Analysis of Vibroacoustic Properties of Dimpled Beams Using a Boundary Value Model

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ANALYSIS OF VIBROACOUSTIC PROPERTIES OF DIMPLED BEAMS USING A BOUNDARY VALUE MODEL

by

Kyle R. Myers

A dissertation submitted to the Graduate College in partial fulfillment of the requirements for the degree of Doctor of Philosophy
Department of Mechanical and Aerospace Engineering
Western Michigan University
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Attention has been given recently to the use of dimples as a means of passively altering the vibroacoustic properties of structures. Because of their geometric complexity, previous studies have modeled dimpled structures using the finite element method. However, the dynamics of dimpled structures are not completely understood. The goal of this study is to provide a better understanding of these structures through the development of a boundary value model (BVM) using Hamilton’s Variational Principle. The focus of this study is on dimpled beams, which represent the simplest form of a dimpled structure.

A general model of a beam with $N$ dimples in free vibration is developed. Since dimples formed via a stamping process do not change the mass of the beam, the dimple thickness is less than that of the straight segments. Differential equations of motion that describe the normal and axial motion of the dimpled beams are derived. Their numerical solution yields the natural frequencies and analytical mode shapes of a dimpled beam. The accuracy of this model is checked against those obtained using the finite element method, as well as the analytical studies on the vibrations of arches, and shown to be accurate.

The effect of dimple placement, dimple angle, its chord length, its thickness, as well as beam boundary conditions on beam natural frequencies and mode shapes are investigated. For beams with axially restrictive boundary conditions, the results
show that a peak in a natural frequency for certain dimple angles corresponds to a changing mode shape within the dimple. Previous studies had suggested that dimple thinning was the cause of this phenomenon. The natural frequencies also exhibit a greater sensitivity to changes in dimple angle for dimples located at regions of highest modal strain energy (MSE) of a uniform beam. The use of MSE as a design strategy for optimum placement of dimples is demonstrated. Finally, using the MSE in combination with the genetic algorithm (GA), single and multiple dimples are shown to alter beam vibroacoustic properties significantly.
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Kyle R. Myers
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NOMENCLATURE

Acronyms

BVM Boundary Value Model
FEA Finite Element Analysis
FEM Finite Element Method
FRF Frequency Response Function
GA Genetic Algorithm
MAC Modal Assurance Criterion
MSE Modal Strain Energy (per length)

General

\[ \mathcal{A} \] coefficient matrix, Eq. (3.61)
\( b \) beam width, Fig. 3.2
\( D_i \) modal strain energy constant, Eq. (5.3)
\( E \) elastic modulus, Eq. (3.5b)
\( \{f\} \) forced response excitation vector, Eq. (4.5)
\( f^* \) natural frequency (uniform beam) [Hz], Eq. (5.1)
\( i \) segment index number (Section 3.2.1); mode number (Eq. (4.5))
\( j \) imaginary unit, \( j = \sqrt{-1} \)
\( k \) summation index
\( L \) total beam length
$\delta \mathcal{L}$ Lagrangian functional, Eq. (3.7b)

$N$ number of dimples

$N_m$ number of modes in forced response, Eq. (4.5)

$n$ number of discretized nodes along beam, Eq. (4.5)

$q$ volume velocity, Eq. (6.1)

$\mathcal{T}$ total kinetic energy functional, Eq. (3.4a)

$t$ time, Eq. (3.3)

$\mathcal{V}$ total strain energy functional, Eq. (3.5a)

$\{X\}$ vector of modal coefficients, Eq. (3.61)

$\{Z\}$ forced response vector, Eq. (4.5)

$\delta$ first order variation, Eq. (3.6)

$\eta_i$ damping loss factor, Eq. (4.5)

$\rho$ beam density, Eq. (3.4b)

$\sigma$ modal strain energy per length, Eq. (5.2)

$\{\phi\}$ mass-normalized mode shape vector, Eq. (4.5)

$\omega$ excitation frequency [rad/s], Eq. (4.5)

$\omega_i$ natural frequency [rad/s] of mode $i$, Eq. (4.5)

**Straight Segment Parameters**

$A_{ik}$ transverse modal coefficient, Eq. (3.43)

$C_{ik}$ longitudinal modal coefficient, Eq. (3.47)

$h$ segment thickness, Fig. 3.2

$I$ area moment of inertia, $bh^3/12$ in this study, Eq. (3.5b)

$l$ segment length, Fig. 3.1

$q$ roots of transverse characteristic equation, Eq. (3.42)

$r$ roots of longitudinal characteristic equation, Eq. (3.47)

$S$ cross-sectional area, $bh$ in this study, Eq. (3.4b)

$T$ kinetic energy functional, Eq. (3.4b)
$U$ longitudinal coordinate (space only)

$u$ longitudinal coordinate (space and time), Fig. 3.1

$v$ strain energy functional, Eq. (3.5b)

$x$ axial position, local coordinate, Fig. 3.1

$Y$ transverse coordinate (space only)

$y$ transverse coordinate (space and time), Fig. 3.1

$\beta$ transverse frequency parameter, Eq. (3.40)

$\gamma$ longitudinal frequency parameter, Eq. (3.46)

**Dimple Parameters** (bar denotes equivalent dimple quantities)

$B_{ik}$ modal coefficient, Eq. (3.57)

$D_{ik}$ modal coefficient, Eq. (3.58)

$\bar{h}$ segment thickness, Fig. 3.2

$\bar{I}$ area moment of inertia, $bh^3/12$ in this study, Eq. (3.5c)

$\bar{l}$ chord length, Fig. 3.1

$\mathcal{M}$ bending moment, Eq. (3.18)

$\mathcal{P}$ tangential force, Eq. (3.20)

$p_i$ extensional vibration parameter, Eq. (3.53)

$Q$ radial shear force, Eq. (3.19)

$R$ radius, Fig. 3.1

$\bar{S}$ cross-sectional area, $bh$ in this study, Eq. (3.4c)

$s$ roots of tangential and radial characteristic equation, Eq. (3.56)

$\bar{T}$ kinetic energy functional, Eq. (3.4c)

$V$ tangential coordinate (space only)

$\bar{v}$ strain energy functional, Eq. (3.5c)

$W$ radial coordinate (space only)

$w$ radial coordinate (space and time), Fig. 3.1

$x_c$ location of dimple center
\( \alpha \)  dimple angle, Fig. 3.1
\( \zeta_i \)  frequency parameters, Eq. (3.52)
\( \theta \)  angular position, local coordinate, Fig. 3.1
\( \lambda_{ik} \)  multiplicative constant for modal coefficient, Eq. (3.60)
\( \nu \)  tangential coordinate (space and time), Fig. 3.1
\( \psi \)  bending slope, Eq. (3.17)
Chapter 1

RESEARCH OVERVIEW

This chapter contains an overview of the research in this dissertation. First, a definition of what is meant by “dimpled beam” is given in Section 1.1. The motivation for this research is discussed in Section 1.2, followed by a broad overview of the chapters that comprise this study in Section 1.3.

1.1 Definition of Dimpled Beams

Dimpled beams referenced in this study refer to the type shown in Fig. 1.1. This is showing a beam with one dimple, although a model of beams with any number of dimples is developed in this dissertation. Each dimple is assumed to be connected to a straight segment, so that if there are $N$ dimples, there are $N+1$ straight segments. Throughout this study, a dimple on a beam is understood to be an arch along the

Figure 1.1: A dimpled beam with one dimple
length of the beam and of uniform height along the width of the beam. Using the terminology of Leissa [6, p.403], we define a dimple as an open circular cylindrical shell with constant radius (i.e., it is generated by partially rotating a line about an axis parallel to the line). It is also noted that the words “dimple” or “arch” can be used interchangeably. What we refer to as “dimples” in this study, older studies may refer to as “arches” or “ring segments.”

1.2 Research Motivation

The original motivation of this dissertation was to extend the work of Alshbatat [2] in order to further study the effect of dimples on the vibroacoustics of beams. Some of the results prompted important questions as to why dimples affected the structural dynamics. Concepts such as the “thinning effect” and the “curvature effect” that were used to explain observed trends in natural frequencies required further study. Thus, a large motivation for this work was to study in depth the vibrations of dimpled beams and to gain a better understanding of the effect of dimples. Although beams are comparatively simple structures, the conclusions made from a study of dimpled beams give insight into the dynamics of more complicated structures (e.g., dimpled plates).

An important finding regarding the modal strain energy arose during study of the dynamics of dimpled beams in this dissertation. Alshbatat originally suggested that the modal strain could explain trends in natural frequencies. This suggestion, along with some new concepts, is investigated further. This study proceeded to show that modal strain energy could also be used as a design strategy. It was discovered that regions of high modal strain energy of a uniform beam could be used to suggest optimal placement of dimples. Dimples placed in these areas would exhibit the largest
effect on the natural frequencies for various dimple angles. In other words, the modal strain energy provided a measure of design sensitivity. This concept had not been used for design of dimpled beams until the completion of this study. It is likely to motivate further investigation for dimple placement on more complex structures.

The first goal of this dissertation is to predict and analyze both the natural frequencies and mode shapes of dimpled beams. This is achieved by using Hamilton’s Variational Principle to develop a boundary value model for a beam with any number of dimples. This model is unique when compared to previous approaches to model dimpled beams, and it provides a better understanding of dimpled beams. The development of this model, and its implementation, is considered a significant contribution of this study.

Finally, what motivates use of Hamilton’s Principle for development of a boundary value model when other methods could be used to model dimpled beams? The approach taken here offers some important advantages over previous models (e.g., finite element method, impedance method). For one, the development of an analytical model to describe dimpled beams gives important insight into their behavior. We show that the behavior of vibrating dimpled beams can be better understood by gaining a better understanding of the vibrations of straight beams and of arches. Developing this model requires a thorough understanding of the necessary assumptions to model a dimple. That is, should dimples be considered inextensional (zero strain along the circumferential axis) or extensional (non-zero strain along the circumferential axis)? What implications does one assumption have over the other? What conditions must be satisfied at the point where a straight segment connects to a dimple? These questions have been addressed through development of the boundary value model.

Development of the boundary value model may also simplify the methods presented in other papers on dimpled beams. One study in particular used a finite
element discretization of the dimple in order to develop a relationship between the forces and velocities on one side of the dimple to the other side [1] (i.e., mobility matrix). This description, however, is closely related to the continuity conditions developed in the boundary value model. It is possible that these conditions could be used to develop an analytical mobility matrix for a dimple, thus eliminating the need for a finite element approach.

1.3 Dissertation Organization

Chapter 2 contains a review of the studies relevant to this dissertation. Then, the boundary value model for a beam with \( N \) dimples is developed in Chapter 3. In Chapter 4, the boundary value model is validated in several ways by comparing its outcome to known or previous results. In Chapter 5, the dynamics of dimpled beams in free vibration is studied. Emphasis is given to explanations of the trends in natural frequencies. In Chapter 6, a design strategy for placement of dimples is suggested by examination of the modal strain energy of uniform beams. Finally, a summary of important results is given in Chapter 7, followed by the author’s view of future topics of study.
Chapter 2

OVERVIEW OF RELEVANT RESEARCH

This chapter contains a review of the literature relevant to this study. Section 2.1 consists of studies that have used Hamilton’s Principle for developing boundary value models. Section 2.2 summarizes papers that have studied the vibration and acoustic properties of various dimpled and beaded structures. Beads are included in the review since they may be thought of as a series of tightly packed dimples. In Section 2.3, studies on dimpled beams are explained in detail. Previous explanations on the effect of dimples on beam vibrations are emphasized. Finally, Section 2.4 contains a review of the vibrations of arches and other related structures. Knowledge of the vibrations of arches is critical to understanding the motion of dimples. A review of the vibrations of uniform beams is omitted since these structures are commonly studied in textbooks on vibrations [7],[8].

2.1 Uses of Hamilton’s Principle

A boundary value model (BVM) is a general description of a system governed by equations of motion that must satisfy prescribed boundary conditions. Many references use Hamilton’s Principle in order to formulate a boundary value model of a structure.
Its use requires knowledge of the kinetic and strain energy of the system. The advantage of Hamilton’s Principle over other methods is that it yields all physically correct boundary conditions, in addition to the differential equations of motion. The reader is referred to Meirovitch [7, p.42] for an excellent background on the underlying theory. Some examples are given on the transverse [7, p.128] and longitudinal [7, p.45] vibrations of uniform beams. These basic examples demonstrate the mathematical tools required for use in Hamilton’s Principle (e.g., integration by parts, use of the variational operator). For a discussion of the meaning of the variational operator in Hamilton’s Principle (and its properties), see Rao [8, p.89].

Rao gives several examples where Hamilton’s Principle is used [8]. In one example, the boundary value model for a shaft in free torsional vibration is derived [8, p.115]. Another example investigates the vibrations of a uniform beam with intermediate supports [8, p.359]. The beam is divided into segments whose domains are defined locally between supports. Continuity conditions are enforced at each support. Then, the natural frequencies are computed by solving simultaneous homogeneous algebraic equations, where the non-trivial solution is found by equating the matrix determinant to zero. The treatment of a single structure as separate substructures, and the formulation of the system equations in this example is similar to the approach taken in this study. The dimpled beam is divided into straight and dimpled segments, and the natural frequencies for a beam with \( N \) dimples are found by solving a set of \( 12N + 6 \) homogeneous algebraic equations.

Kwon et al. [9] derived the boundary value problem for a system of stepped cantilever beams each joined by a rigid mass (Fig. 2.1). The equations of motion and boundary conditions were developed using Hamilton’s Principle. This study is interesting because continuity conditions at the steps were also required. They were determined by inserting geometric boundary conditions into the Lagrangian. The
system was described by 24 homogeneous algebraic equations in 24 unknowns. A similar technique for determining continuity conditions between dimples and straight segments is used in Chapter 3 of this study.

![Figure 2.1: Stepped cantilever beams joined by a rigid mass](image)

2.2 Dimples and Beads in Vibroacoustics

Early experimental and numerical studies focused on the use of beads and dimples to alter the vibroacoustics of automotive panels. One study compared the sound and vibration of dimpled and beaded automotive panels to an unmodified panel [10]. The dimple and bead patterns were not optimized for minimum sound radiation, and some designs actually increased the sound radiation. A systematic method of topology optimization to determine the optimal layout of beads was subsequently used in order to raise the natural frequencies of a simplified vehicle body structure [11]. One drawback was that bead orientation could not be uniquely determined. The orientation problem was addressed later by discretizing a plate into “cells” of orthotropic material. The cells could be rotated, thus converting the optimal bead orientation problem to that of bending-equivalent orthotropic materials [12].

Other authors recognized the utility of using beads and dimples on plates to improve noise and vibration characteristics. Fritze [4] published a study where the frequency-averaged sound power of a plate was minimized using beads with a square
projection (Fig. 2.2). A lumped parameter approximation of sound power was used, which required calculating the volume velocity of discrete sections of the plate using the finite element method. The optimization results showed that sound power could be reduced when the bead was placed near the point of excitation. The drawback to their approach, however, was the large time needed to compute the sound power.

![Figure 2.2: Approximate representation of the square bead used by Fritze [4]](image)

Kulkarni [13] experimentally studied different beading patterns on the floor of an automobile in order to reduce the sound pressure level at the location of the driver and passenger’s ear. The beading patterns were also tested for their ability to reduce the level of floor vibration that would be felt by a passenger’s foot. The patterns were chosen based off of what was already being used for production. Similarly, researchers from Japan designed the floor of a train car using a flat plate mounted on top of a dimpled plate [14] (see Fig. 2.3). Experimentally measured results showed that the floor radiated less sound power per unit force than conventional honeycomb floors.

Park [15] developed a systematic approach using an optimization routine with the finite element method in order to find optimal patterns of grooves on plates. The grooves were formed on the plate in such a way that the fundamental frequency was maximized. Interestingly, initial groove domains were chosen based off of regions of
high modal strain energy calculated from a finite element analysis. Then, the target domains were restricted to neighboring areas around the initial domain. This approach led to a significant reduction in computational effort. A similar design strategy is employed later in this dissertation by considering regions of high modal strain energy of uniform beams as candidate locations for dimples. In the present case, the modal strain energy is computed analytically.

Alshabatat published two studies on the vibration and acoustic properties of dimpled and beaded plates. Optimum dimple patterns were obtained by using a genetic algorithm (GA). The finite element method (FEM) was used for calculation of all vibration characteristics. The first of the two studies formed dimples on a plate in order to maximize the change in fundamental frequency, and to maximize the gap between the first and second natural frequencies [16]. Dimples placed in the corners of a simply-supported plate produced the largest increase in fundamental frequency. Their effect was to reduce the structural half-wavelength of the fundamental mode. Achieving a reduction in the sound power at a fixed frequency and over a broad band was also a focus. The second of the papers focused exclusively on shifting the natural
frequencies of plates using dimples and beads [17]. The GA was used with the FEM as in the previous study. It was suggested that beads placed in regions of high modal strain were the cause of large shifts in natural frequencies. However, plots of modal strain or modal strain energy were not provided.

2.3 Studies on Dimpled Beams

Recent studies involving structural modifications that are functionally equivalent to dimples include forming “V-notches” (Fig. 2.4) on the surface of a beam [18]. V-notches were used in order to shift the beam’s natural frequencies. The V-notch had a reduced thickness so that mass of the beam stayed constant before and after forming the notch. For boundary conditions that restricted axial motion (e.g., clamp, pin), it was observed that the V-notch could increase or decrease the fundamental frequency depending on the location and geometry. The suggested cause for this was an interaction between the shape of the V-notch (“V-shape effect”), and the thinning of the region (“thinning effect”).

Figure 2.4: A single V-notch formed on a beam

In a related study [19], Alshabatat changed the height of a beam at certain key points in order to form a “bent” beam (Fig. 2.5). Changing the shape of the beam reduced the radiated sound power at discrete frequencies and over a broadband. Key points along the beam were chosen as design variables in the finite element model, whereby the optimum out-of-plane heights were found using the GA. The sound power of the structure was computed using a lumped-parameter approximation [20].
Cheng et al. [1] studied the natural frequencies of hinged-roller (i.e. simply-supported) dimpled beams using an impedance matching technique. The goal of the study was to find combinations of dimple angles and locations in order to “tune” the natural frequencies to prescribed values. The dimpled beam was divided into straight and curved segments, where each segment was described by a mobility matrix (mobility is the inverse of impedance). An analytical mobility matrix was used for the straight segments. However, the analytical matrices were unknown for the dimples, so the mobility matrices were derived using a finite element discretization. By matching the impedances at each connection, a $6 \times 6$ global mobility matrix could be constructed for computation of the natural frequencies. A parametric study showed a decrease in the fundamental frequency for all combinations of dimple angle and location. The decrease was attributed to the dimple being thinner than the rest of the beam. It was also noted that the fundamental frequency was most sensitive to changes in dimple angle when dimples were placed at the anti-nodes. The fundamental exhibited the least sensitivity when dimples were placed at nodes. This claim is investigated for other boundary conditions later in this study. The authors did not determine mode shapes using the impedance technique, and it was not suggested how, if possible, structural mode shapes could be extracted using their technique.

Cheng et al. [21] also investigated the radiation efficiency of dimpled beams. Since mode shapes are required for calculation of radiation efficiency, a finite element
model of dimpled beams was used instead of their previously developed impedance technique. Radiation efficiency of the dimpled beam was reduced from that of the undimpled beam by using many dimples to enforce weak radiating mode shapes [22] at a specific excitation frequency. One drawback of their study is that the dimples were allowed to become thicker than the rest of the beam, although the total beam mass was held constant. This presents practical difficulties when manufacturing the optimized beams, since dimples tend to become thinner when formed via the stamping method.

Alshabtat [2], [23] later studied dimpled beams for additional boundary conditions (in addition to dimpled and beaded plates). The focus of his study was primarily to determine optimum patterns of dimples for achieving certain vibroacoustic objectives. His study showed the feasibility of using dimples for noise control while keeping manufacturing limitations in mind. Natural frequencies and mode shapes were computed using the FEM, and optimum dimple patterns and geometries were determined using the GA. The thickness of the dimples was made to decrease with increasing dimple angle so that the total beam mass was unchanged before and after dimpling. It was discovered that boundary conditions that permitted axial motion of the beam (e.g., boundaries with a free end or roller) resulted in decreased natural frequencies after a dimple was introduced. This was attributed to a lack of horizontal constraint and to the thinning of the dimple. For boundary conditions that restricted axial motion (e.g., fixed boundaries), it was observed that the natural frequencies could increase or decrease with respect to the uniform beam frequency, depending on dimple location and angle. To explain these results, an increase in natural frequency was attributed to a “curvature effect,” where the dimple curvature made the beam stiffer. A decrease in natural frequency was attributed to a “thinning effect” of the dimple as dimple angle increased, where a thinner dimple lowered the beam stiffness. The tendency of one effect to dominate the other for certain dimple angles was offered as an explanation.
of the observed trends in natural frequencies.

Various hypotheses were suggested to explain the trends in natural frequencies, and to give support to the proposed thinning-curvature effect. One hypothesis was that dimples placed in regions of high modal strain would have a significant effect on natural frequencies. However, the modal strain energy for different modes and boundary conditions was not calculated and compared to the natural frequencies. Another hypothesis involved a study of the static stiffness of dimpled beams, where the static stiffness ratio was defined as the stiffness of the dimpled beam to the undimpled beam. The analysis was performed for various dimple angles, locations, and boundary conditions. Two examples are illustrated in detail below.

First, the static stiffness ratio was plotted against dimple location assuming a dimple angle of 100 degrees and a transverse force located at the beam center. For the fixed-fixed boundary condition, the ratio was relatively large for dimples placed at the boundaries, and largest for a dimple placed at the center [2, p.77]. However, this observation was in contrast to the fact that the maximum increase in fundamental frequency occurred for dimples placed at the fixed boundaries, not at the center of the beam. It also did not explain the entirely different behavior of higher modes.

In another example, the static stiffness ratio was plotted against dimple angle, assuming the dimple was located at the beam center. For the fixed-fixed boundary condition, the results showed that the dimpled beam had a higher static stiffness than the undimpled beam for any dimple angle [2, p.79]. This would imply a dominant curvature effect regardless of dimple angle, and based on the hypothesis, the dimple would always increase the natural frequencies. This is contradicted by two observations: 1) the fundamental frequency of a fixed-fixed dimpled beam can be lower than that of the uniform beam for a dimple placed at the center and having a large angle, and 2) the second natural frequency decreases monotonically for any dimple
angle when located at the center of a fixed-fixed beam (i.e., the second natural frequency of the dimpled beam is always lower than the second natural frequency of the uniform beam).

Another approach is needed to explain trends in natural frequencies for dimpled beams. One limitation of a static analysis is that it cannot, by definition, address the dynamic behavior of different modes of vibration. In general, different natural frequencies exhibit an entirely different behavior as dimple location and angle is changed. A static stiffness study also misses an important opportunity to investigate the ability of mode shapes to influence the natural frequencies. In Chapter 5 of this study, dynamic concepts such as dynamic stiffness, modal strain energy, and changing mode shapes within the dimple are investigated in order to better understand the trends in natural frequencies.

### 2.4 Vibrations of Curved Structures

A few papers on the vibrations of curved structures were of interest during the development of this study. The reader is referred to reference [24] for a comprehensive collection of 407 studies on the vibrations of curved beams, rings, and arches. A development of the mechanics of curved bars, including the strain of an arch caused by radial and tangential displacements, is given by Henrych [25, p.21].

An early study of the inextensional equation of motion of arches can be found in Love’s classic text [26]. The inextensional assumption neglects the strain of the arch along its circumferential axis. Later, the first two natural frequencies of pinned and clamped arches was considered by Den Hartog [27] using Rayleigh’s energy method. Noteworthy was the comparison of inextensional and extensional modes of vibration. When the arch is assumed to “stretch” along the circumferential axis (i.e., non-zero...
strain), the arch is termed extensional. It was shown that the fundamental mode shape will switch from an extensional mode shape (Fig. 2.6(a)) to an inextensional one (Fig. 2.6(b)) at certain arch angles. The switch was followed by a decrease in the fundamental frequency (Fig. 2.7). This interesting observation motivates some of the work in this study. Namely, it suggests that changing mode shapes within the dimple should be investigated as a cause for decreasing beam natural frequencies.

Figure 2.6: Mode shapes of an arch: extensional symmetric (a) and inextensional anti-symmetric (b). Shapes are approximate representations.

Figure 2.7: Representation of the fundamental frequency vs. dimple angle for an arch with clamped or hinged ends
Extending the analysis of Den Hartog, Nelson [28],[29] studied the inextensional and extensional assumptions for higher modes of hinged arches using the Rayleigh-Ritz technique. The results for the fundamental frequency were in agreement with Den Hartog. The natural frequencies for symmetric and antisymmetric modes were expressed as terms in an infinite series summation, where the polynomial roots yielded the natural frequencies. Results obtained using these equations will serve as a check to the dimpled beam model in Chapter 4, where the dimpled beam is approximated as an arch by letting the straight segment lengths approach zero.

Lang [30] studied the inextensional and extensional vibrations of partial and complete rings, where the equations of motion were developed using an equilibrium approach. His study noted the different forms of characteristic roots of arches, and demonstrated a technique for uncoupling the equations of motion. Both free and forced vibrations were considered. Lang derived the equations of motion using an equilibrium approach, in contrast to the variational approach taken in this study. Despite having developed the extensional equations of motion for an arch, he considered only an example of a complete and inextensional ring. Lang’s work was instrumental to the study of dimples in this dissertation, and it will be cited in detail during the development of the boundary value model in Chapter 3.

Other authors studied the vibrations of a column-arch system [5] shown in Fig. 2.8. This example is interesting due to the similarity of the structure to a dimpled beam. Unlike the dimpled beam, however, the column-arch did not have a discontinuity in slope where the segments joined. The authors used polynomial approximations in the Rayleigh-Schmidt method to determine an upper bound to the fundamental frequency, and a modified Dunkerley approach to determine a lower bound. The results were checked using the FEM. The choice of the Rayleigh-Schmidt method (a variant
Figure 2.8: Schematic of a column-arch system used for study in reference [5]

of Rayleigh-Ritz) is interesting since it is plausible that a similar approach could be adopted for dimpled beams in future study. This method requires an assumption of the mode shape in the form of an admissible function, where an eigenfunction yields an exact natural frequency. However, the eigenfunctions of dimpled beams were unknown prior to the development of the boundary value model in this study. Since the eigenfunctions will be developed after solution to the equations of motion, this may suggest a choice of admissible functions in future studies of dimpled beams where approximate methods are used.
Chapter 3

THE BOUNDARY VALUE MODEL

An analytical model (i.e., a boundary value model, BVM) of a beam with any number of dimples is developed in this chapter. The BVM is used in order to calculate the natural frequencies and mode shapes of a dimpled beam in free transverse and longitudinal vibration. The model describes the transverse and longitudinal vibrations in the straight segments, and the tangential and radial vibrations in the dimples. The effects of shear force and rotary inertia on the natural frequencies are neglected. This is a valid assumption as long as the bending wavelength of the beam is much larger than the beam thickness [31, p.342]. In order to satisfy the force equilibrium between straight segments and dimples, the extension of the dimple along its circumferential axis is included in the formulation (i.e., the dimple is allowed to “stretch”). Because of this inclusion, the dimples are characterized as “extensional”, in contrast to an “inextensional” arch where the extension of the circumferential axis is neglected.

In Section 3.1, Hamilton’s Principle is used in order to develop the equations of motion, boundary conditions, and continuity conditions for a beam consisting of one straight segment connected to one dimple (arch segment). To consider additional
straight segments or dimples during the development is redundant. Instead, the approach here is to generalize the BVM of the two-segment model to a beam with any number of dimples. This generalization is carried out in Section 3.2. In Section 3.3, the general solutions to the equations of motion are given. Then as an example, it is shown in Section 3.4 how to formulate the coefficient matrix of a beam with one dimple. Finally, some concluding remarks are made in Section 3.5.

3.1 Two-Segment Model

Shown in Fig. 3.1 is a schematic of a dimpled beam consisting of one straight segment of length \( l \) connected to an arched segment (dimple) with chord length \( \bar{l} \). Along with the chord length, the dimple is defined by a constant radius \( R \) and dimple angle \( \alpha \).

The straight segment has uniform thickness \( h \) and the dimple has uniform thickness \( \bar{h} \), as shown in Fig. 3.2. The beam width \( b \) is constant. The eigenfunctions in the straight segment (longitudinal and transverse) and the dimple (tangential and radial)

![Figure 3.1: Schematic of a two-segment model: one straight segment and one dimple (arch segment)](image-url)
are defined over a local coordinate system, where downward is considered positive,

\[
\begin{align*}
    u(x, t), y(x, t) & \quad 0 \leq x \leq l, \\
    \nu(\theta, t), w(\theta, t) & \quad 0 \leq \theta \leq \alpha.
\end{align*}
\tag{3.1}
\]

The relative orientations of the eigenfunctions as shown in Fig. 3.1 will be important when the equations of continuity between the straight segment and dimple are written. Note that when the dimple becomes flat, the radial and transverse components align, as do the tangential and longitudinal components.

Dimples are commonly formed on a surface by stamping them via a punch and die. Forming the dimple in this way does not change the mass of the beam, and so the dimple thickness must be smaller than the straight segment thickness. Assuming constant mass and dimple radius, the relationship between the thicknesses is given by [2]

\[
\bar{h} = \frac{2 \sin(\alpha/2)}{\alpha} h,
\tag{3.2}
\]

where the dimple thickness is considered uniform throughout. In a practical manufacturing application, the dimple thickness is non-uniform. In order to simplify the mathematical model, this non-uniformity is neglected. We note that in the limit that \( \alpha \) approaches zero (i.e., the dimple is flattened), the dimple thickness approaches that of the straight segment.

![Figure 3.2: Isometric view of two-segment model](image-url)
3.1.1 Formulation of the Lagrangian

The equations of motion of the model are developed using Hamilton’s Principle [7, p.44],

$$\delta \int_{t_1}^{t_2} (T - V) \, dt = 0,$$

where $T$ and $V$ are kinetic and strain energy functionals, respectively. Hamilton’s Principle states that the integral of the Lagrangian functional (action integral) over any two instants in time is stationary for the true path of the system. In fact, this stationary value is a minimum. The principle requires that all varied paths and the true path coincide at the two instances in time. As an example, imagine a simple pendulum moving in an irregular pattern: swinging, suddenly stopping, starting again, etc., but in a way that does not violate the constraint of the rope. This varied (and fictitious) path would yield a non-minimum value for the action integral. However, the action integral is a minimum for the path that the pendulum actually takes. The advantage of using Hamilton’s Principle is that the equations of motion governing the system dynamics are obtained along with the boundary and continuity conditions.

For the two segment model, the total kinetic energy is a sum of the straight segment [8, p.652] and dimpled kinetic energies [28], given by

$$\mathcal{T}(t) = T(t) + \bar{T}(t),$$

where

$$T(t) = \frac{1}{2} \rho S \int_0^l \left( \frac{\partial y}{\partial t} \right)^2 + \left( \frac{\partial u}{\partial t} \right)^2 \, dx,$$

$$\bar{T}(t) = \frac{1}{2} \rho \bar{S} R \int_0^\alpha \left( \frac{\partial \nu}{\partial t} \right)^2 + \left( \frac{\partial \omega}{\partial t} \right)^2 \, d\theta.$$
\[ \mathcal{V}(t) = v(t) + \bar{v}(t), \]  

(3.5a)

where

\[
v(t) = \frac{1}{2} EI \int_0^l \left( \frac{\partial^2 y}{\partial x^2} \right)^2 \, dx + \frac{1}{2} ES \int_0^l \left( \frac{\partial u}{\partial x} \right)^2 \, dx,
\]

(3.5b)

\[
\bar{v}(t) = \frac{1}{2} \frac{EI}{R^3} \int_0^\alpha \left( \frac{\partial \nu}{\partial \theta} + \frac{\partial^2 w}{\partial \theta^2} \right)^2 \, d\theta + \frac{1}{2} \frac{ES}{R} \int_0^\alpha \left( \frac{\partial \nu}{\partial \theta} - w \right)^2 \, d\theta.
\]

(3.5c)

The constant quantities have been taken outside of the integral. Throughout this dissertation, a “bar” over a quantity indicates the equivalent quantity for the dimple as in the straight segment. In Eq. (3.5b), the strain energy in the straight segment is due to bending (first term) and axial displacement (second term). In Eq. (3.5c), the strain energy in the dimple is caused by bending (first term) and circumferential extension (second term).

In order to formulate the Lagrangian for the two-segment model, Eqs. (3.4)–(3.5) are inserted into Eq. (3.3). Applying the first order variation \( \delta \) to each term yields

\[
\int_{t_1}^{t_2} \left\{ \rho S \int_0^l \frac{\partial y}{\partial t} \frac{\partial}{\partial t} (\delta y) + \frac{\partial u}{\partial t} \frac{\partial}{\partial t} (\delta u) \right\} \, dx \\
+ \rho \bar{S} R \int_0^\alpha \frac{\partial \nu}{\partial \theta} \frac{\partial}{\partial \theta} (\delta \nu) + \frac{\partial w}{\partial \theta} \frac{\partial}{\partial \theta} (\delta w) \right\} \, d\theta \\
- \frac{EI}{R^3} \int_0^\alpha \left( \frac{\partial \nu}{\partial \theta} + \frac{\partial^2 w}{\partial \theta^2} \right) \left( \frac{\partial}{\partial \theta} (\delta \nu) + \frac{\partial^2}{\partial \theta^2} (\delta w) \right) \right\} \, d\theta \\
- \frac{ES}{R} \int_0^\alpha \left( \frac{\partial \nu}{\partial \theta} - w \right) \left( \frac{\partial}{\partial \theta} (\delta \nu) - \partial w \right) \right\} \, d\theta = 0.
\]

(3.6)

The commutative property of variation with differentiation and integration was used above [8, p.89]. Next, each variation is integrated by parts a number of times equal to...
to the order of the differential operator acting on it. For example, in the first integral, \( \delta y \) is integrated by parts with respect to \( t \) once, whereas in the third integral, \( \delta y \) is integrated with respect to \( x \) twice. Integrating Eq. (3.6) and grouping similar variations yields

\[
\int_{t_1}^{t_2} \delta \mathcal{L} \, dt = 0, \tag{3.7a}
\]

where the Lagrangian functional is given by

\[
\delta \mathcal{L} = \int_0^l \left[ -\rho S \frac{\partial^2 y}{\partial t^2} - EI \frac{\partial^3 y}{\partial x^3} \right] \delta y \, dx
\]  
\[
+ \int_0^l \left[ -\rho S \frac{\partial^2 u}{\partial t^2} + ES \frac{\partial^2 u}{\partial x^2} \right] \delta u \, dx
\]  
\[
+ \int_0^\alpha \left[ -\rho \bar{S} R \left( \frac{\partial^2 \nu}{\partial \theta^2} - \frac{\partial^2 \nu}{\partial \theta^2} \right) \right] \delta \nu \, d\theta
\]  
\[
+ \int_0^\alpha \left[ -\rho \bar{S} R \left( \frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial x^2} \right) \right] \delta w \, d\theta
\]  
\[
+ \left[ -EI \frac{\partial^2 y}{\partial x^2} \frac{\partial}{\partial x} (\delta y) + EI \frac{\partial^2 y}{\partial x^3} \delta y - ES \frac{\partial u}{\partial x} \delta u \right] \bigg|_{x=0}
\]  
\[
+ \left[ -\frac{ES}{R} \left( \frac{\partial \nu}{\partial \theta} - w \right) \right] \delta \nu + \frac{EI}{R^3} \left( \frac{\partial^2 \nu}{\partial \theta^2} + \frac{\partial^3 w}{\partial \theta^3} \right) \delta w + \cdots
\]  
\[
- \frac{ES}{R^2} \left( \frac{\partial \nu}{\partial \theta} + \frac{\partial^2 w}{\partial \theta^2} \right) \left( \delta \nu + \frac{\partial}{\partial \theta} (\delta w) \right) \bigg|_{\theta=0}.
\]  

In Eq. (3.7), the variations are zero at \( t_1 \) and \( t_2 \) by definition of Hamilton’s Principle. Using Eq. (3.7), the equations of motion, boundary conditions, and continuity conditions are obtained.
3.1.2 Equations of Motion

The variations, $\delta y$, $\delta u$, $\delta \nu$, $\delta w$, in Eq. (3.7b) are all arbitrarily chosen over their domains and independent from one another. Since Eq. (3.7) must be true for all domains, it follows by inspection of Eq. (3.7b) that the expressions in brackets inside each of the four integrals are all zero. These represent the four differential equations of motion of the two-segment beam:

$$\rho S \frac{\partial^2 y}{\partial t^2} + EI \frac{\partial^4 y}{\partial x^4} = 0,$$

(3.8)

$$\rho S \frac{\partial^2 u}{\partial t^2} - ES \frac{\partial^2 u}{\partial x^2} = 0,$$

(3.9)

$$\rho \bar{S} R \frac{\partial^2 \nu}{\partial t^2} - \frac{EI}{R^3} \left( \frac{\partial^2 \nu}{\partial \theta^2} + \frac{\partial^3 w}{\partial \theta^3} \right) - \frac{ES}{R} \left( \frac{\partial^2 \nu}{\partial \theta^2} - \frac{\partial w}{\partial \theta} \right) = 0,$$

(3.10)

$$\rho \bar{S} R \frac{\partial^2 w}{\partial t^2} + \frac{EI}{R^3} \left( \frac{\partial^3 \nu}{\partial \theta^3} + \frac{\partial^4 w}{\partial \theta^4} \right) - \frac{ES}{R} \left( \frac{\partial \nu}{\partial \theta} - w \right) = 0.$$

(3.11)

Equations (3.8)–(3.9) govern the free transverse and longitudinal vibrations of the straight segment [8, p.236, 325]. Equation (3.8) is the well-known fourth order partial differential equation obtained from Euler-Bernoulli beam theory. This result may have been expected since the effects of shear deformation and rotary inertia were neglected.

Equations (3.10)–(3.11) govern the free tangential and radial vibrations of the dimple. Lang [30] derived the same equations when studying complete circular rings by using an equilibrium approach, as opposed to the variational approach used here. Note that Eqs. (3.10)–(3.11) are coupled differential equations, meaning that both $\nu$ and $w$ appear in each equation. However, the two equations can be uncoupled by algebraic manipulation of the differential operators acting on $\nu$ and $w$ [30]. As a result,
the uncoupled equations are given by

\[
\begin{align*}
\frac{\partial^6 \nu}{\partial \theta^6} + 2 \frac{\partial^4 \nu}{\partial \theta^4} - \frac{\rho R^2}{E} \frac{\partial^2 \nu}{\partial t^2 \partial \theta^2} + \frac{\partial^2 \nu}{\partial \theta^2} &+ \frac{\rho R^2}{E} \frac{\partial^4 \nu}{\partial t^2 \partial \theta^2} + \frac{\rho^2 SR^4}{E^2 I} \frac{\partial^4 \nu}{\partial \theta^4} = 0, \\
\frac{\partial^6 w}{\partial \theta^6} + 2 \frac{\partial^4 w}{\partial \theta^4} - \frac{\rho R^2}{E} \frac{\partial^2 w}{\partial t^2 \partial \theta^2} + \frac{\partial^2 w}{\partial \theta^2} &+ \frac{\rho R^2}{E} \frac{\partial^4 w}{\partial t^2 \partial \theta^2} + \frac{\rho^2 SR^4}{E^2 I} \frac{\partial^4 w}{\partial \theta^4} = 0.
\end{align*}
\]

Note that the form is identical, i.e. the same differential operator acts on \( \nu \) as does \( w \). Each equation has a differential order of six. This would seem to imply that twelve total unknown coefficients exist for the eigenfunctions. It will be shown in Section 3.3.2, however, that only six coefficients are needed since one set of six coefficients are related to the other six by constants of proportionality.

Equations (3.12) and (3.13) account for the extension of the dimple along its centroidal axis, which was a result of including the strain energy due to circumferential displacement given by the second integral in Eq. (3.5c). Many authors neglect this component of strain energy, especially when the thickness is small [8, p.397], [27], [28]. In effect, this assumes that the strain along the centroidal axis is zero, and it follows that the radial and tangential vibrations are related simply by \( w = \partial \nu / \partial \theta \) (this makes the second integral in Eq. (3.5c) equal to zero). The “inextensional” assumption can be viewed as a special case of the more general “extensional” equations presented and used here. Although the inextensional assumption is the mathematically simpler model of the two, it is not used in the formulation of the BVM. As will be shown in Section 3.1.4, this is because the force developed by extension of the dimple is necessary in order to fully describe the coupling between straight segment and dimple.
3.1.3 Boundary Conditions at $x = 0$ and $\theta = \alpha$

The boundary conditions of the two-segment beam model are found by examination of Eq. (3.7). The boundary condition imposed on the straight segment at $x = 0$ is given by

$$\left[ EI \frac{\partial^2 y}{\partial x^2} \frac{\partial}{\partial x} (\delta y) - EI \frac{\partial^3 y}{\partial x^3} \delta y + ES \frac{\partial u}{\partial x} \delta u \right]_{x=0} = 0.$$  

(3.14)

Each of the three terms must be zero due to the fact that the variations are arbitrarily chosen over their domains (including at $x = 0$) and independent of each other. Therefore, two boundary conditions are needed for transverse motion and one boundary condition is required for longitudinal motion:

$$EI \left. \frac{\partial^2 y}{\partial x^2} \frac{\partial}{\partial x} (\delta y) \right|_{x=0} = 0,$$  

(3.15a)

$$EI \left. \frac{\partial^3 y}{\partial x^3} \delta y \right|_{x=0} = 0,$$  

(3.15b)

$$ES \left. \frac{\partial u}{\partial x} \delta u \right|_{x=0} = 0.$$  

(3.15c)

Equation (3.15) states that at $x = 0$, either bending moment $EI(\partial^2 y/\partial x^2)$ or bending slope $\partial y/\partial x$ is zero (Eq. (3.15a)), either shear force $EI(\partial^3 y/\partial x^3)$ or transverse deflection is zero (Eq. (3.15b)), and either axial force $ES(\partial u/\partial x)$ or longitudinal deflection is zero (Eq. (3.15c)).

The three boundary conditions on the dimple at $\theta = \alpha$ are found by setting the last bracketed term in Eq. (3.7) to zero. As before, each of the three terms are zero since the variations are arbitrary over their domains (including at $\theta = \alpha$) and independent of each other. Dimpled beam configurations that require boundary conditions directly on the dimple (e.g. a dimple positioned at one extreme end of the beam).
beam) will not be encountered in this study since, in general, a dimple will be positioned between two straight segments. It is interesting to note, however, that similar physical interpretations exist for the boundary conditions on a dimple as do a straight segment. They are summarized in Eq. (3.16) as follows: zero bending moment $\mathcal{M}$ or bending slope $\psi$ (Eq. (3.16a)), zero radial shear $Q$ or radial deflection (Eq. (3.16b)), and zero tangential force $P$ or tangential deflection (Eq. (3.16c)),

\[
\mathcal{M}\delta\psi\bigg|_{\theta=\alpha} = 0, \quad (3.16a)
\]

\[
Q\delta w\bigg|_{\theta=\alpha} = 0, \quad (3.16b)
\]

\[
P\delta\nu\bigg|_{\theta=\alpha} = 0, \quad (3.16c)
\]

where

\[
\psi = \frac{1}{R}\left(\nu + \frac{\partial w}{\partial \theta}\right), \quad (3.17)
\]

\[
\mathcal{M} = \frac{EI}{R^2}\left(\frac{\partial \nu}{\partial \theta} + \frac{\partial^2 w}{\partial \theta^2}\right), \quad (3.18)
\]

\[
Q = \frac{EI}{R^3}\left(\frac{\partial^2 \nu}{\partial \theta^2} + \frac{\partial^3 w}{\partial \theta^3}\right), \quad (3.19)
\]

\[
P = \frac{ES}{R}\left(\frac{\partial \nu}{\partial \theta} - w\right). \quad (3.20)
\]

It can be shown that in the limit the dimple becomes flat, or $R \to \infty$, then $dx \approx R d\theta$, and all boundary conditions on the dimple reduce to the form given by Eqs. (3.15a)–(3.15c).

---

1In the special case that a dimple lies at one extreme end of the beam, the dimple boundary conditions given by Eq. (3.16) can be used.
3.1.4 Continuity Conditions at $x = l$

The solutions to Eqs. (3.8), (3.9), (3.12), and (3.13) must satisfy conditions of continuity where the straight segment and dimple meet at $x = l$ and $\theta = 0$. These conditions are found from Eq. (3.7) by setting the sum of the bracketed terms evaluated at $x = l$ and $\theta = 0$ to zero,

$$\left[ -EI \frac{\partial^2 y}{\partial x^2} \frac{\partial}{\partial x} (\delta y) + EI \frac{\partial^3 y}{\partial x^3} \delta y - ES \frac{\partial u}{\partial x} \delta u \right] \bigg|_{x=l} + \left[ M \delta \psi - Q \delta w + P \delta \nu \right] \bigg|_{\theta=0} = 0.$$  \hspace{1cm} (3.21)

Equation (3.21) contains six different variations ($\delta u, \delta y, \delta y', \delta \nu, \delta w, \delta w'$), where three can be eliminated by using three geometrical conditions of continuity (i.e. horizontal and vertical deflection, slope). Since the straight segment and dimple must remain connected both horizontally and vertically at all times, it follows by inspection of Fig. 3.1 that

$$\begin{cases} u(l,t) = \begin{bmatrix} \cos(\alpha/2) & \sin(\alpha/2) \end{bmatrix} \begin{bmatrix} \nu(0,t) \\ \psi(0,t) \end{bmatrix}, \\ y(l,t) = \begin{bmatrix} -\sin(\alpha/2) & \cos(\alpha/2) \end{bmatrix} \begin{bmatrix} \nu(0,t) \\ \psi(0,t) \end{bmatrix}, \end{cases}$$ \hspace{1cm} (3.22)

where the half-angle is made at $\theta = 0$ between the vertical and radial line. The square matrix in Eq. (3.22) represents a clockwise rotation of $\alpha/2$ degrees, essentially transforming the displacements in the dimple from a tangential-radial coordinate system to those in a horizontal-vertical system. Furthermore, it is assumed that the connection is rigid, i.e. the slope due to bending is continuous,

$$y'(l,t) \equiv \frac{\partial y(l,t)}{\partial x} = \psi(0,t).$$ \hspace{1cm} (3.23)

Using Eqs. (3.22) and (3.23), the expressions for $y(l,t), y'(l,t)$, and $w(0,t)$ are substituted into Eq. (3.21). Grouping similar remaining variations in $\delta u(l,t), \delta \nu(0,t), \delta w'(0,t)$, and noting as before that these variations are arbitrary and independent,
three equilibrium conditions are obtained:

\[ EI \frac{\partial^2 y(l,t)}{\partial x^2} = \mathcal{M}(0,t) , \]  
(3.24)

\[ ES \frac{\partial u(l,t)}{\partial x} = P(0,t) \cos \left( \frac{\alpha}{2} \right) - Q(0,t) \sin \left( \frac{\alpha}{2} \right) , \]  
(3.25)

\[ EI \frac{\partial^3 y(l,t)}{\partial x^3} = P(0,t) \sin \left( \frac{\alpha}{2} \right) + Q(0,t) \cos \left( \frac{\alpha}{2} \right) . \]  
(3.26)

Equations (3.24)–(3.26) represent an equilibrium of moment, axial force, and shear force at \( x = l \) and \( \theta = 0 \), respectively. In Eqs. (3.25)–(3.26), the radial shear and tangential forces in the dimple are transformed by a rotation matrix into force components in the horizontal and vertical directions. As such, they balance the opposing forces from the straight beam acting on a differential element located where the straight segment and dimple meet.

In Section 3.1.2, the extensional equations of motion governing the dimple were developed. The physical meaning of these equations is now apparent: by accounting for the extension of the dimple along its circumference, a force tangent to the dimple’s circumference is developed. This tangential force, \( P \), appears in Eqs. (3.25)–(3.26), and together with the radial shear force \( Q \), is needed to fully describe the force coupling between the straight segment and dimple. In the inextensional case, \( P = 0 \) since \( w = \partial \nu / \partial \theta \) (Eq. (3.20)), and force equilibrium cannot be satisfied.

In summary of Section 3.1, four equations of motion (two per segment) given by Eqs. (3.8)–(3.11) are needed to describe the two-segment model. The total number of boundary conditions is six, given by Eq. (3.15) for the straight segment, and Eq. (3.16) for the dimple. Six continuity conditions exist at \( x = l \) and \( \theta = 0 \), and these conditions are given by Eqs. (3.22)–(3.26). Therefore, the total number of undetermined coefficients in the general solution is twelve for the two-segment model.
3.2 \( N \)-Dimpled Beam Model

The core mathematical modeling has been completed by consideration of the two-segment model. The purpose of this section is to generalize the equations developed in Section 3.1 to describe a beam with any number of dimples. To illustrate the additional complexity, but similarity between a two-segment model and a model with any number of dimples, consider a beam with \( N = 2 \) dimples as shown in Fig. 3.3. Each longitudinal and transverse displacement, \( u_i \) and \( y_i \), are defined locally over \( 0 \leq x_i \leq l_i \) for the \( i^{th} \) straight segment. Similarly, the tangential and radial displacement, \( \nu_i \) and \( w_i \), are defined locally over \( 0 \leq \theta_i \leq \alpha_i \) for the \( i^{th} \) dimpled segment. Formulation of the Lagrangian for this model now requires that the total kinetic energy include five integrals - three for the straight segments given by the form of Eq. (3.4b), and two for the dimpled segments given by the form of Eq. (3.4c). Similarly, the total strain energy is computed from five integrals of the form given by Eqs. (3.5b)–(3.5c). This means that the Lagrangian, \( \mathcal{T} - \mathcal{V} \), which now has additional terms but an identical form to the two-segment model, will produce similar equations of motion, boundary conditions, and continuity conditions. Each of these are generalized in the following sections, and together, comprise the BVM of a beam with \( N \) dimples.

![Figure 3.3: Schematic of a beam with two dimples](image)
3.2.1 Equations of Motion

For $N$ dimples in general, the equations of motion governing the $i^{th}$ straight segment, where $i = 1, 2, \ldots N+1$, are given by

$$
\rho S \frac{\partial^2 y_i}{\partial t^2} + EI \frac{\partial^4 y_i}{\partial x_i^4} = 0,
$$

(3.27)

$$
\rho S \frac{\partial^2 u_i}{\partial t^2} - ES \frac{\partial^2 u_i}{\partial x_i^2} = 0.
$$

(3.28)

For the $i^{th}$ dimple, where $i = 1, 2, \ldots N$, the uncoupled equations of motion are given by

$$
\frac{\partial^6 v_i}{\partial \theta_i^6} + 2 \frac{\partial^4 v_i}{\partial \theta_i^4} - \frac{\rho R_i^2}{E} \frac{\partial^6 v_i}{\partial t^2 \partial \theta_i^4} + \frac{\partial^2 v_i}{\partial \theta_i^2} + \frac{\rho R_i^2}{E} \frac{\partial^4 v_i}{\partial t^2 \partial \theta_i^2} + \frac{\rho S_i R_i^4}{E} \frac{\partial^4 v_i}{\partial \theta_i^2} = 0,
$$

(3.29)

$$
\frac{\partial^5 w_i}{\partial \theta_i^6} + 2 \frac{\partial^4 w_i}{\partial \theta_i^4} - \frac{\rho R_i^2}{E} \frac{\partial^6 w_i}{\partial t^2 \partial \theta_i^4} + \frac{\partial^2 w_i}{\partial \theta_i^2} + \frac{\rho R_i^2}{E} \frac{\partial^4 w_i}{\partial t^2 \partial \theta_i^2} + \frac{\rho S_i R_i^4}{E} \frac{\partial^4 w_i}{\partial \theta_i^2} = 0.
$$

(3.30)

It is assumed throughout this study that the straight segments have the same thickness. In general, the dimples do not have the same radius or thickness since thickness depends on the dimple angle (see Eq. (3.2)). Material properties of the beam such as density and elastic modulus are assumed uniform. For convenience, the subscripts on $x$ and $\theta$ will subsequently be dropped since the domain is implied by the subscript on the eigenfunction.

3.2.2 Boundary Conditions at $x_1 = 0$ and $x_{N+1} = l_{N+1}$

The dimples considered in this study are assumed to be located between two straight segments. Consequently, the boundary conditions at $x_1 = 0$ and $x_{N+1} = l_{N+1}$ are
those previously derived for the straight segment,

\[
EI \left. \frac{\partial^2 y_1}{\partial x^2} \frac{\partial}{\partial x} (\delta y_1) \right|_{x=0} = 0, \quad (3.31a)
\]

\[
EI \left. \frac{\partial^3 y_1}{\partial x^3} (\delta y_1) \right|_{x=0} = 0, \quad (3.31b)
\]

\[
ES \left. \frac{\partial u_1}{\partial x} \delta u_1 \right|_{x=0} = 0, \quad (3.31c)
\]

\[
EI \left. \frac{\partial^2 y_{N+1}}{\partial x^2} \frac{\partial}{\partial x} (\delta y_{N+1}) \right|_{x=l_{N+1}} = 0, \quad (3.31d)
\]

\[
EI \left. \frac{\partial^3 y_{N+1}}{\partial x^3} (\delta y_{N+1}) \right|_{x=l_{N+1}} = 0, \quad (3.31e)
\]

\[
ES \left. \frac{\partial u_{N+1}}{\partial x} \delta u_{N+1} \right|_{x=l_{N+1}} = 0. \quad (3.31f)
\]

Five boundary conditions (and some of their combinations) are considered in this study. They are summarized below, where primes indicate derivatives with respect to \( x \), and each function should be evaluated at either boundary, \( x_1 = 0 \) or \( x_{N+1} = l_{N+1} \):

1. Pin support: \( u = y = y'' = 0 \)
2. Pin support on roller (horizontal motion): \( u' = y = y'' = 0 \)
3. Pin support on roller (vertical motion): \( u = y'' = y''' = 0 \)
4. Fixed: \( u = y = y' = 0 \)
5. Free: \( u' = y'' = y''' = 0 \)

### 3.2.3 Continuity Conditions at \( x_i = l_i \) and \( x_{i+1} = 0 \)

The continuity conditions where the right end of a straight segment meets the left end of a dimple were derived for the two-segment model in Section 3.1.4 and are applied to the model of a beam with \( N \) dimples at locations \( x_i = l_i \) and \( \theta_i = 0 \) for
\[ i = 1, 2, \ldots, N \]. At these locations, the geometrical and natural conditions are given by

\[
\begin{align*}
\begin{cases}
    u_i(l_i, t) \\
y_i(l_i, t)
\end{cases}
= 
\begin{bmatrix}
    \cos(\alpha_i/2) & \sin(\alpha_i/2) \\
    -\sin(\alpha_i/2) & \cos(\alpha_i/2)
\end{bmatrix}
\begin{cases}
    \nu_i(0, t) \\
w_i(0, t)
\end{cases}, 
\end{align*}
\]

(3.32a)

\[
\frac{\partial y_i(l_i, t)}{\partial x} = \psi_i(0, t), 
\]

(3.32b)

\[
EI \frac{\partial^2 y_i(l_i, t)}{\partial x^2} = M_i(0, t),
\]

(3.32c)

\[
ES \frac{\partial u_i(l_i, t)}{\partial x} = P_i(0, t) \cos(\alpha_i/2) - Q_i(0, t) \sin(\alpha_i/2),
\]

(3.32d)

\[
EI \frac{\partial^3 y_i(l_i, t)}{\partial x^3} = P_i(0, t) \sin(\alpha_i/2) + Q_i(0, t) \cos(\alpha_i/2),
\]

(3.32e)

where

\[
\psi_i = \frac{1}{R_i} \left( \nu_i + \frac{\partial w_i}{\partial \theta} \right),
\]

(3.33)

\[
M_i = \frac{EI_i}{R_i^2} \left( \frac{\partial^2 \nu_i}{\partial \theta^2} + \frac{\partial^2 w_i}{\partial \theta^2} \right),
\]

(3.34)

\[
Q_i = \frac{EI_i}{R_i^3} \left( \frac{\partial^2 \nu_i}{\partial \theta^2} + \frac{\partial^3 w_i}{\partial \theta^3} \right),
\]

(3.35)

\[
P_i = \frac{ES_i}{R_i} \left( \frac{\partial \nu_i}{\partial \theta} - w_i \right).
\]

(3.36)

Another set of six continuity conditions exist on the right end of the dimple since a straight segment is connected to it. These conditions are derived in the same rigorous manner as the previous set of continuity conditions by considering the differential Lagrangian at \( \theta_i = \alpha_i \) and \( x_{i+1} = 0 \) for \( i = 1, 2, \ldots, N \). The derived result is equivalent to replacing \( \alpha_i/2 \) in Eqs. (3.32a), (3.32d), and (3.32e) with \( -\alpha_i/2 \). Physically, this represents a counter-clockwise rotation to transform the displacement and forces in the dimple at \( \theta_i = \alpha_i \) to align with those in the straight segment at \( x_{i+1} = 0 \). The equations for continuity of slope and moment remain unchanged. With
this substitution, the continuity conditions are given by

\[
\begin{align*}
\begin{cases}
  u_{i+1}(0, t) \\
y_{i+1}(0, t)
\end{cases} &=
\begin{bmatrix}
  \cos(\alpha_i/2) & -\sin(\alpha_i/2) \\
  \sin(\alpha_i/2) & \cos(\alpha_i/2)
\end{bmatrix}
\begin{cases}
  v_i(\alpha_i, t) \\
w_i(\alpha_i, t)
\end{cases},
\end{align*}
\]

(3.37a)

\[
\frac{\partial y_{i+1}(0, t)}{\partial x} = \psi_i(\alpha_i, t),
\]

(3.37b)

\[
EI \frac{\partial^2 y_{i+1}(0, t)}{\partial x^2} = M_i(\alpha_i, t),
\]

(3.37c)

\[
ES \frac{\partial^2 u_{i+1}(0, t)}{\partial x^3} = -P_i(\alpha_i, t) \sin\left(\frac{\alpha_i}{2}\right) + Q_i(\alpha_i, t) \cos\left(\frac{\alpha_i}{2}\right).
\]

(3.37d)

\[
EI \frac{\partial^2 y_{i+1}(0, t)}{\partial x^3} = -P_i(\alpha_i, t) \sin\left(\frac{\alpha_i}{2}\right) + Q_i(\alpha_i, t) \cos\left(\frac{\alpha_i}{2}\right).
\]

(3.37e)

In summary of Section 3.2, \(4N+2\) equations of motion are needed to describe a beam with \(N\) dimples, assuming each dimple is positioned between two straight segments (see Fig. 3.3). The total number of boundary conditions is six, and there are \(12N\) continuity conditions (six on each side of the dimple), yielding \(12N+6\) modal coefficients to be determined. In Section 3.3, the solutions to the equations of motion are presented. Then in Section 3.4, the coefficient matrix for a fixed-fixed beam with one dimple is derived as an example.

### 3.3 Solutions to the Equations of Motion

In Section 3.3.1, the general solutions to Eqs. (3.27)-(3.28) governing the transverse and longitudinal motion of the straight segments are presented. Then, the general solutions to Eqs. (3.29)-(3.30) governing the tangential and radial motion of the dimples are given in Section 3.3.2.
3.3.1 Straight Segments

Equation (3.27) is a linear homogenous partial differential equation with constant coefficients that governs the transverse vibrations. Therefore, a solution of the form

$$y_i(x, t) = Y_i(x)e^{j\omega t}$$  

is valid for the \(i\)th segment since Eq. (3.27) is separable, and the vibration is harmonic in time with frequency \(\omega\). Insertion of Eq. (3.38) into Eq. (3.27) yields an ordinary differential equation

$$\frac{d^4Y_i}{dx^4} - \beta^4 Y_i = 0,$$  

where \(\beta\) is the transverse frequency parameter defined as

$$\beta = \left(\frac{\rho S}{EI}\omega^2\right)^{1/4}.$$  

(3.40)

Note this frequency parameter is the same for all straight segments. The solution to Eq. (3.39) is of the form

$$Y_i(x) = Ae^{qx},$$  

where the roots \(q\) are determined from the characteristic equation

$$q^4 - \beta^4 = 0.$$  

(3.42)

Solution of Eq. (3.42) yields four roots, \(q_1, \ldots q_4\), given by \(\pm\beta, \pm j\beta\). Therefore, the general solution to Eq. (3.39) is

$$Y_i(x) = \sum_{k=1}^{4} A_{ik} e^{q_k x},$$  

(3.43)
where the modal coefficients $A_{ik}$ will be determined by the boundary and continuity conditions (to within an arbitrary constant). Using the Euler identity, Eq. (3.43) can be written in terms of trigonometric and hyperbolic functions, i.e.

$$Y_i(x) = A'_{i1} \cos \beta x + A'_{i2} \sin \beta x + A'_{i3} \cosh \beta x + A'_{i4} \sinh \beta x.$$  (3.44)

From an analytical point of view, the exponential form given by Eq. (3.43) is more convenient for formulating the coefficient matrix (see Section 3.4). It is also easier to implement in a computer program.

The solution to Eq. (3.28) governing the longitudinal motion follows in a similar manner. After a separation of variables, an ordinary differential equation is obtained,

$$\frac{d^2 U_i}{dx^2} + \gamma^2 U_i = 0,$$  (3.45)

where the longitudinal frequency parameter $\gamma$ is defined as

$$\gamma = \left( \frac{\rho}{E} \omega^2 \right)^{1/2}. \quad (3.46)$$

This frequency parameter is the same for every straight segment. The general solution is given by

$$U_i(x) = \sum_{k=1}^{2} C_{ik} e^{r_k x}, \quad (3.47)$$

for modal coefficients $C_{ik}$ and roots $r_1 = j \gamma$ and $r_2 = -j \gamma$.

### 3.3.2 Dimpled Segments

The dimple is governed by Eqs. (3.29)–(3.30), where it was noted that the uncoupled equations are identical in form, i.e. the differential operator acting on the tangential
coordinate \( \nu \) and radial coordinate \( w \) is identical. Therefore, the process of obtaining the general solution for each equation is identical. The notation used in this section is borrowed heavily from Lang \cite{30}, although some key differences are discussed at the end of this section.

Considering Eqs. (3.29)–(3.30), the solutions are of the form

\[
\nu_i(\theta, t) = V_i(\theta) e^{j\omega t}, \quad (3.48)
\]

\[
w_i(\theta, t) = W_i(\theta) e^{j\omega t}, \quad (3.49)
\]

where although Eqs. (3.29)–(3.30) are not separable, a harmonic solution can be assumed \cite{30}. Inserting Eq. (3.48) into Eq. (3.29) yields

\[
\frac{d^6 V_i}{d \theta^6} + (2 + p_i \zeta_i) \frac{d^4 V_i}{d \theta^4} + (1 - p_i \zeta_i - \zeta_i) \frac{d^2 V_i}{d \theta^2} + (\zeta_i - p_i \zeta_i^2) V_i = 0, \quad (3.50)
\]

and similarly,

\[
\frac{d^6 W_i}{d \theta^6} + (2 + p_i \zeta_i) \frac{d^4 W_i}{d \theta^4} + (1 - p_i \zeta_i - \zeta_i) \frac{d^2 W_i}{d \theta^2} + (\zeta_i - p_i \zeta_i^2) W_i = 0. \quad (3.51)
\]

Here, the frequency parameter for the \( i^{th} \) dimple is defined as

\[
\zeta_i = \frac{\rho S_i R_i^4}{EI_i \omega^2}, \quad (3.52)
\]

where in general, the parameters are different since each dimple has a different geometry. Furthermore, a parameter associated with the extensional vibration of the dimple is defined as

\[
p_i = \frac{\bar{I}_i}{S_i R_i^2}, \quad (3.53)
\]
which is entirely dependent on the dimple geometry.\footnote{In many studies of arches, this parameter is neglected when the arch thickness is small compared to the radius, and Eq. (3.50) then reduces to the classical inextensional equation [26, p.431]. In the context of dimpled beams however, the general extensional model is required in order to satisfy force equilibrium (see discussion in Section 3.1.4).} The same parameter appears in the element stiffness matrix of a dimple in reference [1].

The general solutions of Eqs. (3.50)-(3.51) are of the form

\begin{equation}
V_i(\theta) = Be^{s_i\theta},
\end{equation}

\begin{equation}
W_i(\theta) = De^{s_i\theta},
\end{equation}

for modal coefficients $B$ and $D$ and characteristic root $s$. Considering the tangential coordinate, insertion of Eq. (3.54) into Eq. (3.50) yields the characteristic equation for the dimple,

\begin{equation}
\begin{align*}
s^6 + (2 + p_i\zeta_i)s^4 + (1 - p_i\zeta_i - \zeta_i)s^2 + (\zeta_i - p_i\zeta_i^2) &= 0.
\end{align*}
\end{equation}

Numerical solution of Eq. (3.56) yields six roots per dimple, $s_{i1}, \ldots, s_{i6}$, which are used to formulate the general solutions to Eqs. (3.50)-(3.51) given by

\begin{equation}
V_i(\theta) = \sum_{k=1}^{6} B_{ik}e^{s_{ik}\theta},
\end{equation}

\begin{equation}
W_i(\theta) = \sum_{k=1}^{6} D_{ik}e^{s_{ik}\theta}.
\end{equation}

In general, $D_{ik}$ and $B_{ik}$ are related by a multiplicative constant $\lambda_{ik}$, i.e. $D_{ik} = \lambda_{ik}B_{ik}$, so that

\begin{equation}
W_i(\theta) = \sum_{k=1}^{6} \lambda_{ik}B_{ik}e^{s_{ik}\theta},
\end{equation}

\(\lambda\) and \(\beta\) are related by a multiplicative constant \(\lambda\), i.e. \(D_{ik} = \lambda B_{ik}\), so that

\begin{equation}
W_i(\theta) = \sum_{k=1}^{6} \lambda \lambda_{ik}B_{ik}e^{s_{ik}\theta}.
\end{equation}
where
\[
\lambda_{ik} = \frac{p_i \zeta_i + (p_i + 1)s_{ik}^2}{s_{ik} - p_i s_{ik}^3}.
\] (3.60)

Because of this relationship, the total number of unknown modal coefficients is six, not twelve. Physically, this is because an arch requires only six boundary conditions (three on each side). See Appendix A for a derivation of Eq. (3.60).

Equations (3.57) and (3.59) represent the mode shapes for the \(i^{th}\) dimple written in exponential form, and they have an interesting behavior due to the nature of the roots. Unlike Eq. (3.42) which always has two real roots and two complex conjugates, Eq. (3.56) has six roots that change form depending on the value of \(p_i\) and \(\zeta_i\). The roots are either purely real, purely imaginary, or complex, and the combinations of these roots fall into four cases (e.g. one particular case has four complex roots, and two imaginary roots. See Appendix B for a detailed description). The physical significance of the four cases is that the dimple mode shapes can be described by equations of four different trigonometric forms, \(^3\) found by converting the exponential solution to the trigonometric equivalent using the Euler identity. This leads to situations where for a given frequency, dimples of different geometry (i.e. different \(p_i\) and \(\zeta_i\)) may have different forms of equations describing the mode shapes. In addition, the same dimple may experience a “form switch” for different natural frequencies (i.e. different \(\zeta_i\)). The form switching is noteworthy because it causes numerical issues during the search for the natural frequencies (see Appendix B).

Finally, it is noted that either the four trigonometric forms or the single exponential form can be used to construct a coefficient matrix in order to obtain the natural frequencies (Section 3.4). However, there is an advantage of economy to choosing the exponential form, which is the approach taken in this study. The trigonometric

\(^3\)By forms, we mean for example that \(\cos \theta\) and \(\cos 2\theta\) are different equations but have the same form, whereas \(\cos \theta \cosh \theta\) is of a different form altogether.
forms of Eqs. (3.57) and (3.59), and in particular, the relationship between the modal coefficients, are very complicated. Lang used the trigonometric forms when deriving the coefficient matrices for arches [30]. Use of this approach produces four coefficient matrices for the extensional case, each with complicated expressions for the matrix elements. It is perhaps for this reason that despite having derived the extensional equations for an arch, a simplified example of a complete and inextensional ring was considered for study. In contrast, this study uses the exponential form to construct the coefficient matrices involving an incomplete and extensional arch (i.e. dimple). Only one coefficient matrix is needed for any given beam since the exponential form is the same regardless of the root types. Formulating the coefficient matrix is the topic of the next section.

3.4 Formulation of the Coefficient Matrix

The process of formulating the coefficient matrix for a beam with $N$ dimples requires using the general solutions developed in Section 3.3 with the boundary and continuity conditions given in Sections 3.2.2–3.2.3. This leads to a system of $12N + 6$ homogeneous algebraic equations given by

$$[\mathcal{A}(\omega)] \{X\} = \{0\},$$

(3.61)

where $[\mathcal{A}(\omega)]$ is a coefficient matrix that is dependent on natural frequency $\omega$, and $\{X\}$ is a column vector of modal coefficients given by $\{A_{ik}, B_{ik}, C_{ik}\}^T$. In order to illustrate the process, the coefficient matrix is derived for a fixed-fixed beam with a single dimple ($N = 1$). In this case, there are 18 conditions to satisfy (i.e., 6 boundary conditions and 12 continuity conditions).
The general solutions for each coordinate are given again below for convenience,

\[
y_i(x,t) = e^{j\omega t} \sum_{k=1}^{4} A_{ik} e^{q_k x},
\]

(3.62)

\[
u_i(x,t) = e^{j\omega t} \sum_{k=1}^{2} C_{ik} e^{r_k x},
\]

(3.63)

\[
u_i(\theta,t) = e^{j\omega t} \sum_{k=1}^{6} B_{ik} e^{s_{ik} \theta},
\]

(3.64)

\[
u_i(\theta,t) = e^{j\omega t} \sum_{k=1}^{6} \lambda_{ik} B_{ik} e^{s_{ik} \theta},
\]

(3.65)

where \( q_k = \pm \beta, \pm j\beta, r_k = \pm j\gamma, \) and the frequency parameters for the straight segments, \( \beta, \gamma, \) are computed using Eqs. (3.40) and (3.46). The dimple roots \( s_{ik} \) are found using Eq. (3.56), where the frequency parameters \( \zeta_i \) and extensional parameters \( p_i \) for each dimple are computed using Eqs. (3.52)–(3.53). The modal parameter \( \lambda_{ik} \) is found using Eq. (3.60). The harmonic time dependence \( e^{j\omega t} \) has been included in the general solutions for completeness, although the term will cancel when the general solutions are inserted into the boundary and continuity conditions.

To begin, the Eqs. (3.62)–(3.65) are inserted into the boundary conditions for the fixed-fixed beam (condition #4 in Section 3.2.2). The six conditions are given by \( u_1(0) = y_1(0) = y_1'(0) = u_2(l_2) = y_2(l_2) = y_2'(l_2) = 0. \) Evaluating each of the six conditions for the fixed-fixed beam yields

\[
\sum_{k=1}^{2} C_{1k} = 0,
\]

(3.66)

\[
\sum_{k=1}^{4} A_{1k} = 0,
\]

(3.67)

\[
\sum_{k=1}^{4} q_k A_{1k} = 0,
\]

(3.68)
\[
\sum_{k=1}^{2} C_{2k} e^{r_{k} l_{2}} = 0, \quad (3.69)
\]
\[
\sum_{k=1}^{4} A_{2k} e^{q_{k} l_{2}} = 0, \quad (3.70)
\]
\[
\sum_{k=1}^{4} q_{k} A_{2k} e^{q_{k} l_{2}} = 0. \quad (3.71)
\]

Next, the general solutions are inserted into twelve continuity conditions given in Section 3.2.3. They are summarized below in Eqs. (3.72)–(3.83).

Continuity of horizontal deflection at \(x_1 = l_1, \theta_1 = 0\), Eq. (3.32a):

\[
\sum_{k=1}^{2} C_{1k} e^{r_{k} l_{1}} - \sum_{k=1}^{6} B_{1k} \left( \cos(\alpha_1/2) + \lambda_{1k} \sin(\alpha_1/2) \right) = 0. \quad (3.72)
\]

Continuity of vertical deflection at \(x_1 = l_1, \theta_1 = 0\), Eq. (3.32a):

\[
\sum_{k=1}^{4} A_{1k} e^{q_{k} l_{1}} - \sum_{k=1}^{6} B_{1k} \left( -\sin(\alpha_1/2) + \lambda_{1k} \cos(\alpha_1/2) \right) = 0. \quad (3.73)
\]

Continuity of slope at \(x_1 = l_1, \theta_1 = 0\), Eq. (3.32b):

\[
\sum_{k=1}^{4} q_{k} A_{1k} e^{q_{k} l_{1}} - \frac{1}{R_1} \sum_{k=1}^{6} B_{1k} (1 + s_{1k} \lambda_{1k}) = 0. \quad (3.74)
\]

Equilibrium of moment at \(x_1 = l_1, \theta_1 = 0\), Eq. (3.32c):

\[
EI \sum_{k=1}^{4} q_{k}^2 A_{1k} e^{q_{k} l_{1}} - \frac{EI_1}{R_1^2} \sum_{k=1}^{6} B_{1k} (s_{1k} + s_{1k}^2 \lambda_{1k}) = 0. \quad (3.75)
\]
Equilibrium of axial force at \( x_1 = l_1, \theta_1 = 0 \), Eq. (3.32d):

\[
ES \sum_{k=1}^{2} r_k C_{1k} e^{r_k l_1} - \sum_{k=1}^{6} B_{1k} \left( \frac{E S_{1}}{R_1} (s_{1k} - \lambda_{1k}) \cos(\alpha_1/2) - \frac{E I_1}{R_1^3} (s_{1k}^2 + s_{1k}^3 \lambda_{1k}) \sin(\alpha_1/2) \right) = 0. \tag{3.76}
\]

Equilibrium of shear force at \( x_1 = l_1, \theta_1 = 0 \), Eq. (3.32e):

\[
EI \sum_{k=1}^{4} q_k A_{1k} e^{q_k l_1} - \sum_{k=1}^{6} B_{1k} \left( \frac{E S_{1}}{R_1} (s_{1k} - \lambda_{1k}) \sin(\alpha_1/2) + \frac{E I_1}{R_1^3} (s_{1k}^2 + s_{1k}^3 \lambda_{1k}) \cos(\alpha_1/2) \right) = 0. \tag{3.77}
\]

Continuity of horizontal deflection at \( x_2 = 0, \theta_1 = \alpha_1 \), Eq. (3.37a):

\[
\sum_{k=1}^{2} C_{2k} - \sum_{k=1}^{6} B_{1k} (\cos(\alpha_1/2) - \lambda_{1k} \sin(\alpha_1/2)) e^{s_{1k} \alpha_1} = 0. \tag{3.78}
\]

Continuity of vertical deflection at \( x_2 = 0, \theta_1 = \alpha_1 \), Eq. (3.37a):

\[
\sum_{k=1}^{4} A_{2k} - \sum_{k=1}^{6} B_{1k} (\sin(\alpha_1/2) + \lambda_{1k} \cos(\alpha_1/2)) e^{s_{1k} \alpha_1} = 0. \tag{3.79}
\]

Continuity of slope at \( x_2 = 0, \theta_1 = \alpha_1 \), Eq. (3.37b):

\[
\sum_{k=1}^{4} g_k A_{2k} - \frac{1}{R_1} \sum_{k=1}^{6} B_{1k} (1 + s_{1k} \lambda_{1k}) e^{s_{1k} \alpha_1} = 0. \tag{3.80}
\]

Equilibrium of moment at \( x_2 = 0, \theta_1 = \alpha_1 \), Eq. (3.37c):

\[
EI \sum_{k=1}^{4} q_k^2 A_{2k} - \frac{E I_1}{R_1^2} \sum_{k=1}^{6} B_{1k} (s_{1k} + s_{1k}^2 \lambda_{1k}) e^{s_{1k} \alpha_1} = 0. \tag{3.81}
\]
Equilibrium of axial force at \( x_2 = 0, \theta_1 = \alpha_1 \), Eq. (3.37d):

\[
ES \sum_{k=1}^{2} r_k C_{2k} - \sum_{k=1}^{6} B_{1k} \left( \frac{ES_1}{R_1} (s_{1k} - \lambda_{1k}) \cos(\alpha_1/2) \right. \\
+ \frac{EI_1}{R_1^3} (s_{1k}^2 + s_{1k}^3 \lambda_{1k}) \sin(\alpha_1/2) \bigg) e^{s_{1k} \alpha_1} = 0.
\] (3.82)

Equilibrium of shear force at \( x_2 = 0, \theta_1 = \alpha_1 \), Eq. (3.37e):

\[
EI \sum_{k=1}^{4} q_k^3 A_{2k} - \sum_{k=1}^{6} B_{1k} \left( - \frac{ES_1}{R_1} (s_{1k} - \lambda_{1k}) \sin(\alpha_1/2) \\
+ \frac{EI_1}{R_1^3} (s_{1k}^2 + s_{1k}^3 \lambda_{1k}) \cos(\alpha_1/2) \bigg) e^{s_{1k} \alpha_1} = 0.
\] (3.83)

Using Eqs. (3.66)–(3.83), the coefficients in front of each modal coefficient \( A_{1k}, A_{2k}, B_{1k}, C_{1k}, C_{2k} \) are assembled into an \( 18 \times 18 \) coefficient matrix \([\mathcal{A}(\omega)]\). This represents the coefficient matrix for a fixed-fixed beam with one dimple. The assembled matrix is not shown here due to space limitations.

To carry out the assembly process by hand for a beam with many dimples is tedious since there are \( 12N + 6 \) conditions to satisfy. Therefore, the coefficient matrix for a beam with \( N \) dimples and any particular boundary condition is assembled using MATLAB\textsuperscript{®}. The matrices can be saved and the appropriate matrix is called for specific cases. It is noted that for different boundary conditions, only the coefficients in six rows of the matrix need to be modified, and all other rows are unchanged.

For beams with \( N \) dimples, Eqs. (3.72)–(3.83) are repeated \( N \) times by replacing the straight segment and dimple number with the next appropriate segment number.

Once the coefficient matrix is assembled, Eq. (3.61) is solved in order to obtain the natural frequencies \( \omega \) and corresponding mode shapes \( \{X\} \) for a given beam.
non-trivial solution of Eq. (3.61) exists if and only if the determinant of the coefficient matrix is zero (i.e., \( \det([A(\omega)]) = 0 \)). Due to the size of the coefficient matrix, a solution is obtained numerically using an iterative procedure: a frequency \( \omega \) is assumed in matrix \([A(\omega)]\) and its determinant is computed. This process is repeated for many frequencies over a specified frequency range, where frequencies that yield a very small determinant correspond to natural frequencies (a numerical issue during the search process caused by changing dimple roots is discussed in Appendix B). The corresponding modal coefficients \( \{X\} \) are used in Eqs. (3.62)–(3.65) for obtaining the mode shapes. In Ch. 4, several examples are presented in order to validate the boundary value model and its numerical implementation.

### 3.5 Summary

This chapter began by using Hamilton’s Principle to derive the boundary value model (i.e. equations of motion, boundary and continuity conditions) for a simple two-segment model consisting of one straight segment connected to a dimple. The transverse and longitudinal components of motion in the straight segments were accounted for. Furthermore, it was shown that in order to satisfy an equilibrium of forces between straight segments and dimples, dimples must be regarded as extensional (i.e. the extension of the central axis is non-zero). The boundary value problem was then generalized to describe a beam with \( N \) dimples in Section 3.2. This model of dimpled beams represents an exact formulation, assuming complicating effects due to rotary inertia and shear can be ignored. Then, using the general solutions given in Section 3.3, the coefficient matrix of a fixed-fixed beam with one dimple was derived as an example in Section 3.4.

Until the completion of this dissertation, only finite element models of dimpled
beams existed [1],[2],[21]. The reason for this may be explained by Cheng [1]: “When the analytical model of a substructure [referring to the dimple] is not available due to complicated geometry or boundary conditions, the experimental measurement (e.g., impact testing) or numerical simulation (e.g., finite element method) is adopted...”

The analytical model of dimpled beams presented in this chapter reveals that their behavior is described by differential equations governing the vibration of straight beams and arches. As it turns out, these differential equations can be found in the literature since the vibration of straight segments and arches have been treated separately. An analytical model of the two structures together, however, has not been considered until now. Fundamentally, the dimpled beam model presented in this dissertation shows how the motion of a straight beam is coupled to the motion of a curved beam through the continuity conditions derived in Section 3.2.3.

Although the treatment of dimpled beams has been analytical up to this point, moving forward requires that the natural frequencies and modal coefficients be determined numerically (due to the large size of the coefficient matrices). The validation of the boundary value model is the topic of the next chapter.
In this chapter, the accuracy of the boundary value model for the dimpled beam is checked in several ways by comparing its free vibration and forced responses to those obtained from other models. A dimpled beam model, as described in the previous chapter, allows the user to explore a variety of models. For example, by setting the dimple angle to zero, one ends up with a straight uniform beam. On the other hand, if one wants to set the lengths of the straight segments to very small values, now the dimple (or arch) is modeled by itself. These extremes provide the user with opportunities to check the model predictions against simpler analytical results widely known.

In Section 4.1, the dimpled beam is approximated as a uniform beam, and results are compared to exact values. In Section 4.2, a beam with one dimple is approximated as a single arch by making the lengths of the straight segments very small. The results are compared to studies concerned with the vibrations of arches. In Section 4.3, a comparison is made to the work published by others. Finally, the discrete forced response of dimpled beams is calculated in Section 4.4, and a comparison is made to results obtained using the finite element method developed by this author.
4.1 Approximation of a Uniform Beam, $\alpha \to 0$

As the dimple angle $\alpha \to 0$, the dimple becomes a flat segment making the dimpled beam approximately a uniform beam. In this case, natural frequencies and mode shapes can be compared to exact results. Uniform beams of three different boundary conditions are considered here: fixed-fixed, simply-supported (pinned supports), and cantilever (fixed-free). For each boundary condition, the first three frequency parameters $\beta$ (see Eq. (3.40)) are computed using the boundary value model, assuming a dimple angle $\alpha = 0.01^\circ$, the beam is unit length, and the dimple is placed in the range $0.3 \leq x \leq 0.4$. During numerical implementation, the angle is set to a very small value in order to avoid numerical difficulties (i.e. division by zero). Actually, the placement and chord length of the dimple makes no difference when the dimple angle is small. This check is also valid for multiple dimples, assuming small dimple angles. A comparison is made in Table 4.1 to exact values [8, p.325], where the agreement is good for all cases. For the simply-supported boundary condition, the frequency parameters are the same regardless if both ends are pinned or one end is a roller (although there is a difference when the dimple angle is non-zero).

Due to good agreement between frequencies, mode shapes are expected to be in agreement with analytical expressions for all boundary conditions. Choosing the simply-supported boundary condition as a representative example, the first three mode shapes are compared to exact expressions in Fig. 4.1 (see Appendix C for a

<table>
<thead>
<tr>
<th></th>
<th>Fixed-Fixed</th>
<th>Simple Supports</th>
<th>Cantilever</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_1$</td>
<td>BVM 4.7297</td>
<td>Exact 4.7300</td>
<td>BVM 3.1414</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>BVM 7.8536</td>
<td>Exact 7.8530</td>
<td>BVM 6.2837</td>
</tr>
<tr>
<td>$\beta_3$</td>
<td>BVM 10.9950</td>
<td>Exact 10.9950</td>
<td>BVM 9.4245</td>
</tr>
</tbody>
</table>
Figure 4.1: First three mode shapes of a simply-supported uniform beam

description of how to plot mode shapes of dimpled beams, and Appendix D for mass-normalizing mode shapes). The exact mode shapes of a simply-supported beam are given by \( Y_n(x) = A_n \sin(n\pi x/L) \) for \( n = 1, 2, 3 \), where the constant \( A \) is arbitrary. The mode shapes obtained from the boundary value model are eigenfunctions defined \textit{locally} within each segment. For the fundamental frequency, the transverse eigenfunction within the first straight segment is given by

\[
Y_1(x) = 1.72 \times 10^{-4} \cos(3.14x) + 3.23 \sin(3.14x) + 1.68 \times 10^{-4} \cosh(3.14x) - 1.68 \times 10^{-4} \sinh(3.14x)
\]

(4.1)
defined locally over \( 0 \leq x \leq 0.3 \). The sine term is dominant (as expected) and the other coefficients are small, but not zero as in the exact case. This is because the modal coefficients are determined numerically. Accuracy can be increased by choosing a smaller search increment within the specified frequency window. For the
second straight segment, the transverse eigenfunction is given by

\begin{equation}
Y_2(x) = 3.07 \cos(3.14x) + 0.997 \sin(3.14x) + 1.78 \times 10^{-4} \cosh(3.14x) - 1.24 \times 10^{-4} \sinh(3.14x),
\end{equation}

defined locally over $0 \leq x \leq 0.6$. Globally, the segment lies in the beam domain $0.4 \leq x \leq 1$, and this is important only for plotting (see Appendix C). For the dimpled segment, the tangential and radial eigenfunctions are given in trigonometric form by

\begin{equation}
V_1(\theta) = -1.06 \times 10^{-3} \cos(1800\theta) + 1.47 \times 10^{-3} \sin(1800\theta) + 9.34 \times 10^{-4} \cosh(1800\theta) + 2.03 \times 10^{-5} \sinh(1800\theta),
\end{equation}

\begin{equation}
W_1(\theta) = 2.65 \cos(1800\theta) + 1.90 \sin(1800\theta) - 7.28 \times 10^{-2} \cos(1.29\theta) + 7.24 \times 10^{-4} \sin(1.29\theta) + 3.66 \times 10^{-2} \cosh(1800\theta) - 5.80 \times 10^{-3} \sinh(1800\theta),
\end{equation}

defined locally over $0 \leq \theta \leq 0.01^\circ$. Globally, the segment lies in the beam domain $0.3 \leq x \leq 0.4$. The large arguments in Eqs. (4.3)-(4.4) might at first glance seem to imply a rapid oscillatory behavior. However, the small domain of $\theta$ ensures a small variation over the domain.

As shown in Fig. 4.1, the piecewise assembly of the eigenfunctions within each segment are in agreement with the exact mode shapes. Plots such as Fig. 4.1 are also useful as a visual check that continuity is being met between segments (i.e., to check that the beam is not “broken”).
4.2 Approximating the Dimpled Beam as an Arch

As shown in Fig. 4.2, a dimpled beam is approximately an arch when the straight segment lengths are made very small compared to the dimple chord length, and their thicknesses are much larger than the dimple thickness. In this case, most of the vibrations are isolated to within the arch, and the dynamic behavior of the approximate arch is expected to match the behavior predicted by analytical models. In the example below, we will see that this is exactly the case.

![Figure 4.2: Dimpled beam approximated as a single arch](image)

Natural frequencies and mode shapes of the approximate arch are calculated using the boundary value model and the results are compared to studies of arches by other authors. Nelson [27] studied the natural frequencies and mode shapes of circular ring segments for hinged boundary conditions (pinned on both sides). Similar to Den Hartog [27], Nelson classified the natural frequencies into two groups: modes of symmetric shape, and modes with anti-symmetric shape. A representation of the first symmetric and first anti-symmetric mode shape is given in Fig. 4.3.

![Figure 4.3: First symmetric (a) and first anti-symmetric (b) mode shape of a hinged ring segment. Shapes are approximate representations.](image)
Nelson then derived formulas for the natural frequencies of each type of mode shape using the Rayleigh-Ritz method. Frequencies obtained using Nelson’s formulas for a hinged ring segment are compared to those obtained for the approximate arch using the boundary value model. The lowest two natural frequencies are plotted in Fig. 4.4 against dimple angle assuming a steel arch with 1 cm thickness\textsuperscript{1} and 10 cm chord length, and straight segments with 10 cm thickness and 0.1 mm length. The results obtained from the boundary value model are in excellent agreement with the lowest symmetric and anti-symmetric natural frequencies predicted by Nelson. It is important to note that each model assumes the arch to be extensible. In fact, the symmetric shape shown in Fig. 4.3(a) is an extensional mode shape, and such a solution cannot be obtained if the arch is assumed to be inextensional (i.e., zero strain along the circumferential axis).

The mode shapes for the first two natural frequencies exhibit a “switching” behavior. Focusing our attention on the fundamental frequency in Fig. 4.4, the fundamental $f_1$ follows the lowest symmetric curve in the range $0^\circ \leq \alpha \leq 60^\circ$. In other words, the mode shape is that shown in Fig. 4.5. The frequency increases until an interesting phenomenon happens at $\alpha = 60^\circ$. At $60^\circ$, $f_1$ begins following the lowest anti-symmetric curve, i.e. the mode shape switches to the type shown in Fig. 4.6. The switch in mode shape is accompanied by a decrease in frequency. The fundamental frequency continues to decrease while vibrating asymmetrically for every dimple angle above $60^\circ$. Again, it is important to note that the fundamental frequency stops increasing after the mode shape switches from that depicted in Fig. 4.5 to the shape in Fig. 4.6.

Focusing our attention now on the second natural frequency in Fig. 4.4, $f_2$ follows the lowest anti-symmetric curve in the range $0^\circ \leq \alpha \leq 60^\circ$ (i.e., the mode

\textsuperscript{1}This is the thickness when $\alpha = 0^\circ$. The thickness then decreases with angle according to Eq. (3.2).
Figure 4.4: First two natural frequencies of a simply-supported arch: boundary value model vs. Rayleigh-Ritz [28]. The fundamental frequency (\(-\circ\)) switches mode shapes at 60° (Fig. 4.5 to Fig. 4.6). The second natural frequency (\(-\square\)) switches mode shapes at 60° (Fig. 4.6 to Fig. 4.5), and again at 100° (Fig. 4.5 to Fig. 4.7).

shape shown in Fig. 4.6). In this range, the frequency is decreasing. Then, the mode shape switches at 60°, where it vibrates with the first symmetric type (Fig. 4.5) throughout the range 60° ≤ α ≤ 100°. In this range, the frequency is increasing. At 100°, the mode shape switches again, but now to the second symmetric type (Fig. 4.7). In this range, the frequency begins decreasing. This means that the lowest symmetric root in Fig. 4.4 is either that shown in Fig. 4.5 or in Fig. 4.7. Nelson points out that the frequency associated with the mode shape in Fig. 4.5 increases with α². In other words, this mode shape is very “stiff”, and will be excited only at very high frequencies when the dimple angle is large.

The natural frequencies and mode-switching behavior or arches predicted by the boundary value model are in exact agreement with the calculations of Nelson [28]
and Den Hartog [27]. It is noted that similar mode-switching behavior is observed for clamped-clamped boundary conditions [27]. An analysis made in the present study shows that mode-switching occurs for any combination of boundary conditions that prevents the arch ends from moving horizontally (e.g., clamp, pin, vertical roller). On the other hand, if the boundary condition is changed such that one end is free or allowed to roll horizontally (i.e., along the chord length), then the behavior is entirely different. In this case, no mode-switching occurs, and every natural frequency decreases monotonically. These observations give us an important clue into the vibration properties of dimpled beams. We will see in Ch. 5 that dimpled beams exhibit a behavior very similar to those of arches.
Figure 4.6: First anti-symmetric mode shape predicted using the BVM ($\alpha = 30^\circ$)

Figure 4.7: Second symmetric mode shape predicted using the BVM ($\alpha = 160^\circ$)
4.3 Dimpled Beam Results of Previous Studies

In this section, results found in other studies are used for comparison to those computed using this boundary value model (BVM). An example of a simply-supported beam with five dimples is considered for comparison to Cheng [1], and a cantilever beam with a single dimple is used for comparison to the work of Alshbatat [2].

4.3.1 Natural Frequencies found by Cheng [1]

Cheng, et al. [1] computed the natural frequencies of pin-pin roller dimpled beams using an impedance method. A mobility matrix for the system was constructed using a combination of analytical and finite element methods, and the natural frequencies were found by equating the matrix determinant to zero. As an example, they used a simply-supported (pin, pin roller) steel beam of length 0.3 m with five dimples, all having a dimple angle of $\alpha = 90^\circ$. Using the same beam, the first three natural frequencies are calculated using the BVM and the FEM, and these are compared in Table 4.2 to those found in Cheng. Details of the finite element models used in this study are found in Appendix E.

<table>
<thead>
<tr>
<th>Frequency [Hz]</th>
<th>BVM</th>
<th>FEA</th>
<th>Cheng [1]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_1$</td>
<td>22.22</td>
<td>22.22</td>
<td>22.22</td>
</tr>
<tr>
<td>$f_2$</td>
<td>88.61</td>
<td>88.64</td>
<td>88.63</td>
</tr>
<tr>
<td>$f_3$</td>
<td>197.97</td>
<td>197.98</td>
<td>197.96</td>
</tr>
</tbody>
</table>

Cheng did not compute the corresponding mode shapes, and it is not clear how it would be possible using their impedance method. In a later paper [21] concerned with achieving weak-radiating mode shapes, the same authors used the finite element for study instead. The ability to calculate mode shapes for beams with any
number of dimples is one advantage of the boundary value model over the impedance method. In Figs. 4.8–4.10, the first three mode shapes of the simply supported beam with five dimples is shown. The results are compared to those found using a finite element analysis performed in the present study (see Appendix E for FEA details). The agreement is very good for all three frequencies.

Upon close inspection of the axial displacement close to $x/L = 1$ (Fig. 4.10), the effect of the roller on the right end of the beam is apparent, where the dimples have pulled the beam leftward. This is due to the fact that the curvature of the dimples tend to couple the transverse and axial motions. This means that in cases where the boundary condition permits axial motion, the bending wavelength changes as the beam vibrates through one complete cycle. This will be explored in greater detail in Section 5.1.
Figure 4.9: Second mode shape of the dimpled beam found in Cheng [1]. BVM (–) vs. FEM (- -)

Figure 4.10: Third mode shape of the dimpled beam found in Cheng [1]. BVM (–) vs. FEM (- -). Note the non-zero displacement near $x/L = 1$. 
4.3.2 Natural Frequencies found by Alshabtat [2]

Alshabtat [2] computed the natural frequencies of several types of dimpled beams using the finite element method. Most of the results were reported in the form of parametric plots except for a few examples where values were tabulated. In one example, the natural frequencies of a cantilevered beam with one dimple (Table 4.3) were compared to experimentally measured values. Using this beam for comparison, the first three natural frequencies are calculated using the BVM. In Table 4.4, a comparison is made to the numerical and experimental values of Alshabtat.

Table 4.3: Parameters for cantilevered dimpled beam found in reference [2]

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>First segment length, $l_1$</td>
<td>5.5 cm</td>
</tr>
<tr>
<td>Dimple chord length, $\bar{l}_1$</td>
<td>3.0 cm</td>
</tr>
<tr>
<td>Second segment length, $l_2$</td>
<td>13.5 cm</td>
</tr>
<tr>
<td>Width, $b$</td>
<td>2.5 cm</td>
</tr>
<tr>
<td>Thickness, $h$</td>
<td>1.16 mm</td>
</tr>
<tr>
<td>Dimple angle, $\alpha$</td>
<td>136.9°</td>
</tr>
<tr>
<td>Elastic modulus, $E$</td>
<td>190 GPa</td>
</tr>
<tr>
<td>Density, $\rho$</td>
<td>7700 kg/m$^3$</td>
</tr>
</tbody>
</table>

Although there is a small discrepancy with Alshabtat’s results, the agreement is generally good between all methods. With respect to the experimentally measured natural frequencies, Alshabtat attributed the discrepancy to the non-uniform thickness of the manufactured dimple, since the dimple thickness was assumed to be uniform (the same assumption is made in this study).

Table 4.4: First three natural frequencies, cantilever beam (Alshabtat [2] vs. BVM)

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_1$</td>
<td>17.0</td>
<td>17.5</td>
<td>16.81</td>
</tr>
<tr>
<td>$f_2$</td>
<td>110.0</td>
<td>111.9</td>
<td>113.02</td>
</tr>
<tr>
<td>$f_3$</td>
<td>281.4</td>
<td>290.0</td>
<td>275.31</td>
</tr>
</tbody>
</table>
4.3.3 Beams with Multiple Dimples

In many cases, multiple dimples are needed for achieving certain vibroacoustic objectives because of the larger effect they can produce together. The idea that multiple dimples can have a larger effect on the vibroacoustics of structures is not surprising, since additional dimples represent a larger modified structural surface area. Use of multiple dimples is somewhat analogous to making the chord length of a single dimple very large with respect to the beam length (see Section 5.4). Since other authors have given examples where multiple dimples have been used to alter structural vibroacoustics, a summary of the results from these studies is given in this section. Then, the BVM is used in order to check the results found in reference [2] for beams with multiple dimples.

Cheng [21] has used 20 dimples on a beam in order to achieve weak radiating mode shapes. The dimples alter the mode shapes of the beam in such a way that the radiation efficiency of the beam is minimized over a frequency band. In a related study [1], the same authors noted that when multiple dimples are used for tuning a natural frequency of a beam to a desired value, the dimple angles do not have to be as large as compared to a single dimple. They also claimed that in order to tune $N$ natural frequencies to a desired value, at least $N$ dimples are needed for a unique solution to exist. If more than $N$ dimples are used, there are infinite solutions (i.e. combinations of dimple location and angles that satisfy the desired criteria). If less than $N$ dimples are used, there is no guarantee that a unique solution exists.

In other studies, Alshbatat [2],[23] used up to four dimples on a beam in order to shift various natural frequencies, and to minimize the radiated sound power at a discrete frequency as well as over a broad frequency band. In one example, the objective was to maximize the fundamental frequency of a fixed-fixed beam [2, p.82].
Using two dimples, a 25% increase with respect to the uniform beam fundamental frequency was achieved. Increasing the number of dimples to 4 yielded a 36.5% increase. The optimum pattern of dimples was found using the genetic algorithm (GA), where the optimization routine placed the dimples close to the beam boundaries (these are regions where the modal strain energy is largest - see Fig. 5.20). Another example sought to minimize the fundamental frequency of a simply-supported beam (pin-pin roller). Numerical optimization via the GA yielded a 41.6% decrease when 2 dimples were placed at the beam center (a region of minimum modal strain energy, Fig. 5.26). When a pattern of 4 dimples was used, all dimples were placed by the optimization algorithm next to each other at the beam center, and a 52.4% reduction was achieved.

Alshbatat [16],[17] also used multiple spherical dimples on plates in order to achieve similar vibroacoustic objectives. Some of the optimized results showed that when multiple dimples were used, the optimum dimple patterns were dimples arranged in a line, as if to imitate the shape of a cylindrical bead. In his dissertation [2], he showed that use of beads could be very effective for improving plate vibroacoustics, since the bead is essentially a long cylindrical modification of the plate area. Generally, the bead represents a greater modified area of the structure and is able to produce a larger effect as a consequence.

Fixed-fixed and simply-supported beams found by Alshbatat [2] are used as examples below in order to compare their natural frequencies to those found using the BVM. Each of the beams have a length of 0.3 m, width 2.5 cm, thickness 1 mm, dimple chord length 0.03 m, elastic modulus 189 GPa, and density 7688 kg/m$^3$. The fundamental frequencies of a fixed-fixed beam with two, three, and four dimples are given in Table 4.5, where the dimple angles and locations are given in Table 4.6. A similar comparison is made in Table 4.7 for a simply-supported beam, with parameters defined in Table 4.8. In both cases, the agreement is excellent.
Table 4.5: Fixed-fixed beam: comparison of fundamental frequencies [Hz] (Alshabtat [2] vs. BVM)

<table>
<thead>
<tr>
<th>N dimples</th>
<th>FEM [2]</th>
<th>BVM</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>71.1</td>
<td>71.1</td>
</tr>
<tr>
<td>3</td>
<td>74.2</td>
<td>74.0</td>
</tr>
<tr>
<td>4</td>
<td>77.4</td>
<td>77.4</td>
</tr>
</tbody>
</table>

Table 4.6: Fixed-fixed beam: dimple angle $\alpha$ [deg], dimple center location $x_c$ [m]

<table>
<thead>
<tr>
<th>N</th>
<th>$\alpha_1(x_{c1})$</th>
<th>$\alpha_2(x_{c2})$</th>
<th>$\alpha_3(x_{c3})$</th>
<th>$\alpha_4(x_{c4})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>56.9(0.0180)</td>
<td>57.0(0.2818)</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>3</td>
<td>67.1(0.0181)</td>
<td>43.5(0.0510)</td>
<td>59.7(0.2820)</td>
<td>-</td>
</tr>
<tr>
<td>4</td>
<td>57.8(0.0180)</td>
<td>28.7(0.0511)</td>
<td>28.3(0.2478)</td>
<td>57.9(0.2819)</td>
</tr>
</tbody>
</table>

Table 4.7: Pin-pin roller beam: comparison of fundamental frequencies [Hz] (Alshabtat [2] vs. BVM)

<table>
<thead>
<tr>
<th>N dimples</th>
<th>FEM [2]</th>
<th>BVM</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>14.6</td>
<td>14.5</td>
</tr>
<tr>
<td>3</td>
<td>12.9</td>
<td>12.8</td>
</tr>
<tr>
<td>4</td>
<td>11.9</td>
<td>11.8</td>
</tr>
</tbody>
</table>

Table 4.8: Pin-pin roller beam: dimple angle $\alpha$ [deg], dimple center location $x_c$ [m]

<table>
<thead>
<tr>
<th>N</th>
<th>$\alpha_1(x_{c1})$</th>
<th>$\alpha_2(x_{c2})$</th>
<th>$\alpha_3(x_{c3})$</th>
<th>$\alpha_4(x_{c4})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>179.2(0.1340)</td>
<td>179.9(0.1670)</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>3</td>
<td>179.7(0.1170)</td>
<td>179.9(0.1500)</td>
<td>178.9(0.1830)</td>
<td>-</td>
</tr>
<tr>
<td>4</td>
<td>179.9(0.1010)</td>
<td>179.7(0.1343)</td>
<td>179.9(0.1674)</td>
<td>179.0(0.2016)</td>
</tr>
</tbody>
</table>
4.4 Forced Response of Dimpled Beams

The previous sections treated the free vibration characteristics (i.e., natural frequencies and mode shapes) of dimpled beams and two extreme variations of its geometry. In practical applications, structures are subject to external excitation, and the response of the structure under these conditions must be known. The structural response is found using its natural frequencies and mode shapes. In this section, the forced response of several dimpled beams is computed using the boundary value model and the finite element method, and the two methods are compared to each other. Such a comparison is important for three reasons: 1) the forced response is used to calculate acoustic properties such as sound power, volume velocity, and radiation efficiency (quantities calculated later in this study), 2) a check of the forced response is essentially a check across a broad band of frequencies, not just at the specific natural frequencies, and 3) a check of the forced response is a simultaneous check of both natural frequencies and mode shapes since these characteristics must be accurate in order to obtain an accurate forced response. Each validation is presented in the form of frequency response plots, that is, the transverse and longitudinal response of the structure across a broad frequency spectrum.

The eigenfunctions obtained from the BVM could be used to calculate the forced response analytically. However, the forced response can also be calculated using a discretized approach. The latter of the two approaches is chosen here since it simplifies the process and yields equivalent results. Computation of the discretized forced response is as follows: the dimpled beam is discretized into \( n \) nodes, and forces at the excitation frequency \( \omega \) are applied to the nodes. The excitation can be a single force or a collection of forces of different magnitudes and directions. In general, this produces a transverse and longitudinal response at every node (i.e., horizontal and vertical responses at every node, including those in the dimples). The responses can
be collected into a vector $\{Z\}$ of size $2n \times 1$, and this defines the forced response of the structure. The discretized forced response is calculated using [32]

$$\{Z(\omega)\} \approx \sum_{i=1}^{N_m} \frac{1}{\omega_i^2 - \omega^2 + j\eta_i \omega_i^2} \{\phi_i\}^\top \{f\},$$

where $\omega_i$ is the $i^{th}$ natural frequency, $\{\phi_i\}$ is the $i^{th}$ discretized mode shape (mass-normalized, see Appendix D), $\eta_i$ is the damping loss factor (assumed 0.02 for all modes in this study), $\omega$ is the excitation frequency, and $\{f\}$ is a $2n \times 1$ excitation force vector. The mode shapes are each arranged in a $2n \times 1$ column vector, and contain both transverse and longitudinal degrees of freedom. The above relationship is approximate because only $N_m$ modes are used to calculate the total response, where in reality an infinite number of modes contribute to the response (although the individual contributions will lessen with increasing frequency). Damping is included in order to prevent an infinite response at resonance. This also makes the response at each node complex, so that the relative nodal phases are not necessarily 0° or 180° as in the undamped case.

Two examples are presented for beams with different boundary conditions and numbers of dimples. All beams considered are 1 m long, have 1 mm thick straight segments, and are made of steel ($\rho = 7688$ kg/m$^3$, $E = 189$ GPa).

### 4.4.1 Example 1: Cantilever Beam, $N = 1$

A single transverse force of magnitude 1 N located at 0.1 m (see '+' marker in Fig. 4.11) excites the cantilever beam with one dimple ($\alpha = 180^\circ$, $\bar{l} = 0.1$ m). The forced response at each excitation frequency is calculated assuming five modes in the summation of Eq. (4.5). The first three mode shapes are shown in Fig. 4.11.
The response of the beam at the location of the force is considered in this example. This is referred to as the driving point frequency response function (FRF). The magnitude and phase in the transverse direction is plotted in Fig. 4.12, and the longitudinal response is plotted in Fig. 4.13. The response is relatively large at the natural frequencies, as shown by the peaks in the FRF. The transverse FRF is several orders of magnitude greater than the longitudinal FRF as expected. In both figures, the results obtained from the boundary value model and the finite element method are in agreement. Similar agreement is observed for the responses at other nodes.

For purposes of comparison in Fig. 4.12, the transverse FRF of the uniform cantilever beam is also shown (note there is no longitudinal response for the uniform beam). In this plot, only four uniform beam natural frequencies are shown in the range considered (four peaks). The effect of the dimple is to lower the natural frequencies, and this is typical of boundary conditions that do not restrict axial motion.
Figure 4.12: Example 1: Transverse FRF (cantilever, $N = 1$)

Figure 4.13: Example 1: Longitudinal FRF (cantilever, $N = 1$)
4.4.2 Example 2: Fixed-Pin Beam, \( N = 2 \)

This example considers a beam with two dimples (\( \alpha_1 = 90^\circ, \bar{l}_1 = 0.1 \text{ m}, \alpha_2 = 150^\circ, \bar{l}_2 = 0.2 \text{ m} \)) and fixed-pin boundary condition. As before, a single transverse force of magnitude 1 N located at 0.1 m (see ‘+’ marker in Fig. 4.14) excites the beam. The forced response at each excitation frequency is calculated assuming five modes in the summation of Eq. (4.5). The first three mode shapes are shown in Fig. 4.14. The driving point FRFs for the transverse and longitudinal motions are shown in Figs. 4.15-4.16. In both figures, the results obtained from the boundary value model and the finite element method are in agreement.

The transverse FRF for the uniform beam is plotted in Fig. 4.15. Unlike the cantilever beam, the dimples on the fixed-pin beam increase some natural frequencies and decrease others. This is typical of boundary conditions that restrict axial motion. We will investigate the reasons for this phenomenon in Ch. 5.

Figure 4.14: Example 2: First 3 mode shapes (fixed-pin, \( N = 2 \)
Figure 4.15: Example 2: Transverse FRF (fixed-pin, $N = 2$)

Figure 4.16: Example 2: Longitudinal FRF (fixed-pin, $N = 2$)
4.5 Summary

The validity of the boundary value model (BVM) has been checked in several ways. First, the dimpled beam was approximated as a uniform beam by making the dimple angles very small. The free vibration response of the approximate uniform beam was then compared to exact results, showing good agreement. A plot of the segment eigenfunctions in the global domain provided a visual check that continuity between segments was met.

Next, a beam with one dimple was approximated as a single arch by making the straight segment lengths very small compared to the dimple chord length. The vibration characteristics of the approximate arch was compared to analytical studies of arches. The approximate arch with pinned boundary conditions on each side exhibited a mode switching behavior as the dimple angle was increased. This behavior, predicted by the BVM, is in agreement with the observations of others. The success of the BVM in this context is because the dimple is assumed to be extensional.

Then, dimpled beam models found in the previous literature were considered for comparison to other methods. Results obtained from the BVM for beams with one or more dimples were in agreement with numerical and experimentally measured values [2], and with results obtained using an impedance matching technique [1]. Mode shapes of the beam found in reference [1] were calculated and shown for the first time in this study.

Finally, the discrete forced response of dimpled beams was calculated using the BVM and the finite element method. The driving point FRF in both transverse and longitudinal directions showed good agreement across the frequency spectrum. This demonstrates the ability of the BVM to accurately calculate the response at any excitation frequency, not only at the natural frequencies.
Chapter 5

VIBRATORY ANALYSIS OF DIMPLED BEAMS

The purpose of this chapter is to gain a better understanding of the effect of dimpling on beam dynamics. Each section takes a basic but methodical approach by addressing a specific question: what is the effect on the dynamics of a dimpled beam when parameter ‘X’ is changed? In the six sections that follow, the effect of boundary condition (Section 5.1), dimple angle (Section 5.2), dimple location (Section 5.3), chord length (Section 5.4), dimple thinning (Section 5.5), and multiple dimples (Section 4.3.3) are investigated for their effect on the natural frequencies and mode shapes. Finally, Section 5.6 presents a summary of the important results.

Unless otherwise noted, the beams chosen for consideration will have the properties given in Table 5.1. The length is 100 times greater than the beam thickness so that effects of shear and rotary inertia are negligible for the first 10 natural frequencies [31, p.342]. Because the developed boundary value model accounts only for motion along the beam length, the width has no effect on calculation of the natural frequencies. However, it is chosen such that the width is 1/20 of the beam length so that the structure more closely resembles a beam than a plate.
Table 5.1: Parameters for dimpled beams in Chapter 5

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total beam length, $L$</td>
<td>100 cm</td>
</tr>
<tr>
<td>Width, $b$</td>
<td>5 cm</td>
</tr>
<tr>
<td>Thickness, $h$</td>
<td>1 cm</td>
</tr>
<tr>
<td>Dimple chord lengths, $\bar{l}_i$</td>
<td>$L/10$</td>
</tr>
<tr>
<td>Elastic modulus, $E$</td>
<td>189 GPa</td>
</tr>
<tr>
<td>Density, $\rho$</td>
<td>7688 kg/m$^3$</td>
</tr>
</tbody>
</table>

5.1 Effect of Boundary Condition

In this section, the fundamental frequency of beams with various boundary conditions is investigated. Seven boundary conditions will be considered: fixed-fixed, fixed-free (cantilever), fixed-pin, fixed-pin roller, fixed-pin vertical roller, pin-pin, and pin-pin roller. A roller denoted by ‘vertical’ allows the beam end to slide in the vertical direction (transverse) but not the horizontal (axial). All other rollers are assumed to allow only horizontal motion. Each of these boundary conditions are defined in Section 3.2.2. The pin-pin roller and pin-pin boundary conditions will be chosen as representative examples for a detailed investigation.

To begin, one dimple is placed at the center of the beam and the dimple angle $\alpha$ is varied from $0^\circ$ to $180^\circ$ in 5 degree increments. For each angle, the frequency of the $i^{th}$ mode of the dimpled beam is compared to the corresponding natural frequency of the uniform beam. The percentage change in frequency $f_i$ with respect to the uniform beam natural frequency $f_i^\ast$ is defined as

$$\Delta f_i \,[\%] = \frac{f_i - f_i^\ast}{f_i^\ast} \times 100 \,,$$

so that a positive change indicates the dimpled beam natural frequency is higher than that of the uniform beam. In Figs. 5.1–5.2, the change in fundamental frequency is plotted against dimple angle. Figure 5.1 groups boundary conditions that allow axial
Figure 5.1: Fundamental frequency vs. dimple angle for permissive boundary conditions (center dimple)

Figure 5.2: Fundamental frequency vs. dimple angle for restrictive boundary conditions (center dimple)
motion, and Fig. 5.2 groups boundary conditions that restrict axial motion. From Fig. 5.1, we see that for boundary conditions that allow axial motion (referred to here as “axially permissive” boundary conditions), the fundamental frequency decreases for all dimple angles. The behavior of the fundamental on the beams with boundary conditions that restrict axial motion (“axially restrictive”) is entirely different, where the fundamental frequency increases initially and then begins to decrease. Each class of boundary condition is considered in the next sections.

5.1.1 Axially Permissive Boundary Conditions

The axially permissive class of boundary condition allows axial motion. With reference to Fig. 5.1, we see that for axially permissive boundary conditions, the fundamental frequency decreases monotonically. Even though only the behavior of the first natural frequency is depicted here, this behavior is characteristic of higher modes, and other dimple locations and geometries. What follows is a detailed analysis of a pin-pin roller beam as a representative example.

The fundamental mode shape of a pin-pin roller beam is shown Fig. 5.3. The dimple pushes and pulls the right end of the beam outward and inward since the roller allows horizontal motion. The coupling can be quantified by plotting the horizontal displacements at points along the beam (i.e., the horizontal component of the mode shape). Such a plot is closely related to the axial load within the beam, which is proportional to the change in horizontal displacement between nodes ($F_x \propto \Delta U/\Delta x$).

Considering the mode shape above the undisplaced beam in Fig. 5.3, the horizontal displacement is plotted in Fig. 5.4. Three observations can be made: 1) the horizontal displacement of the dimple and right segment ($x \geq 0.45$ m) is generally negative because those segments are shifted to the left in Fig. 5.3, 2) the slope is zero at both ends of the beam, meaning that the boundaries apply zero horizontal reaction force.
(and zero axial load within the segments), and 3) equal nodal displacements in the right segment \((0.55 \leq x \leq 1 \text{ m})\) imply that the dimple moves the segment horizontally as one rigid body. This demonstrates the coupling from the dimple and the absence of any restrictive horizontal force. As an interesting side note, examination of Fig. 5.4 at the dimple center shows that the slope is zero. This means that the center of the dimple is rigid while under an axial load, and the relative deflection is small in comparison to say, the horizontal deflection at the right end of the dimple.

Next, it is shown that as a consequence of the axially permissive boundary condition, the stiffness of the pin-pin roller beam decreases as dimple angle is increased. This is demonstrated by plotting the dynamic stiffness in Fig. 5.5. Unlike the static stiffness, the dynamic stiffness is useful for showing the stiffness of different vibration
Figure 5.4: Horizontal component of fundamental mode shape (pin-pin roller, $\alpha = 180^\circ$). The right segment moves as a rigid body.

modes. The dynamic stiffness [3] is calculated as follows: a sinusoidal transverse excitation force of magnitude 1 N is applied to the beam in Fig. 5.3 at $x = 0.1$ m. The excitation frequency is equal to that of the fundamental frequency of the dimpled beam in order to excite the fundamental mode shape (note the excitation frequency changes with dimple angle). Then, the complex transverse displacement of the beam is calculated at the point of excitation using Eq (4.5). Taking the magnitude of the force to response ratio yields the dynamic stiffness at the driving point (driving point response). It is noted that the response could be measured at any point, and the force could be applied anywhere except at nodal locations where the response is zero.

The driving point dynamic stiffness is plotted against dimple angle in Fig. 5.5. The plot shows that dynamic stiffness decreases as dimple angle increases. This may have been expected since the end of the beam at $x = L$ is allowed to slide. In fact,
Figure 5.5: Dynamic stiffness vs. dimple angle (fundamental mode shape, pin-pin roller)

A static analysis of the dimple by itself reveals the same decrease in stiffness if one of the dimple ends is allowed to slide along its chord length. In Fig. 5.6, if a point force is applied in the axial direction to the sliding end, the static stiffness decreases with dimple angle (i.e., the end deflection increases with angle). In other words, the dimple is stiffest when it is a straight beam in pure axial compression. The loss in stiffness (dynamic and static) with increasing dimple angle explains the behavior of the fundamental frequency in Fig. 5.1. It is noted that throughout this example, the dimple thickness is assumed to decrease with increasing angle. However, if the dimple thickness is kept the same as the straight segment thickness (i.e., $h = \bar{h}$), the dynamic stiffness and the natural frequencies still decrease with dimple angle. The effect of dimple thickness will be explored further in Section 5.5.

The above analysis may be summarized conceptually. Ordinarily, an arch is stiffer than a flat structure if its ends are prevented from moving horizontally (i.e.,
Figure 5.6: Deflection of a uniform beam as compared to an arch for the same applied force (pin pin-roller)

along the chord length). Since the arch couples its vertical and horizontal displacements, applying a load downward on top of an arch will tend to “push” the ends out. If the supports prevent any motion from occurring, the result is to stiffen the system. Similarly, the boundary value model of the dimpled beam in Chapter 3 demonstrated the dimple’s tendency to couple horizontal and vertical motion of adjacent segments via continuity conditions. In the case of a dimpled beam with a free end or roller, there is no axially restrictive force on the end of the beam to support the coupling, and the result is a system with lower stiffness than the uniform case. As a consequence, the natural frequencies decrease.

5.1.2 Axially Restrictive Boundary Conditions

Boundary conditions that prevent axial motion are considered in this section. In Fig. 5.2, the behavior of the fundamental frequency of such beams is different as compared to the fundamental frequencies for beams with axially permissive boundary conditions (Fig. 5.1). For the axially restrictive class of boundary condition, the fundamental increases initially, then begins to decrease after the dimple angle surpasses
a specific value (the reason for this is explained in Section 5.2). According to the previous discussion on coupling, the dimple is expected to make the system stiffer because the boundary condition prevents any axial movement.

Choosing the pin-pin beam as an example, the fundamental mode shape is plotted in Fig. 5.7. It is observed that the pins prevent axial motion. Therefore a non-zero horizontal reaction force is expected to develop at both pin locations because of the coupling caused by the dimple. This is shown in a plot of the horizontal component of the mode shape given in Fig. 5.8. The positive slopes at the left and right ends indicate that the dimple is pulling the straight segments inward, creating a tension within the segments. This creates a non-zero horizontal reaction force at both pins. Throughout the vibration cycle, the dimple will tend to push and pull the adjacent straight segments. However, the boundaries will prevent any motion and the beam stiffness will increase. The increase in stiffness is expected to cause the fundamental
frequency to increase. As before, we will show that the dynamic stiffness is correlated to the fundamental frequency.

As an interesting side note, the positive slope at $x = 0.5\, \text{m}$ in Fig. 5.8 means the dimple is being stretched at its center (when in the configuration shown in Fig. 5.7). That is, a point just to the left of the dimple center is displaced to the left (negative displacement) and a point to the right of the center is displaced to the right (positive displacement). This yields a tensile force at the dimple center. This further underscores the need to assume that the dimple extends along its circumferential axis, as was assumed during development of the BVM.

The dynamic stiffness of the fundamental mode shape is plotted in Fig. 5.9 using the same procedure described in the previous section. The similarity to the fundamental frequency in Fig. 5.2 is remarkable. The dynamic stiffness of the pin-pin
beam increases with dimple angle up to 100°, and decreases for \( \alpha > 100^\circ \). Likewise, the fundamental frequency increases up to 100°, and decreases for \( \alpha > 100^\circ \). Similar observations are made in the dynamic stiffness curves for the other three axially restrictive boundary conditions in Fig. 5.2. An important caveat is that for the axially restrictive boundary conditions, the initial rise and fall of the fundamental frequency (as shown in this example) is dependent on the dimple location and geometry. The behavior of the fundamental (and higher modes) will be different for other dimple locations, and this analysis is reserved for Section 5.3. The chord length of the dimple also influences where the peak in frequency and dynamic stiffness occurs (Section 5.4).

![Figure 5.9: Dynamic stiffness vs. dimple angle (fundamental mode shape, pin-pin)](image)

The results presented here show that for both axially permissive and axially restrictive boundary conditions, the fundamental frequency behaves similarly to the dynamic stiffness of the fundamental mode shapes. For the axially permissive case,
the fundamental frequency and dynamic stiffness decreases monotonically with dimple angle. This observation is representative of higher modes and other dimple locations. For the axially restrictive case, the examples show that the fundamental increases initially, and then decreases after a certain dimple angle. An interesting question is raised as to why the dynamic stiffness itself decreases after the dimple angle surpasses a specific value. Unlike the axially permissive boundary condition, this result is difficult to interpret using a static analysis of the dimple. Furthermore, the behavior of higher modes can be entirely different from the fundamental frequency. In the next section, we will take a closer look at the effect of dimple angle on natural frequencies for axially restrictive boundary conditions, and in particular, the ability of a changing mode shape to influence the natural frequencies.

5.2 Effect of Dimple Angle

In this section, the effect of dimple angle on the fundamental frequency and higher modes is studied. In particular, we investigate the reason why the natural frequencies increase up to a certain dimple angle, and decrease beyond this angle. In the previous section, this phenomenon was shown for boundary conditions that restrict axial motion (axially restrictive boundary conditions). Therefore, the examples in the following three sections will be confined to beams with axially restrictive boundary conditions (i.e., pin, fixed). The behavior of the fundamental frequency for a pin-pin beam with one dimple at the center is revisited in Section 5.2.1. In Section 5.2.2, the effect of dimple angle on higher modes is investigated. Finally, Section 5.2.3 investigates the effect of dimple angle on a fixed-fixed beam for three different dimple locations.
5.2.1 Example 1: Pin-Pin Beam, Fundamental Mode

In the previous section, the change in fundamental frequency with respect to the uniform beam natural frequency was shown for various dimple angles. For the pin-pin beam, the dynamic stiffness of the fundamental exhibited a behavior similar to that of the fundamental, where both quantities reached a maximum at $\alpha = 100^\circ$. These quantities are plotted together in Fig. 5.10. Recalling the discussion on arches on p.53, the fundamental frequency of a pin-pin arch was shown to increase as the dimple angle was increased. At a certain dimple angle, the fundamental mode shape suddenly switched, and the frequency decreased with increasing dimple angle. The results presented below shows that there is an analogous situation for dimpled beams.

![Figure 5.10: Change in fundamental and dynamic stiffness, $\alpha = 0^\circ$–$180^\circ$ (pin-pin)](image)

The fundamental mode shapes of the pin-pin beam are plotted in Fig. 5.11 for dimple angles between $0^\circ$–$180^\circ$ in $5^\circ$ increments. Each of the mode shapes have
been mass-normalized (Appendix D). Two observations are made here: 1) there is no apparent change in the overall mode shape as the dimple angle is increased, and 2) there is a slight change in amplitude between the mode shapes, especially in the dimple area. On closer inspection of the dimple (dashed box in Fig. 5.11), we see an interesting mode shape behavior. Figures 5.12–5.13 are plots of only the portion of the mode shapes within the dashed box for the range $\alpha = 0^\circ–100^\circ$ and $\alpha = 100^\circ–180^\circ$. Note that the undisplaced dimple curvature $\bar{Y}_0$ has been removed from Figs. 5.12–5.13 (see Fig. C.1 in Appendix C). This is because the undisplaced curvature is much greater than the mode shape displacement.

Figure 5.12 shows that 1) the amplitude of the mode shape is decreasing, where the amplitude is largest at $\alpha = 0^\circ$ and smallest at $\alpha = 100^\circ$ and 2) the mode shape at $\alpha = 0^\circ$ is of the first-order symmetric type, and begins to “flatten” as dimple angle is increased. The first observation that the amplitude decreases is consistent with the
Figure 5.12: Mode shapes of the dimple at the fundamental frequency, $\alpha = 0^\circ$–100$^\circ$ (pin-pin). Note the flattening of the mode shape and decreasing amplitude. The undeformed dimple profile has been subtracted from the plots.

The fact that the dynamic stiffness is increasing in the range $0^\circ \leq \alpha \leq 100^\circ$ (Fig. 5.10). It is expected that increasing the stiffness of the beam will lower its vibration amplitude, and increase the fundamental frequency. The second observation that the dimple is flattening is analogous to the situation described in Fig. 5.6, where the beam is at its stiffest when it is flat and is under pure compression (or axial loading). Similarly, the dimple has arranged itself in a flattened configuration such that the loading on its ends from the adjacent segments is as close to axial as possible.

Figure 5.13 shows that 1) the amplitude of the mode shape is increasing, where the amplitude is smallest at $\alpha = 100^\circ$ and largest at $\alpha = 180^\circ$ and 2) the mode shape transitions from the “flat” configuration when $\alpha = 100^\circ$ to a second-order symmetric type when $\alpha = 180^\circ$. The first observation that the amplitude increases is consistent with the fact that the dynamic stiffness decreases for $\alpha > 100^\circ$ (Fig. 5.10). Decreasing
the beam stiffness makes the structure more compliant, and the effect is to lower the fundamental frequency. The second observation is further evidence that the dimple makes the beam appear very stiff when the mode shape is relatively flat. In this configuration, the fundamental frequency is maximized.

Figure 5.13: Mode shapes of the dimple at the fundamental frequency, $\alpha = 100^\circ$–$180^\circ$ (pin-pin). Note the mode shape transition and increasing amplitude. Some mode shapes have been removed to increase clarity.

To summarize the observations, the mode shape within the dimple switches from a first order symmetric type shown at $\alpha = 0^\circ$ in Fig. 5.12 to a second-order symmetric type shown at $\alpha = 180^\circ$ in Fig. 5.13. At $\alpha = 100^\circ$, where the fundamental frequency and dynamic stiffness reaches a maximum, the modal amplitude is smallest. It is at this angle that the mode shape is nearly flat, and is in its stiffest configuration. This mode-switching behavior observed here for dimpled beams is analogous to the behavior of arches observed previously. For both cases, the similarity is that the fundamental frequency increases when the mode shape is of the first-order symmetric
type (Fig. 4.5 vs. Fig. 5.12), and decreases when the shape is either anti-symmetric (Fig. 4.6) or a second-order symmetric type (Fig. 4.7 vs. Fig. 5.13).

There are, however, some differences between the cases. For the dimpled beam, the mode shape within the dimple at the fundamental frequency switches from a first-order symmetric type to a second-order symmetric type, much like the arch mode shapes in Figs. 4.5 and 4.7. For the arch, the mode shape at the fundamental frequency switches from a first-order symmetric type (Fig. 4.5) to a first-order anti-symmetric type (Figs. 4.6). The reason for this difference can be stipulated to be that the boundary conditions are different between the two cases - that is, in a fixed or pinned arch, the ends are not allowed to move. On the other hand, placing an arch within a beam allows the ends to move horizontally and vertically (not freely, but not constrained as a fixed or pinned arch). A suggested way forward is to approximate these boundary conditions using a model of an arch with spring supports of variable stiffness. Another plausible explanation notes that the overall fundamental mode shape is symmetric (Fig. 5.11), and the dimple must maintain continuity conditions at the points of connection. However, a mode shape switch within the dimple to an anti-symmetric type may violate some of the continuity conditions (e.g. continuity of bending slope at either side), so that the only admissible switch is to a symmetric type.

The findings in this section improve our understanding of dimpled beams. The results presented here show that the peaks in the natural frequencies for a certain dimple angle correspond to a changing mode shape within the dimple. After the mode shape changes, the natural frequency decreases with increasing dimple angle. Previous studies have suggested that a decreasing natural frequency was caused by the thinning of the dimple with increasing dimple angle [2, p.78]. In fact, this phenomenon occurs even when the dimple is not assumed thinner than the rest of the beam. The effect
of dimple thinning will be studied in Section 5.5.

5.2.2 Example 2: Pin-Pin Beam, Higher Modes

Thus far, the analysis been limited to the fundamental frequency. How do higher modes behave when dimple angle is changed? This example investigates modes 2–4 using the same beam with a dimple located at the center. In Figs. 5.14–5.16, the percentage change in $f_2$, $f_3$ and $f_4$ is plotted against dimple angle. The corresponding driving point dynamic stiffness is plotted with the frequencies.

The results show that the frequency of even modes (Figs. 5.14, 5.16) decreases monotonically with dimple angle, but the frequency of odd modes (Fig. 5.15 and $f_1$) increases initially, then decreases. It is emphasized that this is not true in general, that is, the frequencies of even modes do not always decrease with increasing dimple
angle. The reason for this has to do with the location of the dimple (shown in the next example). In Section 5.3, the effect of dimple location is investigated further. This example does, however, illustrate the complexity of axially restrictive boundary conditions. Depending on the dimple location and the mode, the natural frequencies may either increase and decrease over a range of dimple angles, or decrease monotonically. It is noted that a change in dimple mode shape occurs for mode 3, and the change is similar to that shown for mode 1 in the previous section. For modes 2 and 4, there appears to be no change in mode shape within the dimple. This explains why these frequencies decreases monotonically over the range of dimple angles.

Figure 5.15: Change in $f_3$ and dynamic stiffness (pin-pin)
5.2.3 Example 3: Fixed-Fixed Beam, Fundamental Mode

The change in fundamental frequency of a fixed-fixed beam is plotted against dimple angle in Fig. 5.17. For comparison, three different dimple locations are chosen: \( x_c = 0.055 \text{ m}, \ x_c = 0.22 \text{ m}, \) and \( x_c = 0.5 \text{ m}, \) where \( x_c \) represents the location of the dimple center. Here we observe that for the dimple placed at the boundary \( (x_c = 0.055 \text{ m}), \) there is a rapid rise in the fundamental, followed by a rapid decrease. For the location approximately halfway between the beam center and boundary \( (x_c = 0.22 \text{ m}), \) the fundamental decreases monotonically. For the dimple placed at the beam center \( (x_c = 0.5 \text{ m}), \) there is also an initial increase and decrease of the fundamental, although the peak change in frequency is not as large as compared to the dimple placed at the boundary.

Comparing the sensitivity \( (\max(\Delta f_1) - \min(\Delta f_1)) \) of the fundamental at the three locations, the fundamental is most sensitive to changes in dimple angle when
the dimple is placed at the boundary ($x_c = 0.055$ m). A comparatively moderate sensitivity to dimple angle occurs at the beam center ($x_c = 0.5$ m), and the least sensitivity occurs for the dimple placed approximately halfway between the boundary and center ($x_c = 0.22$ m). This example motivates the analysis in the next section, where the effect of dimple location on the natural frequencies is explored further using the concept of modal strain energy.

### 5.3 Effect of Dimple Location

Examples 1–2 in the previous section suggested that different natural frequencies of a dimpled beam exhibit different sensitivities to changes in dimple angle for the same dimple location. Example 3 suggested that the fundamental frequency has a different sensitivity to dimple angle for different dimple locations. In this section, the
reason for the different sensitivities between modes and for different dimple locations
is examined. Alshabatat’s [2] observation that dimples placed in regions of high modal
strain have a large effect on natural frequencies is investigated further. Fixed-fixed
and pin-pin beams are chosen as examples for analysis in the following sections.

5.3.1 Example 1: Fixed-Fixed Beam

In order to assess the effect of dimple location on the natural frequencies, the dimple
angle is held fixed while the dimple is positioned at different locations on the beam.
The location of the dimple is defined by the position of its center, $x_c$. For example,$x_c = 0.5$ m corresponds to a dimple placed at the beam center, and $x_c = 0.05$ m
means the dimple is placed as close to the left boundary as possible (since the chord
length is 0.1 m). The percentage change in fundamental frequency is plotted against
dimple location in Fig. 5.18. Each of the curves corresponds to a different dimple
angle. The plots are shown only for half of the beam length ($0 \leq x_c \leq 0.5$ m) because
of the symmetry of the fixed-fixed boundary condition.

Several observations can be made by examination of Figure 5.18. For one, it
shows that the largest variation in fundamental frequency occurs when the dimple is
placed at the boundary. For the dimple angles shown, the variation is between $-5\%$
for $\alpha = 180^\circ$, and $7\%$ for $\alpha = 100^\circ$. These changes also represent the largest decrease
and largest increase in the fundamental frequency for any location. In other words,
the fundamental is sensitive to changes in dimple angle for a dimple placed at the
boundary. As the dimple is moved towards $x_c = 0.22$ m, the variation in fundamental
begins to decrease. At $x_c = 0.22$ m, the variation ranges from $-2.3\%$ when $\alpha = 180^\circ$
to no change in frequency. It should be noted that the change in frequency is negative
for all dimple angles (i.e., the fundamental is lower than that of the uniform beam).
Comparing this location of the dimple to the one where the dimple is near the left
boundary, we conclude that the fundamental is relatively insensitive to changes in
dimple angle. Moving the dimple towards the beam center, the variation begins to
increase again. For the dimple placed at the center \( (x_c = 0.5 \, \text{m}) \), the variation is
relatively large, but not as large as when placed at the boundary. Figure 5.18 also
illustrates an important point: placing a dimple on beam does not guarantee a change
in natural frequency. Some combinations of location and dimple angle produce no
change at all.

Figure 5.18 shows that the fundamental is more sensitive to changes in dimple
angle in some regions than in others. The reason for this may be understood by
examining the modal strain energy (MSE) of the fundamental mode along the length
of a uniform beam. The term “modal strain energy” is used for brevity, where we
actually mean “modal strain energy per length,” or “modal strain energy density.”
For a uniform beam in transverse vibration, the MSE is caused entirely by bending since the non-linear interaction with the axial component is neglected. With reference to Eq. (3.5b), the MSE due to bending is proportional to \((Y_i')^2\), where \(Y_i\) is the mode shape for the \(i^{th}\) mode of the uniform beam. The first three mode shapes of a uniform fixed-fixed beam are shown in Fig. 5.19. Using the eigenfunction for the \(i^{th}\) mode of a fixed-fixed beam \([8]\), the MSE (shown as \(\sigma_i(x)\)) for a beam of length \(L\) is given by

\[
\sigma_i(x) = \frac{1}{4}\left[ -\cos \beta_i x - \cosh \beta_i x + D_i (\sin \beta_i x + \sinh \beta_i x) \right]^2 ,
\]

(5.2)

for \(0 \leq x \leq L\) and where the factor of \(1/4\) is chosen such that the maximum MSE is unity. The constant \(D_i\) is given by

\[
D_i = \frac{\cos \beta_i L - \cosh \beta_i L}{\sin \beta_i L - \sinh \beta_i L} .
\]

(5.3)

Figure 5.19: First three modes shapes of a uniform fixed-fixed beam
Non-dimensional frequency parameters and constants $D_i$ have been tabulated in Table 5.2 for the first three modes of a fixed-fixed beam.

<table>
<thead>
<tr>
<th>Mode</th>
<th>$\beta_i L$</th>
<th>$D_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4.730</td>
<td>0.9825</td>
</tr>
<tr>
<td>2</td>
<td>7.853</td>
<td>1.0008</td>
</tr>
<tr>
<td>3</td>
<td>10.996</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

Using Eq. (5.2) the MSE of the first three modes is plotted in Fig. 5.20 over the full beam length. Each plot is symmetric due to the symmetry of the boundary conditions. Observing the MSE for the first mode (MSE 1) in Fig. 5.20, we see that MSE is largest at the boundary and relatively large at the beam center. At $x = 0.22$ m, the MSE is zero. Recall from Fig. 5.18 that the fundamental was most sensitive to changes in dimple angle at the boundary (largest MSE), relatively sensitive for the dimple placed at the beam center (relatively large MSE), and least sensitive when the dimple was located at $x_c = 0.22$ m (zero MSE). Together, Figs. 5.18 and 5.20 suggest that the fundamental exhibits maximum sensitivity to changes in dimple angle when the dimple is placed in a region of maximum MSE. The converse is also true: the fundamental exhibits minimum sensitivity to changes in dimple angle if the dimple is placed in a region of minimum MSE. Although the fundamental exhibits a comparatively moderate sensitivity at the beam center (which is also an anti-node of the fundamental, Fig. 5.19), the greatest sensitivity occurs at the boundary where the MSE is maximum. For other boundary conditions (e.g., pin-pin), the anti-nodes are coincidentally at the locations of maximum MSE (shown in the next example). This explains why Cheng [1] observed that dimples placed at the anti-nodes of a simply-supported beam had a large effect on the fundamental frequency.

Comparing the MSE for modes 2 and 3 in Fig. 5.20, we see that regions of relatively high and low MSE occur at different locations (although the global
maximum occurs at the boundaries for all modes). This is why the behavior between modes can be entirely different for a dimple at the same location. For all modes, the global maximum in MSE occurs at the boundary. Based off of the observations of the fundamental, it is expected that modes 2 and 3 will also be sensitive to changes in dimple angle at the boundaries. Indeed, this is shown for the second natural frequency in Fig. 5.21, and the third natural frequency in Fig. 5.22.
Figure 5.21: Change in $f_2$ vs. dimple location, 5 different dimple angles (fixed-fixed)

Figure 5.22: Change in $f_3$ vs. dimple location, 5 different dimple angles (fixed-fixed)
Why is the fundamental frequency sensitive to changes in dimple angle at regions of high MSE? A region of high MSE corresponds to a region of large curvature ($Y''$), that is, the bending slope is rapidly changing in these regions. Thus, the internal bending moment is large in regions of high MSE. Therefore, it may be expected that the dimple has a large effect on the beam dynamics when it is placed in a region where a large bending moment is experienced. Conversely, the dimple has little potential to influence the natural frequencies if placed in a region where the internal bending moment is small.

The example presented here has demonstrated three points. First, the natural frequencies of all three modes are most sensitive to changes in dimple angle when the dimple is placed in the region of largest MSE. For the fixed-fixed beam, these locations correspond to the boundaries. Second, the natural frequencies for all three modes decrease monotonically with dimple angle when the dimple is placed at a region of zero MSE. Third, for the fundamental frequency, the least amount of variation occurs where the MSE is zero. However, for mode 2, the amount of variation between the region of zero MSE and relatively high MSE is almost the same (Fig. 5.21). For mode 3, the least amount of variation does not occur where the MSE is zero, but where the MSE is relatively large (Fig. 5.22). This is an unexpected result. The reason for this is because the peaks and valleys in the MSE get closer together as frequency increases, and the dimple spans into both of these regions. Use of a smaller dimple chord length would reduce the overlap into regions of large and small MSE. In which case, the least amount of variation occurs where the MSE is zero.

### 5.3.2 Example 2: Pin-Pin Beam

In this section, a pin-pin beam is considered as an example. As before, the dimple location is moved across the beam while the dimple angle is held constant. In
Figs. 5.23–5.24, the change in the first and second natural frequencies are plotted against dimple location for five different dimple angles. From Fig. 5.23, it is apparent that the fundamental is sensitive to changes in dimple angle when the dimple is located at the beam center, and insensitive when placed at the boundaries. This is in contrast to the fixed-fixed beam, where the fundamental frequency was most sensitive to changes in dimple angle when the dimple was placed at the boundary. Interestingly, Fig. 5.23 shows a very small difference between $\alpha = 50^\circ$ and $\alpha = 150^\circ$.

With reference to Fig. 5.24, the second natural frequency has the largest variation at $x = 0.25$ m.

![Figure 5.23: Change in fundamental frequency vs. dimple location, 5 different dimple angles (pin-pin)](image)

For the pin-pin boundary condition, the above observations can be explained by examining either the mode shapes or the MSE. The mode shapes of a uniform pin-pin beam are given by $Y_i(x) = \sin(k\pi x/L)$, the first two of which are plotted in Fig. 5.25. Comparing Fig. 5.23 with the fundamental mode shape in Fig. 5.25, we
see that the fundamental has the largest variability when the dimple is placed at the anti-node (beam center). Similarly, placing a dimple at the anti-node of the second mode shape will have the largest effect on the second natural frequency ($x = 0.25$ m and $x = 0.75$ m in Figs. 5.24 and 5.25). The anti-nodes, however, happen to coincide with regions of large MSE. For the uniform pin-pin beam, the MSE is given by

$$\sigma_k(x) = \sin^2\left(\frac{k\pi x}{L}\right).$$

Using Eq. (5.4), the MSE is plotted in Fig. 5.26 for the first and second natural frequencies. Comparing Fig. 5.26 to Figs. 5.23–5.24, the conclusion is that the natural frequencies are most sensitive to changes in dimple angle when the dimple is placed in a region of high MSE, and least sensitive when placed near a region of zero MSE.
Figure 5.25: First and second modes shapes of a uniform pin-pin beam

Figure 5.26: Modal strain energy, modes 1–2 (pin-pin)
These results help to explain the observations in Section 5.2.2, where for a pin-pin beam with a dimple at its center, frequencies of the odd modes increased initially, and then decreased after a certain dimple angle (the amount of the increase depends on the chord length, as shown in the next section). Frequencies of even modes decreased monotonically with dimple angle. From Fig. 5.26, we now see that the fundamental (and higher odd modes not shown) has maximum MSE at the beam center, and the second natural frequency (and higher even modes not shown) has zero MSE at the center. The MSE concept also helps to generalize the observations made by Cheng [1], who studied the natural frequencies of pin-pin roller beams. A uniform pin-pin roller beam has the same MSE as that of a pin-pin beam. It was observed that a dimple had maximal influence on the fundamental when located at the beam center, or at the anti-node of the fundamental. It is now apparent that this location coincides with a region of high MSE.

5.4 Effect of Dimple Chord Length

Throughout this chapter, the chord length has been assumed to be 1/10 of the beam length, or 0.1 m. The purpose of this section is to study the effect of chord length on the natural frequencies. Two examples are presented below for a pin-pin beam.

In this example, the effect of changing the chord length $\bar{l}$ on the fundamental frequency of a pin-pin beam is investigated. The dimple is placed at the center of the beam. In Fig. 5.27, the percentage change in fundamental frequency with respect to the fundamental frequency of the uniform beam is plotted against dimple angle for six different chord lengths ranging from 0.04 m to 0.12 m. In Fig. 5.28, a similar plot is made for larger chord lengths ranging from 0.15 m to 0.5 m (0.5 m represents half the beam length).
From Figs. 5.27–5.28, the effect of increasing the chord length is to increase the maximum fundamental frequency for a given dimple angle. This is consistent with the optimization results of Alshabtat [2, p.80], who noted that an arch is the optimum shape of a beam in order to maximize the fundamental. It is interesting to compare the peak locations in Figs. 5.27–5.28. As the chord length is increased, notice that the dimple angle corresponding to the maximum change in fundamental frequency is getting smaller (termed the “critical angle”). In Fig. 5.28, for example, the critical angle occurs at $\alpha = 85^\circ$ for $\bar{l} = 0.15$ m, but at $\alpha = 40^\circ$ for $\bar{l} = 0.50$ m. This means that it is difficult to make general conclusions about which dimple angle one should choose in order to maximize the fundamental frequency of a dimpled beam without detailed analysis of each case. This is due to the fact that changing only one dimension of the beam would change its vibration characteristics completely. On the other hand, if one were to scale all dimensions of the beam by the same factor, the critical angle would
not change (although the natural frequencies themselves will change). It is noted that at each critical angle, there is a change in mode shape within the dimple that causes the fundamental frequency to decrease (see Section 5.2.1).

The observations of the dimpled beam described above are similar to that of arches. A shift in the critical angle is also observed in a pin-pin arch when the arch chord length is changed. In Fig. 5.29, the fundamental frequency of the pin-pin arch is plotted against arch angle for three different chord lengths. The fundamental frequency is computed using the equations developed by Nelson [28]. As the chord length of the arch is increased, the critical angle shifts to smaller values (circled). For chord lengths larger than the ones shown, the switch occurs at very small angles. At each critical angle, the mode shape switches from a first-order symmetric type to a first-order anti-symmetric type (see Section 4.2, Fig. 4.3).
Next, the effect of chord length on the first four modes of the pin-pin beam (dimple at the center) is investigated in Fig. 5.30. The first four frequencies are plotted against dimple angle, assuming the same chord length throughout (0.1 m on a beam of length 1 m). In Fig. 5.30, the natural frequency (Hz) is plotted instead of the percentage change in frequency. The effect of the chord length is assessed by comparing the maximum frequency with the minimum frequency across the range of dimple angles. The larger the absolute difference is, the larger the effect of chord length. Comparing the percentage changes among modes is not the best comparison because the reference changes (see Eq. (5.1)).

From Fig. 5.30, the absolute difference is approximately 1.5 Hz (mode 1), 3 Hz (mode 2), 12 Hz (mode 3), and 43 Hz (mode 4). The same dimple has a larger effect on higher modes of vibration as compared to lower modes. These results are typical of
other examples. The reason for this is that for higher modes, the bending wavelength decreases, and the chord length to bending wavelength ratio increases (although the chord length is held constant). Equivalently, for a given mode where the bending wavelength is constant, increasing the chord length has a larger effect on the natural frequency. This is consistent with the observation in the previous example, where increasing the dimple chord length had a larger effect on the fundamental frequency. It is noted that the frequencies of the even modes decrease monotonically because the dimple is placed at the beam center where the MSE is zero for even modes.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Figures/Figure5.30.png}
\caption{Frequency (modes 1-4) vs. dimple angle (pin-pin, center dimple). Note the increasing scale for higher modes.}
\end{figure}

### 5.5 Effect of Dimple Thickness

The effect of dimple thickness on the natural frequencies and mode shapes of dimpled beams is investigated in this section. Throughout this dissertation, the dimple thickness was assumed to decrease with dimple angle according to Eq. (3.2), so that
the thickness of the dimple is always smaller than the thickness of the straight segment ($\bar{h} < h$ for $\alpha > 0$). This assumption was made in order to account for the fact that during the stamping process, the overall structural mass remains the same. The question posed here is: if the dimple thinning is not incorporated in the beam model, and dimple thickness remains the same as the thickness of the straight segments (i.e., $\bar{h} = h$ for all dimple angles $\alpha$), what would be the effect on the natural frequencies and mode shapes? The answer to this question is significant because all previous models of dimpled beams\footnote{Except for reference [21], where dimples were allowed to be thicker than the rest of the beam, although the overall beam mass remained the same} have assumed that the dimples are thinner than the rest of the beam [1],[2],[23].

Three beams with different boundary conditions are chosen for examples (pin-pin roller, pin-pin, and fixed-fixed beam). In each example, the dimple is placed at the center of the beam, where the beam material and geometrical properties are defined in Table 5.1. Three comparisons are made in each of the examples. When the dimple thickness, $\bar{h}$, is assumed to decrease with dimple angle according to Eq. (3.2), this is referred to as “Case 1: $\bar{h} < h$.” When the dimple thickness is the same as the straight segment thickness, $h$, for all dimple angles, this is referred to as “Case 2: $\bar{h} = h$.” In Case 2, the mass of the beam increases with increasing dimple angle (the volume of the dimple increases). Therefore, a third case is considered where a small correction factor is applied to the beam density in order to maintain the mass of the beam as dimple angle increases. This case is referred to as “Case 3: $\bar{h} = h$ ; $m_3 = m_1$”, where the beam mass $m$ in Cases 1 and 3 are equal. For the beam considered in the following examples, the additional mass in Case 2 ($m_2$) is approximately 0.22 kg (i.e., a 6% increase when $\alpha = 180^\circ$). The additional mass will tend to lower the natural frequencies, so Case 3 corrects for this effect. In the examples that follow, the change in fundamental frequency and mode shapes will be compared between all three cases.
5.5.1 Example 1: Pin-Pin Roller Beam

The change in fundamental frequency is plotted against dimple angle in Fig. 5.31 for Cases 1–3. In all cases, the fundamental decreases with respect to the fundamental frequency of the uniform beam. Case 1 yields the largest decrease in fundamental. This means that a thinner dimple is responsible for a larger reduction in the fundamental as compared to a dimple of constant thickness.

Figure 5.31: Effect of dimple thickness: change in $f_1$ vs. dimple angle (pin-pin roller, center dimple)

It is noted that Cases 2–3 also yield lower fundamental frequencies as compared to the uniform beam fundamental frequency. This is noteworthy because the previous literature attributed this effect exclusively to the dimple being thinner than the rest of the beam [1],[2, p.201]. This example shows that the fundamental frequency decreases even when the dimple thickness is the same as the rest of the beam, and when a correction is applied to keep the beam mass from increasing with dimple
angle. In other words, the curvature of the dimple is responsible for lowering the fundamental frequency when the boundary condition allows axial motion. This situation is analogous to that described in Fig. 5.6, where for an arch with a pin-pin roller boundary condition, the arch is at its stiffest configuration when it is nearly flat, and the stiffness decreases when the arch angle is increased.

5.5.2 Example 2: Pin-Pin Beam

The change in fundamental frequency is plotted against dimple angle in Fig. 5.32 for Cases 1–3. In this example, we see again that the effect of the thinner dimple (Case 1) is to lower the fundamental by a larger amount as compared to cases where the dimple thickness is held constant (Cases 2–3). For large dimple angle, the fundamental in Case 1 is lower than that of the uniform beam.

![Graph showing the change in fundamental frequency vs. dimple angle for Cases 1-3.]

Figure 5.32: Effect of dimple thickness: change in $f_1$ vs. dimple angle (pin-pin, center dimple)
In all cases, however, the fundamental frequency reaches a maximum at a certain dimple angle. This example demonstrates that the thinning of the dimple does not cause the fundamental to decrease after the dimple angle is further increased since this occurs even when the dimple thickness is held constant (Cases 2 and 3). In fact, the fundamental frequency stops increasing after the mode shape within the dimple switches (this happens for all cases). Figure 5.33 compares the mode shape at $\alpha = 180^\circ$ for Cases 1 and 2. We see the effect of the thinner dimple is to exaggerate or distort the mode shape within the dimple more than when compared to the mode shape of the thicker dimple (both mode shapes are mass-normalized). This is an indication that the thinner dimple is more flexible than the thicker dimple. As a result, the fundamental frequency is smaller at $\alpha = 180^\circ$ in Case 1 than in other cases.

Figure 5.33: Effect of dimple thickness: comparison of dimple mode shapes at $\alpha = 180^\circ$ (pin-pin, center dimple). Note the thinner dimple mode shape (Case 1) exhibits a larger curvature than the mode shape of the thicker dimple.
5.5.3 Example 3: Fixed-Fixed Beam

The change in fundamental frequency is plotted against dimple angle in Fig. 5.34 for Cases 1–3. We see that for Cases 1 and 2, the fundamental frequency is smaller than that of the uniform beam for large dimple angles (near 180°). In Case 2, the decrease can be attributed, in part, to the larger beam mass (in addition to the mode shape switch). In Case 3, where the mass is held constant, the fundamental reaches a peak and begins to decrease, but the frequency is always higher than the uniform beam fundamental (i.e., positive $\Delta f_1$).

![Figure 5.34: Effect of dimple thickness: change in $f_1$ vs. dimple angle (fixed-fixed, center dimple)](image)

In Case 3, it appears that keeping the dimple thickness (and beam mass) constant prevents the fundamental frequency from becoming lower than that of the uniform beam (dashed line). However, if the chord length of the dimple is made smaller, as in Fig. 5.35, the fundamental frequency becomes lower than the uniform...
beam fundamental (see circled region). This is significant because it demonstrates that there are multiple reasons why the fundamental can become lower than that of the uniform beam (e.g., changing chord length, mode shape switching). The thinning of the dimple cannot be solely responsible for making the natural frequencies of dimpled beam lower than the natural frequencies of uniform beam since this is observed even when the dimple thickness and beam mass is held constant.

Figure 5.35: Effect of dimple thickness: change in $f_1$ vs. dimple angle (fixed-fixed, center dimple, $\bar{l} = 0.05$ m)

In summary, a beam with a thinner dimple generally has a lower fundamental than a beam with a thicker dimple, after correcting for the increased beam mass. Some important observations on the effect of dimple thinning are noted: for a pin-pin roller beam (axial motion allowed), the natural frequencies decrease regardless of the dimple thickness. The curvature of the dimple itself is responsible for the decrease. For a pin-pin beam (no axial motion allowed), dimple thinning does not cause the fundamental frequency to suddenly decrease after the initial increase, as previously thought. This
effect is observed even when the dimple thickness is held constant. The decrease in the fundamental frequency is accompanied by a switch of the mode shape within the dimple, and is consistent with previously observed behavior of arches [28],[27]. For the fixed-fixed beam (no axial motion allowed), a thinner dimple can reduce the fundamental below that of the uniform beam. However, the effect can be reproduced by decreasing the chord length of a dimple with the same thickness as the straight segments. This shows that there are additional variables responsible for reducing the fundamental frequency (e.g., chord length, mode shape switch).

5.6 Summary

The purpose of this chapter was to obtain a better understanding of dimpled beams by studying the effect of changing the parameters of a dimple on beam natural frequencies and mode shapes. To this end, five parameters were considered.

First, it was shown that changing the beam boundary conditions can have a large effect on the fundamental frequency. The ability of the dimple to couple the transverse and longitudinal motions of a beam was shown by analysis of the beam mode shapes. By studying the beam mode shape and separately focusing on the transverse and longitudinal beam displacements, we concluded that the addition of a dimple coupled transverse and longitudinal degrees of freedom. Because of the coupling, the natural frequencies of axially permissive or restrictive boundary conditions had a different behavior. For axially permissive boundary conditions, the fundamental frequency (and higher natural frequencies) of the dimpled beam was found to be lower than the fundamental frequency of a uniform beam, regardless of dimple angle. A plot of the dynamic stiffness against dimple angle confirmed that the beam stiffness decreases as dimple angle increases, thus explaining the trend in the natural
frequencies. For axially restrictive boundary conditions, the examples showed that the fundamental frequency increased initially, and then decreased as the dimple angle increases. Again, a plot of the dynamic stiffness of the fundamental mode showed a trend very similar to that of the fundamental frequency.

Second, the effect of changing the dimple angle on the beam natural frequencies was investigated. The most important discovery in this section was that for axially restrictive boundary conditions, the mode shape within the dimple changes at a specific dimple angle. This change in dimple mode shape caused the beam natural frequencies to begin decreasing as dimple angle was further increased. At the dimple angle for which a natural frequency reached its maximum, the mode shape within the dimple was relatively flat, thus causing the beam to exhibit a maximum dynamic stiffness in this configuration. A comparison was made to the mode-switching behavior of arches, where a decrease in the fundamental frequency of an arch was accompanied by a change in mode shape. However, in the case of an arch, this mode switching is not accompanied by a flattening of the mode shape.

Third, the natural frequencies were found to be sensitive to dimple angle when dimples were placed in locations corresponding to high modal strain energy (MSE). They were relatively insensitive to changes in dimple angle if the dimple was placed in a region of zero MSE. This correlation was shown by computing analytically the MSE over the length of a uniform beam for various boundary conditions. In the case of the fundamental mode of a fixed-fixed beam, the results showed that the largest MSE occurred at the boundaries. The results for the fundamental mode of a pin-pin beam showed that the anti-node happens to coincide with the region of largest MSE. This explained the observations of Cheng [1], who noted the fundamental exhibited greatest sensitivity to changes in dimple angle when when the dimple was placed at the anti-node of a simply-supported beam. Our work showed that Cheng’s observation
was incorrect and it held true only for one specific case.

Fourth, the effect of changing the dimple chord length on the natural frequencies was studied. Two cases were considered where the chord length to bending wavelength ratio was changed. First, the fundamental frequency vs. dimple angle was plotted for dimples of various chord lengths (i.e., variable chord length, constant bending wavelength). It was found that as dimple chord length was increased, the change in fundamental frequency increased. Second, the change in frequency of the first four modes were plotted against dimple angle using the same chord length (i.e., constant chord length, variable bending wavelength). The results showed that higher modes exhibited greater variability in frequency than lower modes. In other words, a larger effect on the natural frequencies was obtained by increasing the dimple chord length to bending wavelength ratio. It was also discovered that increasing the dimple chord length caused the peak in the fundamental frequency to occur at smaller and smaller dimple angles. The peak in the fundamental frequency was accompanied by a change in mode shape within the dimple. Similar behavior was observed in arches.

Fifth, a study of the thickness of the dimple revealed that a beam with a thinner dimple exhibited a lower fundamental frequency than a beam with a dimple whose thickness was the same as the rest of the beam. However, the fundamental frequency of a pin-pin roller beam was found to decrease with increasing dimple angle even as the thickness of the dimple was held constant (i.e., the same as the thickness of the straight segments). This showed that the curvature of the dimple was responsible for the decrease, not the thinning of the dimple as was previously thought. Another example showed that the decrease in the fundamental frequency beyond a certain dimple angle after an initial increase is not caused by the reduced thickness of the dimple (this phenomenon happens regardless of the dimple thickness). In fact, a mode shape switch within the dimple causes the natural frequencies to decrease.
after an initial increase. A third example of a fixed-fixed beam showed that dimple
thinning is not responsible for lowering the fundamental frequency below that of the
uniform beam. The same effect was observed by decreasing the chord length of a
dimple whose thickness was the same as the thickness of the straight segments.
Chapter 6

A DESIGN STRATEGY FOR DIMPLED BEAMS

This chapter presents a strategy for placement of dimples using the modal strain energy (MSE) of uniform beams. In Section 5.3, it was shown that natural frequencies are sensitive to changes in dimple angle when dimples are placed in regions of large MSE, and relatively insensitive to changes in dimple angle when placed at regions of zero MSE. The results of Section 5.3 also suggest that regions of highest MSE are excellent candidate locations for their ability to shift a beam’s natural frequencies and mode shapes. As such, dimples locations can be determined \emph{a priori} through examination of the MSE.

Although use of the MSE as a design strategy for placement of dimples on beams is unique to this study, the strategy has been used elsewhere for placement of other structural modifications on plates. For example, the dampening properties of a plate were improved via Active Constrained Layer Damping (ACLD), where certain damping characteristics were achieved by targeting the regions of highest MSE in an untreated plate [33]. Similarly, Park [15] found optimum patterns of grooves on plates by targeting regions of high MSE for the initial formation of the groove. A similar design approach is taken in this study. The example presented in this chapter will
demonstrate that use of the MSE is a feasible approach for placement of dimples on beams.

In Section 6.1, the fundamental frequency of a fixed-fixed beam is maximized using two dimples. This shift in frequency could be used as a means of noise control at a fixed frequency. Two approaches are compared for their ability to reach the global optimum solution (i.e., finding the dimple angles and locations that maximize the fundamental frequency). In the first approach, optimum dimple angles \textit{and} dimple locations are determined using a genetic algorithm (GA). This is similar to the approach taken by previous authors [2]. In the second approach, the dimple locations are chosen \textit{a priori} based on locations of high MSE, and the optimum dimple angles are determined using the GA. Then, the effect of maximizing the fundamental frequency of the beam on the radiated sound power is shown in Section 6.2. Finally, in Section 6.3, a comparison is made between the mode shapes of beams with a single dimple to those of uniform beams. This is of interest since a change of mode shape affects the radiation properties of such structures over a broad frequency spectrum.

### 6.1 Dimple Placement Using Modal Strain Energy

The following example demonstrates the advantage of using the MSE as a targeted strategy for optimum placement of dimples, as opposed to their determination via an optimization routine. Previous studies used an optimization routine (such as a genetic algorithm (GA)) in order to determine optimum dimple angles \textit{and} dimple locations that would minimize or maximize some vibroacoustic objective [2]. This study takes a different approach. The locations of the dimples are chosen at locations of high MSE of a uniform beam with similar boundary conditions. After the dimple locations are selected, the optimum dimple angles are found using a GA [34]. In order
to compare the two approaches, the fundamental frequency of a fixed-fixed beam with two dimples is maximized. This is a relatively simple structure for which the global optimum is determined easily using an exhaustive search. The purpose here is to compare the two approaches by comparing their convergence towards the global optimum solution after a set number of iterations of the GA.

Before we begin, a brief description of the genetic algorithm (GA) is given [34]. The GA is a method for solving constrained and unconstrained optimization problems, where a “population” evolves towards the optimum solution. The algorithm is designed for finding a global minimum. A population of “individuals” are generated randomly by the algorithm and each individual is evaluated with respect to an objective function. The next “generation” of individuals is created using various techniques such as mutation, crossover, migration, and random generation. The latter of the techniques is used in order to protect against potential convergence on a local minimum. Generally speaking, the population mean of the objective function converges towards the best value as the number of generations is increased. After many generations, there is a high probability of reaching a global minimum. The algorithm terminates once a stopping criteria is reached. This could be an imposed time limit (since a GA is computationally expensive), a limit on the number of generations, or when there is no improvement in the best objective function over successive generations.

In the following plots, the objective is to maximize the percentage change in fundamental frequency, $\Delta f_1 \ [%]$, or equivalently, minimize the negative of the percentage change in fundamental frequency, $-\Delta f_1 \ [%]$. Of the 20 individuals in a generation, each represent a set of dimple angles ($\alpha$) and dimple center locations ($x_c$). Each individual yields a specific value of the objective function ($-\Delta f_1 \ [%]$). The average of the objective functions of all 20 individuals is labeled “Mean $-\Delta f_1 \ [%]$” in the plots. The individual with the best value of the objective function is labeled
“Best $-\Delta f_1 \%$.” The algorithm will terminate after 25 generations, after which we will compare how close the solution is to the global optimum (found using an exhaustive search).

The fundamental frequency of a fixed-fixed beam is maximized using two dimples. The beam geometrical and material properties were given in Table 5.1. In Fig. 6.1, the GA is used to determine all four design variables, that is, two dimple angles and two locations of the dimple centers. As shown in Fig. 6.1, the convergence between the mean and best objective function is relatively slow, where the mean best values converge at the 16th generation. In the absence of knowing the global optimum, this convergence is an indication that the optimum has been reached (or is close to being reached). However, there is still a 0.4% difference between the best objective function and the global optimum after the 25th generation.

Figure 6.1: Results of the GA after searching for 4 variables on the fixed-fixed beam (2 dimple angles, 2 dimple locations)
The locations and angles of the two dimples on the fixed-fixed beam (as found after 25 generations) is shown in Fig. 6.2. The GA has placed the dimples nearest to the boundaries (during the search, adjacent dimple ends are constrained from being within 5 mm of each other or the boundaries). Choice of this location is not a coincidence. As shown previously in Fig. 5.20, the boundaries of a fixed-fixed beam are regions of highest MSE. Perhaps unexpectedly, the dimple angles are different from each other in spite of the symmetry of the boundary conditions. This is because at 25 generations, the beam is near to, but not at the optimum configuration. It is expected that if a larger number of generations was allowed, the two dimples angles would be identical.

![Figure 6.2: Fixed-fixed beam with the highest fundamental frequency after 25 generations (4 variables determined using the GA)](image)

As an alternative and a demonstration of our design strategy, the dimple locations are chosen based on regions of high MSE for the uniform fixed-fixed beam. Examination of Fig. 5.20 shows that the MSE is largest at the boundaries, so one
dimple will be placed at each end of the beam (within 5 mm of the boundary). Of the four variables (two dimple angles, two dimple locations), two have now been eliminated by choosing the locations of the dimples. A third variable can be eliminated if we take advantage of symmetry and constrain the two dimple angles to be equal to each other. Now, the GA is used to determine only one variable, which is the dimple angle. The results of the GA are shown in Fig. 6.3. Two observations can be made by examination of Fig. 6.3. First, of the 20 individuals (dimple angles) in the population, one of the dimple angles yielded the global optimum value at the first generation. Second, the mean of the population converges to the best value very quickly in comparison to Fig. 6.1 (i.e. convergence is reached at the fourth generation). The advantage over the previous approach is now apparent: by using the MSE to choose dimple locations, variables are eliminated from the search space, and the convergence towards the global optimum is much faster. In fact, the global optimum was reached
in this approach, in contrast to the previous approach where the results were very close but not exact. The optimum configuration of a fixed-fixed beam that maximizes the fundamental frequency using two dimples is shown in Fig. 6.4. The fundamental frequency of the optimum beam ($f_1 = 60.6$ Hz) represents a 19% increase from that of the uniform beam ($f_1^* = 51$ Hz). The optimum dimple angles are $87.9^\circ$.

![Figure 6.4: A beam with two dimples optimized to achieve a maximum fundamental frequency ($f_1 = 60.6$ Hz, $f_1^* = 51$ Hz)](image)

This simple example has demonstrated the advantage of using the MSE to choose optimum dimple locations before using the GA. Using only two dimples, the fundamental frequency of a fixed-fixed beam is maximized by placing a dimple at each of its boundaries (regions of highest MSE). By eliminating dimple location from the search space, the solution converges to the global optimum much faster as compared to when dimple location is included in the search space. There are, however, some limitations to the approach. If many dimples are required, it may be difficult to determine the optimum locations of all of them since one dimple may occupy the
entire region of largest MSE (see Fig. 6.5). In this case, further investigation is needed. The best approach may be to place dimples on the beam such that the average of the MSE across all dimples is as large as possible.

Figure 6.5: MSE of third mode (fixed-fixed). Dotted lines indicate potential boundaries of regions occupied by dimples.

6.2 Dimpled Beam Acoustics

Two dimples were used in order to maximize the fundamental frequency of a fixed-fixed beam, where the optimum beam is shown in Fig. 6.4. The fundamental frequency of the optimum beam is 60.6 Hz, which is a 19% increase from that of the uniform beam (51 Hz). Shifting the structural natural frequencies is an indirect method for noise control, where the radiated sound power at a discrete frequency (usually corresponding to a natural frequency) is reduced by shifting the natural frequency away from the original frequency. The question here is, how does the radiated sound power of the optimized dimpled beam compare to that of the uniform beam.
The sound power is computed using a quadratic expression written in terms of the surface normal velocity of the beam [22]. First, the discretized beam response (i.e., nodal displacement) is computed using Eq. (4.5), assuming a transverse point force of magnitude 10 N located at $x = 0.15$ m. Then, the velocity is found by multiplying the response by $j\omega$ since the motion is harmonic. The surface normal velocity (complex) at each node is then used to calculate the radiated sound power. We have assumed that the damping loss factor of the beam is 0.02, and that an accurate representation of the beam response at its fundamental frequency is found by using the first 5 modes in Eq. (4.5).

Results for the sound power (plotted using a dB scale, with a reference of $10^{-12}$ Watts) are shown in Fig. 6.6, where the radiated sound power for the optimized beam (Fig. 6.4) is compared to that of the uniform beam in the range encompassing their fundamental frequencies (40–70 Hz). From Fig. 6.6, the radiated sound power at the fundamental frequency of the uniform beam (indicated with a dotted line in the figure) has been reduced by nearly 30 dB. Therefore, the effect of maximizing the fundamental frequency is to simply shift the peak away from a particular frequency, and as a result, lower the radiated sound power at that frequency. Comparing the sound power level at each of the peaks, the sound power level at the fundamental frequency of the optimized beam (60.6 Hz) is slightly smaller than that at the uniform fundamental frequency (51 Hz). Although the reduction is large at 51 Hz, the dimpled beam may radiate more sound power than the uniform beam at other frequencies (e.g., in the range 56–70 Hz).

The cause of this reduction in sound power at the uniform beam fundamental frequency is attributed to a reduction in the volume velocity of the dimpled beam. The volume velocity is a measure of the volume of air displaced by the beam per
second (units: m$^3$/s). Using Eq. (4.5), the volume velocity, $q$, is given by

$$q(\omega) = j\omega \sum_{i=1}^{n-1} S_i \frac{Z_i + Z_{i+1}}{2},$$

(6.1)

where $n - 1$ is the number of discretized elements, $S_i$ is the surface area of the $i^{th}$ element, and $Z$ is the displacement response normal to the surface at nodes $i$ and $i+1$. The absolute value of Eq. (6.1) is taken in order to determine the volume velocity. In Fig. 6.7, the volume velocity of the optimized dimple beam is compared to that of the uniform beam. The shape of the volume velocity is similar to that of the sound power shown in Fig. 6.6. In other words, the reduction in the sound power is caused by a reduction in volume velocity. This is expected, since by increasing the fundamental frequency of the dimpled beam, the response of the dimpled beam at the fundamental frequency of the uniform beam will be smaller.
Finally, how does maximizing the fundamental frequency of the beam affect its radiation characteristics across a broad frequency spectrum? One measure of this is to compute the radiation efficiency of the beam [22]. The radiation efficiency is a ratio of the sound power radiated by the structure to the sound power radiated by a reference structure (the ratio can be greater than one). The reference structure is taken to be a baffled piston with the same surface area and spatially averaged normal surface velocity as the structure of interest. In Fig. 6.8, the radiation efficiency of the dimpled beam in Fig. 6.4 is compared to that of the uniform beam over the range 40–260 Hz. The sound power level of the two structures is also plotted in the same frequency range. Examination of Fig. 6.8 shows that the difference in radiation efficiency between the two structures is small. Within certain frequency bands, (e.g., 80–190 Hz), the radiation efficiency of the dimpled beam is smaller than that of the uniform beam. However, the radiation efficiency of the dimpled beam is higher than
that of the uniform beam within other frequency bands (e.g., 190–260 Hz). The scale in Fig. 6.8 is much less than one, indicating that both the uniform beam and dimpled beam are both poor radiators of sound.

![Graph showing sound power and radiation efficiency](image)

Figure 6.8: Sound power and radiation efficiency of the beam in Fig. 6.4 compared to that of the uniform beam (40–260 Hz)

Minimizing the radiation efficiency across a frequency spectrum is another form of noise control. One way of accomplishing this is by designing a structure in such a way that the structural mode shapes are inherently poor radiators of sound. Naghshineh [22] calculated the weak-radiating velocity profiles of a beam that produce minimum sound radiation. Then by tailoring the material properties of the structure, the structural mode shape at a target natural frequency can be prescribed as a weak-radiating profile. He showed that other natural frequencies also exhibit weak-radiating characteristics, even though these modes were not specifically targeted during optimization of the material properties. A similar approach has been taken by Cheng [21], who used dimples on a beam in order to minimize the radiation...
efficiency. Their study had one drawback - the thickness of the dimples were allowed to be larger than the straight segments, although the overall beam mass remained the same. Once optimization was completed, the dimpled beam was equivalent to a beam whose mass had been redistributed (i.e. “mass-tailoring”). Their approach could be modified by allowing the dimple thickness to become smaller with dimple angle, as this study has assumed (see Eq. (3.2)). If use of dimples is a feasible option for achieving weak-radiating profiles, however, then we must have knowledge of how much dimples can affect the structural mode shapes. This is the topic of the next section, where the mode shapes of a beam with a single dimple are compared to that of a uniform beam.

6.3 Mode Shapes: Dimpled Beam vs. Uniform Beam

In this section, the mode shapes of a fixed-fixed beam with a single dimple are compared to that of a uniform beam (see beam properties in Table 5.1). The similarity of the two mode shapes are quantified using the Modal Assurance Criterion (MAC) [32], given by

$$\text{MAC}_{ik} = \frac{\left(\{\phi_i\}^\top \{\phi_k\}\right)^2}{\left(\{\phi_i\}^\top \{\phi_i\}\right)\left(\{\phi_k\}^\top \{\phi_k\}\right)}, \quad (6.2)$$

where column vectors $\{\phi_i\}$ and $\{\phi_k\}$ are discretized mode shapes of modes $i$ and $k$. The MAC value is always between 0 and 1, where 0 indicates that the mode shapes are orthogonal, and 1 indicates perfect similarity. Note that a scale factor has no effect on the MAC.

In Figs. 6.9–6.12, the MAC is plotted against dimple location (over half the beam) for the first four mode shapes of the dimpled and uniform beams. Plots are made for five different dimple angles ranging from $50^\circ$ to $180^\circ$. For all modes, the dimple changes the mode shape the most when the dimple angle is $180^\circ$, and when
it is located at the boundary of the fixed-fixed beam. As shown in Fig. 5.20, this is where the modal strain energy is highest. Comparing scales between the figures, the scale is small for the fundamental mode shapes (Fig. 6.9), meaning that a single dimple has very little influence on the fundamental mode shape. The scale increases for higher modes, indicating that a single dimple has a larger effect on the mode shapes of higher modes.

We have shown that the dimpled beam mode shapes (single dimple) are not identical to the mode shapes of a uniform beam. The next question is: can a single dimple be used to change the mode shapes “enough” in order to achieve a weak-radiating velocity profile? To answer this question requires research beyond the scope of this study. However, the preliminary results presented above suggest that dimples with large dimple angles placed in regions of high MSE can have a larger and larger effect on the higher mode shapes. It is likely that by adding multiple dimples, we will change the mode shapes significantly. If this is the case, multiple dimples could be used for achieving the desired weak-radiating mode shapes.
Figure 6.9: Modal Assurance Criterion vs. dimple location for a fixed-fixed beam with one dimple (Mode 1)

Figure 6.10: Modal Assurance Criterion vs. dimple location for a fixed-fixed beam with one dimple (Mode 2)
Figure 6.11: Modal Assurance Criterion vs. dimple location for a fixed-fixed beam with one dimple (Mode 3)

Figure 6.12: Modal Assurance Criterion vs. dimple location for a fixed-fixed beam with one dimple (Mode 4)
6.4 Summary

A design strategy was presented that made use of the MSE of a uniform beam to suggest placement of dimples \textit{a priori}. By choosing dimple locations based on locations of high MSE, design variables were eliminated from the search space, and this yielded a better convergence towards the optimum solution during optimization. In Section 6.1, the concept was demonstrated using a fixed-fixed beam with two dimples, where the objective was to maximize the fundamental frequency. Using the GA to determine dimple locations and dimple angles, a solution close to that of the optimum was reached after 25 iterations. However, when using the MSE to set the dimple locations \textit{a priori}, the optimum solution was reached after one iteration.

In Section 6.2, we showed that a consequence of maximizing the beam fundamental frequency was that the radiated sound power was changed. The sound power level was reduced by nearly 30 dB at the fundamental frequency of the uniform beam. However, the dimpled beam radiated more power at other frequencies, making this a narrow-band solution for quieting a structure. The cause of the reduction (or increase) in sound power was identified to be the change in the volume velocity of the structure. The radiation efficiency was also shown across a broad frequency spectrum. The difference between the radiation efficiency of the uniform beam and that of the dimpled beam were small, indicating that more dimples are needed in order to control the radiated noise over a broad band.

Finally, Section 6.3 examined the effect of introduction of a single dimple on the first four mode shapes of a fixed-fixed beam. The degree of similarity (or dissimilarity) between the mode shapes of a dimpled and uniform beam was quantified using the Modal Assurance Criterion. It was shown that a dimple having a large dimple angle and placed in a region of large MSE (i.e., at the boundaries of the fixed-fixed beam)
can have a large effect on the mode shapes of higher modes. This preliminary study suggests that one or more dimples could change the modes shapes of a beam in such a way that it radiates very little sound (i.e., the beam is a weak-radiator). More work is needed in this area since only the fixed-fixed beam was studied in this case.
Chapter 7

CONCLUSIONS AND FUTURE WORK

7.1 Conclusions

The purpose of this study was to gain a better understanding of the vibration and acoustic properties of dimpled beams. To this end, a boundary value model was developed for a beam with any number of dimples and subject to various boundary conditions. We have shown that formulation of this model and its implementation has improved our understanding of the vibroacoustics of dimpled beam.

A boundary value model (BVM) of a beam with any number of dimples was developed using Hamilton’s Variational Principle. In contrast to other studies that have used the finite element method for modeling dimpled beams, this study was the first to develop an analytical model of such structures. The dimpled beam was modeled as an alternating series of straight beams and arches (i.e., dimples) connected to each other. Since dimples are commonly formed via the stamping process, and no mass is added to the structure during their formation, the dimples were assumed to be thinner than those of the straight segments. The differential equations of motion describing the axial and transverse vibrations were derived. Their solution satisfied
the beam boundary conditions, as well as continuity and equilibrium conditions where the straight segments and dimple meet. An important feature in this model accounted for the non-zero extension of the dimple along its circumferential axis (i.e., the dimples are assumed to be “extensional”). As a consequence of this assumption, equilibrium between the straight segments and dimples could be satisfied.

The accuracy of the BVM was checked by comparing its solution to those obtained from other models. The model was first checked for its ability to predict the natural frequencies and mode shapes of uniform beams and of arches. As the dimple angles were made very small, the dimpled beam could be approximated as a uniform beam. As the straight segments were made very small on a beam with one dimple, the dimple or arch could be modeled by itself. In both of these special cases, the results found using the BVM were in agreement with analytical models of uniform beams and arches. The BVM was also used to calculate the natural frequencies and mode shapes of beams with multiple dimples. Excellent agreement was shown by comparison to the finite element models and experimentally measured results of Alshabtat [2], and to the impedance method of Cheng [1]. One advantage of the BVM over the impedance method was its ability to calculate analytical mode shapes. A discretized forced response was also compared to that obtained using the finite element method, and the results were in agreement.

The effect of dimple placement, dimple angle, chord length, thickness, and beam boundary conditions were investigated for their effect on beam natural frequencies and mode shapes. For beams that do not allow axial motion (i.e., axially restrictive boundary conditions), the natural frequencies were shown to increase initially with increasing dimple angle, and decrease once the dimple angle surpassed a critical value. This study showed that this peak in natural frequency was caused by a changing mode shape within the dimple. Previous studies had suggested that
dimple thinning was responsible for the cause. In fact, a peak in a natural frequency is observed regardless of the dimple thickness. The effect of increasing the dimple chord length was to shift this peak in a natural frequency to smaller and smaller dimple angles, where the peak was accompanied by a change in dimple mode shape. A similar mode-switching behavior and its effect on the natural frequencies of arches has been observed in studies on arches, and the BVM confirms this behavior for these structures as well.

The effect of dimple location on beam vibrations was also studied. Dimples placed in regions of high modal strain energy (MSE) exhibited a comparatively large sensitivity to changes in dimple angle in contrast to regions of zero MSE. This observation was in contrast to other authors who suggested that natural frequencies are sensitive to changes in dimple angle when dimples are placed at beam anti-nodes. Coincidentally for some boundary conditions, the anti-nodes of a beam correspond to regions of highest MSE. The correlation was shown by comparison to analytical calculations of the MSE of undimpled (i.e., uniform) beams.

A design strategy for dimple placement was suggested based on placement of dimples at highest MSE of the undimpled (i.e., uniform) beam. The location of dimples could be chosen \textit{a priori}, thereby avoiding the need for an optimization routine to determine their locations. Dimple angles could then be found using any optimization algorithm (e.g., genetic algorithm). The advantage of using the MSE with the GA was demonstrated through optimization of a fixed-fixed beam with two dimples that maximized its fundamental frequency. The MSE-GA approach, as compared to using only the GA to determine all dimple parameters (i.e., dimple angle and location), yielded faster convergence towards the global optimum, and more accurate results after a fixed number of iterations.

It was shown that by maximizing the fundamental frequency of the beam, the
radiated sound power of the dimpled beam was reduced at the fundamental frequency of the uniform beam (i.e., fixed-frequency noise control). The reduction was caused by a reduction in the volume velocity of the dimple beam. However, the optimized beam did not have a large effect on the radiation efficiency. Since a change in radiation efficiency is dependent on the ability of a dimple to change the beam mode shapes, the effect of a single dimple on beam mode shapes was studied. The Modal Assurance Criterion was used for quantifying the difference in mode shapes between a uniform fixed-fixed beam and beam with one dimple for different dimple locations and angles.

7.2 Future Work

This study has led to a better understanding of the vibroacoustics of dimpled beams. However, it raises some additional questions to be addressed in future studies. In the author’s view, two major areas of study should be addressed.

First, the design strategy using the concept of modal strain energy (MSE) is a work in progress. A relatively simple structure of a fixed-fixed beam with two dimples was used in order to demonstrate its advantage when paired with the genetic algorithm (GA) for determining optimized structures. Regions of highest MSE suggest optimum dimple locations for their potential to influence beam natural frequencies and mode shapes. However, if many dimples are used, where are the optimum locations for the remaining dimples? Should the dimples be placed in regions of “next highest” MSE? Answering this question is very useful from a design standpoint. Increasing the number of dimples increases the number of design variables. Any strategy that eliminates design variables from the search space (e.g., choosing dimple locations \textit{a priori} based on examination of the MSE) will save time in the optimization process. It is noted that introduction of dimples onto a beam changes the modal strain energy.
of the beam. A robust design strategy may account for this change, whereby the first
dimple is placed based on the MSE profile of a uniform beam, and all subsequent
dimples are placed based on the “new” MSE profile.

This study also relied on either exhaustive search or the GA to determine
optimum dimple angles. It is possible that this could be avoided in future studies.
As an example, the fundamental frequency of a fixed-fixed beam was shown to peak
at a certain dimple angle. Can the dimple angle at which the peak occurs be pre-
dicted without an exhaustive search or an optimization routine? By analogy, if the
MSE suggests optimum dimple locations, is there an equivalent method that suggests
optimum dimple angles? There are several hints that this is possible. This study
demonstrated that an increase in the dimple chord length caused the peak location to
shift towards smaller dimple angles. This behavior was observed for the fundamental
frequency of a pinned-pinned (or fixed-fixed) arch. Furthermore, the sudden decrease
in the natural frequency was always accompanied by a changing mode shape within
the dimple. Similarly, this mode switching behavior was observed in the studies of
arches. In other words, the chord length of a dimple (or arch) is known to affect the
dimple angle at which a natural frequency reaches its maximum. It is plausible that
by further study of just an arch, which would be the simpler structure as compared
to a dimpled beam, a better understanding of this effect on dimpled beams could be
obtained.

The ability of dimples to create weak radiating structures is a promising area of
research. This has been addressed, in part, by Cheng [21]. However, Cheng allowed the
dimples to become thicker than the straight segments (while maintaining a constant
overall beam mass). Future research should constrain the dimples to be thinner than
the rest of the structure. Then, the preliminary analysis of this study (which analyzed
only a fixed-fixed beam with one dimple) could be extended in order to examine the
effect of multiple dimples on changing the mode shapes of beams. This should be investigated with other boundary conditions as well.

The second major research area should seek to develop an analytical model for the vibrations of dimpled plates. Such a model would be of interest to industry due to their widespread use for their stiffening and acoustic properties. The boundary value model of dimpled beams suggests a way forward. Since the dimpled beam was divided into straight beams and arches, an analogous approach to dimpled plates would be to divide the structure into a plate with a hole, and a partial spherical shell (this would represent the spherical dimple). Each of these structures have been studied separately for their vibration properties. The challenge would be to determine the continuity and equilibrium requirements where the plate and spherical shell meet.

The MSE design strategy described above for dimpled beams could be applied equally well to dimpled plates, possibly shells, and other complex structures. In the latter case, development of an analytical model for dimpled structures of arbitrary shape is likely to require finite element modeling. By targeting regions of high modal strain energy of the complex structure, dimples could be strategically placed such that they optimize the structure for various vibroacoustic objectives. A similar strategy has been used for application of damping patches, and for creating grooves on the structure in order to optimize the structure for vibroacoustic criteria. Such a numerical study on general dimpled structures would demonstrate the use of dimples as a feasible alternative to these approaches.
REFERENCES


APPENDIX A

RELATIONSHIP BETWEEN $B_{ik}$ AND $D_{ik}$
A derivation of Eq. (3.60) is given in this Appendix. Equation (3.60) expresses the relationship between the modal coefficients $B_{ik}$ and $D_{ik}$ of the general solutions governing the tangential and radial vibrations of the dimple.

Either of the coupled differential equations, Eqs. (3.10)-(3.11), may be used to derive the relationship. Using the notation given in Eqs. (3.52)-(3.53), and assuming a harmonic solution, Eq. (3.10) is given by

$$p_i \zeta_i V_i(\theta) + (p_i + 1)V''_i(\theta) + p_i W'''_i(\theta) - W'_i(\theta) = 0.$$  \hspace{1cm} (A.1)

The general solutions for the tangential and radial vibrations are given by

$$V_i(\theta) = \sum_{k=1}^{6} B_{ik} \text{e}^{s_{ik} \theta},$$  \hspace{1cm} (A.2)

$$W_i(\theta) = \sum_{k=1}^{6} D_{ik} \text{e}^{s_{ik} \theta}.$$  \hspace{1cm} (A.3)

Inserting Eqs. (A.2)-(A.3) into Eq. (A.1) yields

$$\sum_{k=1}^{6} \left[ p_i \zeta_i B_{ik} + (p_i + 1)s_{ik}^2 B_{ik} + p_i s_{ik}^3 D_{ik} - s_{ik} D_{ik} \right] \text{e}^{s_{ik} \theta} = 0.$$  \hspace{1cm} (A.4)

For each $k$, the expression in brackets must be zero since $e^{s_{ik} \theta} \neq 0$. Solving for $D_{ik}$ yields

$$D_{ik} = \left( \frac{p_i \zeta_i + (p_i + 1) s_{ik}^2}{s_{ik} - p_i s_{ik}^3} \right) B_{ik},$$  \hspace{1cm} (A.5)

where the quantity in parenthesis is $\lambda_{ik}$ in Eq. (3.60). Note that for an inextensional arch, the parameter $p$ is zero. Therefore, $D_{ik} = s_{ik} B_{ik}$, and this leads to the relationship $W = dV/d\theta$, as required in the inextensional case.
APPENDIX B

DIMPLE ROOT TYPES
Information about the roots of a polynomial can be found by analyzing the coefficients via computation of the discriminant. The characteristic equation of the dimple is given by Eq. (3.56),

\[ s^6 + (2 + p_i \zeta_i) s^4 + (1 - p_i \zeta_i - \zeta_i) s^2 + (\zeta_i - p_i \zeta_i^2) = 0. \]  

(B.1)

The polynomial is converted into a third order by a change of variables, \( z = s^2 \), yielding

\[ z^3 + (2 + p_i \zeta_i) z^2 + (1 - p_i \zeta_i - \zeta_i) z + (\zeta_i - p_i \zeta_i^2) = 0. \]  

(B.2)

The discriminant \( \Delta \) of a third order polynomial with coefficients \( b, c, d \) is given by [35, p.156]

\[ \Delta = 18 bcd - 4 b^3 d + b^2 c^2 - 4 c^3 - 27 d^2. \]  

(B.3)

The discriminant can be either positive, negative or zero. Analysis of the discriminant shows that the six dimple roots fall into one of the four cases given in Table B.1, where \( \zeta_i \) and \( p_i \) are given by Eqs. (3.52) and (3.53). The parameters \( \zeta_L \) and \( \zeta_H \) represent the positive zero crossings of the discriminant plotted against frequency parameter. They are determined by assuming a particular dimple geometry and frequency (i.e., their values are dependent on \( \zeta_i \) and \( p_i \)).

<table>
<thead>
<tr>
<th>Case 1</th>
<th>Case 2</th>
<th>Case 3</th>
<th>Case 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 &lt; ( \zeta_i &lt; \zeta_L )</td>
<td>( \zeta_L &lt; \zeta_i &lt; \zeta_H )</td>
<td>( \zeta_i &gt; \zeta_H )</td>
<td>( \zeta_i &gt; 0 )</td>
</tr>
<tr>
<td>( s_1 )</td>
<td>( ja_1 )</td>
<td>( ja_2 )</td>
<td>( ja_3 )</td>
</tr>
<tr>
<td>( s_2 )</td>
<td>( -ja_1 )</td>
<td>( -ja_2 )</td>
<td>( -ja_3 )</td>
</tr>
<tr>
<td>( s_3 )</td>
<td>( jb_1 )</td>
<td>( b_2 + jc_2 )</td>
<td>( b_3 )</td>
</tr>
<tr>
<td>( s_4 )</td>
<td>( -jb_1 )</td>
<td>( b_2 - jc_2 )</td>
<td>( -b_3 )</td>
</tr>
<tr>
<td>( s_5 )</td>
<td>( jc_1 )</td>
<td>( -b_2 + jc_2 )</td>
<td>( c_3 )</td>
</tr>
<tr>
<td>( s_6 )</td>
<td>( -jc_1 )</td>
<td>( -b_2 - jc_2 )</td>
<td>( -c_3 )</td>
</tr>
</tbody>
</table>
When the dimple roots change cases, the magnitude of the determinant of the coefficient matrix (Eq. (3.61)) becomes a local minimum at the frequency corresponding to the switch. A local minimum is also produced at a natural frequency (i.e., the determinant is zero, or very small at natural frequencies). This switch produces “false” natural frequencies that must be omitted.
APPENDIX C

PLOTTING MODE SHAPES
This appendix explains how to construct a single discretized mode shape of a dimpled beam using multiple eigenfunctions obtained from the boundary value model. The required steps are detailed below.

First, the undeformed beam is discretized into points, as shown in Fig. C.1. The points $X_0$ are uniformly spaced for convenience, and more points can be used for a better approximation of the dimple curvature. In order to plot the dimple, the height $\bar{Y}_0(\theta)$ at each point is needed. It is convenient to express the height as a function of position $\theta$ rather than local x-coordinate. Therefore, the relationship between the local x-coordinates of the dimple $\bar{x}_0$ and the angular coordinates $\theta$ is written as

$$\bar{x}_0 = \frac{l}{2} - R \sin(\alpha/2 - \theta),$$

(C.1)

where $\bar{l}$ is the chord length, $\alpha$ is the dimple angle, and the dimple radius is given by

$$R = \frac{\bar{l}}{2 \sin(\alpha/2)}.$$  

(C.2)

In Eq. (C.1), the position $\bar{x}_0 = 0$ represents the left end of the dimple, and $\bar{x}_0 = \bar{l}$ is the right end. It does not matter whether $\bar{x}_0$ is chosen first and corresponding angular coordinates $\theta$ are calculated, or vice-versa. In this case, uniformly spaced $\bar{x}_0$ were chosen first. Note that this makes the angular coordinates non-uniformly spaced. Then, the angular coordinates $\theta$ are used to calculate the height of the dimple at each point $\bar{x}_0$. In terms of the maximum dimple height, $H$, the vertical height at any point on the dimple is given by

$$\bar{Y}_0(\theta) = \frac{1}{2} \left( H - \frac{\bar{l}^2}{4H} \right) + R \cos(\alpha/2 - \theta),$$

(C.3)

where

$$H = R \left( 1 - \cos(\alpha/2) \right).$$

(C.4)
Equation (C.3) is used in order to plot the undisplaced dimple, as in Fig. C.1.

Next, a coordinate transformation is needed for the eigenfunctions of the dimple. For a dimpled beam, the straight segments have both longitudinal and transverse displacements, $U_i(x)$ and $Y_i(x)$, where $x$ is the local coordinate system of the $i^{th}$ segment. No transformation is needed for the straight segments. However, the dimple eigenfunctions $V_i(\theta)$ and $W_i(\theta)$ require a transformation into a horizontal-vertical coordinate system. The transformation at each angular coordinate $\theta$ is given by

$$
\begin{bmatrix}
\bar{U}_i(\theta) \\
\bar{Y}_i(\theta)
\end{bmatrix} =
\begin{bmatrix}
\cos \phi & \sin \phi \\
-\sin \phi & \cos \phi
\end{bmatrix}
\begin{bmatrix}
V_i(\theta) \\
W_i(\theta)
\end{bmatrix},
$$

where $\bar{U}_i(\theta)$ and $\bar{Y}_i(\theta)$ are horizontal and vertical displacements measured from points on the undisplaced dimple. The transformation matrix represents a clockwise rotation.
of $\phi = \alpha/2 - \theta$, and assumes that downward is positive. Note $\theta = \alpha/2$ at the dimple center, and no transformation is required (i.e., the matrix is an identity).

Finally, local horizontal and vertical displacements are added to the global coordinates of the undisplaced beam in a piecewise manner. For example, the second straight segment in Fig. C.1 has local coordinates $0 \leq x \leq 0.3$, but global coordinates $0.7 \leq x \leq 1$. Therefore, local eigenfunction $U_2(x)$ must be added to the global $X_0$ vector over $0.7 \leq x \leq 1$. In general, global $x$-coordinates of the displaced beam are found by using $X_0 + U_i(x)$ in the straight segments, and $X_0 + \bar{U}_i(\theta)$ in the dimpled segments. Global $y$-coordinates of the displaced beam are found by using $-Y_i(x)$ in the straight segments, and $\bar{Y}_0(\theta) - \bar{Y}_i(\theta)$ in the dimpled segments. The negative signs make the plots positive upwards since downward positive was originally assumed.
APPENDIX D
MASS-NORMALIZING MODE SHAPES
Although mode-shapes can be scaled in any way, it is common to normalize them according to the beam mass. Mass-normalized mode shapes are assumed when calculating the forced response using Eq. (4.5). A dimpled beam mode shape for the \(i^{th}\) mode can be mass-normalized by multiplying it by a constant \(c_i\). For \(N\) dimples and \(N + 1\) straight segments, the modal constant is given by \(c_i = 1/\sqrt{M_i}\) where

\[
\begin{align*}
\sum_{k=1}^{N+1} & \left[ \int_0^{l_k} \rho S (Y_{ki} Y_{kj} + U_{ki} U_{kj}) \, dx \right] + \\
\sum_{k=1}^N & \left[ \int_0^{\bar{l}_k} \rho \bar{S}_k R_k (V_{ki} V_{kj} + W_{ki} W_{kj}) \, d\theta \right] = \\
& \begin{cases} \\
M_i, & i = j \\
0, & i \neq j
\end{cases} \\
\text{(D.1)}
\end{align*}
\]

The case for modes \(i \neq j\) is zero since mode shapes \(i\) and \(j\) are orthogonal.

For example, the \(i^{th}\) mass-normalized transverse mode shape in the straight segment is given by \(Y_{i}^* = c_i Y_i\), and similarly for the other mode shapes. Inserting mass-normalized mode shapes into Eq. (D.1) yields

\[
\begin{align*}
\sum_{k=1}^{N+1} & \left[ \int_0^{l_k} \rho S (Y_{ki}^* Y_{kj}^* + U_{ki}^* U_{kj}^*) \, dx \right] + \\
\sum_{k=1}^N & \left[ \int_0^{\bar{l}_k} \rho \bar{S}_k R_k (V_{ki}^* V_{kj}^* + W_{ki}^* W_{kj}^*) \, d\theta \right] = \delta_{ij} = \\
& \begin{cases} \\
1, & i = j \\
0, & i \neq j
\end{cases} \\
\text{(D.2)}
\end{align*}
\]
All finite element analysis performed in this study uses ANSYS [36]. The analysis uses a 2D elastic beam element with tension, compression and bending capabilities, and 3 degrees of freedom per node (BEAM3). Each node is separated by 0.01 m in the x-direction, although this is adjusted if the straight segment lengths or chord lengths of the dimples become comparable in size. The thickness of the dimple is assumed to be smaller than the straight segments. A modal analysis (ANTYPE, 2) is used to calculate the natural frequencies. After the solution is complete, two text files are created by ANSYS: one contains the natural frequencies, and the other contains the x-y nodal positions and mode shapes for each individual mode. These are used by MATLAB® [37] for further post-processing.

The listings provided in this appendix are an example of the input file to ANSYS (generated by MATLAB) and the code used by ANSYS for batch processing. The steps below can be repeated in an iterative process as follows:

1. MATLAB generates input text file
2. MATLAB calls ANSYS to run in batch mode
3. ANSYS batch code reads the input file
4. ANSYS creates text output
5. MATLAB reads text output

Sample input file (ANSYSinputfile.txt):

```
*SET, alpha1, 1.745329e+000
*SET, D1, 1.000000e-001
*SET, R1, 6.527036e-002
*SET, hd1, 8.778223e-003
*SET, L1, 4.500000e-001
*SET, L2, 4.500000e-001
*SET, b, 5.000000e-002
*SET, h, 1.000000e-002
*SET, E, 1.890000e+011
*SET, rho, 7688
```

Sample ANSYS code for a fixed-fixed beam with one dimple:

```
/BATCH
```
/PREP7
/INPUT, ' ANSYSinputfile ', ' txt ', ' C:\ ansys ', , 0
*SET, pi, 3.141592653589793
*SET, STinc, 0.01
*SET, ARinc, 0.01

K, 1, 0, 0, 0
K, 2, L1, 0, 0
K, 3, L1+D1, 0, 0
K, 4, R1*sin(alpha1/2)+L1, -R1*cos(alpha1/2), 0
K, 5, L1+D1+L2, 0, 0
L, 1, 2
CIRCLE, 4, R1, 3, alpha1*180/pi, 2
L, 5, 6

KDELE, 4
NUMRG, KP, 0.001, , ,
NUMCMP, KP

ET, 1, BEAM3
KEYOPT, 1, 6, 0
KEYOPT, 1, 9, 0
R, 1, b*h, 1/12*b*h**3, h
R, 2, b*hd1, 1/12*b*hd1**3, hd1
MPTEMP, 1, 0
MPDATA, EX, 1, , E
MPDATA, PRXY, 1, , 0.3
MPDATA, DENS, 1, , rho

LSEL, S, LINE, , 1, 4, 3
LESIZE, ALL, STinc
ALLS
LSEL, S, LINE, , 2, 3
LESIZE, ALL, ARinc
ALLS
LMESH, ALL
LSEL, S, , 2, 3
ESLL, S, ALL
EMODIF, ALL, REAL, 2
ALLS
FINISH

/SOL
ANTYPE, 2
EQSLV, SPAR
MXPAND, 10, , YES
LUMP, 0
PSTRES, 0
MODOPT, LANB, 10, 0, 10000, , OFF

! BC on left end
NSEL, S, LOC, X, −0.1, 0
D, ALL, , 0, , , UX, UY, ROTZ, , , ALLS

! BC on right end
NSEL, S, LOC, X, L1+D1+L2, L1+D1+L2+0.1
D, ALL, , 0, , , UX, UY, ROTZ, , , ALLS

SOLVE
FINISH

/POST1
*GET, f1, ,MODE, 1, FREQ
*CREATE, ansuitmp
*CFOPEN, 'C:\ansys\frequencies', 'txt', '
*VWRITE, f1 (F9.4)
*CFCLOSE
*END
/INPUT, ansuitmp

SET, , , , , , , 1
*GET, Nnodes, NODE, , count
*DIM, AA, ARRAY, Nnodes, 5, 1
*VGET, AA(1, 1), NODE, , NLIST
*VGET, AA(1, 2), NODE, , LOC, X, , , 2
*VGET, AA(1, 3), NODE, , LOC, Y, , , 2
*VGET, AA(1, 4), NODE, , U, X, , , 2
*VGET, AA(1, 5), NODE, , U, Y, , , 2
*CREATE, ansuitmp
*CFOPEN, 'C:\ansys\f1_modal_data', 'txt', '
*VWRITE, AA(1, 1), AA(1, 2), AA(1, 3), AA(1, 4), AA(1, 5)
(F4.0, 5X, 5 ES14.5 E2)
*CFCLOSE
*END
/INPUT, ansuitmp
Sample listing `f1_modal_data.txt`:

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.000E+00</td>
<td>0.000E+00</td>
<td>0.000E+00</td>
<td>0.000E+00</td>
</tr>
<tr>
<td>2</td>
<td>4.500E−01</td>
<td>0.000E+00</td>
<td>−2.693E−03</td>
<td>−7.790E−01</td>
</tr>
<tr>
<td>3</td>
<td>1.000E−02</td>
<td>0.000E+00</td>
<td>−5.987E−05</td>
<td>−1.181E−03</td>
</tr>
<tr>
<td>4</td>
<td>2.000E−02</td>
<td>0.000E+00</td>
<td>−1.197E−04</td>
<td>−4.649E−03</td>
</tr>
<tr>
<td>5</td>
<td>3.000E−02</td>
<td>0.000E+00</td>
<td>−1.796E−04</td>
<td>−1.028E−02</td>
</tr>
</tbody>
</table>