What Allows Teachers to Extend Student Thinking during Whole-Group Discussions

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WHAT ALLOWS TEACHERS TO EXTEND STUDENT THINKING DURING WHOLE-GROUP DISCUSSIONS?

Nesrin Cengiz, Ph.D.
Western Michigan University, 2007

Research indicates that extending students' mathematical thinking during whole-group discussions is challenging, even for the most experienced teachers. That is, it is challenging for teachers to help students move beyond their initial mathematical observations and solutions during whole-group discussions. To better understand this phenomena, the teaching of six experienced elementary school teachers, who had been teaching a Standards-based curriculum for several years and had participated in a multi-year professional development project focused on that curriculum, is explored in this study. In particular, two issues are addressed: what it looks like to extend student thinking during whole-group discussions and how teachers' beliefs and knowledge support them in their efforts to extend student thinking.

Classrooms were observed as teachers taught an investigation (several connected lessons) on number and operations. Segments of whole-group discussions that had the potential to extend student thinking were analyzed to gain insight into the focus issues. Semi-structured interviews—before, during, and after teaching the investigation—were conducted to gain insight into the teachers' thinking processes relevant to their actions and to understand the relationship between teachers' knowledge/beliefs and their instructional actions.
All six teachers participating in this study created opportunities to extend student thinking during whole-group discussions by engaging students in problematizing mathematics, mathematical reflection, and mathematical reasoning. In creating these discussions, teachers utilized various instructional actions. Some of the least frequently occurring instructional actions, providing counterspeculation and making connections among representations and contexts, may be among the most effective instructional actions as they were found in the most powerful episodes.

The evidence from this study also suggests that the teachers’ beliefs about the instructional actions they valued were closely related to the prevalent instructional actions that took place during the extending episodes. However, the presence of tension concerning their role during whole-group discussions seemed to weaken some teachers’ ability to extend student thinking. This suggests it may be necessary to have a reasonably harmonious vision to enact these instructional actions in the classroom. Finally, the extent to which the teachers’ knowledge was developed had a clear impact on the powerfulness of the extending episodes.
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Nesrin Cengiz
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CHAPTER I

INTRODUCTION

Statement of the Problem

The extant research on how students learn mathematics prompted the National Council of Teachers of Mathematics' (NCTM, 1989, 2000) call for teaching based on student thinking. That is, teachers are encouraged to establish classroom environments where students discuss, reflect on, and reason about their mathematical thinking so that they can build connections between their existing knowledge and new mathematical ideas (Ball & Bass, 2000; Cobb, Boufi, McClain, & Whitenack, 1997; Hiebert & Carpenter, 1992; Sherin, 2002). This kind of teaching (henceforth called teaching based on student thinking) looks very different from conventional instruction, in which teachers demonstrate mathematical ideas and rules, and students apply these ideas and rules to solve problems.

Teaching based on student thinking places great demands on teachers, such as accessing students’ thinking, being open to students’ ideas and listening to them carefully, and being willing to deal with uncertainty about student responses (Heaton, 2000). Facilitating whole-group discussions is particularly demanding, because it requires teachers to make on-the-spot decisions as students are engaged with activities and discussions and to find a balance between having a classroom environment where
students are encouraged to discuss their own ideas and where students learn specific mathematical content (Sherin, 2002).

It seems that during whole-group discussions, eliciting student thinking becomes a main instructional tool for accessing students’ mathematical ideas, because teachers first need to know what their students are thinking so that the instruction can be built on that knowledge (Martino & Maher, 1999). After having students share their thinking, it is important, and certainly more challenging, to extend student thinking. Helping students move beyond their initial observations about mathematical situations and make sense of these observations requires teachers to have students reflect on and provide justifications for claims; respond to each other’s arguments; look for similarities as well as differences among mathematical solutions, ideas, and concepts; and make generalizations (Martino & Maher, 1999; NCTM, 2000; Wood, 1999).

Studies show that many elementary school teachers in the U.S. still teach mathematics traditionally (Stigler & Hiebert, 1999). That is, mathematics teaching emphasizes rules, procedures, and memorization rather than making sense of mathematics through reasoning (Cohen & Ball, 1990). This is likely due to the fact that changing the practice of teaching is not as simple as one would hope (Heaton, 2000). Initially, teachers tend to make surface-level changes in their instructional practice, such as having students use manipulatives, work in small groups, and describe their solutions to the group (Kazemi & Stipek, 2001; Schifter & Fosnot, 1993). While these changes can be helpful, they are not sufficient for extending student thinking. It is more challenging for teachers to move beyond these surface-level changes to engage students in mathematical inquiry and encourage them to make sense of mathematics through

Beginning in the early 1990s, the National Science Foundation (NSF) funded several curriculum development projects (henceforth called Standards-based curricula) that would support experienced and novice teachers to change their practice in accordance with the recommendations of the NCTM. Standards-based curricula often include thought-provoking tasks that are designed to help students make sense of mathematical ideas through discussions. They also provide teachers with information about the mathematics involved in the tasks and how students learn the mathematics involved in the tasks. For example, some teacher's guides offer information about common student misconceptions and provide suggestions about how to address these misconceptions. While this kind of information is invaluable in supporting teachers, research has shown that teachers using these materials are still challenged to facilitate discussions that effectively extend student thinking (Fraivillig, Murphy, & Fuson, 1999; Grant, Kline, Crumbaugh, Kim, & Cengiz, in press).

There are several studies that examined what is entailed in extending student thinking in the course of teaching mathematics. However, they usually involved teachers who were also researchers (e.g., Ball, 1993; Heaton, 2000; Lampert, 1990), or included teacher-participants who received a great amount of support from researchers for their daily practice (e.g., Wood, Cobb, & Yackel, 1991). These studies have provided insightful information about the mathematical discourse that took place in these teachers' classrooms and the kind of knowledge elementary school teachers need in order to establish desired classroom environments. What remains unclear, however,
is characterizing what these discussions look like in classrooms where teachers may not have day-to-day support, as well as what it is that allows these teachers to be able to create whole-group discussions that have the potential for extending student thinking.

Purpose of the Study

The current study was designed to explore the nature of the segments of whole-group discussions (henceforth called episodes) that have potential for extending student thinking. In this study, extending student thinking is conceptualized as helping students move beyond their initial mathematical observations and further develop an understanding of a mathematical phenomenon. The characterization of the episodes that have potential for extending student thinking centered on the instructional actions that played an essential role in creating these episodes. An additional goal was to gain insight into what factors might influence a teacher’s ability to extend student thinking by focusing on the relationship between teachers’ knowledge and beliefs and their instructional actions.

This study was designed to address the following research questions:

1. What characterizes the episodes that teachers create during whole-group discussions to extend student thinking? What instructional actions are utilized in support of these episodes?

2. What is the relationship between teacher knowledge/beliefs and instructional actions that take place during whole-group discussions?
CHAPTER II

LITERATURE REVIEW

Two major areas of research were drawn upon in designing this study. First, literature on the nature of mathematics for understanding was reviewed. Following that, studies about the factors that influence teachers' instructional actions were examined. While there can be many factors that impact instructional actions, this study focuses only on teacher knowledge and beliefs. The discussion of the results of research in these areas is organized into three main categories: the nature of teaching mathematics for understanding, teacher beliefs, and teacher knowledge.

The Nature of Teaching Mathematics for Understanding

Studies show that providing students with opportunities to explore mathematical ideas and to construct connections between (and within) their existing mathematical knowledge and new ideas helps students learn mathematics with understanding (Hiebert & Carpenter, 1992). It has also been shown that communication and reflection help students construct these connections and, therefore, develop solid mathematical understanding (Hiebert & Wearne, 1993). In particular, focusing communication and reflection on the reasoning behind important mathematical ideas and concepts plays an important role in learning mathematics with understanding (Ball & Bass, 2000, 2003; Lampert, 2001). However, establishing classroom environments in which students are engaged in meaningful discussions about mathematics is a challenging endeavor.
This section begins with a review of several major studies on what teaching mathematics for understanding looks like and continues with a review of studies that illustrates the feasibility of this way of teaching. Finally, the section concludes with specific characterizations of teaching mathematics for understanding.

What Do We Know About Teaching Mathematics for Understanding in General?

The recent reform movement in mathematics education places great emphasis on engaging students in doing mathematics and participating in mathematical discourse about doing mathematics (NCTM, 1989, 2000). Several researchers, such as Lampert (1990) and Cobb et al. (1997), have provided insightful information on why it is important to engage students in doing mathematics and discussing about doing mathematics.

Following Lakatos’ argument, Lampert (1990) suggests that “mathematics develops as a process of ‘conscious guessing’ about relationships among quantities and shapes, with proof following a ‘zig-zag’ path starting from conjectures and moving to the examination of premises through the use of counterexample or ‘refutations’” (p. 30). Drawing on what Polya (1954) considered “moral qualities” required to do mathematics—intellectual courage (being ready to revise our beliefs); intellectual honesty (changing a belief when there is a good reason to change it); wise restraint (not changing a belief without a serious examination), Lampert developed a classroom pedagogy in which engaging in mathematical argument and discourse is a main goal for both the teacher and students. Therefore, she argued that students should be involved in doing mathematics by engaging in problem solving, communication, and reasoning.
Complementing Lampert’s work, Cobb et al. (1997) introduced the notion of “reflective discourse” which they defined as a social process that involves “repeated shifts such that the students and teacher do in action subsequently becomes an explicit object of discussion” (p. 258). Cobb and colleagues suggested that they might have called this kind of discourse as mathematizing discourse because actions and processes are transformed into mathematical objects. Consider an example given by Cobb et al.

First, students engaged in a problem involving five monkeys on two trees. They generated different arrangements of five monkeys on two trees (i.e., 5 and 0, 4 and 1) and the teacher created a table to record possibilities generated by students. Next, students were asked to check whether they listed all possibilities. This prompt shifted the focus of the discourse from sharing possibilities to reflecting on their prior activity. As students participated in a reflective discourse, they used the table to organize the generated combinations and suggested pairing up possibilities involving the same number combinations. After a few weeks, when students were asked to solve a similar problem with bigger numbers, some of the students were able to use a systematic way of generating combinations by creating a table.

Cobb et al. (1997) claimed that reflective discourse has potential for allowing students to learn mathematics by collectively reflecting on prior activity and by organizing the results of prior mathematical reasoning. They further discussed the importance of the teacher’s role during reflective discourse. In particular, they highlighted certain aspects of that role, such as initiating shifts in discourse in a way that allows reflecting on what was previously done in action, and developing symbolic representations of the children’s contributions.
Lampert also investigated, from the teacher’s perspective, what is involved in developing a classroom environment where the discourse of argument and conjecturing takes place and the authority shifts from teacher to students’ sense making (Lampert, 1990, 2001). In particular, she focused on how the teacher and students interact during the course of doing mathematics. Based on the results of those studies, Lampert suggested that students should be provided with tasks that are familiar to them, that would allow them to engage in mathematical questioning, and that expose students to unfamiliar and important mathematical ideas. In her classrooms, through discourse based on problem solving, Lampert attempted to make knowing mathematics in the classroom similar to knowing mathematics in the discipline. She stated:

Learning in my classroom was a matter of becoming convinced that your strategy and your answer are mathematically legitimate. In mathematics, knowing that something is true requires reasoning from agreed-upon assumptions to their logical conclusions in the context of some given conditions, practical or theoretical. (2001, p. 6)

Like Lampert, Ball (Ball, 1991; Ball & Bass, 2000, 2003) paid attention to creating a classroom culture that allows teaching mathematics for understanding. She purposefully kept sense making of mathematics through reasoning as the focus of her teaching. She worked to establish norms that allowed students to provide conjectures, explain their reasoning, discuss and question their own and each other’s thinking, and argue about mathematical claims. However, keeping discussions focused on mathematics and having students actually reason about mathematical ideas can be challenging.

There is no single way to teach mathematics for understanding given the social and responsive nature of it. As Heaton (2000) pointed out, one of the biggest challenges
of this way of teaching is the uncertainty that takes place when students are provided with thought-provoking open-ended problems and when they are encouraged to discuss their thinking. Heaton suggested that having students share their thinking is only a part of the process. In order for teachers to focus their instruction on student thinking, teachers need to listen carefully to their students. Based on how the teacher interprets what he or she hears, the teacher determines what students have learned, what to offer them next, and what ideas to pursue (Martino & Maher, 1999). Therefore, listening plays a critical role in determining the following instructional steps. After hearing students’ thinking, knowing how to respond to student statements is critical. Heaton (2000) stated that this way of teaching involves “a continuous negotiation of moves determined by the situation rather than defined and prescribed in advance” (p. 142).

Another challenge for teachers is to find a balance between using students’ ideas as a basis for discussions and addressing specific mathematical content during discussions. Sherin (2002) characterized this challenge as a “tension between supporting the process of mathematical discourse on the one hand, and the content of mathematical discourse on the other hand” (p. 209). She defined the term process as the ways in which the teacher and the students interact in discussions and the term content as the mathematical concepts that are the focus of the discussions. Some researchers suggest that some teachers manage this tension by focusing on the discourse process and setting classroom norms in the beginning of the school year and focusing on the content of discourse throughout the year (Wood et al., 1991). Sherin (2002), on the other hand, illustrated a situation where the teacher struggled with finding the balance and shifted his focus between the process and content of discourse in his classroom as
the school year progressed. It appeared that the teacher focused on the process of discourse in the beginning of the year. The class spent time on students’ roles during small group and whole-group discussions by using both nonmathematical and mathematical contexts. After a few weeks, the teacher added content to the process. The teacher was able to keep a balance between process and content of discourse for a few months. However, in January and February the focus shifted back to the process as the teacher was concerned with students’ justifications for their methods. The teacher encouraged students to take some new roles about providing convincing justifications during discussions. Towards the end of the school year another shift occurred when the teacher began a unit on algebra. He focused on content of discourse because of his beliefs about algebra as a highly structured domain, and the parents’ expectations about their kids’ learning of algebra. Thus, it appeared that several factors, such as the teacher’s beliefs about student learning and the content, were highly influential in the way he facilitated discussions.

These studies suggest that the social interaction that takes place during mathematical discourse helps students make the individual thought process public and build off of each other’s ideas. As students explain their thinking, provide evidence to prove that their arguments are mathematically legitimate, and question each others’ claims and ideas, they are likely to engage in reflective thought and reorganize their own thinking (Cazden, 2001; Cobb et al., 1997; Lampert, 2001; Lampert & Cobb, 2003). However, establishing a mathematics discourse community places complex demands on teachers.
The Feasibility of Teaching Mathematics for Understanding

Several of the major studies that have investigated the nature of teaching mathematics for understanding have been teaching experiments—where either the researchers themselves teach and analyze the lessons or the classroom teacher receives day-to-day support from researchers to design, teach, and reflect on the lessons. These studies, some of which were discussed in the previous section, illustrate that it is indeed possible to teach mathematics for understanding.

First consider those studies in which the researcher is the teacher. Two of the most well known teaching experiments are those done by Ball (1993) and Lampert (2001). In these studies Ball and Lampert utilized a narrative case-study approach to their year-long experiences teaching third grade and fifth grade, respectively. They incorporated several kinds of data, such as video recordings of lessons, copies of student work, and their journals to capture their own thinking process about their plans and actions. Unlike the typical elementary school teacher, they taught only mathematics. Both studies showed that the teachers were able to establish classrooms where students reasoned about mathematics and made generalizations about mathematical situations.

Martino and Maher (1999) also conducted a study in which they were both researchers and teachers. In this case, the study took place over 10 years and focused on students’ development of mathematical understandings. In addition to classroom videotapes, student work, and teacher journals, several interviews with students following mathematics sessions were conducted to provide detailed data on student
thinking. Similar to Ball's and Lampert's studies, this study provided examples of classroom episodes where students learned to justify their thinking and generalize ideas.

Other major studies of teaching and learning mathematics with understanding have involved the pairing of classroom teachers with mathematics education researchers. Perhaps some of the most famous of these are Hiebert and Wearne's study with six second-grade classrooms (Hiebert & Wearne, 1993), Cobb and his colleagues' Second Grade Classroom Teaching Project (Cobb, Wood, & Yackel, 1990; Cobb et al., 1991; Wood et al., 1991), and Shifter and her colleagues' SummerMath in-service program (Schifter & Fosnot, 1993; Simon & Schifter, 1996).

Hiebert and Wearne (1993) investigated the relationship between teaching and learning in six second-grade classrooms during 12 weeks of mathematics instruction on multidigit addition and subtraction. An alternative to the conventional textbook approach was implemented in two of the classrooms by two specially trained teachers. Hiebert and Wearne designed problem-based tasks to be implemented in these two classes. The tasks emphasized the relationships between place value and strategies to solve addition and subtraction problems. The researchers also met daily with the two teachers to prepare the lessons. In the other four classes, teachers implemented the conventional textbooks. All six classrooms were observed and audiotaped approximately once a week. Hiebert and Wearne developed written assessments to gather information about student learning of place value and addition and subtraction.

Hiebert and Wearne (1993) provided a comparative analysis relating the instructional tasks and classroom discourse to students' achievement. They analyzed instructional tasks by focusing on the number of problems introduced and average time
spent on each one, the contextual and mathematical features of the problems, and materials that were provided to solve the problems. The classroom discourse was analyzed by calculating who talked during the lesson and how long they talked and by coding the type of the questions teachers asked. Not surprisingly, the results of this study showed that differences in the instructional tasks and classroom discourse occurred together. Teachers in the problem-based tasks classrooms were more able to facilitate students’ mathematical learning. Students in these classrooms showed considerable gains in their understanding of place value and computation strategies for whole number addition and subtraction.

The Second Grade Classroom Teaching Project had a slightly different focus. It was designed to both support teachers in creating classrooms that facilitate children’s mathematical learning and investigate students’ learning in these classrooms. The initial study involved a teaching experiment in which the researchers collaborated with one teacher to implement cognitively-based instructional tasks in her classroom. The researchers developed a set of problem-centered instructional tasks based on a constructivist model of how students develop mathematical concepts at second grade and they studied the teacher’s instructional actions. The results of this study showed that there were dramatic changes in the project teacher’s instructional actions by the end of the year.

As a next step, Cobb and colleagues (1990) designed and implemented an in-service program to help teachers build classrooms that allowed students to learn mathematics by constructing their own understanding. The classrooms of teachers who participated in this project were compared with other classrooms. The results of this
study showed that project teachers were more able to focus their instruction on student thinking. In addition, the project students had higher levels of conceptual understanding about mathematical topics covered and held stronger beliefs about the importance of understanding mathematics and collaborating with their peers.

Similarly, The SummerMath Project was designed to help teachers investigate how learning takes place and to support teachers to establish classroom environments where instruction focuses on enhancing student learning (Schifter & Fosnot, 1993; Simon & Schifter, 1996). A group of researchers, including Confrey, Ferrini-Mundy, Simon, Schifter, and Fosnot, designed and directed this in-service program beginning from the early 1980s. The program was originally designed to provide summer institutes to secondary school teachers but was later expanded to include teachers of grades K–8. Teachers were encouraged to become mathematics learners as they discovered important mathematical concepts through problem solving and reflected on their own learning process. Furthermore, they were expected to synthesize their experiences at the institute by writing essays.

During the summer institute, teachers were introduced to thought-provoking mathematical tasks, but they were not given any set of activities to use in their teaching. Instead they were encouraged to think about ways of transferring the information they gained about learning mathematics to their teaching. Recognizing that it would be challenging for teachers to make that transition, the researchers offered a year-long support system in which one researcher was paired with a teacher. The researcher would visit the teacher’s classroom once a week and meet with the teacher following the lesson. During these visits and meetings, the researchers provided opportunities for
reflection on the teachers' goals for and questions about teaching, and occasionally demonstrated teaching.

In order to study the teachers' development and investigate the impacts of the participation in the summer institute and the classroom follow-up support on teaching, Schifter and Fosnot (1993) conducted case studies. A subgroup of teachers who participated in the follow-up support agreed to be subjects of these studies. The data sources included teachers' own journals and other writings, interview transcripts, videotapes of classrooms, and field notes taken by the researchers during the year which Schifter and Fosnot provided follow-up support. These case studies showed that despite the challenges these teachers faced, they were able to create opportunities for extending student thinking.

Though these studies vary in focus—some concentrating more on what it takes for the teacher to establish an environment in which student understanding is central, while others researched the connection between student understanding and instructional actions, together they provide convincing evidence that it is feasible to teach mathematics for understanding. However, these studies also serve to highlight the complexity of teaching mathematics for understanding and the challenges the teachers face in working to establish these classroom environments. In the next section, what we know about specific characterizations of teaching mathematics for understanding and the instructional actions that allow this way of teaching are discussed.
More Specific Characterizations of Teaching for Understanding

Many of the studies described previously include detailed characterizations of what teaching for understanding entails. While some researchers focused on certain aspects of instruction, such as questioning, some others studied what teaching mathematics for understanding looks like in a broader sense and investigated instructional actions that support this way of teaching. This section begins with a review of studies that provided insightful information on teacher questioning and continues with results from studies on instructional actions. Finally, a synthesis of the types of instructional action leads into the discussion of the development of the framework that was utilized in this study.

Teacher Questioning

As part of their analysis of classroom discourse, Hiebert and Wearne (1993) looked at the type of the questions teachers asked during whole-class discussions. They stated that when students are encouraged to “explain the reasons for their responses or define their positions, they will engage in deeper reflective, integrative thought than if they are asked to recall facts or rules” (p. 397). In their analysis, they categorized questions into four types: recalling facts and procedures, describing strategies, generating problems, and examining underlying features. While the first two types of questions help elicit students’ thinking, the other two types of questions help students move beyond their initial mathematical observations and possibly reach reasonable mathematical generalizations.
During their longitudinal study, Martino and Maher (1999) also developed a framework of teacher questioning and they particularly focused on questioning to promote student justification and generalization. As was the case with Hiebert and Wearne’s categorization scheme, Martino and Maher’s framework also included questions designed to both elicit and extend student thinking, such as encouraging students to explain their solution methods, to think about alternative situations, to focus on incomplete parts of their arguments, to find ways to convince their peers that their strategy makes sense, to reflect on each others’ thinking, and to make connections to the prior work has been done.

In addition to distinguishing among teacher questions, Martino and Maher (1999) also highlighted the role of teachers’ careful listening and observing students as they work on problems in creating opportunities for deepening student thinking. Their data demonstrated that there were strong relationships between

(1) a teacher’s monitoring of the process of a student’s construction of a problem solution and (2) the teacher’s posing a timely question which invites or challenges students to revisit earlier thinking, revise it in light of new experience, and, if appropriate, move forward to deeper, stronger understanding. (p. 74)

In the spring of 2006, a pilot study was conducted to explore the types of questions that teachers ask to extend student thinking during whole-group discussions and to identify the contextual features that enable these questions to occur (Cengiz, 2006). During the identified extending episodes, the teachers usually posed questions like, “Can you share how you solved this problem?” “What do you think about her idea?” or “Can you say more about what you were thinking?” Analysis of these episodes showed that simply asking one of these questions did not necessarily result in
extending student thinking. Teacher questions were only part of the context that allowed instances of extending student thinking. The teachers utilized a series of other instructional actions that made the episodes effective, such as restating student responses, having students work on a claim in groups, and having them try out different numbers for that claim. Thus, the next section considers research on a wider array of instructional actions used by teachers during whole-group discussion.

**Instructional Actions**

Lampert’s (2001) book offers one of the most detailed accounts of classroom teaching in general, with a specific focus on highlighting and analyzing the instructional actions utilized during whole-group discussions. Lampert used these discussions to illustrate the problems she faced, including making decisions about creating visual representations of the ideas being discussed, deciding who to call on to begin and continue the discussion, keeping the discussion on track while allowing students to contribute to the discussion spontaneously, and keeping track of time. Another significant problem for Lampert was to decide how to initiate discussions. She suggested that there were three actions to choose from to begin a discussion: choosing the question to begin a discussion, deciding which student’s idea or solution to discuss, and “choosing to give someone who is bidding for the floor an entrée into the discussion” (p. 175). Making choices about how to begin a discussion helps the teacher consider and identify possible resources available in the classroom environment. This kind of thinking also motivates teachers to be aware of students’ current mathematical development.
However, initiating the discussions is only one of the steps. Given the fact that initiating questions do not, in and of themselves, extend student thinking, the ways in which teachers follow-up these questions becomes crucial. As a response to the initial prompts, a student may not be able to fully articulate his or her thinking or students may share other ideas instead of focusing on a specific idea that has already been offered. In either case, the path of least resistance is to allow students to move on to new questions or new ideas (Fraivillig et al., 1999), never moving beyond a surface level understanding. What is challenging is to have students stay focused on a particular idea and to extend student thinking around that idea. Teachers need to make decisions about how to pursue student thinking based on how students respond to the initial questions.

Lampert (2001) offered several teacher moves that can be made following an initial response from students:

- when an assertion is made, choosing to stay with the student who made it and requesting an explanation;
- when an assertion is made, choosing to stay with the student who made it and suggesting my interpretation;
- when an assertion is made, moving to other students and requesting a counterspeculation; or
- when an assertion is made, moving to other students and requesting an explanation.

Whichever of these actions is chosen, the teacher can then continue by:

- asking additional students to comment on another student’s thinking;
• rephrasing a student’s explanation in more precise mathematical terms and asking him or her to comment; or
• creating a representation of the students’ talk on the chalkboard (p. 175).

These actions provide opportunities for both teachers and students to reflect on what they already know about mathematical ideas being discussed and what kind of connections they can construct.

While all of these actions are valuable, a critical factor is deciding when to utilize which action in the midst of a discussion. Lampert (2001) suggests that in making these decisions, the goal should be:

- teaching students both mathematics and how to study mathematics by asking students to reason, to explain, to attend to and to interpret the assertions of others, and by reasoning, explaining, attending, and interpreting the mathematics herself in concert with their responses. (p. 176)

Lampert’s discussion provided insightful information about her perspective as both a researcher and a teacher.

Whereas Lampert’s book provides detailed and insightful information about the range of instructional actions taken by teachers and the decisions that teachers must make in choosing among these instructional actions, it does not provide suggestions for possible frameworks for categorizing these instructional actions. Thus, the remainder of this section focuses on the work of researchers who have suggested particular types of instructional actions.

Based on her analysis of the ways in which a middle school teacher facilitated meaningful discussions that focused on students’ mathematical ideas, Sherin (2002) identified three phases of this teacher’s whole-group discussion. The teacher initiated group discussions by idea generating, that is, he elicited student thinking by asking
questions such as “What do you think?” Then, still in this idea generation phase, he also asked students to elaborate on their ideas by responding “Why?” and “Can you explain that?” questions. Following that, the teacher invited students to provide other ideas about the original task. The next phase was characterized as comparison and evaluation, in which students were asked to compare different ideas or different solution methods and to explain if they agreed or disagreed with others. In the next part, the class narrowed the focus of the discussion and investigated a few ideas in detail by providing reasoning behind their thinking. Sherin called the final phase filtering. Here, what the teacher considered as worthwhile mathematical ideas to have students focus on played a significant role in determining the direction of the discussion. It appears that engaging students in these three phases of activities—idea generating, comparing and evaluating, and filtering—would have potential for having students move beyond their initial mathematical observations, therefore, for extending student thinking.

In related work, Fraivillig et al. (1999) investigated the instructional actions that help advance student thinking as part of a 5-year longitudinal project charting the mathematics achievement of a cohort of students using a Standards-based curriculum. A total of 18 first-grade teachers with 1 to 4 years of experience teaching that particular curriculum participated in the longitudinal study. Of those 18 teachers, the researchers identified 6 teachers as “skillful teachers” based on two classroom observations and also based on project staff suggestions. Those 6 skillful teachers were observed two additional times. Data on the teachers’ lesson planning, classroom practices, and “on-the-fly” decisions were collected. Each teacher was interviewed immediately following each classroom observation to ascertain their thoughts on their teaching and the
rationale behind their instructional decisions. Two researchers were present during each of the four lesson observations, with one researcher focusing on the teacher and the other on the students.

Based on their in-depth analysis of the instruction of one of the six teachers, and follow-up analysis of the instruction of the other five skillful teachers, the researchers developed a framework for Advancing Children’s Thinking (ACT). The ACT framework consisted of three sections: Eliciting Children’s Methods (eliciting), Supporting Children’s Conceptual Understanding (supporting), and Extending Children’s Thinking (extending). The authors considered the instructional actions of “providing opportunity and necessary encouragement to express their [students’] ideas about mathematics” as eliciting (Fraivillig et al., 1999, p. 154). Several examples of these actions were facilitating students’ responses by eliciting many solution methods for one problem and waiting for and listening to children’s descriptions of their solutions.

In their framework, the actions of providing assisted practice as students work in developing a mathematical idea were placed under the supporting category, such as reminding students of conceptually similar situations or providing a demonstration. Both eliciting and supporting instructional actions were focused on “accessing and facilitating children’s thinking about solution methods with which they are already familiar” (Fraivillig et al., 1999, p. 160). The extending student thinking instructional strategies, on the other hand, involved challenging and further developing student thinking. They included having high standards and expectations for all students, encouraging mathematical reflection, and going beyond initial solution methods by
providing reasoning for their actions. It is important to note that the authors recognized and highlighted the overlap between and among these three components. One example they used to illustrate this is that during an elicitation process, for example, a teacher might also support student thinking by reminding students about a definition of a mathematical term.

Beyond the development of this framework, the findings from the analysis of the instructional actions that took place in the six skillful teachers' classrooms showed that it was challenging for these teachers to both elicit and extend student thinking. In fact, only two of the six teachers illustrated eliciting actions, and only one of these two teachers illustrated extending instructional strategies consistently in their classrooms. Given that supporting actions are more closely related to the notion of didactic teaching and teachers are usually more comfortable with this way of teaching, Fraivillig et al. (1999) suggested that observing supporting actions more often was not surprising. In their study of elementary school teachers' use of a Standards-based curriculum that was being used for the first time, Grant et al. (2006) found similar results regarding how challenging it is to extend student thinking during discussions.

In their work on the Second Grade Classroom Teaching Project, Yackel, Cobb, and Wood identified the importance of actions taken to establish norms for mathematical inquiry and argumentation. Yackel and Cobb (1996) stated that “the development of individuals' reasoning and sense-making processes cannot be separated from their participation in the interactive constitution of taken-as-shared mathematical meanings” (p. 460). For example, the project teacher requested that students share different solutions to computation problems during whole-class discussions. This
initiated “a change in the setting from solving the problem to comparing solutions” and thus “the children’s activity extends beyond listening to, and trying to make sense of, the explanations of others to attempting to identify similarities and differences among various solutions” (p. 464). As this reflective activity had the potential to improve students’ mathematical understanding by considering different ways of thinking about a problem, the notion of what counts as a different solution strategy to a computation problem was also taken-as-shared in the class. Similarly, developing notions about what counts as an explanation and justification for mathematical arguments was also encouraged during whole-class discussions.

Wood (1999) further analyzed instructional actions that reflect practices of teaching mathematics for understanding in a classroom that was part of the Second Grade Classroom Teaching Project. Wood focused on the strategies that enabled the teacher to establish a context for argumentation. The researcher defined argument as “a discursive exchange among participants for the purpose of convincing others through the use of certain modes of thought” and argumentation as “an interactive process of knowing how and when to participate in the exchange” (p. 172). Wood claimed that in classrooms where student thinking and reasoning were central, it was common for students to have conflicting ideas. The findings of that study suggest that in order to create a context for argument in a mathematics class, the teachers need to establish certain expectations for students, such as participating in discussions by explaining their solutions to others and becoming active listeners. According to Wood, active listening involves both following others’ thinking and the reasoning behind their claims, and expressing disagreements and reasons for them. Wood highlighted the importance of
having these expectations as part of the classroom norms. Once these classroom norms have been established, students are able to shift their attention from understanding their social setting to understanding the mathematical experiences.

The majority of the studies presented previously took place in classrooms of teachers who are also researchers or teachers who receive day-to-day support from researchers about how to teach mathematics for understanding. This shows that the types of episodes and instructional actions that take place in classrooms where teachers do not receive day-to-day support need to be further studied.

Summary

The studies where researchers investigated the nature of teaching mathematics for understanding provide evidence that it is feasible to teach mathematics this way. However, the findings from these studies also point out the complexity of teaching mathematics to help students construct their own understanding of the subject through making sense of it.

The general findings about teaching mathematics for understanding provide insightful information about what it looks like to teach this way in classrooms. From the findings of these studies, themes emerge that suggest the types of episodes that have the potential for extending student thinking. These episodes include having students make conjectures, having students provide different solution methods for problems, encouraging mathematical argumentation, and encouraging students to question each other's thinking. Several studies also provide an analysis of specific instructional actions that may help support the development of these episodes. Some of these
instructional actions include initiating whole-group discussions with a student’s idea/solution/question, asking students to provide an explanation for their responses, asking for generalizations, providing an interpretation of what students suggest, asking students to rephrase each other’s explanations, asking for counterexamples, creating mathematical representations of students’ responses, and providing mathematical definitions and terms.

Framework for Instruction

As a tool for designing this study, a framework (see Table 1) was developed by utilizing the work of Grant et al. (in press), Fraivillig et al. (1999), Lampert (2001), Wood (1999), and Schifter and Fosnot (1993). The guiding framework consists of two sections: scenarios and instructional actions. The ways in which findings from these studies were incorporated in this framework follow.

In examining teachers’ use of curriculum materials, Grant et al. (in press) revised the ACT framework of instructional strategies (Fraivillig et al., 1999), incorporating Wood’s (1999) suggestions about extending student thinking by creating a context for argument and considering only the instructional strategies that were appropriate for whole-group discussions. Grant et al.’s framework includes the three categories of instructional actions described earlier, eliciting, supporting, and extending (see Appendix A for their framework). Since the current study focuses only on extending student thinking, this part of the framework has been taken as a basis and has been expanded by incorporating the results of several other studies.
In this study, the instructional actions that were categorized in Grant et al.’s framework are being conceptualized as scenarios for whole-group discussions that have potential for extending student thinking. The term *scenario* is used to capture the nature of the imagined segments of whole-group discussions that had potential for extending student thinking and the instructional actions that were utilized in that segment. The episodes, on the other hand, are the realized opportunities of extending student thinking that fit a certain scenario or that present a new scenario to the framework.
The instructional actions that were highlighted in Lampert’s (2001) analysis of her own teaching, in Schifter and Fosnot’s (1993) case studies, and in the pilot study were listed as potential instructional actions that might also be observed in this present study. These changes resulted in the framework shown in Table 1, which was utilized for both data collection and analysis. This framework was also revised and enhanced throughout the study as the data were analyzed. (See Table 8 on p. 58 for the revised framework.)

It is important to note that the scenarios may overlap; that is, one episode can involve aspects of more than one of these listed scenarios. The nature of these scenario categories are also somewhat ambiguous in that they may refer to themes in overall scenarios, but they may also point to specific instructional actions. This lack of clear demarcation is what led to further categorization once it was utilized to analyze the data. This will be addressed in the methodology chapter where the utilization and adaptation of the framework is described.

Teacher Knowledge and Beliefs

A review of the literature of research on factors that impact teachers’ instructional actions indicates that the most common factors studied were teacher knowledge, beliefs, and conceptions. Given the close connection that exists among these three constructs, the distinctions among them are not always clear. One issue is the overlap in the way these constructs have been defined. Plato defined knowledge as “true belief”; similarly, Wilson and Cooney (2002) suggested that someone knows something if he or she believes it and has reasonable evidence to support it. In both cases, the
question of what constitutes “truth,” or reasonable evidence, is uncertain. Further complicating the issue is the fact that some researchers consider beliefs and knowledge as part of conceptions (Philipp, 2007; Thompson, 1992). For example, Lloyd and Wilson (1998) describe conceptions as “a person’s general mental structures that encompass knowledge, beliefs, understandings, preferences, and views” (p. 249). Similarly, Thompson (1992) states, “A teacher’s conception of the nature of mathematics may be viewed as that teacher’s conscious or subconscious beliefs, concepts, meanings, rules, mental images, and preferences concerning the discipline of mathematics” (p. 132). However, Thompson also uses the terms conceptions and beliefs interchangeably in her review of research on teacher beliefs and conceptions. Tirosh (2000), on the other hand, used the term conception interchangeable with knowledge.

Despite these difficulties in distinguishing between these constructs, many studies have treated beliefs and knowledge separately and have shown that both impact instruction. For example, research indicates that teachers’ beliefs about the roles of teacher and student in a mathematics classroom and their views of mathematics as a subject area impact their instructional practices. Similarly, research shows the impact of specific content knowledge. It appears then that considering both beliefs and knowledge, specifically beliefs about the nature of mathematics and learning mathematics in a general sense, and knowledge of specific mathematical content, may allow for a broader characterization of teachers’ instructional actions and the factors that influence them.

The section that follows begins with a review of the major studies on teacher beliefs and knowledge. Both the section on beliefs and the section on knowledge
conclude with a discussion of the frameworks that were developed as a result of the review, for use in this study.

Teacher Knowledge

It is well established that teacher knowledge is one of the main factors that influences mathematics instruction. However, what kind of knowledge teachers need to have in order to help students learn mathematics with understanding has remained uncertain (Ball, Lubienski, & Mewborn, 2001). There have been several different ways of conceptualizing the knowledge teachers need for teaching.

Shulman (1986, 1987) introduced seven categories of knowledge involved in teaching: content knowledge; general pedagogical knowledge; curriculum knowledge; pedagogical content knowledge; knowledge of learners and their characteristics; knowledge of educational contexts; and knowledge of educational ends, purposes, and values. Three of these were categories of subject-matter knowledge for teaching—content knowledge, pedagogical content knowledge, and curriculum knowledge, and the other four were types of general pedagogical knowledge. Leinhardt and Smith (1985), on the other hand, identified two fundamental kinds of knowledge from a cognitive perspective: subject matter knowledge (concepts, algorithms, connections among different procedures, understanding of student misconceptions, and curriculum) and lesson structure (planning and running a lesson, and providing clear explanations).

These categories have been developed and refined by other researchers in various ways as they work to capture the complexity of the knowledge needed for teaching. For example, Grossman (1990) grouped Shulman’s categories into four:
subject-matter knowledge, general pedagogical knowledge, pedagogical content knowledge, and knowledge of context.

This section begins with a discussion of mathematical content knowledge and pedagogical content knowledge, as they have been the focus of many studies in the field. Then more recent notions of profound understanding of mathematics (Ma, 1999) and mathematical knowledge for teaching (Ball, Bass, & Hill, 2004) are discussed. The section ends with a discussion of the framework used for this study, based heavily on the work of Ball and colleagues.

Mathematical Content Knowledge and Pedagogical Content Knowledge

Mathematical content knowledge consists of an understanding of the facts, concepts, and principles in a discipline, as well as the methods of proof and reasoning (Brown & Borko, 1992). Ball (1991) developed a framework for exploring teachers’ mathematical content knowledge by proposing a distinction between knowledge of mathematics and knowledge about mathematics. While knowledge of mathematics refers to knowledge of concepts, ideas, and procedures, knowledge about mathematics entails understandings about doing mathematics. The former is what many consider as content knowledge. The latter includes

what it means to know and to do mathematics, what is the relative centrality of different ideas, what is arbitrary or conventional versus what is necessary or logical, and what is key to having a sense of the philosophical Lizates within the discipline. (Ball et al., 2001, p. 444)

Many researchers believe that knowledge of the subject is fundamental to being able to help someone else learn it. Furthermore, “How well teachers know mathematics is central to their capacity to use instructional materials wisely, to assess students’
progress, and to make sound judgments about presentation, emphasis, and sequencing” (Ball, Hill, & Bass, 2005, p. 14). Studies with expert teachers support the view that when teachers have an integrated conceptual understanding of mathematics, they are able to establish classrooms where students are able to interact with the conceptual nature of the subject (e.g., Ball, 1991; Carpenter, Fennema, Peterson, & Carey, 1988; Lampert, 1989). Therefore, teachers need to have solid content knowledge in order to help their students learn mathematics with understanding. However, studies also show that understanding the content is not sufficient for teaching mathematics with understanding (Fennema & Franke, 1992; Ma, 1999). In particular, studies suggest that teachers do not always know how to use what they know or what they know may not fully assist them in the act of teaching (Ball, 1993; Ball et al., 2001).

The category of pedagogical content knowledge was introduced to the field by Shulman and colleagues. They linked content and pedagogy in pedagogical content knowledge, which is subject matter knowledge for teaching (Shulman, 1986). It includes knowing

the ways of representing and formulating the subject that make it comprehensible to others . . . [also] an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons. (Shulman, 1986, p. 9)

Many researchers emphasize the critical role of pedagogical content knowledge, particularly teachers’ understanding of student learning, in teaching mathematics for understanding (Ball & Bass, 2000; Carpenter et al., 1988; Fennema & Franke, 1992; Ma, 1999; Sherin, 2002). This groundbreaking view of content-specific knowledge for teaching paved the way for constructs discussed in the next sections.
Profound Understanding of Fundamental Mathematics

Recently, several new definitions and terms have been developed to characterize the knowledge teachers need to have in order to be able to facilitate student learning with understanding. In a highly recognized study where Ma (1999) compared Chinese and U.S. elementary teachers’ mathematical knowledge, she introduced the notion of profound understanding of fundamental mathematics (PUFM). Ma defined PUFM as the awareness of the conceptual structure of mathematics and the ability to provide a foundation for students grasp that conceptual structure. According to Ma, PUFM is a connected, curricularly structured, and coherent knowledge of mathematics. She states:

The teaching of a teacher with PUFM has connectedness, promotes multiple approaches to solving a given problem, revisits and reinforces basic ideas, and has longitudinal coherence. A teacher with PUFM is able to reveal and represent connections among mathematical concepts and procedures to students. He or she appreciates different facets of an idea and various approaches to a solution, as well as their advantages and disadvantages—and is able to provide explanations for students of these various facets and approaches. A teacher with PUFM is aware of the “simple but powerful” basic ideas of mathematics and tends to revisit and reinforce them. He or she has a fundamental understanding of the whole elementary mathematics curriculum, thus is ready to exploit an opportunity to review concepts that students have previously studied or to lay the groundwork for a concept to be studied later. (p. 124)

Ma’s study also provided information about when and how PUFM is attained. Chinese teachers attain PUFM as they study teaching materials intensely, as they learn mathematics from colleagues and from students, as they teach mathematics, and as they do mathematics.
While these studies highlight the role of teacher knowledge in teaching mathematics and help us better understand of the nature of knowledge involved in teaching, what mathematical knowledge is required for teaching mathematics needs further study. Examining the practice of teaching itself has potential to provide information about the nature of the knowledge that teachers need for teaching mathematics. In recent years, several researchers have been working on developing a practice-based theory of mathematical knowledge for teaching (e.g., Ball et al., 2004; Hill, Rowan, & Ball, 2005; Hill, Schilling, & Ball, 2004). For example, Ball and colleagues, including mathematicians, cognitive and social psychologists, educational researchers, and those with expertise in teaching practice, have been analyzing a year’s worth of classroom data in order to develop a theory of mathematical knowledge that is entailed by and used in teaching (Ball et al., 2004).

Ball and colleagues defined mathematical knowledge for teaching as the mathematical knowledge used to carry out the work of teaching mathematics. Examples of this “work of teaching” include explaining terms and concepts to students, interpreting students’ statements and solutions, judging and correcting textbook treatments of particular topics, using representations accurately in the classroom, and providing students with examples of mathematical concepts, algorithms, or proofs. (Hill et al., 2005, p. 373)

They further studied the mathematical knowledge involved in this type of work and where and how it is used in teaching.

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1 Ball, herself as a researcher and a teacher, taught a third grade classroom during the entire 1989-1990 school year. The videotapes and audiotapes of the lessons, transcripts, Ball’s lesson plans and reflection notes, and students’ written work have been analyzed.
According to Ball and colleagues, the work of teaching mathematics requires teachers to have a different kind of mathematical knowledge than other people whose work involves doing mathematics, such as mathematicians. For example, mathematicians may use compressed mathematical information in their work, whereas teachers often need to unpack mathematical knowledge in order to be able to manage the development of students' understanding of mathematics (Ball et al., 2004).

Similar to Shulman's (1986) framework for knowledge, Ball and colleagues developed several categories of knowledge that is required for teaching mathematics. They defined the mathematical knowledge that people who have not taught children mathematics would have as common content knowledge (CCK). For example, in solving a simple subtraction problem such as 307 – 168, being able to carry out a procedure is considered as CCK. They suggest that this kind of knowledge is not sufficient for teaching. Teachers further need to have a specialized knowledge of content (SKC) that includes "building or examining alternative representations, providing explanations, and evaluating unconventional student methods" (Hill et al., 2004, p. 16). This kind of knowledge allows teachers to be able to make sense of mathematical activities. They consider these two types of knowledge being closely related to Shulman’s subject matter knowledge.

The third domain of knowledge for teaching mathematics identified by Ball and colleagues is knowledge of students and content (KSC). It includes knowledge about "common student misconceptions and misconceptions, about what mathematics students find interesting or challenging, and about what students are likely to do with specific mathematics tasks" (Ball, Bass, Sleep, & Thames, 2005, p. 3). They further
suggest that teachers also need to have knowledge of teaching and content (KTC). This last category involves knowledge about "instructional sequencing of particular content, about useful examples for highlighting salient mathematical issues, and about advantages and disadvantages of representations used to teach a specific content idea" (Ball, Bass, et al., 2005). The combination of the former two types of knowledge is close to what Shulman described as the pedagogical content knowledge.

All of these studies highlight the importance of knowledge in teaching mathematics for understanding. Investigating teachers' mathematical knowledge for teaching and the role it plays in planning and enacting lessons may help shed light on the instructional decisions made by teachers as they facilitate whole-group discussions of mathematics.

Framework for Mathematical Knowledge for Teaching

One of the goals of this study is to investigate the relationship between teachers' knowledge and their instructional decisions. In exploring this relationship, it was critical to have a framework for mathematical knowledge for teaching. Therefore, the following framework (Table 2) was developed heavily based on the Ball and colleagues' work related to mathematical knowledge for teaching.

According to this framework, teachers who have knowledge about mathematical ideas and solution methods involved in a content area or task are considered as having content knowledge, which is the combination of what Ball and colleagues defined as CCK and SCK. Teachers who are knowledgeable about what to expect from students in a certain content areas are considered as having knowledge of students and content.
Table 2

*Framework for Mathematical Knowledge for Teaching*

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<th>Aspects of Knowledge</th>
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<tr>
<td>Content Knowledge</td>
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<tr>
<td>Mathematical ideas involved in a content area or task</td>
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<tr>
<td>Solution methods involved in a content area or task</td>
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<tr>
<td>Knowledge of content and students</td>
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<tr>
<td>What students know</td>
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<tr>
<td>Common student misconceptions/struggles</td>
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<tr>
<td>Knowledge of content and teaching</td>
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<tr>
<td>Appropriate mathematical language</td>
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<tr>
<td>What counts as evidence for student understanding</td>
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<tr>
<td>Appropriate tools/representations and connections among them</td>
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<tr>
<td>Connections among lessons/what had been discussed in class</td>
</tr>
</tbody>
</table>

Finally, if teachers know appropriate mathematical language, what counts as evidence for student understanding, appropriate tools/representations and connections among them, and how to sequence lessons, this implies that they have knowledge of teaching and content. Teachers who have strong knowledge in all those areas are probably more able to create opportunities for extending student thinking. Therefore, investigating teachers’ knowledge in these areas can help better interpret instructional actions during discussions and make connections between actions and their knowledge for teaching mathematics.
Teacher Beliefs

Teacher beliefs, like knowledge, impact teachers' goals for teaching and their instructional actions. Studies of teacher beliefs have focused on beliefs about the subject of mathematics and/or beliefs about mathematics teaching and learning. The goals of these studies have varied. Some studies focused on describing the nature of teacher beliefs, others investigated the relationship between teachers' beliefs and instructional practice, and yet another group of studies focused on change in teacher beliefs. The methods used in these studies have also varied from case studies to standardized administration of a belief survey. A variety of types of data was utilized, including Likert-scale questionnaires, interviews, classroom observations, and stimulated recall interviews (Thompson, 1992).

In general, studies suggest that teachers' beliefs about mathematics and how to teach mathematics are shaped by their experiences with learning and teaching mathematics (Brown & Borko, 1992). Research also shows that changes in beliefs about teaching usually require an intervention, such as teachers recognizing their students' struggles with understanding some mathematical ideas (Cobb et al., 1990; Thompson, 1992). The research on the relationship between beliefs about teaching and instructional practices reveals mixed results. While some studies show a high degree of correlation between teachers' professed beliefs of teaching mathematics and their instructional practice, others demonstrate sharp contrasts (Thompson, 1984, 1992). Therefore, it is important to investigate both what teachers profess to believe about discussions that have potential for extending student thinking as well as how their facilitation of whole-group discussions reflect those beliefs.
Several studies about beliefs have been highly influential in understanding the nature of beliefs and the relationship between teacher beliefs and instructional actions. Four of these studies that have had the most impact on the field are discussed here: Green’s (1971) study on metaphorical analysis of belief systems, Perry’s (1970) and Belenky, Clinchy, Goldberger, and Tarule’s (1986) schemes that address reliance on authority for knowing, and Ernest’s (1991) categorization of personal philosophies.

Belief Systems

In understanding the role of beliefs in teaching, it is helpful to look at Green’s (1971) characterization of belief systems. Green identified three dimensions of belief systems and he summarized them as follows:

First, there is the quasi-logical relation between beliefs. They are primary or derivative. Secondly, there are relations between beliefs having to do with their spatial order or their psychological strength. They are central or peripheral. But there is a third dimension. Beliefs are held in clusters, as it were, more or less in isolation from other clusters and protected from any relationship with other sets of beliefs. Each of these characteristics of belief systems has to do not with the content of our beliefs, but with the way we hold them. (pp. 47-48)

According to Green, a belief is not held in total independence of other beliefs; there are central beliefs that are more strongly held than peripheral beliefs, and beliefs can be held in clusters, which makes it is possible to hold conflicting beliefs. Furthermore, belief systems are dynamic in nature, that is, they change as the individuals evaluate them based on their experiences.

In addition, Green (1971) addressed the role of evidence in his analysis of belief systems. He explained the distinction between beliefs nonevidentially held with beliefs evidentially held:
When beliefs are held without regard to evidence, or apart from good reasons or the canons for testing reasons and evidence, then I shall say they are held nonevidentially. It follows immediately that beliefs held nonevidentially cannot be modified by introducing evidence or reasons. They cannot be changed by rational criticism. When beliefs, however, are held on the basis of evidence or reasons, they can be rationally criticized and therefore can be modified in the light of further evidence or better reasons. I shall say that beliefs held in that way are held evidentially. (p. 48)

When beliefs are not held evidentially, the person is not able to defend that belief. Furthermore, in the case of being faced with a conflicting or challenging situation, contradictory belief structures are developed when beliefs are held from a nonevidential perspective. For the purpose of the proposed study, it is important to explore that aspect of teacher beliefs by pursuing the reasoning behind their beliefs—for example, whether teachers evidentially believe in teaching mathematics based on student thinking or may play an important role in how they facilitate whole-group discussions.

_Broad Theories of Adult Knowing_

In examining teacher beliefs about teaching mathematics and the role of authority in teaching and learning, several researchers have found Perry’s scheme of a sequence of stages of development useful. Perry (1970) created a scheme for the development of epistemological and ethical positions of a group of male individuals who were students at Harvard during the 1950s and 1960s. Perry grouped the nine positions of the scheme, which college students moved through in order, into four categories: simple dualism (dualism), complex dualism (multiplicity), relativism, and commitment. A person who holds a _dualistic_ view believes that every question can have a right and a wrong answer and that an authority that lies outside the individual decides which one is right. A person with a _multiplistic_ view, on the other hand, believes that
there can be multiple perspectives on a particular problem, and all are equally legitimate. A person who holds a relativistic view also believes that there are many viewpoints, but different from multiplism, not all opinions are equally legitimate. Quality of opinions matter and beliefs can depend on the context of evaluation. Finally, a person at the stage of commitment believes that there are alternative viewpoints about situations and that knowledge is personal and it is structured to interpret experience.

To better understand the philosophy of teaching mathematics, Ernest (1991) provided a discussion of Perry's scheme by relating it to positions in the philosophy of mathematics. He adapted only the first three categories from that scheme to describe positions. Ernest summarized these three personal philosophies as follows:

**Dualistic views of mathematics** regard it as concerned with facts, rules, correct procedures and simple truths determined by absolute authority. Mathematics is viewed as fixed and exact; it has a unique structure. Doing mathematics is following rules. In **Multiplistic views of mathematics** multiple answers and multiple routes to an answer are acknowledged, but regarded as equally valid, or a matter of personal preference. Not all mathematical truths, the paths to them or their application are known, so it is possible to be creative in mathematics and its application. However, criteria for choosing from this multiplicity are lacking. **Relativistic views of mathematics** acknowledge multiple answers and approaches to mathematical problems, and their evaluation depends on the mathematical system, or its overall context. Likewise mathematical knowledge is understood to depend on the system or frame adopted, and especially on the inner logic of mathematics, which provide principles and criteria for evaluation. (pp. 113-114, emphasis added)

In Perry's (1970) study, the participants were all males and did have the common experience of going to the same university, which might have limited the variety of the responses. Building on Perry's scheme, Belenky et al. (1986) studied how women come to know things and they developed a scheme by using a metaphor of voice. Belenky and colleagues interviewed females that came from a variety of backgrounds and found that women's thinking differed in significant ways from those
suggested by Perry. Based on this research, they developed a new scheme consisting of five categories: “silence—having no voice; received knowledge—listening to the voice of others; subjective knowledge—listening to the voice of inner self and questing for self-identity; procedural knowledge—applying the voice of reason in separate or connected ways; and constructed knowledge—integrating voices” (Cooney, Shealy, & Arvold, 1998, p. 312). While Perry and Belenky et al. looked at two different genders and had different goals for their studies, their schemes highlight the issue of relying on an external authority as a source for determining the truth of an idea and incorporating opinions of others.

In analyzing struggles mathematics teachers experience to make changes in their instruction, Wilson and Goldenberg (1998) also adapted Perry’s scheme. Wilson and Goldenberg questioned the idea of moving in an orderly manner through stages in Perry’s scheme and suggested that the four categories represented “not points on a continuum through which we expect mathematics teachers to move on their way to pedagogical perfection, but rather territories which any given practitioner might visit in a given day of teaching” (p. 274). Similar to Perry’s (1970) and Belenky et al.’s (1986) schemes, Wilson and Goldenberg’s framework also highlights the concept of authority.

*Framework for Beliefs for Teaching*

Wilson and Goldenberg’s (1998) framework of beliefs for teaching was utilized in the current study (see Table 3). They used four categories to capture the nature of the beliefs teachers hold: dualistic, pluralistic, extreme relativistic, and experimentalist. According to Wilson and Goldenberg, a person that holds a dualistic view believes that
there is an absolute right or wrong answer to any question and there is an authority that lies outside the individual; a person that holds a pluralistic view is aware of multiple perspectives and various opinions on any given subject and there is some authority that decides which of these multiple ideas is correct; a person that holds an extreme relativistic view believes that there are multiple viewpoints and no perspective is privileged, therefore there is no authority; and a person that holds an experimentalistic view believes that there are many viewpoints and internally constructed authority makes the choices given the circumstances.

Table 3

<table>
<thead>
<tr>
<th>Framework for Beliefs for Teaching Mathematics</th>
</tr>
</thead>
<tbody>
<tr>
<td>View of solution methods for solving mathematical problems</td>
</tr>
<tr>
<td>Dualistic</td>
</tr>
<tr>
<td>Pluralistic</td>
</tr>
<tr>
<td>Extreme relativistic</td>
</tr>
<tr>
<td>Experimentalist</td>
</tr>
</tbody>
</table>

Therefore, teachers who believe that there is only one correct answer to a mathematical problem or there is only correct way of solving a problem and that either the teacher or the textbook is the authority will be considered as holding dualistic views. The remaining categories all assume that there can be more than one way to
solve a problem or view a particular phenomena. These remaining categories differ in how, and if, one distinguishes among these different views. If a teacher believes that the teacher (or some other external authority) decides which way is the best, then that teacher will be considered as holding pluralistic views. If a teacher believes that all ways are equally valid and no one has the authority to distinguish among them, then this teacher will be considered as extreme relativistic. Finally, teachers who hold experimentalist views believe that there are many choices for any situation and that the individual has the authority and the responsibility to make judgments about which choices make more sense given the particular situation at hand.

Note that one implication of the experimentalist view for teaching is that the students must reason about and distinguish among different solution strategies to consider, for example, the relative efficiency of different methods. Thus, teachers who operate as Wilson and Goldenberg’s experimentalists might be more focused on teaching mathematics for understanding and be more successful at creating episodes of extending student thinking.
CHAPTER III

METHODOLOGY

Participants and Setting

*The Curriculum: Investigations in Number, Data, and Space*

Classrooms in which a NCTM Standards-based curriculum, *Investigations in Number, Data, and Space* (henceforth *Investigations*), was implemented were chosen for this study primarily for two reasons. The first was the strong focus in the curriculum on having students develop their own understanding of mathematics. In *Investigations*, a typical investigation lasts about 5 days. The sessions in the investigations are connected and students develop mathematical understanding by working on related problems or projects throughout an investigation.

A typical lesson in *Investigations* begins with a launch where the goal is to have students engage with the task. Following that, the bulk of a lesson is usually spent with small group work where students have a chance to focus on a mathematical problem and discuss their thinking with one or two peers. Finally, the lesson often ends with a whole-group discussion where students share and examine their solution strategies. During this discussion, students have an opportunity to reflect on each other's thinking and communicate their ideas. While every lesson does not have all of these components, because sometimes students need to spend longer time on exploring the mathematics involved in the tasks in small groups, the curriculum does utilize whole-group
discussions as a main way to enhance student thinking through reflection and communication.

The second reason for choosing *Investigations* was the support provided in the teacher’s guide for engaging with students’ thinking. The lesson plans include sections called “Teacher Notes” and “Dialogue Boxes.” The Teacher Notes are designed to help teachers develop an understanding of the content so that they comprehend the main mathematical ideas and the goals of the lesson. These notes provide information about the content, such as the meaning of operations and strategies to solve problems, as well as information about student thinking, such as common student misconceptions. The Dialogue Boxes are designed to help teachers become familiar with some student responses so that they have a vision of what to expect from students during the lesson. They offer real classroom conversations from field-test sites and illustrate sample student responses to the tasks and teacher questioning. This kind of support can help teachers think more deeply about how to facilitate whole-group discussions and work to extend student thinking.

The mathematical focus of all of the investigations for this study was number and operations. There were two reasons for this choice. First, there is a great deal of research on student learning of number; therefore, the teacher guides for the number units tend to contain more detailed, research-based information. Second, number is the strand that elementary school teachers spend the most time on, and one with which the administration and parents are most concerned. These factors may help teachers be able to focus more on student thinking and create episodes that have potential for extending student thinking.
Participants

A total of six elementary school teachers (grades 1-4) participated in this study (see Table 4). The teachers were chosen from a larger group of teachers who had been teaching *Investigations* for several years, had participated in extensive professional development (both as teachers and as teacher-leaders), and had facilitated professional development sessions themselves. There were several reasons for choosing these teachers for this study. Because of their experience with participating in, and facilitating, professional development, and teaching *Investigations* for several years, these teachers are likely to have strong beliefs about teaching mathematics for understanding and solid mathematical knowledge for teaching. These beliefs and knowledge should enable teachers to provide opportunities for students to make sense of mathematics. Therefore, these classrooms were optimal sites for the current study to

Table 4

Participant Information

<table>
<thead>
<tr>
<th>Teacher</th>
<th>Grade</th>
<th>District Type</th>
<th>Years of Teaching <em>Investigations</em></th>
<th>Years of Teaching the Same Grade Level</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linda</td>
<td>1st</td>
<td>Urban</td>
<td>8</td>
<td>3</td>
</tr>
<tr>
<td>Melissa</td>
<td>1st</td>
<td>Rural/Suburban</td>
<td>11</td>
<td>10</td>
</tr>
<tr>
<td>Kasey</td>
<td>2nd</td>
<td>Rural</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>Judy</td>
<td>3rd</td>
<td>Rural</td>
<td>11</td>
<td>11</td>
</tr>
<tr>
<td>Liz</td>
<td>3rd</td>
<td>Suburban</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>Meredith</td>
<td>4th</td>
<td>Suburban</td>
<td>12</td>
<td>2</td>
</tr>
</tbody>
</table>
gain information on instructional actions that have potential for extending student thinking and also to explore the relationship between teachers' knowledge/beliefs and their instructional actions.

Data Collection

This study was designed to characterize the episodes created by teachers to extend student thinking during whole-group discussions and to explore the relationship between teachers' knowledge/beliefs and their instructional actions. Two kinds of data were collected to address these goals: classroom observations and teacher interviews.

The observational data were collected from six elementary school teachers' classrooms, in which Investigations was taught. The observations occurred throughout the course of teaching one investigation from a number unit. From each investigation, the sessions that had potential for having whole-group discussions were selected for classroom observations. Then each teacher was observed two to five times during teaching an investigation about number and operations, more specifically whole number addition and subtraction in 1st, 2nd, and 3rd grade, and multiplication and division in 4th grade.

Interviews were conducted in order to understand teachers' beliefs and knowledge about mathematics and learning and teaching mathematics, to gain insight into teachers' thinking processes relevant to their actions, and to understand the relationship between teachers' knowledge/beliefs and their instructional actions. There were three kinds of semistructured interviews with teachers: a pre-investigation interview, post-lesson interview(s), and a post-investigation interview. The
semistructured nature of the interviews allowed probing beyond their responses to the prepared questions, therefore gaining richer insights into teachers’ perspectives on episodes of extending student thinking. (See Table 5 for the data collection timeline and Table 6 for the data collection timeline for all six teachers.)

Table 5

Data Collection Timeline for Each Teacher

<table>
<thead>
<tr>
<th></th>
<th>Pre-Investigation Interview</th>
<th>Lesson Observation</th>
<th>Post-Lesson Interview</th>
<th>Post-Investigation Interview</th>
</tr>
</thead>
<tbody>
<tr>
<td>Day 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Day 2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Day 3</td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>Day 4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Day 5</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Day 6</td>
<td></td>
<td></td>
<td>As Needed</td>
<td></td>
</tr>
</tbody>
</table>

Before the data collection began, an initial analysis of the nature of the tasks in the investigations was conducted. This analysis focused on identifying significant mathematical ideas and solution methods involved in the tasks and the sections of the investigations that had the potential for mathematically rich discussions. Consider the third grade lesson, “School Days.” In this lesson, students are to determine whether they spend more days in school or out of school. There is potential for students to discuss several important mathematical ideas and solution methods in this lesson. For example, students become familiar with the concept of annual calendars, that is, they learn that
Table 6

Data Collection Timeline for All Teachers

<table>
<thead>
<tr>
<th>Teacher</th>
<th>Grade</th>
<th>Time</th>
<th>Number of Observations</th>
<th>Unit/Investigation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linda</td>
<td>1&lt;sup&gt;st&lt;/sup&gt;</td>
<td>Nov 19 – Dec 13</td>
<td>5</td>
<td>Building Number Sense Investigation 4: Addition and Subtraction</td>
</tr>
<tr>
<td>Melissa</td>
<td>1&lt;sup&gt;st&lt;/sup&gt;</td>
<td>Nov 9 – Dec 5</td>
<td>5</td>
<td>Building Number Sense Investigation 4: Addition and Subtraction</td>
</tr>
<tr>
<td>Kasey</td>
<td>2&lt;sup&gt;nd&lt;/sup&gt;</td>
<td>March 8 – 15</td>
<td>4</td>
<td>Putting Together Taking Apart Investigation 1: Combining and Separating</td>
</tr>
<tr>
<td>Judy</td>
<td>3&lt;sup&gt;rd&lt;/sup&gt;</td>
<td>Feb 13 – 21</td>
<td>5</td>
<td>Combining and Comparing Investigation 5: Calendar Comparisons</td>
</tr>
<tr>
<td>Liz</td>
<td>3&lt;sup&gt;rd&lt;/sup&gt;</td>
<td>April 18 – 25</td>
<td>3</td>
<td>Collections and Travel Stories Investigation 3: Finding the Difference</td>
</tr>
<tr>
<td>Meredith</td>
<td>4&lt;sup&gt;th&lt;/sup&gt;</td>
<td>April 19 – 26</td>
<td>2</td>
<td>How Many Packages? How Many Groups? Investigation 3: Solving Division Problems</td>
</tr>
</tbody>
</table>

there are 12 months and 365 days in each year (366 days in a leap year). Students also develop addition and subtraction strategies to find the total number of school days and non-school days. While some students might use units of time such as weeks and months to calculate these numbers, some others might count all the school days and non-school days by one. Also, if students recognize that they are either in school or out of school, and therefore that these two sets of days do not have any common days and these two parts make a whole year, then they can easily find the number of non-school
days by subtracting the number of school days from 365 (or vice versa) instead of calculating the number of non-school days separately.

The information about the tasks from this kind of analysis guided both the interviews and classroom observations. The ways in which this analysis was utilized in data collection and data analysis are discussed in the following sections.

Pre-Investigation Interview

Data collection began with a pre-investigation interview with teachers. (See Appendix B for the interview protocol.) The goal of this interview was to gain information about the teachers' teaching background, their beliefs about mathematics and teaching mathematics for understanding, and their mathematical knowledge for teaching about the particular investigation(s) that they were going to teach during classroom observations. The first two parts of this interview (background and beliefs about teaching) were uniform to all teachers, and the last part was specific to the investigation that was taught by individual teachers.

The first section of the interview was designed to capture the teaching background of the teachers and their view of the curriculum. The next section focused on teachers' beliefs about the nature of mathematics and their vision of effective whole-group discussions, as well as their perceived strengths and weaknesses in facilitating discussions. The last section focused on the investigation. The initial task analysis provided the foundation to pursue teacher responses in a way that would explore their mathematical knowledge for teaching. Thus, teachers' knowledge about the important mathematical ideas and solution methods, and student thinking that was expected to
surface during the investigation(s) were explored. This kind of information was critical in interpreting teachers' instructional actions and analyzing the relationship between teachers' beliefs and knowledge and their instructional actions.

Classroom Observations

From each investigation, the sessions that were likely to have whole-group discussions were selected for classroom observations. Then, each teacher was observed and videotaped two to five times while teaching an investigation about number and operations (see Table 6 on page 50 for the number of observations that were made for each teacher). The video camera followed the teacher during the observations to capture what she said and did during whole-group discussions, as well as how the students contributed. Although interactions that had potential to extend student thinking might have occurred throughout the lesson, including times when students were working on the tasks, this study focused only on whole-group discussions.

The initial analysis of the tasks guided the classroom observations. Knowing the goals of the tasks and the potential they had for extending student thinking played an important role in identifying realized opportunities to create desired episodes. This also helped make sense of the teacher actions that took place during whole-group discussions. Notes on the episodes in which student thinking was extended were taken during the classroom observations. These notes were also utilized to help focus post-lesson interviews as well as the post-investigation interviews.
Post-Lesson Interview(s)

Following select classroom observations, brief post-lesson interviews were conducted when it was possible (see Appendix C for the interview protocol). Due to the teachers' schedules, it was not possible to have post-lesson interviews each time an extending episode had been identified during a classroom observation. The number of post-lesson interviews conducted with each teacher is shown in Table 7.

Table 7

<table>
<thead>
<tr>
<th>Teacher</th>
<th>Number of Post-Lesson Interviews</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linda</td>
<td>0</td>
</tr>
<tr>
<td>Liz</td>
<td>0</td>
</tr>
<tr>
<td>Melissa</td>
<td>1</td>
</tr>
<tr>
<td>Meredith</td>
<td>1</td>
</tr>
<tr>
<td>Kasey</td>
<td>2</td>
</tr>
<tr>
<td>Judy</td>
<td>3</td>
</tr>
</tbody>
</table>

Conducting the post-lesson interview provided the opportunity to gain insight into the teacher's perspective right away, rather than waiting till the end of the investigation. While it is impossible to capture the exact thought processes teachers go through in the act of teaching, the goal of these interviews was to gain an understanding of teachers' intentions and rationales behind their instructional actions while these memories were relatively fresh. In conducting these interviews, the notes taken during
observations indicating the episodes where student thinking was pursued were utilized. Teachers were asked to briefly discuss a particular idea or strategy that was pursued, provide their perspective on why that idea/strategy was pursued, and describe briefly which instructional actions they deemed effective in that pursuit. Although post-lesson interviews were planned to be conducted whenever an observed lesson contained an episode that had the potential for extending student thinking, such interviews were not always possible due to teachers' schedules.

*Post-Investigation Interview*

After all classroom observations were completed for a particular teacher, a post-investigation interview was conducted. While some sections of this interview remained the same for all participants and addressed teachers' overall sense of how the investigation went and whether their instructional goals were accomplished, other sections were tailored towards a specific episode chosen by the researcher. (See Appendix D for the interview protocol.)

Several extending episodes occurred in each teacher's classroom; however, an attempt was made to choose only one episode for each teacher to reflect on during the post-investigation interviews due to the time constraints. In choosing those episodes, the researcher considered the nature of the extending episodes and the teachers' responses from the pre-investigation interviews. In terms of the nature of the extending episodes, those that seemed to fall outside of the categories from the original framework for scenarios were deliberately chosen so that the framework could be expanded. In terms of the pre-investigation interviews, an effort was made to choose episodes that included
instructional actions that were closely related (either directly or inversely) to what had been discussed during the pre-investigation interviews. Finally, for two of the teachers, two extending episodes were chosen because the two episodes from each teacher took place during the course of one session and both of the teachers had extra time to reflect on more than one episode during the post-investigation interviews.

The tape of selected episode(s) was reviewed by the teachers during the interview to help them recall their thinking about their instructional actions and the factors that might have impacted their decisions during those episodes. Discussions concentrated on what ideas teachers pursued during these episodes, their reasons for pursuing these ideas, and how they made the decisions about their instructional actions. If teachers suggested that they would like to do things differently, then they were also asked why they thought this didn’t occur to them at the time. The answers to these questions provided valuable information about both the thinking teachers experienced in the midst of whole-group discussions and their beliefs about mathematics and learning and teaching mathematics and their mathematical knowledge for teaching.

Data Analysis

Qualitative analysis methods were utilized to understand the nature of the segments of whole-group discussions that had the potential to extend student thinking and the relationship between teachers’ beliefs and knowledge and the instructional actions that took place during these discussions. The analysis of the instructional tasks and the videotapes of the lessons was used to address the first research question, *What characterizes the episodes that teachers create during whole-group discussions to*
extend student thinking? What instructional actions are utilized in support of these episodes? The interview data were the main source for answering the second research question, *What is the relationship between teacher knowledge/beliefs and instructional actions that take place during whole group discussions?* However, videotapes of classroom observations were also utilized. All interviews and the portions of the classroom videotapes (capturing the extending episodes) were transcribed. Data analysis was an ongoing process throughout and after data collection, and it proceeded in several steps.

*Initial Analysis of the Pre-Investigation and Post-Lesson Interviews, and Videotapes*

Since the pre-investigation and post-lesson interviews were not usually transcribed by the time the post-investigation interviews were conducted, there was an initial level of analysis of these interviews and the videotapes to identify the episodes before the post-investigation interview. After the pre-investigation interviews were conducted, teachers’ beliefs about mathematics and their views on teachers’ and students’ roles during whole-group discussions were briefly noted. Also, teachers’ knowledge about important mathematical ideas and different solution methods involved in the tasks, and the expected student misconceptions or struggles and how teachers would address these issues were recorded.

Right after each classroom observation (and post-lesson interview if it occurred), the researcher reviewed the videotape and identified potential episodes for extending student thinking by utilizing the observation notes and the framework for the scenarios for extending student thinking. Teachers’ comments on student ideas that
were pursued during whole-group discussions from the post-lesson interviews were also noted. These notes were utilized to choose which episode to reflect on with the teacher during the post-investigation interview.

*Analysis of Extending Episodes and Instructional Actions*

As mentioned above, the analysis of the episodes began prior to the completion of data collection. The initial analysis of videotapes was utilized to identify which episodes to discuss with teachers during post-investigation interviews. In identifying the episodes that had potential for extending student thinking, the original framework for scenarios to extend student thinking was used. However, since an episode is composed of a series of exchanges between the teachers and students, parts of episodes sometimes involved aspects of different types of scenarios.

A total of eight extending episodes were identified from six classrooms. At least one episode that fit in the original framework was observed from each of the six teachers’ classrooms. There was only one episode that did not fit in that framework. In order to cover that case, the framework was expanded by adding one more scenario to the existing list: *encouraging students to consider the reasonableness/validity of a claim* (see the revised framework in Table 8).

The teacher actions that took place during extending episodes were coded by using the second column (instructional actions) of the framework. Throughout the coding process, the instructional actions section of the original framework was revised in two ways: by adding new components and by combining some types of instructional actions from the original framework under one category. There were two instructional
actions that took place in the identified extending episodes that were not in the original framework. Therefore, the framework was expanded to include these two actions:

*inviting students to share methods*; and *reminding students of the goal of the discussion, the problem, or other information*. Also, it appeared that having students reflect on a mathematical claim/justification and having them analyze mistakes were not necessarily distinct categories. Therefore, these two types of instructional actions from the original framework, which were *inviting students to reflect on a claim/justification* and *inviting
students to analyze mistakes, were combined, and a new category of instructional action was added to the framework: inviting students to evaluate a claim or an observation.

Another alteration that was made to the original framework was to delete inviting students to analyze different representations and using mathematical representations, and to add introducing different representation/contexts/solution methods instead. There were several reasons for making this final modification. First, the teacher moves of inviting students to analyze different representations were captured by the new component of inviting students to evaluate a claim or an observation. Second, using mathematical representations appeared to be a vague phrase and using it to code instructional actions was not revealing any patterns about the data. Also, the new component, introducing different representation/contexts/solution methods, not only captured the moments where teachers introduced different representations, but it also revealed information about the moments where teachers introduced new contexts and solution methods.

After the coding of the instructional actions was completed, a summary analysis of each episode was written by focusing on the role of each instructional action. It became apparent that there were three categories of instructional actions that occurred in these episodes in terms of the role of the actions: eliciting, extending, and supporting. Finally, these categories were incorporated into the original framework of instructional actions to result in the final revised framework (see the revised framework in Table 8).

It is important to note that these were also the three components from Fraivillig et al.'s (1999) framework for instructional strategies for advancing children’s thinking. However, it is also important to recognize that the way they conceptualized...
instructional strategies differed from the way the instructional actions were described in this study. That is, some of the components of Fraivillig et al.'s framework involved more than one individual teacher move, whereas in this study, each single teacher move was considered as an instructional action. This was what prompted the addition to the framework of the new section called scenarios in the first place.

In sum, three types of individual teacher moves, eliciting, supporting, and extending, took place in the extending episodes, which were segments of whole-group discussions that involved several of those instructional actions. The ways in which these three types of single instructional actions supported extending episodes are discussed in the next chapter.

Once the data were categorized using the revised framework, frequencies of the actions from each episode were compiled along with totals across all episodes in order to search for trends in least and most frequently occurring instructional actions (see Appendix E). In addition, actions that took place in all of the episodes or actions that took place in only one of the episodes were identified in order to make conjectures about the connections between teachers' beliefs and knowledge and their instructional actions.

After categorizing the extending episodes by utilizing the revised framework, the purpose of the whole-group discussions was compared and contrasted across episodes. This process provided a different framework than the earlier one. Two distinct categories emerged: building new connections and addressing student misconceptions or struggles. Interestingly, these two themes, building connections among concepts and utilizing struggles, are also the only two instructional approaches identified by Hiebert.
and Grouws (2007) in their recent review of research that positively influences student achievement.

During the analysis of the nature of the episodes, it also became apparent that in some of the episodes there was convincing evidence for student thinking being extended as the teachers in those episodes created opportunities for student thinking to be publicly shared. Also, in some of the extending episodes teachers illustrated instructional actions in such a way that seemed more powerful in terms of extending student thinking.

Noticing these kinds of distinctions among the identified eight extending episodes led the researcher to rank the episodes. Therefore, the relative powerfulness of the episodes was determined by considering both the nature of the actions teachers utilized in the episodes and the evidence that student thinking was extended. Students' verbalized responses during the discussions were utilized as evidence for thinking being extended. However, ranking all eight episodes at once did not seem possible because of the complex nature of the episodes. Therefore, four episodes in each of the two categories, building new connections and addressing misconceptions and struggles, were ranked. This kind of sorting revealed insightful information about both the types of the instructional actions that helped make an extending episode powerful and also about the ways in which teacher knowledge and beliefs may have supported teachers to utilize particular instructional actions that took place in more powerful episodes.
Analysis of the Interviews

The interviews were analyzed to understand the relationship between teachers’ beliefs/knowledge and their instructional actions during whole-group discussions. All interviews were transcribed. The software HyperResearch was utilized for coding the transcripts. In order to stay consistent across teachers, beliefs and knowledge were coded separately. That is, all teachers’ beliefs about the nature of mathematics were coded first. All teachers’ beliefs about learning and teaching mathematics were coded next. Teachers’ content knowledge, KCS, and KCT were coded in the same manner.

Coding Beliefs

First, the statements related to teachers’ beliefs about mathematics were identified from all interviews. These statements were not content-specific; rather they were about mathematics in a general sense. All statements about beliefs on the nature of mathematics from all interviews were coded by using these two codes: making sense, and more than one way. Statements that implied that there was reasoning behind mathematical concepts were coded as making sense, such as, As a class we try to look at whether their explanation makes sense mathematically, whether it makes sense in situation. Statements about having more than one way to solve problems were coded as more than one way such as, In general, there is one right answer for most problems we do, just lots of different ways to get there.

Next, the statements related to teachers’ beliefs about learning and teaching mathematics and their views on the teacher’s and students’ roles during whole-group discussions were identified. These statements were coded by using the following codes:
how students learn mathematics; role of student thinking; students’ role; teacher’s role; strengths; weaknesses, and challenges of facilitating discussions.

Coding Knowledge for Teaching Mathematics

After statements on beliefs were coded, the statements related to the mathematical content of the investigations were identified. Then, these statements were categorized into two groups: statements merely about content versus statements about learning and teaching that specific content. The statements that were only about the content were coded as content knowledge. They included statements about mathematical ideas and solution methods involved in the investigations.

Next, the transcripts were coded for KCS—that is, the teachers’ knowledge of how students learn mathematics. All transcripts were searched for the content-specific statements that matched the following codes: how students learn specific content; what students know about the specific content; and student struggles and misconceptions.

Finally, teachers’ statements related to KCT was coded. Statements about design of instruction were identified and coded by using the following categories: what to pursue; how to pursue; appropriate level of challenge; appropriate language; evidence for understanding; tools and representations; connections among grade levels; and lessons in the curriculum.

Interpreting the Codes About Beliefs and Knowledge

Following the process of coding statements about teachers’ beliefs and knowledge, the statements that had the same codes were compiled together. The themes
that emerged from these statements were documented in a summary analysis. Statements that supported the identified themes as well as statements that conflicted with them were searched throughout the transcripts. These findings were then organized around the frameworks about teachers' beliefs and mathematical knowledge for teaching and a summary analysis was written about each teacher.

The summary analysis on beliefs had two main parts: view of mathematics and view of learning and teaching mathematics. Teachers' beliefs about the nature of mathematics were summarized around the theme there is more than one solution to solve problems. The summary of their beliefs about learning and teaching mathematics were centered around three themes: students learn mathematics by constructing understanding; instructional actions teachers valued; challenges in enacting instructional actions.

The section on teachers' knowledge included the three domains of knowledge for teaching mathematics and also some subparts for each domain. The teachers' content knowledge was documented into two categories: mathematical ideas and solution methods involved in the investigation.

Teachers' KCS was documented in two sections: what students knew about the specific content and easier and more challenging ideas for students to understand. The codes for KCT were categorized into four groups and a summary analysis was written by using those groups: appropriate language; evidence for student understanding; connections among representations; and connections among lessons.
Interpreting the Relationship Between Teachers’ Beliefs and Knowledge and Actions

To gain an insight into the ways in which teachers’ beliefs and knowledge supported their instructional actions, teachers’ beliefs about valued instructional actions and the most and least frequently observed instructional actions were compared. Teachers’ beliefs about valued instructional actions were also compared and contrasted with the actions that took place in the most powerful episodes. In addition, the instructional actions of the two teachers who expressed a tension in their beliefs about the goals of teaching mathematics and their role as teachers, and their beliefs about valued instructional actions, were compared and contrasted. Finally, the connections between the teachers’ three domains of mathematical knowledge for teaching and the instructional actions that took place during the most and least powerful episodes were identified.
CHAPTER IV

THE NATURE OF EXTENDING EPISODES

This study was designed to explore two key issues about teaching mathematics for understanding: what extending student thinking during whole-group discussions looks like, and what it takes for teachers to be able to create opportunities for extending student thinking during those discussions. The results of this study are presented in two chapters. In this chapter, a discussion of the characteristics of the extending episodes is presented, with a focus on the commonalities and differences among instructional actions that took place in support of creating these episodes. In the next chapter, the results about the relationship between teacher knowledge and beliefs and their instructional actions are discussed. While this chapter provides a picture of the episodes and instructional actions portrayed from the researcher’s perspective, the next chapter includes the teachers’ perspectives on the episodes and their instructional actions.

What Does Extending Student Thinking Look Like?

In this study, extending student thinking is conceptualized as helping students to move beyond their initial mathematical observations and to make sense of mathematical situations through mathematical reflection and reasoning. In the analysis of the observational data, the segments of whole-group discussions where there was potential for extending student thinking were classified as extending episodes.
Types of Extending Episodes

Earlier studies suggest that extending episodes are not frequently observed in elementary classrooms. In Fraivillig et al.’s (1999) study, only a few teachers were able to illustrate instructional actions that had potential for extending student thinking. In contrast to the results from Fraivillig et al.’s study, it is significant to note that all six teachers who participated in this study created at least one episode where there was potential for extending student thinking in their classrooms.

For this study, the classrooms were observed when teachers were teaching number units. As a result of this, the discussions in the extending episodes focused on mathematical ideas and strategies about whole number computation. The students in the first, second, and third grade classrooms were solving addition and subtraction problems, and the students in the fourth grade classroom were solving division problems. Most of the episodes involved contextualized problems that allowed students to come up with different solution methods and to discuss these methods.

A total of eight extending episodes, which fit in five different types of scenarios, were identified. (Note that there were more extending episodes identified in the classrooms, but only one or two were chosen for the post-investigation interview and detailed analysis.) Four of the extending episodes illustrated cases where teachers encouraged mathematical reflection. Two episodes focused on going beyond initial solutions or solution methods, and two episodes were cases of encouraging students to reason about mathematics. The types of identified episodes are shown in Table 9 (bolded and italicized). In the sections that follow, the episodes are briefly described, highlighting
the mathematical task being discussed and the pivotal issue or question that defines each extending episode.

Table 9

<table>
<thead>
<tr>
<th>Episode Type</th>
<th>Main Task</th>
<th>Pivotal Issue or Question</th>
<th>Grade</th>
<th>Teacher</th>
</tr>
</thead>
<tbody>
<tr>
<td>Putting Together:</td>
<td>Rosa had 6 toy cars. Her mom gave her 6 more. How many cars does Rosa have now?</td>
<td>I want to talk about the way you solved your problems</td>
<td>1</td>
<td>Linda</td>
</tr>
<tr>
<td>Encouraging Mathematical Reflection</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Using multiple solutions to promote reflection. (1.3)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>School Days:</td>
<td>How many school days and non-school days are there in a year?</td>
<td>Can more than one of them be correct?</td>
<td>3</td>
<td>Judy</td>
</tr>
<tr>
<td>Distance Riddles:</td>
<td>The difference between 100 and me is 45. What numbers can I be?</td>
<td>I think we might be stuck in that word difference.</td>
<td>3</td>
<td>Liz</td>
</tr>
<tr>
<td>Encouraging students to consider the reasonableness or validity of a solution/claim. (1.4)</td>
<td>Separating Situations:</td>
<td>How do you know that would be very low?</td>
<td>2</td>
<td>Kasey</td>
</tr>
<tr>
<td></td>
<td>Yesterday at the park, I counted 69 pigeons. When a big dog walked by, 47 of them flew away. How many were still there?</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Goes Beyond Initial Solution Methods</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pushing students to try alternative solution methods for one problem. (2.1)</td>
<td>Sharing Cards:</td>
<td>Do you have to figure out how many cards are in all 3 decks?</td>
<td>4</td>
<td>Meredith</td>
</tr>
<tr>
<td></td>
<td>You have 3 decks of cards with 52 cards in each deck. How many cards will each person get if the 3 decks are dealt out evenly to 6 people?</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Promoting use of more efficient solution methods for all students. (2.2)</td>
<td>Making 8:</td>
<td>How can one of these ways to make 8 help us make 9?</td>
<td>1</td>
<td>Melissa</td>
</tr>
<tr>
<td></td>
<td>How can we make 8 by using dot cards?</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Encouraging students to offer a justification for their solutions/claims. (3.1)</td>
<td>Separating Situations:</td>
<td>Why would you add when it's a subtraction problem?</td>
<td>2</td>
<td>Kasey</td>
</tr>
<tr>
<td></td>
<td>Yesterday at the park, I counted 69 pigeons. When a big dog walked by, 47 of them flew away. How many were still there?</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>School Days:</td>
<td>How many school days and non-school days are there in a year?</td>
<td>Are there some other kinds of days that wouldn't fit in school days or non-school days?</td>
<td>3</td>
<td>Judy</td>
</tr>
</tbody>
</table>
Using Multiple Solutions to Promote Reflection

As seen in Table 9, in three of the extending episodes, teachers used multiple solutions to promote reflection. In these episodes, students were encouraged to participate in a reflective discourse by stepping back to critically examine the solutions or solution methods that were offered by a peer or by the whole group. All three of these episodes took place towards the end of addition and subtraction units. While one of them was from a first grade classroom, the other two were from two different third grade classrooms. A brief description of these extending episodes follows.

In the Putting Together lesson, the first grade students were solving addition and subtraction story problems such as “Rosa had $6$ toy cars. Her mom gave her $6$ more. How many toy cars does Rosa have now?” Up to this point, the unit included mostly games and tasks that involved breaking numbers into parts in different ways and comparing quantities to develop number sense. Students had been solving story problems during the few days prior to the Putting Together lesson. Just before the extending episode from this lesson, the teacher recorded three solution methods students shared on the board: counting by ones, adding on, and using known facts. During the extending episode, the teacher encouraged the whole group to look at those strategies carefully and to decide which of these strategies they used to solve their problem. Having this kind of mathematical discussion had potential for extending student thinking in several ways. First of all, it allowed students to realize that there was more than one way to solve a single problem. Also, it encouraged them to focus on the strategies and compare and contrast their strategies with the presented ones, and to group their strategy with one of the other categories.
In the School Days lesson, students were asked to find the difference between the number of school days and non-school days in an entire calendar year. After students worked on the problem in pairs, the teacher created a table on the board in which she listed the students’ results: the number of school days, the number of non-school days, and the difference between them. All number combinations were distinct and varied. The extending episode focused on reflecting on the list of numbers and examining the idea of having more than one correct answer for a single problem. This episode was considered an extending one because of the potential it had for helping students move beyond their initial experience of solving the problem and realize that not all of the solutions listed could be correct.

The main goal of the Distance Riddle lesson was to help students understand the concept of difference between 2- and 3-digit numbers and 100 on a number line. Students were first asked to solve riddles such as “The difference between 100 and me is 45. What numbers can I be?” After they found the two numbers (e.g., 55 and 145 in that case), they were asked to find the difference between those two numbers. In this lesson, students were having a hard time understanding the meaning of the term difference and representing that difference on a number line. The teacher encouraged students to represent the difference between 55 and 145 on the 300 chart and also consider a story problem about candies. She created a whole-group discussion that had potential for extending student thinking by encouraging students to reflect on different representations and a story problem to illustrate the mathematical term difference and to recognize the connections between the number line representation, 300 chart representation, and the story problem.
Encouraging Students to Consider the Reasonableness or Validity of a Solution/Claim

There was one episode where the teacher encouraged students to evaluate the validity of a solution through reflection. The Separating Situations lesson took place towards the beginning of the unit on addition and subtraction. The second grade students were to solve a story problem about a separating situation: *Yesterday at the park, I counted 69 pigeons. When a big dog walked by, 47 of them flew away. How many were still there?* After several students shared their solution methods for that problem, the teacher encouraged students to reflect on an erroneous solution (having 4 as an answer to $69 - 47$) and to examine the reasonableness of it. This episode had the potential for helping students develop the habit of mind of checking the reasonableness of their answers for problems and also to improve their understanding of subtraction.

Pushing Students to Try Alternative Solution Methods for One Problem Situation

There was one episode in this category, the Sharing Cards lesson from a unit on multiplication and division. Prior to this lesson, students had been solving multiplication problems and were recently introduced to two different types of division problems: partitive division (knowing the number of groups sharing a number of objects and finding out the number of objects each group receives) and measurement division (knowing the number of objects each group receives and finding out the number of groups that share the total number of objects). The students were solving division problems in the Sharing Cards lesson. The majority of the students were solving one of the problems—*You have 3 decks of cards with 52 cards in each deck. How many cards will each person get if the 3 decks are dealt out evenly to 6 people?*—by using one particular method. They were
figuring out the total number of cards in all three decks and then they were dividing that number by 6. One student solved the problem simply by dividing 52 by 2, which was keeping the ratio of the number of decks and the people sharing them the same. When the teacher recognized this, she encouraged students to consider this solution method, one that differed from their initial method. This allowed students to move beyond their initial solution methods and to be exposed to one they might not have considered otherwise.

_Promoting Use of More Efficient Solution Methods for All Students_

Another episode about moving beyond initial solution methods involved encouraging students to use the solution methods they discussed earlier to solve another problem. The Making 8 lesson took place towards the end of the unit on addition and subtraction. In this lesson, first grade students were to think about different ways of composing numbers. They first discussed different ways of making an 8 and a 9, such as 4 + 4 and 5 + 3 for 8, and 5 + 4 and 3 + 6 for 9. Following that, the teacher invited students to use the ways of making 8 to make 9: _How can one of these ways to make 8 help us make 9?_ This encouraged students to use ways they had already generated for making 8 to make 9 and to recognize the relationship between consecutive numbers. It further helped students make a generalization about the addends and the sum in addition problems, such as, _If you want to increase the number by one then you also need to increase one of the addends by one_. This episode had potential for extending student thinking because it allowed students to recognize the relationship between numbers and to have a better understanding of addition.
Encouraging Students to Offer a Justification for Their Solutions/Claims

Two of the extending episodes were categorized as encouraging mathematical reasoning. In both cases, the teachers encouraged students to offer justifications for their mathematical observations and solution methods.

One of these episodes was observed in the Separating Situations lesson that is discussed above. After three strategies of solving that problem were shared on the board, the teacher recorded another strategy of breaking numbers apart by using place value: 

\[
69 - 47 = 22, \quad 60 - 40 = 20, \quad 9 - 7 = 2, \quad 20 + 2 = 22.
\]

Then she invited students to explain why they needed to add 20 and 2 for the last step of this strategy. Encouraging students to provide a rationale for this solution method had the potential for extending student thinking because it allowed students to make sense of the steps and to build connections between single steps of a method and the whole problem.

In the second episode from the School Days lesson, described earlier, the students were encouraged to engage in justifying the part-part-whole relationship of school days, non-school days, and an entire year. The teacher invited students to consider if there was any day that didn’t fit in either of these two sets. Inviting students to examine how these two sets, school days and non-school days, are complementary to each other and how the union of these two sets makes a whole year had potential for extending student thinking.

Episodes Versus Actions

As discussed in Chapter II, some of the components in Fraivillig et al.’s (1999) framework were individual teacher moves, whereas others involved more than individual teacher moves. For example, one of the components from their framework was
“encouraging students to consider and discuss interrelationships among concepts.” In the sample exchange Fraivillig et al. provided to illustrate this component, the discussion focused on the following question: Zero plus zero is zero, then is negative five plus zero negative five? The teacher asked several questions to elicit student thinking and also provided interpretations of students’ claims. Therefore, there were several individual teacher moves involved in that exchange. On the other hand, another component from their framework was listing multiple solution methods on the chalkboard to promote reflection. In contrast with the previous component, this one seems to refer to one fairly specific type of teacher move.

In this study, components that involved a series of teacher moves were conceptualized as scenarios that have potential for extending student thinking. If a segment of a whole-group discussion fits in one of the scenarios, then it was called an extending episode. For the purposes of this study, only single teacher moves were considered as instructional actions.

This distinction between episodes and instructional actions is a helpful one in understanding how single actions support extending episodes. In the episode from the Distance Riddle lesson, the teacher utilized seven different types of instructional actions in the course of one extending episode. For example, one instructional action was reminding students about what they previously discussed. This action supported the episode of using multiple solutions to promote reflection because it allowed students to recall and reflect on their existing knowledge and their earlier observations.

The analysis of the observational data showed that teachers utilized a variety of instructional actions that supported extending episodes. When the role of the instructional
actions was considered, the same three categories of instructional actions from Fraivillig et al.'s (1999) study emerged: eliciting, supporting, and extending. Eliciting actions allowed teachers to access students' existing thinking and to make it public. Supporting actions assisted students to remember or visualize what they already knew, and to consider new information. Extending actions encouraged students to move beyond their initial mathematical activity through reflection and reasoning.

The types of observed instructional actions and the frequency of each instructional action are presented in Table 10. In interpreting the information listed in this table, it is important to note that the length of the episodes varied from approximately 2 to 6 minutes. The main goal of this table is to provide a sense of the different types of actions that took place in the episodes. In addition, the data presented in the “Overall” column illustrate the least and most prevalent actions that took place across the eight episodes. This information is critical in interpreting the relationship among teachers’ beliefs, knowledge, and these actions, and considering why these particular actions are observed less or more frequently than others.

*The Role of Eliciting Actions*

Eliciting actions provide students with opportunities to express their existing thinking about their mathematical activity or a mathematical phenomenon. They also allow teachers to become knowledgeable about their students' existing thinking so that they can decide which ideas to pursue based on this information. During classroom observations, eliciting actions took place in all six classrooms. These actions included
Table 10

Instructional Actions That Supported Extending Episodes

<table>
<thead>
<tr>
<th>Type of Instructional Actions</th>
<th>The Frequency of Instructional Actions</th>
<th>Overall</th>
<th>Episodes of Encouraging Mathematical Reflection</th>
<th>Episodes of Going Beyond Initial Solution Methods</th>
<th>Episodes of Encouraging Mathematical Reasoning</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eliciting Actions</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Inviting students to share methods</td>
<td></td>
<td>3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Extending Actions</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Inviting students to:</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Evaluate a claim or an observation</td>
<td></td>
<td>24</td>
<td>15</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>Provide reasoning for a claim</td>
<td></td>
<td>18</td>
<td>7</td>
<td>2</td>
<td>9</td>
</tr>
<tr>
<td>Compare different methods</td>
<td></td>
<td>7</td>
<td>6</td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>Use same method for new problems</td>
<td></td>
<td>3</td>
<td></td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Provide counterspeculation for a claim</td>
<td></td>
<td>2</td>
<td></td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>Supporting Actions</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Suggesting an interpretation of a claim/observation</td>
<td>26</td>
<td>11</td>
<td>7</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>Reminding students of goal of the discussion, the problem, or other information</td>
<td>13</td>
<td>9</td>
<td>1</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>Repeating a claim</td>
<td></td>
<td>10</td>
<td>3</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>Recording student thinking</td>
<td></td>
<td>9</td>
<td></td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>Introducing different representations/contexts</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

inviting students to share their solution methods by posing questions such as, *How did you solve the problem?*

In this study, extending episodes were usually chosen from the part of whole-group discussions that occurred right after students and/or teachers shared solution methods and ideas. As a result, not many eliciting actions were observed during extending episodes. Only one of the extending episodes, from the Making 8 lesson, included several eliciting actions, because the extending episode itself revolved around students sharing different solution methods. In the beginning of this episode, the teacher invited students to use the solutions they had for making 8 to make 9: *How can one of these ways to make 8 help us make 9?* Then, inviting students to share their methods for
making 9 occurred several times throughout the episode. Consider the following exchange from that episode.

T: So 4 and 4 helps us [plus] 5 and 4. All right. Do you have another idea, Renee?
S: Uh huh. . . . 5 and the 3. If you just switch the 3 to a 4 and you can make 9.
T: Oh, make the 3 into a 4, make the 8 into a 9. Make that one number bigger. Makes our answer bigger. Cool. Okay. Lacey, [got another idea] for us?
S: 2 + 2 --- 2 + 2 + 2 + 1.
T: How many 2’s?
S: Um, 4.
T: 4? Okay. 2 + 2 + 2 + 1? 9. So that’s this, with an extra 1? ’Cause we need one more? Cool. Okay. One more idea for us, Becky?
S: Um, if you have 9, but 8 and then you put 1 more, 8 and then 9. And that’s 8 and then there’s 1 more.

As students shared their methods, the teacher recorded them on the board and she sometimes suggested an interpretation of what students offered. As a result, the entire episode was considered an extending episode, but one that necessarily included a significant number of eliciting actions.

The Role of Extending Actions

The extending actions are critical in creating opportunities for extending student thinking. After student thinking is elicited, teachers need to utilize instructional actions that allow students to further develop connections among ideas and solution methods and to move beyond their existing mathematical knowledge. In this study, five types of extending actions supported extending episodes. The most commonly observed extending action was inviting students to evaluate a claim or an observation. Teachers encouraged students to reflect on their shared ideas and solution methods by usually asking questions...
such as, *What do you think? Do you agree? Do you think it’s true?* Another prevalent extending action was inviting students to provide reasoning for their claims and solution methods by posing questions such as, *What makes you say that? How do you know? Why do you suppose that?*

The third type of extending action from this study was inviting students to compare different solution methods. Teachers encouraged students to reflect on shared solution methods and sometimes to compare and contrast them by stating, *I want you to look at what Jenny did. Did she do it like he did?* Inviting students to solve a new problem by applying a solution method that they used earlier also helped extending student thinking. Posing questions such as, *How can we use one of these ways to make one of those?* encouraged students to consider efficient ways of solving problems.

In this study, the least frequently observed extending action was inviting students to provide a counterspeculation for their claims. Asking questions such as, *Are there some other kinds of days that wouldn’t fit in school days or non-school days? Can you think of any kind?* encouraged students to consider possibilities that might contradict with their generalization of school days and non-school days making a whole year.

One might assume that the greater the number of extending actions in an episode, the more powerful the episode. However, the data in this study indicate that the number of extending actions that occurred in the extending episodes did not appear to be related to the powerfulness of the episodes. For example, in the episode from the Distance Riddle lesson, while 16 instructional actions took place, only 3 of them were considered as extending actions, and yet, it was one of the most powerful episodes. The issue of powerfulness of an episode is discussed later in this chapter.
The Role of Supporting Actions

Supporting actions could be viewed as less desirable in extending episodes because they involve more teacher telling. However, the data indicate that supporting actions play a significant role in making extending episodes powerful. Students do not always make reasonable connections between their existing knowledge and new ideas, reason through their mathematical observations, or thoroughly articulate their thinking. And therefore, it seems they need assistance doing these things in order to focus their mathematical reflection and reasoning on important mathematical concepts.

In this study, teachers utilized five different types of supporting instructional actions that assisted students during extending episodes. The teachers sometimes shared their own interpretations of their observations and of students' claims. This allowed students to become aware of important mathematical issues they were discussing and thus stay focused on those issues, and to hear more thorough interpretations of their peers' claims. For example, in the Distance Riddle lesson, the teacher helped students become aware of what they were struggling with by saying, I think we might be stuck in that word "difference." Then she allowed them to recognize the connection between the representation of the difference between 55 and 145 on a number line and on the 300 chart by saying, Let's look at this chart. You know this is just kind of like that number line we just did. We went back 45 and we went forward 45.

Reminding students of information that is related to the problem they were solving or to the ideas they were discussing helped students form connections between their new observations and what they already knew. Again, in the Distance Riddle lesson, the teacher supported student thinking by reminding them of the original problem and by
labeling the numbers that they found earlier on the 300 chart: *We decided that it was 55, the difference [between 100 and me] is 45, and 145. All right. So, that's our number line. And so, it says that the difference is 45 and we landed on 55 and we landed on 145.*

Repeating students' claims or having students repeat each other's claims was also useful for having students engage in discussions or stay focused on ideas being discussed. An example of this action was stating, *So, the calendars we marked were all the same, so you're saying the answers have to be the same.* Another instructional action that supported the extending episodes was recording student thinking on the board. Having a recording of shared ideas or solution methods allowed students to collectively reflect on their thinking.

The fifth supporting action, and the one that was observed least frequently, was introducing representations or contexts that were familiar to students. This supported students to both recognize and form connections between their new observations and already established knowledge. For example, in the Distance Riddle lesson, students were struggling with understanding the meaning of the word *difference,* and the teacher provided them with a story problem that was familiar to the students: *If I had 55 pieces of candy in my hand, but if I really needed 145 pieces of candy, how many pieces of candy do I need to add?* All of these supporting actions helped the extending episodes become powerful and effective.

*Reflections on Instructional Actions*

As illustrated in Table 10, the number of occurrences of the instructional actions that took place during the extending episodes varied from 2 to 26. The extending
instructional action that occurred most frequently (24 instances) was inviting students to evaluate a claim or an observation. Not only did it occur most frequently, but it was also observed in seven of the eight extending episodes. This action provided students with an opportunity to reflect on their shared mathematical ideas or solution methods. Engaging students in this kind of collective reflection process allowed them to recognize connections between their existing knowledge and their observations. It also allowed teachers to become aware of what their students were thinking about particular mathematical issues. Given the fact that this instructional action was observed often indicates teachers’ interest in keeping student thinking central during whole-group discussions. Inviting students to provide reasoning was another commonly observed extending instructional action. It was observed in all classrooms higher than first grade. These two commonly observed extending actions were usually generic actions and they were variations of these two basic questions: What do you think? and Why do you think?

The most frequently observed supporting action was suggesting an interpretation of a claim or an observation. Teachers seemed comfortable with this instructional action. It was observed during all eight episodes. Suggesting an interpretation of a claim or an observation may seem more consistent with traditional notions of didactic teaching, because the teachers share their own opinions rather than having students share their thinking. The episodes observed in this study showed that this instructional action can play an important role in making extending episodes effective. For example, in the School Days lesson, after one of the students stated that If you put the school days and non-school days together, you should get 365 or 366, the teacher provided an interpretation of that conjecture, If we add the school days plus non-school days, Caleb
says that this should be all the days in a year. In her interpretation, by including the phrase “all the days in a year” the teacher created an opportunity for students to focus on the meaning of Caleb’s statement and to recognize the notion of two quantities making a whole year rather than merely thinking about the numbers 365 or 366.

The least frequently observed extending and supporting actions were inviting students to provide counterspeculation for a claim and introducing different representations or contexts, respectively. Considering counterspeculations for mathematical claims or conjectures is related to providing proofs for mathematical claims or conjectures. In the episode from the School Days lesson, the discussion focused on proving that school days and non-school days make a whole year. These two sets of days are complementary of each other, that is, they don’t have any common days (as it is implied by the names of the sets—school days and non-school days) and their union is the universal set, which is an entire year. While recognizing that relationship might be trivial for adults, it was clearly challenging for students. During the episode, the teacher invited students to examine the existence of counterexamples for that claim: Are there some other kinds of days that wouldn’t fit in school days or non-school days? Can you think of any kind? By utilizing this particular action, the teacher allowed students to focus on the crux of the issue of recognizing that these two sets of days are complementary sets and there aren’t any other days left over in an entire year.

The other least frequently observed action, introducing different representations or contexts, was also observed in only one episode: the Distance Riddle episode. Students were struggling with representing and finding the difference between 55 and 145 on a number line. The teacher first introduced the 300 chart because she knew that students
were comfortable with using this tool to represent distances between numbers. She then encouraged students to recognize the connections between these two representations: the number line and the 300 chart. She next allowed them to have better understanding of the word *difference* by offering a contextualized problem that would be familiar to the students: *If I had 55 pieces of candy in my hand, but if I really needed 145 pieces of candy, how many pieces of candy do I need to add?*

In both cases in which the two teachers utilized these infrequently observed instructional actions, the teachers' beliefs about what learning mathematics entails and how students learn mathematics may have supported them to utilize these instructional actions as well as their knowledge about the content and with what students might struggle. This idea will be discussed in more detail in the next section on the findings about the teachers' beliefs and knowledge.

*Themes Across Eight Episodes*

In all of the extending episodes, the centrality of student thinking was significant. The teachers were able to have their students discuss worthwhile mathematical ideas and solution strategies, and the students seemed comfortable sharing and discussing their thinking. All eliciting, supporting, and extending instructional actions observed during the episodes showed genuine interest in student thinking. Both supporting and extending actions encouraged students to reflect on and reason about their mathematical observations and they focused discussions on what students had to offer. The commonalities around this focus on student thinking seemed to center on two themes: the use of representations and the use of context.
The Use of Representations and Records of Student Thinking

In all of the episodes except for one, teachers recorded student thinking either on the board or on several different representations such as number lines and the 300 chart. This provided both students and teachers with a reference to reflect upon during discussions. For example, during the Making 8 lesson, the teacher wrote the number sentences suggested by the students (i.e., $3 + 5, 4 + 4$). Having a record of ways of making 8 provided students with a reference to reflect on and helped students visualize ways of making 9.

The only lesson in which student thinking wasn’t recorded on the board was the Sharing Cards lesson. This discussion took place in the midst of a small group work. As the teacher was walking around the classroom, she realized that the majority of the students were solving the problem by using a particular method. When the teacher observed that only one of the students used a different solution method, she interrupted students’ group work to provide an opportunity for students to be exposed to the idea of solving the problem by using a different solution method. Since the major goal of this discussion was to help students realize that there could be a different, and possibly an easier, way to solve the problem, recording the solution method didn’t seem critical in this episode. (Note: The student who used the different solution method had only two number sentences on his notebook: $52 + 2 = 26$ and $50 + 2 = 25$.)

The Use of Context

During most of the episodes, both the teachers and the students referred back to the context of the tasks during whole-group discussions. Using contexts supports students
to focus on the meaning of the operations and mathematical terms. In this study, there were three different uses of context: (1) the task is already in a context and the teacher and students utilize the context during their discussions; (2) the task is already in a context, but the teacher and the students leave it out of their discussions; and (3) the task is not in context, and the teacher creates a context at some point during the discussion.

For the first case, consider the first episode from the School Days lesson. Both the students and the teacher referred back to the context of the problem throughout the episode. They could have easily looked at the solutions provided on the board and simply analyzed the numbers of days and non-school days to figure out if the differences were correct. However, it was only by keeping the context of the problem as part of the conversation that students could consider whether the actual combinations were correct and probably increased students’ awareness of the fact that there could be only one correct number combination of school days and non-school days.

The second episode from the Separating Situations lesson was an example where the original task was a contextualized problem, but the teacher and students left out the context in discussing solution methods. The original task that students were solving was a contextualized problem about pigeons. The students were struggling to make sense of the last step of the solution method where you add the partial differences together (69 − 47 = 22, 60 − 40 = 20, 9 − 7 = 2, 20 + 2 = 22). Had the teacher and the students focused on the “take-away” meaning of subtraction in the context of the problem, it might have been easier for them to justify. For example, they could have provided a justification, such as the following one: First I separated the pigeons into two groups: 60 pigeons on one side of the park, and 9 pigeons on the other. I took 40 of the 60 pigeons away (or 40 of the
pigeons flew first), and then I took 7 of the 9 pigeons away (or 7 of the 9 pigeons flew next). There were 20 and 2 pigeons left in the park. Note that the context of the problem makes it clear that a total of 22 pigeons were left in the park. Without the context of the problem, or an explicit tie to the take-away meaning of subtraction, it would be extremely difficult for the students to provide reasoning for their thinking.

Only one teacher introduced a context when the original task did not involve one. During the episode from the Distance Riddle lesson, the students were struggling with grasping the meaning of the term difference. The teacher came up with a story about candies: *If I have 55 candies in my hand, and if I want to have 145 candies total. How many more candies do I need to get?*” This appeared to be a powerful way of engaging students with the concept of difference and further allowed students to make connections between the story and their representations of the difference between 55 and 145 on the 300 chart and a number line. Using situations with which students are familiar helped them focus on the meaning of the operations and visualize mathematical situations. It also helped students recognize the connections between stories and solution strategies.

*Degrees of Extending Student Thinking*

Although this is a select group of teachers (they were teacher leaders, they participated in a great amount of professional development as a teacher and as a facilitator, and they have taught Investigations for many years), it was striking to observe that there was variability in the degree to which they appeared successful in extending students’ thinking. While the purpose of this study was not to do a ranking of the extending episodes, it did seem productive to analyze their relative powerfulness. The
evidence within the episodes that student thinking was being extended and the number of particular kinds of instructional actions were taken in consideration for this analysis. While the former analysis was productive, the second one did not reveal any connections.

The analysis of the extending episodes suggested that a different organization of the episodes (other than grouping them into three major categories: encouraging mathematical reflection, going beyond initial solution methods, and encouraging mathematical reasoning) could be useful in understanding the nature and powerfulness of the extending episodes. The observational data showed that there were two different types of extending episodes in terms of the purpose of the whole-group discussions: the ones that were created to help students recognize and build connections among mathematics concepts that they might not have done so on their own, and the ones that were created to address misconceptions or to help students overcome their struggles with a mathematical concept. As stated earlier, these two were the only two instructional approaches that were highlighted in Hiebert and Grouws' (2007) recent review of research that positively impact student learning.

*Building New Connections*

There were four extending episodes where the goal of the whole-group discussions seemed to be to help students move beyond their initial mathematical observations by building new connections between their existing knowledge and new ideas (see Table 11). In those episodes, students either were encouraged either to reflect on their own solution methods or were introduced to a new solution method.
Table 11

The Episodes of Building New Connections

<table>
<thead>
<tr>
<th>Less Powerful</th>
<th>More Powerful</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Main Task</strong></td>
<td><strong>Pivotal Issue or Question</strong></td>
</tr>
<tr>
<td>Putting Together:</td>
<td>I want to talk about the way you solved your problems.</td>
</tr>
<tr>
<td>Rosa had 6 toy cars. Her mom gave her 6 more.</td>
<td>How do you know that would be very low?</td>
</tr>
<tr>
<td>How many cars does Rosa have now?</td>
<td>How can one of these ways to make 8 help us make 9?</td>
</tr>
<tr>
<td>Separating 1:</td>
<td>Sharing Cards:</td>
</tr>
<tr>
<td>Yesterday at the park, I counted 69 pigeons.</td>
<td>You have 3 decks of cards with 52 cards in each deck. How many cards will each person get if the 3 decks are dealt out evenly to 6 people?</td>
</tr>
<tr>
<td>When a big dog walked by, 47 of them flew away. How many were still there?</td>
<td>Making 8: How can we make 8 by using dot cards?</td>
</tr>
</tbody>
</table>

In the Putting Together lesson, the first grade students were encouraged to compare and contrast their own methods for solving an addition problem with the three methods that were recorded on the board. This was the shortest episode of all. The teacher encouraged students to look at their own solution methods carefully and then decide which of the three listed methods matched with theirs. Throughout the episode, the teacher reminded students what the three solution methods were, and how they were different from each other.

T: Did you do it like Connor, where you drew whatever the first number was, and then you drew whatever the second number was? Raise your hand if you did it like Connor. You did. You’re right, Emily. You did, you’re right. You did, you’re right. Connor, you did it, right? . . . Raise your hand. Kara, you didn’t do it that way, right? You did it that way, Nancy. Mike, you did it that way. Kean, is that the way you did it, too?

S: . . .

T: But did you draw ALL of the things that you were talking about? Okay. So put your hands down. And you drew, you’re right. Now, raise your hand if you did it like Harry, where you started with the number and then you added up.
There were two aspects of this episode that indicated that it was less powerful than the other episodes. First, students in this episode only raised their hands to show which solution method they used and they didn’t discuss their own thinking about what connections they were identifying between their method and the one on the board. Second, the teacher didn’t seem to be utilizing the connections between their strategies and the ones on the board to make any further points herself. Thus, while the students’ thinking may have been extended by having to make connections between their thinking and the thinking of a peer, there was not much evidence of the kinds of connections being made by the students, or even which ones were being fostered by the teacher.

The first episode from the Separating Situations lesson was also considered to be less powerful than the others in this category. Recall that this episode focused on helping students develop the habit of mind of considering the reasonableness of their answers for mathematical problems. The teacher encouraged students to reflect on an erroneous answer to a subtraction problem, 69 – 47 = 4, and she invited students to reason about why it didn’t make sense to have 4 as an answer for this problem. However, as discussed earlier, the lack of reference to the context of the problem and the meaning of the operation seemed to make the discussion stay at an abstract level and students were not able to provide reasoning for their thinking. Thus, it was not clear to what degree student thinking was extended, and therefore this episode was also considered to be a less powerful one.

Two of the episodes in this category were considered to be more powerful. In the episode from the Making 8 lesson, the format of the discussion was similar to the one from the Putting Together lesson, where the teacher posed a question and students
responded, and then the teacher repeated the original question to allow different
responses to be shared. In the Putting Together lesson, there wasn’t much pursuing of an
idea or having students focus on a particular solution method; rather, the discussion was
more about sharing different methods and then possibly reaching a general conclusion
about them, whereas in the Making 8 lesson the first graders were able to come up with
different ways of making 9 (i.e., $4 + 5$, $3 + 6$) by using ways of making 8 (i.e., $4 + 4$,
$3 + 5$). Students noticed that if you want to increase the number by 1, then you need to
increase one of the addends by 1, too. As there was evidence that student thinking was
extended in this episode, it was considered a more powerful one.

Finally, the episode from Sharing Cards about exposing students to a solution
method that they might not have considered before was deemed to be more powerful than
the other three episodes. During this episode, the teacher invited students to evaluate one
of their peer’s solution methods for the Sharing Cards problem: sharing 1 deck of cards
between two people (which has the same ratio with sharing 3 decks of cards among 6
people). While students struggled with engaging in the discussion about this new method,
the teacher encouraged them to focus on it, and a few of the students were able to
articulate the reasoning behind their claims during this episode (e.g., by stating, *Since
there are three decks of cards and you put two people to each deck. Two times, two
people for each deck and there are three decks, that would be six people for the total*).
This episode was considered as the most powerful one of the four building new
connections episodes because of the ways the teacher persisted with one solution method
and also because of the evidence that student thinking was, in fact, extended.
Addressing Misconceptions and Struggles

There were four extending episodes where teachers created opportunities for addressing student misconceptions or struggles (see Table 12). The teachers were knowledgeable about students' common misconceptions or the challenges that they pursued.

Table 12

The Episodes of Addressing Misconceptions or Struggles

<table>
<thead>
<tr>
<th>Main Task</th>
<th>Less Powerful</th>
<th>More Powerful</th>
</tr>
</thead>
<tbody>
<tr>
<td>Separating 2:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Yesterday at the park,</td>
<td>School Days 1 &amp; 2:</td>
<td>Distance Riddles:</td>
</tr>
<tr>
<td>I counted 69 pigeons.</td>
<td>How many school days and non-school days</td>
<td>The difference</td>
</tr>
<tr>
<td>When a big dog walked by, 47 of them flew away. How many were still there?</td>
<td>are there in a year?</td>
<td>between 100 and me is 45. What numbers can I be?</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Pivotal Issue or Question</th>
<th>Less Powerful</th>
<th>More Powerful</th>
</tr>
</thead>
<tbody>
<tr>
<td>Why would you add when it's a subtraction problem?</td>
<td>Can more than one of them be correct?</td>
<td>I think we might be stuck in that word difference.</td>
</tr>
<tr>
<td>Are there some other kinds of days that wouldn't fit in school days or non-school days?</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In the second episode from Separating Lesson, the teacher invited students to provide reasoning for the last step of a strategy (breaking numbers apart by using place value, \(69 - 47 = 22, 60 - 40 = 20, 9 - 7 = 2, 20 + 2 = 22\)). It is important to note that students often use a similar strategy to solve addition problems by first adding the tens and the ones separately, and then by putting them together. Thus, focusing students' attention on whether to subtract the 2 from the 20 or add the 2 to the 20 was of particular importance in their understanding of this approach to the problem. While the teacher utilized a variety of instructional actions and tried to provide support to help students reason about the last step of the strategy, as discussed previously, the lack of reference to
the story of the problem and the meaning of the operations during this lesson seemed to hinder students from recognizing the connection between the last step of the strategy and the whole problem. Thus, this episode was considered to be a less powerful one in this category.

In contrast, the other episodes in this category were considered more powerful. For example, in the first episode from the School Days lesson, the students were struggling with recognizing the fact that there could be only one correct answer to the school days problem. The teacher encouraged them to examine the numbers listed on the board by posing several focusing questions.

T: Everybody ok with the rest of them [the list of the numbers]? (pause)
Sts: Yeah. No.
T: All right. So we have a ton of different answers. Whose is correct? (pause)
Can more than one of them be correct?
Sts: Yeah. No.
T: Would somebody talk about whether more than one of them could be correct or not? Mia?
S: I think there can't be more than one right answer, because we all worked with the same information, uhm, calendar.
T: So, the calendars we marked were all the same, so you're saying the answers have to be the same. What do the rest of you think about Mia's reasoning there? (Pause) [Some students nodded their heads.]

Both Mia's response and the teacher's instructional actions indicated that this was a more powerful extending episode. It appeared that by posing focusing questions, the teacher created an opportunity for extending student thinking, and Mia clearly stated that there could be only one correct answer since they all used the same information. The fact that the teacher first provided an interpretation of Mia's claim and then invited students to
evaluate it made this episode even more powerful because it further provided an opportunity for students to understand her claim.

The second episode from the School Days lesson was also considered as one of the more powerful episodes for similar reasons. There was evidence that student thinking was extended and also the teacher utilized instructional actions that allowed her to capitalize on significant mathematical ideas. The teacher initiated the discussion with a question that encouraged students to consider their existing knowledge.

T: I'm wondering if there's something that we talked about yesterday that might help here. (pause) Caleb?
C: How many days in a year.
T: How many, how could that help us Caleb?
C: If you put the school days and non-school days together, you should get 365 or 366.

After one of the students suggested that the total of school days and non-school days should be 365, the teacher encouraged students to evaluate this claim, and as a group they decided that the total should be 365 because this is not a leap year. Then the teacher focused students on justifying the claim about the relationship between school days, non-school days, and an entire year.

T: Let's evaluate Caleb's statement here. (pause) If we add the school days plus non-school days, Caleb says that this should be all the days in a year. Is that true? Are there some other kinds of days that wouldn't fit in school days or non-school days? Can you think of any kind?

Although this was a difficult concept for students to grasp and students were struggling with understanding it, the teacher stayed persistent with this idea and encouraged students to examine the correctness of it by providing possible counterspeculations.

The episode from the Distance Riddles was also considered as a more powerful episode. During this episode, the teacher had students stay focused on the main
mathematical issues and reflect on and reason about them. Utilizing instructional actions, such as reminding students related information, introducing different representations and contexts, and suggesting interpretations of their observations, each supported students to make connections between what they were familiar with and to build on that knowledge. Thus, it seemed that there was a greater opportunity for student thinking to be extended.

Another feature of the episode that made it powerful was the evidence of student understanding. Not only did students come up with solution methods to find the difference between 55 and 145, but they also explained how their solutions made sense.

T: What is the difference between these two numbers? (Pause) (Most of the people raised their hands) What do you think Keegan?
S: 90.
T: What makes you say that?
S: Because I started at 55 and I counted by 10 down until I landed on 145, and I got 90.
T: Let’s see. 10, 20, 30, 40, 50, 60, 70, 80, 90. Is that the difference between 55 and 145?
Sts: Yes.
T: If I had 55 pieces of candy in my hand, but if I really needed 145 pieces of candy, how many pieces of candy do I need to add?
Sts: 90.
T: I need to add 90 to this hand. So, if you start at 55 the difference between these two numbers is 90. Jack?
J: There is 55 and you count 45 and we already land on 100. And then jump another 45 and you land on 145. You just add the 45 and 45 together. (Teacher drew arches between 55 and 100 and 100 and 145 on the 300 chart as Jack talked.)
T: Let’s think about what Jack just said. Let’s look at this chart. You know this is just kind of like that number line we just did. We went back 45 and we went forward 45. And what is it?
Sts: 90.
Jack provided a thorough explanation for his solution method, which was different from the previous one, but it was similar to the one they used on the number line. Although it wasn’t clear from his explanation that he made that connection, it could be possible. The teacher then capitalized on Jack’s method and highlighted the connection between the two representations, the number line and the 300 chart.

**Summary**

Earlier studies indicate that extending student thinking is not commonly observed in elementary classrooms. Thus, it is significant to note that extending episodes were observed at least once in each of the six classrooms in this study. All six teachers recognized opportunities for considering important mathematical issues and proceeded to engage students in pursuing those issues. The analysis of instructional actions that were utilized in support of these extending episodes yielded further insight into the role of supporting and extending actions in extending episodes. While extending actions were critical in extending student thinking as they allowed students to move beyond their initial mathematical observations through reflection and reasoning, supporting actions were also important as they assisted students in remembering and visualizing their existing knowledge, and in considering new mathematical information. The results of this study showed that it was the nature of the actions that made episodes powerful rather than the frequency of them.

It appeared that there were two main types of extending episodes in terms of the purpose of the discussions: helping students build new connections, and addressing misconceptions and struggles. In episodes from the first category, the teachers
encouraged students to reflect on mathematical ideas and solution methods so that students could build new connections that they might not have done on their own. In episodes from the second category, the teachers created opportunities to help students focus on, and make sense of, particular mathematical issues that were challenging for them to grasp.
CHAPTER V

TEACHER BELIEFS, KNOWLEDGE, AND INSTRUCTIONAL ACTIONS

Teachers' beliefs and knowledge about mathematics and learning mathematics greatly influence the way they teach mathematics. Recall that the teachers who participated in this study were likely to have a solid knowledge of the mathematical content and the curriculum, as well as a strong belief about teaching mathematics for understanding because of their experience with participating in and facilitating professional development, and teaching *Investigations* for several years, ranging from 8 to 12 years. However, just which aspects of their beliefs and knowledge would play a prominent role in their teaching was unclear.

The results of the analysis of the teachers' beliefs and knowledge and the connections between those and their instructional actions are discussed in this chapter. The first section focuses on the teachers' stated beliefs about mathematics and their views of learning and teaching mathematics. Following that, the results on the teachers' mathematical knowledge for teaching are presented. Finally, the relationship between the teachers' beliefs and knowledge and their instructional actions is discussed. The author recognizes that one can also find evidence of both beliefs and knowledge in practice. For the purposes of this chapter, the connections between stated and enacted beliefs, as well as the connections between the mathematical knowledge for teaching that the teachers presented during interviews and during teaching, is discussed in the context of the final
section, on the relationship between teachers’ beliefs and knowledge, and their instructional actions.

Beliefs

Teachers’ beliefs about the nature of mathematics, how students learn, and teaching mathematics impact their instructional actions. For this study, pre-investigation, post-lesson, and post-investigation interviews were analyzed to identify teachers’ stated beliefs about the nature of mathematics and their views on learning and teaching mathematics.

There were several themes that emerged during the analysis of teachers’ beliefs about mathematics and learning and teaching mathematics. All six teachers believed that there is more than one solution method for mathematical problems, that students learn mathematics by constructing their own understanding of mathematics, and that they must be active participants in class discussions. Furthermore, there was some consensus on the types of instructional actions that teachers should employ as they teach mathematics in order to facilitate student understanding.

These results imply that all six teachers who participated in this study held experimentalist views (Wilson & Goldenberg, 1998). That is, they held beliefs that supported creating classroom environments where students are encouraged to think about mathematics from different viewpoints and to critically consider the reasonableness of their mathematical thoughts and arguments. Having said that, it is important to note that some of the teachers appeared to hold these beliefs more strongly than others.
**There Is More Than One Solution Method for Problems**

During the interviews, all six teachers suggested that doing mathematics is about solving problems. They further stated that there is more than one method to solve problems. For example, during the pre-investigation interview, one teacher expressed her view by stating:

In general, there is one right answer for most of the problems we do, just lots of different ways to get there.

The teachers also elaborated on what this belief meant in terms of an aspect of their own understanding of mathematics. They stated that they, as teachers, needed to be aware of different ways of solving problems.

You’ve got to be able to look at a problem or a story problem or whatever it is you’re working on, you may have to look at it in more than one way.

—Linda, Pre-Investigation Interview

I need to be aware that there are different ways to solve problems and still be, you know, still [come up with that] correct answer, and to be more flexible in my thinking than perhaps that person in a different profession.

—Meredith, Pre-Investigation Interview

Believing that there is more than one solution method for problems has potential implications for the way teachers teach mathematics. It allows teachers to welcome different views and solution methods about problems. Holding this belief could lead some teachers to intentionally create opportunities for students to consider and to make sense of different and more efficient ways of thinking about problems and solution methods for them. Engaging students in discussions of mathematical reflection and reasoning about different solution methods would have potential for extending student thinking because it would allow students to move beyond their initial activity of problem solving and their initial solution method.
Teachers’ beliefs about how students learn mathematics play a critical role in how they teach mathematics and, therefore, how they facilitate whole-group discussions. In this study, all six teachers believed in the power of making sense of mathematics and building connections between existing knowledge and new ideas in learning mathematics. They suggested that students learn mathematics by constructing their own understanding of mathematical concepts as they engage in a thinking process about solving problems and actively participate in discussions.

I don’t think you can learn mathematics deeply by just being passive and sitting back and I can’t make that stuff go on in their head without them engaging in it too, so they need to be active participants... As a class we try to look at whether their explanation makes sense mathematically, whether it makes sense in a situation.

—Judy, Pre-Investigation Interview

All six teachers were explicit about the students’ role in reflecting on and making sense of presented ideas and solutions. They stated that students should participate in discussions by sharing their thinking, listening carefully to each other’s thinking, and reflecting on each other’s ideas and solution methods.

I see their role as listening carefully to other people’s ideas, agreeing or disagreeing with evidence. So, you know, if someone is sharing an idea, it’s a student-the other students’ job is to listen intently to see if they can understand what that student’s mathematical idea is, and if they agree, to agree with evidence; if they disagree, to disagree with evidence.

—Liz, Pre-Investigation Interview

The four teachers who were teaching second, third, and fourth grade also believed that students should not only share their thinking, they should also provide evidence for
their claims. They believed that every time students agreed or disagreed with an argument or a claim, they should be expected to explain why they agreed or disagreed.

   They have to be able to explain why they disagree. And one of the things that I tell them too is you cannot say “I disagree because I got this answer.” [You have to] explain why you think their answer is incorrect. You have to explain why you think it’s incorrect.

   —Kasey, Pre-Investigation Interview

The two first grade teachers, on the other hand, stated that students at this age have enough difficulty simply expressing what they did, and thus they tend to spend most of their time focusing on this aspect of student thinking.

   But also communication is a big piece of these lessons, but to solve the problem they also have to show how they solved the problem, and being able to explain what they did, and that’s the challenge for most of the kids, is they can find an answer but they can’t tell you how or why necessarily.

   —Melissa, Pre-Investigation Interview

This will be further discussed in the section on challenges.

The teachers’ beliefs about learning mathematics and about the role students should take on during whole-group discussions can be closely connected to the instructional actions the teachers value, which is the focus of the following section.

*Instructional Actions Teachers Valued*

   During the interviews teachers were asked to describe their views on the role of the teacher during whole-group discussions, as well as to evaluate their strengths and weaknesses as a facilitator. The analysis of their responses to these questions yielded a set of instructional actions that this group of teachers valued.

   All of the teachers highlighted the importance of reflecting on and making sense of presented ideas and different solution methods as a whole group in learning.
mathematics with understanding. In clarifying this idea, they described the responsibilities of teachers in this process. They suggested that they, as teachers, should encourage students to share thinking and they should listen carefully to what students offer so that they can have a clear understanding of their students’ thinking. This would then allow them to further pursue students’ ideas through discussions to help them construct an understanding of mathematical concepts.

[One of my main roles is] Definitely to be a good listener. To question the kids, to ask them great kinds of questions to push their thinking and to make them explain what they’re doing.

—Kasey, Pre-Investigation Interview

We really make the kids think hard and as teachers we need to not make assumptions about what they know but we pursue and ask them to explain you know verbally and in writing so that we know really deeply what they know . . . I need to understand it well enough that I can help kids, you know to push their thinking.

—Judy, Pre-Investigation Interview

Another teacher summarized how she views her role as a facilitator during whole-group discussion by stating:

As a facilitator, first of all, you’re listening carefully to what kids are saying. You’re listening intently, you’re . . . asking good questions, you know, at the right time. To listen to ideas, pull out big ideas. . . . So as a facilitator, I would say my role is to kind of manage the conversation by carefully placing questions at the right time.

—Liz, Pre-Investigation Interview

As it is stated in this quotation, the teachers acknowledged the importance of pursuing student thinking during whole-group discussions and they considered questioning student thinking as a significant part of their role.

The teachers also talked about how when students are in the midst of developing new ideas, their thinking is usually not well formed and it is not uncommon for students...
to struggle with articulating their thinking clearly. Furthermore, they noted that it is sometimes difficult for students to stay focused on the issues being discussed.

Sometimes the kids have a hard time explaining... I think sometimes they leave out parts. So I make them back track, and I’ll ask them more specific questions. “Well, what did you mean? Did you mean this?”

—Kasey, Post-Investigation Interview

“Well, did you hear Adam when he was explaining?” and such and such. “Did you understand him?” Or sometimes I’ll just, I’ll just very bluntly say, you know, “So and so, will you repeat what Adam just explained? Can you do that in your own words?” So they, they begin to know that “You know what? I’ve got to listen so I can figure out what they’re saying.”

—Meredith, Pre-Investigation Interview.

The teachers believed that they, as teachers, should encourage students to clarify their thinking and repeat each other’s claims during whole-group discussions. While having students clarify their own thinking can help them have a more thorough understanding of their own ideas, it gives the rest of the students another chance to hear it. Also, having students repeat others’ thinking encourages students to listen carefully to others and to participate in discussions.

As discussed above, the four teachers in second, third, and fourth grade believed that providing reasoning is a critical part of learning mathematics. The same four teachers then highlighted the importance of encouraging students to provide reasoning for their thinking during whole-group discussions.

[When I ask a question and I get a response, I will say “What makes you say that?” or something like that, that I want to have the child go deeper with their thinking and they’re explaining to generalize and to justify and clarify and do all of those parts of their thinking, to reason through their math.

—Meredith, Pre-Investigation Interview

It appeared that encouraging students to reason through their own thinking was important to those teachers.
Encouraging students to reflect on their ideas and solution methods and reason about them has the potential for helping students move beyond their initial observations and make connections between their existing knowledge and new ideas and solution methods. Therefore, valuing these actions can support teachers in extending student thinking.

Challenges in Enacting Instructional Actions

The teachers' beliefs about the centrality of student thinking in their teaching were apparent in their discussions of the challenges of facilitating whole-group discussions. They suggested that since they needed to make instructional decisions based on what students offer during lessons, they could not plan their lessons in a detailed manner ahead of time. They suggested that their teaching involved making on-the-spot decisions about what ideas to pursue and how to pursue them, depending on students' responses. The teachers found this aspect of teaching mathematics challenging. One explanation that illustrates the teachers' viewpoint was:

All those decisions you need to make on the spot. You know, what should I pursue, uhm, you know, all the mistakes or the strategies or it's always trying to decide where to go with that . . . , it's hard because you don't know ahead of time what they're going to come up with. So, it's not like I can really plan it out very well, uhm, and it always depends on how they're explaining things or on-the-spot sorts of things that make it hard to decide.

—Judy, Post-Investigation Interview

In addition, both of the first grade teachers suggested that verbalizing thinking, listening to others' thinking carefully, and staying focused on the task might be more difficult for younger students. Therefore, they believed that they, as teachers, might need
to revoice student thinking more often than teachers in other grade levels. One of them stated:

They need to clarify their own thinking. And that’s hard for first graders. . . . A lot of times you can get them to say it in a way that I understand and they understand but everyone else doesn’t understand, so then you kind of have to rephrase, but you still let the child have ownership of it. So it’s like, “What I hear you saying is . . . ,” you know, and “Is that right?”

—Melissa, Pre-Investigation Interview

While the examples above illustrate an awareness of challenges one might expect from experienced teachers, two of the teachers seemed to go beyond that by describing some of their challenges as actual tensions in their beliefs. The two teachers were one of the first grade teachers and the second grade teacher. One of them talked about her tension over telling students information rather than letting students do so. She suggested that sometimes she presents information or explains students’ solutions to the whole group when there is not enough time for having students discuss mathematical ideas or solution methods.

I’m very guilty of when they’re not getting where I want them to go, to get impatient, to just, you know, come on . . . and giving them what I want them to get, because some of the things you don’t, you only have so much time, and we need to make sure we get to this certain point by a certain time of the year.

—Melissa, Pre-Investigation Interview

The other teacher suggested that having learned and taught mathematics in a traditional way for many years still impacts how she teaches mathematics now. According to her, she sometimes explains students’ thinking to the whole group instead of allowing students to clarify their thinking.
I try to have the kids explain like their solutions, their strategies and everything, then I think sometimes that I do too much talking after the kids and I, you know, when I see questioning looks on kids' faces that they don’t understand, I kind of take over and I kind of explain what the student has done.

—Kasey, Pre-Investigation Interview.

The fact that these two teachers expressed conflicting beliefs about their valued instructional actions may be an indication that their beliefs that students learn mathematics by constructing their own understanding may not be as strongly held as the other teachers. One potential impact of this conflict on their teaching could be a tendency to assert more control during whole-group discussions by doing things like asserting themselves as the authority on correctness, and, in general, validating student thinking themselves rather than engaging students in reflecting on, and evaluating their own thinking.

Despite the tension indicated by these two teachers, all six teachers’ stated beliefs about their valued instructional actions for facilitating whole-group discussions indicates that they had a clear vision of what teaching mathematics based on student thinking should look like, along with a recognition of the challenges as well. The ways in which these stated beliefs aligned with observed instructional actions during the extending episodes are discussed in the last section of this chapter. What follows is the discussion of the teachers’ mathematical knowledge for teaching.

Mathematical Knowledge for Teaching

It is well established that having a solid understanding of mathematical concepts is important for effective teaching. Furthermore, knowing how students learn specific content and knowing how to design instruction are critical. In this study, teachers'
content knowledge, knowledge of content and students, and knowledge of content and teaching were analyzed to understand the nature of teachers' mathematical knowledge for teaching. While data from the interviews, in which teachers were asked questions about an entire investigation (an average of five lessons per investigation), were utilized in this analysis, the focus is on the teachers' mathematical knowledge for teaching regarding the identified extending episodes. As shown in Table 13, there were similarities in the teachers' mathematical knowledge for teaching.

Table 13

*Teachers' Mathematical Knowledge for Teaching*

<table>
<thead>
<tr>
<th>Aspects of Knowledge</th>
<th>Strong</th>
<th>Less Strong</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Content Knowledge</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mathematical ideas involved in a content area or task</td>
<td>5 teachers</td>
<td>1 teacher (Kasey)</td>
</tr>
<tr>
<td>Solution methods involved in a content area or task</td>
<td>5 teachers</td>
<td>1 teacher (Kasey)</td>
</tr>
<tr>
<td><strong>Knowledge of content and students</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>What students know</td>
<td>6 teachers</td>
<td></td>
</tr>
<tr>
<td>Common student misconceptions/struggles</td>
<td>5 teachers</td>
<td>1 teacher (Kasey)</td>
</tr>
<tr>
<td><strong>Knowledge of content and teaching</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Appropriate mathematical language</td>
<td>5 teachers</td>
<td>1 teacher (Kasey)</td>
</tr>
<tr>
<td>What counts as evidence for student understanding</td>
<td>6 teachers</td>
<td></td>
</tr>
<tr>
<td>Appropriate tools/representations and connections among them</td>
<td>6 teachers</td>
<td></td>
</tr>
<tr>
<td>Connections among lessons/what had been discussed in class</td>
<td>6 teachers</td>
<td></td>
</tr>
</tbody>
</table>
Content Knowledge

Mathematical Ideas

Recall that the observed lessons all involved number and operation. During the interviews, all teachers described important mathematical ideas and solution methods about number and operations, in general, as well as discussing particular ideas and solution methods related to the content of the extending episodes. While there are many different mathematical ideas involved in the content of number and operations, such as composing and decomposing numbers, one important mathematical idea that was relevant to virtually every lesson is the meaning of the operation.

During the interviews, all six teachers explicitly verbalized the meaning of the operations they were teaching. They used phrases such as “putting together” for addition, “taking away” or “finding the difference” for subtraction, and “knowing the number of objects in each group and finding the number of groups” (measurement meaning) or “knowing the number of groups and finding the number of objects in each group” (partitive meaning) for division.

Although the second grade teacher displayed some knowledge of mathematical ideas relevant to the observed lessons, she often lacked the degree of specificity of thinking displayed by the others. For example, her references to the meaning of the operations were often vague. Although she used the phrase “putting together” for addition and “take-away” for subtraction, her discussion of subtraction stayed at an abstract level. For example, in discussing her goals for the investigation, she focused on generalizations that could be interpreted as rules (e.g., *If this is adding, then my answer*
needs to be larger. Subtracting, my answer needs to be smaller. Kasey, Pre-Investigation Interview). This could possibly cause students to develop misconceptions about adding and subtracting integers, rather than highlighting the mathematical reasoning behind these generalizations. Because of that, her content knowledge of mathematical ideas involved in a content area or a task was considered to be not as strong as the other teachers.

Solution Methods

Similar to the teachers’ knowledge about the mathematical ideas, it is not surprising that all six teachers were knowledgeable about different solution methods for the operations they were teaching. Considering the fact that these teachers have been teaching Investigations for approximately 10 years, it is very likely that they have seen students using a variety of solution methods for these problems. The first, second, and third grade teachers described several strategies for solving addition and subtraction problems, such as counting by ones, counting on, using known facts, using landmark numbers, and breaking numbers down to easier numbers. The fourth grade teacher also knew several strategies for solving multiplication and division problems, such as counting by groups of numbers, breaking numbers down to easier parts, and using easier problems and then compensating.

While five of the teachers seemed to have well-developed knowledge about how and why certain solution methods worked, the second grade teacher’s explanations for why specific solution methods worked lacked clarity. Therefore, her content knowledge about solution methods was considered to be not as strong as the others. During the
second episode from the Separating lesson, when Kasey invited students to reason about the last step of a solution method \((69 - 47 = 22, 60 - 40 = 20, 9 - 7 = 2, 20 + 2 = 22)\). Students suggested that they knew that they needed to add 20 and 2, because they wouldn’t get the right answer otherwise. In discussing this solution method during the post-investigation interview, Kasey recognized that what students suggested was not a legitimate justification for this step.

> [W]hat I’m trying to get at is, what if you didn’t know the answer and he went ahead and did that [subtracted 2 from 20]? . . . I want them to understand the strategy, not focus so much on the answer. I want them to understand what was going on there.

—Kasey, Post-Investigation Interview

Although Kasey knew that explaining why it makes sense to put the two numbers together was an essential piece of the justification, her own discussion of the solution method during the post-interview was somewhat vague. She didn’t focus on the meaning of subtraction when she explained why the strategy was mathematically legitimate. She simply suggested that for subtraction the last two parts should be put together because they were what was left over.

> We’ve split the number up into tens and ones [69 into 60 and 9; 47 into 40 and 7]. We had two separate things going on in here [60 – 40; 9 – 7], but we couldn’t just leave those [20 and 2] by themselves. We had to put them back together.

—Kasey, Post-Investigation Interview

The teachers’ content knowledge about both number and operations in general, and about the mathematical ideas and solution methods that were specific to teaching situations, was probably one of the major factors that allowed teachers to realize opportunities for extending student thinking. That kind of knowledge may have also impacted the extent to which student thinking was extended during whole-group discussions. The knowledge teachers had about the content and students may have also

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allowed teachers to realize opportunities to extend thinking. This knowledge will be discussed in the next section.

Knowledge of Content and Students

Understanding what students know about the content, what kind of misconceptions they might develop as they learn the content, and what kind of ideas might be easier or more challenging for them to grasp can definitely help teachers be prepared for teaching and help them make effective decisions in the midst of teaching mathematics. The analysis of teachers’ mathematical knowledge for teaching showed that most of them were knowledgeable about what their students knew and didn’t know about the important mathematical ideas and strategies, and what kind of misconceptions they might have about these ideas and solution methods.

What Students Know

All of the teachers were knowledgeable about what their students knew about the content. Both first grade teachers knew that students usually use one of the following solution methods to solve addition problems: counting by ones, counting on, and using known facts. In the Putting Together lesson, Linda also knew that most of the students were not aware of others’ solution methods. When students had shared their solution methods, it usually was more about sharing their own solution methods rather than comparing and contrasting theirs with others’. She knew that it was important for students to recognize different solution methods and be exposed to more sophisticated ones.
[Some of them] are still drawing everything and they are still pulling those crayons every single time. I can’t allow, I really can’t allow them to continue to do that all year long. You know at some point, you got to nudge them up to that next level. . . . There really aren’t many ways. This is it, this three ways, so again, we always want kinds to figure out, here you know you always ask them, did anybody else did it that way? And yet part of it is everybody gets to share but it’s also for them, ok I fit in this group . . .

—Linda, Post-Investigation Interview

In the Making 8 lesson, Melissa knew that some of her students could use their understanding of the relationships among numbers to better understand relationships among addends. Furthermore, she recognized the importance in facilitating the development of this kind of thinking in order for students to develop strong number sense. For example, when students use counting on to solve $6 + 3$, either they start with 6 and count on three numbers, 7, 8, 9, or they start with 3 and count on 6 numbers, 4, 5, 6, 7, 8, 9. In either case, when they then want to make 10, they can use the fact that 10 is 1 more than 9, and add 1 more to one of the addends.

The idea is to start building it on the, you know, adding on, adding 1, you know, adding 1’s to make it a strategy. . . . That they’re gonna say, “Oh, I know that $6 + 3$ is 9, oh, and $6 + 4$ is gonna be 10.” Some of them can see that . . . they know that they can just add 1 more.

—Melissa, Post-Investigation Interview

Easier and More Challenging Ideas

All of the teachers except for Kasey were knowledgeable about which ideas were easier and which ideas were more challenging for their students to grasp. As an example, in the School Days lesson, Judy knew that accuracy would be one of the main challenges for some of the students when they worked on the school days problem. As a result of not computing the numbers accurately, she expected to see many different number combinations for the school days and non-school days when she listed the numbers on the
board. She also knew that students do not always realize that there could be only one right answer to computation problems. She suggested that since students solve problems by using different strategies, they sometimes assume that there could be several correct answers to computation problems.

Kids are very accepting these days about, I think they know that there is lots of different ways to solve things so they have in mind that there is lots of right answers for things. And in general, there is one right answer for most of the problems we do, just lots of different ways to get there. So, they’ll look at all kinds of different answers and they’ll go “Yep.”

—Judy, Pre-Investigation Interview

Another challenge that she expected her students might have was recognizing the part-part-whole relationship between the school days, non-school days, and a whole year. She knew that it was not easy for students to see that we are either in school or not in school; therefore, the combination of these two groups makes a whole year.

And some of the kids are very unclear and very unsure that there might not be some other possibilities. But they have to come to an understanding that we are either in school or not in school. You know, if we are not in school we are doing other things but we are not in school. So, that whole concept for some kids, just to see that there is only two possibilities is difficult.

—Judy, Pre-Investigation Interview

Judy also knew that applying this piece of information would make the problem simpler. She knew that it could help students eliminate some of the number combinations they had on the board. They could add the suggested numbers together to see if they make 365, and if not, they could simply eliminate them.

If they can see it as, one of two choices, school or non-school, and they know the total, then it’s a whole much simpler problem for them.

—Judy, Pre-Investigation Interview

My goal kind of was to be able to eliminate the ones that don’t total 365.

—Judy, Post-Investigation Interview
In the Sharing Cards lesson, Meredith understood the strategy that one of her
students was using. She knew that his way of thinking about the problem was different
from others’ thinking and it was much simpler than other strategies. While the majority
of the students were finding the total number of cards in the three decks and then dividing
that by 6, the one student simply divided 52 by 2, thereby keeping the ratio the same.

Orion’s immediate thought was, if you just take two of those 6 and divide 1 deck,
you’re going to be able to find the answer. . . . I thought it was not the most
important, but it was an idea out there to use, you know, if you’ve got the big
numbers like what they [had in this problem] and to make it a simpler problem.
—Meredith, Post-Lesson Interview

The second grade teacher, on the other hand, had a less strong understanding of
what was easier or more challenging with respect to addition and subtraction situations.
While she first suggested that students would easily identify addition and subtraction
situations in story problems, she then contradicted herself by stating the opposite.

I think a lot of times these story problems are like real life problems, and I think
that they know, you know, by reading the problem if the answer will be more or if
it’ll be less, and I think that this will be the easiest, just knowing if they add or
subtract . . . the only other thing is just not knowing if it’s an addition or
subtraction situation. Maybe not totally understanding what addition means and
what subtraction means.
—Kasey, Pre-Investigation Interview

It seems that having a solid knowledge of content and students, being aware of
what students know and what they find difficult to understand, could support teachers in
creating whole-group discussions that have the potential for extending student thinking.
One can assume that there is a close connection between knowledge about what students
know or don’t know and what kinds of ideas might be easier or more challenging for
them and the amount of experience one has teaching particular content and tasks.
this is probably the case with most of the teachers in this study, two of the teachers somewhat contradict this generalization.

Meredith was only in her second year of using *Investigations* in fourth grade after using it in kindergarten for 10 years, and she was using a new unit from the revision for the first time. However, she still recognized that the majority of students were using the same solution method, while only one student was using a more efficient and more mathematically sophisticated solution than the others. It may be that her genuine interest in student thinking and her close observations of students’ solution methods as she was walking around the classroom enabled her to recognize different solution methods student were utilizing. Also, the professional development in which she has participated, particularly as a teacher-leader and national trainer for *Investigations* where you necessarily work across all grade levels, may have supported her in developing this knowledge. Finally, the information provided in the curriculum materials on the different types of solution methods students use for division may have also impacted her knowledge of students and content.

Kasey, on the other hand, had been teaching second grade for 12 years using *Investigations*. However, she was the only teacher who lacked pieces of KCS. It may have been that her less strong content knowledge hindered her from recognizing these aspects of her students’ mathematical knowledge.

**Knowledge of Content and Teaching**

Teachers’ KCT was a critical piece of knowledge that supported teachers in creating opportunities for extending student thinking. It helped teachers decide what ideas
and solution methods to pursue and how to pursue them. The analysis of the interviews provided information on the teachers' KCT regarding appropriate mathematical language, what counts as evidence for student understanding, and connections among tools and representations.

**Appropriate Language**

It is important both for the teacher and the students to have a common understanding of the terms that are being used during whole-group discussions. Five of the teachers were knowledgeable about how to help students clarify and explain their thinking. For example, Liz knew that it could be challenging for students to understand the meaning of the term *difference*. She also knew how to help students learn the meaning of this term, which is discussed in detail later in this chapter.

Only the second grade teacher was considered as having a less strong knowledge of appropriate language, because it appeared that she was narrowly using the terms *subtraction* and *addition* to describe problem contexts. This is problematic when one considers that the methods students use to solve some subtraction problems may involve addition. Consider the example that Kasey provided:

I would give them a situation with 27 students in class, and 14 more came in. How many would we have? And I would ask them, have them explain if they think it's addition or subtraction,

—Kasey, Pre-Investigation Interview

While this problem is clearly an addition situation, consider the next example Kasey provided:
If I give them a problem, story problem that talks about, let’s say, John is 27 and Jill is 14. How much older is John than Jill? Some kids will subtract 14 from 27. Some kids will start out with 14 and count up until they get to 27.

—Kasey, Pre-Investigation Interview

In this case, it may not be so important for students to be able to label this problem as subtraction or addition, and may, in fact, cause confusion because the solution methods can involve either addition or subtraction. In the second episode of Kasey’s Separating Situations lesson, there was evidence that this caused confusion. Students struggled with making sense of why they needed to add the last two numbers when the original problem was a subtraction problem. Kasey didn’t seem to have well-developed knowledge about the terms, such as distinctions between names of the operations versus methods for solving problems involving those operations, and she struggled with helping students make sense of these ideas.

Evidence for Student Understanding

All of the teachers were knowledgeable about what counts as evidence for student understanding. As an example, for the School Days problem, Judy recognized that since students all used the same calendar, there should be only one correct combination for the number of school days and non-school days as evidence for understanding the problem. She also knew that students might not realize this on their own. According to her, this was an important mathematical issue and she would pursue that idea by having students think about the reasonableness of having different answers for the school day problem. She specifically suggested that after creating the list of answers from different pairs, she would pose questions like “What do you think? Does this make sense?”
With the school days, a big part of showing the understanding is when we put up all their results on the chart of school days and the non-school and the difference, and then the discussion we have about, you know, can all these possibly be right answers? And can you prove somebody else that you are correct and then they start going through kind of explaining the way went through it.

—Judy, Pre-Investigation Interview

[They are not really critical thinkers a lot of times about things. They just, “Ok, whatever,” and so that’s another thing I really, I try not to be the in charge of the correctness or not correctness of things. I want them to look at, you know, “What do you think? Does this all makes sense to you?”

—Judy, Pre-Investigation Interview

Connections Among Representations

All of the teachers were knowledgeable about the connections among representations that students might use. For example, in the Distance Riddles lesson, Liz knew that students could have a hard time understanding how to use the number line and understanding the meaning of difference. She knew that using tools or representations, such as the 300 chart model and contextualized problems with which the students were already familiar, would help them understand the meaning of the word difference.

I think they’ll start thinking more along a number line model to help them solve problems. I notice a lot of kids are moving back and forth from the number line to a 300 chart. Some kids are using the 300 chart to help them do number lines. . . . I just look at this experience as building another understanding of the model that they can picture in their head and be able to move around on this number line, to add, you know, and subtract numbers.

—Liz, Pre-Investigation Interview

I was just listening to the conversation and thinking to myself, “There’s still a lot of people that are really confused. What could I do that they’re familiar with that they could make the same moves and maybe it be more of a, more of a visual?” Because I think that 300 chart is a pretty big visual and they were used to moving around on the capture board. So that’s why I thought about that. And then the candy thing—it’s just that idea of naked numbers. How can you put a context around that so they can understand it?

—Liz, Post-Investigation Interview

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Although Liz hadn’t taught that specific lesson before, she had taught similar lessons at the third level for many years from *Investigations*. It seems that her knowledge about the connections among representations and what tools are familiar to students allowed her to effectively pursue the concept *difference*.

**Connections Among Lessons**

All six teachers knew how the sessions they were teaching were connected to the prior and the following sessions. They had a clear sense of what was discussed and what was coming next. Judy, for example, knew the connections between a problem students solved the day before the School Days lesson and the School Days problem, and she knew how she could utilize this knowledge in her teaching to allow students to make that connection also.

The only answer I have for that is that because I have taught that unit many times, I started to see where those connections can be made. I think just knowing where you’re going is so big . . . I guess probably I’ve started to see over the years how important those connections are and how the kids don’t make them on their own. So, that’s kind of always on my mind is helping them make those connections.

—Judy, Pre-Investigation Interview

As Judy stated, being experienced at teaching the same lessons for several years and being aware of the connections among them could support teachers to develop a solid knowledge of content and teaching.

**The Relationship Between Teacher Beliefs/Knowledge and Instructional Actions**

In each of the six teachers' classrooms, episodes of extending student thinking were observed. What follows is the discussion of the ways in which the teachers' beliefs and knowledge supported the instructional actions that took place during extending
episodes. Throughout this section, specific examples of instructional actions are presented from extending episodes.

*The Ways in Which Teachers' Beliefs Supported Instructional Actions*

Recall that the central beliefs common among all six teachers were: that there is more than one solution method for mathematical problems, that students learn mathematics by constructing their own understanding of mathematics, and that they must be active participants in class discussions. Although many teachers might make statements similar to these, what these statements mean for actual classroom practice can vary greatly by teacher. The beliefs of these six teachers were further clarified by their descriptions of the specific types of instructional actions that teachers should employ as they teach mathematics in order to facilitate student understanding. This specificity in valued instructional actions was reflected in their classroom practice.

Consider the teachers' statements about valued instructional actions from the pre-interviews. The teachers' statements about their role in whole-group discussion focused on encouraging students to:

- share and clarify their thinking;
- listen carefully to what students offer;
- restate the thinking of peers;
- reflect on each other's ideas and solution methods; and
- provide reasoning for their thinking.

In the next section, the connection between these stated beliefs and their observed actions is explored.
Connections Between Beliefs About Valued Instructional Actions and Observed Instructional Actions

The instructional actions that took place in the extending episodes (see Table 10 on page 76) appeared to align closely with teachers' experimentalist views. They included actions that supported teachers' beliefs about having more than one solution method for mathematical problems and having students learn mathematics by constructing their own understanding of the subject. Furthermore, teachers' instructional actions from the episodes paralleled their stated beliefs about valued actions.

Inviting students to evaluate a claim or an observation was the most commonly observed extending instructional action across the extending episodes. All teachers except for Melissa utilized this instructional action during the chosen episodes. The teachers either first allowed students to share their general reactions to a mathematical phenomenon, or they directed students' attention to particular mathematical issues to reflect upon and to make sense of them. During the interviews, the teachers clearly stated that encouraging students to reflect on each other's ideas and solution methods was an instructional action that they valued. Therefore, there seems to be a close connection between their beliefs about this particular instructional action and the observed ones.

Another commonly observed extending instructional action was inviting students to provide reasoning for their thinking. The four teachers at second, third, and fourth grade utilized this instructional action in the episodes. These teachers identified encouraging students to provide reasoning for their thinking as a valued action during the pre-investigation interviews. And in the episodes, they often asked students to elaborate
on what they offered by asking questions such as *What makes you say that?* and *Why do you suppose that?*

The most commonly observed supporting action was suggesting an interpretation of a claim or an observation. This action was observed in each of the eight extending episodes. Teachers sometimes provided a summary of what had been discussed or they repeated or paraphrased some ideas. Interestingly, this action was not explicitly mentioned as a valued instructional action during the interviews. One possible interpretation of this discrepancy is that the teachers may have viewed themselves as a member of the classroom community and, thus, they may have indirectly included themselves in the valued action of reflecting on each other’s ideas and solution methods. Another possible interpretation is that this action may tend to occur when teachers feel a tension about their role in whole-group discussions in terms of how much control they need to exert in making sure issues are resolved in specific ways and in a timely manner. This issue of potential tensions will be further discussed in the next section.

Another commonly observed supporting instructional action was reminding students of the goal of the discussion, the problem, or other piece of knowledge that students already established. Five of the teachers utilized this instructional action during the episodes. Although the teachers didn’t explicitly state that they valued this particular instructional action during the interviews, it seems closely related to teachers’ beliefs about teaching mathematics based on students’ existing knowledge.
Relationship to Most Powerful Episodes

The least commonly observed supporting instructional action was introducing different representations and contexts, and the extending action was inviting students to provide counterspeculation for a claim. Interestingly, these two actions were observed in the two most powerful extending episodes. Neither of these two instructional actions was explicitly stated as a valued action. However, the former one seemed to be implied by teachers’ beliefs about how students learn mathematics as they make connections between the representation and contexts they are familiar with and their new mathematical observations. The latter one requires that students not only reflect on the claim, but also consider the reasoning for that claim, and thus it appears to be related to two valued actions: reflecting on each other’s ideas and providing reasoning for their thinking.

Overall, the analysis of the instructional actions from the most powerful episodes demonstrates a close alignment between those two teachers’ valued instructional actions from the interviews and the observed instructional actions. This result suggests that two teachers, Liz and Judy, had a well-developed and detailed vision of teaching mathematics based on student thinking. This might also suggest that their beliefs about the nature of mathematics and how students learn mathematics were more strongly held than other beliefs.

Level of Commitment to Beliefs

As discussed earlier, two of the teachers held conflicting beliefs about the teacher’s role during whole-group discussions. One of the least powerful episodes was
observed in one of these teacher's classrooms. Recall that during the pre-investigation
interview Kasey mentioned how she sometimes does too much talking and she
considered this as a weakness of her teaching. Consider the following exchange, which
took place towards the end of the second episode from the Separating Situations lesson.

T: So, who can tell me why he added there rather than subtracted. Marcus?
S: Because he wouldn't want to get the wrong answer?
T: But he wouldn't know, he wouldn't know that was the wrong answer.
S: Because it wouldn't make sense?
T: Why wouldn't it make sense? What wouldn't make sense about it? Boys and
girls, I want you to tell me why he added.
S: Because those are the two answers that he got when he took away.
T: But he could have said 20 take away 2 equals 18. Those would be the two
answers too and he could be subtracting. I want you to look at what Jenny
did. She subtracted the tens. Correct? And she got how many left over?
S: 20.
T: Then she subtracted the ones? Jenny, why did you put those together?
S: Because those were the numbers left over.
T: Those were the numbers left over. She had 20 and she had 2. That was what
was left over. And she put them together. What does putting together mean?
Sts: Adding.
T: Remember our chart from yesterday? What does putting together mean?
Sts: Adding.
T: Adding. Well, you know what, this is exactly what Don did too. He did what
Jenny did, he did with his numbers, he split it up. She kind of did the same
thing, only she splitted hers into tens and ones. So he had 20 left over, she
had 20 left over. He had 2 left over, she had 2 left over. You take what was
left over and put them together. Do you understand?
Sts: Yeah.
For the first part of this episode, Kasey utilized several instructional actions that aligned closely with the teachers’ valued instructional actions. However, as students continued to struggle with reasoning about the strategy, Kasey appeared to get impatient and she began asking questions that were not helpful for students to think more deeply about mathematics. This seemed to conflict with the teachers’ valued instructional actions that supported teaching mathematics based on student thinking.

Melissa was the other teacher who seemed to have conflicting goals for teaching mathematics and beliefs about her role during whole-group discussions. Although the extending episode from her classroom was considered as a relatively powerful one, the types of instructional actions she utilized differed from other teachers in several ways. This was the only episode in which the instructional action of inviting students to evaluate a claim or an observation was not observed. The students only shared their solutions and the teacher suggested an interpretation of their solutions as she recorded student thinking on the board. The students were not encouraged to engage with each other’s thinking. When one of the students made a mistake, instead of allowing students to reflect on it, Melissa illustrated how putting 5 twos together doesn’t make 9.

T: How can one of these ways to make 8 help us make 9, Betty?
S: . . . 2 + 2 + 2 + 2 . . . [five of the 2’s].
T: So if I had 2 + 2 + 2 + 2 + 2? Five 2’s? Let’s see. I need five 2’s. So count 5 times 2, 4, 6, 8—
S: 10.
T: Oh, wait a minute. That doesn’t make 9, does it? You may be on the right track, though. Let’s see if we can—how can we use one of these ways to help us make one of these? Sarah?

Thus, in both cases, there is some indication that the tensions that the teachers indicated in their beliefs about their roles in the classroom impacted their efforts to
facilitate extending situations. In both cases, the need for teacher control over the situation took precedence over the pursuit of student thinking. This control was manifested by instructional actions like asking very specific questions, for which a one-word answer was sufficient and choosing to provide their own interpretations of a situation. Although these actions were present in other extending episodes, the difference in these cases was that they were not accompanied by instructional actions that encouraged further student thinking.

*The Ways in Which Teachers' Knowledge Supported Instructional Actions*

The analysis of the interviews and observational data showed that teachers' content knowledge played a critical role in the way they realized the opportunities for extending student thinking. In all of the episodes, teachers utilized their content knowledge about the meaning of the operations and different solution methods for problems. It helped them recognize worthwhile mathematical ideas and strategies to pursue for whole-group discussions, understand their students' thinking, recognize the validity of students' ideas and strategies, and comprehend the reasoning behind them so that they could facilitate the discussions in a way that allowed them to extend student thinking.

*Relationship to Most Powerful Episodes*

Three of the most powerful extending episodes were from Judy's and Liz's classrooms. Recall that two of the most infrequent instructional actions took place during two of those episodes and these two teachers had strong content knowledge, KCS, and
KCT. As discussed earlier, both teachers had specific plans for what mathematical ideas and solution methods they would pursue during teaching the chosen sessions.

Judy knew the relationship between the two sets of days and the fact that school days and non-school days are complementary sets. She also knew that it would be difficult for students to recognize this relationship and that she would pursue this relationship by encouraging students to make connections between their earlier observations about the number of days in an entire year. Similarly, Liz knew that students would have a hard time understanding the concept of difference and representing it on a number line. She knew that she would use the representations and stories that were easier for students to use to help them grasp the meaning of the term difference.

What is striking about these two cases is the degree to which they displayed specific knowledge during the pre-investigation interview that could be tied to their extending episodes. While one cannot say that the other teachers did not have similar kinds of specific knowledge, the fact that these teachers had the most powerful extending episodes makes one wonder that perhaps their knowledge was more well established, or well connected, and that they were better able to make specific ties between this knowledge and what their students brought into discussions.

Relationship to Least Powerful Episodes

During the extending episodes from Kasey’s classroom, Kasey utilized a variety of instructional actions. As discussed earlier, while Kasey certainly had some important knowledge for teaching, she lacked some key pieces of content, KCS, and KCT, which seemed to hinder her ability to allow her to extend student thinking. Her content
knowledge about how the steps of the solution methods for subtraction were related to the whole problem was not strong. While she recognized that it was a worthwhile mathematical issue to pursue, her discussion of the ways the solution method was connected to the story problem was not clear. Although she encouraged students to provide reasoning for the solution method eight times during the episode (8 out of 24 total actions in one episode), the students were not able to reason about it, and she seemed unable to provide them with the kind of support they needed to do so. It seems likely that if she had had a stronger understanding of the solution method herself, that she might have been better able to support students in their efforts to understand it.

Summary

All six teachers believed that there is more than one solution method to solve mathematical problems, and they believed that students should be encouraged to examine the reasonableness of their thinking about problems and their solution methods. There were several themes that emerged during the analysis of teachers' beliefs about valued instructional actions and some of the most frequent instructional actions from the extending episodes were closely connected to teachers' stated valued instructional actions. In particular, two teachers expressed tensions in their beliefs about their goals for teaching mathematics and their roles in whole-group discussions and, not surprisingly, their teaching reflected those tensions.

The analysis of teachers' mathematical knowledge of teaching showed that all six teachers had generally solid content knowledge, knowledge of content and students, and knowledge of content and teaching. In particular, two of the teachers demonstrated
specific content knowledge, KCS, and KCT about the mathematical issues they would pursue and how they would pursue them during the lessons they taught. Interestingly, the two of the most powerful episodes took place in these classrooms. Furthermore, the instructional actions that seemed critical to making these two episodes powerful were the least frequently observed instructional actions. One of the teachers had less strong knowledge than other teachers in some domains of knowledge, such as what mathematical ideas would be easier and more challenging for students, and appropriate mathematical language. The lack of these aspects of her knowledge hindered her efforts to fully capitalize on students' ideas and solution methods during the extending episodes.
Teaching mathematics based on student thinking requires teachers to create classroom environments where students are provided with opportunities to engage in thought-provoking mathematical tasks and participate in rich discussions involving mathematical reflection and reasoning (Ball & Bass, 2003; Cobb et al., 1997; Lampert, 2001). To facilitate these discussions in a way that allows students to build meaningful connections between their existing knowledge and new ideas, teachers need to elicit, support, and extend student thinking (Fraivillig et al., 1999). While all three components are critical in teaching based on student thinking, extending student thinking seems to be more challenging than the other two components.

The desire to understand what extending student thinking entails and what it takes for teachers to be able to extend student thinking prompted this study. In this chapter, the findings from this study will be summarized and connections to past research and implications for future research will be discussed.

Types of Extending Episodes

In designing this study, the author adopted Grant et al.'s (2006) suggestions about the types of extending episodes that are relevant to elementary mathematics classrooms. Building off of the work from Fraivillig et al. (1999) and Wood (1999), they proposed three categories of extending episodes: engaging in mathematical reflection, going
beyond initial solution methods, and engaging in mathematical reasoning. These categories are consistent with suggestions made by other mathematics educators for what whole-group discussions of mathematics should entail (e.g., Cobb et al., 1997).

It is noteworthy that all of the extending episodes observed in this study fell into these three main categories as well. This suggests that this framework may account for all types of extending episodes that teachers create at the elementary level. In terms of the subtypes of scenarios for extending student thinking, however, the eight extending episodes that were observed in this study fell into only five of the eight types of extending scenarios from the framework. This might be due to the limited amount of data collected from each classroom, or it may mean that these five types of scenarios are more likely to occur in elementary classrooms.

It seems that using this framework may not be the most useful way of categorizing these episodes. The analysis of the episodes in this study suggests that a different framework for categorizing extending episodes might also be productive: *building new connections* and *addressing misconceptions/struggles*.

This new framework has several potential advantages over the original framework. First, the two categories in the new framework are more easily distinguishable than those of the original framework. The distinction between *building new connections* and *addressing misconceptions/struggles* seems to be more straightforward than distinguishing between *mathematical reflection* and *mathematical reasoning*. Second, the new framework nicely coincides with the results of a recent review of research on the effects of teaching mathematics on students' learning. In this review, Hiebert and Grouws (2007) identified the only two features of teaching
mathematics that facilitate students’ conceptual development as “explicit attention to connections among ideas, facts, and procedures, and engagement of students in struggling with important mathematics” (p. 391). Thus, this new framework, based on the two distinct categories, may provide the most potential for focusing future research on the ways in which teachers extend student thinking.

This study also provides significant information about the ways in which the powerfulness of extending episodes may be characterized. Students’ verbal expressions that showed that their thinking was indeed being extended, and the ways in which certain instructional actions allowed opportunities for students to move beyond their initial mathematical observations were considered as criteria for ranking the extending episodes in this study. While these criteria provided beginning notions about the powerfulness of extending episodes, further research would be necessary to develop a framework that more fully captures the complex nature of discussions that take place in mathematics classrooms.

Instructional Actions

The finding in this study that some of the most powerful episodes had the least frequent instructional actions was telling about the nature of those instructional actions and the challenges of enacting them in the classroom. Two of the least frequent instructional actions were making connections among representations and contexts, and using counterspeculation. In her writings about her experience teaching third grade, Ball (e.g., Ball & Bass, 2000) often refers to a quintessential example of students “doing” mathematics where one student is arguing that the number 6 can uniquely be both even
and odd because it can be divided into pairs with nothing leftover and has an odd number of pairs. Another student in the class provides a counterexample to his claim that 6 is unique by suggesting the number 10 and explaining that it has an odd number of pairs as well, and that if it is also even and odd, then many numbers would be considered so, and emphatically states that there would then be no point in having the discussion at hand.

What is so poignant about this example is the fact that argumentation through counterexample existed and that it was spontaneously offered by a student. If counterspeculation is such a powerful way to extend student thinking, one wonders why it was such an uncommon occurrence in this study given the teachers' beliefs and knowledge. This suggests that additional focus may need to be placed on such instructional actions as counterspeculation and making connections in order to promote their more frequent utilization in the classroom.

Using single instructional actions, then, as the unit of analysis in this study was useful in understanding how single instructional actions supported creating opportunities for extending student thinking during whole-group discussions. However, this focus did have limitations as well. Since teachers sometimes utilized a series of instructional actions at once, and since these actions were usually closely connected to each other, trying to determine the role of each single action was challenging. Consider the following example from the Distance Riddle lesson.

T: It says the difference between 100 and me is 45, what can it be? (The teacher brings out the 300s chart.) Think about using this 300s chart. What number are we using for our benchmark?

Sts: 100.
T: 100. Here is 100, right? (She marks 100 on the chart.) And it says the difference between me, 100 and me is 45. So what number did we decide that was?

In this short exchange, the teacher utilized one of the least frequent instructional actions, introducing a different representation, but she also utilized the action of reminding students of the problem. This makes it difficult to determine the role of the individual actions. Furthermore, when you focus on individual actions, some actions may seem more or less powerful than others. For example, reminding students of the problem may not seem a powerful action in and of itself; however, in the context of this episode, it helped set the stage for students to make connections between the number line representation and the 300 chart representation.

Finally, in some cases particular instructional actions were effective and in some others they were not. This makes it difficult to generalize about the potential of individual actions to extend student thinking. For example, while in some episodes inviting students to provide reasoning for a claim allowed students to move beyond merely sharing a claim and students did in fact justify their claims, in some other episodes, this single action was not sufficient for helping students provide reasoning for their claims. This showed that each instructional action may not result in extending student thinking on its own. In fact, it was often the case that utilizing a combination of those actions was what made some of the extending episodes more powerful than others. This suggests that future research should include the development of a more nuanced framework in order to capture the impact of not only single instructional actions, but collections of them as well.
Having a Vision for Extending Student Thinking

The teachers who participated in this study held generally experimentalist views about the nature of mathematics and learning and teaching mathematics (Wilson & Goldenberg, 1998). Furthermore, it appeared that all six teachers had fairly well-developed visions of classrooms where teaching is based on student thinking as well as what extending student thinking looks like. In all six classrooms, students seemed comfortable with sharing their ideas and solution methods and discussing them. In some classrooms, students also developed the habit of providing reasoning for their claims even when they were not specifically asked. Finally, the teachers’ stated beliefs about the instructional actions they valued seemed to be connected to the instructional actions they utilized in creating opportunities for extending student thinking.

Despite all those similarities in beliefs and valued instructional actions, there were differences in their practice that in some cases seemed to be related to differences in the depth or conviction of these beliefs, and/or tensions among beliefs. For example, all six teachers discussed the challenges of teaching mathematics based on student thinking. For the most part, these challenges appeared consistent with what Heaton (2000) called the uncertainty that is involved in anticipating what students have to offer during discussions. They believed that the uncertainty required teachers to make a lot of on-the-spot decisions and it made planning their lessons ahead of time challenging for them. However, for two of the teachers, it seemed that the tensions they expressed about the goal of teaching mathematics and their roles in whole-group discussion impacted the way they dealt with this uncertainty. Inconsistencies or tensions among beliefs have been documented previously and are consistent with Green’s (1971) theory about belief...
systems in which he suggested that since beliefs can be held in clusters, one can hold conflicting beliefs, especially if the beliefs are not held evidentially. This study provides explicit details on the ways in which these inconsistencies impact teachers as they work to extend student thinking during whole-group discussions. What remains to be seen is what might happen if there is little alignment between teachers' stated beliefs about instruction and observed instruction, or if there are other types of tensions in beliefs that may hinder teachers' ability to extend student thinking.

Having the Mathematical Knowledge for Teaching for Extending Student Thinking

The results from this study showed that all three domains of mathematical knowledge for teaching supported teachers to utilize instructional actions that are critical in creating opportunities of extending student thinking. The teachers' mathematical knowledge for teaching supported them in deciding which mathematical ideas to pursue and how to pursue them during whole group discussions. Particularly, three teachers' well-developed and detailed knowledge appeared to support them to create the more powerful episodes. These results not only support Ball and colleagues' (2005) argument that the work of teaching requires teachers to have knowledge of all these domains, but provide detailed accounts of the ways in which this knowledge is utilized during whole-group discussions. Furthermore, this study provides evidence of how a lack of certain aspects of knowledge can negatively impact a teacher's pursuit of student thinking.

The finding that these teachers generally had solid knowledge of content and teaching may not be surprising given the experience they have had with teaching Investigations and participating in and facilitating professional development about how to
implement *Investigations* effectively. The teachers were generally knowledgeable about what ideas and strategies to pursue and how to pursue them in the content area. That said, it is interesting to note the areas in which experience did not play the expected role. For example, the second grade teacher had a great amount of experience teaching second grade and yet still had some undeveloped knowledge across all domains. Also, the fourth grade teacher had a great deal of experience with the curriculum, but mostly at the kindergarten level. Even though it was only her second year of teaching fourth grade, she demonstrated a solid knowledge across all domains and she created one of the more powerful extending episodes. Clearly extensive experience at a particular grade level may not be necessary, nor sufficient, to predict a teacher's mathematical knowledge for teaching.

For future research, it would be helpful to include teachers with a more diverse knowledge base in order to make conclusions about the relative importance of the variety of aspects across all the domains of knowledge. While one teacher in this study had a lack of knowledge related to certain aspects, such as appropriate language, one can conjecture that the impact on instructional actions might be even more pronounced if a teacher had a lack of knowledge of key aspects of content, such as different solution methods and their connections.

**The Curriculum and the Tasks**

Curriculum plays a significant role in the way teachers teach mathematics and what students have opportunity to learn. The *Investigations* curriculum was designed to “support students to make sense of mathematics” through problem solving, to “emphasize
reasoning about mathematical ideas,” and to support students to develop their own solution methods and justify them (Russell, 2007, p. 24). In particular, whole-group discussion of student ideas is a significant feature of Investigations lessons. For these reasons, the Investigations curriculum provided an ideal context for this study.

Support Information

Investigations is one of the curricula that provides a great amount of support for teachers to teach mathematics based on student thinking. The lesson plans provide explicit suggestions how one can teach each particular lesson. For example, they include specific questions for teachers to pose during discussions. There are also Teacher Notes that focus on the content of the lessons and how students learn that content. Finally, there are Dialogue Boxes that provide examples of discussions with students from field-test classrooms.

The authors of Investigations provide information about what role teachers should play in their classrooms and how student thinking should guide instructional decisions. That kind of information supports teachers in developing a vision for teaching mathematics based on student thinking. For example, for the School Days lesson, the authors of Investigations suggest that teachers encourage students to share their findings and how they found their numbers. They further suggest that teachers should create a list of numbers on the board with three columns: number of school days, number of non-school days, and the difference. Next, they suggest having students reflect on the whole list and explain which numbers they agree with and why. This kind of explanation for how a teacher should facilitate whole-group discussion implies that teachers should
facilitate student learning by encouraging students to share their thinking, to reflect on their thinking, and provide reasons for their ideas/solutions.

The authors also provide information about the mathematical content, how students learn specific mathematical content based on research on student learning, and possible ways of pursuing students’ ideas and solution methods. In short, they help teachers develop mathematical knowledge for teaching. For example, in the School Days lesson, the authors of *Investigations* help teachers develop KCS by suggesting that they should not expect every student to have found the same numbers for school days and non-school days. They further suggest that the fact that there are 365 days in a year may come up in the discussion and teachers should encourage this kind of thinking.

In this study, the ways in which the support information provided in *Investigations* might help teachers create opportunities for extending student thinking was not explored. It seems that the support provided in the curriculum most likely impacted both the teacher’s knowledge as well as the way she facilitated the whole-group discussion in the School Days lesson. A detailed analysis of the curriculum materials would be useful in determining the ways in which the information that is provided in the curriculum materials supports teachers in creating opportunities for extending student thinking.

*The Tasks*

The analysis of the extending episodes in this study suggests that the nature of the tasks themselves was also an important factor in creating opportunities for interesting and sometimes conflicting mathematical ideas and solution methods to surface. This seemed
to be the case whether or not problems were contextualized, and whether or not they had more than one answer. It appeared that the expectations that students find and share different ways of solving a problem, make sense of different and sometimes conflicting ideas, and reach a consensus as a whole group was what made these tasks powerful. In other words, the tasks helped both the teachers and the students to problematize mathematics (Hiebert et al., 1996).

Consider the School Days problem, which had the potential for extending student thinking for several reasons. First, while the task may seem to be a simple addition problem, it involved large numbers and required fairly sophisticated strategies to count across months on the calendar. Second, it provided an opportunity for students to recognize the part-part-whole relationship between the number of school days, non-school days, and a whole year. Therefore, students could simply find the number of school days, then subtract this number from 365 to find the number of non-school days. Finally, because students might make simple calculation mistakes in their strategies, it provided an opportunity to create a conflict about the solutions provided by students and to challenge students to make sense of these solutions as a group.

Now consider the problem from the Making 8 lesson. While it didn’t have a real life context, it was still closely connected to the games (e.g., using dot cards to compose numbers) that students had been playing. The fact that there was more than one way to make 8 created an opportunity for students to share and discuss different solution methods. Also, encouraging students to build on the ways of making 8 in composing 9 further allowed students to recognize the relationship between numbers. For future research, it would be beneficial to study the relationship between the nature of tasks,
perhaps using a framework for analyzing tasks developed by the QUASAR project (Stein, Grover, & Henningsen, 1996), and teachers’ ability to extend student thinking. In other words, in what ways might the instructional actions be necessarily different when tasks do not inherently problematize mathematics?

Final Thoughts on Implications for Future Research

This study helped to provide much needed detail on what extending student thinking looks like, which instructional actions were instrumental in achieving the potential of extending episodes, and the ways in which teachers’ knowledge and beliefs supported their teaching. While this study suggests a new and more distinct framework for studying extending episodes, there also remain questions about what would be a helpful framework for capturing the impact of instructional actions. In addition, it is also important to remember that the teacher participants in this study were essential to providing a context within which to study extending student thinking. As mentioned before, the teachers who participated in this study were not typical teachers. In general, they had fairly well-developed mathematical knowledge for teaching related to the lessons that were the focus of the episodes, and they also held beliefs that aligned with teaching mathematics based on student thinking. In addition, examining the relationship between teachers’ knowledge and beliefs was beyond the scope of this study. To broaden the results of this study, it would be necessary to expand this research to other types of teachers, across a wide range of knowledge and beliefs, in order to understand what supports or hinders their ability to extend student thinking, and also to understand the relationship between beliefs and knowledge.
Finally, in this study, the only evidence of student thinking was what was observed during the episodes. Students’ written work was not collected, nor were interviews with students conducted. Therefore, it was not possible to get a sense of the extent to which the thinking of all of the students was extended. Collecting student work or having interviews with students could provide further insight into how extending episodes and teachers’ instructional actions impact student learning. This source of data might also perhaps help contribute to the development of a new framework, discussed previously, that would better capture the impact of single instructional actions and collections of them.

Implications Beyond Research

In support of earlier studies, the results of this study suggest that teaching mathematics based on student thinking places great demands on teachers. In particular, facilitating whole-group discussions so that they have the potential for extending student thinking is challenging. It seems that having a well-developed vision for teaching mathematics based on student thinking and a well-connected mathematical knowledge for teaching play a critical role in capitalizing on opportunities to create discussions that extend student thinking. The question then becomes, how do we help teachers develop this kind of vision and knowledge?

The results from this study suggest that one way to assist teachers in developing the desired vision may be to share with them the two different frameworks for extending scenarios: (1) encouraging mathematical reflection, going beyond initial solution methods, and encouraging mathematical reasoning; and (2) building new connections,
and addressing misconception/struggles. Further, providing explicit information about instructional actions that have the potential for supporting extending episodes can also be useful, along with the caveat that because of the context-bounded nature of discussions, there is no recipe for which instructional actions to utilize when in the midst of discussions. These decisions in the midst of discussions are clearly influenced by teachers’ mathematical knowledge for teaching.

In regards to developing this knowledge, providing information about the three domains of knowledge that are specific to the tasks can help teachers be better prepared for creating opportunities to extend student thinking. For example, the support provided in curriculum materials could be organized explicitly around the three types of knowledge—content, KCS, and KCT—necessary for teaching each task. While it is not possible to provide information about each and every possible mathematical issue that might come up during a discussion, a balanced approach to including this information seems critical for helping teachers navigate the challenges of extending student thinking.
REFERENCES


Appendix A

An Existing Framework for Instructional Actions
<table>
<thead>
<tr>
<th>ELICITING</th>
<th>SUPPORTING</th>
<th>EXTENDING</th>
</tr>
</thead>
<tbody>
<tr>
<td>Facilitates students' responding</td>
<td>Supports describer’s thinking</td>
<td>Encourages mathematical reflection</td>
</tr>
<tr>
<td>Elicits many solution methods for one problem from the entire class</td>
<td>Assists individual students in clarifying their own solution methods</td>
<td>Encourages students to analyze, compare, and generalize mathematical concepts</td>
</tr>
<tr>
<td>Waits for and listens to students’ descriptions of solution methods</td>
<td>Supports listeners’ thinking</td>
<td>Encourages students to consider and discuss interrelationships among concepts</td>
</tr>
<tr>
<td>Encourages elaboration of students’ descriptions</td>
<td>Provides teacher-led instant replays</td>
<td>Lists multiple solution methods on the chalkboard to promote reflection</td>
</tr>
<tr>
<td>Conveys accepting attitude toward students’ errors and problem-solving efforts</td>
<td>Demonstrates teacher-selected solution methods without endorsing the adoption of a particular method</td>
<td>Goes beyond initial solution methods</td>
</tr>
<tr>
<td>Promotes collaborative problem solving</td>
<td>Supports describer’s and listeners’ thinking</td>
<td>Pushes individual students to try alternative solution methods for one problem situation</td>
</tr>
<tr>
<td></td>
<td>Records symbolic representation of each solution method on the chalkboard</td>
<td>Promotes use of more efficient solution methods for all students</td>
</tr>
<tr>
<td></td>
<td>Asks a different student to explain a peer’s method</td>
<td>Encourages mathematical reasoning</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Encourages students to offer a justification for their solutions/claims</td>
</tr>
<tr>
<td>Orchestrates classroom discussions</td>
<td></td>
<td>Encourages students to engage with each others’ justifications by questioning, challenging, and elaborating</td>
</tr>
<tr>
<td>Uses students’ descriptions of solution methods for lesson’s content</td>
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<td></td>
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<tr>
<td>Monitors students’ levels of engagement</td>
<td></td>
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<tr>
<td>Decides which students need opportunities to speak publicly or which methods should be discussed</td>
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Grant, T. J., Kline, K., Crumbaugh, C., Kim, O., & Cengiz, N. (2006, p. 7)
Appendix B

Pre-Investigation Interview Protocol
Background

1. How long have you been teaching mathematics? How long at this grade level?
2. How long have you been teaching Investigations?
3. What do you think about Investigations?
4. How would you describe your own understanding of mathematics?
   Do you think you need a different understanding of mathematics as a teacher
   than someone else might? In what ways?

Teaching: Using Whole Group Discussions

<table>
<thead>
<tr>
<th>Main Questions</th>
<th>Probing/Follow-up questions</th>
</tr>
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<tbody>
<tr>
<td>1. What would you say is your primary focus when teaching mathematics?</td>
<td>If they say something like “I want to teach mathematics for understanding” then:</td>
</tr>
<tr>
<td></td>
<td>a. What does it mean to teach mathematics for understanding? Why is it important to do that?</td>
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<td></td>
<td>If they say something like “to help students make sense of mathematics” then:</td>
</tr>
<tr>
<td></td>
<td>b. Why is it important to do that?</td>
</tr>
<tr>
<td></td>
<td>If they say something like “I want them to enjoy mathematics” then:</td>
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<tr>
<td></td>
<td>c. Why is it important to do that? What other goals do you have for teaching mathematics?</td>
</tr>
<tr>
<td></td>
<td>To follow a, b, and c:</td>
</tr>
<tr>
<td></td>
<td>How is mathematics similar to and different from other subjects that you teach?</td>
</tr>
<tr>
<td>2. What do you see as the teacher’s role during whole group discussions?</td>
<td>a. What role does student understanding play in your decisions during whole group discussions?</td>
</tr>
<tr>
<td>3. What do you see as the students’ role during whole group discussions?</td>
<td>a. How do you convey those roles to the students?</td>
</tr>
<tr>
<td>4. Can you describe the strengths and weaknesses of how you facilitate whole group discussions?</td>
<td>a. How did you come to recognize those strengths and weaknesses?</td>
</tr>
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<td></td>
<td>b. How do they affect your whole group discussions?</td>
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</table>
### Investigation

<table>
<thead>
<tr>
<th>Main Questions</th>
<th>Probing/Follow-up questions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. What are your overarching goals for first grade students when you are teaching about number and operations?</td>
<td>a. What do you really want students to learn about numbers and addition and subtraction at this grade level?</td>
</tr>
</tbody>
</table>
| 2. What do you see as the main goals of this investigation? | If they list more than one, then:  
a. Can you prioritize them?  |
| 3. What important mathematical ideas do you expect/hope to come out during this investigation?  
[This may lead to a discussion of mathematical ideas and/or strategies.] Which are more important? | If no mathematical ideas are listed, then:  
You have been focusing on strategies that you would like students to explore during this investigation.  
a. Are there any ideas that you hope your students will learn during this investigation?  
There are two ways to think about this:  
b. First, are there some ideas students might need to be able to use the strategies you described?  
c. Or second, what ideas might they learn from using the strategies?  
If no strategies then:  
d. What are strategies do you expect/hope to arise during this investigation?  
Probes for both Ideas/Strategies:  
a. Do you view some as more important than others? Why?  
b. Which would be the easiest for students to learn during this investigation? Why?  
c. Which might be challenging for them? Why?  |
d. What would count as evidence of students understanding? [That is, what would you like them to say that would convince you that they understand a particular idea/strategy?]

<table>
<thead>
<tr>
<th>4. In what ways do tools/representations/manipulatives support student understanding in this investigation? Are some of them easier to use than others?</th>
</tr>
</thead>
<tbody>
<tr>
<td>e. Are there some common errors or misconceptions that you might expect?</td>
</tr>
<tr>
<td>i. What might be the reasons for these errors/challenges/misconceptions?</td>
</tr>
<tr>
<td>ii. How do you think you can help students to fix them?</td>
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</table>
Appendix C

Post-Lesson Interview Protocol
<table>
<thead>
<tr>
<th>Main Questions</th>
<th>Probing/Follow-up questions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Why did you choose to pursue this [particular idea (strategy)] during whole group discussion?</td>
<td>a. What made this idea stand out? Why did you choose this idea over others to pursue?</td>
</tr>
<tr>
<td>2. Why did you choose to pursue the ideas (strategies) in the ways that you did? Why did you take that action?</td>
<td>a. Why did you ask this question?</td>
</tr>
<tr>
<td></td>
<td>b. What were you hoping to hear?</td>
</tr>
<tr>
<td>3. Do you have a sense that students understood these ideas/strategies? What kinds of things did students say or do that convinced you that they understood?</td>
<td>a. If not, what would you have liked to have heard?</td>
</tr>
</tbody>
</table>
Appendix D

Post-Investigation Interview Protocol
<table>
<thead>
<tr>
<th>Main Questions</th>
<th>Probing/Follow-up questions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. During the pre-investigation interview you suggested that your goals for this investigation were…</td>
<td>a. To what extent do you feel the goals have been met? What seemed to contribute to meeting those goals?</td>
</tr>
<tr>
<td>2. Now, I’d like you to think about the whole group discussions. How would you describe them? How did they go? Were you satisfied with the ways you facilitated them and your students’ participation in them?</td>
<td>a. What did you find most challenging in facilitating whole group discussions in this investigation?</td>
</tr>
</tbody>
</table>
Examining one episode where student thinking is extended.

Now, I'd like you to reflect on one of the episodes that I have chosen. First, I'll show you the clip and then I'd like us to talk about particular instructional decisions you made during the episode.

Show clip

<table>
<thead>
<tr>
<th>Main Questions</th>
<th>Probing/Follow-up questions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. What were you thinking about at this point in the lesson? What was on your mind just prior to the start of this episode?</td>
<td>a. What was your overall goal in this particular episode?</td>
</tr>
<tr>
<td>2. What do you see as the critical/important decisions you made during this episode?</td>
<td>Pursuing Particular Ideas/Strategies</td>
</tr>
<tr>
<td>• What motivated that particular decision?</td>
<td>a. What important mathematical ideas were pursued? [What student strategy did you choose to focus on?]</td>
</tr>
<tr>
<td>• What was your decision based on?</td>
<td>b. Why did you pursue this particular idea (strategy)? What made it important? In what ways did you think doing this might have helped improving students’ understanding of the concept?</td>
</tr>
<tr>
<td>3. What do you see as the critical/important contributions of the students during this episode?</td>
<td>Method of Pursuit</td>
</tr>
<tr>
<td></td>
<td>c. Why did you choose to pursue it the way you did? [Why did you think that this was a good thing to do?]</td>
</tr>
<tr>
<td></td>
<td>d. Why did you choose to ask this question? [Why did you think that this was a good question to ask?]</td>
</tr>
<tr>
<td></td>
<td>Use of Tools</td>
</tr>
<tr>
<td></td>
<td>e. In what ways were the representations or tools beneficial or not beneficial?</td>
</tr>
<tr>
<td></td>
<td>f. In what ways could representations/tools have been beneficial?</td>
</tr>
<tr>
<td>4. Are there things that you would do differently next time? Explain.</td>
<td>a. Why would you do those differently?</td>
</tr>
<tr>
<td></td>
<td>b. Why do you think they didn’t occur to you in the moment?</td>
</tr>
<tr>
<td>Supporting Actions</td>
<td>Eliciting Actions</td>
</tr>
<tr>
<td>--------------------</td>
<td>------------------</td>
</tr>
<tr>
<td><strong>Using multiple solutions to promote reflection</strong></td>
<td><strong>Using multiple solutions to promote reflection</strong></td>
</tr>
<tr>
<td>I want to talk about the way you solved your problems (1st grade)</td>
<td>Can more than one of them be correct? (3rd grade)</td>
</tr>
<tr>
<td><strong>Encouraging students to try alternative solution methods</strong></td>
<td><strong>Promoting use of more efficient solutions</strong></td>
</tr>
<tr>
<td>Do you have to figure out how many cards are in all 3 decks? (4th grade)</td>
<td>How can one of these ways to make 8 help us make 9? (1st grade)</td>
</tr>
<tr>
<td><strong>Are there some other kinds of days that wouldn't fit in s. days or n-s days? (3rd grade)</strong></td>
<td><strong>Supporting Actions</strong></td>
</tr>
<tr>
<td><strong>Reminding students about goal of the discussion, the problem, or other information</strong></td>
<td><strong>Inviting students to share methods</strong></td>
</tr>
<tr>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td><strong>Recording student thinking</strong></td>
<td><strong>1</strong></td>
</tr>
<tr>
<td><strong>Repeating a claim</strong></td>
<td><strong>2</strong></td>
</tr>
<tr>
<td><strong>Suggesting an interpretation of a claim/observation</strong></td>
<td><strong>1</strong></td>
</tr>
<tr>
<td><strong>Introducing different representations/contexts</strong></td>
<td><strong>1</strong></td>
</tr>
<tr>
<td><strong>2</strong></td>
<td><strong>4</strong></td>
</tr>
<tr>
<td><strong>1</strong></td>
<td><strong>4</strong></td>
</tr>
<tr>
<td><strong>2</strong></td>
<td><strong>4</strong></td>
</tr>
<tr>
<td><strong>2</strong></td>
<td><strong>4</strong></td>
</tr>
</tbody>
</table>

| **Encouraging students to offer a justification for their solutions** |
| Are there some other kinds of days that wouldn't fit in s. days or n-s days? (3rd grade) |
| 2 | 1 | 4 |
| 1 | 1 | 4 |
| 1 | 1 | 4 |
| 4 | 1 | 1 |
Appendix F

Human Subjects Institutional Review Board
Letter of Approval
Date: September 26, 2006

To: Kate Kline, Principal Investigator
   Terry Grant, Co-Principal Investigator
   Nesrin Cengiz, Student Investigator for dissertation

From: Amy Naugle, Ph.D., Chair

Re: HSIRB Project Number: 06-09-10

This letter will serve as confirmation that your research project entitled "What Allows Teachers to Extend Student Thinking During Whole Group Discussions?" has been approved under the expedited category of review by the Human Subjects Institutional Review Board. The conditions and duration of this approval are specified in the Policies of Western Michigan University. You may now begin to implement the research as described in the application.

Please note that you may only conduct this research exactly in the form it was approved. You must seek specific board approval for any changes in this project. You must also seek reapproval if the project extends beyond the termination date noted below. In addition if there are any unanticipated adverse reactions or unanticipated events associated with the conduct of this research, you should immediately suspend the project and contact the Chair of the HSIRB for consultation.

The Board wishes you success in the pursuit of your research goals.

Approval Termination: September 26, 2007