Hamiltonicity and Connectivity in Distance-Colored Graphs

Kyle C. Kolasinski
Western Michigan University

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Hamiltonicity and Connectivity
in Distance-Colored Graphs

by
Kyle C. Kolasinski

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Hamiltonicity and Connectivity in Distance-Colored Graphs

Kyle Kolasinski, Ph.D.

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For a connected graph $G$ and a positive integer $k$, the $k$th power $G^k$ of $G$ is the graph with $V(G^k) = V(G)$ where $uv \in E(G^k)$ if the distance $d_G(u, v)$ between $u$ and $v$ is at most $k$. The edge coloring of $G^k$ defined by assigning each edge $uv$ of $G^k$ the color $d_G(u, v)$ produces an edge-colored graph $G^k$ called a distance-colored graph.

A distance-colored graph $G^k$ is Hamiltonian-colored if $G^k$ contains a properly colored Hamiltonian cycle. The minimum $k$ for which $G^k$ is Hamiltonian-colored is the Hamiltonian coloring exponent $hce(G)$. It is shown that for each pair $k, d$ of integers with $4 < k < d$, there exists a tree $T$ with $hce(T) = k$ and diam$(T) = d$. Hamiltonian coloring exponents are determined for several well-known classes of graphs. It is also shown that for each integer $k \geq 2$, there exists a tree $T_k$ with $hce(T_k) = k$ such that every properly colored Hamiltonian cycle in the $k$th power of $T_k$ must use all colors $1, 2, \ldots, k$. For a grid $G = P_n \square P_m$ with $n, m \geq 2$, it is shown that $G^2$ is Hamiltonian-colored if and only if $nm \equiv 0 \pmod{4}$ and a complete solution is presented for restricted Hamiltonian-colored cycles in $G^3$.

A distance-colored graph is properly $p$-connected if two distinct vertices $u$ and $v$ in the graph are connected by $p$ internally disjoint properly colored $u-v$ paths. For a connected graph $G$ and an integer $k \geq 2$, the color-connectivity of $G^k$ is
the maximum positive integer $p$ for which $G^k$ is properly $p$-connected. The color-connectivities are determined for some well-known classes of graphs. It is shown that $G^2$ is properly 2-connected for every 2-connected graph that is not complete, a double star is the only tree $T$ for which $T^2$ is properly 2-connected and $G^3$ is properly 2-connected for every connected graph $G$ of diameter at least 3. All pairs $k, n$ of positive integers for which $P_n^k$ is properly $k$-connected are determined and other related results are presented.

Various color-related distance parameters and rainbow concepts in distance-colored graphs are introduced and studied. Results and open problems are also presented in this area of research.
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Chapter 1

Introduction

1.1 History and Motivation

A fundamental problem in graph theory deals with whether a given graph $G$ contains a Hamiltonian cycle or a Hamiltonian path or possesses various other Hamiltonian properties such as $G$ being Hamiltonian-connected, where every two vertices of $G$ are connected by a Hamiltonian path.

Over the years, a number of operations defined on a graph $G$ have been studied. Among the best known of these are:

1. The subdivision graph $S(G)$, where each edge of $G$ is subdivided exactly once,

2. The $k$th power $G^k$ where $V(G^k) = V(G)$ and $uv \in E(G^k)$ if the distance $d_G(u,v)$ between $u$ and $v$ (the length of a shortest $u-v$ path in $G$) is at most $k$. The graph $G^2$ is called the square of $G$ and $G^3$ is the cube of $G$,

3. The line graph $L(G)$ of $G$ whose vertices can be put in one-to-one correspondence with the edges of $G$ where two vertices of $L(G)$ are adjacent if the
corresponding edges of $G$ are adjacent;

(4) The total graph $T(G)$ of $G$ whose vertices can be put in one-to-one correspondence with the vertices and the edges of $G$ where two vertices of $T(G)$ are adjacent if the corresponding elements of $G$ are adjacent or incident.

When each of the operations (2)–(4) is applied to a Hamiltonian graph $G$, a Hamiltonian graph results; while $S(G)$ is Hamiltonian if and only if $G = C_n$ for some $n \geq 3$.

In [6] Chartrand and Kronk showed that every path in a Hamiltonian graph $G$ of order $n \geq 3$ with a given but arbitrary vertex $v$ can be extended to a Hamiltonian cycle with initial vertex $v$ of $G$ if and only if $G = C_n$, $G = K_n$ or $G = K_{n/2,n/2}$ where $n$ is even.

The terms Hamiltonian path, Hamiltonian cycle and Hamiltonian graph are of course named for the famous mathematician Sir William Rowan Hamilton who considered spanning cycles on the dodecahedron in connection with the Icosian Game he developed in 1856. Months earlier, however, the Reverend Thomas Kirkman had considered spanning cycles on a variety of polyhedra. Hamilton not only observed that the dodecahedron contains a spanning cycle but in the instructions accompanying his Icosian Game, Hamilton wrote that every path of order 5 could be extended to a spanning cycle on the dodecahedron, that is, for every path $P = (v_1, v_2, v_3, v_4, v_5)$ in the graph of the dodecahedron, there is a spanning cycle $C = (v_1, v_2, v_3, v_4, v_5, u_1, u_2, \ldots, u_{15}, u_{16} = v_1)$ in this graph.

Hamilton’s observation led Chartrand to define a related concept. Let $G$ be
a Hamiltonian graph of order $n$ and let $k$ be an integer with $3 \leq k \leq n$. Then $G$ is $k$-ordered Hamiltonian if for every sequence $s : v_1, v_2, \ldots, v_k$ of $k$ vertices of $G$, there is a Hamiltonian cycle of $G$ encountering these $k$ vertices of $s$ in the order listed. While every Hamiltonian graph is 3-ordered Hamiltonian, not every Hamiltonian graph is 4-ordered Hamiltonian. Chartrand defined a Hamiltonian-connected graph $G$ to be $k$-ordered Hamiltonian-connected ($k \geq 3$) if for every sequence $s : v_1, v_2, \ldots, v_k$ of $k$ vertices of $G$ there exists a Hamiltonian $v_1 - v_k$ path encountering the vertices of $s$ in the given order.

Milan Sekanina (1931-1987) was a mathematician from Czechoslovakia whose research interests comprised many fields of mathematics including algebra, topology and graph theory. Much of Sekanina’s research concerned ordered sets. In 1960 Sekanina [27] proved that for every two distinct vertices $u$ and $v$ in a connected graph $G$ of order $n$, there is an ordering $u = v_1, v_2, \ldots, v_n = v$ of the vertices of $G$ such that $d_G(v_i, v_{i+1}) \leq 3$ for $i = 1, 2, \ldots, n - 1$. Using current terminology, he showed that the cube of every connected graph of order at least 3 is Hamiltonian-connected and therefore Hamiltonian as well. The square $G^2$ is not necessarily Hamiltonian however. For example, for the graph $G$ of Figure 1.1, its square $G^2$ is not Hamiltonian. In 1968 Karaganis [21] independently proved that the cube of every connected graph is Hamiltonian-connected. The square, cube and powers of a graph were evidently terms coined by Ian Ross and Frank Harary [26] in 1960.

Three years later, while attending the 1963 Graph Theory Symposium in Smolenice (now in Slovakia), Sekanina [28] suggested the research problems of determining the class of graphs $G$ such that $G^2$ is Hamiltonian and the class of
graphs $G$ such that $G^2$ is Hamiltonian-connected. The smallest graph $G$ of order at least 3 such that $G^2$ is not Hamiltonian is $S(K_{1,3})$, that is, $(S(K_{1,3}))^2 = T(K_{1,3})$ is not Hamiltonian (see Figure 1.1). In 1964, Sekanina’s student František Neuman [23] determined those trees $T$ and pairs $u, v$ of vertices of $T$ for which the graph $T^2$ has a Hamiltonian $u - v$ path. The following theorem is a consequence of Neuman’s work but was later proved more directly and more simply by Frank Harary and Allen Schwenk [18].

**Theorem 1.1.1** The square $T^2$ of a tree $T$ of order at least 3 is Hamiltonian if and only if $T$ is a caterpillar.

The graph $G$ of Figure 1.1 contains cut-vertices, that is, vertices $v$ for which $G - v$ is disconnected. A connected graph without cut-vertices is a 2-connected graph. At the 1966 Graph Theory Conference in Tihany, Hungary, Crispin Nash-Williams [22] asked whether the square of every 2-connected graph is Hamiltonian. Michael D. Plummer had independently made the same conjecture. Herbert Fleischner [11] and Hansjoachim Walther [29] independently showed, using a theorem of Petersen (guaranteeing the existence of a 1-factor), that if $G$ is a cubic bridgeless graph,
then $G^2$ is Hamiltonian

During the period September 1970 – January 1971, Herbert Fleischner worked on this conjecture and was finally successful in proving it [12]

**Theorem 1.1.2** (Fleischner’s Theorem) *The square of every 2-connected graph is Hamiltonian*

Fleischner’s proof was contained in four papers, all of which were submitted to the *Journal of Combinatorial Theory*. According to Fleischner [14], the referee “gave hell to the author” and the four papers never appeared in their original form but in two revised papers that were much clearer and much shorter than the original (thanks to the referee and, later, to Nash-Williams). These two papers were finally published in 1974 [13].

Prior to the publication of Fleischner’s two papers, his theorem had become widely known. The technique and the theorem itself were useful in proving that the square of a 2-connected graph had even stronger properties. In 1973 Hobbs [20] proved that the square of every 2-connected graph is in fact vertex-pancylic. In 1974 Chartrand, Hobbs, Jung, Kapoor and Nash-Williams [5] showed that the square of every 2-connected graph is Hamiltonian-connected. In 1976 Faudree and Schelp [10] proved that the square of every 2-connected graph is panconnected.

More recently, shorter proofs of Fleischner’s theorem have been given— in 1991 by Řuha [25] and in 2009 by Georgakopoulos [16]. In fact, Georgakopoulos showed that for every 2-connected graph $G$ and every vertex $v$ of $G$, the square $G^2$ has a Hamiltonian cycle whose two edges incident with $v$ belong to $G$. 
In 1964 Chartrand [3] showed (see also [4]) that if $G$ is a connected graph that is not a path, then for sufficiently large integers $r$, the iterated line graph $L^r(G)$ is Hamiltonian, where $L^2(G) = L(L(G))$ and for $r \geq 3$, $L^r(G) = L(L^{r-1}(G))$.

In 1973 Chartrand and Wall [8] defined the Hamiltonian index $h(G)$ of $G$ as the minimum $r$ for which $L^r(G)$ is Hamiltonian. They showed that if $\delta(G) \geq 3$, then $h(G) \leq 2$. In 1965 Harary and Nash-Williams [17] characterized those graphs having a Hamiltonian line graph. A dominating circuit in a connected graph $G$ is a circuit $C$ in $G$ in which every edge of $G$ is incident with a vertex of $C$.

**Theorem 1.1.3** (Harary and Nash-Williams) *Let $G$ be a connected graph. Then $L(G)$ is Hamiltonian if and only if $G = K_n^n$ for some $n \geq 3$ or $G$ contains a dominating circuit.*

The first paper on $k$-ordered Hamiltonian graphs was authored by Ng and Schultz [24] in 1997. They gave sufficient conditions for a graph to be $k$-ordered Hamiltonian. Several other papers dealing with this concept have been written since then. The following results also appear in the paper of Ng and Schultz.

1. Every $(k + 1)$-ordered Hamiltonian-connected graph, $k \geq 3$, is $k$-ordered Hamiltonian.

2. For each integer $k \geq 4$, there is a $k$-ordered Hamiltonian-connected graph that is not $k$-ordered Hamiltonian.

In 2008, Chebikin [9] showed that for each integer $k \geq 4$ and every connected graph $G$ of order at least $k$, the graph $G^{[3k/2]}$ is $k$-ordered Hamiltonian. Chebikin
also showed that for \( n \geq 5 \), the graph \( C_n^4 \) is 5-ordered Hamiltonian. He also showed that for \( r \geq 3 \) and \( n \) sufficiently large, there exists a sequence \( s \) of \( 2r \) vertices in \( C_n \) such that no cycle in \( C_n^r \) contains the vertices of \( s \) encountered in the given order.

In 2009 Hartke and Ponto [19] showed for a graph \( G \) that if \( L^r(G) \) is \( k \)-ordered Hamiltonian for a sufficiently large integer \( r \), then \( L^{r+1}(G) \) is \( (k+1) \)-ordered Hamiltonian. They also showed that if \( G \) is a connected graph that is not a path, cycle or \( K_{1,3} \), then there exists an integer \( N \) such that \( L^{N+(k-3)}(G) \) is \( k \)-ordered Hamiltonian for \( k \geq 3 \). Hartke and Ponto also showed that for \( k = 4 \) and \( k = 5 \), if \( G \) is a \( k \)-ordered Hamiltonian graph, then so is \( L(G) \). They asked if this is true for \( k \geq 6 \).

In the past few decades, graph labelings have been growing in popularity because of their applications to many other areas of research in graph theory. The origin of the study of graph labelings as a major area of graph theory can be traced to a 1967 research paper by Alexander Rosa [30]. Among the labelings he introduced was a vertex labeling he referred to as a \( \beta \)-valuation. Let \( G \) be a graph of order \( n \) and size \( m \). A one-to-one function \( f : V(G) \to \{0,1,2,\ldots,m\} \) is called a \( \beta \)-valuation (or a \( \beta \)-labeling) of \( G \) if \( \{|f(u) - f(v)| \mid uv \in E(G)\} = \{1,2,\ldots,m\} \). In order for a graph to possess a \( \beta \)-labeling, it is necessary that \( m \geq n - 1 \). Five years later (in 1972), Golomb [?] referred to a \( \beta \)-labeling as a graceful labeling and a graph possessing a graceful labeling as a graceful graph. Eventually, it was this terminology that became standard. Gallian [15] has written a survey on labelings of graphs that includes an extensive discussion of graceful labelings and their applications. One of the major conjectures in graph theory concerns graceful
graphs and is due to Kotzig (see Rosa [30]) which is known as the Graceful Tree Conjecture.

The Graceful Tree Conjecture Every tree is graceful.

1.2 Definitions and Notation

In this section, we introduce some basic definitions and notation in graph theory. We refer to the book [7] for graph theory notation and terminology not described in this work. We assume that all graphs under consideration are connected.

- In a connected graph $G$, the distance $d_G(u, v)$ (or simply $d(u, v)$ if the graph $G$ under consideration is clear) between two vertices $u$ and $v$ in $G$ is the length of a shortest $u-v$ path in $G$. A $u-v$ geodesic is a $u-v$ path of length $d(u, v)$. The eccentricity $e(v)$ of a vertex $v$ in a connected graph $G$ is

$$e(v) = \max \{d(v, x) : x \in V(G)\}.$$

The radius of a connected graph $G$ is

$$\text{rad}(G) = \min \{e(v) : v \in V(G)\},$$

while the diameter of $G$ is

$$\text{diam}(G) = \max \{e(v) : v \in V(G)\}.$$

Thus, the diameter of $G$ is the greatest distance between any two vertices of $G$. The center $C(G)$ of a connected graph $G$ is the subgraph of $G$ induced
by those vertices of \( G \) having eccentricity \( \text{rad}(G) \), while the **periphery** \( P(G) \) of \( G \) is the subgraph of \( G \) induced by the vertices of \( G \) having eccentricity \( \text{diam}(G) \).

- A **proper vertex coloring** (or simply a **coloring**) of the vertices of a graph \( G \) is an assignment of colors (the elements of some set) to the vertices of \( G \), one color to each vertex, such that adjacent vertices are assigned distinct colors. The minimum number of colors in a proper vertex coloring of \( G \) is the *chromatic number* \( \chi(G) \) of \( G \).

- A **coloring** of the edges of a graph \( G \) is an assignment of colors (the elements of some set) to the edges of \( G \), one color to each edge. A graph with a coloring of its edges is called an **edge-colored graph**. If adjacent edges are assigned distinct colors, then the coloring is a **proper edge coloring**. The minimum number of colors in a proper edge coloring of \( G \) is the *chromatic index* \( \chi'(G) \) of \( G \).

- Recall that a cycle containing every vertex of a graph \( G \) (necessarily exactly once) is called a **Hamiltonian cycle** of \( G \). A graph \( G \) is **Hamiltonian** if \( G \) contains a Hamiltonian cycle. A path containing every vertex of a graph \( G \) is called a **Hamiltonian path** of \( G \). A graph \( G \) is **traceable** if \( G \) contains a Hamiltonian path. A graph \( G \) is **Hamiltonian-connected** if for every pair \( u, v \) of vertices of \( G \), there is a Hamiltonian \( u-v \) path in \( G \). Necessarily, every Hamiltonian-connected graph of order 3 or more is Hamiltonian (but the converse is not true).
• A circuit $C$ of $G$ (a cyclic walk which never repeats an edge) that contains every edge of $G$ (necessarily exactly once) is an **Eulerian circuit**. A connected graph $G$ is called **Eulerian** if $G$ contains an Eulerian circuit.

• A graph $G$ of order $n \geq 3$ is called **pancyclic** if $G$ contains a cycle of every possible length, that is, $G$ contains a cycle of length $\ell$ for each $\ell$ with $3 \leq \ell \leq n$. Thus if $G$ is pancyclic, then $G$ is Hamiltonian. A connected graph $G$ of order $n$ is said to be **panconnected** if for each pair $u, v$ of distinct vertices of $G$, there exists a $u - v$ path of length $\ell$ for each integer $\ell$ satisfying $d(u, v) \leq \ell \leq n - 1$. If a graph is panconnected, then it is Hamiltonian-connected.

• The **connectivity** $\kappa(G)$ of a graph $G$ is the minimum number of vertices whose removal from $G$ results in a disconnected or trivial graph. If $\kappa(G) \geq t$ for some positive integer $t$, then $G$ is $t$-connected. By a well-known theorem of Whitney [31], a graph $G$ is $t$-connected if and only if $G$ contains $t$ internally disjoint $u - v$ paths for every two distinct vertices $u$ and $v$ of $G$. That is, $G$ contains $u - v$ paths $P_1, P_2, \ldots, P_t$ such that $V(P_i) \cap V(P_j) = \{u, v\}$ (and so $E(P_i) \cap E(P_j) = \emptyset$) for all distinct integers $i$ and $j$ with $1 \leq i, j \leq t$. In particular, if $\kappa(G) = \ell$, then $G$ contains $\ell$ internally disjoint $u - v$ paths for every pair $u, v$ of vertices of $G$, but $G$ does not contain $\ell + 1$ internally disjoint $x - y$ paths for some pair $x, y$ of vertices of $G$.

• A graph $G$ of size $m$ is **graceful** if it is possible to label the vertices of $G$ with distinct elements from the set $\{0, 1, \ldots, m\}$ in such a way that the induced
edge labeling, which prescribes the integer $|i - j|$ to the edge joining vertices labeled $i$ and $j$, assigns the distinct labels from the set $\{1, 2, \ldots, m\}$ to the edges of $G$. A labeling that accomplishes this is a \textit{graceful labeling} (This was called a $\beta$-valuation by Rosa [30]).

- A graph $G$ of size $m$ is \textit{bi-graceful} if the vertices of $G$ can be labeled with distinct elements from the set $\{0, 1, \ldots, 2m\}$ in such a way that the set $S$ of induced edge labels that prescribes the integer $|i - j|$ to the edge joining vertices labeled $i$ and $j$ has the property that for each set $S_i = \{i, 2m + 1 - i\}$ with $1 \leq i \leq m$, $|S \cap S_i| = 1$. A labeling with this property is called a \textit{bi-graceful labeling}. That is, a bi-graceful labeling of a graph $G$ is a one-to-one function $f : V(G) \to \{0, 1, 2, \ldots, 2m\}$ such that the set $S = \{|f(u) - f(v)| \mid uv \in E(G)\}$ has the property that $|S \cap S_i| = 1$ for all $i$ with $1 \leq i \leq m$. (This was called a $\rho$-valuation by Rosa [30]). A graph possessing a bi-graceful labeling is a \textit{bi-graceful graph}. If $G$ is a graceful graph and $f$ is a graceful labeling of $G$, then $S \cap S_i = \{i\}$ for all $i$ ($1 \leq i \leq m$) and so $f$ is a bi-graceful labeling of $G$. Therefore, every graceful graph is bi-graceful.
Chapter 2

Color-Hamiltonicity

2.1 Distance-Colored Graphs

We have mentioned that Sekanina proved that if $G$ is a connected graph of order at least 3, then $G^3$ is Hamiltonian. Consequently, if $G$ is any connected graph of order $n \geq 3$, then there is a cyclic ordering

$$s : v_1, v_2, \ldots, v_{n-1}, v_n, v_{n+1} = v_1$$

of the vertices of $G$ such that $1 \leq d_G(v_i, v_{i+1}) \leq 3$ for every integer $i$ with $1 \leq i \leq n$.

Thus for each consecutive pair $v_i, v_{i+1}$ of vertices in $s$, the distance $d_G(v_i, v_{i+1})$ is 1, 2 or 3. This of course says that for each integer $i$ with $1 \leq i \leq n$, $v_iv_{i+1}$ is an edge of $G^3$. This also suggests the idea of assigning to the edge $v_iv_{i+1}$ of $G^3$ the distance $d_G(v_i, v_{i+1})$, which is either 1, 2 or 3. We can interpret such an assignment of the integers 1, 2 and 3 to the edges of $G^3$ as a 3-edge coloring of $G^3$, which may not be a proper edge coloring of $G^3$.

Such an assignment of the integers 1, 2 and 3 to the edges of $G^3$ is not entirely new. For example, consider the triangle constructed on three of the seven vertices
labeled 0, 1, ..., 6 shown in Figure 2.1, where the seven vertices are those of a regular 7-gon and the edges of the triangle are assigned the integers 1, 2 and 3.

\[
\begin{array}{c}
\text{Figure 2.1} \quad \text{A triangle in } K_7
\end{array}
\]

Of course, with the vertices of the triangle (having order \( n = 3 \) and size \( m = 3 \)) in Figure 2.1 labeled with three of the four labels 0, 1, 2, 3 = \( n \) and each edge incident with vertices labeled \( i \) and \( j \) is itself labeled \( |i - j| \), producing the edge labels 1, 2, 3 = \( m \), we have obtained the well-known fact that \( K_3 \) is graceful.

If this triangle is rotated clockwise through an angle of \( 2\pi/7 \) radians a total of six times, where the color of each rotated edge is maintained, then we obtain the 3-edge coloring of \( K_7 \) in Figure 2.2. This illustrates another well-known fact for a graceful graph \( G \) of size \( m \), namely that there is a decomposition of \( K_{2m+1} \) into \( 2m + 1 \) copies of \( G \). In fact, there is a *rainbow* \( K_3 \)-decomposition of \( K_7 \) (that is, all three edges of \( K_3 \) are colored differently).

The 3-edge coloring of \( K_7 \) shown in Figure 2.2 can be obtained in another manner. Suppose that \( G = C_7 \). By Sekanina's theorem, \( G^3 = K_7 \) contains a Hamiltonian cycle. Of course, this is a rather trivial observation as we are beginning with a cycle. Also, \( K_n \) is Hamiltonian for every integer \( n \geq 3 \). (In fact, \( C_7 \) itself is a
Hamiltonian cycle of $G^3$.) For distinct vertices $u$ and $v$ in $G$, assign $uv$ the color $d_G(u, v)$. Thus every edge of $K_7$ is colored with one of the colors 1, 2 and 3. This edge coloring of $K_7$ suggests the problem of determining whether $K_7$ contains a properly colored Hamiltonian cycle. It does, as Figure 2.3 shows. This of course is equivalent to the statement that there is a cyclic sequence $s : v_1, v_2, \ldots, v_7, v_8 = v_1$ of the seven vertices of $G = C_7$ such that $d_G(v_i, v_{i+1}) \neq d_G(v_{i+1}, v_{i+2})$ for every integer $i$ with $1 \leq i \leq 6$. More generally, suppose that $G = C_n$, where $n \geq 3$, and we consider $G^3$. An
edge \( uv \) of \( G^3 \) is colored \( r \in \{1, 2, 3\} \) if \( d_G(u, v) = r \) Since \( \text{diam}(C_n) = \lceil n/2 \rceil \), it follows that \( G^3 = K_n \) if \( n \leq 7 \) The question here then is whether \( G^3 \) contains a properly colored Hamiltonian cycle If \( G = C_3 \), then every edge of \( K_3 \) is colored 1, while if \( G = C_5 \), then every edge of \( K_5 \) is colored 1 or 2 Since \( \chi'(C_3) = \chi'(C_5) = 3 \), there is no properly colored Hamiltonian cycle in \( C_n^3 \) for either \( n = 3 \) or \( n = 5 \) There is a properly colored Hamiltonian cycle in both \( C_4^3 \) and \( C_6^3 \), however, as is shown in Figure 2.4

![Figure 2.4 Properly colored Hamiltonian cycles in \( C_4^3 \) and \( C_6^3 \)](image)

Since the diameter of \( C_4 \) is 2, not only is \( C_4^3 = K_4 \) but \( C_4^2 = K_4 \) Consequently, every edge of \( K_4 \) is colored 1 or 2 and there is a properly colored Hamiltonian cycle in \( K_4 \), using the colors 1 and 2 Of course, it is only possible for \( C_n^2 \) to have a properly colored Hamiltonian cycle if \( n \) is even While such a Hamiltonian cycle exists in \( C_4^2 \), no such Hamiltonian cycle exists in \( C_6^2 \) In fact, we have the following result

**Theorem 2.1.1** Let \( G = C_n \) where \( n \geq 4 \) and \( n \) is even Assign the edge \( uv \) in \( G^2 \) the color \( r \in \{1, 2\} \) if \( d_G(u, v) = r \) where \( 1 \leq r \leq 2 \) Then \( G^2 \) has a properly colored Hamiltonian cycle if and only if \( n \equiv 0 \pmod 4 \)
Proof. Let \( C_n = (v_1, v_2, \ldots, v_n, v_1) \) and suppose that \( G^2 \) has a properly colored Hamiltonian cycle \( C \). We may assume that the initial (and terminal) vertex of \( C \) is \( v_1 \) and the initial edge incident with \( v_1 \) is colored 2 and that this edge is \( v_1v_3 \). The next two vertices following \( v_1v_3 \) thus have even subscripts. Hence the vertices of \( C \) can be described in terms of a sequence of 2-element subscripts, namely \( E_1, E_2, \ldots, E_k, E_1 \) where \( E_i \) consists of two odd integers if \( i \) is odd and \( E_i \) consists of two even integers if \( i \) is even. Hence \( k \) is even, say \( k = 2\ell \) and each set \( E_i \) \((1 \leq i \leq 2\ell)\) is a 2-element set; so \( n = 2(2\ell) \equiv 0 \pmod{4} \).

If \( n \equiv 0 \pmod{4} \), say \( n = 4\ell \), then the cyclic sequence

\[
1, 3, 2, 4, 5, 7, 6, 8, \ldots, n - 3, n - 1, n - 2, n, 1
\]

describes a properly colored Hamiltonian cycle in \( G^2 \). \( \blacksquare \)

Before proceeding further, we introduce a more general concept. Let \( G \) be a connected graph of order \( n \) with diameter \( d \) and let \( k \) be an integer with \( 1 \leq k \leq d \). An edge \( uv \) of \( G^k \) is colored \( i \) \((1 \leq i \leq k)\) if \( d_G(u, v) = i \). This edge coloring of \( G^k \) produces what is called a distance-colored graph \( G^k \). Suppose that \( G^k \) contains a subgraph isomorphic to a graph \( H \). We say that \( G^k \) is \( H \)-colored if \( G^k \) contains a properly colored copy of \( H \). Certainly, this is only possible if \( \chi'(H) \leq k \). If \( H = C_n \), then \( G^k \) is called a Hamiltonian-colored graph. By Theorem 2.1.1 the distance-colored graph \( C_n^2 \) is Hamiltonian-colored if and only if \( n \equiv 0 \pmod{4} \).

As an illustration, let \( P \) be the Petersen graph containing two disjoint 5-cycles

\[
(u_1, u_2, u_3, u_4, u_5, u_1) \text{ and } (v_1, v_3, v_5, v_2, v_4, v_1)
\]
such that \( u, v \in E(P) \) for \( 1 \leq r \leq 5 \) Then \( P^2 \) contains a properly colored Hamiltonian cycle as shown in Figure 2.5 and so \( P^2 \) is Hamiltonian-colored.

![Figure 2.5 A properly colored Hamiltonian cycle in \( P^2 \)](image)

If \( G \) is a connected graph of order at least 3 such that \( G \) contains a vertex \( u \) that is adjacent to every vertex of \( G \), then \( d(u, v) = 1 \) for all \( v \in V(G) - \{u\} \), which implies that \( u \) does not belong to any properly colored cycle in \( G^k \) for every integer \( k \geq 2 \) Thus we have the following observation

**Observation 2.1.2** If \( G \) is a connected graph of order \( n \geq 3 \) with maximum degree \( \Delta(G) = n - 1 \), then the distance-colored graph \( G^k \) is not Hamiltonian-colored for any integer \( k \geq 2 \) In particular, if \( G = K_n \) or \( G = K_{1,n-1} \), then \( G^k \) is not Hamiltonian-colored for any integer \( k \geq 2 \)

Figure 2.6 shows a connected graph \( G \) of order \( n \geq 3 \) with \( \Delta(G) = n - 2 \) such that \( G^3 \) contains the properly colored Hamiltonian cycle

\[
(v_1, v_2, v_3, v_4, v_5, v_6, v_1)
\]
and so $G^3$ is Hamiltonian colored. For this graph $G$, its square $G^2$ is not Hamiltonian colored, however, for such a properly colored Hamiltonian cycle $C$ would have to contain the path

$$(v_1, v_6, v_3, v_2),$$

say, implying that $v_2v_1$ is an edge of $C$, which is impossible.

![Figure 2.6 A connected graph $G$ of order 6 and $\Delta(G) = 4$ such that $G^3$ contains a properly colored Hamiltonian cycle.](image)

We now consider the case where $G$ is the path $P_n$ of order $n$. A properly colored Hamiltonian cycle is shown in each of distance colored graphs $P^3_n$, $P^3_5$, and $P^3_6$ in Figure 2.7, where only the edges of the cycle are shown. While the Hamiltonian cycle in $P^3_4$ is properly colored, this Hamiltonian cycle is also properly colored in $P^2_4$. There is no integer $n \geq 3$ with $n \neq 4$ such that $P^2_n$ has a properly colored Hamiltonian cycle. The situation for $P^3_n$ is different, however.

![Figure 2.7 Properly colored Hamiltonian cycles in $P^3_n$ for $4 \leq n \leq 6$.](image)

**Theorem 2.1.3** For each integer $n \geq 4$, the distance colored graph $P^3_n$ is Hamiltonian colored.
Proof. We have already seen that $P_n^3$ is Hamiltonian-colored for $n = 4, 5, 6$. Thus, we now assume that $n \geq 7$. Let $P_n = (v_1, v_2, \ldots, v_n)$. We show by induction on $n \geq 7$ that the distance-colored graph $P_n^3$ has a properly colored Hamiltonian cycle $C$ containing an edge $e$ incident with $v_n$ that is colored 1 such that the two edges of $C$ adjacent to $e$ are both colored 2. That the statement is true when $7 \leq n \leq 16$ is shown in Figure 2.8.

![Figure 2.8: Properly colored Hamiltonian cycles in $P_n^3$ for $7 \leq n \leq 16$](image)

Assume that the statement is true for every integer $n$ with $7 \leq n \leq k$, where $k \geq 16$, and consider the distance-colored graph $P_{k+1}^3$. By the induction hypothesis, there exists a properly colored Hamiltonian cycle $C'$ in the distance-colored graph
$P^3_{k-9}$ such that an edge $e$ of $C'$ incident with $v_{k-9}$ is colored 1 and the two edges of $C'$ adjacent to $e$ are colored 2. Thus $e = v_{k-10}v_{k-9}$.

Let $P$ be the properly colored Hamiltonian path in $P^3_{10}$ shown in Figure 2.9 where

$$P = (v_{k-7}, v_{k-8}, v_{k-5}, \ldots, v_{k-1}).$$

Delete the edge $e = v_{k-10}v_{k-9}$ from $C'$ and add the edges $v_{k-9}v_{k-6}$ and $v_{k-10}v_{k-7}$. This produces a properly colored Hamiltonian cycle in the distance-colored graph $P^3_{k+1}$ in which the edge $f = v_{k+1}v_k$ is colored 1 and the two edges of $C$ adjacent to $f$ are colored 2.

Figure 2.9: A step in the proof of Theorem 2.1.3

For $3 \leq n \leq 5$, $C^2_n = C^3_n$ and so $C^3_4$ is Hamiltonian-colored and $C^3_3$ and $C^3_4$ are not Hamiltonian-colored. For each $n \geq 6$, the properly colored Hamiltonian cycle of $P^3_n$ described in the proof of Theorem 2.1.3 is also a properly colored Hamiltonian cycle of $C^3_n$.

**Theorem 2.1.4** *For each integer $n \geq 6$, the distance-colored graph $C^3_n$ is Hamiltonian-colored.*
2.2 Hamiltonian Coloring Exponents of Graphs

We’ve seen that for some connected graphs $G$, there is no positive integer $k$ such that the distance-colored graph $G^k$ is Hamiltonian-colored. For example, for every graph $G$ of odd order and diameter 2, the distance-colored graph $G^k$ is not Hamiltonian-colored for any $k$. On the other hand, if for a given connected graph $G$, there is a positive integer $k$ for which the distance-colored graph $G^k$ is Hamiltonian-colored, then it is natural to ask for the smallest integer $k$ for which $G^k$ is Hamiltonian-colored. This gives rise to the following concept. Let $G$ be a connected graph for which $G^k$ is Hamiltonian-colored for some positive integer $k$.

The *Hamiltonian coloring exponent* $\text{hce}(G)$ of $G$ is the minimum $k$ for which $G^k$ is Hamiltonian-colored. Thus if $\text{hce}(G) = k$, then $G^k$ is Hamiltonian-colored but $G^{k-1}$ is not Hamiltonian-colored. In this section, we show that for every integer $k \geq 2$, there exists a connected graph $G$ such that $\text{hce}(G) = k$. Of course, we have already seen that there are graphs with Hamiltonian coloring exponents 2 and 3, namely $C_4$ and $C_6$, respectively. We begin with the following lemma.

**Lemma 2.2.1** For each integer $n \geq 3$, let $P_n = (r_1, r_2, \ldots, r_n)$ be a path of order $n$. Then the distance-colored graph $P_n^3$ contains a properly colored Hamiltonian $r_2 - r_n$ path in which the edge incident with $r_2$ is colored 1 and the edge incident with $r_n$ is colored 1 or 2.

**Proof.** We show that the distance-colored graph $P_n^3$ has a properly colored Hamiltonian $r_2 - r_n$ path $Q$ such that (1) the edge incident with $r_2$ is colored 1
and (2) if \( n \geq 3 \) is odd, then the edge incident with \( x_n \) in \( Q \) is colored 2 and \( Q \) ends with the subpath \( (x_{n-1}, x_{n-2}, x_n) \), while if \( n \geq 4 \) is even, then the edge incident with \( x_n \) in \( Q \) is colored 1 and \( Q \) ends with the subpath \( (x_{n-2}, x_{n-3}, x_{n-1}, x_n) \).

Figure 2.10 shows a properly colored Hamiltonian \( x_2 - x_n \) path having the desired properties in each distance-colored graph \( P^3_n \) for \( 3 \leq n \leq 8 \), where only the edges of the path are shown.

![Figure 2.10: Hamiltonian \( x_2 - x_n \) paths in \( P^3_n \) for \( 3 \leq n \leq 8 \)](image)

The properly colored Hamiltonian \( x_2 - x_n \) paths \( Q \) of \( P^3_n \) shown in Figure 2.10 for \( 3 \leq n \leq 8 \) illustrate a general pattern. For \( n \geq 9 \), we consider the path \( Q \) in \( P^3_n \) that begins with \( (x_2, x_1, x_4, x_3, x_6, x_5, \ldots, x_{r-1}, x_{r-2}) \), where \( r - 2 \) is the largest odd integer that is less than \( n - 1 \). Thus \( r - 2 = n - 2 \) if \( n \) is odd, while \( r - 2 = n - 3 \) if \( n \) is even. The path \( Q \) is then completed as follows:

\[
Q = (x_2, x_1, x_4, x_3, x_6, x_5, \ldots, x_{n-1}, x_{n-2}, x_n) \text{ if } n \text{ is odd},
\]

\[
Q = (x_2, x_1, x_4, x_3, x_6, x_5, \ldots, x_{n-2}, x_{n-3}, x_{n-1}, x_n) \text{ if } n \text{ is even}.
\]

In both cases, \( Q \) is a properly colored Hamiltonian \( x_2 - x_n \) path whose initial edge is colored 1 and whose final edge is (i) colored 2 if \( n \) is odd and (ii) colored 1 if \( n \)
is even

We are now prepared to present the following result

**Theorem 2.2.2**  For every integer $k \geq 2$, there exists a connected graph $G$ such that

$$\text{hce}(G) = k$$

**Proof.** Since $C_4^n$ is Hamiltonian-colored if and only if $n \equiv 0 \pmod{4}$ by Theorem 2.1.1, $\text{hce}(C_n) = 2$ if $n \equiv 0 \pmod{4}$. Also, $C_4^n$ is Hamiltonian colored for all $n \geq 6$ by Theorem 2.1.4 and so $\text{hce}(C_n) = 3$ if $n \geq 6$ and $n \not\equiv 0 \pmod{4}$. Thus we may assume that $k \geq 4$.

For $k \geq 4$, let $F$ and $H$ be two copies of $K_k$, where

$$V(F) = \{u, u_1, u_2, \ldots, u_k\} \text{ and } V(H) = \{w, w_1, w_2, \ldots, w_k\},$$

and let $P = (v_0, v_1, v_2, \ldots, v_{k-3}, v_k)$ be a path of order $k - 1$. Let $G_k$ be the graph obtained from $F, H$ and $P$ where $u$ and $v_0$ are identified and $w$ and $v_{k-2}$ are identified. Figure 2.11 shows the graphs $G_k$ for $k = 4$ and $k = 5$. Observe that $\text{diam}(G_k) = k$.

Let $G = G_k$. We first show that $G^k$ is Hamiltonian colored. By Lemma 2.2.1, for the path $P = (u, v_1, v_2, \ldots, v_{k-3}, w)$ of order $k - 1 \geq 3$, the distance colored graph $P^3$ contains a properly colored Hamiltonian $v_1 - w$ path $Q$ in which the edge incident with $v_1$ is colored 1 and the edge incident with $w$ is colored 2 if $k - 1$ is odd and colored 1 if $k - 1$ is even.
Next, we construct a properly colored $v_1 - w$ path $Q^*$ of $G^k$ such that $V(Q) \cup V(Q^*) = V(G)$ and $V(Q) \cap V(Q^*) = \{v_1, w\}$.

If $k - 1$ is odd, then let

$$Q^* = (v_1, w_1, u_1, u_2, w_2, w_3, u_3, u_4, w_4, w_5, \ldots, w_{k-2}, w_{k-1}, u_{k-1}, w).$$

The color of $v_1w_1$ is $k - 2 \geq 2$ and the color of $u_{k-1}w$ is $k - 1 \geq 3$. For example, if $k = 4$, then the path

$$Q^* = (v_1, w_1, u_1, u_2, w_2, w_3, u_3, w_4, w_5)$$

for the distance-colored graph $G^4$ of the graph $G = G_4$ is shown in Figure 2.11(a).

If $k - 1$ is even, then let

$$Q^* = (v_1, u_1, w_1, w_2, u_2, w_3, u_3, w_4, \ldots, u_{k-3}, u_{k-2}, w_{k-2}, w_{k-1}, u_{k-1}, w).$$
The color of \(v_1u_1\) is 2 and the color of \(u_{k-1}w\) is \(k - 1 \geq 3\). For example, if \(k = 5\), then the path

\[ Q^* = (v_1, u_1, u_1, w_2, u_2, u_3, w_3, u_4, u_4, w) \]

for the distance-colored graph \(G^5\) of the graph \(G = G_5\) is shown in Figure 2.11(b).

The union of the \(v_1 - w\) paths \(Q^*\) and \(Q\) in \(G^k\) produces a properly colored Hamiltonian cycle in \(G^k\). Therefore, \(G^k\) is Hamiltonian-colored.

It remains to show that \(G^{k-1}\) is not Hamiltonian-colored. Assume, to the contrary, that \(G^{k-1}\) contains a properly colored Hamiltonian cycle \(C\). Let

\[
U = V(F) - \{u\} = \{u_1, u_2, \ldots, u_{k-1}\} \text{ and } W = V(H) - \{w\} = \{w_1, w_2, \ldots, w_{k-1}\}.
\]

As we proceed cyclically about \(C\) in some direction, we encounter at most two consecutive vertices belonging to \(U\) or to \(W\). Let \(X_1, X_2, \ldots, X_p\) be the maximal subsets of consecutive vertices in \(U\) or in \(W\) that are encountered on \(C\) in the given order. Then \(1 \leq |X_i| \leq 2\) for \(1 \leq i \leq p\). Since \(|U| = |W| = k - 1\), it follows that

\[
|\bigcup_{i=1}^{p} X_i| = |U \cup W| = 2k - 2.
\]

Thus \(p \geq k - 1\). Since (1) \(X_i\) cannot be immediately followed by \(X_{i+1}\) if \(X_i \cup X_{i+1} \subseteq U\) or \(X_i \cup X_{i+1} \subseteq W\) from the maximality of these sets and (2) \(X_i\) cannot be immediately followed by \(X_{i+1}\) if one of \(X_i\) and \(X_{i+1}\) is a subset of \(U\) and the other is a subset of \(W\) as no vertex in \(U\) is adjacent to a vertex of \(W\) in \(G^{k-1}\), it follows that a vertex of \(P\) must lie between \(X_i\) and \(X_{i+1}\) for each \(i\) (\(1 \leq i \leq p\)) where
$X_{p+1} = X_1$. Since the order of $P$ is $k - 1$ and $p \geq k - 1$, it follows that $p = k - 1$ and that $|X_i| = 2$ for $i = 1, 2, \ldots, p = k - 1$. This implies that exactly one vertex of $P$ lies between $X_i$ and $X_{i+1}$ for $i = 1, 2, \ldots, k - 1$. This, however, implies that the vertex $u$ (as well as $u'$) lies between $X_j$ and $X_{j+1}$ for some $j$ ($1 \leq j \leq k - 1$). If either $X_j \subseteq U$ or $X_{j+1} \subseteq U$, then, since $|X_j| = |X_{j+1}| = 2$, two consecutive edges on $C$ are colored 1, a contradiction. Otherwise, both $X_j$ and $X_{j+1}$ are subsets of $W$, implying that two consecutive edges on $C$ are colored $k - 1$, again a contradiction. Therefore, $G^{k-1}$ is not Hamiltonian-colored.

2.3 Graphs with Hamiltonian Coloring

Exponent 2

There is a concept that is closely related to powers of a graph. For a connected graph $G$ of diameter $d$ and each integer $i$ with $1 \leq i \leq d$, the $i$-step graph $G[i]$ of $G$ is the graph whose vertex set is $V(G)$ and two vertices $u$ and $v$ of $G$ are adjacent in $G[i]$ if $d_G(u, v) = i$. In particular, $G[1] = G$. Thus if $i$ and $k$ are integers with $1 \leq i \leq k \leq d$, then the $i$-step graph $G[i]$ is a spanning subgraph of the distanced-colored graph $G^k$ and each edge of $G[i]$ is colored $i$. Furthermore, the set \{ $E(G[i]), E(G[2]), \ldots, E(G[k])$ \} of color classes of $G^k$ is a partition of $E(G^k)$.

If $G$ is a connected graph of order $n \geq 3$ such that $G^2$ is Hamiltonian-colored (so $hce(G) = 2$), then every properly colored Hamiltonian cycle of $G^2$ gives rise to two 1-factors $F_1$ and $F_2$ such that $F_i$ is a 1-factor of $G[i]$ for $i = 1, 2$. The converse of this statement is not true, however. For example, if $G = P_{4k}$ for some integer
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\[ k \geq 2, \text{ then each of } G \text{ and } G^{[2]} \text{ has a 1-factor but } hce(G) = 3 \text{ by Theorem 2.1.3} \]

The following two observations related to a connected graph \( G \) with \( hce(G) = 2 \) will be useful to us.

**Observation 2.3.1** If \( G \) is a connected graph of order \( n \geq 3 \) such that \( hce(G) = 2 \), then \( n \) is even.

**Observation 2.3.2** If \( G \) is a connected graph of order at least 3 such that \( hce(G) = 2 \), then \( G \) cannot contain two vertices \( u \) and \( w \) such that there is a unique vertex \( v \) for which \( d(u, v) = d(w, v) = r \) for \( r = 1 \) or \( r = 2 \).

As an immediate consequence of Observation 2.3.2, if \( G \) is a star of order at least 3 or a double star (a tree of diameter 3) of order at least 5, then \( G^2 \) is not Hamiltonian-colored.

By Observation 2.3.1, if \( G \) is a connected graph of odd order having diameter 2, then \( G^k \) is not Hamiltonian-colored for each integer \( k \geq 2 \). A well-known class of graphs of diameter 2 are the complete multipartite graphs. Let \( K_{n_1, n_2, \ldots, n_k} \) be the complete \( k \)-partite graph of \( n \geq 5 \), where \( n_1 \leq n_2 \leq \ldots \leq n_k \) and \( n = \sum_{i=1}^{k} n_i \).

Then it is known that

\[ K_{n_1, n_2, \ldots, n_k} \text{ is Hamiltonian if and only if } n_k \leq \sum_{i=1}^{k-1} n_i \quad (2.1) \]

By (2.1), if \( n_1, n_2, \ldots, n_k \) are \( k \geq 2 \) positive even integers, then

\[ K_{n_1, n_2, \ldots, n_k} \text{ is Hamiltonian if and only if } K_{\frac{n_1}{2}, \frac{n_2}{2}, \ldots, \frac{n_k}{2}} \text{ is Hamiltonian} \quad (2.2) \]
Theorem 2.3.3  Let $G = K_{n_1, n_2, \ldots, n_k}$ be the complete $k$-partite graph of order $\sum_{i=1}^{k} n_i \geq 5$, where $n_1 \leq n_2 \leq \cdots \leq n_k$. The distance-colored graph $G^2$ is Hamiltonian-colored if and only if $n_i$ is even for $1 \leq i \leq k$ and $G$ is Hamiltonian.

Proof. First, suppose that the distance-colored graph $G^2$ is Hamiltonian-colored. Then $G^2$ has a properly colored Hamiltonian cycle $C$ whose edges are alternately colored 1 and 2. We saw that $C$ produces two 1-factors $F_1$ and $F_2$ in $G^2$ such that $F_i$ is a 1-factor of the $i$-step graph $G^{[i]}$ of $G$ for $i = 1, 2$. Since $G^{[2]} = K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_k}$, it follows that each $K_{n_i}$ $(1 \leq i \leq k)$ is 1-factorable and so $n_i$ is even. Next, we show that $G$ is Hamiltonian. Contracting each edge colored 2 in the properly colored Hamiltonian cycle $C$ in $G^2$ produces a Hamiltonian cycle in the complete $k$-partite graph $H = K_{\frac{n_1}{2}, \frac{n_2}{2}, \ldots, \frac{n_k}{2}}$. Thus $H$ is Hamiltonian. It then follows by (2.2) that $G$ is Hamiltonian.

For the converse, suppose that $n_i$ is even for $1 \leq i \leq k$ and $G$ is Hamiltonian. Then $G^{[2]} = K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_k}$ contains 1-factors. Let $F$ be a 1-factor of $G^{[2]}$. We now construct the complete $k$-partite graph $H = K_{\frac{n_1}{2}, \frac{n_2}{2}, \ldots, \frac{n_k}{2}}$ by identifying every pair $u, v$ of vertices of $G$ if $uv$ is an edge in $F$ and labeling the resulting vertex by $u$. Since $G$ is Hamiltonian, it follows by (2.2) that $H$ is Hamiltonian. Let $C'$ be a Hamiltonian cycle of $H$. Replacing each vertex $u$ of $C'$ by the vertices $u$ and $v$ and the edge $uv$ in $F$ produces a properly colored Hamiltonian cycle in $G^2$. Therefore, $G^2$ is Hamiltonian-colored.

Corollary 2.3.4  Let $G = K_{r, s}$ be a complete bipartite graph of order $r + s \geq 3$. The distance-colored graph $G^2$ is Hamiltonian-colored if and only if $r = s = 2t$ for
some positive integer \(t\)

Let \(G\) be a connected graph of order \(n\) and diameter 2. Then \(\text{hce}(G)\) exists if and only if \(n\) is even and \(\text{hce}(G) = 2\). Note that if the diameter of \(G\) is 2, then \(G^{[2]} = \overline{G}\). Thus, if \(G\) is a connected graph of diameter 2 and \(\text{hce}(G) = 2\), then each of \(G\) and \(\overline{G}\) contains 1-factors.

### 2.4 Hamiltonian Coloring Exponents of Trees

In this section, we investigate Hamiltonian coloring exponents of trees. It is known that for each integer \(k \geq 2\), there exists a connected graph whose Hamiltonian coloring exponent is \(k\) (see Theorem 2.2.2). For each integer \(k \geq 4\), the graph \(G\) constructed in the proof of Theorem 2.2.2 has clique number \(k\) and \(\text{diam}(G) = \text{hce}(G)\). We now show that not only does there exist a tree with Hamiltonian coloring exponent \(k\) but the tree can have an arbitrarily large diameter. We begin with a lemma, which is a consequence of the proof of Theorem 2.1.3.

**Lemma 2.4.1** For each integer \(n \geq 4\), let \(P_n\) be a path of order \(n\). Then the distance-colored graph \(P_n^{d}\) has a properly colored Hamiltonian cycle \(C\) in which an edge \(e\) incident with one of the end vertices of \(P_n\) is colored 1 such that the two edges of \(C\) adjacent to \(e\) are both colored 2.

**Theorem 2.4.2** For each pair \(k, d\) of integers with \(4 \leq k \leq d\), there exists a tree \(T\) with \(\text{hce}(T) = k\) and \(\text{diam}(T) = d\).

**Proof.** Let \(F\) and \(H\) be two copies of the star \(K_{1,k-1}\) where
\[ V(F) = \{u, u_1, u_2, \ldots, u_{k-1}\}, \quad V(H) = \{w, w_1, w_2, \ldots, w_{k-1}\}, \]

\( u \) is the central vertex of \( F \) and \( w \) is the central vertex of \( H \). Next, let \( P = (v_0, v_1, v_2, \ldots, v_{k-3}, v_{k-2}) \) be a path of order \( k - 1 \). Construct a tree \( T'_k \) from \( F, H \) and \( P \) by (i) identifying \( u \) and \( v_0 \) and labeling the identified vertex by \( v_0 \) and (ii) identifying \( w \) and \( v_{k-2} \) and labeling the identified vertex by \( v_{k-2} \). If \( d = k \), let \( T_{k,k} = T'_k \), while if \( d \geq k + 1 \), then \( T_{k,d} \) is obtained from \( T'_k \) and the path \((z_1, z_2, \ldots, z_{d-k})\) of order \( d - k \) by joining \( z_1 \) to the vertex \( w_{k-1} \) in \( T'_k \). Observe that \( \text{diam}(T_{k,d}) = d \).

Let \( T = T_{k,d} \). We first show that \( T^k \) is Hamiltonian-colored. By Lemma 2.2.1, for the path \( P = (v_0, v_1, v_2, \ldots, v_{k-3}, v_{k-2}) \) of order \( k - 1 \geq 3 \), the distance-colored graph \( P^3 \) contains a properly colored Hamiltonian \( v_1 - v_{k-2} \) path \( Q \) in which the edge incident with \( v_1 \) is colored 1 and the edge incident with \( v_{k-2} \) is colored 1 or 2. Next, we construct a properly colored \( v_1 - v_{k-2} \) path \( Q^* \) of \( T^k \) such that

(i) \( V(Q) \cup V(Q^*) = V(T) \) and \( V(Q) \cap V(Q^*) = \{v_1, v_{k-2}\} \) and

(ii) the edge incident with \( v_1 \) in \( Q^* \) is colored \( a \) where \( a \geq 2 \) and the edge incident with \( v_{k-2} \) in \( Q^* \) is colored \( b \) where \( b \geq 3 \).

We begin with a \( v_1 - v_{k-2} \) path in \( T^k \) according to whether \( k \) is even or \( k \) is odd. If \( k \) is even, then let

\[ Q_0 = (v_1, w_1, u_1, u_2, w_2, u_3, u_4, w_4, u_5, \ldots, w_{k-2}, w_{k-1}, u_{k-1}, v_{k-2}). \] \hspace{1cm} (2.3)

If \( k \) is odd, then let

\[ Q_1 = (v_1, u_1, w_1, u_2, w_2, u_3, w_3, w_4, \ldots, u_{k-3}, u_{k-2}, w_{k-2}, w_{k-1}, u_{k-1}, v_{k-2}). \] \hspace{1cm} (2.4)
In (2.3) and (2.4), the color of $v_1w_1$ is $k - 2 \geq 2$ and the color of $u_{k-1}v_{k-2}$ is $k - 1 \geq 3$. In (2.4), the color of $v_1u_1$ is 2 and the color of $u_{k-1}v_{k-2}$ is $k - 1 \geq 3$. We now consider four cases, according to the values of $d$ and $k$.

**Case 1.** $d = k$. If $k$ is even, then let $Q^* = Q_0$ as described in (2.3); while if $k$ is odd, then let $Q^* = Q_1$ as described in (2.4).

**Case 2.** $d = k + 1$. If $k$ is even, then let $Q^*$ be the $v_1 - v_{k-2}$ path obtained from the path $Q_0$ in (2.3) by replacing the edge $w_{k-2}w_{k-1}$ in $Q_0$ by the path $(w_{k-2}, z_1, w_{k-1})$; while if $k$ is odd, then let $Q^*$ be the $v_1 - v_{k-2}$ path obtained from the path $Q_1$ in (2.4) by replacing the edge $w_{k-2}w_{k-1}$ in $Q_1$ by the path $(w_{k-2}, z_1, w_{k-1})$.

**Case 3.** $d = k + 2$. If $k$ is even, then let $Q^*$ be the $v_1 - v_{k-2}$ path obtained from the path $Q_0$ in (2.3) by replacing the edge $w_{k-2}w_{k-1}$ in $Q_0$ by the path $(w_{k-2}, z_1, z_2, w_{k-1})$; while if $k$ is odd, then let $Q^*$ be the $v_1 - v_{k-2}$ path obtained from the path $Q_1$ in (2.4) by replacing the edge $w_{k-2}w_{k-1}$ in $Q_1$ by the path $(w_{k-2}, z_1, z_2, w_{k-1})$.

**Case 4.** $d \geq k + 3$. In this case, the order of the path $P_{d-k+1} = (w_{k-1}, z_1, z_2, \ldots, z_{d-k})$ is $d - k + 1 \geq 4$. By Lemma 2.4.1, the distance-colored graph $P_{d-k+1}^3$ has a properly colored Hamiltonian cycle $C$ in which $(z_2, w_{k-1}, z_1, z_3)$ is a subpath of $C$. Let $P^* = C - w_{k-1}z_1$ be the $z_1 - w_{k-1}$ subpath of $C$. Then $P^*$ is a Hamiltonian $z_1 - w_{k-1}$ path of $P_{d-k+1}^3$ whose initial and terminal edges (the edges incident with $z_1$ and $w_{k-1}$) are both colored 2. If $k$ is even, then let $Q^*$ be the $v_1 - v_{k-2}$ path.
obtained from the path $Q_0$ in (2.3) by replacing the edge $w_{k-2}w_{k-1}$ in $Q_0$ by the path $(w_{k-2}, P^*)$; while if $k$ is odd, then let $Q^*$ be the $v_1 - v_{k-2}$ path obtained from the path $Q_1$ in (2.4) by replacing the edge $w_{k-2}w_{k-1}$ in $Q_1$ by the path $(w_{k-2}, P^*)$. Since $d(w_{k-2}, z_1) = 3$ and $d(w_{k-1}, u_{k-1}) = k \geq 4$, the path $Q^*$ is properly colored.

In each case, the $v_1 - v_{k-2}$ path $Q^*$ has the prescribed properties. Thus the union of the $v_1 - v_{k-2}$ paths $Q^*$ and $Q$ in $T^k$ produces a properly colored Hamiltonian cycle in $T^k$. Therefore, $T^k$ is Hamiltonian-colored.

It remains to show that $T^{k-1}$ is not Hamiltonian-colored. Assume, to the contrary, that $T^{k-1}$ contains a properly colored Hamiltonian cycle $C$. Let $U = \{u_1, u_2, \ldots, u_{k-1}\}$ and let $V = \{v_0, v_1, \ldots, v_{k-2}\}$. Since at most two vertices from $U$ can be consecutive vertices in $C$, it follows that in $C$, each vertex in $U$ is adjacent to a vertex not in $U$. Furthermore, $N_{T^{k-1}}(u) = U \cup V(P)$ for each $u \in U$ and $|V(P)| = |U| = k - 1$. Thus, we may assume, without loss of generality, that $u_i v_{i-1} \in E(C)$ for $1 \leq i \leq k - 1$. Since $d_T(u_2, v_1) = 2$, it follows that $u_2 u_i \notin E(C)$ for all $i$ with $1 \leq i \leq k - 1$ and $i \neq 2$. This implies that $u_2 v_j \in E(C)$ for some $j$ with $2 \leq j \leq k - 2$. However then, either $(u_2, v_j, u_j)$ or $(u_j, v_j, u_2)$ is a subpath of $C$, each of whose two edges is colored 2, which is impossible. Therefore, $T^{k-1}$ is not Hamiltonian-colored, as claimed. □

### 2.5 Some Well-Known Classes of Trees

We continue our study of Hamiltonian-colored powers of trees by determining the Hamiltonian coloring exponents of trees belonging to some much-studied classes.
We saw in Observation 2.1.2 that if $T$ is a star (a tree of diameter 2), then $T^k$ is Hamiltonian for no integer $k \geq 2$. In this section, we determine the Hamiltonian coloring exponents of double stars and the so-called cubic caterpillars.

### 2.5.1 Double Stars

If $T$ is a double star, then $T$ has exactly two non-end-vertices called the central vertices of $T$. For integers $p, q \geq 2$, let $S_{p,q}$ be the double star having central vertices $A$ and $B$ such that $\deg A = p$ and $\deg B = q$. Then the order of $S_{p,q}$ is $n = p + q \geq 4$. We saw by Observation 2.3.2 that the square of a double star of order $n \geq 4$ is Hamiltonian-colored if and only if $n = 4$. Next, we determine all double stars whose cube is Hamiltonian-colored. In order to do this, we present two additional definitions. For a Hamiltonian cycle

$$C = (v_1, v_2, \ldots, v_n, v_1)$$

in a graph $G$, the distance sequence of $C$ is defined as

$$s_C = (d_G(v_1, v_2), d_G(v_2, v_3), \ldots, d_G(v_{n-1}, v_n), d_G(v_n, v_1)).$$

Thus if $C$ is a properly colored Hamiltonian cycle, then every two consecutive terms in $s_C$ are distinct.

**Theorem 2.5.1** Let $G$ be the double star $S_{p,q}$ where $p, q \geq 2$. Then the distance-colored graph $G^3$ is Hamiltonian-colored if and only if

$$|p - q| \leq 1 \text{ and } \{p, q\} \neq \{2, 3\}, \{3, 3\}.$$
Proof. Let \( G = S_{p,q} \) whose central vertices are \( A \) and \( B \), where \( \deg A = p \) and \( \deg B = q \) and \( n = p + q \) is the order of \( G \). Let \( V_A = N(A) - \{B\} \) and \( V_B = N(B) - \{A\} \). We denote any vertex in \( V_A \) by \( a \) and any vertex in \( V_B \) by \( b \) (see Figure 2.12). For \( u, v \in V(G) \), it follows that \( d(u, v) = 3 \) if and only if \( \{u, v\} = \{a, b\} \) and \( d(u, v) \leq 2 \) if \( \{u, v\} \cap \{A, B\} \neq \emptyset \).

![Figure 2.12: Labeling the vertices of \( G = S_{p,q} \)](image_description)

Suppose first that \( G^3 \) contains a properly colored Hamiltonian cycle

\[
C = (v_1, v_2, \ldots, v_n, v_1).
\]

We consider three cases, according to the distance \( d_C(A, B) \) of \( A \) and \( B \) on the cycle \( C \).

Case 1. \( d_C(A, B) = 1 \). We may assume that \( A = v_i \) and \( B = v_{i+1} \), where \( 1 \leq i \leq n - 1 \). Then the sequence \( b, A, B, a \) appears clockwise on \( C \), the sequence \( 2, 1, 2 \) appears clockwise on \( s_C \) and each of other terms in \( s_C \) is either 2 or 3. This implies that

\[
C = (b, A, B, a, b, a, b, a, b, a, \ldots, b, b, a, a, b)
\]

and so the distance sequence of \( C \) is \( s_C = (2, 1, 2, 3, 2, 3, \ldots, 2, 3) \). Since at least one subsequence \( b, b, a, a \) must appear in \( C \), it follows that \( |V_A| = |V_B| \geq 3 \) and so \( p = q \geq 4 \). Therefore, \( |p - q| \leq 1 \) and \( \{p, q\} \neq \{2, 3\}, \{3, 3\} \).
Case 2. $d_C(A, B) = 2$. We may assume that $A = v_i$ and $B = v_{i+2}$, where $1 \leq i \leq n - 2$. Then either the sequence $a, A, b, B, a$ or the sequence $b, A, a, B, b$ appears clockwise on $C$. Assume, without loss of generality, that $a, A, b, B, a$ appears clockwise on $C$. Then the sequence $1, 2, 1, 2$ appears clockwise on $s_C$ and each of the remaining terms in $s_C$ is either $2$ or $3$. Thus either

$$C = (a, A, b, B, a, b, a, b, a, a, a, b, b, a, a, a, a, b, b, a, a)$$
or

$$C = (a, A, b, B, a, b, a, b, a, a, a, b, b, a, a, a, b, b, a, a),$$

where at least one subsequence $b, b, a, a$ must appear in $C$. Hence either

$$s_C = (1, 2, 1, 2, 3, 2, \ldots, 3, 2)$$
or

$$s_C = (1, 2, 1, 2, 3, 2, \ldots, 3, 2, 3).$$

Thus either $|V_A| = |V_B| + 1 \geq 3$ or $|V_A| = |V_B| \geq 3$, that is, either $p = q + 1 \geq 4$ or $p = q \geq 4$. Therefore, $|p - q| \leq 1$ and $\{p, q\} \neq \{2, 3\}, \{3, 3\}$.

Case 3. $d_C(A, B) \geq 3$. We may assume that $A = v_i$ and $B = v_j$, where $1 \leq i < j \leq n$, and that $a, A, b$ appears clockwise on $C$. Thus either $b, B, a$ or $a, B, b$ appears clockwise on $C$ and exactly two terms in $s_C$ are $1$. We consider these two subcases.

Subcase 3.1. The sequence $b, B, a$ appears clockwise on $C$. Let

$$C = (a, A, b, s_1, b, B, a, s_2, a),$$

where $s_1$ and $s_2$ are subsequences of $C$. The length of each of $s_1$ and $s_2$ at least $2$. In fact, $s_1$ is either
where it is possible that $s_1$ is $a, a, b$ or $a, a$. Similarly, $s_2$ is either

\[ b, b, a, a, b, a, a, b, \ldots, b, b, a, a, b, a, a, b, \] where it is possible that $s_2$ is $b, b, a$ or $b, b$. In each case then, $|V_A| \geq 4$, $|V_B| \geq 4$, and $||V_A| - |V_B|| \leq 1$. Therefore, $|p - q| \leq 1$ and $\{p, q\} \neq \{2, 3\}, \{3, 3\}$.

**Subcase 3.2.** *The sequence $a, B, b$ appears clockwise on $C$.* Let

\[ C = (a, A, b, s_1, a, B, b, s_2, a), \]

where $s_1$ and $s_2$ are subsequences of $C$. Then

\[ s_1 : a, a, b, b, a, a, b, b, \ldots, a, a, b, b, \]

where the subsequence $a, a, b, b$ must appear at least once in $s_1$. Thus the length of $s_1$ is $4\ell$ for some integer $\ell \geq 1$. On the other hand, $s_2$ is one of the following:

1. $a$,
2. $b$,
3. $b, a$,
4. $b, a, a, b, b, a, a, b, b, \ldots, a, a, b, b, a$,
5. $b, a, a, b, b, a, a, b, b, \ldots, a, a, b, b$ or
6. $a, a, b, b, a, a, b, b, \ldots, a, a, b, b$. 
In each of (4), (5) and (6), the subsequence $a, a, b, b$ must appear at least once. If (3), (4) or (6) occurs, then $p = q$; while if (1), (2) or (5) occurs, then $|p - q| = 1$. In each case, $|V_A| \geq 4$, $|V_B| \geq 4$. Therefore, $|p - q| \leq 1$ and $\{p, q\} \neq \{2, 3\}, \{3, 3\}$.

For the converse, assume that $|p - q| \leq 1$ and $\{p, q\} \neq \{2, 3\}, \{3, 3\}$. If $p = q$, then

$$C = (b, A, B, a, b, a, b, a, b, a, a, \ldots, b, b, a, a, b)$$

is a properly colored Hamiltonian cycle in $G^3$; while if $p = q + 1$, then

$$C = (a, A, b, B, a, b, a, b, a, a, b, b, a, a, \ldots, b, b, a, a)$$

is a properly colored Hamiltonian cycle in $G^3$. Thus $G^3$ is Hamiltonian-colored. ■

Since a double star has diameter 3, the following is an immediate consequence of Theorem 2.5.1.

**Corollary 2.5.2** Let $G$ be the double star $S_{p,q}$ where $p, q \geq 2$. Then hce($G$) exists if and only if

$$|p - q| \leq 1 \text{ and } \{p, q\} \neq \{2, 3\}, \{3, 3\}.$$ 

The subdivision graph $S(F)$ of a connected graph $F$ is obtained from $F$ by subdividing each edge of $F$ exactly once. By Observation 2.3.2, if $F$ is a star or a double star, then the square of $S(F)$ is not Hamiltonian-colored. On the other hand, the cube of $S(F)$ is, in fact, Hamiltonian-colored.
Proposition 2.5.3 If $F$ is a star of order at least 3 or a double star of order at least 4 and $G = S(F)$, then the distance-colored graph $G^3$ is Hamiltonian-colored.

Proof. First, let $F = K_{1,t}$ for some integer $t \geq 2$ with $V(F) = \{u, v_1, v_2, \ldots, v_t\}$, where $\deg u = t$. Suppose that $G = S(F)$ is obtained from $F$ by subdividing the edges of $F$ with $t$ new vertices $w_1, w_2, \ldots, w_t$ such that $w_i$ is adjacent to $u$ and $v_i$ for $1 \leq i \leq t$. Then $G^3$ contains the properly colored Hamiltonian cycle

$$C = (v_1, w_2, v_2, w_3, v_3, \ldots, w_{t-1}, v_{t-1}, w_t, u, v_1),$$

where the resulting cyclic distance sequence of $C$ is $s_C = (3, 1, 3, 1, \ldots, 3, 1, 2)$.

Figure 2.13 shows such a properly colored Hamiltonian cycle $C$ for $t = 4$, where only edges on the cycle $C$ are drawn.

Next, let $F = S_{p,q}$ for some integers $p, q \geq 2$. We may assume that $p \geq q$. If $p = q = 2$, then the result follows by Theorem 2.13. Thus, we may assume that $p \geq 3$. Suppose that the central vertices of $S_{p,q}$ are $u$ and $v$ such that $u$ is adjacent to $v, u_1, u_2, \ldots, u_{p-1}$ and $v$ is adjacent to $u, v_1, v_2, \ldots, v_{q-1}$. Let $G$ be the subdivision graph of $F$ obtained by subdividing the edges of $F$ with new vertices $w, \tau_1, \tau_2, \ldots, \tau_{p-1}$ and $y_1, y_2, \ldots, y_{q-1}$ such that (1) $w$ is adjacent to $u$ and $v$, (2)
$r_i$ is adjacent to $u$ and $u_i$ for $1 \leq i \leq p - 1$ and (3) $y_j$ is adjacent to $v$ and $v_j$ for $1 \leq j \leq q - 1$. If $q = 2$, then $G^3$ contains the properly colored Hamiltonian cycle

$$C = (u_1, w, y_1, v_1, v, r_{p-1}, u_{p-1}, r_{p-2}, u_{p-2}, \ldots, r_2, u_2, r_1, u, u_1),$$

(see Figure 2.14(a) for $p = 3$), while if $q \geq 3$, then $G^3$ contains the properly colored Hamiltonian cycle

$$C = (u_1, w, y_1, v_1, y_2, v_2, \ldots, y_{q-1}, v_{q-1}, v, r_{p-1}, u_{p-1}, r_{p-2}, u_{p-2}, \ldots, r_2, u_2, r_1, u, u_1),$$

as desired.

![Figure 2.14](image)

**Figure 2.14** Illustrating the proof of Proposition 2.5.3

### 2.5.2 Cubic Caterpillars

A tree $T$ is called **cubic** if the degree of each non-end-vertex of $T$ is 3. A **caterpillar** is a tree of order 3 or more, the removal of whose end-vertices produces a path called
the spine of the caterpillar. For each integer \( d \geq 2 \), let \( T_d \) be the cubic caterpillar of diameter \( d \). Thus \( T_2 \) is the star \( K_{1,3} \) and \( T_3 \) is the double star \( S_{3,3} \). We have seen that \( T_d^k \) is not Hamiltonian-colored for each \( k \geq 2 \) by Observation 2.1.2, while \( T_d^k \) is Hamiltonian-colored when \( k \geq 3 \) by Theorem 2.5.1. Thus, we may assume that \( d \geq 4 \). By Observation 2.3.2, for each integer \( d \geq 4 \), the distance-colored graph \( T_d^2 \) is not Hamiltonian-colored. On the other hand, \( T_d^3 \) is Hamiltonian-colored for each integer \( d \geq 4 \), as we show next. Figure 2.15 shows a properly colored Hamiltonian cycle in \( T_d^3 \) for each integer \( d \) with \( 4 \leq d \leq 11 \) and so these graphs are Hamiltonian-colored.

Theorem 2.5.4 For each integer \( d \geq 4 \), the distance-colored graph \( T_d^3 \) is Hamiltonian-colored.

Proof. Let \( P = (v_1, v_2, \ldots, v_{d+1}) \) be a geodesic of length \( d \) in \( T_d \), where the vertex \( v_i \) is adjacent to the end-vertex \( u_i \) for \( 2 \leq i \leq d \). We have seen (in Figure 2.15) that \( T_d^3 \) and \( T_5^3 \) are Hamiltonian-colored. We now show by induction on \( d \geq 6 \) that the distance-colored graph \( T_d^3 \) has a properly colored Hamiltonian cycle \( C \) such that \( C \) contains one of the two paths shown in Figure 2.16. That the statement is true when \( 6 \leq d \leq 11 \) is verified in Figure 2.15.

Assume that the statement is true for every integer \( d' \) with \( 6 \leq d' \leq d \), where \( d \geq 11 \), and consider the distance-colored graph \( T_{d+1}^3 \). By the induction hypothesis, there exists a properly colored Hamiltonian cycle \( C' \) in the distance-colored graph \( T_{d-4}^3 \) such that \( C' \) contains one of the two paths as shown in Figure 2.16 in which the subscript \( i \) of each vertex \( (u_i \) or \( v_i) \) is replaced by \( i-4 \). We consider two cases.
Case 1. The cycle \( C' \) contains the path as shown in Figure 2.16(a). Let

\[
P = (v_{d-4}, v_{d-1}, v_{d+1}, v_{d+2}, u_{d+1}, u_d, v_d, u_{d-1}, u_{d-2}, v_{d-3}, u_{d-3}, v_{d-3})
\]

be the properly colored path in \( T^3_{d+1} \). (See Figure 2.17(a).) Replace the edge \( v_{d-4}v_{d-3} \) from \( C' \) by the path \( P \). This produces a properly colored Hamiltonian cycle \( C \) in the distance-colored graph \( T^3_{d+1} \) such that \( C \) contains the path in Figure 2.16(a) in which the subscript of each vertex \( (u_i \text{ or } v_i) \) is replaced by \( i + 1 \).
Case 2. The cycle $C'$ contains the path as shown in Figure 2.16(b). Let

$$Q = \{v_{d-4}, v_{d-2}, v_{d+1}, v_{d+2}, u_{d+1}, u_{d}, v_{d}, u_{d-1}, v_{d-1}, u_{d-2}, u_{d-3}, v_{d-3}\}$$

be the properly colored path in $T_{d+1}^3$. (See Figure 2.17(b).) Replace the edge $v_{d-4}v_{d-3}$ from $C'$ by the path $Q$. This produces a properly colored Hamiltonian cycle $C$ in the distance-colored graph $T_{d+1}^3$ such that $C$ contains the path in Figure 2.16(b), in which the subscript $i$ of each vertex ($u_i$ or $v_i$) is replaced by $i+1$. ■
The following is an immediate consequence of Observation 2.3.2 and Theorem 2.5.4

**Corollary 2.5.5**  For each integer \( d \geq 3 \), \( hce(T_d^k) = 3 \)

If \( G \) is a connected graph with \( hce(G) = k \), then necessarily every properly colored Hamiltonian cycle in \( G^k \) must use the color \( k \). On the other hand, it is possible that \( G^k \) contains a properly colored Hamiltonian cycle that does not use all \( k \) colors \( 1, 2, \ldots, k \).

**Theorem 2.5.6**  For each integer \( k \geq 3 \), there exists a connected graph \( G_k \) with \( hce(G_k) = k \) and a properly colored Hamiltonian cycle in the \( k \)th power of \( G_k \) whose edges are colored with fewer than \( k \) colors.

**Proof.**  For \( k = 3 \), let \( G_3 = C_6 = (v_1, v_2, \ldots, v_6 = v_1) \). We have seen that \( hce(G_3) = 3 \) and the properly colored Hamiltonian cycle \((v_1, v_2, v_5, v_6, v_3, v_4, v_1)\) in the cube of \( G_3 \) uses two colors, namely 1 and 3. For each integer \( k \geq 4 \), let \( G_k \) be the tree \( T_{k,k} \) constructed in Case 1 when \( \text{diam}(G_k) = k \) in the proof of Theorem 2.4.2 and so \( hce(G_k) = k \). To simplify notation, let \( G = G_k \). For \( k \in \{4,5,6\} \), Figure 2.18 shows a properly colored Hamiltonian cycle \( C \) in each graph \( G^k \) whose edges are colored with less than \( k \) colors. In Figure 2.18, the edges of \( G \) are drawn as dashed lines, the edges of \( C \) are drawn as solid lines and the remaining edges in \( G^k \) are not drawn. (Notice that the color 3 is not used in any of these cases and that the cycle in \( G^6 \) uses only four colors.)
For $k \geq 7$, let $C$ be the properly colored Hamiltonian cycle in $G^k$ described in the proof of Theorem 2.4.2; that is, $C$ consists of the $v_1-v_{k-2}$ paths $Q$ and $Q^*$, where the path $Q$ is described in Lemma 2.2.1 and the path $Q^*$ is described in (2.3) or (2.4) of Theorem 2.4.2 according to the parity of $k$. The path $Q$ uses the colors 1, 2, 3 and the path $Q^*$ uses the colors 1, 2, $k$ and at most two other colors (namely, $k-2$ and $k-1$). Observe that for $k \geq 5$, the color $[k/2]$ is never used and for $k \geq 8$, none of the colors in the set $\{5, 6, \ldots, k-3\}$ are used. Thus, the cycle $C$ uses at most six different colors for each integer $k \geq 7$.

Theorem 2.5.6 suggests another concept. For a connected graph $G$ with $hce(G) = k$, let $\mu(G)$ denote the minimum number of colors used in a properly colored
Hamiltonian cycle of $G^k$. Thus, if $hce(G) = 2$, then $\mu(G) = hce(G) = 2$, while if $hce(G) = 3$ and the order of $G$ is odd, then $\mu(G) = hce(G) = 3$. Since $\mu(G) \leq hce(G)$ for every connected graph $G$, it follows that

$$0 < \frac{\mu(G)}{hce(G)} \leq 1$$

(25)

For each integer $k \geq 4$, the graph $G_k$ constructed in the proof of Theorem 2.5.6 is a tree. Thus the following corollary is an immediate consequence of the proof of Theorem 2.5.6.

**Corollary 2.5.7** There exists a sequence $\{T_k\}_{k=4}^\infty$ of trees such that

$$\lim_{k \to \infty} \frac{\mu(T_k)}{hce(T_k)} = 0$$

Next we show for each integer $k \geq 2$, that there exists a tree $T_k$ such that $\mu(T_k) = hce(T_k)$ and so the upper bound in (25) holds.

**Theorem 2.5.8** For each integer $k \geq 2$, there exists a tree $T_k$ with $hce(T_k) = k$ such that every properly colored Hamiltonian cycle in the $k$th power of $T_k$ must use all of the colors $1, 2, \ldots, k$.

**Proof.** For $k = 2$, let $T_2 = P_4$, while for $k = 3$, let $T_3 = P_5$. For each fixed integer $k \geq 4$, let $P = (u_1, u_2, \ldots, u_k)$ be a path of order $k$ and let $H$ be a copy of star $K_{1,k-1}$ with $V(H) = \{v, v_1, v_2, \ldots, v_{k-1}\}$, where $v$ is the central vertex of $H$. The tree $T_k$ is constructed from $P$ and $H$ by identifying $u_1$ and $v$ and labeling the identified vertex by $u_1$. We claim that $hce(T_k) = k$. Let $T = T_k$. We first show that $T^k$ is Hamiltonian-colored. For each even integer $k \geq 4$, let
\[ C = (u_1, v_1, u_2, v_2, v_3, u_3, v_3, u_4, v_4, v_5, u_5, v_5, \ldots, u_{k-2}, u_{k-1}, v_{k-2}, v_{k-1}, u_k, u_1) \]

Figures 2 19(a), (b) show the cycle \( C \) for \( k = 4 \) and \( k = 6 \), respectively, where only the edges on \( C \) are drawn. Since \( d_T(u_1, v_1) = 1 \), \( d_T(v_1, u_2) = 2 \) and \( d_T(v_i, u_{i+1}) = \tau + 1 \) for \( 2 \leq \tau \leq k - 1 \), it follows that \( C \) is a properly colored Hamiltonian cycle in \( T^k \). For each odd integer \( k \geq 5 \), let

\[ C = (u_1, v_1, u_2, v_2, u_3, v_3, v_4, u_4, u_5, v_5, v_6, \ldots, u_{k-2}, u_{k-1}, v_{k-2}, v_{k-1}, u_k, u_1) \]

Figures 2 19(c), (d) show the cycle \( C \) for \( k = 5 \) and \( k = 7 \), respectively, where only the edges on \( C \) are drawn. Since \( d_T(u_1, v_1) = 1 \), \( d_T(v_1, v_2) = 2 \), \( d_T(u_3, v_3) = 3 \), \( d_T(u_4, v_2) = 4 \) and \( d_T(v_i, u_{i+1}) = \tau + 1 \) for \( 4 \leq \tau \leq k - 1 \), it follows that \( C \) is a properly colored Hamiltonian cycle in \( T^k \).

Figure 2 19 Illustrating the properly colored Hamiltonian cycles of \( T^k \) (4 ≤ \( k \) ≤ 7) in the proof of Theorem 2 5 8.
Next, we show that $T^{k-1}$ is not Hamiltonian-colored. Assume, to the contrary, that $T^{k-1}$ contains a properly colored Hamiltonian cycle $C$. Let

$$U = \{u_1, u_2, \ldots, u_k\}$$

and let

$$V = \{v_1, v_2, \ldots, v_{k-1}\}$$

Since at most two vertices from $V$ can be consecutive vertices in $C$, it follows that each vertex in $V$ is adjacent to a vertex in $U$ in $C$. Furthermore, no two vertices in $V$ can be adjacent to a common vertex in $U$ and no vertex in $V$ can be adjacent to the vertex $u_k \in U$ in $T^{k-1}$. Thus, we may assume, without loss of generality, that $v_i, u_r \in E(C)$ for $1 \leq r \leq k-1$. In particular, $v_2, u_2 \in E(C)$. Since $d_T(v_2, u_2) = 2$, it follows that $v_2, u_j \notin E(C)$ for all $1 \leq j \leq k-1$ and $j \neq 2$. This implies that $v_2, u_{\ell} \in E(C)$ for some $\ell$ with $1 \leq \ell \leq k-1$ and $\ell \neq 2$. However, then, either $(v_\ell, u_\ell, v_2)$ or $(v_2, u_\ell, v_\ell)$ is a subpath of $C$, each of whose edges is colored $\ell$, which is impossible. Therefore, $T^{k-1}$ is not Hamiltonian-colored.

Therefore, $hce(T) = k$, as claimed. It remains to show that every properly colored Hamiltonian cycle in $T^k$ must use all colors $1, 2, \ldots, k$, that is, $\mu(T) = k$.

Assume, to the contrary, that there is a properly colored Hamiltonian cycle $C$ in $T^k$ such that $C$ avoids the color $j$ for some $j$ with $1 \leq j \leq k-1$. If $j = 2$, then no two vertices in $V$ can be consecutive vertices in $C$. Thus, for each integer $r$ with $1 \leq r \leq k-1$, the vertex $v_i$ must be adjacent to two vertices in $U$. Since $k \geq 4$, $|V| = k-1$ and $|U| = k$, it follows that some vertex of $U$ is adjacent to two vertices of $V$ in $C$, which is a contradiction. Thus $C$ must use the color 2 and so $j \neq 2$. This implies that no vertex in $V$ can be adjacent to the vertex $u_j \in U$. Furthermore, recall that in $C$, each vertex in $V$ is adjacent to a vertex in $U$, no two
vertices in $V$ can be adjacent to a common vertex in $U$ and no vertex in $V$ can be adjacent to the vertex $u_k \in U$. Since $|V| = k - 1$ and $|U - \{u_j, u_k\}| = k - 2$, this is impossible. Therefore, as claimed, every properly colored Hamiltonian cycle in $T^k$ must use all colors $1, 2, \ldots, k$ and so $\mu(T) = k$. ■

The following is an immediate consequence of the proof of Theorem 2.5.8.

**Corollary 2.5.9** There exists a sequence $\{T_k\}_{k=2}^{\infty}$ of trees such that

$$\lim_{k \to \infty} \frac{\mu(T_k)}{hce(T_k)} = 1.$$
Chapter 3

Colored Bridge Problems

3.1 The Königsgberg Bridge Problem

The city of Königsgberg was founded in 1255 when it was the capital of German East Prussia and home of the Prussian Royal Castle. The River Pregel flowed through Königsgberg, separating it into four land regions. Seven bridges were built over the river. A map of Königsgberg, showing the four land regions (labeled A, B, C, D) and the location of the river and the seven bridges (labeled a, b, c, d, e, f, g), is shown in Figure 3.1.

Königsberg played an interesting role in the origin of graph theory. The story goes that during the 1730s the citizens of Königsgberg enjoyed strolling about the city. Some citizens wondered whether it was possible to go for a walk in the city and pass over each bridge exactly once. This problem eventually became known as the Königsgberg Bridge Problem.

Undoubtedly, the greatest mathematician of the 18th century was Leonhard
Euler from Switzerland. Euler was known to correspond with many people, mathematicians and non-mathematicians alike. One of the people with whom Euler corresponded was Carl Leonhard Gottlieb Ehler, who was the mayor of the city of Danzig in Prussia. Danzig is located approximately 80 miles west of Königsberg. Ehler was aware of the Königsberg Bridge Problem, which had become known to many people in the area. It is believed that it was Ehler who wrote to Euler mentioning this problem to him. In fact, in a letter dated March 9, 1736, Ehler asked Euler if he would send him a solution of the problem.

Initially, Euler did not think that this problem was particularly mathematical in nature but when he discovered a method for solving not only the Königsberg Bridge Problem but a generalization of the problem, he wrote a paper on this subject, describing the problem and its generalization and presenting his proof. In particular, Euler showed, mathematically, that it was not possible to walk about Königsberg and pass over each bridge exactly once.

Euler’s proof, indeed Euler’s paper, neither mentioned graphs nor even hinted at
graphs. Nevertheless, Euler’s reasoning was graph theoretic in nature. In fact, the term “graph” as used in this context was evidently not introduced until 1878 when the British mathematician James Joseph Sylvester first used this word. That the map of Königsberg can be represented by a graph (actually a multigraph since some pairs of vertices are joined by more than one edge) is apparent (see Figure 3.2).

![Figure 3.2: A multigraph representing Königsberg](image)

In 1736 Euler proved the following result, which we state in the language of graph theory. Suppose that there is a city consisting of land regions, some pairs of which are joined by one or more bridges. Then this town can be represented by a graph or multigraph $G$ whose vertices are the land regions where every two vertices of $G$ are joined by a number of edges equal to the number of bridges joining the corresponding land regions. Then there is a walk in the town passing over each bridge exactly once if and only if $G$ is connected and every vertex of $G$ has even degree.
3.2 A Colored Bridges Problem

There is another problem concerning traveling about a town with bridges that also has a graph theory representation and was inspired by a problem in geographical analysis and transportation networks and the Königsberg Bridge Problem.

The Colored Bridges Problem A town has a network of walkways running through it, each of which is colored gray. A red bridge is constructed between every two walkway intersections that are two blocks apart and a blue bridge is constructed between every two intersections that are three blocks apart. Is it possible to take a round trip about the town that passes through every intersection in the town exactly once and such that during this trip the walkway or bridge used to enter every intersection is of a different color than that used to exit it?

As one would probably expect, the answer to the question posed in the Colored Bridges Problem depends on the structure of the network of walkways in the town being considered. For example, the business area of a town might consist of a collection of rectangular blocks with two parallel walkways running east and west and three parallel walkways running north and south. Figure 3.3(a) shows such a situation, where A, B, C, D, E, F are the six intersections in this case. Figure 3.3(b) shows the six red bridges that are constructed between each pair of intersections that are two blocks apart and Figure 3.3(c) shows the two blue bridges that are constructed between each pair of intersections that are three blocks apart. At the end, the bridges shown in Figure 3.3(b) and in Figure 3.3(c) will all be present in
the town.

Figure 3.3: A portion of a town
This problem can be stated as a problem in graph theory. Consider a network \( N \) of walkways in a town. This can be represented by a connected graph \( G \) whose vertices are the intersections in \( N \) and where two vertices \( u \) and \( v \) of \( G \) are joined by an edge if \( u \) and \( v \) correspond to two intersections connecting a walkway passing through no intermediate intersection. Color each edge of \( G \) gray. If the distance between two vertices \( u \) and \( v \) of \( G \) is 2, then add an edge between \( u \) and \( v \) colored red. If the distance between \( u \) and \( v \) is 3, then add an edge between \( u \) and \( v \) colored blue. Hence the resulting graph is \( G^3 \), where each edge of \( G^3 \) is colored gray, red or blue. For this town, the Colored Bridges Problem has a solution if and only if the distance-colored graph \( G^3 \) has a properly colored Hamiltonian cycle.

Perhaps the most common structure for a network of walkways in a town is when all blocks in town are rectangular, having \( n \) walkways running north and south, say, and \( m \) walkways running east and west, that is, the walkways in the town form a grid. In the particular case of the town shown in Figure 3.3(a), \( n = 3 \) and \( m = 2 \). The graph \( G \) representing the network of walkways in this town is shown in Figure 3.4(a), while Figure 3.4(b) shows the graph \( G^2 \) representing the town with the six red bridges in Figure 3.3(b). Figure 3.4(c) shows the graph \( G^3 \) representing the town with both the red and blue bridges. Here, \( g \), \( r \) and \( b \) indicate the colors gray, red and blue, respectively.

The Colored Bridges Problem has a solution for the town in Figure 3.3 since there is a properly colored Hamiltonian cycle in the graph of Figure 3.4(c). Such a cycle is shown in Figure 3.5.
3.3 The Colored Bridges Problem for Grids

The Colored Bridges Problem has a connection with distance-colored graphs. We have already seen that the graph $G^3$ in Figure 3.4(c) is Hamiltonian-colored and
so the Colored Bridges Problem has a solution for the town in Figure 3.3(a). The graph $G$ in Figure 3.4(a) is often denoted by $P_3 \square P_2$ and called the *Cartesian product* of the paths $P_3$ and $P_2$. We now show that the Colored Bridges Problem always has a solution whenever the walkways in the town form a grid and so the graph representation of the town is $P_n \square P_m$ for some integers $n, m \geq 2$.

Before verifying that the distance-colored graph $(P_n \square P_m)^3$ is Hamiltonian-colored for all integers $n, m \geq 2$, we first observe (in Figure 3.6) that this is the case when $n, m \in \{2, 3, 5, 6\}$, where as in Figure 3.6, a bold line represents a gray edge, a solid line represents a red edge and a dashed line represents a blue edge. (We have already observed this in Figure 3.5 when $n = 3$ and $m = 2$.)

We have seen that for each integer $n \geq 4$, the distance-colored graph $P_n^3$ is Hamiltonian-colored. In the following lemma, we provide an alternative proof of this result for $n \geq 7$ that shows that $P_n^3$ has a properly colored Hamiltonian cycle possessing four specified edges.

**Lemma 3.3.1** For each integer $n \geq 7$, let the path $P_n = (v_1, v_2, \ldots, v_n)$. Then the distance-colored graph $P_n^3$ has a properly colored Hamiltonian cycle containing the four edges $v_1v_2$, $v_{n-3}v_{n-1}$, $v_{n-2}v_n$ and $v_{n-1}v_n$.

**Proof.** We begin by observing that for each $n \in \{4, 7, 9, 10\}$, the graph $P_n^3$ has a properly colored Hamiltonian cycle containing the four edges $v_1v_2$, $v_{n-3}v_{n-1}$, $v_{n-2}v_n$ and $v_{n-1}v_n$ (see Figure 3.7(a)). Next, observe that $P_8^3$ has this property as well. This can be seen by adding the four vertices $v_5, v_6, v_7$ and $v_8$ to the properly colored
Thus the distance-colored graphs $P^3_7$, $P^3_8$, $P^3_9$, and $P^3_{10}$ possess properly colored Hamiltonian cycles with the specified edges. Assume, for an integer $n \geq 11$, that all distance-colored graphs $P^3_k$, where $7 \leq k < n$ possess properly colored Hamiltonian cycles containing the specified edges. Consider the distance-colored graph $P^3_n$. By the induction hypothesis, $P^3_{n-4}$ possesses a properly colored Hamiltonian cycle
Figure 3.7: Properly colored Hamiltonian cycles in the distance-colored graph $P^3_n$ for $n \in \{4, 7, 8, 9, 10\}$

$C$ containing the four edges $v_1v_2, v_{n-7}v_{n-5}, v_{n-6}v_{n-4}$ and $v_{n-5}v_{n-4}$. Proceeding as above, we construct a properly colored Hamiltonian cycle $C'$ in the distance-colored graph $P^3_n$ by deleting the edge $v_{n-5}v_{n-4}$ from $C$ and adding the edges $v_{n-5}v_{n-2}$, $v_{n-4}v_{n-3}$, $v_{n-3}v_{n-1}$, $v_{n-2}v_n$ and $v_{n-1}v_n$. $\blacksquare$

We now verify the Colored Bridges Problem for all towns represented by a graph $P_n \square P_m$ for all integers $n, m \geq 2$.

**Theorem 3.3.2** For every two integers $n, m \geq 2$, the distance-colored graph $(P_n \square P_m)^3$ is Hamiltonian-colored.

**Proof.** We have already seen that this is true when $n, m \in \{2, 3, 5, 6\}$. Thus, we may assume that at least one of $n$ and $m$ does not belong to $\{2, 3, 5, 6\}$, say $n \notin \{2, 3, 5, 6\}$. Therefore, either $n = 4$ or $n \geq 7$. Hence $P_n \square P_m$ contains $m$ copies of $P_n = (v_1, v_2, \ldots, v_n)$ and $(P_n \square P_m)^3$ contains $m$ copies $G_i$ ($1 \leq i \leq m$) of $P^3_n$. By Lemma 3.3.1, $G_i$ possesses a properly colored Hamiltonian cycle $C_i$. 
containing the four edges $v_1v_2, v_{n-3}v_{n-1}, v_{n-2}v_n$ and $v_{n-1}v_n$. Next, we delete the edge $v_1v_2$ from each cycle $C_i$, unless $m$ is odd in which case the edge $v_1v_2$ is not deleted from $C_m$. In addition, the edge $v_{n-1}v_n$ is deleted from $C_i$ for $1 < i < m$ and from $C_m$ as well if $m$ is odd. For each odd integer $i$ with $1 \leq i < m$, the two vertices $v_1$ and the two vertices $v_2$ are joined by an edge in $C_i$ and $C_{i+1}$; while for each even integer $i$ with $2 \leq i < m$, the two vertices $v_{n-1}$ and the two vertices $v_n$ are joined by an edge in $C_i$ and $C_{i+1}$. (See Figure 3.8 for the cases when $n = 7$ and $m = 6$.) This produces a properly colored Hamiltonian cycle in $(P_n \Box P_m)^3$. ■

\[ \text{Figure 3.8: A properly colored Hamiltonian cycle in the distance-colored graph } (P_7 \Box P_6)^3 \]

3.4 Restricted Colored Bridge Problems

As a consequence of Theorem 3.3.2, whenever a town can be represented by the graph $P_n \Box P_m$ for some $n, m \geq 2$, then it is always possible to take a round trip about the town that encounters each intersection in the town exactly once such
that the walkway or bridge entering every intersection is of a different color than that used to exit it. In fact there are some circumstances when no blue bridges, or no red bridges or only bridges are needed to solve the Colored Bridges Problem. In this section, we present necessary and sufficient conditions for such a round trip to exist when (1) no blue bridge is used, (2) no red bridge is used and (3) only bridges are used.

3.4.1 Restricted Colored Bridge Problem I

We now show that if a town can be represented by a certain grid, then it is always possible to take a round trip about the town that encounters each intersection in the town exactly once such that the walkway or bridge entering every intersection is of a different color than that used to exit it and no blue bridges are needed. We begin with a lemma.

Lemma 3.4.1 For each even integer \( n \geq 2 \), the square of \( P_n \square P_2 \) is Hamiltonian-colored.

Proof. Let \( G_n = P_n \square P_2 \). Suppose that \( G_n \) is constructed from two copies \((u_1, u_2, \ldots, u_n)\) and \((v_1, v_2, \ldots, v_n)\) of \( P_n \) by adding the edges \( u_iv_i \) for \( 1 \leq i \leq n \). First, we show by induction on even integers \( n \geq 2 \) that \( G_n^2 \) contains a properly colored Hamiltonian cycle \( C \) such that either

\[
(u_{n-1}, u_n, v_{n-1}, v_n) \text{ or } (u_n, u_{n-1}, v_n, v_{n-1})
\]

is a subpath of \( C \). This statement is true for \( 2 \leq n \leq 6 \), as shown in Figure 3.9.
Assume for some even integer $k \geq 6$ that the statement is true for $G_k^2$ and consider the distance-colored graph $G_{k+2}^2$. By the induction hypothesis, $G_k^2$ contains a properly colored Hamiltonian cycle $C'$ such that either $(u_{k-1}, u_k, v_{k-1}, v_k)$ or $(u_k, u_{k-1}, v_k, v_{k-1})$ is a subpath of $C'$. We now construct a properly colored Hamiltonian cycle $C$ of $G_{k+2}^2$ from the cycle $C'$ as follows:

(a) If $(u_{k-1}, u_k, v_{k-1}, v_k)$ is a subpath of $C'$, then $C$ can be constructed from $C'$ by replacing the edge $u_kv_{k-1}$ by the path $(u_k, u_{k+2}, u_{k+1}, v_{k+2}, v_{k+1}, v_{k-1})$. Then $(u_{k+2}, u_{k+1}, v_{k+2}, v_{k+1})$ is a subpath of $C$.

(b) If $(u_k, u_{k-1}, v_k, v_{k-1})$ is a subpath of $C'$, then $C$ can be constructed from $C'$ by replacing the edge $u_{k-1}v_k$ by the path $(u_{k-1}, u_{k+1}, u_{k+2}, v_{k+1}, v_{k+2}, v_k)$. Then $(u_{k+1}, u_{k+2}, v_{k+1}, v_{k+2})$ is a subpath of $C$.

Therefore, $G_{k+2}^2$ is Hamiltonian-colored.

**Theorem 3.4.2** For integers $n$ and $m$ with $n, m \geq 2$, the square of $P_n \square P_m$ is Hamiltonian-colored if and only if $nm \equiv 0 \pmod{4}$. 
\textbf{Proof}. Let $G = P_n \square P_m$. First, we show that if $nm \not\equiv 0 \pmod{4}$, then $G^2$ is not Hamiltonian-colored. Since $nm \not\equiv 0 \pmod{4}$, it follows that

(i) $n \not\equiv 0 \pmod{4}$ and $m \not\equiv 0 \pmod{4}$ and

(ii) at least one of $n$ and $m$ is odd.

If $n$ and $m$ are both odd, then the order $nm$ of $G$ (and $G^2$) is odd. Since the chromatic index of an odd cycle is 3, it follows that $G^2$ is not Hamiltonian-colored. Thus, exactly one of $n$ and $m$ is even, say $n$ is odd and $m$ is even. Then $n = 2p + 1$ and $m = 4q + 2$ where $p \geq 1$ and $q \geq 0$. Then $nm = 8pq + 4p + 4q + 2$. Assume, to the contrary, that $G^2$ is Hamiltonian-colored. Let $C$ be a properly colored Hamiltonian cycle in $G^2$. Thus the edges of $C$ are alternately colored 1 and 2. Since $G$ is bipartite, the vertex set of $G$ can be partitioned into two sets $X$ and $Y$, where then

$$|X| = |Y| = 4pq + 2p + 2q + 1$$

is odd. Suppose that

$$C = (z_1, z_2, \ldots, z_{nm}, z_{nm+1} = z_1).$$

We may assume that $z_1 \in X$ and $z_1z_2$ is colored 2. Thus $z_iz_{i+1}$ is colored 2 if $i$ is odd and $z_iz_{i+1}$ is colored 1 if $i$ is even, where $1 \leq i \leq nm$. This implies that $z_i \in X$ if $i \equiv 1, 2 \pmod{4}$ and $z_i \in Y$ if $i \equiv 3, 4 \pmod{4}$. Since $nm = 8pq + 4p + 4q + 2$, it follows that $nm + 1 \equiv 3 \pmod{4}$ and so $z_{nm+1} \in Y$, contradicting the fact that $z_{nm+1} = z_1 \in X$. Therefore, $G^2$ is not Hamiltonian-colored.
For the converse, suppose that \( nm \equiv 0 \pmod{4} \). We show that \( G^2 \) is Hamiltonian-colored. Since \( nm \equiv 0 \pmod{4} \), at least one of \( n \) and \( m \) is even, say \( m \) is even. Thus either \( m \equiv 0 \pmod{4} \) or \( m \equiv 2 \pmod{4} \). By Lemma 3.4.1, we may assume that \( m \geq 4 \). Hence either \( m = 4q \) or \( m = 4q + 2 \) for some positive integer \( q \). We consider these two cases.

**Case 1.** \( m = 4q \) where \( q \geq 1 \). By Lemma 3.4.1, we may assume that \( n \geq 3 \). Suppose that \( G = P_n \square P_m \) is constructed from \( m \) copies \( F_1, F_2, \ldots, F_m \) of \( P_n \) where

\[
F_i = (v_{i,1}, v_{i,2}, \ldots, v_{i,n})
\]

for \( 1 \leq i \leq m \) by adding the edges

\[
v_{i,1} v_{i+1,1}, v_{i,2} v_{i+1,2}, \ldots, v_{i,n} v_{i+1,n}
\]

for \( 1 \leq i \leq m - 1 \). For each integer \( j \) with \( 1 \leq j \leq p - 1 \), let \( H_j = P_n \square P_{4p} \) be the induced subgraph of \( G \) with

\[
V(H_j) = V(F_{4j}) \cup V(F_{4j+1}) \cup V(F_{4j+2}) \cup V(F_{4j+3}).
\]

For each integer \( j \) with \( 1 \leq j \leq p - 1 \), define a \((4n)\)-cycle \( C_j \) in the square of \( H_j \) by

\[
C_j = (v_{4j,1}, v_{4j,2}, v_{4j+1,1}, v_{4j+1,2}, v_{4j+2,1}, v_{4j+2,2}, v_{4j+3,1}, v_{4j+3,2}, v_{4j+2,3}, v_{4j+3,3}, v_{4j+2,4}, v_{4j+3,4}, \ldots, v_{4j+2,n}, v_{4j+3,n}, v_{4j+1,n}, v_{4j,n}, v_{4j+1,n-1}, v_{4j,n-1}, \ldots, v_{4j+1,3}, v_{4j,3}, v_{4j+1,1}).
\]
Then $C_j$ is a properly colored Hamiltonian cycle in $H_j^2$ for $1 \leq j \leq p - 1$. Figure 3.10(a) shows such a properly colored Hamiltonian cycle in the square of $P_n \square P_4$ for each $n \in \{3, 9\}$. Then a properly colored Hamiltonian cycle $C$ in $G^2$ can be constructed from these $p - 1$ cycles $C_1, C_2, \ldots, C_{p-1}$ by

(1) deleting the edges $v_{4j+3,2}v_{4j+2,3}$ and $v_{4j+4,1}v_{4j+4,3}$ for $1 \leq j \leq p - 1$ and

(2) adding the edges $v_{4j+4,1}v_{4j+3,2}$ and $v_{4j+4,3}v_{4j+2,3}$ for $1 \leq j \leq p - 1$.

For each $n \in \{3, 9\}$, Figure 3.10(b) shows such a properly colored Hamiltonian cycle in the square of $P_n \square P_8$ that is constructed from the two properly colored Hamiltonian cycles in the squares of the two copies of $P_n \square P_4$. In Figure 3.10(b), the dashed lines indicate the deleted edges and the bold lines indicate the added edges between the squares of the two copies of $P_n \square P_4$ for each $n \in \{3, 9\}$.

**Case 2.** $m = 4q + 2$ where $q \geq 1$. Since $nm \equiv 0 \pmod{4}$, it follows that $n$ is even. By Lemma 3.4.1, we may assume that $n \geq 4$. Suppose that $G = P_n \square P_m$ is constructed from $m$ copies $F_1, F_2, \ldots, F_m$ of $P_n$, as described in Case 1. Let $H = P_n \square P_2$ be the induced subgraph of $G$ with $V(H) = V(F_1) \cup V(F_2)$. Construct a properly colored Hamiltonian path $P$ in $H^2$ with the initial vertex $v_{2,2}$ and the terminal vertex $v_{2,4}$ such that

(a) if $n = 4$, then $P = (v_{2,2}, v_{2,1}, v_{1,2}, v_{1,1}, v_{1,3}, v_{2,3}, v_{1,4}, v_{2,4})$;
Figure 3.10: Properly colored Hamiltonian cycles in the squares of $P_n \square P_4$ and $P_n \square P_8$ for $n \in \{3, 9\}$

(b) if $n \geq 6$, then

$$P = \langle v_{2,2}, v_{2,1}, v_{1,2}, v_{1,1}, v_{1,3}, v_{2,3}, v_{1,4}, v_{1,5}, v_{2,6}, v_{2,7},$$

$$v_{1,8}, v_{1,9}, v_{2,10}, v_{2,11}, \ldots, v_{1,n-2}, v_{1,n-1}, v_{2,n}, v_{1,n}$$

$$v_{2,n-1}, v_{2,n-2}, v_{1,n-3}, v_{1,n-4}, \ldots, v_{1,7}, v_{1,6}, v_{2,5}, v_{2,4} \rangle.$$

Figure 3.11 shows such a properly colored Hamiltonian path $P$ in the square of $P_n \square P_2$ for $n \in \{4, 6, 8, 10\}$. 
Next, let $H' = P_n \square P_{4p}$ be the induced subgraph of $G$ with

$$V(H') = V(F_3) \cup V(F_4) \cup \cdots \cup V(F_{4p+2})$$

and let $C$ be the properly colored Hamiltonian cycle in the square of $H'$ that is constructed as described in Case 1. Then a properly colored Hamiltonian cycle in $G^2$ can be constructed from $P$ and $C$ by (1) deleting the edge $v_{3,1}v_{3,3}$ and (2) adding the two edges $v_{3,1}v_{2,2}$ and $v_{3,3}v_{2,4}$. For each $n \in \{4, 10\}$, Figure 3.12 shows such a properly colored Hamiltonian cycle in the square of $P_n \square P_6$ that is constructed from the properly colored Hamiltonian path $P$ in the square of $P_n \square P_2$ and the properly colored Hamiltonian cycle $C$ in the square of $P_n \square P_4$. In Figure 3.12, the dashed lines indicates the deleted edges and the bold lines indicate the added edges. Therefore, $G^2$ is Hamiltonian-colored. \[\blacksquare\]
3.4.2 Restricted Colored Bridge Problem II

We now show that whenever a town can be represented by the graph $P_n \square P_m$ for some $n, m \geq 2$ under certain conditions, then it is possible to take a round trip about the town that encounters each intersection in the town exactly once such that the walkway or bridge entering every intersection is of a different color than that used to exit it and no red bridges are needed.

**Theorem 3.4.3**  For integers $n, m \geq 2$, the distance-colored graph $(P_n \square P_m)^3$ contains a properly colored Hamiltonian cycle using only the colors 1 and 3 if and only if

$$nm \text{ is even unless } n = m = 2 \text{ or } \{n, m\} = \{2, 3\} \text{ or } \{2, 7\}.$$

**Proof.** First, observe that if $nm$ is odd, then no Hamiltonian cycle in $(P_n \square P_m)^3$ can be properly colored with only two colors. If $n = m = 2$, then the diameter of $P_n \square P_m$ is 2 and no edge in $(P_n \square P_m)^3$ is colored 3. If $\{n, m\} = \{2, 3\}$,
then there are only two edges colored 3 in \((P_n \square P_m)^3\) and so no properly colored Hamiltonian cycle using the colors 1 and 3 exists in \((P_n \square P_m)^3\).

Next, we show the distance-colored graph \((P_2 \square P_7)^3\) does not contains a properly colored Hamiltonian cycle using only the colors 1 and 3. Assume, to the contrary, that such a properly colored Hamiltonian cycle \(C\) exists. Let the vertices of \(P_2 \square P_7\) be labeled as shown in Figure 3.13. Necessarily, each vertex of \(C\) is incident with an edge colored 1 and an edge colored 3. We consider two cases, according to the edge of \(C\) colored 3 that is incident with the vertex \(x_1\).

Figure 3.13: The graph \(P_2 \square P_7\)

**Case 1. The edge \(x_1x_4\) is on \(C\).** (See Figure 3.14, where we always represent an edge colored 1 by a solid edge and an edge colored 3 by a dotted edge.) Since the only possible edges on \(C\) that are colored 3 and incident with \(y_2\) are \(y_2x_4\) and \(y_2y_5\), the cycle \(C\) must contain \(y_2y_5\). Since the only possible edges on \(C\) that are colored 3 and incident with \(x_7\) are \(x_4x_7\) and \(y_5x_7\), a contradiction is produced.

Figure 3.14: The situation where \(C\) contains the edge \(x_1x_4\)
Since Case 1 cannot occur, none of the edges \( x_1x_4, y_1y_4, x_4x_7 \) and \( y_4y_7 \) can belong to \( C \).

**Case 2. All of the edges** \( x_1y_3, y_1x_3, x_5y_7 \) **and** \( y_5x_7 \) **belong to** \( C \). (See Figure 3.15.)

Since the only possible edges on \( C \) that are colored 3 and incident with \( x_2 \) are \( x_2x_5 \) and \( x_2y_4 \), the edge \( x_2y_4 \) must belong to \( C \). Similarly, the only possible edges on \( C \) that are colored 3 and incident with \( x_6 \) are \( x_3x_6 \) and \( y_4x_6 \). Therefore, the edge \( y_4x_6 \) must belong to \( C \), which produces a contradiction. 

\[ \begin{array}{cccccccc}
  x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\
  \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc \\
  y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 
\end{array} \]

**Figure 3.15:** The situation where \( C \) contains the edges \( x_5y_7, x_1y_3, y_1y_3 \), and \( y_5y_7 \)

For the converse, assume that \( n \geq 2 \) and \( m \geq 2 \) are integers such that \( nm \) is even and such that neither \( n = m = 2 \) nor \( \{n, m\} = \{2, 3\} \) or \( \{2, 7\} \) occurs. We show that the distance-colored graph \( (P_n \square P_m)^3 \) contains a properly colored Hamiltonian cycle using only the colors 1 and 3. Since \( nm \) is even, we may assume that \( n \) is even. The graph \( P_n \square P_m \) consists of \( n \) paths \( P_m \) which we denote by

\[ P_{m,i} = (v_{i,1}, v_{i,2}, \ldots, v_{i,m}) \]

for \( 1 \leq i \leq n \) such that \( v_{i,t} \) is adjacent to \( v_{j,t} \) (\( 1 \leq t \leq m \)) when \( |i - j| = 1 \). With this labeling, the graph \( P_2 \square P_3 \) is shown in Figure 3.16.

We first show by induction on \( m \geq 4 \) with \( m \neq 7 \) that the distance-colored graph \( (P_2 \square P_m)^3 \) has a properly colored Hamiltonian cycle using only the colors
1 and 3 and containing the edges $v_{1,m}v_{2,m}$, $v_{1,m-2}v_{1,m-1}$ and $v_{2,m-2}v_{2,m-1}$. First, observe that this is true for $m \in \{4, 5, 6, 11\}$ as shown in Figure 3.17. (Again, the solid edges are colored 1 and dotted edges are colored 3.)

For $m \geq 4$ with $m \neq 7$, assume that $(P_2 \Box P_m)^3$ has a properly colored Hamiltonian cycle $C''$ using the colors 1 and 3 and containing the edges $v_{1,m}v_{2,m}$, $v_{1,m-2}v_{1,m-1}$ and $v_{2,m-2}v_{2,m-1}$. For the properly colored Hamiltonian cycle $C''$ in $(P_2 \Box P_4)^3$ shown in Figure 3.18, we delete the edge $v_{1,m}v_{2,m}$ from $C'$ and the edge $v_{1,m+1}v_{2,m+1}$ from $C''$ and add the edges $v_{1,m}v_{1,m+1}$ and $v_{2,m}v_{2,m+1}$ to produce a properly colored Hamiltonian cycle $C$ in $(P_2 \Box P_{m+4})^3$ containing the edges $v_{1,m+4}v_{2,m+4}$, $v_{1,m+2}v_{1,m+3}$ and $v_{2,m+2}v_{2,m+3}$. 
Therefore, \((P_2 \boxtimes P_m)^3\) contains a properly colored Hamiltonian cycle using the colors 1 and 3 and containing the edges \(v_{1,m}v_{2,m}, v_{1,m-2}v_{1,m-1}\) and \(v_{2,m-2}v_{2,m-1}\) for every integer \(m \geq 4\) with \(m \neq 7\).

Next, we show by induction on even integers \(n \geq 2\) that except for \(n = m = 2\) and \(\{n,m\} = \{2,3\}, \{2,7\}\) the distance-colored graph \((P_n \boxtimes P_m)^3\) has a properly colored Hamiltonian cycle using the colors 1 and 3 and containing the edge \(v_{n,m-2}v_{n,m-1}\). We have seen that this is true for \(n = 2\). Assume first that this is true for an even integer \(n \geq 2\) and an integer \(m \geq 2\) where \(m \notin \{2,3,7\}\). Then there exists a properly colored Hamiltonian cycle \(C'\) in \((P_n \boxtimes P_m)^3\) using the colors 1 and 3 and containing the edge \(v_{n,m-2}v_{n,m-1}\). Consider the graph \(P_2 \boxtimes P_m\) whose vertices are labeled as in Figure 3.19.
We have seen that when \( m \not\in \{2, 3, 7\} \), there is a properly colored Hamiltonian cycle \( C'' \) in \((P_2 \square P_m)^3 \) using only the colors 1 and 3 and containing the edges \( v_{n+1,m-2}v_{n+1,m-1} \) and \( v_{n+2,m-2}v_{n+2,m-1} \). We now delete the edge \( v_{n,m-2}v_{n,m-1} \) of \( C' \) and the edge \( v_{n+1,m-2}v_{n+1,m-1} \) of \( C'' \) and add the two edges \( v_{n,m-2}v_{n+1,m-2} \) and \( v_{n,m-1}v_{n+1,m-1} \), producing a properly colored Hamiltonian cycle \( C \) in \((P_{n+2} \square P_m)^3 \) using only the colors 1 and 3 and containing the edge \( v_{n+2,m-2}v_{n+2,m-1} \). Consequently, for every even integer \( n \geq 2 \) and every integer \( m \geq 2 \) with \( m \not\in \{2, 3, 7\} \), there is a properly colored Hamiltonian cycle using only the colors 1 and 3 in \((P_n \square P_m)^3 \).

To complete the proof, it remains to show that \((P_n \square P_3)^3 \) and \((P_n \square P_7)^3 \) contain properly colored Hamiltonian cycles using only the colors 1 and 3 for each even integer \( n \geq 4 \). We verify by induction on even integers \( n \geq 4 \) that such a cycle exists containing the edge \( v_{n,m-2}v_{n,m-1} \) for \( m = 3 \) and \( m = 7 \). That this is true for \( n = 4 \) and \( n = 6 \) is shown in Figure 3.20 (where the edges \( v_{n,m-2}v_{n,m-1} \) are drawn in bold).

Assume for an even integer \( k \geq 6 \) that \((P_3 \square P_3)^3 \) and \((P_3 \square P_7)^3 \) contain properly colored Hamiltonian cycles using only the colors 1 and 3 and containing the edge \( v_{j,m-2}v_{j,m-1} \) for every even integer \( j \) with \( 4 \leq j \leq k \) and \( m \in \{3, 7\} \). We show that \((P_{k+2} \square P_3)^3 \) and \((P_{k+2} \square P_7)^3 \) contain properly colored Hamiltonian cycles using only the colors 1 and 3 and containing the edge \( v_{k+2,m-2}v_{k+2,m-1} \) for \( m \in \{3, 7\} \). From the induction hypothesis, it follows that \((P_{k-2} \square P_3)^3 \) and \((P_{k-2} \square P_7)^3 \) contain properly colored Hamiltonian cycles \( C'_1 \) and \( C'_2 \), respectively,
using only the colors 1 and 3 and containing the edge \( v_{n,m-2}v_{n,m-1} \) for four pairs \((n, m)\).

Each of the graphs \((P_4 \Box P_3)^3\) and \((P_4 \Box P_7)^3\) consists of four paths \( P_m \) \((m \in \{3, 7\})\) which we denote by

\[
P_{m,i} = (v_{k+i+2,1}, v_{k+i+2,2}, \ldots, v_{k+i,m})
\]

for \(-1 \leq i \leq 2\) such that \( v_{i,t} \) is adjacent to \( v_{j,t} \) \((1 \leq t \leq m)\) when \(|i - j| = 1\). Then \((P_4 \Box P_3)^3\) and \((P_4 \Box P_7)^3\) contain properly colored Hamiltonian cycles \( C_i'' \) and \( C_2'' \), respectively, using only the colors 1 and 3 and containing the edge \( v_{k-1,m-2}v_{k-1,m-1} \) and \( v_{k+2,m-2}v_{k+2,m-1} \) for \( m \in \{3, 7\} \). Deleting the edge \( v_{k-2,m-2}v_{k-2,m-1} \) of \( C_i' \) and the edge \( v_{k-1,m-2}v_{k-1,m-1} \) of \( C_i'' \) for \( i = 1, 2 \) and adding the edges \( v_{k-2,m-2}v_{k-1,m-2} \)
and $v_{k-2,m-1}v_{k-1,m-1}$ result in properly colored Hamiltonian cycles $C_1$ and $C_2$ in $(P_{k+2} \square P_3)^3$ and $(P_{k+2} \square P_2)^3$, respectively, using only the colors 1 and 3 and containing the edge $v_{k+2,m-2}v_{k+2,m-1}$ for $m \in \{3, 7\}$, completing the proof.

### 3.4.3 Restricted Colored Bridge Problem III

We now show that if a town can be represented by the graph $P_n \square P_m$ for some $n, m \geq 2$ under certain conditions, then it is possible to take a round trip about the town that encounters each intersection in the town exactly once such that the walkway or bridge entering every intersection is of a different color than that used to exit it and only bridges are used.

**Theorem 3.4.4** For integers $n, m \geq 2$, the distance-colored graph $(P_n \square P_m)^3$ contains a properly colored Hamiltonian cycle using only the colors 2 and 3 if and only if $nm \equiv 0 \mod 4$ unless $n = m = 2$.

**Proof.** Suppose first that $(P_n \square P_m)^3$ contains a properly colored Hamiltonian cycle $C$ using only the colors 2 and 3. Since the diameter of $P_2 \square P_2$ is 2, it is impossible that $n = m = 2$. Let

$$C = (v_1, v_2, \ldots, v_{nm-1}, v_{nm}, v_1).$$

Since $P_n \square P_m$ is bipartite, it has two partite sets $A$ and $B$. We may assume that $v_1 \in A$ and that $v_1v_2$ on $C$ is colored 2. Thus the distance between $v_1$ and $v_2$ in $P_n \square P_m$ is 2 and $v_2 \in A$ as well. Since $v_2v_3$ is colored 3 and $v_3v_4$ is colored 2, both $v_3$ and $v_4$ belong to $B$. Because $v_4v_5$ is colored 3, it follows that $v_5 \in A$. This
implies that the $nm$ vertices of $P_n \boxtimes P_m$ are encountered cyclically on $C$ in groups of 4, the first pair of which belongs to $A$ and the second pair of which belongs to $B$. Thus $nm \equiv 0 \pmod{4}$.

For the converse, let $n, m \geq 2$ be integers with $nm \equiv 0 \pmod{4}$ such that $(n, m) \neq (2, 2)$. We show that the distance-colored graph $(P_n \boxtimes P_m)^3$ contains a properly colored Hamiltonian cycle using only the colors 2 and 3. Since $nm$ is even, we may assume that $n$ is even. The graph $P_n \boxtimes P_m$ consists of $n$ paths of order $m$, which we denote by

$$P_{m,i} = (v_{i,1}, v_{i,2}, \ldots, v_{i,m})$$

for $1 \leq i \leq n$ such that $v_{i,t}$ is adjacent to $v_{j,t}$ ($1 \leq t \leq m$) when $|i - j| = 1$.

We first show by induction on the even integers $m \geq 4$ that the distance-colored graph $(P_2 \boxtimes P_m)^3$ contains a properly colored Hamiltonian cycle using only the colors 2 and 3 and containing the edges $v_{1,1}v_{2,2}$ and $v_{1,m-1}v_{2,m}$ colored 2. That this is true for $m = 4$ and $m = 6$ is shown in Figure 3.21 (where the solid edges are colored 2 and dotted edges are colored 3).

![Figure 3.21: Properly colored Hamiltonian cycles in $(P_2 \boxtimes P_m)^3$ for $m = 4, 6$ using only the colors 2 and 3](image)

Assume for an even integer $m \geq 6$ that for every even integer $t$ with $4 \leq t \leq m$, the distance-colored graph $(P_2 \boxtimes P_t)^3$ contains a properly colored Hamiltonian cy-
We claim that the distance-colored graph \((P_2 \sqcap P_{m+2})^3\) contains a properly colored Hamiltonian cycle using only the colors 2 and 3 and containing the edges \(v_{1,1}v_{2,2}\) and \(v_{1,t-1}v_{2,t}\). By the induction hypothesis, the distance-colored graph \((P_2 \sqcap P_{m-2})^3\) contains a properly colored Hamiltonian cycle \(C'\) using only the colors 2 and 3 and containing the edges \(v_{1,1}v_{2,2}\) and \(v_{1,m+1}v_{2,m+2}\). The graph \((P_2 \sqcap P_4)^3\) consists of two paths \(P_4\) which we denote by

\[
(v_{1,m-1}, v_{1,m}, v_{1,m+1}, v_{1,m+2}) \quad \text{and} \quad (v_{2,m-1}, v_{2,m}, v_{2,m+1}, v_{2,m+2}),
\]

respectively, as shown in Figure 3.22 where the vertices of \((P_2 \sqcap P_4)^3\) are drawn as solid vertices. Let \(C''\) be the cycle in \((P_2 \sqcap P_4)^3\) also shown in Figure 3.22. We now construct a properly colored Hamiltonian cycle \(C\) from \(C'\) and \(C''\) by removing the edge \(v_{1,m-3}v_{2,m-2}\) from \(C'\) and the edge \(v_{1,m-1}v_{2,m}\) from \(C''\) and adding the edges \(v_{1,m-3}v_{2,m-1}\) and \(v_{2,m-2}v_{2,m}\). The cycle \(C\) uses only colors 2 and 3 and contains the edges \(v_{1,1}v_{2,2}\) and \(v_{1,m+1}v_{2,m+2}\). This verifies the claim.

Next, we show by induction on the even integers \(n \geq 2\) that for every even integer \(m \geq 2\), except for \(m = 2\) when \(n = 2\), the distance-colored graph \((P_n \sqcap P_m)^3\) contains a properly colored Hamiltonian cycle using only the colors 2 and 3 and containing the edge \(v_{n,1}v_{n,3}\) colored 2. We have already seen that this is true for \(n = 2\). Assume that this is true for an even integer \(n \geq 2\) and every even integer \(m \geq 2\) where \((n, m) \neq (2, 2)\). Then for each even integer \(m \geq 2\), there is a properly colored Hamiltonian cycle \(C''\) in \((P_n \sqcap P_m)^3\) using only the colors 2 and 3 and
Figure 3.22: Constructing a properly colored Hamiltonian cycle in \((P_2 \square P_{m+2})^3\) using only the colors 2 and 3 and containing the edges \(v_{1,1}v_{2,2}\) and \(v_{1,m+1}v_{2,m+2}\) containing the edge \(v_{n,1}v_{n,3}\). For this integer \(m\), let \(P_2 \square P_m\) consist of two paths \(P_m\) which we denote by

\[(v_{n+1,1}, v_{n+1,2}, \ldots, v_{n+1,m}) \quad \text{and} \quad (v_{n+2,1}, v_{n+2,2}, \ldots, v_{n+2,m}),\]

respectively, and \(v_{n+1,i}\) is adjacent to \(v_{n+2,i}\) for \(1 \leq i \leq m\). We have seen that there is a properly colored Hamiltonian cycle \(C''\) in \((P_2 \square P_m)^3\) using only the colors 2 and 3 and containing the edges \(v_{n+1,2}v_{n+1,4}\) and \(v_{n+2,1}v_{n+2,3}\). The cycle in \((P_{n+2} \square P_m)^3\) obtained by deleting the edges \(v_{n,1}v_{n,3}\) and \(v_{n+1,2}v_{n+1,4}\) from \(C'\) and \(C''\), respectively, and adding the edges \(v_{n,1}v_{n+1,2}\) and \(v_{n,3}v_{n+1,4}\) produces a properly colored Hamiltonian cycle using only the colors 2 and 3 and containing the edge \(v_{n+2,1}v_{n+2,3}\). It therefore follows that for every two even integers \(n \geq 2\) and \(m \geq 2\) with \((n, m) \neq (2, 2)\), the distance-colored graph \((P_n \square P_m)^3\) contains a properly colored Hamiltonian cycle using only the colors 2 and 3.

To complete the proof, it remains to show that the distance-colored graph \((P_n \square P_m)^3\) contains a properly colored Hamiltonian cycle using only the colors 2
and 3 when \( n \equiv 0 \pmod{4} \) and \( m \geq 3 \) is odd or equivalently, when \( n \geq 3 \) is odd and \( m \equiv 0 \pmod{4} \). We verify this latter formulation of the statement.

First, we show by induction on the integers \( m \geq 4 \) with \( m \equiv 0 \pmod{4} \) that the distance-colored graph \((P_3 \ □ \ P_m)^3\) has a properly colored Hamiltonian cycle using only the colors 2 and 3. We use the same notation as before for the vertex set of \( P_3 \ □ \ P_m \). In fact, we show that \((P_3 \ □ \ P_m)^3\) has a properly colored Hamiltonian cycle using only the colors 2 and 3 and containing the two edges \( v_{1,1}v_{2,2} \) and \( v_{1,m-1}v_{2,m} \), both colored 2. That this is true for \( m = 4 \) is shown in Figure 3.23.
Assume, for an integer \( m \geq 4 \) with \( m \equiv 0 \pmod{4} \) that the distance-colored graph \((P_3 \boxtimes P_m)^3\) has a properly colored Hamiltonian cycle \( C' \) using only the colors 2 and 3 and containing the edges \( v_{1,1}v_{2,2} \) and \( v_{1,3}v_{2,4} \). Let \( P_3 \boxtimes P_4 \) consist of three paths \( P_4 \) which we denote by

\[
(v_{i,m+1}, v_{i,m+2}, v_{i,m+3}, v_{i,m+4})
\]

for \( i = 1, 2, 3 \). We have seen that the distance-colored graph \((P_3 \boxtimes P_4)^3\) has a properly colored Hamiltonian cycle \( C'' \) using only the colors 2 and 3 and containing the edges \( v_{1,m+1}v_{2,m+2} \) and \( v_{1,m+3}v_{2,m+4} \). Then the cycle \( C \) in \((P_3 \boxtimes P_{m+4})^3\) constructed by deleting the edges \( v_{1,m-1}v_{2,m} \) and \( v_{1,m+1}v_{2,m+2} \) from \( C' \) and \( C'' \), respectively, and adding the edges \( v_{1,m-1}v_{1,m+1} \) and \( v_{2,m}v_{2,m+2} \) (both colored 2) is a properly colored Hamiltonian cycle using only the colors 2 and 3 and containing the edges \( v_{1,1}v_{2,2} \) and \( v_{1,m+3}v_{2,m+4} \). This verifies the statement. Furthermore, observe that the cycle \( C \) constructed also contains the edge \( v_{3,1}v_{3,3} \), which is colored 2.

We now show by induction on the odd integers \( n \geq 3 \) that for every integer \( m \geq 4 \) with \( m \equiv 0 \pmod{4} \) there exists a properly colored Hamiltonian cycle.
in the distance-colored graph \((P_n \boxplus P_m)^3\) that uses only the colors 2 and 3 and contains the edge \(v_{n,1}v_{n,3}\). We have already seen that this is true when \(n = 3\).

Assume for an odd integer \(n \geq 3\) and for every integer \(m \geq 4\) with \(m \equiv 0 \pmod{4}\) there exists a properly colored Hamiltonian cycle \(C'\) in the distance-colored graph \((P_n \boxplus P_m)^3\) that uses only the colors 2 and 3 and contains the edge \(v_{n,1}v_{n,3}\).

For an integer \(m \geq 4\) with \(m \equiv 0 \pmod{4}\), let \(P_2 \boxplus P_m\) consist of two paths \(P_m\) which we denote by

\[
P_{m,i} = (v_{i,1}, v_{i,2}, \ldots, v_{i,m})
\]

for \(i = n + 1, n + 2\). We have seen that \((P_2 \boxplus P_m)^3\) has a properly colored Hamiltonian cycle \(C''\) using only the colors 2 and 3 and containing the edges \(v_{n+1,2}v_{n+1,4}\) and \(v_{n+2,1}v_{n+2,3}\).

By deleting the edges \(v_{n,1}v_{n,3}\) and \(v_{n+1,2}v_{n+1,4}\) from \(C'\) and \(C''\), respectively, and adding the edges \(v_{n,1}v_{n+1,2}\) and \(v_{n,3}v_{n+1,4}\) (both colored 2), we obtain a properly colored Hamiltonian cycle \(C\) in the distance-colored graph \((P_{n+2} \boxplus P_m)^3\) using only the colors 2 and 3 and containing the edge \(v_{n+2,1}v_{n+2,3}\). This completes the proof.

In Theorem 3.4.2, all pairs \(n, m \geq 2\) of integers are determined for which the distance-colored graphs \((P_n \boxplus P_m)^3\) and \((P_n \boxplus P_m)^2\) contain a properly colored Hamiltonian cycle. In the second case, this is equivalent to determining all pairs \(n, m \geq 2\) of integers for which the distance-colored graph \((P_n \boxplus P_m)^3\) has a properly colored Hamiltonian cycle the colors of whose edges follow the permutation \((1 2)\).

Theorems 3.4.3 and 3.4.4 answered the question for the permutations \((1 3)\) and \((2 3)\) respectively. This might suggest another question: For which pairs \(n, m \geq 2\)
of integers, does the distance-colored graph $G(P_n \boxtimes P_m)^3$ contain a properly colored Hamiltonian cycle the colors of whose edges follow the permutation $(1 2 3)$? Of course, this is only possible if $nm \equiv 0 \pmod{3}$. Suppose that such a cycle $C = (v_1, v_2, \ldots, v_{nm-1}, v_{nm}, v_1)$ exists. Since $P_n \boxtimes P_m$ is bipartite, it has two partite sets $A$ and $B$. If $nm$ is even, then $|A| = |B|$; while if $nm$ is odd, then one of $|A|$ and $|B|$ exceeds the other by 1, say $|B| = |A| + 1$. We may assume that $v_1 \in A$ and that $v_1v_2$ is colored 1, which implies that $v_2 \in B$. Thus $v_2v_3$ is colored 2, $v_3v_4$ is colored 3 and $v_4v_5$ is colored 1. Therefore, $v_3 \in B$, $v_4 \in A$ and $v_5 \in B$. This implies that two-thirds of the vertices of $P_n \boxtimes P_m$ belong to $B$ and one-third to $A$, that is, $|B| = 2|A|$. This is impossible if $|A| = |B|$. If $|B| = |A| + 1$, then $|B| = 2$, $|A| = 1$ and $(P_2 \boxtimes P_1)^3$ is not Hamiltonian. Hence there are no integers $n, m \geq 2$ for which the distance-colored graph $(P_n \boxtimes P_m)^3$ has a properly colored Hamiltonian cycle the colors of whose edges follow the permutation $(1 2 3)$.
Chapter 4

Color Distance and Connectivity

4.1 Color-Connection Exponent and Color-Distance

For a connected graph $G$ and a positive integer $k$, the distance-colored graph $G^k$ is called properly color-connected, or simply color-connected, if every two vertices $u$ and $v$ in $G^k$ are connected by a properly colored $u-v$ path in $G^k$. If the diameter of $G$ is $d$, then $G^d$ is complete and so every two vertices $u$ and $v$ in $G^d$ are connected by the path $(u, v)$ of length 1 in $G^d$ and so $G^d$ is trivially color-connected. It is natural, therefore, to seek the smallest integer $k$ for which $G^k$ is color-connected. This gives rise to the following concept. The color-connection exponent $\text{cce}(G)$ of $G$ is defined as the minimum $k$ for which $G^k$ is color-connected. Thus $\text{cce}(G) = 1$ if and only if $G$ is a complete graph; while if $\text{cce}(G) = k \geq 2$, then $G^k$ is color-connected but $G^{k-1}$ is not color-connected. First, we make a useful observation.

Observation 4.1.1 Let $G$ be a connected graph that is not complete. If a distance-
colored graph $G^k$ is Hamiltonian-colored, then $G^k$ is color-connected and so

$$2 \leq cce(G) \leq hce(G).$$

In particular, if the diameter of $G$ is 2, then $cce(G) = 2$.

By Observation 4.1.1, if the diameter of a connected graph $G$ is 2, then $cce(G) = 2$. In fact, we show that $cce(G) = 2$ for all connected graphs $G$ of diameter at least 2. In order to do this, we first verify this fact for paths by establishing a stronger property possessed by the square of $P_n$.

**Proposition 4.1.2** For each integer $n \geq 3$, the distance-colored graph $P_n^2$ contains a properly colored Hamiltonian path.

**Proof.** By Observation 4.1.1, we may assume that $n \geq 4$. Let $G = P_n = (v_1, v_2, \ldots, v_n)$. We consider four cases, according to whether $n$ is congruent to 0, 1, 2 or 3 modulo 4.

**Case 1.** $n \equiv 0 \pmod{4}$. Then $n = 4k$ for some positive integer $k$. We show by induction on $k$ that $G^2$ contains a properly colored Hamiltonian $v_1 - v_n$ path, in which each of $v_1$ and $v_n$ is incident with an edge colored 2. If $k = 1$, then $(v_1, v_3, v_2, v_4)$ has the desired properties. Assume, for $H = P_{4k}$, where $k \geq 1$, that the distance-colored graph $H^2$ contains a properly colored Hamiltonian $v_1 - v_{4k}$ path $Q$ such that each of $v_1$ and $v_n$ is incident with an edge colored 2. Now for $G = P_{4k+4}$, the distance-colored graph $G^2$ contains the properly colored Hamiltonian $v_1 - v_{4k+4}$ path $(Q, v_{4k+1}, v_{4k+3}, v_{4k+2}, v_{4k+4})$, in which each of $v_1$ and $v_{4k+4}$ is incident with an edge colored 2.
Case 2. \( n \equiv 1 \pmod{4} \). Let \( F = G - v_n \). By Case 1, \( F^2 \) contains a properly colored Hamiltonian \( v_1 - v_{n-1} \) path \( Q \) such that each of \( v_1 \) and \( v_{n-1} \) is incident with an edge colored 2. Then \( (Q, v_n) \) has the desired properties.

Case 3. \( n \equiv 2 \pmod{4} \). Let \( F = G - \{v_1, v_n\} \). By Case 1, \( F^2 \) contains a properly colored Hamiltonian \( v_2 - v_{n-1} \) path \( Q \) such that each of \( v_2 \) and \( v_{n-1} \) is incident with an edge colored 2. Then \( (v_1, Q, v_n) \) has the desired properties.

Case 4. \( n \equiv 3 \pmod{4} \). Let \( F = G - \{v_{n-2}, v_{n-1}, v_n\} \). By Case 1, \( F^2 \) contains a properly colored Hamiltonian \( v_1 - v_{n-3} \) path \( Q \) such that each of \( v_1 \) and \( v_{n-3} \) is incident with an edge colored 2. Then the properly colored Hamiltonian \( v_1 - v_{n-1} \) path \( (Q, v_{n-2}, v_n, v_{n-1}) \) has the desired properties.  

The following is then a consequence of Proposition 4.1.2.

**Corollary 4.1.3** If \( G \) is a non-complete connected graph, then \( cce(G) = 2 \).

**Proof.** For two nonadjacent vertices \( u \) and \( v \) of \( G \), let \( P \) be a \( u - v \) geodesic in \( G \). By Proposition 4.1.2, the distance-colored graph \( P^2 \) contains a properly colored Hamiltonian path and so \( P^2 \) is color-connected. Since each properly colored \( u - v \) path in \( P^2 \) is also a properly colored \( u - v \) path in \( G^2 \), it follows that \( G^2 \) is color-connected.

By Corollary 4.1.3, we know that for every two vertices \( u \) and \( v \) in a connected graph \( G \), there is a properly colored \( u - v \) path in \( G^2 \). In fact, there may be several properly colored \( u - v \) paths in \( G^2 \). A *properly colored \( u - v \) geodesic* is a properly
colored $u - v$ path of minimum length in $G^2$ and its length is the \textit{properly colored distance} between $u$ and $v$ in $G^2$ or simply the \textit{color-distance} between $u$ and $v$ and is denoted by $cd(u, v)$. Once we know the distance $d(u, v)$ between two vertices $u$ and $v$ in a connected graph $G$, the color-distance $cd(u, v)$ between two vertices $u$ and $v$ in $G^2$ can be immediately determined.

**Proposition 4.1.4** Let $u$ and $v$ be two vertices in a connected graph $G$. If $d(u, v) = \ell$, then $cd(u, v) = \left\lfloor \frac{2\ell - 1}{3} \right\rfloor$.

**Proof.** First, we show that $cd(u, v) \geq \left\lfloor \frac{2\ell - 1}{3} \right\rfloor$. Let $P$ be a $u - v$ geodesic (of length $\ell$) in $G$ and let $Q$ be a properly colored $u - v$ geodesic in $G^2$. Then $\ell = 3q + r$, where $0 \leq r \leq 2$. Suppose first that $r = 0$. Since every two consecutive edges of $Q$ are colored 1 and 2, these two edges produce a subpath of $Q$ of length 2 in $G^2$ that covers at most three edges of $P$ in $G$. Thus $Q$ contains at least $2q$ edges of $G^2$ and at least $2q$ edges of $G^2$ are needed to produce $Q$; that is, $cd(u, v) \geq 2q = \left\lfloor \frac{2\ell - 1}{3} \right\rfloor$. If $r > 0$, then $Q$ contains more than $2q$ edges of $G^2$ and so $cd(u, v) \geq 2q + 1 = \left\lfloor \frac{2\ell - 1}{3} \right\rfloor$.

It remains to show that $cd(u, v) \leq \left\lfloor \frac{2\ell - 1}{3} \right\rfloor$. To verify this, it suffices to exhibit a properly colored $u - v$ path of length $\left\lfloor \frac{2\ell - 1}{3} \right\rfloor$ in $G^2$. Suppose that $P = (v_0, v_1, \ldots, v_{\ell})$ is a $u - v$ geodesic in $G$. If $\ell = 3q$, then the properly colored $u - v$ path $Q_0 = (v_0, v_1, v_2, v_3, v_4, v_5, \ldots, v_{3i}, v_{3i+1}, v_{3i+2}, v_{3i+3}, \ldots, v_{3q-3}, v_{3q-2}, v_{3q})$ has length $\left\lfloor \frac{2\ell - 1}{3} \right\rfloor$. If $\ell = 3q + 1$, then $Q_1 = (Q_0, v_{3q+1})$ has length $\left\lfloor \frac{2\ell - 1}{3} \right\rfloor$. If $\ell = 3q + 2$, then $Q_2 = (v_0, v_2, v_3, v_5, v_6, \ldots, v_{3i}, v_{3i+2}, v_{3i+3}, \ldots, v_{3q-3}, v_{3q-1}, v_{3q}, v_{3q+2})$ has length $\left\lfloor \frac{2\ell - 1}{3} \right\rfloor$.  

It is well known that the distance $d$ is a metric on the vertex set of a connected graph $G$. This, however, is not the case for the color-distance $cd$. While $cd(u, v) \geq$
0 for all \( u, v \in V(G) \), \( \text{cd}(u, v) = 0 \) if and only if \( u = v \) and \( \text{cd}(u, v) = \text{cd}(v, u) \) for all \( u, v \in V(G) \), the color-distance \( \text{cd} \) does not satisfy the triangle inequality. For example, suppose that \( P \) is a \( u-w \) geodesic of length 10 in a connected graph \( G \) and \( v \) is the central vertex of \( P \). Then \( d(u, v) = d(v, w) = 5 \) and \( d(u, w) = 10 \). By Proposition 4.1.4, \( \text{cd}(u, v) = \text{cd}(v, w) = \left\lfloor \frac{2 \cdot 5 - 1}{3} \right\rfloor = 3 \) while \( \text{cd}(u, w) = \left\lfloor \frac{2 \cdot 10 - 1}{3} \right\rfloor = 7 \). So \( \text{cd}(u, v) + \text{cd}(v, w) < \text{cd}(u, w) \).

**Theorem 4.1.5**  For every three vertices \( u, v \) and \( w \) in a connected graph \( G \),

\[
\text{cd}(u, w) \leq \text{cd}(u, v) + \text{cd}(v, w) + 1. \quad (4.1)
\]

*Equality holds in (4.1) if and only if \( d(u, v) \equiv 2 \pmod{3} \), \( d(v, w) \equiv 2 \pmod{3} \) and \( d(u, w) = d(u, v) + d(v, w) \).*

**Proof.** Suppose that \( d(u, v) = a \), \( d(v, w) = b \) and \( d(u, w) = c \). We first establish (4.1). Since \( a + b \geq c \), it follows by Proposition 4.1.4 that

\[
\text{cd}(u, v) + \text{cd}(v, w) = \left\lfloor \frac{2a - 1}{3} \right\rfloor + \left\lfloor \frac{2b - 1}{3} \right\rfloor \geq \frac{2a - 1}{3} + \frac{2b - 1}{3} = \frac{2(a + b) - 2}{3} \geq \frac{2c - 2}{3} \geq \left\lfloor \frac{2c - 4}{3} \right\rfloor = \left\lfloor \frac{2c - 1}{3} \right\rfloor - 1 = \text{cd}(u, w) - 1
\]

and so (4.1) holds.

Next, assume that \( a \equiv 2 \pmod{3} \), \( b \equiv 2 \pmod{3} \) and \( c = a + b \). Then \( a = 3p + 2 \) and \( b = 3q + 2 \) for some nonnegative integers \( p \) and \( q \). By Proposition 4.1.4,
\[
\text{cd}(u, v) = \left\lceil \frac{2a - 1}{3} \right\rceil = 2p + 1, \quad \text{cd}(v, w) = \left\lceil \frac{2b - 1}{3} \right\rceil = 2q + 1 \text{ and }
\]
\[
\text{cd}(u, w) = \left\lceil \frac{2c - 1}{3} \right\rceil = \left\lceil \frac{2(a + b) - 1}{3} \right\rceil = \left\lceil \frac{2(3p + 3q + 4) - 1}{3} \right\rceil
\]
\[
= \left\lceil \frac{6p + 6q + 7}{3} \right\rceil = 2p + 2q + 3 = \text{cd}(u, v) + \text{cd}(v, w) + 1.
\]

For the converse, first suppose that \(a \not\equiv 2 \pmod{3}\) or \(b \not\equiv 2 \pmod{3}\), say the former. Since \(c \leq a + b\), it follows by Proposition 4.1.4 that
\[
\text{cd}(u, w) = \left\lceil \frac{2c - 1}{3} \right\rceil \leq \left\lceil \frac{2(a + b) - 1}{3} \right\rceil.
\]
(4.2)

It suffices to show that
\[
\left\lceil \frac{2(a + b) - 1}{3} \right\rceil \leq \left\lceil \frac{2a - 1}{3} \right\rceil + \left\lceil \frac{2b - 1}{3} \right\rceil.
\]
(4.3)

Let \(a = 3q_1 + r_1\) and \(b = 3q_2 + r_2\), where \(0 \leq r_1, r_2 \leq 2\) and at least one of \(r_1\) and \(r_2\) is not 2. Without loss of generality, we consider the following cases.

- If \(r_1 = 0\) and \(r_2 = 0\), then \(\left\lceil \frac{2a - 1}{3} \right\rceil = 2q_1\) and \(\left\lceil \frac{2b - 1}{3} \right\rceil = 2q_2\) and \(\left\lceil \frac{2(a + b) - 1}{3} \right\rceil = 2q_1 + 2q_2\).

- If \(r_1 = 0\) and \(r_2 = 1\), then \(\left\lceil \frac{2a - 1}{3} \right\rceil = 2q_1\) and \(\left\lceil \frac{2b - 1}{3} \right\rceil = 2q_2 + 1\) and \(\left\lceil \frac{2(a + b) - 1}{3} \right\rceil = 2q_1 + 2q_2 + 1\).

- If \(r_1 = 0\) and \(r_2 = 2\), then \(\left\lceil \frac{2a - 1}{3} \right\rceil = 2q_1\) and \(\left\lceil \frac{2b - 1}{3} \right\rceil = 2q_2 + 1\) and \(\left\lceil \frac{2(a + b) - 1}{3} \right\rceil = 2q_1 + 2q_2 + 1\).

- If \(r_1 = 1\) and \(r_2 = 1\), then \(\left\lceil \frac{2a - 1}{3} \right\rceil = 2q_1 + 1\) and \(\left\lceil \frac{2b - 1}{3} \right\rceil = 2q_2 + 1\) and \(\left\lceil \frac{2(a + b) - 1}{3} \right\rceil = 2q_1 + 2q_2 + 1\).
• If \( r_1 = 1 \) and \( r_2 = 2 \), then \( \left\lceil \frac{2a-1}{3} \right\rceil = 2q_1 + 1 \) and \( \left\lceil \frac{2b-1}{3} \right\rceil = 2q_2 + 1 \) and \( \left\lceil \frac{2(a+b)-1}{3} \right\rceil = 2q_1 + 2q_2 + 2 \).

In each case, \( \text{cd}(u, w) \leq \text{cd}(u, v) + \text{cd}(v, w) \). Finally, suppose that \( a \equiv 2 \pmod{3} \) and \( b \equiv 2 \pmod{3} \) but \( c \leq a + b - 1 \). Then \( a = 3q_1 + 2 \) and \( b = 3q_2 + 2 \). Thus \( \left\lceil \frac{2a-1}{3} \right\rceil = 2q_1 + 1 \) and \( \left\lceil \frac{2b-1}{3} \right\rceil = 2q_2 + 1 \). Since \( c \leq 3q_1 + 3q_2 + 3 \), it follows by Proposition 4.1.4 that

\[
\text{cd}(u, w) = \left\lceil \frac{2c-1}{3} \right\rceil \leq \left\lceil \frac{2(3q_1 + 3q_2 + 3) - 1}{3} \right\rceil = 2q_1 + 2q_2 + 2
\]

\[
= \left\lceil \frac{2a-1}{3} \right\rceil + \left\lceil \frac{2b-1}{3} \right\rceil = \text{cd}(u, v) + \text{cd}(v, w).
\]

Thus, strict inequality holds in (4.1).

4.2 Color-Eccentricity, Color-Radius and Color-Diameter

For a vertex \( v \) in a connected graph \( G \), the \textit{color-eccentricity} \( \text{ce}(v) \) of \( v \) is the color-distance between \( v \) and a vertex farthest from \( v \) in \( G^2 \). The minimum color eccentricity among the vertices of \( G \) is its \textit{color-radius} and the maximum eccentricity is its \textit{color-diameter}, which are denoted by \( \text{crad}(G) \) and \( \text{cdiam}(G) \), respectively. A vertex in \( G \) whose color-eccentricity equals \( \text{crad}(G) \) is a \textit{color-central vertex} of \( G \) and a vertex whose color-eccentricity equals \( \text{cdiam}(G) \) is a \textit{color-peripheral vertex} of \( G \). Two vertices \( u \) and \( v \) of \( G \) with \( \text{cd}(u, v) = \text{cdiam}(G) \) are \textit{color-antipodal vertices} of \( G \). Necessarily, if \( u \) and \( v \) are color-antipodal vertices in \( G \), then each of \( u \) and \( v \) is a color-peripheral vertex.


**Proposition 4.2.1** For every nontrivial connected graph $G$,

\[
\text{crad}(G) \leq \text{cdiam}(G) \leq 2\text{crad}(G) + 1.
\]

**Proof.** The inequality $\text{crad}(G) \leq \text{cdiam}(G)$ is immediate from the definitions. Let $u$ and $w$ be two vertices such that $\text{cd}(u, w) = \text{cdiam}(G)$ and let $v$ be a color-central vertex of $G$. Therefore, $\text{ce}(v) = \text{crad}(G)$. By Theorem 4.1.5,

\[
\text{cdiam}(G) = \text{cd}(u, w) \leq \text{cd}(u, v) + \text{cd}(v, w) + 1 \leq 2\text{ce}(v) + 1 = 2\text{crad}(G) + 1,
\]
as desired. $lacksquare$

For a vertex $u$ in a connected graph $G$, a vertex $v$ is an eccentric vertex of $u$ if $v$ is a vertex that is farthest from $u$ in $G$ and the set of all eccentric vertices of $u$ in $G$ is denoted by $\text{Ecc}(u)$. As expected, a vertex $w$ is a color-eccentric vertex of $u$ if $w$ is a vertex that is farthest from $u$ in $G^2$, and the set of all color-eccentric vertices of $u$ in $G^2$ is denoted by $\text{CEcc}(u)$. First, we present three preliminary results on the eccentric vertices and color-eccentric vertices of a given vertex in a graph.

**Lemma 4.2.2** Let $G$ be a nontrivial connected graph and $u \in V(G)$. If $x, y \in \text{CEcc}(u)$, then

\[
|d(u, x) - d(u, y)| \leq 1.
\]

**Proof.** Suppose that $d(u, x) = a$ and $d(u, y) = b$. Since $\text{ce}(u) = \left\lceil \frac{2a-1}{3} \right\rceil = \left\lceil \frac{2b-1}{3} \right\rceil$, it follows that $|a - b| \leq 1$. $lacksquare$
Lemma 4.2.3  For each vertex $u$ in a nontrivial connected graph $G$,

$$\text{Ecc}(u) \subseteq \text{CEcc}(u).$$

Proof. Let $x$ be a vertex of $G$ such that $x \not\in \text{CEcc}(u)$. Let $y \in \text{CEcc}(u)$. Thus $cd(u, y) = ce(u)$. Then $cd(u, y) > cd(u, x)$. Suppose that $d(u, x) = a$ and $d(u, y) = b$. Since $ce(u) = \lceil \frac{2b-1}{3} \rceil > cd(u, x) = \lceil \frac{2a-1}{3} \rceil$, it follows that $d(u, y) = b > a = e(u)$ and $x \not\in \text{CEcc}(u)$. 

The reverse set inclusion of that stated in Lemma 4.2.3 does not hold in general. For example, consider the graph $G = P_{10}$ in Figure 4.1. Since $d(u, v) = 4$ and $d(u, w) = 5$, it follows that $ce(u) = cd(u, v) = cd(u, w) = 3$. In this case, $e(u) = 5$ and so $v$ is not an eccentric vertex of $u$. While $v$ and $w$ are both color-eccentric vertices of $u$, only $w$ is an eccentric vertex of $u$.

![Figure 4.1: A color-eccentric vertex of $u$ that is not an eccentric vertex of $u$](image)

Lemma 4.2.4  For two vertices $u$ and $v$ in a nontrivial connected graph,

- if $e(u) \geq e(v)$, then $ce(u) \geq ce(v)$,

- if $ce(u) \geq ce(v)$, then $e(u) \geq e(v) - 1$.

Proof. Let $x \in \text{Ecc}(u)$ and $y \in \text{Ecc}(v)$. Suppose that $e(u) = d(u, x) = a$ and $e(v) = d(v, y) = b$. By Lemma 4.2.3, $x \in \text{CEcc}(u)$ and $y \in \text{CEcc}(v)$. Then $ce(u) =$
\[ \text{cd}(u, r) = \left\lceil \frac{2a-1}{3} \right\rceil \] and \[ \text{ce}(v) = \text{cd}(v, y) = \left\lceil \frac{2b-1}{3} \right\rceil \]

If \( a \geq b \), then \( \left\lceil \frac{2a-1}{3} \right\rceil \geq \left\lceil \frac{2b-1}{3} \right\rceil \),

while if \( \left\lceil \frac{2a-1}{3} \right\rceil \geq \left\lceil \frac{2b-1}{3} \right\rceil \), then either \( a \geq b \) or \( b = a + 1 \).

In particular, if \( a \equiv 1 \) (mod 3) and \( b \equiv 2 \) (mod 3), then \( b = a + 1 \) and \( \left\lceil \frac{2a-1}{3} \right\rceil = \left\lceil \frac{2b-1}{3} \right\rceil \).

For a nontrivial connected graph \( G \) and \( u, v \in V(G) \), let \( Q \) be a properly colored \( u - v \) path in \( G^2 \). An underlying walk \( W_Q \) of \( Q \) is a \( u - v \) walk in \( G \) constructed from \( Q \) by replacing each edge of \( Q \) that is colored 2 by a path of length 2 in \( G \).

A \( u - v \) path of minimum length in \( W_Q \) is called an underlying path of \( Q \) and is denoted by \( P_Q \).

**Theorem 4.2.5** Let \( G \) be a nontrivial connected graph.

(a) If \( u \) and \( v \) are adjacent vertices of \( G^2 \), then \( |\text{ce}(u) - \text{ce}(v)| \leq 2 \)

(b) If \( u \) and \( v \) are adjacent vertices of \( G \), then \( |\text{ce}(u) - \text{ce}(v)| \leq 1 \)

(c) If \( k \) is an integer with \( \text{crad}(G) \leq k \leq \text{cdiam}(G) \), then there is a vertex \( w \) such that \( \text{ce}(w) = k \)

(d) If \( k \) is an integer with \( \text{crad}(G) < k \leq \text{cdiam}(G) \), then there are at least two vertices of \( G \) with color-eccentricity \( k \)

**Proof.** First, we verify (a). Let \( u \) and \( v \) be adjacent vertices of \( G^2 \). Assume, without loss of generality, that \( \text{ce}(u) \geq \text{ce}(v) \). Suppose that \( \text{ce}(u) = \text{cd}(u, r) \). By Theorem 4.1.5,

\[
\text{ce}(u) = \text{cd}(u, r) \leq \text{cd}(u, v) + \text{cd}(v, r) + 1 \leq 2 + \text{ce}(v)
\]
Hence $\text{ce}(u) \leq 2 + \text{ce}(v)$ and so $0 \leq \text{ce}(u) - \text{ce}(v) \leq 2$. Therefore, $|\text{ce}(u) - \text{ce}(v)| \leq 2$.

Second, we verify (b). Let $u$ and $v$ be adjacent vertices of $G$. Let $x \in \text{Ecc}(u)$ and $y \in \text{Ecc}(v)$. Suppose that $e(u) = d(u, x) = a$ and $e(v) = d(v, y) = b$. By Lemma 4.2.3 and Proposition 4.1.4, $\text{ce}(u) = \text{cd}(u, x) = \lceil \frac{2a - 1}{3} \rceil$ and $\text{ce}(v) = \text{cd}(v, y) = \lceil \frac{2b - 1}{3} \rceil$. Since $|e(u) - e(v)| \leq 1$, either $a = b$, $a = b + 1$ or $b = a + 1$.

If $a = b$, then $\text{ce}(u) = \text{ce}(v)$; while if $a = b + 1$ or $b = a + 1$, say the former, then $\text{ce}(u) = \lceil \frac{2a - 1}{3} \rceil = \lceil \frac{2(b + 1) - 1}{3} \rceil = \lceil \frac{2b + 1}{3} \rceil$. Since $\lceil \frac{2b + 1}{3} \rceil - \lceil \frac{2b - 1}{3} \rceil \leq 1$, it follows that $0 \leq \text{ce}(u) - \text{ce}(v) \leq 1$ and so $|\text{ce}(u) - \text{ce}(v)| \leq 1$.

Next, we verify (c). Let $u$ and $v$ be adjacent vertices of $G$ such that $\text{ce}(v) = \text{crad}(G)$ and $\text{ce}(u) = \text{cdiam}(G)$ and let $P$ be a $u-v$ path in $G$, say $P = (u = v_1, v_2, \ldots, v_k = v)$. By (b), the color-eccentricities of $v_i$ and $v_{i+1}$ ($1 \leq i \leq k - 1$) differ by at most 1. Hence some vertex $v_i$ of $P$ has color-eccentricity $k$.

Finally, we verify (d). Let $k$ be an integer with $\text{crad}(G) < k \leq \text{cdiam}(G)$. By (c), there is a vertex $w$ such that $\text{ce}(w) = k$. Let $u$ be a vertex with $\text{cd}(w, u) = \text{ce}(w) = k$. For a color-central vertex $v$ of $G$, let $Q$ be a properly colored $u-v$ path of length $\text{cd}(u, v)$ in $G^2$. Thus $\text{cd}(u, v) \leq \text{ce}(v) = \text{crad}(G)$. Since $\text{cd}(u, w) = k$, it follows that $\text{ce}(v) < k \leq \text{ce}(u)$. Let $P$ be the underlying $u-v$ path of $Q$ in $G$ where say $P = (u = v_1, v_2, \ldots, v_\ell = v)$ where $\ell \geq k$. By (b), the color-eccentricities of $v_i$ and $v_{i+1}$ ($1 \leq i \leq \ell - 1$) differ by at most 1. There is a vertex $x$ on $P$ such that $\text{ce}(x) = k$. Because $\text{cd}(u, x) \leq \text{cd}(u, v) < k$ and $\text{cd}(w, u) = k$, it follows that $x \neq w$. \[\Box\]
For two adjacent vertices \( u \) and \( v \) in \( G^2 \), it is possible that \(|ce(u) - ce(v)| = i\) for each \( i \in \{0, 1, 2\} \) even when \( u \) and \( v \) are not adjacent in \( G \). This fact is illustrated by the graph \( G = P_8 = (v_1, v_2, v_3, \ldots, v_8) \) of Figure 4.2, where each vertex of \( G \) is labeled by its color-eccentricity. In particular, \( v_1 \) and \( v_3 \) are adjacent vertices of \( G^2 \). Since \( d(v_1, v_8) = 7 \) and \( d(v_3, v_8) = 5 \), it follows that \( ce(v_1) = cd(v_1, v_8) = \lceil \frac{2 \cdot 7 - 1}{3} \rceil = 5 \) and \( ce(v_3) = cd(v_3, v_8) = \lceil \frac{2 \cdot 5 - 1}{3} \rceil = 3 \). Thus, \(|ce(v_1) - ce(v_3)| = 2\).

\( P_8 : \hspace{1cm} 5 \quad 4 \quad 3 \quad 3 \quad 3 \quad 4 \quad 5 \)

Figure 4.2: The color-eccentricities of \( P_8 \)

Next, we present a necessary and sufficient condition for two adjacent vertices \( u \) and \( v \) in \( G^2 \) to satisfy the equality \(|ce(u) - ce(v)| = 2\).

**Theorem 4.2.6** Let \( G \) be a nontrivial connected graph and let \( u \) and \( v \) be adjacent vertices of \( G^2 \). Suppose that \( \max\{e(u), e(v)\} = a \) and \( \min\{e(u), e(v)\} = b \). Then \(|ce(u) - ce(v)| = 2\) if and only if \( a \) and \( b \) satisfy one of the following conditions:

(i) \( a = b + 2 \) and \( b \equiv 2 \pmod{3} \),

(ii) \( a = b + 3 \),

(iii) \( a = b + 4 \) and \( b \equiv 1 \pmod{3} \).

**Proof.** Assume, without loss of generality, that \( ce(u) \geq ce(v) \). By Lemma 4.2.4, \( e(u) \geq e(v) \) or \( e(v) = e(u) + 1 \). Let \( e(u) = a \) and \( e(v) = b \).

First, assume that \( ce(u) - ce(v) = 2 \). By Theorem 4.2.5(b), \( u \) and \( v \) are not adjacent vertices of \( G \). Let \( x \in Ecc(u) \) and \( y \in Ecc(v) \). Then \( e(u) = d(u, x) = a \)
and \( e(v) = d(v, y) = b \). By Lemma 4.2.3, \( ce(u) = cd(u, r) = \left\lfloor \frac{2a-1}{3} \right\rfloor \) and \( ce(v) = cd(v, y) = \left\lfloor \frac{2b-1}{3} \right\rfloor \). We consider five cases, according to the value of \( a - b \).

**Case 1.** \( |a - b| \leq 1 \). It then follows by the proof of Theorem 4.2.5(b) that \( 0 \leq ce(u) - ce(v) \leq 1 \). Thus Case 1 is impossible.

**Case 2.** \( a - b = 2 \). Then \( ce(u) = \left\lfloor \frac{2a-1}{3} \right\rfloor = \left\lfloor \frac{2(b+2)-1}{3} \right\rfloor = \left\lfloor \frac{2b}{3} \right\rfloor + 1 \) and so

\[
ce(u) - ce(v) = \left\lfloor \frac{2b}{3} \right\rfloor - \left\lfloor \frac{2b - 1}{3} \right\rfloor + 1.
\]

Suppose that \( b = 3k + \tau \) where \( \tau = 0, 1, 2 \). If \( b = 3k \), then \( \left\lfloor \frac{2b}{3} \right\rfloor = \left\lfloor \frac{2b - 1}{3} \right\rfloor = 2k \), if \( b = 3k + 1 \), then \( \left\lfloor \frac{2b}{3} \right\rfloor = \left\lfloor \frac{2b - 1}{3} \right\rfloor = 2k + 1 \), and if \( b = 3k + 2 \), then \( \left\lfloor \frac{2b}{3} \right\rfloor = 2k + 2 \) and \( \left\lfloor \frac{2b - 1}{3} \right\rfloor = 2k + 1 \). Thus Case 2 is possible only when \( b \equiv 2 \pmod{3} \).

**Case 3.** \( a - b = 3 \). Then \( ce(u) = \left\lfloor \frac{2a-1}{3} \right\rfloor = \left\lfloor \frac{2(b+3)-1}{3} \right\rfloor = \left\lfloor \frac{2b}{3} \right\rfloor + 2 = ce(v) + 2 \). Thus Case 3 is possible.

**Case 4.** \( a - b = 4 \). Then \( ce(u) = \left\lfloor \frac{2a-1}{3} \right\rfloor = \left\lfloor \frac{2(b+4)-1}{3} \right\rfloor = \left\lfloor \frac{2b+1}{3} \right\rfloor + 2 \) and so

\[
ce(u) - ce(v) = \left\lfloor \frac{2b + 1}{3} \right\rfloor - \left\lfloor \frac{2b - 1}{3} \right\rfloor + 2.
\]

Suppose that \( b = 3k + \tau \) where \( \tau = 0, 1, 2 \). If \( b = 3k \), then \( \left\lfloor \frac{2b+1}{3} \right\rfloor = 2k + 1 \) and \( \left\lfloor \frac{2b}{3} \right\rfloor = 2k \), if \( b = 3k + 1 \), then \( \left\lfloor \frac{2b+1}{3} \right\rfloor = \left\lfloor \frac{2b - 1}{3} \right\rfloor = 2k + 1 \), and if \( b = 3k + 2 \), then \( \left\lfloor \frac{2b+1}{3} \right\rfloor = 2k + 2 \) and \( \left\lfloor \frac{2b - 1}{3} \right\rfloor = 2k + 1 \). Thus Case 4 is possible only when \( b \equiv 1 \pmod{3} \).

**Case 5.** \( a - b \geq 5 \). Then \( ce(u) = \left\lfloor \frac{2a-1}{3} \right\rfloor = \left\lfloor \frac{2(b+5)-1}{3} \right\rfloor \geq \left\lfloor \frac{2b-1}{3} \right\rfloor + 3 \). Thus Case 5 is impossible.
For the converse, if \( a \) and \( b \) satisfy one of the conditions (i)-(m), then, by the arguments in Cases 1 4, we see that \( ce(u) - ce(v) = 2 \)

By Theorem 4 2 1, if \( G \) is a nontrivial connected graph such that \( \text{crad}(G) = r \) and \( \text{cdiam}(G) = d \), then \( r \leq d \leq 2r+1 \) Next, we determine all pairs \( r, d \) of positive integers with \( r \leq d \leq 2r+1 \) for which there is a connected graph with color radius \( r \) and color-diameter \( d \) First we present a well-known realization result on the radius and diameter of a graph (see [1])

**Lemma 4.2.7** For each pair \( a, b \) of positive integers with \( a \leq b \leq 2a \), there exists a connected graph \( G \) with \( \text{rad}(G) = a \) and \( \text{diam}(G) = b \)

**Theorem 4.2.8** Let \( r \) and \( d \) be positive integers with \( r \leq d \leq 2r+1 \) Then there exists a connected graph \( G \) with \( \text{crad}(G) = r \) and \( \text{cdiam}(G) = d \) if and only if (i) \( r \leq d \leq 2r \) or (ii) \( d = 2r+1 \) and \( r \) is odd

**Proof.** Suppose that \( r \) and \( d \) are positive integers with \( r \leq d \leq 2r+1 \) that satisfies (i) or (ii) First, we make an observation If \( G \) is a connected graph with \( \text{rad}(G) = a \) and \( \text{diam}(G) = b \), then by Lemma 4 2 3 and Proposition 4 1 4, \( \text{crad}(G) = \left[ \frac{2a-1}{3} \right] \) and \( \text{cdiam}(G) = \left[ \frac{2b-1}{3} \right] \) By Lemma 4 2 7, it suffices to show that there exist positive integers \( a \) and \( b \) such that \( a \leq b \leq 2a \), \( r = \left[ \frac{2a-1}{3} \right] \) and \( d = \left[ \frac{2b-1}{3} \right] \) We consider four cases

**Case 1** \( r \) and \( d \) are both even Then \( r \leq d \leq 2r \) Let \( a = \frac{3r}{2} \) and \( b = \frac{3d}{2} \) Then \( a \leq b = \frac{3d}{2} \leq \frac{3(2r)}{2} = 3r = 2a, \left[ \frac{2a-1}{3} \right] = \left[ \frac{3r-1}{3} \right] = r \) and \( \left[ \frac{2b-1}{3} \right] = \left[ \frac{3d-1}{3} \right] = d \)
Case 2  
**r** is even and **d** is odd

Then \( r < d \leq 2r - 1 \) Let \( a = \frac{3r}{2} \) and  
\( b = \frac{3d+1}{2} \)  
Observe that \( a \leq b = \frac{3d+1}{2} \leq \frac{3(2r-1)+1}{2} = 3r - 1 = 2a - 1 \)  
Furthermore,  
\[ \left\lfloor \frac{2a-1}{3} \right\rfloor = \left\lfloor \frac{3r-1}{3} \right\rfloor = r \] and  
\[ \left\lfloor \frac{2b-1}{3} \right\rfloor = \left\lfloor \frac{3d}{3} \right\rfloor = d \]

Case 3  
**r** is odd and **d** is even

Then \( r < d < 2r \) Let \( a = \frac{3r+1}{2} \) and \( b = \frac{3d}{2} \)  
Since \( r \leq d - 1 \), it follows that  
\[ a = \frac{3r+1}{2} \leq \frac{3(d-1)+1}{2} = \frac{3d-2}{2} < \frac{3d}{2} = b \]  
\[ b = \frac{3d}{2} \leq \frac{3(2r)}{2} = 3r = 2a - 1 < 2a \]  
Furthermore,  
\[ \left\lfloor \frac{2a-1}{3} \right\rfloor = r \] and  
\[ \left\lfloor \frac{2b-1}{3} \right\rfloor = d \]

Case 4  
**r** and **d** are both odd

Then \( r \leq d \leq 2r + 1 \) If \( d = 2r + 1 \), let \( a = \frac{3r+1}{2}, b = 3r + 1 \)  
Then \( b = 2a \)  
Furthermore,  
\[ \left\lfloor \frac{2a-1}{3} \right\rfloor = r \] and  
\[ \left\lfloor \frac{2b-1}{3} \right\rfloor = d \]  
Thus we may assume that \( r \leq d \leq 2r - 1 \) Then \( a = \frac{3r+1}{2} \) and \( b = \frac{3d+1}{2} \)  
Then \( a \leq b \leq 2a - 2, \)  
\[ \left\lfloor \frac{2a-1}{3} \right\rfloor = r \] and  
\[ \left\lfloor \frac{2b-1}{3} \right\rfloor = d \]

For the converse, we show that there is no connected graph \( G \) with \( \text{crad}(G) = r \) and \( \text{cdiam}(G) = d \) such that \( r \) is even and \( d = 2r + 1 \) Assume, to the contrary, that such a graph \( G \) exists Then \( r = 2k \) for some positive integer \( k \) and so \( d = 2r + 1 = 4k + 1 \) Suppose that \( u \) and \( v \) are color-antipodal vertices of \( G \) Then \( \text{cd}(u, v) = 4k + 1 \) Suppose that \( \text{d}(u, v) = a \) Then  
\[ \text{cd}(u, v) = \left\lfloor \frac{2a-1}{3} \right\rfloor = 4k + 1 \]  
(4.4)  
Now \( \left\lfloor \frac{2a-1}{3} \right\rfloor \) is either \( \frac{2a-1}{3}, \frac{2a}{3} \) or \( \frac{2a+1}{3} \)  
If \( \left\lfloor \frac{2a-1}{3} \right\rfloor = \frac{2a-1}{3} \), then \( a = 6k + 2, \) if \( \left\lfloor \frac{2a-1}{3} \right\rfloor = \frac{2a}{3} \), then \( 2a = 12k + 3 \), which is impossible, if \( \left\lfloor \frac{2a-1}{3} \right\rfloor = \frac{2a+1}{3} \), then \( a = 6k + 1 \) Hence  
\[ \text{either } a = 6k + 2 \text{ or } a = 6k + 1 \]  
(4.5)
Let \( w \) be a color-central vertex of \( G \). Then \( cc(w) = crad(G) = r = 2k \) and so \( cd(u, w) \leq 2k \) and \( cd(v, w) \leq 2k \). If \( cd(u, w) < 2k \) or \( cd(v, w) < 2k \), it then follows by Theorem 4.1.5 that \( cd(u, v) \leq cd(u, w) + cd(v, w) + 1 < 2k + 2k + 1 = 4k + 1 \), which contradicts (4.4). Thus
\[
    cd(u, w) = 2k \quad \text{and} \quad cd(v, w) = 2k. \tag{4.6}
\]

Suppose that \( d(u, w) = b \). Then \( cd(u, w) = \left\lfloor \frac{2b-1}{3} \right\rfloor = 2k \) by (4.6). However, \( \left\lfloor \frac{2b-1}{3} \right\rfloor \) is one of \( \frac{2b-1}{3} \), \( \frac{2b}{3} \) and \( \frac{2b+1}{3} \). If \( \left\lfloor \frac{2b-1}{3} \right\rfloor = \frac{2b-1}{3} \), then \( 2b - 1 = 6k \), which is impossible; if \( \left\lfloor \frac{2b-1}{3} \right\rfloor = \frac{2b}{3} \), then \( 2b = 6k \) and \( b = 3k \); if \( \left\lfloor \frac{2b-1}{3} \right\rfloor = \frac{2b+1}{3} \), then \( 2b + 1 = 6k \), which is impossible. Therefore, \( d(u, w) = b = 3k \). Similarly, \( d(v, w) = b = 3k \). However then, by the triangle inequality, \( d(u, v) \leq d(u, w) + d(v, w) \leq 3k + 3k = 6k \), which contradicts (4.5). Therefore, as claimed, there is no connected graph \( G \) with \( crad(G) = r \) and \( cdiam(G) = d \) such that \( r \) is even and \( d = 2r + 1 \).

### 4.3 Color-Center and Color-Periphery

The subgraph of a connected graph \( G \) induced by its color-central vertices is the **color-center** \( CC(G) \) of \( G \) and the subgraph induced by the color-peripheral vertices of \( G \) is the **color-periphery** of \( G \) and is denoted by \( CP(G) \).

In an observation first made by Hedetniemi, there is no restriction on which graphs can be the center of some graph.

**Theorem 4.3.1** [2] Every graph is the center of some connected graph.

There is an analogous result for color-centers, the proof of which uses the same construction as employed by Hedetniemi.
Theorem 4.3.2  Every graph is the color center of some connected graph.

Proof. Let $G$ be a graph. We construct a graph $H$ from $G$ by first adding two new vertices $u$ and $v$ to $G$ and joining them to every vertex of $G$ but not to each other. The construction of $H$ is completed by adding two additional vertices $u_1$ and $v_1$ and the two new edges $u_1u$ and $v_1v$. (This construction is illustrated in Figure 4.3.) Since $ce(u) = ce(v) = 2$, $ce(u_1) = ce(v_1) = 3$ and $ce(r) = 1$ for every vertex $r$ in $G$, it follows that $V(G)$ is the set of the color central vertices of $H$ and so $CC(H) = G$.

![Figure 4.3](image)

Figure 4.3 A graph with a given color-center

If every vertex of $G$ is a color-central vertex, then $CC(G) = G$ and $G$ is self color centered. Thus, if $G$ is self-color-centered, then $crad(G) = cdiam(G)$.

Theorem 4.3.3  Let $G$ be a connected graph with $rad(G) = a$ and $diam(G) = b$. Then $G$ is self color centered if and only if (i) $b = a$ or (ii) $b = a + 1$ and $a \equiv 1 \pmod{3}$.

Proof. First, suppose that $a$ and $b$ satisfy (i) or (ii). By Lemma 4.2.3, $crad(G) = \lceil \frac{2a-1}{3} \rceil$ and $cdiam(G) = \lceil \frac{2b-1}{3} \rceil$. If $a = b$, then $\lceil \frac{2a-1}{3} \rceil = \lceil \frac{2b-1}{3} \rceil$. If $b = a + 1$ and $a \equiv 1 \pmod{3}$, then let $a = 3k + 1$ for some integer $k$. Then $\lceil \frac{2a-1}{3} \rceil = \lceil \frac{2b-1}{3} \rceil = 2k + 1$. Thus $G$ is self-color centered.
For the converse, suppose that neither (i) nor (ii) holds. We consider two cases.

Case 1. \( b = a + 1 \) and \( a \not\equiv 1 \mod 3 \). Thus \( a = 3k + r \) where \( r = 0 \) or \( r = 2 \). If \( r = 0 \), then \( \left\lceil \frac{2a-1}{3} \right\rceil = 2k \) and \( \left\lceil \frac{2b-1}{3} \right\rceil = \left\lceil \frac{2a+1}{3} \right\rceil = 2k + 1 \), if \( r = 2 \), then \( \left\lceil \frac{2a-1}{3} \right\rceil = 2k + 1 \) and \( \left\lceil \frac{2b-1}{3} \right\rceil = \left\lceil \frac{2a+1}{3} \right\rceil = 2k + 2 \). Thus \( \text{crad}(G) < \text{cdiam}(G) \).

Case 2. \( b \geq a + 2 \). Then \( \left\lceil \frac{2b-1}{3} \right\rceil \geq \left\lceil \frac{2(a+2)-1}{3} \right\rceil = \left\lceil \frac{2a+3}{3} \right\rceil = \left\lceil \frac{2a}{3} \right\rceil + 1 \geq \left\lceil \frac{2a-1}{3} \right\rceil + 1 \) and so \( \text{crad}(G) < \text{cdiam}(G) \).

In either case, \( G \) is not self-color centered.

Let \( v \) be any vertex in the center of \( G \). Then by Lemma 4.2.3, \( \text{Ecc}(v) \subseteq \text{CEcc}(v) \). Hence, it follows that \( \text{Cen}(G) \subseteq \text{CC}(G) \) for every nontrivial connected graph \( G \). The graph \( H \) constructed in the proof of Theorem 4.3.2 has the property that \( \text{Cen}(H) = \text{CC}(H) = G \). This gives rise to the following question.

**Problem 4.3.4** Suppose that \( F \) and \( G \) are two graphs where \( F \) is a proper induced subgraph of \( G \). Under what conditions does there exist a connected graph \( H \) such that \( \text{Cen}(H) = F \) and \( \text{CC}(H) = G \)?

Although the answer to Problem 4.3.4 is not known, we present two necessary conditions for such a connected graph \( H \).

**Proposition 4.3.5** Suppose that \( F \) and \( G \) are two graphs where \( F \) is a proper induced subgraph of \( G \). If \( H \) is a connected graph such that \( \text{Cen}(H) = F \) and \( \text{CC}(H) = G \), then \( H \) must satisfy the following two conditions:
(i) There is a positive integer $a$ with $a \equiv 1 \pmod{3}$ such that $e_H(x) = a$ for all $x \in V(F)$, $e_H(y) = a + 1$ for all $y \in V(G) - V(F)$, and $e_H(z) \geq a + 2$ for all $z \in V(H) - V(G)$.

(ii) No vertex in $F$ is adjacent to any vertex in $V(H) - V(G)$.

**Proof.** We first verify (i). Since $\text{Cen}(H) = F$, there is a positive integer $a$ such that $e_H(x) = a$ for all $x \in V(F)$. Let $y \in V(G) - V(F)$. Then $e_H(y) = b$ for some integer $b \geq a + 1$. Since $\text{CC}(H) = G$, there is a positive integer $r$ such that $ce_H(x) = r$ for all $x \in V(G)$. Since $r = \left\lceil \frac{2a+1}{3} \right\rceil = \left\lceil \frac{2b-1}{3} \right\rceil$, it follows that $a \equiv 1 \pmod{3}$ and $b = a + 1$. Thus $e_H(y) = a + 1$ for all $y \in V(G) - V(F)$. If there is $z \in V(H) - V(G)$ such that $a < e_H(z) < a + 1$, then $ce_H(z) = r$. On the other hand, $ce_H(z) \geq r + 1$ for all $z \in V(H) - V(G)$, which is impossible.

We next verify (ii). If there is $x \in V(F)$ such that $x$ is adjacent to some vertex $z$ in $V(H) - V(G)$, then $|e_H(x) - e_H(z)| \leq 1$. Thus $a \leq e_H(z) \leq a + 1$ and so $ce_H(z) = r$, which is impossible. 

By Proposition 4.3.5, if $F = P_k$ and $G = P_{k+1}$, then there is no connected graph $H$ such that $\text{Cen}(H) = F$ and $\text{CC}(H) = G$. To see this, assume that there is a connected graph $H$ such that $\text{Cen}(H) = P_k$ and $\text{CC}(H) = P_{k+1}$. Let $G = P_{k+1} = (v_1, v_2, \ldots, v_{k+1})$ and $F = P_k = P_{k+1} - v_{k+1}$. By Proposition 4.3.5(ii), no vertex $v_i$ ($1 \leq i \leq k$) is adjacent to any vertex in $V(H) - V(G)$. Thus $v_k$ is the only vertex of $F$ that is adjacent to a vertex in $V(G) - V(F) = \{v_{k+1}\}$. This implies that $e_H(v_i) > e_H(v_k)$ for each $i$ with $1 \leq i \leq k - 1$, which contradicts Proposition 4.3.5(i).
Suppose that $F$ and $G$ are two graphs where $F$ is a proper induced subgraph of $G$. The $F$-augmenting graph $\tilde{G}$ of $G$ is obtained from $G$ by adding two new vertices $x$ and $y$ and joining each of $x$ and $y$ to every vertex of $F$.

**Theorem 4.3.6** If $F$ and $G$ are two graphs where $F$ is a proper induced subgraph of $G$, then there is a connected graph $H$ such that $\text{Cen}(H) = F$ and $\text{CC}(H) = \tilde{G}$.

**Proof.** Let $U = V(G) - V(F) = \{u_1, u_2, \ldots, u_p\}$. We begin with two copies of $K_p$ with vertex sets $V = \{v_1, v_2, \ldots, v_p\}$ and $W = \{w_1, w_2, \ldots, w_p\}$, respectively. Let $Q_1 = (x_1, x_2, x_3)$ and $Q_2 = (y_1, y_2, y_3)$ be two copies of the path $P_3$. The graph $H$ is constructed by (1) joining $x$ to each vertex in $V(F) \cup U \cup \{x_1\}$, (2) joining $y$ to each vertex in $V(F) \cup W \cup \{y_1\}$ and (3) joining $u_i$ to $v_i$ and $w_i$ for $1 \leq i \leq p$. (See Figure 4.4).

![Figure 4.4: A graph $H$ with given center and color-center](image)

Since $e_H(v) = 4$ if $v \in V(F)$, $e_H(v) = 5$ if $v \in U \cup \{x, y\}$, $e_H(v) = 6$ if $v \in V \cup W$ and $e_H(x_i) = e_H(y_i) = 5 + i$ for $1 \leq i \leq 3$, it follows that $\text{Cen}(H) = F$. 

Furthermore, $ce_H(v) = 3$ if $v \in V(\tilde{G})$, $ce_H(v) = 4$ if $v \in V \cup W$, $ce_H(x_i) = ce_H(y_i) = 3 + i$ for $i = 1, 2$ and $ce_H(x_3) = ce_H(y_3) = 5$. Thus $CC(H) = \tilde{G}$.

Bielak and Syslo [1] verified that not every graph is the periphery of some graph.

**Theorem 4.3.7** A nontrivial graph $G$ is the periphery of some graph if and only if every vertex of $G$ has eccentricity 1 or no vertex of $G$ has eccentricity 1.

We now present a necessary and sufficient condition for a graph to be the color-periphery of some graph.

**Theorem 4.3.8** A nontrivial graph $G$ is the color-periphery of some graph if and only if every vertex of $G$ has color-eccentricity 1 or no vertex of $G$ has color-eccentricity 1.

**Proof.** First, suppose that every vertex of $G$ has color-eccentricity 1. Then $G^2$ is complete and $CP(G) = G$. Next, assume that no vertex of $G$ has color-eccentricity 1. Let $V(G) = \{u_1, u_2, \ldots, u_n\}$ and let $F = K_n$ with $V(F) = \{v_1, v_2, \ldots, v_n\}$. The graph $H$ is obtained from $G$ and $F$ by joining $v_i$ to $u_i$ for $1 \leq i \leq n$. Since $ce(u_i) = 1$ for $1 \leq i \leq n$ and $ce(u_i) = 2$ for $1 \leq i \leq n$. Thus $CP(H) = G$.

For the converse, suppose that there is a graph $G$ for which some but not all vertices have color-eccentricity 1 and $G$ is the color-periphery of a connected graph $H$. Then $G$ is a proper induced subgraph of $H$ and $cdiam(H) \geq 2$. Let $u$ be a vertex
of $G$ having color eccentricity 1 in $G$, that is, $ce_G(u) = 1$. Then $cd_G(u, w) = 1$ for all $w \in V(G) - \{u\}$. Consequently, $u$ is adjacent in $G^2$ to all vertices $w \in V(G) - \{u\}$. Let $v$ be a vertex of $H$ such that $ce_H(u) = cd_H(u, v) = cdiam(H) \geq 2$. Therefore, $ce_H(v) = cdiam(H) \geq 2$ and so $v$ is a color peripheral vertex of $H$. On the other hand, since $cd_H(v, u) > 2$, the vertex $v$ is not adjacent to $u$ in $H^2$ and so $v$ is not a vertex of $G$, which is a contradiction.

4.4 Color Connectivities of Graphs

By Theorem 4.1.3, the distance-colored graph $G^2$ is color-connected and so for every two vertices $u$ and $v$ of $G^2$, there is at least one properly colored $u-v$ path in $G^2$. This gives rise to another concept. If $G$ is a connected graph with connectivity $\kappa(G) = \kappa$, then it follows from a well-known theorem of Whitney [31] that for every two distinct vertices $u$ and $v$ of $G$, the graph $G$ contains $\kappa$ internally disjoint $u-v$ paths. A graph $G$ is $p$-connected if $\kappa(G) \geq p$. For a connected graph $G$ and an integer $k \geq 2$, the distance-colored graph $G^k$ is properly colored $p$-connected or simply properly $p$-connected if for every two distinct vertices $u$ and $v$ of $G^k$, there are $p$ internally disjoint properly colored $u-v$ paths in $G^k$. If $G$ is a complete graph, then $G^k$ is not properly $p$-connected for all integers $k, p$ with $k \geq 1$ and $p \geq 2$. Thus we consider only non-complete connected graphs. For a connected graph $G$ and an integer $k \geq 2$, the color-connectivity of $G^k$ is the maximum positive integer $p$ for which $G^k$ is properly $p$-connected. In Section 4.4.1, we study the color-connectivities of the square and the cube of a connected graph,
while in Section 4.4.2 we determine all pairs $k, n$ of positive integers for which $P^k_n$ is properly $k$-connected.

### 4.4.1 Properly 2-Connected Graphs

For a connected graph $G$ of order 3 or more, it is well known that $G^2$ is 2-connected and so every two distinct vertices $u$ and $v$ of $G$ are connected by two internally disjoint $u - v$ paths in $G^2$. By Theorem 4.1.3, the distance-colored graph $G^2$ is color-connected. However, $G^2$ need not be properly 2-connected, that is, it is possible that for some pair $u, v$ of vertices of $G^2$, two internally disjoint properly colored $u - v$ paths do not exist in $G^2$. For example, if $G = K_{1,n-1}$ where $n \geq 3$, then $G^2 = K_n$ is 2-connected but not properly 2-connected (as any two end-vertices of $G$ are not connected by two internally disjoint properly colored paths in $G^2$).

On the other hand, if $G$ is 2-connected, then $G^2$ is properly 2-connected.

**Theorem 4.4.1** If $G$ is a 2-connected graph that is not complete, then $G^2$ is properly 2-connected.

**Proof.** Let $u$ and $v$ be two distinct vertices of $G$. Since $G$ is 2-connected, $G$ contains at least two internally disjoint $u - v$ paths. Hence, $u$ and $v$ lie on a common cycle. Among all cycles in $G$ that contain $u$ and $v$, let $C$ be one of shortest length. Construct two internally disjoint $u - v$ paths in $G$, $P$ and $P'$, by traversing $C$ in opposite directions. Say $P = (u = x_1, x_2, \ldots, x_s = v)$ and $P' = (u = y_1, y_2, \ldots, y_t = v)$. By the defining property of $P$ and $P'$, it follows that $d_P(x_i, x_{i+2}) = 2 = d_G(x_i, x_{i+2})$ for $1 \leq i \leq s-2$ and $d_{P'}(y_j, y_{j+2}) = 2 = d_G(y_j, y_{j+2})$.
for $1 \leq j \leq t-2$. By Lemma 4.1.2, there is a properly colored $u - v$ path $Q$ in the square of $P$ and a properly colored $u - v$ path $Q'$ in the square of $P'$. Since $P$ and $P'$ are internally disjoint, so are $Q$ and $Q'$. Therefore, $G^2$ is properly 2-connected.

For a connected graph $G$, its square can be properly 2-connected without $G$ being 2-connected, however. In the case of trees, for example, we know precisely those trees whose square is properly 2-connected. A double star is a tree of diameter 3. The double stars are the only trees $T$ for which $T^2$ is properly 2-connected.

**Theorem 4.4.2** Let $T$ be a tree of order at least 3. Then $T^2$ is properly 2-connected if and only if $T$ is a double star.

**Proof.** First, suppose that $T$ is a double star whose central vertices are $x$ and $y$. Let $X = \{x_1, x_2, \ldots, x_r\}$ and $Y = \{y_1, y_2, \ldots, y_s\}$ be the sets of end-vertices of $T$ such that $x$ is adjacent to every vertex in $X$ and $y$ is adjacent to every vertex in $Y$, where then $r, s \geq 1$. Let $u, v \in V(T)$. We show that $u$ and $v$ are connected by two internally disjoint properly colored paths. If $\{u, v\} = \{x, y\}$, say $u = x$ and $v = y$, then $(u, v)$ and $(u, x_1, v)$ are two internally disjoint properly colored $u - v$ paths in $T^2$. If $\{u, v\} \cap \{x, y\} = \emptyset$, then we may assume that $u \in X$. If $v \in X$, say $v = x_2$, then $(u, v)$ and $(u, x, y_1, y, v)$ are two internally disjoint properly colored $u - v$ paths in $T^2$. If $v \in Y$, say $v = y_1$, then $(u, v)$ and $(u, y, v)$ are two internally disjoint properly colored $u - v$ paths in $T^2$. If $|\{u, v\} \cap \{x, y\}| = 1$, then we may assume that $u = x$ and $v \neq y$. If $v \in X$, say $v = x_1$, then $(u, v)$ and $(u, y, v)$ are two internally disjoint properly colored $u - v$ paths in $T^2$. If $v \in Y$, say $v = y_1$,
then \((u, v)\) and \((u, x_1, y, v)\) are two internally disjoint properly colored \(u - v\) paths in \(T^2\).

For the converse, assume that \(T\) is not a double star. Let \(d = \text{diam}(T)\) and so \(d \neq 3\). If \(d = 2\), then \(T\) is a star and we saw that \(T^2\) is not properly 2-connected. Thus we may assume that \(d \geq 4\). Let \(u\) be an end-vertex of \(T\) whose eccentricity \(e_T(u)\) is \(d\) and let \(v\) be a vertex of \(T\) such that \(d_T(u, v) = 4\). We show that there are no two internally disjoint properly colored \(u - v\) paths in \(T^2\). Suppose that \((u, v_1, v_2, v_3, v)\) is the \(u - v\) path in \(T\). Assume to the contrary that \(T^2\) contains two properly colored \(u - v\) paths, \(P_1\) and \(P_2\). By an extensive case-by-case analysis, it can be shown that each properly colored \(u - v\) path in \(T^2\) must use at least two vertices in \(\{v_1, v_2, v_3\}\). Hence, \(P_1\) and \(P_2\) must each contain at least two vertices from \(\{v_1, v_2, v_3\}\). But this is impossible since we assumed that \(P_1\) and \(P_2\) were internally disjoint.

We have seen that if \(G\) is a connected graph, then \(G^2\) is 2-connected. Since \((G^2)^2 = G^4\) for each connected graph \(G\), the following is an immediate consequence of Theorem 4.4.1.

**Corollary 4.4.3** If \(G\) is a connected graph such that \(G^2\) is not complete, then \(G^4\) is properly 2-connected.

By Corollary 4.4.3, if \(G\) is a connected graph of diameter at least 3, then \(G^4\) is properly 2-connected. This gives rise to a natural question: For a connected graph \(G\) of diameter at least 3, what is the minimum \(k\) such that \(G^k\) is properly
2-connected? By Theorem 4.4.2 and Corollary 4.4.3, either \( k = 3 \) or \( k = 4 \). Next, we show that \( k = 3 \).

**Theorem 4.4.4** If \( G \) is a connected graph of diameter at least 3, then \( G^3 \) is properly 2-connected.

**Proof.** Since \( \text{diam}(G) \geq 3 \), it follows that the order of \( G \) is at least 4. Let \( u \) and \( v \) be two distinct vertices of \( G \). We show that \( u \) and \( v \) are connected by two internally disjoint properly colored paths in \( G^3 \). We consider two cases, according to whether \( 1 \leq d_G(u, v) \leq 2 \) or \( d_G(u, v) \geq 3 \).

*Case 1. \( 1 \leq d_G(u, v) \leq 2 \).* Suppose that \( u \) has a neighbor \( x(\neq v) \) that is not a neighbor of \( v \). Then \((u, v)\) and \((u, x, v)\) are two internally disjoint properly colored \( u - v \) paths in \( G^3 \). Similarly, \( G^3 \) contains two internally disjoint \( u - v \) paths in \( v \) has a neighbor \( x(\neq u) \) that is not a neighbor of \( u \). We may therefore assume that \( N_G(u) - v = N_G(v) - u \). Since \( \text{diam}(G) \geq 3 \), it follows that \( \text{rad}(G) \geq 2 \) and \( e_G(y) \geq 2 \) for all \( y \in V(G) \). If \( e_G(u) \geq 3 \) or \( e_G(v) \geq 3 \), say the former, then there is \( w \in V(G) \) such that \( d_G(u, w) = 3 \). Let \((u, x_1, x_2, w)\) be a \( u - w \) geodesic in \( G \). Since \( N_G(u) - v = N_G(v) - u \), it follows that \( v \) is adjacent to \( x_1 \) (and \( v \) is not adjacent to \( x_2 \)). Then \((u, v)\) and \((u, w, x_1, v)\) are two internally disjoint properly colored \( u - v \) paths in \( G^3 \). Thus we may assume that \( e_G(u) = e_G(v) = 2 \). Then \( \text{rad}(G) = 2 \) and \( 3 \leq \text{diam}(G) \leq 4 \). Since \( N_G(u) - v = N_G(v) - u \) and \( e_G(u) = e_G(v) = 2 \), it follows that \( G \) contains a subgraph isomorphic to one of the four graphs in Figure 4.5 such that \( d(w, w') = 3 \). A dashed line between \( u \) and \( v \) in Figure 4.5 indicates that \( u \)
and \(v\) are not adjacent. Thus neither \(u\) nor \(v\) is adjacent to \(w'\). Then \((u, v)\) and \((u, w, w', v)\) are two internally disjoint properly colored \(u - v\) paths in \(G^3\).

**Figure 4.5: A step in the proof of Theorem 4.4.4**

**Case 2.** \(d_G(u, v) \geq 3\). Let \(d = d_G(u, v)\) and let \(P = (\begin{array}{c} u = v_0, v_1, v_2, \ldots, v_d = v \end{array})\) be a \(u - v\) geodesic in \(G\). Thus \(d_P(x, y) = d_G(x, y)\) for all \(x, y \in V(P)\). If \(d = 3\), then \((u, v)\) and \((u, v_2, v)\) are two internally disjoint properly colored \(u - v\) paths in \(G^3\). If \(d = 4\), then \((u, v_1, v)\) and \((u, v_3, v)\) are two internally disjoint properly colored \(u - v\) paths in \(G^3\). Thus, we may assume that \(d \geq 5\). Suppose that \(d \equiv i \pmod{4}\) where \(i = 0, 1, 2, 3\) and let \(d = 4k + i\) for some positive integer \(k\). In each case, \(G^3\) contains two internally disjoint properly colored \(u - v\) paths \(Q_1\) and \(Q_2\) as follows: For \(d = 4k\),

\[
Q_1 = (u, v_1, v_4, v_5, v_8, v_9, \ldots, v_{4k-3}, v_{4k})
\]

\[
Q_2 = (u, v_2, v_3, v_6, v_7, v_{10}, \ldots, v_{4k-2}, v_{4k}).
\]
For $d = 4k + 1$,

\[
Q_1 = (u, v_1, v_4, v_5, v_8, v_9, \ldots, v_{4k}, v_{4k+1})
\]
\[
Q_2 = (u, v_2, v_3, v_6, v_7, v_{10}, \ldots, v_{4k-1}, v_{4k+1}).
\]

For $d = 4k + 2$,

\[
Q_1 = (u, v_1, v_4, v_5, v_8, v_9, \ldots, v_{4k}, v_{4k+2})
\]
\[
Q_2 = (u, v_2, v_3, v_6, v_7, v_{10}, \ldots, v_{4k-1}, v_{4k+2}).
\]

For $d = 4k + 3$,

\[
Q_1 = (u, v_1, v_4, v_5, v_8, v_9, \ldots, v_{4k+1}, v_{4k+3})
\]
\[
Q_2 = (u, v_2, v_3, v_6, v_7, v_{10}, \ldots, v_{4k+2}, v_{4k+3}).
\]

Therefore, $G^3$ is properly 2-connected.

Theorem 4.4.4 brings up another question: For a connected graph $G$ of diameter at least 3, is $G^3$ properly 3-connected? We will see in the next section that this is not true in general.

### 4.4.2 Color-Connectivities of the Powers of a Path

We have seen in the proofs of Theorems 4.4.1 and 4.4.4 that the color-connectivity of the distance-colored graph $P_n^k$ of a path $P_n$ of order $n$ plays an important role in determining the color-connectivity of a connected graph. Thus in this section, we investigate the color-connectivities of the distance-colored graph $P_n^k$ for integers $n$ and $k$ with $n \geq k + 1 \geq 3$. By Theorem 4.4.2, the distance-colored graph $P_n^2$
is properly 2-connected if and only if $n = 4$. By Theorem 4.4.4, $P_n^3$ is properly 2-connected for all $n \geq 4$. However, $P_n^3$ is not properly 3-connected in general. In fact, for $n \geq 4$, it can be verified that the distance-colored graph $P_n^3$ is properly 3-connected if and only if $n = 6$. Next, we determine all pairs $k, n$ of integers with $n \geq k + 1 \geq 4$ for which $P_n^k$ is properly $k$-connected. In order to do this, we first present two lemmas.

**Lemma 4.4.5** For each even integer $k \geq 2$, the distance-colored graph $P_{k+2}^k$ is properly $k$-connected.

**Proof.** We proceed by induction on even integers $k \geq 2$. By Theorem 4.4.2, $P_4^2$ is properly 2-connected and so the statement holds for $k = 2$. Suppose that $P_{k+2}^k$ is properly $k$-connected for some even integer $k \geq 2$. Let $P_{k+4}^k = (v_0, v_1, v_2, \ldots, v_{k+3})$ and let $u$ and $v$ be two distinct vertices of $P_{k+4}^k$. We show that there are $k+2$ internally disjoint properly colored $u-v$ paths in $P_{k+4}^{k+2}$. We consider two cases.

**Case 1.** $u$ and $v$ are not end-vertices of $G$. Then $u, v \in \{v_1, v_2, \ldots, v_{k+2}\}$. Let $P_{k+2} = (v_1, v_2, \ldots, v_{k+2})$. Since $P_{k+2}^k$ is properly $k$-connected by the induction hypothesis, there are $k$ internally disjoint properly colored $u-v$ paths $Q_1, Q_2, \ldots, Q_k$ in $P_{k+2}^k$. Let $Q_{k+1} = (u, v_0, v)$ and $Q_{k+2} = (u, v_{k+3}, v)$. Therefore, $Q_1, Q_2, \ldots, Q_{k+2}$ are $k+2$ internally disjoint properly colored $u-v$ paths in $P_{k+4}^{k+2}$.

**Case 2.** At least one of $u$ and $v$ is an end-vertex of $G$, say $u = v_0$. If $v = v_{k+3}$, then let $Q_i = (v_0, v_i, v_{k+3})$ for $1 \leq i \leq k+2$, producing $k+2$ internally disjoint properly colored $v_0-v_{k+3}$ paths in $P_{k+4}^{k+2}$. Thus, we may assume that $v \neq v_{k+3}$. If
v = v_{2\ell+1} for some nonnegative integer \ell, where 1 \leq 2\ell + 1 \leq k + 1, then construct k + 2 internally disjoint properly colored \(v_0 - v_{2\ell+1}\) paths \(Q_1, Q_2, \ldots, Q_{k+2}\) in \(P_{k+4}^{k+2}\) as follows:

\[
Q_i = \begin{cases} 
(v_0, v_i, v_{2\ell+1}) & \text{if } 1 \leq i \leq 2\ell \\
(v_0, v_{2\ell+1}) & \text{if } i = 2\ell + 1 \\
(v_0, v_i, v_{2\ell+1}) & \text{if } 2\ell + 2 \leq i \leq k + 2.
\end{cases}
\]

If \(v = v_{2\ell}\) for some positive integer \ell where 2 \leq 2\ell \leq k + 2, then construct \(k + 2\) internally disjoint properly colored \(v_0 - v_{2\ell}\) paths \(Q_1, Q_2, \ldots, Q_{k+2}\) in \(P_{k+4}^{k+2}\) as follows:

\[
Q_i = \begin{cases} 
(v_0, v_i, v_{2\ell}) & \text{if } 1 \leq i \leq k + 2 \text{ and } i \neq \ell, 2\ell \\
(v_0, v_{2\ell}) & \text{if } i = 2\ell \\
(v_0, v_\ell, v_{k+3}, v_{2\ell}) & \text{if } i = \ell.
\end{cases}
\]

Therefore, \(P_{k+4}^{k+2}\) is properly \((k + 2)\)-connected. 

Lemma 4.4.6 For each odd integer \(k \geq 3\), the distance-colored graph \(P_{k+3}^k\) is properly \(k\)-connected.

Proof. We proceed by induction on odd integers \(k \geq 3\). Figure 4.6 shows that \(P_5^3\) is properly 3-connected.

Assume that \(P_{k+3}^k\) is properly \(k\)-connected for some odd integer \(k \geq 3\). We show that \(P_{k+5}^{k+2}\) is properly \((k + 2)\)-connected. Let \(P_{k+5} = (v_0, v_1, \ldots, v_{k+4})\) and let \(u\) and \(v\) be two distinct vertices of \(P_{k+5}\). We may assume that \(u = v_s\) and \(v = v_t\) where \(0 \leq s < t \leq k + 4\). We consider two cases.

Case 1. \(u\) and \(v\) are not end-vertices of \(P_{k+5}\). Then \(u, v \in \{v_1, v_2, \ldots, v_{k+3}\}\).

First, suppose that \(u \neq v_1\) and \(v \neq v_{k+3}\). By the induction hypothesis, there are
$k$ internally disjoint properly colored $u - v$ paths $Q_1, Q_2, \ldots, Q_k$ whose vertices belong to $\{v_1, v_2, \ldots, v_{k+3}\}$. Let $Q_{k+1} = (u, v_0, v)$ and $Q_{k+2} = (u, v_{k+4}, v)$. Then $Q_1, Q_2, \ldots, Q_{k+2}$ are $k + 2$ internally disjoint properly colored $u - v$ paths in $P^{k+2}_{k+5}$ and so $P^{k+2}_{k+5}$ is properly $(k + 2)$-connected.

Next, suppose that either $u = v_1$ or $v = v_{k+3}$. We may assume, without loss of generality, that $u \in \{v_1, v_2, \ldots, v_{k+2}\}$ and $v = v_{k+3}$. Let $d_{P^{k+5}_{k+5}}(u, v) = p$. Suppose that $u = v_i$, where $1 \leq i \leq k + 2$, and so $p = k + 3 - i$. If $p$ is odd, then define

$$Q_j = \begin{cases} (v_i, v_j, v_{k+3}) & \text{if } j \in \{1, 2, \ldots, k + 2\} - \{i\} \\ (v_i, v_{k+3}) & \text{if } j = i. \end{cases}$$

If $p$ is even, then define

$$Q_j = \begin{cases} (v_i, v_j, v_{k+3}) & \text{if } j \in \{1, 2, \ldots, k + 2\} - \{i, \frac{i+k+3}{2}\} \\ (v_i, v_{k+4}, v_{k+3}) & \text{if } j = \frac{i+k+3}{2}. \end{cases}$$

In each case, $Q_1, Q_2, \ldots, Q_{k+2}$ are $k + 2$ internally disjoint properly colored $u - v$ paths in $P^{k+2}_{k+5}$ and so $P^{k+2}_{k+5}$ is properly $(k + 2)$-connected.

**Case 2.** At least one of $u$ and $v$ is an end-vertex of $P_{k+5}$, say $u = v_0$. Observe that the neighborhood of $v_0$ in $P^{k+2}_{k+5}$ is $\{v_1, v_2, \ldots, v_{k+2}\}$. 
• If $v = v_t$ and $1 \leq t \leq k + 2$, then define

$$Q_1 = \begin{cases} (v_0, v_t) & \text{if } t \in \{1, 2, \ldots, k + 2\} - \{t, \frac{t}{2}\} \\ (v_0, v_t, v_{k+3}, v_t) & \text{if } t = \frac{t}{2} \end{cases}$$

• If $v = v_{k+3}$, then define $Q_1 = (v_0, v_t, v_{k+3})$ for $1 \leq t \leq k + 2$ and $t \neq \frac{k+3}{2}$ and

$$Q_{\frac{k+3}{2}} = (v_0, v_{\frac{k+3}{2}}, v_{k+4}, v_{k+3})$$

• If $v = v_{k+4}$, then define $Q_1 = (v_0, v_1, v_{k+3}, v_{k+4})$ and $Q_2 = (v_0, v_1, v_{k+4})$ for $2 \leq t \leq k + 2$

In each case, $Q_1, Q_2, \ldots, Q_{k+2}$ are $k + 2$ internally disjoint properly colored $u - v$ paths in $P_{k+5}^{k+2}$ and so $P_{k+5}^{k+2}$ is properly $(k + 2)$ connected.

We are now prepared to determine all pairs $k, n$ of integers with $n \geq k + 1 \geq 4$ for which $P_n^k$ is properly $k$-connected.

**Theorem 4.4.7** Let $k$ and $n$ be integers where $n \geq k + 1 \geq 4$. Then $P_n^k$ is properly $k$-connected if and only if $n$ is even and either (i) $k$ is odd and $k + 3 \leq n \leq 2k$ or (ii) $k$ is even and $k + 2 \leq n \leq 2k$.

**Proof.** First, suppose that $n$ is even and we show that $P_n^k$ is properly $k$-connected if either (i) $k$ is odd and $k + 3 \leq n \leq 2k$ or (ii) $k$ is even and $k + 2 \leq n \leq 2k$. We consider two cases.

**Case 1** $k$ is odd and $k + 3 \leq n \leq 2k$. We proceed by a finite induction on integers $n$ to show that, for each odd integer $k \geq 3$, if $k + 3 \leq n \leq 2k$, then $P_n^k$
is properly $k$-connected. By Lemma 4.4.6, this statement is true for $n = k + 3$. Assume, for each odd integer $k \geq 3$, that $P_n^k$ is properly $k$-connected for an even integer $n$ with $k + 3 \leq n \leq 2k - 2$. We show that $P_{n+2}^k$ is properly $k$-connected. Let $P_{n+2} = (v_0, v_1, \ldots, v_{n+1})$ and let $u$ and $v$ be two distinct vertices of $P_{n+2}$. First, suppose that $d_{P_{n+2}}(u, v) < n$. We may assume, without loss of generality, that $u, v \in \{v_0, v_1, \ldots, v_{n-1}\}$. By the induction hypothesis, there are $k$ internally disjoint properly colored $u - v$ paths in $P_n^k$ where $P_n = (v_0, v_1, \ldots, v_{n-1})$ and so in $P_{n+2}^k$ as well. Next, suppose that $n < d_{P_{n+2}}(u, v) \leq n + 1$. We may assume that $u = v_0$. If $v = v_n$, then define

$$Q_i = \begin{cases} (v_0, v_i, v_{i+k}, v_n) & \text{if } 1 \leq i < n - k \text{ and } i \neq \frac{n}{2} \\ (v_0, v_i, v_n) & \text{if } n - k \leq i \leq k \text{ and } i \neq \frac{n}{2} \\ (v_0, v_{n-1}, v_n) & \text{if } i = \frac{n}{2}. \end{cases}$$

If $v = v_{n+1}$, then define

$$Q_i = \begin{cases} (v_0, v_i, v_{i+k}, v_{n+1}) & \text{if } 1 \leq i \leq n - k \\ (v_0, v_i, v_{n+1}) & \text{if } n - k < i \leq k. \end{cases}$$

In each case, $Q_1, Q_2, \ldots, Q_k$ are $k$ internally disjoint properly colored $u - v$ paths in $P_{n+2}^k$ and so $P_{n+2}^k$ is properly $k$-connected.

**Case 2. $k$ is even and $k + 2 \leq n \leq 2k$.** We proceed by a finite induction on integers $n$ to show that, for each even integer $k \geq 4$, if $k + 2 \leq n \leq 2k$, then $P_n^k$ is properly $k$-connected. By Lemma 4.4.5, this statement is true for $n = k + 2$. Assume for each even integer $k$ that $P_n^k$ is properly $k$-connected for an even integer $n$ with $k + 2 \leq n \leq 2k - 2$. An argument similar to the one in Case 1 shows that $P_{n+2}^k$ is properly $k$-connected.
To verify the converse, we show that if $P_n^k$ is properly $k$-connected where $n \geq k + 1 \geq 4$, then $n$ is even and either (i) $k$ is odd and $k + 3 \leq n \leq 2k$ or (ii) $k$ is even and $k + 2 \leq n \leq 2k$. We consider two cases, according to $n$ is odd or $n$ is even. In the case when $n$ is odd, we show that $P_n^k$ is not properly $k$-connected for all $k$ and $n$ with $k \geq 2$ and $n \geq k + 1$. In the case when $n$ is even, we show that either (i) $k$ is odd and $k + 3 \leq n \leq 2k$ or (ii) $k$ is even and $k + 2 \leq n \leq 2k$.

Case 1. $n$ is odd. We show that $P_n^k$ is not properly $k$-connected for all $k$ and $n$ with $k \geq 2$ and $n \geq k + 1$. Assume, to the contrary, that for some integer $k \geq 2$ there is an odd integer $n \geq k + 1$ such that $P_n^k$ is properly $k$-connected. By Theorem 4.4.2, $k \geq 3$. Let $n = 2\ell + 1$ for some integer $\ell \geq 2$ and let $P_n = (v_0, v_1, \ldots, v_{2\ell})$. Since $P_n^k$ is properly $k$-connected, there are $k$ internally disjoint properly colored $v_0 - v_{2\ell}$ paths $Q_1, Q_2, \ldots, Q_k$ in $P_n^k$. Since $\text{deg}_{P_n^k} v_0 = k$ and $v_0$ is adjacent to $v_1, v_2, \ldots, v_k$ in $P_n^k$, we may assume that $v_0$ is adjacent to $v_i$ in $Q_i$ for $1 \leq i \leq k$. Since these $k$ paths are internally disjoint, this implies that $v_1 \in V(Q_1)$ must be adjacent to $v_{k+1}$ in $Q_1$ and so $v_2 \in V(Q_2)$ must be adjacent to $v_{k+2}$ in $Q_2$. Continuing in this manner, we obtain that $v_i \in V(Q_i)$ must be adjacent to $v_{k+i}$ in $Q_i$ for $1 \leq i \leq k$. However then, $Q_\ell = (v_0, v_{\ell}, v_{2\ell})$ is not properly colored, which is a contradiction.

Case 2. $n$ is even. In this case, for an odd integer $k \geq 3$, we show that if $n = k + 1$ or $n \geq 2k + 2$, then $P_n^k$ is not properly $k$-connected; while for an even integer $k \geq 2$, we show that if $n \geq 2k + 2$, then $P_n^k$ is not properly $k$-connected. Therefore, it suffices to show that if $n = k + 1$ or $n \geq 2k + 2$, then $P_n^k$ is not
properly \( k \)-connected. Assume, to the contrary, that \( P^k_n \) is properly \( k \)-connected for some integers \( k \geq 2 \) and \( n \) such that \( n = k + 1 \) or \( n \geq 2k + 1 \). We consider these two subcases.

Subcase 2.1. \( n = k + 1 \). Let \( P_{k+1} = (v_0, v_1, \ldots, v_k) \). Since \( P^k_{k+1} \) is properly \( k \)-connected, there are \( k \) internally disjoint properly colored \( v_0 - v_2 \) paths \( Q_1, Q_2, \ldots, Q_k \) in \( P^k_{k+1} \). Since \( \deg_{P^k_{k+1}} v_0 = k \) and \( v_0 \) is adjacent to \( v_1, v_2, \ldots, v_k \) in \( P^k_{k+1} \), it follows that \( v_0 \) is adjacent to \( v_i \) (\( 1 \leq i \leq k \)) in exactly one of these \( k \) paths. We may assume that \( v_0 \) is adjacent to \( v_i \) in \( Q_i \) for \( 1 \leq i \leq k \). However then, \( Q_1 = (v_0, v_1, v_2) \) is not properly colored, which is a contradiction.

Subcase 2.2. \( n \geq 2k + 2 \). Let \( P_n = (v_0, v_1, \ldots, v_{n-1}) \). Since \( P^k_n \) is properly \( k \)-connected, there are \( k \) internally disjoint properly colored \( v_0 - v_{2k} \) paths \( Q_1, Q_2, \ldots, Q_k \) in \( P^k_n \). Since \( \deg_{P^k_n} v_0 = k \) and \( v_0 \) is adjacent to \( v_1, v_2, \ldots, v_k \) in \( P^k_n \), we may assume that \( v_0 \) is adjacent to \( v_i \) in \( Q_i \) for \( 1 \leq i \leq k \). Thus, \( v_i \in V(Q_i) \) must be adjacent to \( v_{k+i} \) in \( Q_i \) for \( 1 \leq i \leq k \). However then, \( v_0v_k \) and \( v_kv_{2k} \) are adjacent edges in \( Q_k \), which is impossible.

While it follows from Theorem 4.4.7 that the distance-colored graph \( P^k_n \) is not properly \( k \)-connected whenever \( n \geq 2k + 1 \geq 7 \), the following is believed to be true.

**Conjecture 4.4.8** For each integer \( k \geq 3 \), the distance-colored graph \( P^{k+1}_n \) is properly \( k \)-connected for all \( n \geq 2k + 1 \).
Conjecture 4.48 is true for \( k = 3 \), however, as we now verify. First, we introduce some additional definitions and notation. For a path \( Q \) of a connected graph \( H \) and \( v \in V(H) - V(Q) \), let \( t(Q, v) \) denote the distance between \( v \) and the terminal vertex of \( Q \). For a set \( Q \) of paths in a connected graph \( H \), let \( V(Q) \) be the set of vertices of \( H \) that belong to some path in \( Q \). For each \( v \in V(H) - V(Q) \), define \( T(Q, v) = \max\{t(Q, v) \mid Q \in Q\} \). That is, \( T(Q, v) \) is the maximum distance between \( v \) and the terminal vertex of a path in \( Q \).

For a connected graph \( G \), let \( Q \) be a properly colored \( u - v \) path in \( G^k \) for some integer \( k \geq 2 \) and let \( w \) be a vertex of \( G^k \) that does not belong to \( Q \). Then \( Q \) is extendable to \( w \) if \( (Q, w) \) is a properly colored \( u - w \) path in \( G^k \). In this case, we can extend \( Q \) to \( w \) in \( G^k \) and the path \( (Q, w) \) is then called the extension of \( Q \) to \( w \) in \( G^k \).

Let \( P_n = (v_0, v_1, \ldots, v_{n-1}) \) be a path of order \( n \geq 7 \). Next, we present an algorithm that produces three internally disjoint properly colored paths \( Q_1, Q_2 \) and \( Q_3 \) in \( P_n^4 \) with initial vertex \( v_0 \) such that the distance between \( v_{n-1} \) and the terminal vertex of each path \( Q_i \) \((1 \leq i \leq 3)\) is at most 4. That is, if \( Q = \{Q_1, Q_2, Q_3\} \), then \( T(Q, v_{n-1}) \leq 4 \).

**Algorithm 1** For \( P_n = (v_0, v_1, \ldots, v_{n-1}) \) where \( n \geq 7 \), this algorithm produces three internally disjoint properly colored paths \( Q_1, Q_2 \) and \( Q_3 \) in \( P_n^4 \) with initial vertex \( v_0 \) such that the distance between \( v_{n-1} \) and the terminal vertex of each path \( Q_i \), \((1 \leq i \leq 3)\) is at most 4.

**Input** An integer \( n \geq 7 \) and a path \( P_n = (v_0, v_1, \ldots, v_{n-1}) \).
**Step 1** The first path begins at \(v_0\) and moves to \(v_1\), the second path begins at \(v_0\) and moves to \(v_2\) and the third path begins at \(v_0\) and moves to \(v_4\).

At the end of Step 1, we obtain three paths

\[ Q^1_1 = (v_0, v_1), \ Q^1_2 = (v_0, v_2), \ Q^1_3 = (v_0, v_4) \]

Let \(Q^1 = \{Q^1_1, Q^1_2, Q^1_3\}\)

**Step 2** If \(T(Q^1, v_{n-1}) \leq 4\), then stop, while if \(T(Q^1, v_{n-1}) > 4\), then do the following

Let \(Q^1_p \in Q^1\) (1 \(\leq p \leq 3\)) such that \(t(Q^1_p, v_{n-1}) = T(Q^1, v_{n-1})\). Suppose that \(j\) is the smallest integer such that \(v_j\) does not belong to any path in \(Q^1\) and \(Q^1_p\) is extendable to \(v_j\) in \(P^n\). Let \(Q^2_p = (Q^1_p, v_j)\) and rename the remaining two paths \(Q^1_q\) and \(Q^1_{q'}\) (where \(q, q' \in \{1, 2, 3\} - \{p\}\)) in \(Q^1\) as \(Q^2_q\) and \(Q^2_{q'}\), respectively. Let \(Q^2 = \{Q^2_1, Q^2_2, Q^2_3\}\).

If \(T(Q^2, v_{n-1}) \leq 4\), then stop, while if \(T(Q^2, v_{n-1}) > 4\), then repeat this procedure above. In general, at the step \(\tau\) for an integer \(\tau \geq 1\), let \(Q^\tau = \{Q^\tau_1, Q^\tau_2, Q^\tau_3\}\).

**Step \(\tau+1\)** If \(T(Q^\tau, v_{n-1}) \leq 4\), then stop, while if \(T(Q^\tau, v_{n-1}) > 4\), then do the following

Let \(Q^\tau_p \in Q^\tau\) (1 \(\leq p \leq 3\)) such that \(t(Q^\tau_p, v_{n-1}) = T(Q^\tau, v_{n-1})\). Suppose that \(j\) is the smallest integer such that \(v_j\) does not belong to any path in \(Q^\tau\).
Q^i and Q^p is extendable to v_j in P^4_n. Let Q^{i+1} = (Q^i, v_j) and rename the remaining two paths Q^q and Q^{q'} (where q, q' ∈ \{1, 2, 3\} \{p\}) in Q^i as Q^{q+1} and Q^{q'+1}, respectively. Let Q^{i+1} = \{Q^{i+1}, Q^{i+1}, Q^{i+1}\}

Output Three internally disjoint properly colored paths Q_1, Q_2, and Q_3 in P^4_n with internal vertex v_0 such that T(\{Q_1, Q_2, Q_3\}, v_{n-1}) ≤ 4

For a properly colored path Q = (r_1, r_2, \ldots, r_\ell) in G^k, the distance sequence of Q is defined as d(Q) = d_1, d_2, \ldots, d_{\ell-1} where d_i = d_G(r_i, r_{i+1}) for 1 ≤ i ≤ \ell - 1

Lemma 4.4.9 For each integer n ≥ 7, let P_n = (v_0, v_1, \ldots, v_{n-1}). Then the distance colored graph P^4_n contains three internally disjoint properly colored v_0 - v_{n-1} paths

Proof. First, suppose that 7 ≤ n ≤ 9

- For n = 7, let Q_1 = (v_0, v_1, v_3, v_6), Q_2 = (v_0, v_2, v_5, v_6) and Q_3 = (v_0, v_4, v_6),
- For n = 8, let Q_1 = (v_0, v_1, v_3, v_7), Q_2 = (v_0, v_2, v_5, v_7) and Q_3 = (v_0, v_4, v_7),
- For n = 9, let Q_1 = (v_0, v_1, v_3, v_7, v_8), Q_2 = (v_0, v_2, v_5, v_8) and Q_3 = (v_0, v_4, v_5, v_8)

We now assume that n ≥ 10. By Algorithm 1, we obtain three internally disjoint properly colored paths Q_1, Q_2 and Q_3 in P^4_n for an arbitrarily large integer n such that T(\{Q_1, Q_2, Q_3\}, v_{n-1}) ≤ 4 and the distance sequences of these three paths are

\[d(Q_1) = 1, 2, 3, 2, 4, 3, 4, 3, 2, 4, 3, 4, 3, 2,\]
\[d(Q_2) = 2, 3, 4, 1, 3, 4, 1, 3, 4, 1, 3, 4, 1, 3, 4,\]
\[d(Q_3) = 4, 3, 4, 3, 2, 4, 3, 4, 3, 2, 4, 3,\]
Let

\[ s_1 : 4, 3, 4, 3, 2, \ s_2 : 1, 3, 4, 1, 3, 4 \text{ and } s_3 : 4, 3, 2, 4, 3. \]  \quad (4.7)

Then the three internally disjoint properly colored paths \( Q_1, Q_2 \) and \( Q_3 \) obtained by Algorithm 1 have the distance sequences as follows:

\[
\begin{align*}
    d(Q_1) & : 1, 2, 3, 2, s_1, s_1, \ldots \\
    d(Q_2) & : 2, 3, 4, s_2, s_2, \ldots \\
    d(Q_3) & : 4, 3, s_3, s_3, \ldots 
\end{align*}
\]

Observe that the sum of integers in \( s_i \) in (4.7) is 16 for \( 1 \leq i \leq 3 \). Let

\[ Q_1^0 = (v_0, v_1, v_3, v_6, v_8), \ Q_2^0 = (v_0, v_2, v_5, v_9) \text{ and } Q_3^0 = (v_0, v_4, v_7). \]

Then \( Q_1^0, Q_2^0 \) and \( Q_3^0 \) are internally disjoint properly colored paths in \( P_n^4 \) for a sufficiently large integer \( n \). For each integer \( i \geq 0 \), the three paths \( Q_1^{i+1}, Q_2^{i+1} \) and \( Q_3^{i+1} \) are constructed from \( Q_1^i, Q_2^i \) and \( Q_3^i \), respectively, as follows:

1. the path \( Q_1^{i+1} \) is obtained from \( Q_1^i \) and the path

\[ X_i = (v_{8+16i}, v_{8+16i+4}, v_{8+16i+7}, v_{8+16i+11}, v_{8+16i+14}, v_{8+16i+17}) \]

by identifying the the vertex \( v_{8+16i} \) in \( Q_1^i \) and \( X_i \), respectively.

2. the path \( Q_2^{i+1} \) is obtained from \( Q_2^i \) and the path

\[ Y_i = (v_{9+16i}, v_{9+16i+4}, v_{9+16i+8}, v_{9+16i+9}, v_{9+16i+12}, v_{9+16i+13}) \]

by identifying the the vertex \( v_{9+16i} \) in \( Q_2^i \) and \( Y_i \), respectively.

3. the path \( Q_3^{i+1} \) is obtained from \( Q_3^i \) and the path

\[ Z_i = (v_{7+16i}, v_{7+16i+4}, v_{7+16i+7}, v_{7+16i+9}, v_{7+16i+13}, v_{7+16i+17}) \]

by identifying the the vertex \( v_{7+16i} \) in \( Q_3^i \) and \( Z_i \), respectively.
Observe that $Q^i_1$, $Q^i_2$ and $Q^i_3$ are internally disjoint properly colored paths in $P^4_n$ for all $i \geq 0$ (where $n$ is sufficiently large). We now verify the following claim.

**Claim.** For each $n \geq 10$, three internally disjoint properly colored $v_0 - v_{n-1}$ paths $Q_1$, $Q_2$ and $Q_3$ can be constructed from the paths $Q^i_1$, $Q^i_2$ and $Q^i_3$ for some integer $i$ with $i \geq \lceil \frac{n-3}{16} \rceil$ by an appropriate modification.

**Proof of Claim.** The distance sequences of $Q^0_1$, $Q^0_2$ and $Q^0_3$ are

$$d(Q^0_1) : 1, 2, 3, 2, \quad d(Q^0_2) : 2, 3, 4, \quad \text{and} \quad d(Q^0_3) : 4, 3.$$ 

In general, for each integer $i \geq 0$,

$$d(Q^{i+1}_1) : d(Q^i_1), s_1, \quad d(Q^{i+1}_2) : d(Q^i_2), s_2, \quad \text{and} \quad d(Q^{i+1}_3) : d(Q^i_3), s_3$$

where $s_i$ are shown in (4.7) for $1 \leq i \leq 3$. Since (i) the sum of integers in $s_i$ is 16 for $1 \leq i \leq 3$ and (ii) the terminal terms in $d(Q^i_1)$, $d(Q^i_2)$, $d(Q^i_3)$ are 2, 4, 3 for all $i \geq 0$, it follows that, to verify the claim, it suffices to show that for each $n$ with $10 \leq n \leq 25$, three internally disjoint properly colored $v_0 - v_{n-1}$ paths $Q_1$, $Q_2$ and $Q_3$ can be obtained from $Q^1_1$, $Q^1_2$ and $Q^1_3$ in $P^4_n$. This is verified by the following table, where a path $(v_{i_1}, v_{i_2}, \ldots, v_{i_s})$ is denoted by $(i_1, i_2, \ldots, i_s)$.
The result then follows from the claim. □

We are now prepared to show that $P^4_n$ is properly 3-connected for each $n \geq 7$.

**Theorem 4.4.10**  For each integer $n \geq 7$, the distance-colored graph $P^4_n$ is properly 3-connected.

**Proof.**  Let $P_n = (v_0, v_1, \ldots, v_{n-1})$. We proceed by induction on $n$. For $n = 7$, it is straightforward to verify that $P^4_7$ is properly 3-connected. Assume that $P^4_k$ is properly 3-connected for some integer $k \geq 7$. Now let $u$ and $v$ be two distinct vertices of $P^4_{k+1}$. First, suppose that $\{u, v\} \neq \{v_0, v_k\}$. We may assume, without loss of generality, that $v_k \notin \{u, v\}$. Let $P_k = P_{k+1} - v_k$. Then $u, v \in V(P_k)$ and

<table>
<thead>
<tr>
<th>$n = 10$</th>
<th>$Q_1$: (0, 1, 3, 7, 9)</th>
<th>$Q_2$: (0, 2, 6, 9)</th>
<th>$Q_3$: (0, 4, 5, 9)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 18$</td>
<td>$Q_1$: (0, 1, 3, 6, 8, 12, 15, 17)</td>
<td>$Q_2$: (0, 2, 5, 9, 10, 13, 17)</td>
<td>$Q_3$: (0, 4, 7, 11, 14, 16, 17)</td>
</tr>
<tr>
<td>$n = 11$</td>
<td>$Q_1$: (0, 1, 3, 6, 10)</td>
<td>$Q_2$: (0, 2, 5, 9, 10)</td>
<td>$Q_3$: (0, 4, 7, 8, 10)</td>
</tr>
<tr>
<td>$n = 19$</td>
<td>$Q_1$: (0, 1, 3, 6, 8, 12, 15, 16, 18)</td>
<td>$Q_2$: (0, 2, 5, 9, 10, 13, 17, 18)</td>
<td>$Q_3$: (0, 4, 7, 11, 14, 18)</td>
</tr>
<tr>
<td>$n = 12$</td>
<td>$Q_1$: (0, 1, 3, 6, 8, 11)</td>
<td>$Q_2$: (0, 2, 5, 9, 11)</td>
<td>$Q_3$: (0, 4, 7, 11)</td>
</tr>
<tr>
<td>$n = 20$</td>
<td>$Q_1$: (0, 1, 3, 6, 8, 12, 15, 19)</td>
<td>$Q_2$: (0, 2, 5, 9, 10, 13, 17, 19)</td>
<td>$Q_3$: (0, 4, 7, 11, 14, 16, 19)</td>
</tr>
<tr>
<td>$n = 13$</td>
<td>$Q_1$: (0, 1, 3, 6, 8, 12)</td>
<td>$Q_2$: (0, 2, 5, 9, 12)</td>
<td>$Q_3$: (0, 4, 7, 11, 12)</td>
</tr>
<tr>
<td>$n = 21$</td>
<td>$Q_1$: (0, 1, 3, 6, 8, 12, 15, 19, 20)</td>
<td>$Q_2$: (0, 2, 5, 9, 10, 13, 17, 20)</td>
<td>$Q_3$: (0, 4, 7, 11, 14, 16, 20)</td>
</tr>
<tr>
<td>$n = 14$</td>
<td>$Q_1$: (0, 1, 3, 6, 8, 12, 13)</td>
<td>$Q_2$: (0, 2, 5, 9, 10, 13)</td>
<td>$Q_3$: (0, 4, 7, 11, 13)</td>
</tr>
<tr>
<td>$n = 22$</td>
<td>$Q_1$: (0, 1, 3, 6, 8, 12, 15, 19, 21)</td>
<td>$Q_2$: (0, 2, 5, 9, 10, 13, 17, 21)</td>
<td>$Q_3$: (0, 4, 7, 11, 14, 16, 21)</td>
</tr>
<tr>
<td>$n = 15$</td>
<td>$Q_1$: (0, 1, 3, 6, 8, 12, 14)</td>
<td>$Q_2$: (0, 2, 5, 9, 10, 13, 14)</td>
<td>$Q_3$: (0, 4, 7, 11, 14)</td>
</tr>
<tr>
<td>$n = 23$</td>
<td>$Q_1$: (0, 1, 3, 6, 8, 12, 15, 19, 22)</td>
<td>$Q_2$: (0, 2, 5, 9, 10, 13, 17, 22)</td>
<td>$Q_3$: (0, 4, 7, 11, 14, 16, 22)</td>
</tr>
<tr>
<td>$n = 16$</td>
<td>$Q_1$: (0, 1, 3, 6, 8, 12, 15)</td>
<td>$Q_2$: (0, 2, 5, 9, 10, 13, 15)</td>
<td>$Q_3$: (0, 4, 7, 11, 14, 15)</td>
</tr>
<tr>
<td>$n = 24$</td>
<td>$Q_1$: (0, 1, 3, 6, 8, 12, 15, 19, 22, 23)</td>
<td>$Q_2$: (0, 2, 5, 9, 10, 13, 17, 21, 23)</td>
<td>$Q_3$: (0, 4, 7, 11, 14, 16, 20, 23)</td>
</tr>
<tr>
<td>$n = 17$</td>
<td>$Q_1$: (0, 1, 3, 6, 8, 12, 15, 16)</td>
<td>$Q_2$: (0, 2, 5, 9, 10, 14, 16)</td>
<td>$Q_3$: (0, 4, 7, 11, 13, 16)</td>
</tr>
<tr>
<td>$n = 25$</td>
<td>$Q_1$: (0, 1, 3, 6, 8, 12, 15, 19, 21, 24)</td>
<td>$Q_2$: (0, 2, 5, 9, 10, 13, 17, 18, 22, 24)</td>
<td>$Q_3$: (0, 4, 7, 11, 14, 16, 20, 23, 24)</td>
</tr>
</tbody>
</table>
By the induction hypothesis, $P_k^4$ contains three internally disjoint properly colored $u-v$ paths and these three paths are also properly colored paths in $P_{k+1}^4$. Next, suppose that $\{u, v\} = \{v_0, v_k\}$. It then follows by Lemma 4.4.9 that $P_{k+1}^4$ contains three internally disjoint properly colored $v_0-v_k$ paths.

### 4.5 On H-Colored and H-Chromatic Graphs

In Section 4.4, we investigated connected graphs $G$ of $\text{diam}(G) = d$ for which $G^k$ is properly $p$-connected for some integers $k, p \geq 2$. If $G^k$ is properly $p$-connected with $p \leq k \leq d$, then $G^k$ contains a properly colored subdivision of $K_{2p}$ as a subgraph. In fact, there is no restriction on what properly colored subgraphs that $G^k$ can possess.

**Theorem 4.5.1** For every connected graph $H$, there exists a connected graph $G$ and a positive integer $k$ such that the distance-colored graph $G^k$ contains a properly colored copy of $H$ using $\chi'(H)$ colors.

**Proof.** Since the result is obvious if $H = K_2$, we may assume that $H$ has order at least 3. Let $\chi'(H) = \chi \geq 2$ and let $c$ be a proper $\chi$-edge coloring of $H$ using the colors $1, 2, \ldots, \chi$. We consider two cases.

**Case 1** $\chi = 2$. For each $e \in E(H)$ such that $c(e) = 2$, subdivide the edge $e$ exactly once. Denote the resulting graph by $G$. Then the distance-colored graph $G^2$ contains a copy of $H$ and a proper 2-edge coloring of $H$ using colors 1 and 2.
Case 2. \( \chi \geq 3 \). Let \( M = \chi - 3 \). For each \( e \in E(H) \), subdivide the edge \( e \) a total of \( M + c(e) - 1 \) times, resulting in a path of length \( M + c(e) \). Denote the resulting graph by \( G \). Suppose that \( e \) joins \( u \) and \( v \) in \( H \). Since there is a path of length \( M + c(e) \) connecting \( u \) and \( v \) in \( G \), it follows that \( d_G(u, v) \leq M + c(e) \). We claim that \( d_G(u, v) = M + c(e) \). Suppose that \( d_G(u, v) < M + c(e) \). Then there exists a \( u-v \) path \( P \) in \( G \) of length less than \( M + c(e) \). From the way in which \( G \) is constructed, the length of such a path must be at least \( (M+p) + (M+q) \) for some positive integers \( p \) and \( q \) such that \( p+q \geq 3 \). Since \( (M+p) + (M+q) \geq 2M + 3 \), it follows that \( 2M + 3 < M + c(e) \leq M + \chi \) and so \( M < \chi - 3 \), which is impossible. Thus \( G^{M+\chi} \) contains a copy of \( H \) and a proper \( \chi \)-edge coloring of \( H \) using colors \( M + 1, M + 2, \ldots, M + \chi \).

Theorem 4.5.1 raises the question of determining, for a given connected graph \( H \), the smallest order of a connected graph \( G \) such that there is a positive integer \( k \) for which the distance-colored graph \( G^k \) contains a properly colored copy of \( H \) and a proper edge coloring of \( H \) using \( \chi'(H) \) colors. Let \( H \) be a given connected graph. A connected graph \( G \) is \( H \)-colored if there is a positive integer \( k \) such that the distance-colored graph \( G^k \) contains a properly edge-colored copy of \( H \) with \( \chi'(H) \) colors. If the properly edge-colored copy of \( H \) in an \( H \)-colored graph \( G \) is produced by a \( \chi'(H) \)-edge coloring \( c \), then \( G \) is an \( H \)-colored graph with respect to \( c \). The minimum order of such a graph \( G \) is called the \( H \)-color-order of \( G \) or the color-order of \( H \), denoted by \( \text{co}(H) \). An \( H \)-colored graph of order \( \text{co}(H) \) is called an \( H \)-chromatic graph. Thus if \( H \) is a nontrivial connected graph of order \( n \) and
$G$ is an $H$-colored graph of order $n'$, then $n \leq \text{co}(H) \leq n'$

Let $H$ be a nontrivial connected graph of size $m$ with $E(H) = \{e_1, e_2, \ldots, e_m\}$ and $\chi'(H) = \chi$. For a $\chi$-edge coloring $c$ of $H$, define the $\sigma$-number of $c$ by

$$\sigma(c) = \sum_{i=1}^{m} (c(e_i) - 1)$$

Let $C(H)$ be the set of all $\chi$ colorings of $H$. The minimum $\sigma$-number (or simply $\sigma$-number) of $H$ is defined by

$$\sigma(H) = \min \{\sigma(c) \mid c \in C(H)\}$$

First, we establish bounds for the color order of a connected graph in terms of its order, size, chromatic index and minimum $\sigma$-number

**Theorem 4.5.2** If $H$ is a nontrivial connected graph of order $n$ with $\chi'(H) \geq 3$, then

$$n + \sigma(H) \leq \text{co}(H) \leq n + m(\chi'(H) - 3) + \sigma(H)$$

**Proof.** The upper bound for $\text{co}(H)$ is a consequence of the proof of Theorem 4.5.1, in which the $H$-colored graph has order $n + m(\chi'(H) - 3) + \sigma(H)$ if we choose a $\chi$-edge coloring whose $\sigma$-number is $\sigma(H)$. Thus, it remains to verify the lower bound. Let $c$ be a $\chi$-edge coloring of $H$ using the colors $1, 2, \ldots, \chi$ and let $G$ be an $H$-colored graph with respect to $c$. Thus there is a positive integer $k$ such that the distance-colored $G^k$ contains a properly edge-colored copy of $H$ produced by the coloring $c$. Let $E(H) = \{e_1, e_2, \ldots, e_m\}$. For each $i$ with $1 \leq i \leq m$, let
Suppose that \( Q = (x_1 = v_0, v_1, \ldots, v_m = y_1) \). Since \( e_i \) is an edge of \( H \), each of the vertices \( v_1, v_2, \ldots, v_{m-1} \) belongs to \( V(G) - V(H) \). This implies that each edge \( e_i \) (1 \( \leq i \leq m \)) colored \( c_i \) contributes a total \( c_i - 1 \) to the order of \( G \). Since \( V(H) \subseteq V(G) \), the order of \( G \) is at least \( n + \sum_{i=1}^{m}(c_i - 1) \geq n + \sigma(H) \), giving the desired result. \( \blacksquare \)

Next, we describe a class of connected graphs \( H \) for which \( \chi'(H) = n + \sigma(H) \).

The girth \( g(H) \) of a graph \( H \) having a cycle is the length of a smallest cycle in \( H \).

**Proposition 4.5.3** Let \( H \) be a nontrivial connected graph of order \( n \). If \( H \) is a tree or \( \chi'(H) \leq [3(g(H) - 1)/2] \), then \( \chi'(H) = n + \sigma(H) \).

**Proof.** Suppose that \( \chi'(H) = \chi \). Let \( c \) be a \( \chi \)-edge coloring of \( H \) using the colors 1, 2, \ldots, \( \chi \) such that \( \sigma(c) = \sigma(H) \). For each \( e \in E(H) \), subdivide the edge \( e \) a total of \( c(e) - 1 \) times, resulting in a path of length \( c(e) \). Denote the resulting graph by \( G \). Thus the order of \( G \) is \( n + \sigma(c) \). It remains to show that \( G \) is \( H \)-colored with respect to the coloring \( c \). Suppose that \( e \) joins \( u \) and \( v \) in \( H \). Since there is a \( u - v \) path of length \( c(e) \) connecting \( u \) and \( v \) in \( G \), it follows that \( d_G(u, v) \leq c(e) \). We claim that \( d_G(u, v) = c(e) \).

First, suppose that \( H \) is a tree. Since \( G \) is a subdivision of a tree, \( G \) is a tree as well. Thus the \( u - v \) path of length \( c(e) \) is the only \( u - v \) path in \( G \). Hence \( d_G(u, v) = c(e) \) if \( H \) is a tree. Next, suppose that \( H \) is not a tree. Assume, to the contrary, that \( d_G(u, v) < c(e) \). Then there exists a \( u - v \) path \( Q \) in \( G \) of length less than \( c(e) \). Let \( P \) be a \( u - v \) path in \( H \). Since the girth of \( H \) is \( g \)
and $uv \in E(H)$, it follows that the length of $P$ is either 1 or is at least $g - 1$. If
the length of $P$ is 1, then $P = (u, v)$ and $P$ gives rise to a $u - v$ path of length $c(e)$ in $G$. Thus, we may assume that the length of $P$ is $\ell$ and so $\ell \geq g - 1$. Let
$P = (u = x_0, x_1, x_2, \ldots, x_\ell = v)$. Since $P$ is properly colored by $c$, it follows from
the way in which $G$ is constructed that $P$ gives rise to a $u - v$ path in $G$ whose
length is at least

$$
\sum_{i=0}^{\ell-1} c(x_i x_{i+1}) \geq \left\lceil \frac{g-1}{2} \right\rceil + 2 \left\lceil \frac{g-1}{2} \right\rceil = \left\lceil \frac{3(g-1)}{2} \right\rceil.
$$

This implies that the length of $Q$ is at least $\lceil 3(g-1)/2 \rceil$. However then,

$$
c(e) > d_G(u, v) \geq \left\lceil \frac{3(g-1)}{2} \right\rceil \geq \chi'(H) \geq c(e),
$$

which is a contradiction. Therefore, $G^x$ contains a properly edge-colored copy of $H$ using $\chi$ colors and $G$ is $H$-colored. 

The following is an immediate consequence of Theorem 4.5.2 and Proposition 4.5.3.

**Corollary 4.5.4** If $H$ is a nontrivial connected graph of order $n$ such that $\chi'(H) = 2$ or $\chi'(H) = 3$, then $\co(H) = n + \sigma(H)$. 


Chapter 5

Rainbow Connected Graphs

5.1 Introduction

As we described in Chapter 1, graph labelings have grown in popularity in recent decades because of their applications to many other areas of research in graph theory. The origin of the study of graph labelings as a major area of graph theory can be traced to a 1967 research paper by Alexander Rosa [30]. Among the labelings he introduced was a vertex labeling he referred to as a $\beta$-valuation. Let $G$ be a graph of order $n$ and size $m$. A one-to-one function $f : V(G) \rightarrow \{0, 1, 2, \ldots, m\}$ is called a $\beta$-valuation (or a $\beta$-labeling) of $G$ if $\{|f(u) - f(v)| : uv \in E(G)\} = \{1, 2, \ldots, m\}$. In order for a graph to possess a $\beta$-labeling, it is necessary that $m \geq n - 1$. In 1972, Golomb [?] referred to a $\beta$-labeling as a graceful labeling and a graph possessing a graceful labeling as a graceful graph. Eventually, it was this terminology that became standard. Gallian [15] has written a survey on labelings of graphs that includes an extensive discussion of graceful labelings and their applications. One of the major conjectures in graph theory concerns graceful graphs and is due to
Kotzig (see Rosa [30]) which is known as the *Graceful Tree Conjecture*

The Graceful Tree Conjecture  *Every tree is graceful*

The following is a consequence of Rosa’s work [30]

**Theorem 5.1.1**  *If G is a graceful graph of size m, then the complete graph K_{2m+1} is cyclically G decomposable*

There is a connection between distance-colored graphs and graceful trees and it can be made even more specific in the following setting. For a nontrivial connected graph G and a positive integer k, a subgraph H in the distance colored graph G^k is a *rainbow subgraph* if no two edges of H are colored the same. First, we show that every connected graph H can be a rainbow subgraph for some distance-colored graph

**Theorem 5.1.2**  *For every connected graph H, there exists a connected graph G and a positive integer k such that the distance-colored graph G^k contains a rainbow H*

**Proof.** Since the result is obvious if H = K_2, we may assume that H has order at least 3. Suppose that V(H) = \{v_1, v_2, \ldots, v_n\} and E(H) = \{e_1, e_2, \ldots, e_m\}. So n \geq 3 and m \geq 2. For \(i = 1, 2, \ldots, m\), subdivide the edge \(e_i\) a total of \(m - 3 + i\) times, resulting in a path of length \(m - 2 + i\). Denote the resulting graph by G. Suppose that the edge \(e_i\) (1 \leq i \leq m) joins \(v_r\) and \(v_s\) in H. Since there is a path of length \(m - 2 + i\) connecting \(v_r\) and \(v_s\) in G, it follows that
We claim that \( d_G(v_r,v_s) = m - 2 + r \) Suppose that \( d_G(v_r,v_s) < m - 2 + r \) Then there exists a \( v_r - v_s \) path in \( G \) of length less than \( m - 2 + r \) From the way in which \( G \) is constructed, the length of such a path must be at least \( (m - 2 + p) + (m - 2 + q) \) for some positive integers \( p \) and \( q \) such that \( p + q \geq 3 \) Since \( (m - 2 + p) + (m - 2 + q) \geq 2m - 1 \), it follows that \( 2m - 1 < m - 2 + r \) and so \( r \geq m + 2 \), which is impossible Thus \( G^{2m} \) contains a rainbow \( H \) in which each edge \( e_i \) of \( H \) is colored \( m - 2 + r \) for \( 1 \leq r \leq m \)

The following result illustrates a connection between rainbow subgraphs in distance colored graphs and graceful trees

**Theorem 5.1.3** A nontrivial tree \( T \) of size \( m \) is graceful if and only if there exists a rainbow \( T \) in the distance-colored graph \( P_{m+1}^m \)

**Proof.** Let \( T \) be a nontrivial tree of size \( m \) and let \( P_{m+1} = (v_0,v_1,\ldots,v_m) \) First, suppose that a tree \( T \) is graceful and that \( f : V(T) \to \{0,1,2,\ldots,m\} \) is a graceful labeling of \( T \) Since the order of \( T \) is \( m + 1 \), it follows that \( f \) is in fact a bijection We show that \( P_{m+1}^m \) contains a rainbow tree \( T \) Define a function \( f^* : V(T) \to V(P_{m+1}) \) by \( f^*(v) = vf(v) \) for each \( v \in V(G) \) Since \( f \) is one-to-one, it follows that \( f^* \) is also one-to-one Furthermore, for each edge \( uv \in V(T) \),

\[
\begin{align*}
\quad d_{P_{m+1}}(f^*(u),f^*(v)) &= d_{P_{m+1}}(uf(u),vf(v)) \\
&= |f(u) - f(v)| = f'(uv)
\end{align*}
\]

Since \( f \) is a graceful labeling, \( f' \) is one-to-one and so

\[
\{d_{P_{m+1}}(f^*(u),f^*(v)) \mid uv \in E(T)\} = \{f'(uv) \mid uv \in E(T)\} = \{1,2,\ldots,m\}
\]
This implies that \( f^* \) induces a rainbow tree \( T \) in the distance-colored graph \( P_{m+1}^m \).

For the converse, suppose that the distance-colored graph \( P_{m+1}^m \) contains a rainbow tree \( T \) of size \( m \). Thus \( V(T) = V(P_{m+1}^m) = \{v_0, v_1, \ldots, v_m\} \). If \( v_i v_j \in E(T) \) where \( 0 \leq i \neq j \), then \( v_i v_j \) is colored by

\[
d_{P_{m+1}^m}(v_i, v_j) = |i-j| \in \{1, 2, \ldots, m\}
\]

in \( P_{m+1}^m \). Since \( T \) is a rainbow tree of size \( m \), it follows that \( \{|i-j| : v_i v_j \in E(T)\} = m \) and so

\[
\{|i-j| : v_i v_j \in E(T)\} = \{1, 2, \ldots, m\}. \tag{5.1}
\]

Now define a labeling \( g : V(T) \to \{0, 1, 2, \ldots, m\} \) by \( g(v_i) = i \) for \( 0 \leq i \leq m \). Then the induced edge labeling \( g' : E(G) \to \{1, 2, \ldots, m\} \) of \( g \) defined by

\[
g'(v_i v_j) = |g(v_i) - g(v_j)| = |i-j|
\]

is one-to-one by (5.1). Therefore, \( g \) is graceful and so \( T \) is graceful.

By Theorems 5.1.1 and 5.1.3, we have the following corollary.

**Corollary 5.1.4** Let \( T \) be a nontrivial tree of size \( m \). If there exists a rainbow \( T \) in the distance-colored graph \( P_{m+1}^m \), then \( K_{2m+1} \) is cyclically \( T \)-decomposable.

Next we illustrate a connection between distance-colored graphs and another well-known graph labeling. Let \( G \) be a graph of size \( m \) and let \( S_i = \{i, 2m+1-i\} \) for \( 1 \leq i \leq m \). A bi-graceful labeling of \( G \) is a one-to-one function \( f : V(G) \to \{0, 1, 2, \ldots, 2m\} \) that induces the edge labeling \( f' : E(G) \to \{1, 2, \ldots, 2m\} \) defined by
\[ f'(e) = |f(u) - f(v)| \] for each edge \( e = uv \) of \( G \)

such that the set \( S = \{ f'(e) : e \in E(G) \} \) has the property that \( |S \cap S_i| = 1 \) for all \( i \) (\( 1 \leq i \leq m \)). A bi-graceful labeling was called a \( \rho \)-valuation by Rosa in [30]. A graph admitting a bi-graceful labeling is a \( \rho \)-graceful graph. In particular, if \( S \cap S_i = \{ i \} \) for all \( i \) (\( 1 \leq i \leq m \)), then \( G \) is graceful and so every graceful graph is bi-graceful. Rosa [30] proved the following:

**Theorem 5.1.5**  
A nonempty graph \( G \) of size \( m \) is bi-graceful if and only if the complete graph \( K_{2m+1} \) is cyclically \( G \)-decomposable.

We now show a connection between rainbow subgraphs in distance-colored graphs and bi-graceful labelings.

**Theorem 5.1.6**  
Let \( G \) be a nonempty graph of size \( m \). Then \( G \) is bi-graceful if and only if there exists a rainbow \( G \) in the distance-colored graph \( C_{2m+1}^m \).

**Proof.** Let \( C_{2m+1} = (v_0, v_1, \ldots, v_{2m}, v_0) \) be a \((2m+1)\)-cycle. First, suppose that \( G \) is bi-graceful. Then there is an injective function \( f : V(G) \to \{0, 1, 2, \ldots, 2m\} \) such that the induced edge labeling \( f' : E(G) \to \{1, 2, \ldots, 2m\} \) defined by \( f'(e) = |f(u) - f(v)| \) for each edge \( e = uv \) of \( G \) uses exactly one label in \( \{i, 2m + 1 - i\} \) for each \( i \) with \( 1 \leq i \leq m \). That is, if \( S = \{ f'(e) : e \in E(G) \} \) and \( S_i = \{i, 2m + 1 - i\} \) for \( 1 \leq i \leq m \), then \( |S \cap S_i| = 1 \). Now define a function \( f^* : V(G) \to V(C_{2m+1}) \) by \( f^*(v) = v_{f(v)} \) for each \( v \in V(G) \). If \( uv \in E(G) \) such that \( |f(u) - f(v)| \in \{i, 2m + 1 - i\} \) for some \( i \) with \( 1 \leq i \leq m \), then

\[ d_{C_{2m+1}}(f^*(u), f^*(v)) = d_{C_{2m+1}}(u_{f(u)}, v_{f(v)}) = i \]
This implies that $f^*$ describes a rainbow copy of $G$ in the distance-colored graph $C_{2m+1}^m$.

For the converse, suppose that $C_{2m+1}^m$ contains a rainbow $G$. We show that $G$ is bi-graceful. Since $\text{diam}(C_{2m+1}) = m$, each edge of $G$ is assigned a color from the set $\{1, 2, \ldots, m\}$. Let $V(G) = \{v_{i_1}, v_{i_2}, \ldots, v_{i_n}\} \subseteq V(C_{2m+1})$, where $0 \leq i_1 < i_2 < \cdots < i_n \leq 2m$. Define a labeling $g : V(G) \to \{0, 1, 2, \ldots, 2m\}$ by $g(v_{i_j}) = i_j$ for all $j$ with $1 \leq j \leq n$. For each pair $v_{i_j}, v_{i_j'}$ of vertices of $G$ where $1 \leq j \neq j' \leq n$, since

$$d_{C_{2m+1}}(v_{i_j}, v_{i_j'}) \in \{1, 2, \ldots, m\},$$

either $d_{C_{2m+1}}(v_{i_j}, v_{i_j'}) = |i_j - i_j'|$ or $d_{C_{2m+1}}(v_{i_j}, v_{i_j'}) = (2m - 1) - |i_j - i_j'|$. Now consider the induced edge labeling $g' : E(G) \to \{1, 2, \ldots, 2m\}$ of $g$ defined by

$$g'(v_{i_j}, v_{i_j'}) = |g(v_{i_j}) - g(v_{i_j'})|$$

for each edge $v_{i_j}, v_{i_j'}$ of $G$ where $1 \leq j \neq j' \leq n$. Since $G$ is a rainbow subgraph of size $m$ in $C_{2m+1}^m$, it follows that

$$|\{d_{C_{2m+1}}(u, v) : u, v \in V(G)\}| = m$$

and so

$$\{d_{C_{2m+1}}(u, v) : u, v \in V(G)\} = \{1, 2, \ldots, m\}.$$

This implies that each edge of $G$ uses exactly one label in $\{i, 2m + 1 - i\}$ for each $i$ with $1 \leq i \leq m$. Therefore, $g$ is a bi-graceful labeling and so $G$ is bi-graceful. \qed
To illustrate Theorem 5.1.6, consider the 5-cycle $C_5 = (u,v,w,x,y,u)$ of size $m = 5$. A bi-graceful labeling of the cycle $C_5$ is shown in Figure 5.1(a). In this case, the set of induced edge labels is $S = \{1, 4, 6, 8, 9\}$, $S_1 = \{1,10\}$, $S_2 = \{2,9\}$, $S_3 = \{3,8\}$, $S_4 = \{4,7\}$ and $S_5 = \{5,6\}$. Observe that $|S \cap S_i| = 1$ for $1 \leq i \leq 5$. Thus $C_5$ is bi-graceful. By Theorem 5.1.6, the distance-colored graph $C_{11}^5$ contains a rainbow $C_5$ induced by the labeling $f^*$ as described in the proof of Theorem 5.1.6. In this case, $f^*(u) = v_0$, $f^*(v) = v_1$, $f^*(w) = v_{10}$, $f^*(x) = v_2$, $f^*(y) = v_6$. The 5-cycle in $C_{11}^5$ is $(v_0, v_1, v_{10}, v_2, v_6, v_0)$ as shown in Figure 5.1(b). The colors assigned to the edges of this 5-cycles are $1, 2, 3, 4, 5$ and so this 5-cycle is a rainbow $C_5$ in $C_{11}^5$.

Figure 5.1: A bi-graceful labeling of $C_5$ and a rainbow $C_5$ in $C_{11}^5$

The following is an immediate consequence of Theorems 5.1.5 and 5.1.6.

**Corollary 5.1.7** Let $G$ be a nonempty graph of size $m$. Then the complete graph $K_{2m+1}$ is cyclically $G$-decomposable if and only if there exists a rainbow $G$ in the distance-colored graph $C_{2m+1}^m$. 
5.2 Rainbow Pancyclic Graphs

Recall that a graph $G$ of order $n \geq 3$ is called \textit{pancyclic} if $G$ contains a cycle of every possible length, that is, $G$ contains a cycle of length $\ell$ for each integer $\ell$ with $3 \leq \ell \leq n$. Let $G$ be a connected graph of order $n \geq 4$ and diameter $d \geq 3$. The distance colored graph $G^d = K_n$ is called \textit{rainbow pancyclic} if $G^d$ contains a rainbow cycle of length $\ell$ for every $\ell$ with $3 \leq \ell \leq d$. First, we show that rainbow pancyclic graphs exist.

**Theorem 5.2.1** For each integer $d \geq 3$, there is a connected graph $G$ of diameter $d$ such that $G^d$ is rainbow pancyclic.

**Proof.** For $d \in \{3, 4\}$, let $G_d = P_{d+1} = (v_0, v_1, \ldots, v_d)$. If $d = 3$, then $(v_0, v_1, v_2, v_0)$ is a rainbow triangle in $G_3^3$, while if $d = 4$, then $(v_0, v_1, v_3, v_0)$ and $(v_0, v_2, v_1, v_4, v_0)$ are rainbow 3-cycle and 4-cycle, respectively, in $G_4^4$. Thus the result holds for $d \in \{3, 4\}$. Furthermore, the order of a connected graph of diameter $d$ is at least $d + 1$.

For each integer $d \geq 5$, let $G_d$ be the graph obtained from $P_{d+1} = (v_0, v_1, \ldots, v_d)$ by adding a new vertex $u$ and joining $u$ to the vertices $v_0$ and $v_1$ in $P_{d+1}$. Then $G_d$ has order $d + 2$ and diameter $d$. We proceed by induction on $d \geq 5$ to show that $G_d^d$ is rainbow pancyclic. Since

$$(v_0, v_1, v_3, v_0), (v_0, v_2, v_1, v_4, v_0) \text{ and } (v_0, v_3, v_1, v_5, u, v_0)$$

are rainbow $k$-cycles for $k = 3, 4, 5$, respectively, in $G_5^5$, the result holds for $d = 5$. Assume that the distance-colored graph $G_k^k$ is rainbow pancyclic for some integer
$k \geq 5$, that is, $G_k^k$ contains a rainbow cycle of length $\ell$ for every $\ell$ with $3 \leq \ell \leq k$.

We show that $G_{k+1}^{k+1}$ is rainbow pancyclic. Since $G_k^k$ is a subgraph of $G_{k+1}^{k+1}$, it follows that $G_{k+1}^{k+1}$ contains a rainbow cycle of length $\ell$ for every $\ell$ with $3 \leq \ell \leq k$. Thus, it remains to find a rainbow $(k+1)$-cycle in $G_{k+1}^{k+1}$. Suppose that $k + 1 \equiv r \pmod{4}$ where $r \in \{0, 1, 2, 3\}$. We consider these four cases.

**Case 1** $k + 1 \equiv 0 \pmod{4}$ Let $k + 1 = 4p$ for some positive integer $p$. Then
\[
C = \langle v_0, v_2p, v_2p-1, v_2p+1, v_2p-2, v_2p+2, v_p+1, v_3p-1, v_p, v_3p+1, v_p-1, v_3p+2, v_p-2, v_3p+3, v_1, v_4p, v_0 \rangle
\]
Note that $C$ does not contain the vertices $u$ and $v_{3p}$ (See Figure 5.2(a) for $k + 1 = 8$).

**Case 2** $k + 1 \equiv 1 \pmod{4}$ Let $k + 1 = 4p + 1$ for some positive integer $p$. Then
\[
C = \langle v_0, u, v_4p+1, v_2p-1, v_2p+1, v_2p-2, v_2p+2, v_p+1, v_3p-1, v_p, v_3p, v_p-1, v_3p+2, v_p-2, v_3p+3, v_1, v_4p, v_0 \rangle
\]
Note that $C$ does not contain the vertices $v_{2p}$ and $v_{3p+1}$ (See Figure 5.2(b) for $k + 1 = 9$).

**Case 3** $k + 1 \equiv 2 \pmod{4}$ Let $k + 1 = 4p + 2$ for some positive integer $p$. Then
\[
C = \langle v_0, v_2p, v_2p+2, v_2p-1, v_2p+3, v_2p-2, v_2p+4, v_p+1, v_3p+1, v_2p+1, v_1, v_4p+2, u, v_0 \rangle
\]
Note that $C$ does not contain the vertices $v_{2p+1}$ and $v_{3p+1}$ (See Figure 5.2(c) for $k + 1 = 10$)

**Case 4** $k + 1 \equiv 3 \pmod{4}$ Let $k + 1 = 4p + 3$ for some positive integer $p$ Then

$$C = (v_0, v_{3p}, v_{3p+1}, v_{3p-1}, v_{3p+2}, v_{3p-2}, v_{3p+3}, v_{3p+2}, v_{3p}, v_{3p+1}, v_{3p+2}, v_{3p+3}, v_0)$$

Note that $C$ does not contain the vertices $u$ and $v_{p+1}$ (See Figure 5.2(d) for $k + 1 = 11$)

In each case, $G_{k+1}^{k+1}$ contains a rainbow $(k + 1)$ cycle and so $G_{k+1}^{k+1}$ is rainbow pancyclic

Next we determine the minimum order of a connected graph $G$ of diameter $d$ for which $G^d$ is rainbow pancyclic

**Theorem 5.2.2** For each integer $d \geq 3$, the minimum order $n_d$ of a connected graph $G$ of diameter $d$ for which $G^d$ is rainbow pancyclic is

$$n_d = \begin{cases} 
  d + 1 & \text{if } d \equiv 0 \pmod{4} \text{ or } d \equiv 3 \pmod{4} \\
  d + 2 & \text{if } d \equiv 1 \pmod{4} \text{ or } d \equiv 2 \pmod{4}
\end{cases}$$

**Proof.** Figure 5.3 shows that $n_d = d + 1$ for $d \in \{3, 4\}$ Thus we may assume $d \geq 5$ We consider two cases

**Case 1** $d \equiv 0 \pmod{4}$ or $d \equiv 3 \pmod{4}$ We show that $n_d = d + 1$ in this case It suffices to show that $P_{d+1}^d$ is rainbow pancyclic In the proof of Theorem 5.2.1,
we saw that $P_{d+1}^d$ contains a rainbow cycle of length $d$ for $d \equiv 0 \pmod{4}$ or $d \equiv 3 \pmod{4}$. Since the distance-colored graph $P_{k+1}^k$ is a subgraph of the distance-colored graph $P_{k'+1}^{k'}$ for every two positive integers $k$ and $k'$ where $k' \geq k$, we only need show that if $d \equiv 3 \pmod{4}$, then $P_{d+1}^d$ contains rainbow cycles of length $d-1$ and $d-2$, respectively. For $d = 7$, Figure 5.4(a) shows a rainbow 6-cycle in $P_8^7$ and Figure 5.4(b) shows a rainbow 5-cycle in $P_8^7$. Thus the result holds for $d = 7$. Thus, we may assume that $d \geq 8$. 

Figure 5.2: Illustrating the proof of Theorem 5.2.1
Let $d = 4k + 3$ for some integer $k \geq 2$ and let

$$P_{4k+3} = (v_0, v_1, v_2, \ldots, v_{4k+3}).$$

The distance-colored graph $P_d^{d+1}$ contains a rainbow cycle of length $d - 1 \geq 7$, namely

$$C_{d-1} = (v_0, v_{4k+3}, v_1, v_{4k+2}, v_2, \ldots, v_{3k+3}, v_{k+1}, v_{3k+1}, v_{k+2}, v_{3k}, v_{k+3}, $$

$$\ldots, v_{2k+3}, v_{2k}, v_{2k+1}, v_0).$$

Observe that the distance sequence of $C_{d-1}$ is

$$4k + 3, 4k + 2, 4k + 1, \ldots, 2k + 2, 2k, 2k - 1, \ldots, 4, 3, 1, 2k + 1.$$ 

So the color 2 is the only color that is not used in $C_{d-1}$. Figure 5.5 shows a rainbow $C_{10}$ in $P_{12}^{14}$ and a rainbow $C_{14}$ in $P_{16}^{15}$, respectively.
The distance-colored graph $P_d^{d+1}$ contains a rainbow cycle of length $d - 2 \geq 6$, namely

$$C_{d-2} = (v_0, v_{4k+3}, v_1, v_{4k+2}, v_2, \ldots, v_k, v_{3k+3}, v_{k+2}, v_{3k+2}, v_{k+3}, v_{3k+1},$$

$$\ldots, v_{2k+5}, v_{2k}, v_{2k+4}, v_{2k+2}, v_0).$$

Observe that the distance sequence of $C_{d-2}$ is

$$4k + 3, 4k + 2, 4k + 1, \ldots, 2k + 3, 2k + 1, 2k, \ldots, 6, 5, 4, 2, 2k + 2.$$ 

So the colors 1 and 3 are the only colors that are not used in $C_{d-2}$. Figure 5.6 shows a rainbow $C_9$ in $P_{12}^{11}$ and a rainbow $C_{13}$ in $P_{16}^{15}$, respectively.

**Case 2.** $d \equiv 1 \pmod{4}$ or $d \equiv 2 \pmod{4}$. We show that $n_d = d + 2$ for each $d \geq 5$. Since the graph $G_d$ described in the proof of Theorem 5.2.1 has order $d + 2$, it follows that $n_d \leq d + 2$. Thus it suffices to show that $P_d^{d+1}$ is not rainbow
Figure 5.6: Rainbow cycles of length $d - 2$ in $P_{d+1}^d$ for $d = 11$ and $d = 15$

pancyclic. Assume, to the contrary, that there is an integer $d \geq 5$ such that $d \equiv 1 \pmod{4}$ or $d \equiv 2 \pmod{4}$ for which $P_{d+1}^d$ is rainbow pancyclic. Thus there is a rainbow cycle $C$ of length $d$ in $P_{d+1}^d$. Let $P_{d+1}^d = (v_0, v_1, v_2, \ldots, v_d)$. Observe that $C$ must contain an edge colored $i$ for each color $i$ with $1 \leq i \leq d$. Necessarily, $v_0$ belongs to $C$. We start at $v_0$, traverse around $C$, say clockwise, and partition the edge set $C$ into two sets $S_1$ and $S_2$ as follows: Let $e = v_iv_j \in E(C)$, where $0 \leq i, j \leq d + 1$. If $i < j$, then let $e \in S_1$; while if $i > j$, then $e \in S_2$. Since $C$ starts and ends at $v_0$, it follows that the sum of colors of edges in $S_1$ equals the sum of colors of edges in $S_2$, that is,

$$\sum_{u \in S_1} d_{P_{d+1}^d}(u, v) = \sum_{u \in S_2} d_{P_{d+1}^d}(u, v). \quad (5.2)$$

Since $d \equiv 1 \pmod{4}$ or $d \equiv 2 \pmod{4}$, the number of odd numbers between 1 and $d$ is odd. Thus either there is an odd number of odd colors used in the edges of $S_1$
or there is an odd number of odd colors used in the edges of $S_2$, say the former. Then the number of odd colors used in the edges of $S_2$ is even. This contradicts (5.2). Therefore, $n_d = d + 2$ for $d \equiv 1 \pmod{4}$ or $d \equiv 2 \pmod{4}$.

\section{5.3 Rainbow Connectivity}

Let $G$ be a nontrivial connected graph of order $n$ and diameter $d$. For an integer $k$ with $1 \leq k \leq d$, a path $P$ in the distance-colored graph $G^k$ is a rainbow path if no two edges of $P$ are colored the same. The graph $G^k$ is called rainbow-connected if every two vertices $u$ and $v$ in $G^k$ are connected by a rainbow $u - v$ path in $G^k$. Since $G^d = K_n$, the graph $G^d$ is certainly rainbow-connected. Thus there is a smallest positive integer $k$ such that $G^k$ is rainbow-connected. This gives rise to the following concept. The rainbow-connection exponent $rce(G)$ of $G$ is defined as the minimum $k$ for which $G^k$ is rainbow-connected. In order to determine the rainbow-connection exponent of an arbitrary connected graph, we first present a lemma.

\textbf{Lemma 5.3.1} Let $k \geq 2$ be an integer. If $\ell$ is an integer with $k < \ell \leq \binom{k+1}{2}$, then there exist integers $k_1, k_2, \ldots, k_{p_\ell}$, where $k > k_1 > k_2 > \cdots > k_{p_\ell} \geq 1$ and $p_\ell \geq 1$, such that $\ell = k + k_1 + k_2 + \cdots + k_{p_\ell}$.

\textbf{Proof.} We proceed by induction on $k$. If $k = 2$, then $2 < \ell \leq \binom{3}{2} = 3$ and so $\ell = 3 = 2 + 1$. Suppose that the result is true for all integers $\ell$ with $k < \ell \leq \binom{k+1}{2}$ for some integer $k \geq 2$. Let $\ell$ be an integer with $k + 1 < \ell \leq \binom{k+2}{2}$. First, suppose
that $k + 1 < \ell \leq \binom{k+1}{2}$. Then $k < \ell \leq \binom{k+1}{2}$ By the induction hypothesis, 
\[ \ell = k + k_1 + k_2 + \ldots + k_{p_\ell} \]
where $k > k_1 > k_2 > \ldots > k_{p_\ell} \geq 1$ for some positive integer $p_\ell$. If $p_\ell \geq 2$, then \[ \ell = (k + 1) + k_1 + k_2 + \ldots + k_{p_\ell - 1} + (k_{p_\ell} - 1), \]
where $k + 1 > k_1 > k_2 > \ldots > k_{p_\ell - 1} > k_{p_\ell} - 1$. If $k_{p_\ell} = 1$, then since $\ell > k + 1$, it follows that $p_\ell \geq 2$ and $k_{p_\ell - 1} \geq 2$. Thus \[ \ell = (k + 1) + k_1 + k_2 + \ldots + k_{p_\ell - 1}, \]
where $k + 1 > k_1 > k_2 > \ldots > k_{p_\ell - 1}$ and $p_\ell - 1 \geq 1$. Next, suppose that $\binom{k+1}{2} < \ell \leq \binom{k+2}{2}$. If $\ell = \binom{k+2}{2}$, then \[ \ell = (k + 1) + k + \ldots + 2 + 1 \]
and the result follows. Thus we may assume that $\ell < \binom{k+2}{2}$. Since $\binom{k+2}{2} = \binom{k+1}{2} + (k + 1)$, it follows that \[ \ell = \binom{k+1}{2} + p \]
where $1 \leq p \leq k$ and so 
\[ \ell = \binom{k+1}{2} + p = k + (k - 1) + (k - 2) + \ldots + 2 + 1 + p \]
\[ = (k + 1) + k + [k - (p - 1) + 1] + (k - p) + (k - p - 1) + \ldots + 2 + 1, \]
producing the desired result.

Theorem 5.3.2 If $G$ is a connected graph of diameter $d \geq 2$, then the rainbow connection exponent $rce(G)$ of $G$ is the unique positive integer $k$ for which 
\[ \binom{k}{2} < d \leq \binom{k+1}{2} \]

Proof. We first show that $rce(G) \leq k$. For two vertices $u$ and $v$ of $G$, we show that there is a rainbow $u - v$ path in $G^k$. Suppose that $d(u, v) = \ell$ where $1 \leq \ell \leq d$. Let $P = (u = v_0, v_1, v_2, \ldots, v_\ell = v)$ be a $u - v$ geodesic in $G$. If $1 \leq \ell \leq k$, then $u v$ is an edge in $G^k$ and so $u$ and $v$ are connected by a rainbow path in $G^k$, namely $(u, v)$. Thus we may assume that $\ell > k$. Since $d \leq \binom{k+1}{2}$, it follows that $k < \ell \leq \binom{k+1}{2}$. By
Lemma 5.3.1, \( \ell = k + k_1 + k_2 + \ldots + k_p \) such that \( k > k_1 > k_2 > \cdots > k_p \geq 1 \) for some positive integer \( p_\ell \). Let \( u_1 = v_k \) and, for \( 2 \leq r \leq p_\ell \), let \( u_r = v_{k+k_1+k_2+\ldots+k_{r-1}} \).

Then \( Q = (u, u_1, u_2, \ldots, u_p, v) \) is a rainbow \( u - v \) path in which the edge \( u_1u \) is colored \( k \), the edge \( u_ru_{r+1} \) is colored \( k_r \) for \( 1 \leq r \leq p_\ell - 1 \) and the edge \( u_pv \) is colored \( k_p \).

Next, we show that \( rce(G) \geq k \). Assume, to the contrary, that the distance-colored graph \( G^j \) is rainbow-connected for some positive integer \( j \) with \( j < k \). Let \( x \) and \( y \) be two antipodal vertices of \( G \), that is, \( d(x, y) = d \). Since \( G^j \) is rainbow-connected, there exists a rainbow \( x - y \) path in \( G^j \) using each color in \( \{1, 2, \ldots, j\} \) at most once. This implies that \( d = d(x, y) \leq 1 + 2 + \cdots + j \leq 1 + 2 + \cdots + (k - 1) = \binom{k}{2} < d \), which is impossible. Therefore, \( rce(G) \geq k \) and so \( rce(G) = k \). 

The following is an immediate consequence of Theorem 5.3.2.

**Corollary 5.3.3** If \( G \) is a connected graph of diameter \( d \geq 2 \), then

\[
    rce(G) = \left\lceil \frac{-1 + \sqrt{1 + 8d}}{2} \right\rceil.
\]

**Proof.** Let \( k \) be the minimum positive integer such that \( \binom{k+1}{2} \geq d \). Then \( k^2 + k - 2d \geq 0 \), which implies that \( k \geq \frac{-1 + \sqrt{1 + 8d}}{2} \). The result then follows by Theorem 5.3.2.

By Corollary 5.3.3, if \( G \) is a connected graph with \( rce(G) = k \geq 2 \), then every two vertices \( u \) and \( v \) in \( G \) are connected by at least one rainbow \( u - v \) path in \( G^k \). In Chapter 4, the concept of color-connectivity was introduced in terms of
internally disjoint properly colored paths in distance-colored graphs. In the view of rainbow paths, this gives rise to another concept.

For a connected graph $G$ and positive integers $k$ and $\ell$, the distance-colored graph $G^k$ is \textit{rainbow $\ell$-connected} if for every two distinct vertices $u$ and $v$ of $G^k$, there are $\ell$ internally disjoint rainbow $u-v$ paths in $G^k$. In particular, if $\ell = 1$, then $G^k$ is rainbow-connected. Thus, we now assume that $\ell \geq 2$. If $G$ is a complete graph, then $G^k$ is not rainbow $\ell$-connected for all positive integers $k$ and $\ell$ with $\ell \geq 2$. Hence we only consider non complete connected graphs. For a connected graph $G$ and an integer $k \geq 2$, the rainbow connectivity of $G^k$ is the maximum positive integer $\ell$ for which $G^k$ is rainbow $\ell$-connected. We begin by considering rainbow 2-connected graphs $G^k$ where $G$ is a path.

**Theorem 5.3.4** For each integer $k \geq 3$, let $P^{(k+1)}_{(\ell+1)+1} = (v_0, v_1, v_2, \ldots, v_{(k+1)})$ be the path of order $(k+1) + 1$. Then the distance colored graph $P^{k}_{(\ell+1)+1}$ is properly 2-rainbow connected. Furthermore, for each pair $v_i, v_j$ of vertices of $P^{k}_{(\ell+1)+1}$, where $0 \leq i < j \leq (k+1)$ and $j - i \geq 3$, there are two internally disjoint rainbow $v_i-v_j$ paths $Q_1$ and $Q_2$ in $P^{k}_{(\ell+1)+1}$ such that

$$V(Q_1) \cup V(Q_2) \subseteq \{v_i, v_{i+1}, \ldots, v_j\}$$

**Proof.** We proceed by induction on $k \geq 3$. If $k = 3$, then $(k+1) + 1 = (\ell) + 1 = 7$. Figure 5.3.4 shows that the statement is true for $P^3_7$. Now assume that the statement is true for the distance colored graph $P^{k}_{(\ell+1)+1}$ for some integer $k \geq 3$.

Let $m = (k+2)$ and $P_{m+1} = (v_0, v_1, v_2, \ldots, v_m)$ Suppose that $v_i, v_j \in V(P_{m+1})$, where $0 \leq i < j \leq m$. If $1 \leq j - i \leq 2$, then the result is true (see Figure 5.3.4).
Thus we may assume that $j - i \geq 3$.

First, assume that $i = 0$ and $3 \leq j \leq m$. If $j \leq \binom{k+1}{2}$, then let

$$P_{(k+1)^{+1}} = (v_0, v_1, v_2, \ldots, v_{(k+1)})$$

Since $P_{(k+1)^{+1}}$ is a subgraph of $P_{m+1}^{k+1}$, the result follows by the induction hypothesis.

Thus we may assume that $j \geq \binom{k+1}{2} + 1$. Now let $t = j - (k+1)$. Since $(k+1) + 1 \leq j \leq \binom{k+2}{2}$, it follows that

$$t \leq \binom{k+2}{2} - (k + 1) = \binom{k+1}{2}$$

and since $k \geq 3$,

$$t \geq \binom{k+1}{2} + 1 - (k + 1) = \binom{k}{2} \geq 3.$$  

Thus $3 \leq t \leq \binom{k+1}{2}$. By the induction hypothesis there are two internally disjoint rainbow $v_0 - v_t$ paths $Q'_1$ and $Q'_2$ in $P_{(k+1)^{+1}}$ (and so in $P_{m+1}^{k+1}$) such that

$$V(Q'_1) \cup V(Q'_2) \subseteq \{v_0, v_1, \ldots, v_t\}.$$  

Let

$$Q'_1 = (v_0, v_{p_1}, v_{p_2}, \ldots, v_{p_r} = v_t)$$

and

$$Q'_2 = (v_0, v_{q_1}, v_{q_2}, \ldots, v_{q_s} = v_t).$$
Since \( j - t = k + 1 \), we can construct two internally disjoint rainbow \( v_0 - v_j \) paths \( Q_1 \) and \( Q_2 \) from \( Q'_1 \) and \( Q'_2 \) by

\[
Q_1 = (v_0, v_{p_1}, v_{p_2}, v_{p_{r-1}}, v_{p_{r-1}+(k+1)}, v_{p_{r-1}+(k+1)} = v_j)
\]

\[
Q_2 = (v_0, v_{q_1}, v_{q_2}, v_{q_{s-1}}, v_{q_{s-1}+(k+1)}, v_{q_{s-1}+(k+1)} = v_j)
\]

Next, assume that \( 1 < t < j \leq m \). We first construct two internally disjoint rainbow \( v_0 - v_{j-1} \) paths \( Q_1^* \) and \( Q_2^* \) in \( P_{m+1}^{k+1} \) with the desired properties, where say

\[
Q_1^* = (v_0, v_{a_1}, v_{a_2}, \ldots, v_{a_r} = v_{j-1})
\]

\[
Q_2^* = (v_0, v_{b_1}, v_{b_2}, \ldots, v_{b_s} = v_{j-1})
\]

Then

\[
Q_1 = (v_0+t, v_{a_1+t}, v_{a_2+t}, \ldots, v_{a_r+t} = v_j)
\]

\[
Q_2 = (v_0+t, v_{b_1+t}, v_{b_2+t}, \ldots, v_{b_s+t} = v_j)
\]

have the desired properties.

The proof of Theorem 5.3.4 also establishes the following result:

**Theorem 5.3.5** For each integer \( m \geq 3 \), let \( k \) be the smallest positive integer for which \( m+1 \leq \binom{k+1}{2} \). Then the distance-colored graph \( P_{m+1}^k \) is properly 2-rainbow connected.

**Theorem 5.3.6** Let \( G \) be a connected graph with \( \text{rad}(G) \geq 3 \) and \( \text{diam}(G) = d \). If \( k \) is the minimum positive integer for which \( d \leq \binom{k+1}{2} \), then the distance-colored graph \( G^k \) is 2-rainbow connected.
Proof. Let $u$ and $v$ be two distinct vertices of $G$. We show that there are two internally disjoint rainbow $u - v$ paths in $G^k$. If $d_{G^k}(u, v) = p \geq 3$, then let $P = (u = v_0, v_1, \ldots, v_p = v)$ be a $u - v$ geodesic in $G$. By Theorem 5.3.5, there are two internally disjoint rainbow $u - v$ paths $Q_1$ and $Q_2$ in the subgraph $P^k$ of $G^k$ and so $Q_1$ and $Q_2$ are internally disjoint rainbow $u - v$ paths in $G^k$ as well. Thus we may assume that $d_{G^k}(u, v) \leq 2$. Since rad$(G) \geq 3$, there is a vertex $w \in V(G)$ such that $d_{G^k}(u, w) = 3$. Let $P' = (u, x, y, w)$ be $u - w$ geodesic in $G$. We consider two cases, according to $d_{G^k}(u, v) = 1$ or $d_{G^k}(u, v) = 2$.

Case 1. $d_{G^k}(u, v) = 1$. Let $Q_1 = (u, v)$. If $vy \notin E(G)$, then let $Q_2 = (u, y, v)$, while if $vy \in E(G)$, then let $Q_2 = (u, w, v)$.

Case 2. $d_{G^k}(u, v) = 2$. Let $Q_1 = (u, v)$. To construct $Q_2$, we consider two subcases.

Subcase 2.1. $vx \in E(G)$. Necessarily, $(u, x, v)$ is a $u - v$ geodesic. If $vy \in E(G)$, then let $Q_2 = (u, y, v)$, while if $vy \notin E(G)$, then let $Q_2 = (u, y, w, v)$, whose edges are colored 1, 2 and 3.

Subcase 2.2. $vx \notin E(G)$. Let $z \in V(G)$ such that $(u, z, v)$ is a $u - v$ geodesic in $G$. Since $vx \notin E(G)$, let $Q_2 = (u, x, v)$ is a rainbow path whose edges are colored 1 and 2 or 1 and 3 (depending on whether $xz \in E(G)$ or not).

In each case, there are two internally disjoint rainbow $u - v$ paths in $G^k$ and so $G^k$ is properly 2-rainbow connected.

In general, we have the following conjecture.
Conjecture 5.3.7 Let $\ell \geq 2$ be any integer and let $G = P_{(2^\ell-1)+1}$. Then $G^{2\ell-2}$ is $\ell$-rainbow connected.

Figure 5.8 shows that $P_{11}^4$ is a properly 3-rainbow connected and so Conjecture 5.3.7 is true for $\ell = 3$.

Figure 5.8: Three internally disjoint rainbow paths in $P_{11}^4$
Chapter 6

Topics for Further Study

6.1 Questions on Color-Hamiltonicity

1. The complete graph $K_7$ can be decomposed into properly colored Hamiltonian cycles $C_1$, $C_2$ and $C_3$ where

\[
C_1 = (v_1, v_2, v_5, v_7, v_4, v_6, v_3, v_1)
\]

\[
C_2 = (v_1, v_6, v_2, v_4, v_3, v_7, v_1)
\]

\[
C_3 = (v_1, v_4, v_5, v_3, v_2, v_7, v_6, v_1)
\]

Such a deposition is shown in Figure 6.1.

This gives rise to the following question: Which complete graphs of odd order can be decomposed into properly colored Hamiltonian cycles?

2. Let $G$ be a connected graph for which $G^k$ is Hamiltonian-colored for some positive integer $k$. Recall that the Hamiltonian coloring exponent $hce(G)$ of $G$ is the minimum $k$ for which $G^k$ is Hamiltonian-colored. For a connected
graph $G$ with $hce(G) = k$, let $\mu(G)$ denote the minimum number of colors used in a properly colored Hamiltonian cycle of $G^k$. The following results were presented in Chapter 2.

**Theorem 6.1.1** For each integer $k \geq 2$, there exists a tree $T_k$ with $hce(T_k) = k$ such that every properly colored Hamiltonian cycle in the $k$th power of $T_k$ must use all of the colors $1, 2, \ldots, k$.

**Theorem 6.1.2** For each integer $k \geq 3$, there exists a connected graph $G_k$ with $hce(G_k) = k$ and a properly colored Hamiltonian cycle in the $k$th power of $G_k$ whose edges are colored with fewer than $k$ colors.

**Corollary 6.1.3** There exists a sequence $\{T_k\}_{k=1}^{\infty}$ of trees such that

$$\lim_{k \to \infty} \frac{\mu(T_k)}{hce(T_k)} = 0.$$
Corollary 6.1.4  There exists a sequence \( \{T_k\}_{k=2}^{\infty} \) of trees such that
\[
\lim_{k \to \infty} \frac{\mu(T_k)}{\text{hce}(T_k)} = 1.
\]

Problem 6.1.5  For which pairs \( a, b \) of integers with \( 2 \leq a \leq b \), does there exist a tree \( T \) with \( \mu(T) = a \) and \( \text{hce}(T) = b \)?

Corollaries 6.1.3 and 6.1.4 give rise to the following concepts. For a connected graph \( G \) with \( \text{hce}(G) = k \), the color range \( \rho(G) \) of \( G \) is defined as the set of all colors \( j \) that are used in some properly colored Hamiltonian cycle of \( G^k \), while the color spectrum \( \sigma(G) \) of \( G \) is defined as the set of all colors \( j \) that are used in every properly colored Hamiltonian cycle of \( G^k \). Thus \( \sigma(G) \subseteq \rho(G) \subseteq \{1, 2, \ldots, k\} \) and \( k \in \sigma(G) \cap \rho(G) \). In particular, if \( k = 2 \), then \( \sigma(G) = \rho(G) = \{1, 2\} \). For each integer \( k \geq 3 \), the graph \( G_k \) constructed in Theorem 6.1.2 has \( \sigma(G_k) \not\subset \rho(G_k) \), while the graph \( H_k \) constructed in Theorem 6.1.1 has \( \sigma(H_k) = \rho(H_k) \).

Problem 6.1.6  Study the relationship between \( \sigma(G) \) and \( \rho(G) \).

6.2 Questions on Color-Distance

1. For which rational numbers \( r \) with \( 0 \leq r \leq 1 \), does there exist a graph \( G \) such that the order of \( \text{CC}(G) \) is \( k \), the order of \( \text{Cen}(G) = \ell \) and \( \frac{\ell}{k} = r \)?

2. Let \( F \) and \( H \) be graphs where every vertex of \( H \) has color-eccentricity 1 or none do. Does there exist a graph \( G \) such that \( \text{CC}(G) \cong F \) and \( \text{CP}(G) \cong H \)?
3 If a graph $G$ has a sufficiently large diameter and $k$ is sufficiently large, is $G^k$ properly 3-connected?

6.3 Questions on Rainbow Subgraphs

Let $G$ be a connected graph of order $n$ and diameter $d$. Then each edge of the distance-colored graph $G^d = K_n$ is colored with one of the colors $1, 2, \ldots, d$

1 Which subgraphs $H$ of size $d$ in $G^d = K_n$ are rainbow subgraph of $K_n$?

For each rainbow subgraph $H$ of $K_n$, define the *rainbow quotient* $rq(H)$ of $H$ as the maximum number of pairwise edge-disjoint copies of a rainbow $H$ in $K_n$. Thus

$$0 \leq rq(H) \leq \left\lfloor \frac{n}{2} \right\rfloor / d$$

Find examples of graphs $H$ for which $rq(H) = \left\lfloor \frac{n}{2} / d \right\rfloor$.

2 Let $G$ be a connected graph of diameter $d \geq 3$ and let $3 \leq k \leq d$ be an integer. For each unicyclic graph $H$ of size $k$, does there exist a rainbow $H$ in $G^d$? If this is not true, then which connected graphs have this property?

3 Let $v$ be a vertex of $G^d = K_n$. Let $P$ be a path with initial vertex $v$ constructed by taking any edge incident with $v$ colored 1, say $vv_1$, followed by any edge incident with $v_1$ whose color is the smallest color greater than 1, etc. Among all paths constructed in this manner, let $r\ell(v)$ denote the maximum length of such a path. Determine the minimum and maximum values
of \( r\ell(v) \) over all vertices \( v \) in \( K_n \). [Note: Instead of encountering the edges whose colors in increasing order, we could just demand that the colors be different.]

4. Does the distance-colored graph \( G^d = K_n \) contain a rainbow path \( P \) of length \( d \) the colors of whose edges follow in the order 1, 2, 3, \ldots, \( d \)?

For each ordering \( \alpha \) of the elements in the set \( \{1, 2, 3, \ldots, d\} \), is there a path of length \( d \) the colors of whose edge follow \( \alpha \)?

5. Study decompositions of distance-colored graphs \( G^d = K_n \) into prescribed rainbow subgraphs.

### 6.4 Questions on Rainbow Connectivity

Recall that the rainbow-connection exponent \( \text{rce}(G) \) of a connected graph \( G \) is defined as the minimum \( k \) for which the distance-colored graph \( G^k \) is rainbow-connected in Chapter 5. We have seen that if \( G \) is a connected graph of diameter \( d \geq 2 \), then

\[
\text{rce}(G) = \left\lceil \frac{-1 + \sqrt{1 + 8d}}{2} \right\rceil. \tag{6.1}
\]

Suppose that \( G \) is a connected graph with \( \text{rce}(G) = k \geq 2 \). By (6.1), we know that for every two vertices \( u \) and \( v \) in a connected graph \( G \), there is a rainbow \( u - v \) path in \( G^k \). In fact, there may be several rainbow \( u - v \) paths in \( G^k \). A rainbow \( u - v \) path of minimum length in \( G^k \) is a rainbow \( u - v \) geodesic in \( G^k \) and its length is the rainbow-distance between \( u \) and \( v \) in \( G^k \) and is denoted by \( \text{rd}(u, v) \).
We have the following conjecture

**Conjecture 6.4.1** Suppose that $G$ is a connected graph with $rce(G) = k \geq 2$. If $u$ and $v$ are two vertices in $G$ such that $d(u, v) = \ell$, then $rd(u, v)$ is the maximum positive integer $p$ for which $\binom{k+1}{2} - \binom{p}{2} \geq \ell$

For a fixed integer $\ell$, define the number $rce(G, \ell)$ to be the minimum positive integer $k$ for which $G^k$ is $\ell$-rainbow connected. Thus $rce(G, 1) = rce(G)$. Recall the following result in Chapter 5

**Theorem 6.4.2** Let $G$ be a connected graph with $rad(G) \geq 3$ and $diam(G) = d$. If $k$ is the minimum positive integer for which $d \leq \binom{k+1}{2}$, then the distance-colored graph $G^k$ is $2$-rainbow connected.

The following is a consequence of Theorem 6.4.2

**Corollary 6.4.3** Let $G$ be a connected graph with $rad(G) \geq 3$ and $diam(G) = d$. If $k$ is the minimum positive integer for which $d \leq \binom{k+1}{2}$, then $rce(G, 2) = k$.

**Problem 6.4.4** Study $rce(G, \ell)$ in general.

Let $G$ be a connected graph of order $n$ and diameter $d$. A tree $T$ in $G^d = K_n$ is a *rainbow tree* if no two edges of $T$ are colored the same. For an integer $k$ with $2 \leq k \leq n$, the distance-colored graph $G^d$ is called $k$-rainbow tree-connected if for every set $S$ of $k$ vertices of $G$, there exists a rainbow tree in $G^d$ containing the vertices of $S$.

**Problem 6.4.5** Determine conditions under which $G^d$ is $k$-rainbow tree-connected.
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