = n$ if and only if $G$ is a modular edge-graceful graph of order $n$. Thus, if $G$ is not modular edge-graceful, then $\text{meg}(G) \geq n + 1$. As in the case of the gracefulfulness of
a graph, the modular edge-gracefulness of a graph $G$ is a measure of how close $G$
is to being modular edge-graceful. In this section, we show that $\text{meg}(G) = n + 1$
for every connected graph $G$ of order $n$ that is not modular edge-graceful. Thus
we show that $\text{meg}(G) = n + 1$ for every connected graph $G$ of order $n$ with $n \equiv 2$
(mod 4). We begin with two lemmas

**Lemma 3.6.1** If $H$ is a connected spanning subgraph of a graph $G$ of order at
least 3, then $\text{meg}(G) \leq \text{meg}(H)$

**Proof.** Suppose that $\text{meg}(H) = k$. Let $f_H: E(H) \rightarrow \mathbb{Z}_k$ be an edge labeling of
$H$ such that the induced vertex labeling $f_H': V(H) \rightarrow \mathbb{Z}_k$ is one-to-one Define
an edge labeling $f_G: E(G) \rightarrow \mathbb{Z}_k$ by $f_G(e) = f_H(e)$ if $e \in E(H)$ and $f_G(e) = 0$
if $e \in E(G) - E(H)$. Since the induced vertex labeling $f_G': V(G) \rightarrow \mathbb{Z}_k$ has the
property that $f_G'(v) = f_H'(v)$ for all $v \in V(G)$, it follows that $f_G'$ is one-to-one.
Thus $\text{meg}(G) \leq k = \text{meg}(H)$. ■

**Lemma 3.6.2** Let $G$ be a connected graph of order at least 3, let $\mathcal{P}$ be a partition
of $V(G)$ into two or more independent sets and let $C_o(G)$ be the odd path closure
of $G$ with respect to $\mathcal{P}$. Then $\text{meg}(G) = \text{meg}(C_o(G))$

**Proof.** Since $G$ is a connected spanning subgraph of a graph $C_o(G)$, then $\text{meg}(G) \leq$
$\text{meg}(C_o(G))$ by Lemma 3.6.1. On the other hand, an argument similar to the proof
of Lemma 3.5.1 shows that $\text{meg}(C_o(G)) \leq \text{meg}(G)$ and so $\text{meg}(C_o(G)) = \text{meg}(G)$ ■

In view of Lemmas 3.5.3, 3.6.1 and 3.6.2, we first determine the modular edge-
gracefulness of a star or a double star.
Theorem 3.6.3  If $G$ is a star or a double star of order $n \geq 6$ with $n \equiv 2$ (mod 4), then \( \text{meg}(G) = n + 1 \).

Proof. First suppose that $G = K_{1,n-1}$ is a star with its central vertex $v$ that is adjacent to $v_1, v_2, \ldots, v_{n-1}$. Define a labeling $f : E(G) \rightarrow \mathbb{Z}_{n+1}$ by

$$f(e_v) = \begin{cases} 0 & \text{if } i = 1 \\ -\frac{1}{2} & \text{if } i \text{ is even and } 2 \leq i \leq n - 1 \\ \frac{i+1}{2} & \text{if } i \text{ is odd and } 3 \leq i \leq n - 3 \\ \frac{n+2}{2} & \text{if } i = n - 1. \end{cases}$$

Thus \( \{f(e_v) : 1 \leq i \leq n - 1\} = \{0, -1, \pm 2, \pm 3, \ldots, \pm \frac{n-2}{2}, \frac{n+2}{2}\} \). Since

$$f'(v) = \frac{n}{2}$$

$$f'(e_v) = \begin{cases} 0 & \text{if } i = 1 \\ -\frac{1}{2} & \text{if } i \text{ is even and } 2 \leq i \leq n - 1 \\ \frac{i+1}{2} & \text{if } i \text{ is odd and } 3 \leq i \leq n - 3 \\ \frac{n+2}{2} & \text{if } i = n - 1, \end{cases}$$

it follows that $f' : V(G) \rightarrow \mathbb{Z}_{n+1}$ is one-to-one and so $f$ is a modular edge-graceful labeling. Therefore, $G$ is modular edge-graceful.

Next, suppose that $G$ is a double star with central vertices $u$ and $v$ where $u$ is adjacent to $u_1, u_2, \ldots, u_r$ and $v$ is adjacent to $v_1, v_2, \ldots, v_s$. Thus $n = r + s + 2$ and so $r + s \equiv 0$ (mod 4). We consider two cases.
Case 1. Either \( r \equiv 0 \pmod{4} \) and \( s \equiv 0 \pmod{4} \) or \( r \equiv 2 \pmod{4} \) and \( s \equiv 2 \pmod{4} \). Define an edge labeling \( f : E(G) \to \mathbb{Z}_{n+1} \) by

\[
f(uu_i) = \begin{cases} 
0 & \text{if } i = 1 \\
1 & \text{if } i = 2 \\
\frac{r+1}{2} & \text{if } i \text{ is odd and } 3 \leq i \leq r - 1 \\
-\frac{r}{2} & \text{if } i \text{ is even and } 4 \leq i \leq r \\
\end{cases}
\]

\[
f(vv_i) = \begin{cases} 
\frac{r+i+1}{2} & \text{if } i \text{ is odd and } 1 \leq i \leq s - 1 \\
-\frac{r+3}{2} & \text{if } i \text{ is even and } 2 \leq i \leq s \\
\end{cases}
\]

\[
f(uv) = \frac{r+s+2}{2}.
\]

Observe that

\[
\{f(uu_i) : 1 \leq i \leq r\} = \left\{0, 1, \pm 2, \pm 3, \ldots, \pm \frac{r}{2}\right\}
\]

\[
\{f(vv_i) : 1 \leq i \leq s\} = \left\{\pm \frac{r+2}{2}, \pm \frac{r+4}{2}, \ldots, \pm \frac{r+s}{2}\right\}
\]

Hence \( \{f'(x) : x \in V(G)\} = \{0, 1, \pm 2, \pm 3, \ldots, \pm \frac{r+s}{2}, \frac{r+s}{2} + 1, \frac{r+s}{2} + 2\} \). Thus the induced vertex labeling \( f' : V(G) \to \mathbb{Z}_{n+1} \) is one-to-one.

Case 2. Either \( r \equiv 1 \pmod{4} \) and \( s \equiv 3 \pmod{4} \) or \( r \equiv 3 \pmod{4} \) and \( s \equiv 1 \pmod{4} \), say the former. That is, we assume that \( r \equiv 1 \pmod{4} \) and \( s \equiv 3 \pmod{4} \). Then \( r \geq 1 \) and \( s \geq 3 \). We consider two subcases, according to whether \( r = 1 \) or \( r \geq 5 \).
Subcase 2.1. $r = 1$. Define an edge labeling $f : E(G) \to \mathbb{Z}_{n+1}$ by

$$f(u_{u_1}) = \frac{1 + s}{2}$$

$$f(v_{v_i}) = \begin{cases} 
\frac{1 + i}{2} & \text{if } i \text{ is odd and } 1 \leq i \leq s - 2 \\
-\frac{i}{2} & \text{if } i \text{ is even and } 2 \leq i \leq s - 1 \\
\frac{1 + s}{2} + 1 & \text{if } i = s 
\end{cases}$$

$$f(uv) = 2.$$  

Figure 3.15 shows the edge labeling $f$ in each case when $s = 3$ and $s = 7$. Observe that $\{f'(x) : x \in V(G)\} = \{\pm 1, \pm 2, \ldots, \pm \frac{s-1}{2}, \frac{s+1}{2}, \frac{s+1}{2} + 1\}$. Thus the induced vertex labeling $f'' : V(G) \to \mathbb{Z}_{n+1}$ is one-to-one.

Figure 3.15: The labelings in Subcase 2.1 for $s = 3$ and $s = 7$
Subcase 2.2. \( r \geq 5 \). Define an edge labeling \( f : E(G) \to \mathbb{Z}_{n+1} \) by

\[
f(uu_i) = \begin{cases} 
\frac{r+1}{2} & \text{if } i \text{ is odd and } 1 \leq i \leq r - 2 \\
-\frac{1}{2} & \text{if } i \text{ is even and } 2 \leq i \leq r - 1 \\
\frac{r+s}{2} & \text{if } i = r
\end{cases}
\]

\[
f(vv_i) = \begin{cases} 
\frac{r+1}{2} & \text{if } i \text{ is odd and } 1 \leq i \leq s - 2 \\
-\frac{r+1-1}{2} & \text{if } i \text{ is even and } 2 \leq i \leq s - 1 \\
\frac{r+s+2}{2} & \text{if } i = s
\end{cases}
\]

\[f(uv) = 2.\]

Observe that

\[\{f(uu_i) : 1 \leq i \leq r\} = \left\{\pm 1, \pm 2, \ldots, \pm \frac{r-1}{2}, \frac{r+s}{2}\right\}\]

\[\{f(vv_i) : 1 \leq i \leq s\} = \left\{\pm \frac{r+1}{2}, \pm \frac{r+3}{2}, \ldots, \pm \frac{r+s-2}{2}, \frac{r+s+2}{2}\right\}\]

Hence \( \{f'(x) : x \in V(G)\} = \{\pm 1, \pm 2, \ldots, \pm \frac{r+s-2}{2}, \frac{r+s}{2}, \frac{r+s+2}{2} + 1\} \). Thus the induced vertex labeling \( f' : V(G) \to \mathbb{Z}_{n+1} \) is one-to-one.

In each case, \( f \) is a modular edge-graceful labeling of \( G \) and so \( G \) is modular edge-graceful.

We are now prepared to show that \( \text{meg}(T) = n + 1 \) for every tree \( T \) that is not modular edge-graceful.

**Theorem 3.6.4** If \( T \) is a tree of order \( n \geq 6 \) with \( n \equiv 2 \pmod{4} \), then

\[\text{meg}(T) = n + 1.\]
Proof. Suppose that the partite sets of $T$ are $U$ and $W$ with $|U| = r$ and $|W| = s$.
Then $n = r + s \equiv 2 \pmod{4}$. By Lemma 3.5.3, the odd path closure $C_o(G)$ of $G$ with respect to the partition $\{U, W\}$ is $K_{r,s}$. If $r = 1$ or $s = 1$, then $\text{meg}(K_{r,s}) = n + 1$ by Theorem 3.6.3. Thus we may assume that $r \geq 2$ and $s \geq 2$. Then the double star $S_{r,s}$ is a spanning subgraph of $K_{r,s}$. Since $K_{r,s}$ is not modular edge-graceful, $\text{meg}(K_{r,s}) \geq n + 1$. On the other hand, $\text{meg}(S_{r,s}) = n + 1$ by Theorem 3.6.3. It then follows by Lemma 3.6.1 that $\text{meg}(K_{r,s}) \leq \text{meg}(S_{r,s}) = n + 1$ and so $\text{meg}(K_{r,s}) = n + 1$. Therefore, $\text{meg}(T) = n + 1$ by Lemma 3.6.2.

As a consequence of Lemma 3.6.2 and Theorem 3.6.4, we have the following

Corollary 3.6.5 If $G$ is a connected graph of order $n \geq 6$ with $n \equiv 2 \pmod{4}$, then

$$\text{meg}(G) = n + 1$$
Chapter 4

Nowhere-Zero Modular Labelings

4.1 Introduction

In Chapter 3, we established the following two results, namely Corollaries 3 5 5 and 3 6 5

Modular Edge-Graceful Graphs Theorem Let $G$ be a connected graph of order $n \geq 3$. Then $G$ is a modular edge-graceful graph if and only if $n \not\equiv 2 \pmod{4}$.

Modular Edge-Gracefulness Theorem If $G$ is a connected graph of order $n \geq 6$ with $n \equiv 2 \pmod{4}$, then $\text{meg}(G) = n + 1$.

We saw in Proposition 3.3.2 that if $G$ is a modular edge-graceful connected graph, then every graph $H$ containing $G$ as a spanning subgraph is also modular edge-graceful. In the proof of this result, a modular edge-graceful labeling of $G$ can be extended to a modular edge-graceful labeling of $H$ by assigning 0 to each
edge of $H$ that does not belong to $G$. We also saw that Proposition 3.3.2 plays an important role in establishing the Modular Edge-Graceful Graphs Theorem and the Modular Edge-Gracefulness Theorem. In other words, modular edge-graceful labelings of a graph that assign 0 to some edges of the graph play an important role in establishing these two results. For this reason, we now investigate those modular edge-graceful labelings in which 0 is not permitted.

For a connected graph $G$ of order $n \geq 3$ let $f : E(G) \to \mathbb{Z}_n - \{0\}$, where $f$ need not be one-to-one. Let $f' : V(G) \to \mathbb{Z}_n$, where

$$f'(u) = \sum_{v \in N(u)} f(uv)$$

(4.1)

and where the sum is computed in $\mathbb{Z}_n$. If $f'$ is one-to-one, then $f$ is called a nowhere-zero modular edge-graceful labeling (or simply nowhere-zero meg-labeling) and $G$ is a nowhere-zero modular edge-graceful graph (or simply nowhere-zero meg-graph). In this chapter, all graphs under consideration are connected graphs of order at least 3.

To illustrate these concepts, consider the graph $G$ of order 11 in Figure 4.1, where two modular edge-graceful labelings of $G$ are shown. In each labeling, an edge of $G$ is labeled with an element in $\mathbb{Z}_{11} = \{0, 1, \ldots, 10\}$ and each vertex of $G$ is labeled with its induced label. The modular edge-graceful labeling of $G$ in Figure 4.1(a) is a nowhere-zero meg-labeling, while the modular edge-graceful labeling in Figure 4.1(b) is not.

In this chapter, we establish the following two main results.
• If $G$ is a connected graph of order $n \geq 3$ where $n \not\equiv 2 \pmod{4}$, then there is a modular edge-graceful labeling $f : E(G) \to \mathbb{Z}_n$ such that $f(e) \neq 0$ for all $e \in E(G)$ with at most one exception.

• A connected graph $G$ of order $n \geq 3$ is nowhere-zero modular edge-graceful if and only if (i) $n \not\equiv 2 \pmod{4}$, (ii) $G \neq K_3$ and (iii) $G$ is not a star of even order.

4.2 Preliminary Results

First, we present some useful results on modular edge-graceful labelings and nowhere-zero modular edge-graceful labelings of graphs.

Proposition 4.2.1 Let $G$ be a connected regular graph of order $n \geq 3$ that is modular edge-graceful, where then $n \not\equiv 2 \pmod{4}$. If there is a modular edge-graceful labeling $f : E(G) \to \mathbb{Z}_n$ and an integer $a \in \mathbb{Z}_n$ such that $f(e) \neq a$ for
every edge $e$ of $G$, then $G$ has a nowhere-zero modular edge-graceful labeling.

**Proof.** Suppose that $G$ is a connected $r$-regular graph where then $r \geq 2$. If $a = 0$, then $f$ is a nowhere-zero modular edge-graceful labeling of $G$. Thus, we may assume that $a \neq 0$. Define the labeling $g : E(G) \to \mathbb{Z}_n$ by $g(e) = f(e) - a$ for each $e \in E(G)$. Since $f(e) \neq a$ for each $e \in E(G)$, it follows that $g(e) \neq 0$ and so $g : E(G) \to \mathbb{Z}_n - \{0\}$. Furthermore, $g'(v) = f'(v) - ra$ for each $v \in V(G)$. Since $f'$ is one-to-one, so is $g'$. Therefore, $g$ is a nowhere zero modular edge-graceful labeling of $G$.

**Theorem 4.2.2** Let $G$ be a connected modular edge-graceful graph of order $n \geq 4$ containing an even cycle $C$ and let $f$ be a modular edge-graceful labeling. Suppose that there is $a \in \mathbb{Z}_n$ that satisfies one of the following two conditions

1. $f(e) \neq \pm a$ for each $e \in E(C)$ and

2. if $a \neq -a$, then $f(e) = a$ for exactly one $e \in E(C)$ and $f(e) \neq -a$ for each $e \in E(C)$

Then $G$ has a modular edge-graceful labeling $g$ for which

1. $g(e) = f(e)$ for each $e \notin E(C)$,

2. $g(e) \neq 0$ for all $e \in E(C)$ and

3. $g'(v) = f'(v)$ for all $v \in V(G)$.
Proof. If $a = 0$, then $g = f$ satisfies conditions (i)-(iii). Thus, we may assume that $a \neq 0$. Let $C = (v_1, v_2, \ldots, v_k, v_{k+1} = v_1)$ where $k \geq 4$ is even. First, suppose that condition (1) holds, that is, $f(e) \neq \pm a$ for each $e \in E(C)$. Define a labeling $g : E(G) \to \mathbb{Z}_n$ by

$$g(e) = \begin{cases} f(e) & \text{if } e \notin E(C), \\ f(e) + a & \text{if } e = v_i v_{i+1} \text{ and } i \text{ is odd}, \\ f(e) - a & \text{if } e = v_i v_{i+1} \text{ and } i \text{ is even}. \end{cases} \quad (4.2)$$

Thus $g(e) = f(e) \pm a \neq 0$ for each $e \in E(C)$. If $v \notin V(C)$, then

$$g'(v) = \sum_{uv \in E(v)} g(uv) = \sum_{uv \in E(v)} f(uv) = f'(v).$$

If $v \in V(C)$, then

$$g'(v) = \sum_{uv \in E(v)} g(uv) = \sum_{uv \in E(v) \cap V(C)} g(uv) + \sum_{uv \in E(v) - V(C)} g(uv)$$

$$= (f(v_1 v) \pm a) + (f(v_{i+1} v) \mp a) + \sum_{uv \in E(v) - V(C)} f(uv)$$

$$= f'(v).$$

Thus $g$ satisfies conditions (i)-(iii). Next suppose that (2) holds, that is, $f(e) = a$ for exactly one $e \in E(C)$. We may assume, without loss of generality, that $e = v_0 v_1$. Then the labeling $g$ is defined in (4.2) satisfies conditions (i)-(iii).

Theorem 4.2.3 Let $G$ be a connected modular edge-graceful graph of order at least 4 that contains an even cycle $C$. Let $g$ be any modular edge-graceful labeling
of G. Then there is a modular edge-graceful labeling \( f : E(G) \to \mathbb{Z}_n \) of G such that \( f(e) = g(e) \) for each \( e \in E(G) - E(C) \) and \( f(e) \neq 0 \) for each \( e \in E(C) \).

**Proof.** Suppose that the order of G is \( n \geq 4 \). Let \( C = (v_1, v_2, \ldots, v_k, v_{k+1} = v_1) \) be an even cycle in G, where then \( k \geq 4 \) is even and let \( g : E(G) \to \mathbb{Z}_n \) be a modular edge-graceful labeling of G. Since G is modular edge-graceful, \( n \neq 2 \pmod{4} \).

First, assume that \( n \) is odd. Since \( k \) is even, \( k < n \). For each integer \( i \) with \( 0 \leq i \leq \frac{n-1}{2} \), let \( S_i \) be the set of edges \( e \) in \( C \) for which \( g(e) = i \) or \( g(e) = -i \). If \( S_0 = \emptyset \), then \( f = g \). Thus we may assume that \( S_0 \neq \emptyset \). We claim that there is an integer \( i \) with \( 1 \leq i \leq \frac{n-1}{2} \) such that \( |S_i| \leq 1 \). Assume, to the contrary, that \( |S_i| > 2 \) for all \( i \) with \( 1 \leq i \leq \frac{n-1}{2} \). Since \( |S_0| \geq 1 \), it follows that

\[
k = \sum_{i=0}^{\frac{n-1}{2}} |S_i| \geq 2 \cdot \left( \frac{n-1}{2} \right) + 1 = n.
\]

Since \( k < n \) in this case, a contradiction is produced. Thus, as claimed, there is an integer \( i \) with \( 1 \leq i \leq \frac{n-1}{2} \) such that \( |S_i| \leq 1 \). It then follows by Theorem 4.2.2 that there is a modular edge-graceful labeling \( f \) of G such that \( f(e) \neq 0 \) for each \( e \in E(C) \) and \( f(e) = g(e) \) for each \( e \notin E(C) \).

Next, assume that \( n \) is even. For each integer \( i \) with \( 0 \leq i \leq \frac{n}{2} - 1 \), let \( S_i \) be the set of edges \( e \) in \( C \) for which \( g(e) = i \) or \( g(e) = -i \) and let \( S_{\frac{n}{2}} \) be the set of edges \( e \) in \( C \) for which \( g(e) = \frac{n}{2} \). If \( S_0 = \emptyset \), then let \( f = g \). Thus we may assume that \( S_0 \neq \emptyset \) and so \( |S_0| \geq 1 \). We consider two cases, according to whether \( k < n \) or \( k = n \).
Case 1. $k < n$. We claim that there is an integer $i$ with $1 \leq i < \frac{n}{2}$ such that $|S_i| \leq 1$ or $|S_{\frac{n}{2}}| = 0$. If this is not the case, then $|S_i| \geq 2$ for all $i$ with $1 \leq i < \frac{n}{2}$, $|S_{\frac{n}{2}}| \geq 1$, and $|S_0| \geq 1$. However then,

$$k = \sum_{i=0}^{\frac{n}{2}} |S_i| \geq 1 + \sum_{i=1}^{\frac{n}{2}-1} |S_i| + 1$$

$$\geq 2 + 2\left(\frac{n}{2} - 1\right) = n$$

which is impossible. Thus, as claimed, if $k < n$, then $|S_i| \leq 1$ for some integer $i$ with $1 \leq i < \frac{n}{2}$ or $|S_{\frac{n}{2}}| = 0$. Hence by Theorem 4.2.2, there is a modular edge-graceful labeling $f$ of $G$ such that $f(e) \neq 0$ for each $e \in E(C)$ and $f(e) = g(e)$ for each $e \notin E(C)$.

Case 2. $k = n$. Then

$$C = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$$

is a Hamiltonian cycle of $G$. By the discussion above, if $|S_0| \neq 0$, $|S_{\frac{n}{2}}| \neq 0$ and $|S_{\frac{n}{2}}| \geq 2$ for all $i$ with $1 \leq i \leq \frac{n}{2} - 1$, then $|S_0| = |S_{\frac{n}{2}}| = 1$ and $|S_{\frac{n}{2}}| = 2$ for all $i$ with $1 \leq i \leq \frac{n}{2} - 1$. Assume, without loss of generality, that $g(v_1v_2) = 0$. Consider the set

$$A = \{g(v_i v_{i+1}) : i \text{ is odd and } 1 \leq i \leq n - 1\}.$$  

Notice that $|A| \leq \frac{n}{2}$. Since $0 \in A$, there exists $a \in \mathbb{Z}_n - A$ such that $1 \leq a \leq \frac{n}{2}$. Define a labeling $f : E(G) \to \mathbb{Z}_n$ by

$$f(e) = \begin{cases} 
    g(e) & \text{if } e \notin E(C) \text{ or } e = v_i v_{i+1}, i \text{ is even and } 1 \leq i \leq n \\
    g(e) - a & \text{if } e = v_i v_{i+1}, i \text{ is odd and } 1 \leq i \leq n - 1.
\end{cases}$$
Clearly, $f(e) = g(e)$ for each $e \notin E(C)$. Since $C$ is a Hamiltonian cycle of $G$, $f'(v) = g'(v) - a$ for each $v \in V(G)$ and so $f$ is a modular edge-graceful labeling of $G$. Furthermore, since $g(v_i v_{i+1}) \neq a$ for all odd integers $i$ with $1 \leq i \leq n - 1$, it follows that $f(v_i v_{i+1}) = g(v_i v_{i+1}) - a \neq 0$ for all odd integers $i$ with $1 \leq i \leq n - 1$. Finally, since $|S_0| = 1$, we have $g(v_i v_{i+1}) = 0$ if and only if $i = 1$. Hence $f(e) \neq 0$ for each $e \in E(C)$.

The following corollary is an immediate consequence of Theorem 4.2.3.

**Corollary 4.2.4** If $G$ is a connected modular edge-graceful graph of order at least 3 and $g : E(G) \to \mathbb{Z}_n$ is a modular edge-graceful labeling of $G$, then there is a modular edge-graceful labeling $f$ of $G$ such that $f(e) \neq 0$ for each edge $e$ that lies on an even cycle of $G$ and $f(e) = g(e)$ for each edge $e$ that does not lie on an even cycle of $G$.

### 4.3 Some Well-Known Classes of Graphs

In this section, we determine several well-known classes of graphs that are nowhere-zero modular edge-graceful, beginning with paths, cycles and complete graphs.

#### 4.3.1 Paths, Cycles and Complete Graphs

**Theorem 4.3.1** For each integer $n \geq 3$, the path $P_n$ is nowhere-zero modular edge-graceful if and only if $n \not\equiv 2 \pmod{4}$.

**Proof.** By the Modular Edge-Graceful Graphs Theorem, it suffices to show that if $n \not\equiv 2 \pmod{4}$, then $P_n$ is nowhere-zero modular edge-graceful.
First, assume that \( n \) is even. Since \( n \not\equiv 2 \pmod{4} \), it follows that \( n \equiv 0 \pmod{4} \) and so \( n = 4k \) for some integer \( k \geq 1 \). Let \( P_{4k} = (v_1, v_2, \ldots, v_{4k}) \). Define a labeling

\[
f : E(P_{4k}) \to \mathbb{Z}_{4k} - \{0\}
\]

by

\[
f(v_i, v_{i+1}) = \begin{cases} 
  \gamma & \text{if } 1 \leq \gamma \leq 2k \\
  2\left\lfloor \gamma/2 \right\rfloor & \text{if } 2k + 1 \leq \gamma \leq 4k - 1
\end{cases}
\]

Figure 4.2 shows the labelings \( f \) and \( f' \) for \( k = 1, 2, 3 \).

First, assume that \( 1 \leq \gamma \leq 2k \). Observe that

\[
f'(v_1) = 1 \text{ and } f'(v_{i+1}) = f'(v_i) + 2 \text{ for } 1 \leq \gamma \leq 2k - 1.
\]

Thus \( f'(v_i) = 2(\gamma - 1) + 1 \) for \( 1 \leq \gamma \leq 2k \), that is,

\[
\{f'(v_i) : 1 \leq \gamma \leq 2k\} = \{1, 3, 5, \ldots, 4k - 1\}.
\]

Next, assume that \( 2k + 1 \leq \gamma \leq 4k \). Observe that
\[ f'(v_{2k+1}) = 4k \equiv 0 \pmod{4k} \text{ and } f'(v_{i+1}) = f'(v_i) + 2 \text{ for } 2k + 1 \leq i \leq 4k - 1. \]

Thus \( f'(v_i) = 2(i - 2k - 1) \) for \( 2k + 1 \leq i \leq 4k \), that is,

\[
\{ f'(v_i) : 2k + 1 \leq i \leq 4k \} = \{0, 2, 4, \ldots, 4k - 2\}. 
\]

Therefore, \( f \) is a nowhere-zero modular edge-graceful labeling of \( P_{4k} \).

Next, assume that \( n \) is odd. Let \( n = 2k + 1 \) for some integer \( k \geq 1 \). Let

\[ P_{2k+1} = (v_1, v_2, \ldots, v_{2k+1}). \]

Define a labeling

\[ f : E(P_{2k+1}) \to \mathbb{Z}_{2k+1} - \{0\} \]

by

\[ f(v_{i-1}, v_i) = i \text{ for } 1 \leq i \leq 2k. \]

In \( \mathbb{Z}_{2k+1} \), \( f'(v_1) = 1 \),

\[ f'(v_i) = f(v_{i-1}, v_i) + f(v_i, v_{i+1}) = (i - 1) + i = 2i - 1 \]

for \( 2 \leq i \leq 2k - 1 \) and \( f'(v_{2k+1}) = 2k = -1 = 2 \cdot (2k + 1) - 1 \). Thus \( f'(v_i) = 2i - 1 \) for \( 1 \leq i \leq 2k + 1 \). Figure 4.2 shows the labelings \( f \) and \( f' \) for \( k = 1, 2, 3 \). For \( 1 \leq i \leq k \), since \( f'(v_i) = 2i - 1 \), it follows that

\[ \{ f'(v_i) : 1 \leq i \leq k \} = \{1, 3, \ldots, 2k - 1\}. \]
For $k + 1 \leq r \leq 2k + 1$, let $r = k + j$ where then $1 \leq j \leq k + 1$. Thus for $k + 1 \leq r \leq 2k + 1$,

$$f'(v_i) = 2r - 1 = 2(k + j) - 1 = 2k + 2j - 1 = (2k + 1) + 2(j - 1)$$

$$= 2(j - 1) \text{ in } \mathbb{Z}_{2k+1}$$

This implies that

$$\{f'(v_i) \mid k + 1 \leq r \leq 2k + 1\} = \{0, 2, 4, \ldots, 2k\}$$

Therefore, $f'$ is one-to-one and so $f$ is a nowhere-zero modular edge-graceful labeling.

**Theorem 4.3.2** For each integer $n \geq 3$, the cycle $C_n$ is nowhere-zero modular edge-graceful if and only if $n \geq 4$ and $n \not\equiv 2 \pmod{4}$

**Proof.** We first show that $C_3$ is not nowhere-zero modular edge-graceful. Let $C_3 = (v_1, v_2, v_3, v_1)$ and let $f : E(C_3) \rightarrow \mathbb{Z}_3 - \{0\}$ be any labeling of $C_3$. Thus
\( f(e) = 1 \) or \( f(e) = 2 \) for each edge \( e \) of \( C_3 \). This implies that at least two edges of \( C_3 \) must be labeled the same by \( f \). We may assume that \( f(v_1v_2) = f(v_2v_3) = 1 \) or \( f(v_1v_2) = f(v_2v_3) = 2 \). In each case, \( f'(v_1) = f'(v_3) \) and so the induced labeling \( f' \) is not one-to-one. Therefore, \( f \) is not a nowhere-zero modular edge-graceful labeling and so \( C_3 \) is not nowhere-zero modular edge-graceful. If \( n \equiv 2 \pmod{4} \), then \( C_n \) is not modular edge-graceful by the Modular Edge-Graceful Graphs Theorem.

For the converse, assume that \( n \geq 4 \) and \( n \not\equiv 2 \pmod{4} \). We show that \( C_n \) is nowhere-zero modular edge-graceful. First, assume that \( n \) is even. Since \( n \equiv 2 \pmod{4} \), it follows that \( n \equiv 0 \pmod{4} \) and so \( n = 4k \) for some integer \( k \geq 1 \).

Let \( C_{4k} = (v_1, v_2, \ldots, v_{4k}, v_{4k+1} = v_1) \). We first construct a modular edge-graceful labeling \( f^* \) of \( C_{4k} \) from the nowhere-zero modular edge-graceful \( f \) of \( P_{4k} \) in the proof of Theorem 4.3.1. Recall that \( f: E(P_{4k}) \to \mathbb{Z}_{4k} - \{0\} \) of \( P_{4k} \) is defined by

\[
  f(v_i v_{i+1}) = \begin{cases} 
    i & \text{if } 1 \leq i \leq 2k \\
    2\lfloor i/2 \rfloor & \text{if } 2k + 1 \leq i \leq 4k - 1. 
  \end{cases}
\]

Then the modular edge-graceful labeling \( f^* \) of \( C_{4k} \) can be defined by \( f^*(e) = f(e) \) if \( e \neq v_1v_{4k} \) and \( f^*(e) = 0 \) if \( e = v_1v_{4k} \). Since \( f^*(e) \neq 2k + 1 \) for each \( e \in E(C_{4k}) \), it follows by Proposition 4.2.1 that \( C_{4k} \) has a nowhere-zero modular edge-graceful labeling.

Next, assume that \( n \) is odd. Then \( n \equiv 1 \pmod{4} \) or \( n \equiv 3 \pmod{4} \). Thus \( n = 4k + i \) for some positive integer \( k \) where \( i \in \{1, 3\} \). We consider these two cases.
Case 1 \( n = 4k + 1 \) Let \( C_{4k+1} = (v_1, v_2, \ldots, v_{4k+1}, v_{4k+2} = v_1) \) We first construct a modular edge-graceful labeling \( f \) of \( C_{4k+1} \) such that 1 is not used for any edge of \( C_{4k+1} \). Define the labeling \( f \) \( E(C_{4k+1}) \rightarrow \mathbb{Z}_{4fc+i} \) by

\[
f(v_i, v_{i+1}) = \begin{cases} 
-\tau & \text{if } 1 \leq \tau \leq 2k - 1 \\
\tau + 1 & \text{if } 2k \leq \tau \leq 4k \text{ and } \tau \text{ is even} \\
\tau & \text{if } 2k + 1 \leq \tau \leq 4k + 1 \text{ and } \tau \text{ is odd}
\end{cases}
\]

Figure 4.4 shows the labeling \( f \) of \( C_9 \) where \( k = 2 \), where 1 is not used, while 0 is used twice. We claim that

\[
f'(v_i) = -2i + 1 \text{ in } \mathbb{Z}_{4k+1} \text{ for } 1 \leq i \leq 2k
\]

If \( i = 1 \), then

\[
f'(v_1) = f'(v_1) = f(v_{4k+1}v_{4k+2}) + f(v_1v_2)
\]

\[
= (4k + 1) + (-1) = -1 = (-2) 1 + 1 \text{ in } \mathbb{Z}_{4k+1}
\]
If $1 < i < 2k - 1$, then

\[
\begin{align*}
    f'(v_i) &= f(v_{i-1}v_i) + f(v_ivi+1) \\
           &= -(i-1) - i = -2i + 1 \text{ in } \mathbb{Z}_{4k+1}.
\end{align*}
\]

If $i = 2k$, then

\[
\begin{align*}
    f'(v_i) &= f'(v_{2k}) = f(v_{2k-1}v_{2k}) + f(v_{2k}v_{2k+1}) \\
           &= -(2k-1) + (2k+1) = 2 = -2(2k) + 1 \text{ in } \mathbb{Z}_{4k+1}.
\end{align*}
\]

Thus (4.3) holds. Observe that for $1 \leq i \leq 2k$,

\[
    f'(v_i) = -2i + 1 = (4k + 1) - 2i + 1 = 4k + 2 - 2i \text{ in } \mathbb{Z}_{4k+1}.
\]

This implies that

\[
    \{f'(v_i) : 1 \leq i \leq 2k\} = \{2, 4, \ldots, 4k\}.
\]

Furthermore, for $2k + 1 \leq i \leq 4k + 1$,

\[
\begin{align*}
    f'(v_i) &= f(v_{i-1}v_i) + f(v_ivi+1) \\
            &= \begin{cases} 
            (i-1) + 1 + i & \text{if } 2k + 1 \leq i \leq 4k + 1 \text{ and } i \text{ is odd} \\
            (i-1) + i + 1 & \text{if } 2k + 1 \leq i \leq 4k + 1 \text{ and } i \text{ is even} 
            \end{cases} \\
            &\quad = 2i.
\end{align*}
\]

For each $i$ with $2k + 1 \leq i \leq 4k + 1$, let $i = 2k + j$ where $1 \leq j \leq 2k + 1$. Then

\[
    f'(v_i) = 2i = 2(2k + j) = 4k + 2j = 2j - 1 \text{ in } \mathbb{Z}_{4k+1}.
\]
Therefore,

\[
\{f'(v_i) : 2k + 1 \leq i \leq 4k + 1\} = \{1, 3, 5, \ldots, 4k + 1 = 0\}.
\]

Thus \( f \) is indeed a modular edge-graceful labeling of \( C_{4k+1} \). Since \( f(e) \neq 1 \) for each \( e \in E(C_{4k+1}) \) and \( C_{4k+1} \) is 2-regular, it then follows by Proposition 4.2.1 that \( C_{4k+1} \) has a nowhere-zero modular edge-graceful labeling.

Case 2. \( n = 4k + 3 \). Let \( C_{4k+3} = (v_0, v_1, v_2, \ldots, v_{4k+2}, v_{4k+3} = v_0) \). Again, we first construct a modular edge-graceful labeling \( f \) of \( C_{4k+3} \) such that 1 is not used for any edge of \( C_{4k+3} \). Define the labeling \( f : E(C_n) \rightarrow \mathbb{Z}_{4k+3} \) by

\[
f(v_i, v_{i+1}) = \begin{cases} 
  i & \text{if } 0 \leq i \leq 2k \text{ and } i \text{ is even} \\
  i + 1 & \text{if } 0 \leq i \leq 2k + 1 \text{ and } i \text{ is odd} \\
  -i & \text{if } 2k + 2 \leq i \leq 4k + 2 
\end{cases}
\]

Figure 4.5 shows the labeling \( f \) of \( C_{4k+3} \) where \( k = 2 \) where 1 is not used while 0 is used.

For each \( i \) with \( 1 \leq i \leq 2k + 1 \),

\[
f'(v_i) = f(v_{i-1}, v_i) + f(v_i, v_{i+1})
\]

\[
= \begin{cases} 
  (i - 1) + (i + 1) & \text{if } i \text{ is odd} \\
  (i - 1) + 1 + i & \text{if } i \text{ is even.}
\end{cases}
\]

This implies that

\[
\{f'(v_i) : 1 \leq i \leq 2k + 1\} = \{2, 4, \ldots, 4k + 2\}.
\]
Figure 4.5: The labeling $f$ of $C_{11}$ where $k = 2$

Notice that in $\mathbb{Z}_{4k+3}$,

$$f'(v_{2k+2}) = f(v_{2k+1}v_{2k+2}) + f(v_{2k+2}v_{2k+3})$$

$$= (2k + 1) + 1 - (2k + 2) = -2(2k + 2) + 1 \text{ in } \mathbb{Z}_{4k+3}.$$  

For each $i$ with $2k + 3 \leq i \leq 4k + 2$,

$$f'(v_i) = f(v_{i-1}v_i) + f(v_iv_{i+1})$$

$$= -(i - 1) + (-i) = -2i + 1 \text{ in } \mathbb{Z}_{4k+3}.$$  

For $i = 4k + 3$,

$$f'(v_{4k+3}) = f'(v_0) = f(v_0v_1) + f(v_{4k+2}v_0)$$

$$= 0 + -(4k + 2) = 1 = -2 \cdot 0 + 1 \text{ in } \mathbb{Z}_{4k+3}.$$  

This implies that $f'(v_i) = -2i + 1$ for $2k + 2 \leq i \leq 4k + 3$, that is,

$$\{f'(v_i) : 2k + 2 \leq i \leq 4k + 3\} = \{1, 3, 5, \ldots, 4k + 1, 4k + 3 = 0\}.$$
Thus \( f \) is indeed a modular edge-graceful labeling of \( C_n \). Since \( f(e) \neq 1 \) for each \( e \in E(C_n) \) and \( C_n \) is 2-regular, it then follows by Proposition 4.2.1 that \( C_n \) has a nowhere-zero modular edge-graceful labeling.

With the aid of Theorem 4.3.2, we have the following more general result on regular Hamiltonian graphs of order at least 4.

**Theorem 4.3.3** Let \( G \) be a regular Hamiltonian graph of order \( n \geq 4 \). Then \( G \) is nowhere zero modular edge-graceful if and only if \( n \equiv 2 \pmod{4} \).

**Proof.** By the Modular Edge-Graceful Graphs Theorem, it suffices to show that if \( n \geq 4 \) and \( n \equiv 2 \pmod{4} \), then \( G \) is nowhere zero modular edge graceful.

Suppose that \( G \) is \( r \)-regular, where then \( r \geq 2 \). Let \( V(G) = \{v_1, v_2, \ldots, v_n\} \), where then \( n \equiv 2 \pmod{4} \). We may assume that \( C_n = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1) \) is a Hamiltonian cycle of \( G \). By Theorem 4.3.2, \( C_n \) is nowhere-zero modular edge graceful. Let \( f : E(G) \to \mathbb{Z}_n - \{0\} \) be a nowhere-zero modular edge graceful labeling of \( C_n \). Define a labeling \( g : E(G) \to \mathbb{Z}_n - \{0\} \) of \( G \) by \( g(e) = f(e) \) if \( e \in E(C_n) \) and \( g(e) = 1 \) if \( e \notin E(C_n) \). Observe that \( g'(v_i) = f'(v_i) + (r - 2) \) for \( 1 \leq i \leq n \). Since \( f' \) is one-to-one, so is \( g' \). Thus \( g \) is a nowhere-zero modular edge-graceful labeling of \( G \).

The following is a consequence of Theorem 4.3.3.

**Corollary 4.3.4** The complete graph \( K_n \) of order \( n \geq 4 \) is nowhere zero modular edge-graceful if and only if \( n \equiv 2 \pmod{4} \).
4.3.2 Complete Multipartite Graphs

By the Modular Edge-Graceful Graphs Theorem, Theorems 4.3.1 and 4.3.3, if \( G \) is a path of order at least 3 or a regular Hamiltonian graph of order at least 4, then \( G \) is nowhere-zero modular edge-graceful if and only if \( G \) is modular edge-graceful. This is not the case for complete multipartite graphs. As an example, we consider stars.

**Proposition 4.3.5** A star of order \( n \geq 3 \) is nowhere-zero modular edge-graceful if and only if \( n \) is odd.

**Proof.** Let \( G = K_{1,s} \) be a star of order \( n = 1 + s \geq 3 \) and let \( V(G) = \{u, v_1, v_2, \ldots, v_s\} \) where \( u \) is the central vertex of \( G \). We show that \( G \) is nowhere-zero modular edge-graceful if and only if \( s \) is even. First, we make an observation. If \( f : E(G) \to \mathbb{Z}_n - \{0\} \) is a nowhere-zero modular edge-graceful labeling, then \( f'(v_i) = f(u v_i) \) for \( 1 \leq i \leq s \). Hence if \( f \) is a nowhere-zero modular edge-graceful labeling of \( G \), then \( f(u v_i) \neq f(u v_j) \) for all \( i, j \) with \( 1 \leq i \neq j \leq s \) and \( f(u v_i) \neq 0 \) for \( 1 \leq i \leq s \). Thus, up to isomorphism, there is only one possible such labeling, that is \( f(u v_i) = i \) for \( 1 \leq i \leq s \) and so \( f'(v_i) = i \). This implies that \( f'(u) = 0 \). On the other hand,

\[
f'(u) = 1 + 2 + \ldots + s = \binom{s+1}{2}.
\]
If $s$ is even, then $f'(u) \equiv 0 \pmod{s + 1}$, while if $s$ is odd, then
\[
f'(u) = \frac{(s + 1)}{2} = \frac{s(s + 1)}{2} = \frac{s + 1}{2} \neq 0 \pmod{s + 1}
\]
Therefore, $f$ is nowhere-zero modular edge-graceful if and only if $s$ is even.

By Proposition 4.3.5, if $G$ is a star of order $n \geq 4$ with $n \equiv 0 \pmod{4}$, then $G$ is modular edge-graceful but not nowhere-zero modular edge-graceful. Certainly, each star is a connected bipartite graph with bridges. On the other hand, Corollary 4.2.4 gives us the following

**Theorem 4.3.6** Let $G$ be a connected bridgeless bipartite graph of order $n \geq 3$ Then $G$ is nowhere-zero modular edge-graceful if and only if $n \equiv 2 \pmod{4}$

**Proof.** By the Modular Edge-Graceful Graphs Theorem, it suffices to show that if $n \not\equiv 2 \pmod{4}$, then $G$ is nowhere-zero modular edge-graceful. Since every edge lies on an even cycle of $G$, it follows by Corollary 4.2.4 that $G$ is nowhere zero modular edge graceful.

In order to establish a characterization of complete multipartite graphs that are nowhere-zero modular edge-graceful, we first present a lemma, which determines a special class of complete 3-partite graphs that are nowhere-zero modular edge-graceful

**Lemma 4.3.7** Let $G = K_{1,1,p}$ be the complete 3-partite graph of order $n = 2 + p \geq 4$ Then $G$ is nowhere-zero modular edge-graceful if and only if $n \not\equiv 2 \pmod{4}$
Proof. By the Modular Edge-Graceful Graphs Theorem, it suffices to show that if \( n \not\equiv 2 \pmod{4} \), then \( G \) is nowhere-zero modular edge-graceful. Suppose that the three partite sets of \( G \) are \( V_1 = \{u\} \), \( V_2 = \{v\} \) and \( V_3 = \{v_1, v_2, \ldots, v_i\} \), where then \( \deg u = \deg v = p + 1 \) and \( \deg v_i = 2 \) for \( 1 \leq i \leq p \). We consider two cases, according to whether \( n \) is even or \( n \) is odd.

Case 1: \( n \) is even. Since \( n \not\equiv 2 \pmod{4} \), it follows that \( n \equiv 0 \pmod{4} \). Define a labeling \( f : E(G) \to \mathbb{Z}_n - \{0\} \) by

\[
f(e) = \begin{cases} 
\frac{n}{2} & \text{if } e = uv \text{ or } e = uv_i \text{ and } 1 \leq i \leq p \\
\frac{n}{4} + 1 & \text{if } e = uv_i \text{ and } \frac{n}{4} \leq i \leq p \\
v & \text{if } e = vv_i \text{ and } 1 \leq i \leq p 
\end{cases}
\]

Figure 4.6 shows the labelings \( f \) and \( f' \) for \( G = K_{11b} \) where then \( n = 8 \).

Figure 4.6 The labelings \( f \) and \( f' \) for \( G = K_{11b} \).
For the induced vertex labeling $f'$ of $G$, observe that in $\mathbb{Z}_n$

\[
f'(u) = (n-1) \cdot \frac{n}{2} = \frac{n^2}{2}.
\]

\[
f'(v) = \left( \sum_{i=1}^{p+1} i \right) - \frac{n}{4} + \frac{n}{2} = \left( \sum_{i=1}^{n} i \right) + \frac{n(n-1)}{2} + \frac{n}{4} = \frac{3n}{4}.
\]

\[
f'(v_i) = \frac{n}{2} + i \quad \text{if } 1 \leq i \leq \frac{n}{4} - 1
\]

\[
f'(v_i) = \frac{n}{2} + i + 1 \quad \text{if } \frac{n}{4} \leq i \leq p
\]

Observe that

\[
\{f'(v_i) : 1 \leq i \leq \frac{n}{4} - 1\} = \left\{ \frac{n}{2} + 1, \frac{n}{2} + 2, \ldots, \frac{3n}{4} - 1 \right\}
\]

\[
\{f'(v_i) : \frac{n}{4} \leq i \leq p\} = \left\{ \frac{3n}{4} + 1, \frac{3n}{4} + 2, \ldots, 0, 1, \ldots, \frac{n}{2} - 1 \right\}.
\]

Thus $f$ is a nowhere-zero modular edge-graceful of $G$.

**Case 2. $n$ is odd.** Define a labeling $g : E(G) \rightarrow \mathbb{Z}_n - \{0\}$ by

\[
g(e) = \begin{cases} 
1 & \text{if } e = uv_i, 1 \leq i \leq p - 1 \\
2 & \text{if } e = uv_p \\
3 & \text{if } e = uv \\
i + 1 & \text{if } e = vv_i, 1 \leq i \leq p - 1 \\
n - 2 & \text{if } e = vv_p
\end{cases}
\]

Figure 4.7 shows the labelings $f$ and $f'$ for $G = K_{1,1,5}$ where then $n = 7$. 
For the induced vertex labeling $f'$ of $G$, observe that in $\mathbb{Z}_n$

$$g'(u) = (p - 1) \cdot 1 + 3 + 2 = (n - 3) + 5 = 2$$

$$g'(v) = \left( \sum_{i=1}^{p-1} (i + 1) \right) + 3 + (n - 2) = \left( \sum_{i=1}^{n-3} (i + 1) \right) + (n - 2) + 3$$

$$= \left( \sum_{i=1}^{n-3} i \right) + (n - 3) + (n - 2) + 3 = \left( \sum_{i=1}^{n-1} i \right) + 1 = 1$$

$$g'(v_i) = (i + 1) + 1 = i + 2 \quad \text{if } 1 \leq i \leq k - 1$$

$$g'(v_k) = (n - 2) + 2 = 0$$

Observe that

$$\{f'(v_i) : 1 \leq i \leq k - 1\} = \{3, 4, \ldots, n - 1\}.$$  

Thus $f$ is a nowhere-zero modular edge-graceful of $G$.

We are now prepared to present a characterization of complete multipartite graphs that are nowhere-zero modular edge-graceful.
Theorem 4.3.8  Let $G$ be a complete $k$-partite graph of order $n \geq 3$ where $k \geq 2$ and $n \neq 2 \pmod{4}$. Then $G$ is nowhere-zero modular edge-graceful if and only if $G$ is neither $K_3$ nor a star of even order at least 4.

Proof. By Proposition 4.3.5 and Theorem 4.3.2, it remains to show that if $G$ is neither $K_3$ nor a star of even order at least 4, then $G$ is nowhere-zero modular edge-graceful. Let $G = K_{n_1, n_2, \ldots, n_k}$ be the complete $k$-partite graph that is neither $K_3$ nor a star of even order at least 4, where $k \geq 2$, $n = n_1 + n_2 + \ldots + n_k \geq 3$ and $n \neq 2 \pmod{4}$. First, suppose that $k = 2$. If $G$ is a star of odd order, then $G$ is nowhere-zero modular edge-graceful by Proposition 4.3.5, while if $G$ is not a star, then $G$ is a bridgeless bipartite graph of order $n \geq 3$ and the result follows by Theorem 4.3.6. Next, suppose that $k = 3$ and assume that $n_1 \leq n_2 \leq n_3$. Since $G \neq K_3$, it follows that $n_3 \geq 2$. If $n_2 \geq 2$, then every edge of $G$ lies on an even cycle of $G$ and so the result follows by Corollary 4.2.4. Thus, we may assume that $n_1 = n_2 = 1$. Thus $G = K_{1, n_3}$, where $n_3 \geq 2$. Then $G$ is nowhere-zero modular edge-graceful by Lemma 4.3.7. Finally, suppose that $k \geq 4$. Then each edge of $G$ lies on an even cycle of $G$ and so $G$ is nowhere-zero modular edge-graceful by Corollary 4.2.4.

4.4 One Zero is Sufficient

In this section, we show that if $G$ is a connected graph of order $n \geq 3$ where $n \neq 2 \pmod{4}$ that is not nowhere-zero modular edge-graceful, then there is a modular edge-graceful labeling $f : E(G) \to \mathbb{Z}_n$ such that $f(e) \neq 0$ for all $e \in E(G)$.
with at most one exception, that is, one zero is sufficient. Furthermore, for each prescribed edge \( e^* \) of \( G \), there is a modular edge graceful labeling \( f^* : E(G) \to \mathbb{Z}_n \) such that \( f^*(e) \neq 0 \) for all \( e \in E(G) \) with possibly one exception \( e^* \). In order to establish these results, we first present a lemma

**Lemma 4.4.1** Let \( G \) be a connected modular edge-graceful graph of order \( n \geq 3 \), where \( n \not\equiv 2 \) (mod 4) and let \( f : E(G) \to \mathbb{Z}_n \) be a given modular edge-graceful labeling of \( G \). If \( P_k = (v_1, v_2, \ldots, v_k) \) is a path of order \( k \geq 3 \) in \( G \), then there is a modular edge-graceful labeling \( g : E(G) \to \mathbb{Z}_n \) of \( G \) that satisfies the following four conditions

1. \( g(e) = f(e) \) for all \( e \notin E(P_k) \),
2. \( g'(v) = f'(v) \) for all \( v \notin V(P_k) \),
3. \( \{ g'(v_i) \mid 1 \leq i \leq k \} = \{ f'(v_i) \mid 1 \leq i \leq k \} \),
4. \( g(v_i v_{i+1}) \neq 0 \) for all \( i \) with \( 1 \leq i \leq k - 2 \)

**Proof.** We proceed by induction on \( k \). For \( k = 3 \), let \( P_3 = (v_1, v_2, v_3) \). If \( f(v_1 v_2) \neq 0 \), then let \( g = f \). Thus, we may assume that \( f(v_1 v_2) = 0 \) Suppose that \( f'(v_1) = a, f'(v_2) = b \) and \( f'(v_3) = c \). Define a labeling \( g : E(G) \to \mathbb{Z}_n \) by

\[
g(e) = \begin{cases} 
f(e) & \text{if } e \notin E(P_3) \\
f(e) + (c - a) & \text{if } e = v_1 v_2 \\
f(e) - (c - a) & \text{if } e = v_2 v_3
\end{cases}
\]

By the definition of \( g \), conditions (1) and (2) hold. Since \( g(v_1 v_2) = f(v_1 v_2) + (c - a) = 0 + c - a = c - a \) and \( a \neq c \), it follows that \( g(v_1 v_2) \neq 0 \) and so (3) holds...
Furthermore,

\[ g'(v_1) = f'(v_1) + (c - a) = a + (c - a) = c \]
\[ g'(v_2) = f'(v_2) + (c - a) - (c - a) = f'(v_2) = b \]
\[ g'(v_1) = f'(v_1) - (c - a) = c - (c - a) = a \]

and so (4) holds.

Assume for some integer \( k \geq 4 \) that the result holds for all paths of order \( k' \) in \( G \) where \( 3 \leq k' < k \). Let \( P_k = (v_1, v_2, \ldots, v_k) \) be a path of order \( k \geq 4 \) in \( G \). First, consider the subpath \( P_{k-1} = (v_1, v_2, \ldots, v_{k-1}) \) of \( P_k \). By the induction hypothesis, there is a modular edge-graceful labeling \( h : E(G) \to \mathbb{Z}_n \) of \( G \) that satisfies the following four conditions:

1. \( h(e) = f(e) \) for all \( e \notin E(P_{k-1}) \),
2. \( h'(v) = f'(v) \) for all \( v \notin V(P_{k-1}) \),
3. \( \{ h'(v_i) \mid 1 \leq i \leq k \} = \{ f'(v_i) \mid 1 \leq i \leq k - 1 \} \),
4. \( h(v_{i}v_{i+1}) \neq 0 \) for all \( i \) with \( 1 \leq i \leq k - 3 \)

If \( h(v_{k-2}v_{k-1}) \neq 0 \), then let \( g = h \). Thus we may assume that \( h(v_{k-2}v_{k-1}) = 0 \).

Now consider the subpath \( P_3 = (v_{k-2}, v_{k-1}, v_k) \) of \( P_k \). Applying the induction hypothesis to \( P_3 \) and the modular edge-graceful labeling \( h \) of \( G \), we conclude that there is a modular edge-graceful labeling \( g : E(G) \to \mathbb{Z}_n \) such that:

1. \( g(e) = h(e) \) for all \( e \notin E(P_3) \),
\( 2^* \) \( g'(v) = h'(v) \text{ for all } v \notin V(P_3), \)

\( 3^* \) \( \{g'(v_{k-2}), g'(v_{k-1}), g'(v_k)\} = \{h'(v_{k-2}), h'(v_{k-1}), h'(v_k)\}, \)

\( 4^* \) \( g(v_{k-2}v_{k-1}) \neq 0 \)

Observe that for each integer \( j \) with \( 1 \leq j \leq 4 \), conditions \( (j') \) and \( (j^*) \) give rise to condition \( (j) \). Therefore, \( g \) and \( f \) satisfy conditions (1)-(4).

**Theorem 4.4.2** Let \( G \) be a connected modular edge graceful graph of order \( n \geq 3 \), where \( n \neq 2 \mod 4 \). Then there is a modular edge graceful labeling \( f : E(G) \to \mathbb{Z}_n \) such that \( f(e) \neq 0 \) for all \( e \in E(G) \) with at most one exception.

**Proof.** Assume, to the contrary, that this statement is not true. Among all modular edge-graceful labelings of \( G \), let \( f : E(G) \to \mathbb{Z}_n \) be one for which the set

\[ S = \{e \in E(G) \mid f(e) = 0\} \]

has the smallest possible cardinality. Then \( |S| \geq 2 \). Let \( e_1, e_2 \in S \). Since \( G \) is connected, there exists a path \( P_k = (v_1, v_2, \ldots, v_k) \) of order \( k \geq 3 \) such that \( e_1 = v_1v_2 \) and \( e_2 = v_{k-1}v_k \). By Lemma 4.4.1, there is a modular edge-graceful labeling \( g : E(G) \to \mathbb{Z}_n \) of \( G \) that satisfies (i) \( g(e) = f(e) \) for all \( e \notin E(P_k) \) and (ii) \( g(v_iv_{i+1}) \neq 0 \) for all \( i \) with \( 1 \leq i \leq k-2 \). Let

\[ S' = \{e \in E(G) \mid g(e) = 0\} \]

Then \( S' \subseteq S \). Since \( g(e_1) \neq 0 \), it follows that \( |S'| < |S| \), which contradicts the defining property of \( S \). 

\[ \square \]
Theorem 4.4.3 Let $G$ be a connected modular edge-graceful graph of order $n \geq 3$, where $n \not\equiv 2 \pmod{4}$. For a fixed edge $e^*$ of $G$, there is a modular edge-graceful labeling $f : E(G) \to \mathbb{Z}_n$ such that $f(e) \neq 0$ for all $e \in E(G) - \{e^*\}$.

Proof. By Theorem 4.4.2, there is a modular edge-graceful labeling $f : E(G) \to \mathbb{Z}_n$ of $G$ such that $f(e) \neq 0$ for all $e \in E(G)$ with at most one exception. If $f(e) \neq 0$ for all $e \in E(G) - \{e^*\}$, then the result is proved. Thus, we may assume that $f(e) = 0$ for some $e \neq e^*$. Since $G$ is connected, there exists a path $P_k = (v_1, v_2, \ldots, v_k)$ of order $k \geq 3$ such that $e = v_1v_2$ and $e^* = v_{k-1}v_k$. By Lemma 4.4.1, there is a modular edge-graceful labeling $g : E(G) \to \mathbb{Z}_n$ of $G$ that satisfies (i) $g(e) = f(e)$ for all $e \notin E(P_k)$ and (ii) $g(v_iv_{i+1}) \neq 0$ for all $i$ with $1 \leq i \leq k - 2$. Therefore, $g(e) \neq 0$ for all $e \in E(G) - \{e^*\}$. ■

4.5 A Characterization of Nowhere-Zero Meg-Trees

Let $T$ be a tree of order $n \geq 3$ with $n \not\equiv 2 \pmod{4}$ having diameter $d \geq 2$. If $d = 2$, then $T$ is a star and, by Proposition 4.3.5, $T$ is nowhere-zero modular edge-graceful if and only if $n$ is odd. In this section we establish a characterization of nowhere-zero modular edge-graceful trees. More precisely, we show that if $T$ is a tree of order $n \geq 3$ with $n \not\equiv 2 \pmod{4}$, then $T$ is nowhere-zero modular edge-graceful if and only if $T$ is not a star of odd order. We first show that if $T$ is a modular edge-graceful graph of diameter at least 4, then $T$ is nowhere-zero modular edge-graceful.
Theorem 4.5.1 If $T$ is a tree of order $n \geq 5$ with $n \not\equiv 2 \pmod{4}$ and $\text{diam}(T) \geq 4$, then $T$ is nowhere-zero modular edge-graceful.

Proof. Assume, to the contrary, that there is a tree $T$ of order $n \geq 5$ with $n \not\equiv 2 \pmod{4}$ and $\text{diam}(T) \geq 4$ but $T$ is not nowhere-zero modular edge-graceful. Let $v_0$ be an end-vertex of $T$ for which $e(v_0) \geq 4$ and let $P = (v_0, v_1, v_2, v_3, v_4)$ be a $v_0 - v_4$ path in $T$. By Theorem 4.4.3, there is a modular edge-graceful labeling $f : E(T) \rightarrow \mathbb{Z}_n$ such that $f(e) \neq 0$ for all $e \in E(T) - \{v_0v_1\}$. Since $T$ is not nowhere-zero modular edge-graceful, $f(v_0v_1) = 0$ and so $f'(v_0) = 0$. Suppose that $f'(v_i) = x_i$ for $1 \leq i \leq 4$ (see Figure 4.8). Then $x_i \neq 0$ for $1 \leq i \leq 4$.

![Figure 4.8: The labelings $f$ and $f'$ on $P$](image)

We now construct a sequence of four edge labelings $g, h, i, j$ of $T$ recursively as follows. First, define $g : E(T) \rightarrow \mathbb{Z}_n$ from the labeling $f$ by

$$g(e) = \begin{cases} x_2 & \text{if } e = v_0v_1 \\ f(e) - x_2 & \text{if } e = v_1v_2 \\ f(e) & \text{otherwise.} \end{cases}$$

Because $g'(v_0) = x_2 = f'(v_2)$, $g'(v_2) = 0 = f'(v_0)$ and $g'(v) = f'(v)$ for all $v \in V(G) - \{v_0, v_1\}$, it follows that $g$ is a modular edge-graceful labeling of $T$. Since $f(e) = g(e)$ for all $e \in E(T) - \{v_0v_1, v_1v_2\}$ and $g(v_0v_1) = x_2 \neq 0$, it follows that $g(e) \neq 0$ for all $e \in E(T) - \{v_1v_2\}$. Again, since $T$ is not nowhere-zero
modular edge-graceful, \( g(v_1v_2) = f(v_1v_2) - \tau_2 = 0 \), implying that \( f(v_1v_2) = \tau_2 \) (see Figure 4.9)

\[
\begin{array}{c c c c c}
3 & f(v_1v_4) & 2 & f(v_2v_1) & 0 \\
v_4 & v_3 & v_2 & v_1 & v_0
\end{array}
\]

Figure 4.9 The labelings \( g \) and \( g' \) on \( P \)

Secondly, define \( h : E(T) \to \mathbb{Z}_n \) from the labeling \( g \) by

\[
h(e) = \begin{cases} 
\tau_3 - \tau_1 & \text{if } e = v_1v_2 \\
g(e) - (\tau_3 - \tau_1) & \text{if } e = v_2v_3 \\
g(e) & \text{otherwise}
\end{cases}
\]

Then \( h'(v) = g'(v) = f'(v) \) for all \( v \in V(G) - \{v_1, v_2, v_3\} \) and

\[
\begin{align*}
h'(v_1) &= g'(v_1) + (\tau_3 - \tau_1) = \tau_1 + (\tau_3 - \tau_1) = \tau_3 = g'(v_3) \\
h'(v_2) &= g'(v_2) + (\tau_3 - \tau_1) - (\tau_3 - \tau_1) = g'(v_2) = 0 \\
h'(v_3) &= g'(v_3) - (\tau_3 - \tau_1) = \tau_3 - (\tau_3 - \tau_1) = \tau_1 = g'(v_1)
\end{align*}
\]

Hence \( h \) is a modular edge-graceful labeling of \( T \). Since \( h(e) \neq 0 \) for all \( e \in E(T) - \{v_2v_3\} \) and \( T \) is not nowhere-zero modular edge graceful, \( h(v_2v_3) = 0 \) This implies that

\[
h(v_2v_3) = g(v_2v_3) - (\tau_3 - \tau_1) = f(v_2v_3) - (\tau_3 - \tau_1) = 0
\]

and so \( f(v_2v_3) = \tau_3 - \tau_1 \) (see Figure 4.10)
Next, define $\pi : E(T) \to \mathbb{Z}_n$ from the labeling $h$ by

$$
\pi(e) = \begin{cases} 
\tau_4 & \text{if } e = v_2v_3 \\
 h(e) - \tau_4 & \text{if } e = v_3v_4 \\
 h(e) & \text{otherwise}
\end{cases}
$$

Then $\pi'(v) = h'(v)$ for all $v \in V(G) - \{v_2, v_4\}$ and

$$
\pi'(v_2) = h'(v_2) + \tau_4 = 0 + \tau_4 = \tau_4 = h'(v_4)
$$

$$
\pi'(v_4) = h'(v_4) - \tau_4 = \tau_4 - \tau_4 = 0 = h'(v_2)
$$

Hence $\pi$ is a modular edge-graceful labeling of $T$. Since $\pi(e) \neq 0$ for all $e \in E(T) - \{v_3v_4\}$ and $T$ is not nowhere-zero modular edge-graceful, $\pi(v_3v_4) = 0$. This implies that

$$
\pi(v_3v_4) = h(v_3v_4) - \tau_4 = f(v_3v_4) - \tau_4 = 0
$$

and so $f(v_3v_4) = \tau_4$. Therefore, $f(v_1v_2) = \tau_2$, $f(v_2v_3) = \tau_3 - \tau_1$ and $f(v_3v_4) = \tau_4$ and so we have the situation as shown in Figure 4.11.

Since $f$ is a modular edge-graceful labeling of $T$, $\tau_1 \neq \tau_4$ and hence $\tau_4 - \tau_1 \neq 0$. Thus $\tau_3 + \tau_4 - \tau_1 \neq \tau_3$, which implies that $\deg v_3 \neq 2$. Let $v_5 \in V(T) - V(P)$ that
is adjacent to \( v_3 \) and let \( f'(v_5) = r_5 \). Applying the same argument to the path 
\((v_0, v_1, v_2, v_3, v_5)\) and the modular edge-graceful labeling \( f \) in which \( f(e) \neq 0 \) for all \( e \in E(T) - \{v_0v_1\} \), we obtain that \( f(v_3v_5) = r_5 \) (see Figure 4.12)

Figure 4.12 A step in the proof of Theorem 4.5.1

Now observe that at most one of \( r_3 - r_1 - r_2 + r_4 \) and \( r_3 - r_1 - r_2 + r_5 \) is 0. We may assume, without loss of generality, that \( r_3 - r_1 - r_2 + r_4 \neq 0 \). We now define a labeling \( j : E(T) \to \mathbb{Z}_n \) from the labeling \( f \) by

\[
j(e) = \begin{cases} 
    r_4 & \text{if } e = v_0v_1 \\
    r_2 - r_4 & \text{if } e = v_1v_2 \\
    r_4 + r_3 - r_2 - r_1 & \text{if } e = v_2v_3 \\
    r_2 & \text{if } e = v_3v_4 \\
    f(e) & \text{otherwise}
\end{cases}
\]

Since \( r_3 - r_1 - r_2 + r_4 \neq 0 \) by assumption and \( f(e) \neq 0 \) for all \( e \in E(T) - E(P) \), it follows that \( j(e) \neq 0 \) for all \( e \in E(T) \). Furthermore, \( j'(v) = f'(v) \) for all
Let $v \in V(T) - V(P)$ and

\[
\begin{align*}
j'(v_0) &= \tau_4 \\
j'(v_1) &= f'(v_1) - \tau_4 + \tau_4 = f'(v_1) = \tau_1 \\
j'(v_2) &= f'(v_2) - \tau_4 + (\tau_4 - \tau_2) = \tau_2 - \tau_4 + (\tau_4 - \tau_2) = 0 \\
j'(v_3) &= f'(v_3) + (\tau_2 - \tau_4) + (\tau_4 - \tau_2) = f'(v_3) = \tau_3 \\
j'(v_4) &= f'(v_4) + (\tau_2 - \tau_4) = \tau_4 + (\tau_2 - \tau_4) = \tau_2
\end{align*}
\]

Thus \( \{j'(v) \mid v \in V(P)\} = \{f'(v) \mid v \in V(P)\} = \{0, \tau_1, \tau_2, \tau_3, \tau_4\} \) and so \( \{j'(v) \mid v \in V(T)\} = \{f'(v) \mid v \in V(T)\} \). Therefore, \( j \) is a nowhere-zero modular edge-graceful labeling of \( T \), which is a contradiction. 

By Proposition 4.3.5 and Theorem 4.5.1, it remains to determine which double stars are nowhere-zero modular edge-graceful.

**Theorem 4.5.2** If \( T \) is a double star of order \( n \geq 4 \) with \( n \not\equiv 2 \) (mod 4), then \( T \) is nowhere-zero modular edge-graceful.

**Proof.** Let \( T = S_{a,b} \) be the double star of order \( n = a + b + 2 \geq 4 \) whose central vertices are \( u \) and \( v \) where \( \deg u = a + 1 \) and \( \deg v = b + 1 \). Let \( u_1, u_2, \ldots, u_a \) be end-vertices of \( T \) that are adjacent to \( u \) and let \( v_1, v_2, \ldots, v_b \) be end-vertices of \( T \) that are adjacent to \( v \).
First, suppose that \( n \geq 5 \) is odd. We may assume, without loss of generality, that \( a \) is odd and \( b \) is even. Define a labeling \( f : E(T) \rightarrow \mathbb{Z}_n - \{0\} \) by

\[
f(e) = \begin{cases} 
\frac{\alpha + 1}{2} & \text{if } e = uu_i, \ 1 \leq i \leq a \text{ and } i \text{ is odd} \\
-\frac{\beta}{2} & \text{if } e = uu_i, \ 1 \leq i \leq a \text{ and } i \text{ is even} \\
-\frac{\alpha + 1}{2} & \text{if } e = uv \\
\frac{\alpha + 1}{2} + \frac{\gamma + 1}{2} & \text{if } e = vv_j, \ 1 \leq j \leq b \text{ and } j \text{ is odd} \\
-\left(\frac{\alpha + 1}{2} + \frac{\gamma}{2}\right) & \text{if } e = vv_j, \ 1 \leq j \leq b \text{ and } j \text{ is even}.
\end{cases}
\]

Figure 4.13 shows the labelings \( f \) and \( f' \) for \( S_{3,2} \) and \( S_{5,4} \).

![Labelings for S3,2 and S5,4](image)

Figure 4.13: The labelings for \( S_{3,2} \) and \( S_{5,4} \) in Theorem 4.5.2

Observe that

\[
f'(u) = \sum_{i=1}^{a} f(uu_i) + f(uv) = \frac{a + 1}{2} - \frac{a + 1}{2} = 0
\]

\[
f'(v) = \sum_{j=1}^{b} f(vv_j) + f(uv) = 0 - \frac{a + 1}{2} = -\frac{a + 1}{2}
\]
and
\[ \{f'(u_i) : 1 \leq i \leq a\} = \left\{ \pm 1, \pm 2, \ldots, \pm \frac{a-1}{2}, \pm \frac{a+1}{2}\right\} \]

\[ \{f'(v_j) : 1 \leq j \leq b\} = \left\{ \pm \left(\frac{a+1}{2} + 1\right) \pm \left(\frac{a+1}{2} + 1\right), \ldots, \pm \frac{n-1}{2}\right\}. \]

Therefore, \( f' \) is one-to-one and so \( f \) is a nowhere-zero modular edge-graceful labeling.

First, suppose that \( n \geq 4 \) is even. Since \( n = a + b + 2 \), it follows that \( a \) and \( b \) are of the same parity. We consider two cases.

Case 1. \( a \) and \( b \) are both odd. We may assume, without loss of generality, that \( a \leq b \). First, suppose that \( a = 1 \). Then \( n = b + 3 \). If \( b = 1 \), then \( T = P_4 \) and \( T \) is nowhere-zero modular edge-graceful by Theorem 4.3.1. Thus, we may assume that \( b \neq 1 \). Since \( n = b + 3 \equiv 0 \pmod{4} \), it follows that \( b \geq 5 \). Now define a labeling \( f : E(T) \to \mathbb{Z}_n - \{0\} \) such that (i)

\[ \{f(\varepsilon_1) : 2 \leq i \leq b\} = \left\{ \pm 1, \pm 2, \ldots, \pm \frac{b+1}{2}\right\} - \left\{ \pm \frac{n}{4}\right\} \]

(where then \( f(\varepsilon_1) \neq \pm \frac{n}{2} \) for \( 2 \leq i \leq b \)) and (ii)

\[ f(e) = \begin{cases} \frac{n}{4} & \text{if } e = uu_1 \\ \frac{n}{2} & \text{if } e = uv \text{ or } e = vv_1 \end{cases} \tag{4.4} \]

Figure 4.14 shows the labelings \( f \) and \( f' \) for \( S_{1,5} \), where then \( a = 1, b = 5 \) and \( n = 8 \). By the definition of \( f \),

\[ \{f'(v_i) : 2 \leq i \leq b\} = \left\{ \pm 1, \pm 2, \ldots, \pm \frac{b+1}{2}\right\} - \left\{ \pm \frac{n}{4}\right\} \text{ in } \mathbb{Z}_n \]
and $f'(u) = 0, f'(u_1) = \frac{n}{4}, f'(v) = \frac{n}{4} + \frac{n}{2} = \frac{3n}{4} = -\frac{n}{4}$ and $f'(v_1) = \frac{n}{2}$ in $\mathbb{Z}_n$. Thus $f$ is a nowhere-zero modular edge-graceful labeling of $T$.

Next, suppose that $a \geq 3$. Let $a = 2p + 1$ and $b = 2q + 1$ for some positive integers $p$ and $q$ where $p \leq q$. Now define a labeling $f : E(T) \to \mathbb{Z}_n - \{0\}$ such that (i)

\[
\{f(u_{i}) : 2 \leq i \leq a\} = \{\pm 1, \pm 2, \ldots, \pm p\}
\]

\[
\{f(v_{j}) : 2 \leq j \leq b\} = \{\pm(p + 1), \pm(p + 3), \ldots, \pm(p + q + 1)\} - \{\pm \frac{n}{4}\}.
\]

(where then $f(u_{i}) \neq \pm \frac{n}{2}$ and $f(v_{j}) \neq \pm \frac{n}{2}$ for $2 \leq i \leq a$ and $2 \leq j \leq b$) and (ii)$f(u_1) = \frac{n}{4}$ and $f(u) = f(v_1) = \frac{n}{2}$ as described in (4.4). Figure 4.14 shows the labelings $f$ and $f'$ for $S_{5,5}$ and $S_{3,7}$. By the definition of $f$,

\[
\{f'(u_{i}) : 2 \leq i \leq a\} \cup \{f'(v_{j}) : 2 \leq j \leq b\} = \mathbb{Z}_n - \left\{0, \frac{n}{4}, \frac{3n}{4} = -\frac{n}{4}\right\} \text{ in } \mathbb{Z}_n.
\]

Furthermore, $f'(u_1) = \frac{n}{4}$ and $f'(v_1) = \frac{n}{2}$ in $\mathbb{Z}_n$

\[
f'(u) = \sum_{u \in N(u)} f(uw) = f(uv) + f(uu_1) + \left(\sum_{i=2}^{a} f(u_i)ight)
\]

\[= f(uv) + f(uu_1) + 0 = \frac{n}{2} + \frac{n}{4} = \frac{3n}{4} = -\frac{n}{4},
\]

\[
f'(v) = \sum_{v \in N(v)} f(vw) = f(uv) + f(vv_1) + \left(\sum_{i=2}^{b} f(v_i)ight)
\]

\[= f(uv) + f(vv_1) + 0 = \frac{n}{2} + \frac{n}{2} = 0.
\]
Thus $f$ is a nowhere-zero modular edge-graceful labeling of $T$.

Case 2. $a$ and $b$ are both even. Because $n \equiv 0 \pmod{4}$ and $n = a + b + 2$, we may assume, without loss of generality, that $a \equiv 0 \pmod{4}$ and $b \equiv 2 \pmod{4}$. Since $a > 0$ and $b > 0$, it follows that $a \geq 4$ and $b \geq 2$. Define the sets $U$ and $W$ of edges of $T$ by

$$U = \{e = uv_i : 3 \leq i \leq a\} \text{ and } W = \{e = uv_i : 1 \leq i \leq b\}.$$  

Then $|U| = a - 2$ and $|W| = b$ are both even and so $|U \cup W| = a + b - 2 = n - 4$. Furthermore, let

$$S = \mathbb{Z}_n - \left\{0, \frac{n}{4}, \frac{n}{2}, \frac{3n}{4}\right\}$$

and so $|S| = n - 4 = |U \cup W|$. Let $g : U \cup W \rightarrow S$ be any bijective function with the property that $g(uv_i) = r \in S$ where $3 \leq i \leq a$ if and only if $g(uv_i) = -r \in S$.
for some \( j \) with \( r \neq j \) and \( 3 \leq j \leq a \). This implies that \( g(vv_j) = r' \in S \) where \( 1 \leq r \leq b \) if and only if \( g(vv_j) = -r' \in S \) for some \( j \) with \( r \neq j \) and \( 1 \leq j \leq b \).

Now define a labeling \( f : E(T) \to \mathbb{Z}_n \) in terms of \( g \) by

\[
f(e) = \begin{cases} 
\frac{n}{2} & \text{if } e = uu_1 \\
\frac{n}{4} & \text{if } e = uu_2 \\
0 & \text{if } e = uv \\
g(e) & \text{if } e \in U \cup W
\end{cases}
\]

Figure 4.15 shows the labelings \( f \) for \( S_{4,2} \) and \( S_{4,6} \) together with the induced edge labeling \( f' \) in each case.

By the definitions of \( f \) and \( g \), it follows that

\[
\{f'(u_i) : 3 \leq i \leq a\} \cup \{f'(v_j) : 1 \leq j \leq b\} = S = \mathbb{Z}_n - \left\{0, \frac{n}{4}, \frac{n}{2}, \frac{3n}{4}\right\}.
\]
Furthermore, \( f'(u_1) = \frac{n}{2}, f'(u_2) = \frac{n}{4} \) and

\[
f'(u) = \left( \sum_{i=1}^{a} f(u_i) \right) + f(uv) = f(uu_1) + f(uu_2) + \left( \sum_{i=3}^{a} f(u_i) \right) + f(uv)
\]

\[
= \frac{n}{2} + \frac{n}{4} + 0 + 0 = \frac{3n}{4} = -\frac{n}{4} \text{ in } \mathbb{Z}_n
\]

\[
f'(v) = \left( \sum_{i=1}^{b} f(v_i) \right) + f(uv) = 0 + 0 = 0.
\]

Thus \( f \) is a modular edge-graceful labeling of \( T \) but \( f \) is not nowhere-zero modular edge-graceful labeling. [Note that the labeling \( f \) in Case 1 when \( a \geq 3 \) could also be defined in this fashion.]

We now construct a nowhere-zero modular edge-graceful labeling \( h \) of \( T \) from \( f \) as follows. Suppose that \( f(vv_1) = s \) for some \( s \in S \). It follows by the definition of the set \( S \) that \( s \neq \frac{3n}{4} \) in \( \mathbb{Z}_n \). Define \( h : E(T) \to \mathbb{Z}_n - \{0\} \) by

\[
h(e) = \begin{cases} 
  f(e) & \text{if } e \neq uv \text{ and } e \neq vv_1 \\
  s - \frac{3n}{4} & \text{if } e = uv \\
  \frac{3n}{4} & \text{if } e = vv_1.
\end{cases}
\]

For example, in the labeling \( f \) in Figure 4.15 for \( S_{4,2} \), \( f(vv_1) = 3 \) and \( n = 8 \). Thus \( h(uv) = 3 - \frac{38}{4} = 3 - 6 = -3 = 5 \) and \( h(vv_1) = 6 \) in \( \mathbb{Z}_8 \). In the labeling \( f \) in Figure 4.15 for \( S_{4,0} \), \( f(vv_1) = 2 \) and \( n = 12 \). Thus \( h(uv) = 2 - \frac{312}{4} = 2 - 9 = -7 = 5 \) and \( h(vv_1) = 9 \) in \( \mathbb{Z}_{12} \). Figure 4.16 shows the labelings \( h \) for \( S_{4,2} \) and \( S_{4,0} \) together.
with the induced edge labeling $h'$ in each case. Observe that

$$h'(u) = f'(u) + s - \frac{3n}{4} = \frac{3n}{4} + s - \frac{3n}{4} = s$$

$$h'(v) = f'(v) + \left(s - \frac{3n}{4}\right) - s + \frac{3n}{4} = f'(v) = 0$$

$$h'(v_1) = \frac{3n}{4}$$

$$h'(w) = f'(w) \text{ if } w \neq u, v_1$$

![Diagram](image)

Figure 4.16 The labelings $h$ for $S_{4,2}$ and $S_{4,6}$ in Case 2

Therefore, $h$ is a nowhere-zero modular edge-graceful labeling of $T$.

Combining Proposition 4.3.5 and Theorems 4.5.1 and 4.5.2, we establish the following characterization of all trees that are nowhere zero modular edge-graceful.

**Theorem 4.5.3** If $T$ is a tree of order $n \geq 3$ with $n \not\equiv 2 \pmod{4}$, then $T$ is nowhere-zero modular edge-graceful if and only if $T$ is not a star of even order.
4.6 A Characterization of Nowhere-Zero Meg-Graphs

We have seen in Theorem 4.5.3 that if T is a tree of order $n \geq 3$ with $n \not\equiv 2 \pmod{4}$, then T is nowhere-zero modular edge-graceful if and only if T is not a star of even order. We now consider connected graphs of order $n \geq 3$ that are not trees, that is, we consider connected graphs with cycles. With the aid of the proof of Theorem 4.2.3 and Theorem 4.3.3, we first establish the following result on connected graphs with even cycles.

**Theorem 4.6.1** If G is a connected modular edge-graceful graph of order $n \geq 4$ that contains an even cycle, then G is nowhere-zero modular edge-graceful.

**Proof.** Let G be a connected modular edge-graceful graph of order $n \geq 4$, where $n \not\equiv 2 \pmod{4}$, such that G contains an even cycle C and let $e^* \in E(C)$. By Theorem 4.4.3, there is a modular edge-graceful labeling $g : E(G) \to \mathbb{Z}_n$ such that $g(e) \not\equiv 0$ for all $e \in E(G) - \{e^*\}$. By Corollary 4.2.4, there is a modular edge-graceful labeling $f$ of G such that $f(e) \not\equiv 0$ for each edge $e$ that lies on C and $f(e) = g(e)$ for each edge $e$ that does not lie on C. Since $g(e) \not\equiv 0$ for all $e \in E(G) - \{e^*\}$, it follows that $f(e) \not\equiv 0$ for all $e \in E(G)$. Hence G is nowhere-zero modular edge-graceful. 

We now consider connected graphs in general. First, we establish a lemma. Let G be a modular edge-graceful graph of order $n \geq 3$ that is not nowhere-zero modular edge-graceful. By Theorem 4.4.3, for each $e^* \in E(G)$, there is a modular
edge-graceful labeling $f: E(G) \rightarrow \mathbb{Z}_n$ such that $f(e) \neq 0$ for all $e \in E(G) - \{e^*\}$ and $f(e^*) = 0$. In fact, more can be said about such a modular edge-graceful labeling $f$ and its induced vertex labeling $f'$, as we show next.

**Lemma 4.6.2** Let $G$ be a modular edge graceful graph of order $n \geq 3$ that is not nowhere zero modular edge-graceful. Let $v_1v_2$ be an edge of $G$ and let $f: E(G) \rightarrow \mathbb{Z}_n$ be a modular edge-graceful labeling of $G$ such that $f(v_1v_2) = 0$ and $f(e) \neq 0$ for all $e \in E(G) - \{v_1v_2\}$. If $P_k = (v_1, v_2, \ldots, v_k)$ is a path of order $k \geq 3$ in $G$ such that $f'(v_i) = \tau_i$ for $1 \leq i \leq k$ and $f(v_iv_{i+1}) = y_i$ for $1 \leq i \leq k - 1$ where $y_1 = 0$, then

$$y_i = \begin{cases} \tau_{i+1} - \tau_1 & \text{if } 2 \leq i \leq k - 1 \text{ and } i \text{ is even} \\ \tau_{i+1} - \tau_2 & \text{if } 3 \leq i \leq k - 1 \text{ and } i \text{ is odd} \end{cases} \quad (4.5)$$

Furthermore, there is a modular edge-graceful labeling $g: E(G) \rightarrow \mathbb{Z}_n$ of $G$ that satisfies the following conditions

1. $g(e) \neq 0$ for all $e \in E(G) - \{v_{k-1}v_k\}$ and $g(v_{k-1}v_k) = 0$,

2. $g(e) = f(e)$ for all $e \in E(G) - E(P_k)$, and

3. If $k$ is odd, then $g'(v_k) = \tau_1$ and $g'(v_{k-1}) = \tau_2$, while if $k$ is even, then $g'(v_k) = \tau_2$ and $g'(v_{k-1}) = \tau_1$.

**Proof.** We proceed by induction on $k$. We first consider the two base cases when $k = 3$ and $k = 4$. First, suppose that $k = 3$. Let $f: E(G) \rightarrow \mathbb{Z}_n$ be a modular edge-graceful labeling of $G$ such that $f(v_1v_2) = 0$ and $f(e) \neq 0$ for all $e \in E(G) - \{v_1v_2\}$ and let $P_3 = (v_1, v_2, v_3)$ Suppose that $f'(v_i) = \tau_i$ for $1 \leq i \leq 3$ (see Figure 4.17). We show that $f(v_2v_3) = y_2 = \tau_3 - \tau_1$. 

...
Define a labeling \( g : E(G) \to \mathbb{Z}_n \) of \( G \) by

\[
g(e) = \begin{cases} 
  f(e) & \text{if } e \neq v_1v_2, v_2v_3 \\
  \tau_3 - \tau_1 & \text{if } e = v_1v_2 \\
  y_2 - (\tau_3 - \tau_1) & \text{if } e = v_2v_3 
\end{cases}
\]

We now consider the induced vertex labeling \( g' : V(G) \to \mathbb{Z}_n \). Observe that

\[
g'(v) = f'(v) \text{ if } v \in V(G) - \{v_1, v_2, v_3\} \text{ and }
\]

\[
g'(v_1) = f'(v_1) + (\tau_3 - \tau_1) = \tau_1 + (\tau_3 - \tau_1) = \tau_3 \\
g'(v_2) = f'(v_2) + (\tau_3 - \tau_1) - (\tau_3 - \tau_1) = f'(v_2) = \tau_2 \\
g'(v_3) = f'(v_3) - (\tau_3 - \tau_1) = \tau_3 - (\tau_3 - \tau_1) = \tau_1
\]

Figure 4.18 shows the labelings \( g \) together with \( g' \) on \( P_3 \). Thus \( \{g'(v_i) \mid 1 \leq i \leq 3\} = \{f'(v_i) \mid 1 \leq i \leq 3\} = \{\tau_1, \tau_2, \tau_3\} \) and so \( g \) is a modular edge graceful labeling of \( G \). Since \( g(e) = f(e) \) for all \( e \in E(G) - E(P_3) \), it follows that \( g(e) \neq 0 \) if \( e \neq v_1v_2, v_2v_3 \). Also, \( \tau_3 \neq \tau_1 \) and so \( g(v_1v_2) = \tau_3 - \tau_1 \neq 0 \). Since \( G \) is not nowhere-zero modular edge-graceful, \( g(v_2v_3) = y_2 - (\tau_3 - \tau_1) = 0 \) and so \( y_2 = \tau_3 - \tau_1 \). Thus (4.5) holds. Furthermore, the modular edge-graceful labeling \( g : E(G) \to \mathbb{Z}_n \) of \( G \) satisfies the following conditions

1. \( g(e) \neq 0 \) for all \( e \in E(G) - \{v_2v_3\} \) and \( g(v_2v_3) = 0 \),

2. \( g(e) = f(e) \) for all \( e \in E(G) - E(P_3) \), and
(3) \( g'(v_3) = r_1 \) and \( g'(v_2) = r_2 \)

Hence the result holds for \( k = 3 \)

\[ \begin{array}{cccc}
  v_1 & x_1 & x_1 & z_1 \\
  v_2 & y_2 = (x_1 - x_1) & v_1 & v_4
\end{array} \]

Figure 4.18 The labelings \( g \) and \( g' \) for \( P_3 \)

Next suppose that \( k = 4 \) Let \( f : E(G) \to \mathbb{Z}_n \) be a modular edge-graceful labeling of \( G \) such that \( f(v_1v_2) = 0 \) and \( f(e) \neq 0 \) for all \( e \in E(G) - \{v_1v_2\} \) and let \( P_4 = (v_1, v_2, v_3, v_4) \) Again, suppose that \( f'(v_i) = \tau_i \), for \( 1 \leq i \leq 4 \) and \( f(v_i, v_{i+1}) = y_i \), for \( 1 \leq i \leq 3 \), where then \( y_1 = 0 \) (see Figure 4.19)

\[ \begin{array}{cccc}
  v_1 & y_1 = 0 & v_2 & y_2 \\
  v_2 & x_2 & v_1 & v_3 \\
  v_3 & x_1 & v_4 \\
  v_4 & y_1 \\
\end{array} \]

Figure 4.19 The labelings \( f \) and \( f' \) for \( P_4 \)

By the case when \( k = 3 \), there is a modular edge graceful labeling \( g : E(G) \to \mathbb{Z}_n \) of \( G \) such that \( g(e) \neq 0 \) for all \( e \in E(G) - \{v_2v_3\} \), \( g(v_3v_4) = y_3 = f(v_3v_4) \), \( g'(v_1) = \tau_3 \), \( g'(v_3) = \tau_1 \), and \( g'(v) = f'(v) \) for all \( v \in V(G) - \{v_1, v_3\} \). The labelings \( g \) and \( g' \) for \( P_4 \) are shown in Figure 4.20

\[ \begin{array}{cccc}
  v_1 & x_1 - x_1 & 0 & x_4 \\
  v_2 & v_1 & v_1 & v_4
\end{array} \]

Figure 4.20 The labeling \( g \) and \( g' \) for \( P_4 \)
Define a labeling \( h : E(G) \to \mathbb{Z}_n \) of \( G \) from the modular edge-graceful labeling \( g \) of \( G \) by

\[
h(e) = \begin{cases} 
g(e) & \text{if } e \neq v_2v_3, v_3v_4 \\
x_4 - x_2 & \text{if } e = v_2v_3 \\
y_3 - (x_4 - x_2) & \text{if } e = v_3v_4.\end{cases}
\]

Observe that \( h'(v) = g'(v) \) if \( v \in V(G) - \{v_2, v_3, v_4\} \) and

\[
\begin{align*}
h'(v_2) &= g'(v_2) + (x_4 - x_2) = x_2 + (x_4 - x_2) = x_4 \\
h'(v_3) &= g'(v_3) + (x_4 - x_2) - (x_4 - x_2) = g'(v_3) = x_1 \\
h'(v_4) &= g'(v_4) - (x_4 - x_2) = x_4 - (x_4 - x_2) = x_2.
\end{align*}
\]

Figure 4.21 shows the labelings \( h \) and \( h' \) on \( P_4 \) where \( g'(v_1) = x_3 \).

Thus \( \{h'(v_i) : 2 \leq i \leq 4\} = \{g'(v_i) : 2 \leq i \leq 4\} = \{x_1, x_2, x_4\} \) and so \( h \) is a modular edge-graceful labeling of \( G \). Since \( h(e) = g(e) \) for all \( e \in E(G) - \{v_2v_3, v_3v_4\} \), it follows that \( h(e) \neq 0 \) if \( e \notin \{v_2v_3, v_3v_4\} \). Also, \( x_4 \neq x_2 \) and so \( h(v_2v_3) = x_4 - x_2 \neq 0 \). Since \( G \) is not nowhere-zero modular edge-graceful, 

\( h(v_3v_4) = y_3 - (x_4 - x_2) = 0 \) and so \( y_3 = x_4 - x_2 \). Thus (4.5) holds. Furthermore, the modular edge-graceful labeling \( h : E(G) \to \mathbb{Z}_n \) of \( G \) satisfies the following conditions:

1. \( h(e) \neq 0 \) for all \( e \in E(G) - \{v_3v_4\} \) and \( h(v_3v_4) = 0 \),
(2) \( h(e) = g(e) = f(e) \) for all \( e \in E(G) - E(P_4) \), and

(3) \( h'(v_4) = x_2 \) and \( h'(v_3) = x_1 \).

Hence the result holds for \( k = 4 \).

Now suppose that the result holds for some integer \( k \geq 4 \). Let \( f : E(G) \to \mathbb{Z}_n \) be a modular edge-graceful labeling of \( G \) such that \( f(v_1v_2) = 0 \) and \( f(e) \neq 0 \) for all \( e \in E(G) - \{v_1v_2\} \) and

\[
P_{k+1} = (v_1, v_2, \ldots, v_{k+1})
\]

be a path of order \( k + 1 \geq 5 \) in \( G \). Suppose that \( f'(v_i) = x_i \) for \( 1 \leq i \leq k + 1 \) and \( f(v_i, v_{i+1}) = y_i \) for \( 1 \leq i \leq k \) where \( y_1 = 0 \). Let \( P_k = (v_1, v_2, \ldots, v_k) \) be the subpath of order \( k \) in \( P_{k+1} \). We consider two cases, according to whether \( k \) is even or \( k \) is odd.

**Case 1. \( k \) is even or \( k + 1 \) is odd.** By the induction hypothesis of \( f \) on the path \( P_k \), we have

\[
y_i = \begin{cases} x_{i+1} - x_1 & \text{if } 2 \leq i \leq k - 2 \text{ and } i \text{ is even} \\ x_{i+1} - x_2 & \text{if } 3 \leq i \leq k - 1 \text{ and } i \text{ is odd.} \end{cases}
\]

Furthermore, there is a modular edge-graceful labeling \( g : E(G) \to \mathbb{Z}_n \) of \( G \) such that

(1) \( g(e) \neq 0 \) for all \( e \in E(G) - \{v_{k-1}v_k\} \) and \( g(v_{k-1}v_k) = 0 \),

(2) \( g(e) = f(e) \) for all \( e \in E(G) - E(P_k) \), and

(3) \( g'(v_k) = x_2 \) and \( g'(v_{k-1}) = x_1 \).
Hence \( g(v_kv_{k+1}) = y_k = f(v_kv_{k+1}) \). This is shown in Figure 4.22.

Define a labeling \( h : E(G) \rightarrow \mathbb{Z}_n \) of \( G \) from the modular edge-graceful labeling \( g \) of \( G \) by

\[
    h(e) = \begin{cases} 
        g(e) & \text{if } e \neq v_{k-1}v_k, v_kv_{k+1} \\
        x_{k+1} - x_1 & \text{if } e = v_{k-1}v_k \\
        y_k - (x_{k+1} - x_1) & \text{if } e = v_kv_{k+1}.
    \end{cases}
\]

Observe that (i) \( h'(v) = g'(v) \) if \( v \in V(G) - \{v_{k-1}, v_k, v_{k+1}\} \) and (ii) by (4.6)

\[
    h'(v_{k-1}) = g'(v_{k-1}) + (x_{k+1} - x_1) = x_1 + (x_{k+1} - x_1) = x_{k+1} \\
    h'(v_k) = g'(v_k) + (x_{k+1} - x_1) - (x_{k+1} - x_1) = g'(v_k) = x_2 \\
    h'(v_{k+1}) = g'(v_{k+1}) - (x_{k+1} - x_1) = x_{k+1} - (x_{k+1} - x_1) = x_1.
\]

The labelings \( h \) and \( h' \) for \( P_{k+1} \) are shown in Figure 4.23.

Thus \( \{h'(v_i) : k - 1 \leq i \leq k + 1\} = \{g'(v_i) : k - 1 \leq i \leq k + 1\} \) and so \( h \) is a modular edge-graceful labeling of \( G \). Since \( h(e) = g(e) \) for all \( e \in E(G) - \{v_{k-1}v_k, v_kv_{k+1}\} \), it follows that \( h(e) \neq 0 \) if \( e \neq v_{k-1}v_k, v_kv_{k+1} \). Also, \( x_{k+1} \neq x_1 \) and so \( h(v_{k-1}v_k) = x_{k+1} - x_1 \neq 0 \). Since \( G \) is not nowhere-zero modular edge-graceful,
$h(v_kv_{k+1}) = y_k - (x_{k+1} - x_1) = 0$ and so $y_k = x_{k+1} - x_1$. Hence (4.5) holds. Furthermore, the modular edge-graceful labeling $h : E(G) \rightarrow \mathbb{Z}_n$ of $G$ satisfies the following conditions:

1. $h(e) \neq 0$ for all $e \in E(G) - \{v_kv_{k+1}\}$ and $h(v_kv_{k+1}) = 0$,

2. $h(e) = g(e) = f(e)$ for all $e \notin E(P_{k+1})$, and

3. $h'(v_{k+1}) = x_1$ and $h'(v_k) = x_2$.

Hence the result holds for $k$ is even (or $k + 1$ is odd).

**Case 2. $k$ is odd or $k + 1$ is even.** By the induction hypothesis of $f$ on the path $P_k$, we have

$$y_i = \begin{cases} x_{i+1} - x_1 & \text{if } 2 \leq i \leq k - 1 \text{ and } i \text{ is even} \\ x_{i+1} - x_2 & \text{if } 3 \leq i \leq k - 2 \text{ and } i \text{ is odd.} \end{cases} \tag{4.7}$$

Furthermore, there is a modular edge-graceful labeling $g : E(G) \rightarrow \mathbb{Z}_n$ of $G$ such that

1. $g(e) \neq 0$ for all $e \in E(G) - \{v_{k-1}v_k\}$ and $g(v_{k-1}v_k) = 0$,

2. $g(e) = f(e)$ for all $e \in E(G) - E(P_k)$, and

3. $g'(v_k) = x_1$ and $g'(v_{k-1}) = x_2$.

Hence $g(v_kv_{k+1}) = y_k = f(v_kv_{k+1})$. This is shown in Figure 4.24.
Define a labeling $h : E(G) \to \mathbb{Z}_n$ of $G$ from the modular edge-graceful labeling $g$ of $G$ by

$$h(e) = \begin{cases} 
g(e) & \text{if } e \neq v_{k-1}v_k, v_kv_{k+1} \\
(t_{k+1} - t_2) & \text{if } e = v_{k-1}v_k \\
y_k - (t_{k+1} - t_2) & \text{if } e = v_kv_{k+1} \end{cases}$$

Observe that (i) $h'(v) = g'(v)$ if $v \in V(G) - \{v_{k-1}, v_k, v_{k+1}\}$ and (ii) by (4.7),

$$h'(v_{k-1}) = g'(v_{k-1}) + (t_{k+1} - t_2) = t_2 + (t_{k+1} - t_2) = t_{k+1}$$

$$h'(v_k) = g'(v_k) + (t_{k+1} - t_2) - (t_{k+1} - t_2) = g'(v_k) = t_1$$

$$h'(v_{k+1}) = g'(v_{k+1}) - (t_{k+1} - t_2) = t_{k+1} - (t_{k+1} - t_2) = t_2$$

The labelings $h$ and $h'$ for $P_{k+1}$ are shown in Figure 4.25

Thus $\{h'(v_i) : k - 1 \leq i \leq k + 1\} = \{g'(v_i) : k - 1 \leq i \leq k + 1\}$ and so $h$ is a modular edge-graceful labeling of $G$. Since $h(e) = g(e)$ for all $e \in E(G) - \{v_{k-1}v_k, v_kv_{k+1}\}$, it follows that $h(e) \neq 0$ if $e \neq v_{k-1}v_k, v_kv_{k+1}$. Also, $t_{k+1} \neq t_2$ and so $h(v_{k-1}v_k) = t_{k+1} - t_2 \neq 0$. Since $G$ is not nowhere-zero modular edge-graceful, $h(v_kv_{k+1}) = y_k - (t_{k+1} - t_2) = 0$ and so $y_k = t_{k+1} - t_2$. Hence (4.5) holds
Furthermore, the modular edge-graceful labeling \( h : E(G) \rightarrow \mathbb{Z}_n \) of \( G \) satisfies the following conditions:

1. \( h(e) \neq 0 \) for all \( e \in E(G) - \{v_kv_{k+1}\} \) and \( h(v_kv_{k+1}) = 0 \),

2. \( h(e) = g(e) = f(e) \) for all \( e \notin E(P_{k+1}) \), and

3. \( h'(v_{k+1}) = x_2 \) and \( h'(v_k) = x_1 \).

Hence the result holds for \( k \) is odd (or \( k + 1 \) is even). 

**Theorem 4.6.3** If \( G \) is a connected modular edge-graceful graph of order \( n \geq 4 \) that is not a star, then \( G \) is nowhere-zero modular edge-graceful.

**Proof.** Assume, to the contrary, that there is a connected modular edge-graceful graph \( G \) of order \( n \geq 4 \) that is not a star such that \( G \) is not nowhere-zero modular edge-graceful. By Theorem 4.5.3, \( G \) is not a tree. By Theorem 4.6.1, \( G \) does not contain an even cycle. By Theorem 4.3.2, \( G \) is not an odd cycle. Since \( G \) is connected, it follows that \( G \) contains a unicyclic subgraph \( H \) that is obtained from an odd cycle by adding a pendant edge. We may assume that \( V(H) = \{v_1, v_2, v_3, \ldots, v_{2k+2}\} \), where \( C_{2k+1} = (v_2, v_3, \ldots, v_{2k+2}, v_2) \) is the odd cycle of \( H \) and \( v_1v_2 \) is the pendant edge of \( H \) (see Figure 4.26).

Since \( G \) is not nowhere-zero modular edge-graceful, by Lemma 4.6.2, there is a modular edge-graceful labeling \( f : E(G) \rightarrow \mathbb{Z}_n \) of \( G \) such that \( f(v_1v_2) = 0 \) and \( f(e) \neq 0 \) for all \( e \in E(G) - \{v_1v_2\} \). Furthermore, if \( P_{2k+2} = (v_1, v_2, v_3, \ldots, v_{2k+2}) \)
such that $f'(v_i) = x_i$ for $1 \leq i \leq 2k + 2$ and $f(v_iv_{i+1}) = y_i$ for $1 \leq i \leq 2k + 1$ where $y_1 = 0$, then

$$y_i = \begin{cases} 
  x_{i+1} - x_i & \text{if } 2 \leq i \leq 2k \text{ and } i \text{ is even} \\
  x_{i+1} - x_2 & \text{if } 3 \leq i \leq 2k + 1 \text{ and } i \text{ is odd.}
\end{cases}$$

Figure 4.27 shows the labelings $f$ and $f'$ on $H$.

Now consider the two paths $Q_1$ and $Q_2$ with initial edge $v_1v_2$ in $H$, namely,

$$Q_1 = (v_1, v_2, v_3, v_4, \ldots, v_{k+2}, v_{k+3})$$
$$Q_2 = (v_1, v_2, v_{2k+2}, v_{2k+1}, v_{2k}, \ldots, v_{k+3}, v_{k+2}).$$

Thus, $|V(Q_1)| = |V(Q_2)| = k + 3$ and $E(Q_1) \cap E(Q_2) = \{v_1v_2, v_{k+2}v_{k+3}\}$. We consider two cases, according to whether $k + 2$ is odd or $k + 2$ is even.

**Case 1.** $k + 2$ is odd. By traversing along the path $Q_1$ and applying (4.8), we obtain $y_{k+2} = x_{k+3} - x_1$. On the other hand, by traversing along the path $Q_2$ and
applying Lemma 4 6 2, we obtain $y_{k+2} = r_{k+2} - r_1$. This implies that $r_{k+3} = r_{k+2}$, which is a contradiction.

Case 2  \( k + 2 \) is even  By traversing along the path $Q_1$ and applying (4 8), we obtain $y_{k+2} = r_{k+3} - r_2$. On the other hand, by traversing along the path $Q_2$ and applying Lemma 4 6 2, we obtain $y_{k+2} = r_{k+2} - r_2$. This implies that $r_{k+3} = r_{k+2}$, which is a contradiction.

Combining Theorems 4 3 2, 4 5 3, 4 6 1 and 4 6 3, we establish the following characterization of connected nowhere-zero modular edge-graceful graphs.

**Theorem 4.6.4**  Let $G$ be a connected graph of order $n \geq 3$. Then $G$ is nowhere zero modular edge-graceful if and only if \( (i) \) $n \not\equiv 2 \pmod{4}$, \( (ii) \) $G \not\equiv K_3$ and \( (iii) \) $G$ is not a star of even order.

### 4.7 Nowhere-Zero Modular Edge-Gracefulness

For every connected graph $G$ of order $n$, there is a smallest integer $k \geq n$ for which there exists an edge labeling $f\colon E(G) \to \mathbb{Z}_k - \{0\}$ such that the induced vertex labeling $f'\colon V(G) \to \mathbb{Z}_k$ defined by

$$f'(v) = \sum_{u \in N(v)} f(uv),$$

where the sum is computed in $\mathbb{Z}_n$, is one-to-one. This number $k$ is referred to as the **nowhere zero modular edge-gracefulness** (or simply **nowhere-zero gracefulness**) of $G$. 
and is denoted by \( nzg(G) \) of \( G \). Thus \( nzg(G) = n \) if and only if \( G \) is nowhere-zero modular edge-graceful.

We first consider connected graphs of order \( n \geq 3 \) where \( n \neq 2 \) (mod 4). By Theorem 4.6.4, it suffices to determine the nowhere-zero gracefulness of a connected graph that is \( K_3 \) or a star of even order.

**Proposition 4.7.1** \( nzg(K_3) = 4 \).

**Proof.** By Theorem 4.6.4, \( nzg(K_3) \geq 4 \). On the other hand, let \( V(K_3) = \{v_1, v_2, v_3\} \) and define an edge labeling \( f : E(K_3) \rightarrow \mathbb{Z}_4 - \{0\} \) by \( f(v_1v_2) = 1, f(v_2v_3) = 2 \) and \( f(v_3v_4) = 3 \). Then \( f'(v_1) = 0, f'(v_2) = 3 \) and \( f'(v_3) = 1 \). Thus \( nzg(K_3) \leq 4 \) and so \( nzg(K_3) = 4 \). □

**Theorem 4.7.2** For each odd integer \( s \geq 3 \), \( nzg(K_{1,s}) = s + 3 \).

**Proof.** By Theorem 4.6.4, \( nzg(K_{1,s}) \geq s + 2 \) for all odd integers \( s \geq 3 \). Assume, to the contrary, that there is an edge labeling \( f : E(K_{1,s}) \rightarrow \mathbb{Z}_{s+2} - \{0\} \) such that the induced vertex labeling \( f' : V(K_{1,s}) \rightarrow \mathbb{Z}_{s+2} \) is an injective function. Since every edge of \( K_{1,s} \) is incident with an end-vertex, it follows that \( f \) is injective as well. Because \( |\mathbb{Z}_{s+2} - \{0\}| = s + 1 = |E(K_{1,s})| + 1 \), there is a unique \( a \in \mathbb{Z}_{s+2} - \{0\} \) such that \( f(e) \neq a \) for all \( e \in E(K_{1,s}) \). Since \( s + 2 \) is odd, \( a \neq -a \) for all \( a \in \mathbb{Z}_{s+2} \). This implies that there is \( e = uv \in E(K_{1,s}) \) such that \( f(e) = -a \). Suppose that \( u \) is an end-vertex of \( K_{1,s} \) and \( v \) is the central
vertex of $K_{1,s}$. Then $f'(u) = -a$ and

$$f'(v) = \left( \sum_{i=1}^{s+1} i - a \right) = \frac{(s + 1)(s + 2)}{2} - a = -a \quad \text{in } \mathbb{Z}_{s+2}.$$ 

Thus $f'(u) = f'(v)$, which is a contradiction. Therefore, $\text{nzg}(K_{1,s}) \neq s + 2$ and so $\text{nzg}(K_{1,s}) \geq s + 3$.

To show that $\text{nzg}(K_{1,s}) \leq s + 3$, we define an edge labeling $g : E(K_{1,s}) \to \mathbb{Z}_{s+3} - \{0\}$ such that the induced vertex labeling $g' : V(K_{1,s}) \to \mathbb{Z}_{s+3}$ is an injective function. Let $V(K_{1,s}) = \{v, v_1, v_2, \ldots, v_s\}$ where $v$ is the central vertex of $K_{1,s}$.

Figure 4.28 shows such a labeling $g$ together with its induced vertex labeling $g'$ for each of $s = 3$ and $s = 5$.

![Figure 4.28: The labeling $g$ for $K_{1,3}$ and $K_{1,5}$](image)

For $s \geq 7$, let $g : E(K_{1,s}) \to \mathbb{Z}_{s+3} - \{0\}$ be defined by by

$$g(vv_i) = \begin{cases} 
  i + 1 & \text{if } 1 \leq i \leq \frac{s-1}{2} - 2 \\
  i + 2 & \text{if } \frac{s-1}{2} - 1 \leq i \leq s.
\end{cases}$$

Figure 4.29 shows such a labeling $g$ together with its induced vertex labeling $g'$ for $s = 7$.

Hence

$$\{g'(v_i) : 1 \leq i \leq s\} = \left\{ 2, 3, \ldots, \frac{s-1}{2} - 1, \frac{s-1}{2} + 1, \ldots, s + 2 \right\}.$$
Furthermore, in $\mathbb{Z}_{s+3}$

$$g'(v) = \sum_{i=1}^{s} f(vu_i) = \left(\sum_{i=1}^{s+2} r\right) - 1 - \frac{s-1}{2}$$

$$= \frac{(s+2)(s+3)}{2} - 1 - \frac{s-1}{2} = -s + \frac{3}{2} - \left(\frac{s-1}{2} + 1\right)$$

$$= -\frac{s+3}{2} - \frac{s+1}{2} = -(s+2) = 1$$

Thus $g'(v) \notin \{g'(v_i) : 1 \leq i \leq s\}$ and so $g'$ is injective. Therefore, $\text{nzg}(K_{1,s}) = s + 3$

The following is an immediate consequence of Proposition 4.7.1 and Theorem 4.7.2

**Corollary 4.7.3** If $G$ is a connected graph of order $n \geq 3$ where $n \not\equiv 2 \pmod{4}$, then

$$\text{nzg}(G) \in \{n, n+1, n+2\}$$

Furthermore, $\text{nzg}(G) = n$ if and only if $G$ is nowhere-zero modular edge-graceful, $\text{nzg}(G) = n + 1$ if and only if $G = K_3$ and $\text{nzg}(G) = n + 2$ if and only if $G$ is a star of even order.
Finally, we consider connected graphs of order $n \geq 6$ where $n \equiv 2 \pmod{4}$. In this case, $G$ is not modular edge-graceful and so $G$ is not nowhere-zero modular edge-graceful. Therefore, $\text{nzg}(G) \geq n + 1$. Applying an argument similar to the one used in the case when $n \not\equiv 2 \pmod{4}$, we can verify the following.

**Theorem 4.7.4** If $G$ is a connected graph of order $n \geq 6$ where $n \equiv 2 \pmod{4}$, then

$$\text{nzg}(G) \in \{n + 1, n + 2\}.$$ 

Furthermore, $\text{nzg}(G) = n + 2$ if and only if $G$ is a star.
Chapter 5

Topics for Further Study

1. Proper Modular Vertex Colorings

For a graph $G$ without isolated vertices, let $c : V(G) \rightarrow \mathbb{Z}_k$ ($k \geq 2$) be a proper vertex coloring of $G$. The color sum $\sigma(v)$ of a vertex $v$ of $G$ is defined as the sum in $\mathbb{Z}_k$ of the colors of the vertices in $N(v)$, that is,

$$\sigma(v) = \sum_{u \in N(v)} c(u).$$

The coloring $c$ is called a proper modular $k$-coloring of $G$ if $\sigma(x) \neq \sigma(y)$ in $\mathbb{Z}_k$ for all pairs $x, y$ of adjacent vertices of $G$. A coloring $c$ is a proper modular coloring if $c$ is a modular $k$-coloring for some integer $k \geq 2$. The proper modular chromatic number or simply the pmc-number of $G$ is the minimum $k$ for which $G$ has a proper modular $k$-coloring.

2. Proper Modular Edge Colorings

For a graph $G$ without isolated vertices, let $c : E(G) \rightarrow \mathbb{Z}_k$ ($k \geq 2$) be a proper edge coloring of $G$. The color sum $s_c(v)$ or $s(v)$ of a vertex $v$ of $G$ is
defined as the sum in \( \mathbb{Z}_k \) of the colors of the edges incident with \( v \), that is, if \( E_v \) is the set of edges incident with \( v \) in \( G \), then

\[
s(v) = \sum_{e \in E_v} c(e),
\]

that is, the sum (in \( \mathbb{Z}_k \)) is taken over all edges incident with \( v \). A proper edge coloring \( c \) is a *proper modular \( k \)-edge coloring* of \( G \) if \( s(u) \neq s(v) \) in \( \mathbb{Z}_k \) for all pairs \( u, v \) of adjacent vertices of \( G \). A proper edge coloring \( c \) is a *proper modular edge coloring* if \( c \) is a proper modular \( k \)-edge coloring for some integer \( k \geq 2 \). The *proper modular chromatic index* or simply the *pmc index* of \( G \) is the minimum \( k \) for which \( G \) has a proper modular \( k \)-edge coloring.

3 Proper Graceful Labelings

Let \( G \) be a graph of order \( n \) and size \( m \). If \( f : V(G) \to \{0, 1, 2, \ldots, m\} \) is a function, where \( f \) need not be one-to-one, such that

\[
\{|f(u) - f(v)| : uv \in E(G)\} = \{1, 2, \ldots, m\},
\]

then \( f \) is a *proper graceful labeling* of \( G \). A graph possessing a proper graceful labeling is a *proper graceful graph*. If \( f \) is a proper graceful labeling of \( G \), then \( f(u) \neq f(v) \) for every two adjacent vertices of \( G \), for otherwise, \( |f(u) - f(v)| = 0 \). Therefore, \( f \) is a proper coloring of \( G \). Certainly, every graceful graph is a proper graceful graph.

(a) Is there a proper graceful graph that is not graceful?
(b) If $G$ is a proper graceful graph, then what is the minimum number of colors needed in a proper graceful labeling $f$ of $G$?