6-2007

The Fock Space and Related Bergman Type Integral Operators

Ovidiu Furdui
Western Michigan University

Follow this and additional works at: https://scholarworks.wmich.edu/dissertations

Part of the Applied Mathematics Commons, and the Mathematics Commons

Recommended Citation
Furdui, Ovidiu, "The Fock Space and Related Bergman Type Integral Operators" (2007). Dissertations. 862.
https://scholarworks.wmich.edu/dissertations/862

This Dissertation-Open Access is brought to you for free and open access by the Graduate College at ScholarWorks at WMU. It has been accepted for inclusion in Dissertations by an authorized administrator of ScholarWorks at WMU. For more information, please contact maira.bundza@wmich.edu.
THE FOCK SPACE AND RELATED BERGMAN TYPE INTEGRAL OPERATORS

by

Ovidiu Furdui

A Dissertation
Submitted to the
Faculty of The Graduate College
in partial fulfillment of the
requirements for the
Degree of Doctor of Philosophy
Department of Mathematics
Dr. John Srdjan Petrovic, Advisor

Western Michigan University
Kalamazoo, Michigan
June 2007
Pentru mica si ticu
ACKNOWLEDGEMENTS

At the endpoint of this important chapter of my life, I would like to offer my thanks to the wonderful people surrounding me.

First, I would like to thank my adviser, Prof. Srdjan Petrovic for his encouragement and good suggestions during my graduate studies in this department. I thank the committee members Professors Paul Eenigenburg, Jim Zhu and Yan-chun James Tung for their time and effort each spent reading my dissertation.

I would also like to thank the other people in the Mathematics Department of Western Michigan University, especially all the professors I have taken classes with, for making my stay a most memorable one. I cannot forget my sister Diana who has always supported me even though I was far away from her.

I dedicate my dissertation to my parents, Maria and Traian, for all of their support especially for the great effort they made to keep me in school. I cannot forget the long winter nights my mother was knitting in order to have enough money to keep me in school. Without her efforts this thesis would have not been written. Above all, I would like to thank God, Iisus Hristos, who gives me strength and makes all things possible.

Ovidiu Furdui
## TABLE OF CONTENTS

ACKNOWLEDGEMENTS ................................................................................. ii

CHAPTER 1 ....................................................................................................... 1
1.1 Introduction ............................................................................................... 1
1.2 Notation and terminology ........................................................................... 2
1.3 Some integration lemmas .......................................................................... 4

CHAPTER 2 ....................................................................................................... 7
2.1 Reproducing functional Hilbert spaces .................................................... 7
2.2 The Fock space .......................................................................................... 8
2.3 Vukotić's inequality .................................................................................. 16
2.4 The reproducing formula ......................................................................... 23

CHAPTER 3 ..................................................................................................... 26
3.1 Bergman type integral operators on $L^p(C^n, dv_a)$ ................................... 26
3.2 Necessary conditions for the boundedness of $S_{b,c}$ .................................. 33
3.3 Sufficient conditions for the boundedness of $T_{b,c}$ .................................. 36
3.4 Proofs of the main results ......................................................................... 37
3.5 Applications ............................................................................................. 44
3.6 Open problems and remarks ................................................................... 50

BIBLIOGRAPHY .......................................................................................... 52
Chapter 1

1.1 Introduction

The Segal-Bargmann space, or the so-called Fock space, $F^p_f = H(C^n) \cap L^p(C^n, d\nu_t)$, is the analogue of the Bergman space in the context of the $n$-dimensional complex Euclidean space $\mathbb{C}^n$. It is an $L^p$ space which consists of holomorphic functions in $\mathbb{C}^n$. In particular, in the Hilbert space setting of $L^2(C^n, d\nu_t)$, there exists a unique orthogonal projection $P_t : L^2(C^n, d\nu_t) \rightarrow F^2_f$, also known as the Bergman projection. It turns out that $P_t$ is an integral operator defined by

$$P_t f(z) = \int_{\mathbb{C}^n} e^{\langle z, w \rangle} f(w) d\nu_t(w),$$

where $d\nu_t(z) = (\frac{1}{\pi})^n e^{-|z|^2} d\nu(z)$ is the Gaussian probability measure on $\mathbb{C}^n$. In 1987 Janson, Petree, and Rochberg (cf., [4, Corollary 9.1]) proved that, surprisingly, the operator $P_t$ is bounded on $L^p(C^n, d\nu_t)$ if and only if $p = 2$. This is quite different than the situation in the Bergman spaces, where it is known that the Bergman projection $P_{\alpha}$ is bounded on $L^p(B_n, d\nu_{\alpha})$ if and only if $p > 1$, (cf., [13, Theorem 2.11]). In this thesis we study the boundedness of the following class of integral operators on $L^p(C^n, d\nu_t)$

$$S_{a,b,c} f(z) = \int_{\mathbb{C}^n} e^{az^2 + b\langle z, w \rangle + c|w|^2} f(w) d\nu_t(w),$$

and

$$T_{a,b,c} f(z) = \int_{\mathbb{C}^n} |e^{az^2 + b\langle z, w \rangle + c|w|^2}| f(w) d\nu_t(w),$$

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
where \( a, b, \) and \( c \) are real parameters. We prove that when \( p > 1 \), respectively \( p = 1 \), these operators are bounded provided that \( p \) satisfies a quadratic, respectively a linear inequality.

The organization of the thesis is as follows. In Chapter 1 we review the notation and terminology and we prove some integration lemmas that we are going to use throughout the thesis. In Chapter 2 we introduce the Fock space as a reproducing functional Hilbert space and we pay a special attention to the basic properties of Fock spaces, including Vukotic’s inequality and the reproducing formula. In Chapter 3 we study the action of \( S_{a,b,c} \) and \( T_{a,b,c} \) on \( L^p(C^n, d\nu) \) and we determine exactly when these operators are bounded on \( L^p(C^n, d\nu) \). In particular, we obtain as an application of our results that, contrary to the situation in Bergman and Besov spaces, the Fock space cannot be characterized by membership of partial derivatives in \( L^p \) spaces. In the last section of Chapter 3 we give some remarks and open questions.

1.2 Notation and terminology

Let \( n \) be a positive integer and let \( C^n = C \times C \times \cdots \times C \) denote the \( n \)-dimensional complex Euclidean space. For any two points \( z = (z_1, z_2, \ldots, z_n) \) and \( w = (w_1, w_2, \ldots, w_n) \) in \( C^n \), we write

\[
(z, w) = z_1 \overline{w_1} + \cdots + z_n \overline{w_n},
\]

for the inner product in \( C^n \) and

\[
|z| = \sqrt{|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2},
\]

for the norm of \( z \). The space \( C^n \) becomes an \( n \)-dimensional Hilbert space when endowed with the inner product above. The standard basis for \( C^n \) consists of the following vectors:

\[
e_1 = (1, 0, \ldots, 0), \quad e_2 = (0, 1, 0, \ldots, 0), \quad \ldots, \quad e_n = (0, 0, \ldots, 1).
\]

An ordered \( n \)-tuple

\[
m = (m_1, m_2, \ldots, m_n),
\]
where each \( m_i, 1 \leq i \leq n, \) is a nonnegative integer is called a \textit{multi-index} of nonnegative integers. The following notations are standard:

\[
|m| = m_1 + m_2 + \cdots + m_n, \quad m! = m_1!m_2! \cdots m_n!.
\]

Also, if \( z = (z_1, z_2, \cdots, z_n) \), we write \( z^m = z_1^{m_1}z_2^{m_2} \cdots z_n^{m_n}. \) In particular, the following \textit{multinomial formula} holds:

\[
(z_1 + z_2 + \cdots + z_n)^N = \sum_{|m|=N} \frac{N!}{m!} z^m. \tag{1.1}
\]

Let \( \Omega \) be an open subset of \( \mathbb{C}^n \). A function \( f : \Omega \to \mathbb{C} \) is said to be \textit{holomorphic} in \( \Omega \), if for every \( z \in \Omega \), and for every \( k \in \{1, 2, \cdots, n\} \) the limit

\[
\lim_{\lambda \to 0} \frac{f(z + \lambda e_k) - f(z)}{\lambda}
\]

exists and is finite, where \( \lambda \in \mathbb{C} \). When \( f \) is holomorphic in \( \Omega \), we use notation \( \frac{\partial f}{\partial z_k}(z) \) to denote the above limit and call it the \textit{partial derivative} of \( f \) with respect to \( z_k \). The class of all holomorphic functions in \( \Omega \) will be denoted by \( H(\Omega) \).

If \( m = (m_1, m_2, \cdots, m_n) \) is a multi-index of nonnegative integers and \( f \) is a holomorphic function in \( \Omega \), we write

\[
\partial^m f(z) = \frac{\partial^{m_1} f(z)}{\partial z_1^{m_1} \partial z_2^{m_2} \cdots \partial z_n^{m_n}}
\]

for the \textit{partial derivative} of \( f \) at \( z \). Let \( r = (r_1, r_2, \cdots, r_n) \in \mathbb{R}^n_+ \) and let \( a \in \mathbb{C}^n \). The set

\[
P_r(a) = \{ z \in \mathbb{C}^n : |z_j - a_j| < r_j \quad \text{for all} \quad j = 1, \cdots, n \}
\]

is called the \textit{open polycylinder} or the \textit{polydisc} with \textit{center} \( a \) and \textit{polyradius} \( r \) and the set

\[
T_r(a) = \{ z \in \mathbb{C}^n : |z_j - a_j| = r_j \quad \text{for all} \quad j = 1, \cdots, n \}
\]
is called the polytorus with center $a$ and polyradius $r$.

Let $f \in H(\Omega)$ and suppose that $\Omega$ contains the closure of some polydisc $P^p_r(a)$. Then

$$f(z) = \sum_{m \in \mathbb{N}^n} \frac{\partial^{|m|} f(a)}{m!} (z - a)^m, \quad z \in P^p_r(a),$$

i.e., every holomorphic function is locally the sum of a convergent power series. All unexplained notations are as in [13].

1.3 Some integration lemmas

Let $n$ be a fixed positive integer. The open unit ball in $\mathbb{C}^n$ is the set

$$B_n = \{ z \in \mathbb{C}^n : |z| < 1 \}.$$

The boundary of $B_n$ will be denoted by $S_n$ and is called the unit sphere in $\mathbb{C}^n$. Thus,

$$S_n = \{ z \in \mathbb{C}^n : |z| = 1 \}.$$

We let $dv$ denote the volume measure on $\mathbb{C}^n = \mathbb{R}^{2n}$, and $dV$ the normalized volume measure on $\mathbb{C}^n$ so that $V(B_n) = 1$. A calculation shows that,

$$dx_1 dy_1 \cdots dx_n dy_n = dv = \frac{\pi^n}{n!} dV. \quad (1.2)$$

The surface measure on $S_n$ will be denoted by $d\sigma$. Once again, we normalize $\sigma$ so that $\sigma(S_n) = 1$. The next lemma, also known as integration in polar coordinates, will be used several times later on.

**Lemma 1.1.** The measures $V$ and $\sigma$ are related by the formula

$$\int_{\mathbb{C}^n} f(z) dV(z) = 2n \int_0^\infty r^{2n-1} dr \int_{S_n} f(r \xi) d\sigma(\xi). \quad (1.3)$$
Proof. For a proof of the lemma see ([8, Proposition 1.4.3]).

We will also make use of the following integration formulas on the unit sphere, the first of which is called integration by slices.

**Lemma 1.2.** For $f \in L^1(S_n, d\sigma)$ we have

\[
\begin{align*}
\text{a)} & \quad \int_{S_n} f \, d\sigma = \int_{S_n} d\sigma(\xi) \frac{1}{2\pi} \int_{0}^{2\pi} f(e^{i\theta}) \, d\theta. \\
\text{b)} & \quad \text{If } n > 1 \text{ then } \int_{S_n} f \, d\sigma = \int_{B_{n-1}} d\nu_{n-1}(z) \frac{1}{2\pi} \int_{0}^{2\pi} f(z, \sqrt{1 - |z|^2} e^{i\theta}) \, d\theta.
\end{align*}
\]

**Proof.** For a proof of the lemma see ([13, Lemma 1.10]).

As a consequence of part b) of Lemma 1.2 we easily check that if $m$ and $l$ are multi-indices of nonnegative integers with $m \neq l$ then,

\[
\int_{S_n} \xi^m \overline{\xi}^l \, d\sigma(\xi) = 0. \quad (1.4)
\]

The next lemma, which is used for calculating norms of certain monomials, will be used several times later on.

**Lemma 1.3.** Let $p$ be a positive real number and let $m = (m_1, m_2, \ldots, m_n)$ be a multi-index of nonnegative integers. Then

\[
\int_{S_n} |\xi|^p \, d\sigma(\xi) = \frac{(n-1)! \prod_{k=1}^{n} \Gamma \left( \frac{m_k p}{2} + 1 \right)}{\Gamma \left( n + \frac{|m| p}{2} \right)}.
\]
Proof. The proof of the lemma is indirectly stated in the proof of Lemma 1.11 of [13]. However, we include it here for the sake of completeness. Let

$$I = \int_{C^n} |z^n| p e^{-|z|^2} dV(z).$$

We evaluate $I$ by two different methods. Integration in polar coordinates shows that

$$I = 2n \int_0^{2\pi} \left( \int_{S_n} |(r\xi)|^m p e^{-r^2} d\sigma(\xi) \right) dr = n \left( 2 \int_0^{2\pi} r^{2n+m-p-1} e^{-r^2} dr \right) \int_{S_n} |\xi|^m p d\sigma(\xi).$$

A calculation shows that

$$\int_0^{2\pi} r^{2n+m-p-1} e^{-r^2} dr = \frac{1}{2} \Gamma \left( n + \frac{m-p}{2} \right).$$

Thus,

$$I = n \Gamma \left( n + \frac{m-p}{2} \right) \int_{S_n} |\xi|^m p d\sigma(\xi). \quad (1.5)$$

Another calculation using (1.2) shows that

$$I = \frac{n!}{\pi^n} \int_{C^n} |z^n| p e^{-|z|^2} dV(z) = \frac{n!}{\pi^n} \prod_{k=1}^n \int_{R^2} z_k^n p e^{-|z_k|^2} dx_k dy_k$$

$$= \frac{n!}{\pi^n} \prod_{k=1}^n \int_0^{2\pi} \int_0^\infty (r^{m_k})^p e^{-r^2} r dr d\theta = \frac{n!}{\pi^n} \prod_{k=1}^n \left[ 2\pi \int_0^{r^{m_k+1}} e^{-r^2} dr \right] \quad (1.6)$$

$$= \frac{n!}{\pi^n} \prod_{k=1}^n \left( \pi \Gamma \left( \frac{m_k p}{2} + 1 \right) \right) = n! \prod_{k=1}^n \Gamma \left( \frac{m_k p}{2} + 1 \right).$$

Combining (1.5) and (1.6) the desired result follows. \qed
2.1 Reproducing functional Hilbert spaces

In this section we briefly review the reproducing functional Hilbert spaces of which the Fock space is a particular case.

**Definition 2.1.** Let $\Omega$ be an open subset of $\mathbb{C}^n$. A reproducing functional Hilbert space on $\Omega$ is a Hilbert space $\mathcal{H}$ of functions on $\Omega$ such that for every $w \in \Omega$ the linear functional $f \rightarrow f(w)$, also known as point-evaluation functional, is bounded on $\mathcal{H}$.

If $\mathcal{H}$ is a reproducing functional Hilbert space on $\Omega$, then by the Riesz Representation Theorem for every $w \in \Omega$ there is a unique element $K_w \in \mathcal{H}$ for which

$$f(w) = \langle f, K_w \rangle, \quad \text{for all } f \in \mathcal{H}. \quad (2.1)$$

The function $K_w$ is called the reproducing kernel at $w$ and equation (2.1) the reproducing formula. The following lemma shows us how to calculate the kernel function in a reproducing functional Hilbert space.

**Lemma 2.2.** If $\{e_j : j \in J\}$ is an orthonormal basis for the reproducing functional Hilbert space $\mathcal{H}$ of functions on an open set $\Omega \subset \mathbb{C}^n$, then

$$K_w = \sum_{j \in J} e_j(w)e_j,$$

where the convergence is in $\mathcal{H}$. In particular,

$$K_w(z) = \sum_{j \in J} e_j(w)e_j(z), \quad z \in \Omega.$$
Proof. For a proof see Proposition 1.2 of [5] or Proposition 1.1.1 of [9]. □

As a consequence of Lemma 2.2 we get that $K_z(w) = \overline{K_w(z)}$. By writing $K(z, w) = K_w(z)$, we get that $K(w, z) = \overline{K(z, w)}$, for all $z, w \in \Omega$. A calculation based on (2.1) shows that the norm of the kernel function is given by:

$$||K_w||^2 = \langle K_w, K_w \rangle = K_w(w) = K(w, w).$$

The normalized reproducing kernel at $w$ is the function defined by:

$$k_w = \frac{K_w}{||K_w||} = \frac{K_w}{\sqrt{K(w, w)}}.$$  \hfill (2.2)

It is worth mentioning that this function plays a crucial role in the study of the boundedness of the Toeplitz operator on $\mathcal{H}$.

2.2 The Fock space

For $s > 0$ we consider the Gaussian probability measure $d\nu_s(z) = \left(\frac{s}{\pi}\right)^n e^{-s|z|^2} dv(z)$ on $\mathbb{C}^n$, where $dv$ is the ordinary Lebesque measure on $\mathbb{C}^n$. The Fock space, $F^p_s$, also known as the Segal-Bargmann space, is the space of all holomorphic functions on $\mathbb{C}^n$ which belong to $L^p(\mathbb{C}^n, dv_s)$. Thus, for $0 < p < \infty$,

$$F^p_s = L^p(\mathbb{C}^n, dv_s) \cap H(\mathbb{C}^n).$$

For $p > 0$ and $s > 0$ we write

$$||f||_{s,p} = \left(\int_{\mathbb{C}^n} |f(z)|^p d\nu_s(z)\right)^{\frac{1}{p}},$$

for the norm of $f$ and, when $p = 2$,

$$\langle f, g \rangle_s = \int_{\mathbb{C}^n} f(z)\overline{g(z)} d\nu_s(z),$$
for the inner product of $f$ and $g$.

If $m = (m_1, m_2, \cdots, m_n)$ is a multi-index of nonnegative integers then, a calculation based on Lemma 1.1, Lemma 1.3, and (1.2), shows that

$$||z^m||_{s,p}^p = \int_{C^n} |z^m|^p d\nu_p(z) = \left(\frac{s}{\pi}\right)^n \int_{C^n} |z^m|^p e^{-s|z|^2} d\nu(z) = \frac{s^n}{n!} \int_{C^n} |z^m|^p e^{-s|z|^2} dV(z)$$

$$= \frac{s^n}{n!} 2n \int_0^\infty r^{2n-1} dr \int_{S_n} |(r\xi)^m|^p e^{-s|\xi|^2} d\sigma(\xi)$$

$$= \frac{s^n}{n!} 2n \left( \int_0^\infty r^{2n-1+|m|p} e^{-sr^2} dr \right) \left( \int_{S_n} |\xi|^m|^p d\sigma(\xi) \right) = \prod_{k=1}^n \frac{\Gamma\left(\frac{m_k^2}{2} + 1\right)}{\frac{|m_k|^2}{2}},$$

since

$$\int_0^\infty r^{2n-1+|m|p} e^{-sr^2} dr = \frac{1}{2^{n+\frac{|m|p}{2}}} \Gamma\left(n + \frac{|m|p}{2}\right).$$

In particular, when $p = 2$, we have

$$||z^m||_{s,2}^2 = \frac{m!}{s^{|m|}}.$$

It follows that all monomials $z^m$ belong to $F^p_s$, and thus all polynomials, hence the Fock space is nontrivial, i.e., it does not consist of only the constant functions. Next, we prove that $F^2_s$ is a reproducing functional Hilbert space and we calculate the kernel function $K_w$. A calculation, based on Lemma 1.1 and (1.4), shows that if $m$ and $n$ are two multi-indices of nonnegative integers such that $m \neq n$ then

$$\langle z^m, z^n \rangle_s = 0.$$
If $f \in F^2_s$ and $f(z) = \sum a_m z^m$, we have

$$||f||^2_{s,2} = \sum_m |a_m|^2 ||z^m||^2_{s,2} = \sum_m |a_m|^2 \frac{m!}{s|m|}.$$  

Next, we prove that point-evaluation functionals are bounded on $F^2_s$. We fix $z \in \mathbb{C}^n$. Using Hölder's inequality we get that

$$|f(z)| \leq \sum_m |a_m| |z^m| = \sum_m |a_m| \sqrt{\frac{m!}{s|m|}} |z^m| \sqrt{\frac{s|m|}{m!}} \leq \sqrt{\sum_m |a_m|^2 \frac{m!}{s|m|}} \sqrt{\sum_m \frac{|z^m|^2 s|m|}{m!}} = ||f||_{s,2} \sqrt{\sum_m |z^m|^2 s|m|}. \tag{2.3}$$

A calculation based on (1.1) shows that

$$\sum_{|m|=k} \frac{|z^m|^2 k!}{m!} = (|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2)^k = |z|^{2k}.$$  

Thus,

$$\sum_m \frac{|z^m|^2 s|m|}{m!} = \sum_{k=0}^{\infty} \sum_{|m|=k} \frac{|z^m|^2 s|m|}{m!} = \sum_{k=0}^{\infty} \frac{s^k}{k!} \sum_{|m|=k} \frac{|z^m|^2 k!}{m!} = \sum_{k=0}^{\infty} \frac{s^k}{k!} |z|^{2k} = e^{s|z|^2}. \tag{2.4}$$

Combining (2.3) and (2.4) we get that for $z \in \mathbb{C}^n$,

$$|f(z)| \leq e^{s|z|^2} ||f||_{s,2},$$

and hence, $F^2_s$ is a reproducing functional Hilbert space. Next, we turn our attention to the calculation of the kernel function. We note that the set \( \left\{ z^m \sqrt{\frac{s|m|}{m!}}, m \right\} \) is an
orthonormal basis for $F^2_s$. Using Lemma 2.2 we get that

$$K_w(z) = \sum_m \frac{s^m}{m!} \overline{w^m} z^m = \sum_k \left( \sum_{|m|=k} \frac{s^m}{m!} \overline{w^m} z^m \right) = \sum_k \frac{s^k}{k!} \left( \sum_{|m|=k} \frac{k!}{m!} \overline{w^m} z^m \right).$$

An application of (1.1) shows that

$$\sum_{|m|=k} \frac{k!}{m!} \overline{w^m} z^m = (z_1 \overline{w_1} + \cdots + z_n \overline{w_n})^k = (z, w)^k.$$

It follows that

$$K_w(z) = \sum_k \frac{s^k}{k!} (z, w)^k = e^{s(z, w)}, \quad z, w \in \mathbb{C}^n. \quad (2.5)$$

Thus, the reproducing kernel function of $F^2_s$ is given by $K_w(z) = e^{s(z, w)}$.

We get, as a consequence of (2.1), that for $w \in \mathbb{C}^n$ and $f \in F^2_s$ the following formula holds:

$$f(w) = \int_{\mathbb{C}^n} f(z) K_w(z) dv_s(z) = \int_{\mathbb{C}^n} f(z) e^{s(z, w)} dv_s(z). \quad (2.6)$$

We will prove in Section 2.4 that equation (2.6) holds for all $f \in F^2_s$, $p > 1$.

Since $F^2_s$ is a closed subspace of $L^2(\mathbb{C}^n, dv_s)$ there exists a unique orthogonal projection $P_s : L^2(\mathbb{C}^n, dv_s) \rightarrow F^2_s$, also known as the Bergman projection, which maps $L^2(\mathbb{C}^n, dv_s)$ onto $F^2_s$. If $f \in L^2(\mathbb{C}^n, dv_s)$, we express $P_s$ in terms of the reproducing kernel $K_w$ as follows:

$$P_s f(w) = \langle P_s f, K_w \rangle = \langle f, K_w \rangle = \int_{\mathbb{C}^n} f(z) K_w(z) dv_s(z) = \int_{\mathbb{C}^n} f(z) K(w, z) dv_s(z).$$

Using (2.5), we get that $P_s : L^2(\mathbb{C}^n, dv_s) \rightarrow F^2_s$ is an integral operator defined by:

$$P_s f(w) = \int_{\mathbb{C}^n} f(z) e^{s(w, z)} dv_s(z), \quad f \in L^2(\mathbb{C}^n, dv_s).$$
In Chapter 3 we will study the action of a general class of integral operators on $L^p(\mathbb{C}^n, dv_x)$ which generalizes the Bergman projection $P_s$. In particular, we get that $P_s$ is bounded on $L^p(\mathbb{C}^n, dv_x)$ if and only if $p = 2$.

The next lemma, which gives the fundamental integral identity for powers of kernel functions on Fock spaces, will be used several times later on.

**Lemma 2.3.** Let $s > 0$ and let $t$ be a real number. Then,

$$\int_{\mathbb{C}^n} |e^{t(x,a)}| dv_x(z) = e^{t^2 |a|^2 / 4s},$$

for all $a \in \mathbb{C}^n$.

**Proof.** The lemma can be proved by an application of the reproducing formula, (cf., [1, Lemma 3]), however we present an elementary argument. Let $a = \alpha + i\beta, \ z = x + iy$, and let

$$I = \int_{\mathbb{R}^2} e^{t|z|^2} e^{-s|z|^2} \, dx \, dy.$$ 

Then,

$$I = \int_{\mathbb{R}^2} e^{t(x+iy)} e^{-s(x^2+y^2)} \, dx \, dy = \int_{\mathbb{R}} e^{-s(x-i\beta)^2} + \frac{t^2 \beta^2}{4s} \, dx \int_{\mathbb{R}} e^{-s(y-i\alpha)^2} + \frac{t^2 \alpha^2}{4s} \, dy$$

$$= e^{t^2 |z|^2 / 4s} \int_{\mathbb{R}} e^{-s(x-i\beta)^2} \, dx \int_{\mathbb{R}} e^{-s(y-i\alpha)^2} \, dy = \frac{\pi}{s} e^{t^2 |z|^2 / 4s}.$$

It follows that,

$$I = \int_{\mathbb{R}^2} e^{t|z|^2} e^{-s|z|^2} \, dx \, dy = \frac{\pi}{s} e^{t^2 |z|^2 / 4s}. \quad (2.7)$$
Using (2.7) we get that,

\[
\int_{\mathbb{C}^n} |e^{t(z,a)}| dv_\nu(z) = \int_{\mathbb{R}^{2n}} |e^{t(z,a)}| \left( \frac{s}{\pi} \right)^n e^{-s|z|^2} dx_1 dy_1 \cdots dx_n dy_n
\]

\[
= \left( \frac{s}{\pi} \right)^n \prod_{i=1}^n \int_{\mathbb{R}^2} |e^{t_i(z)}| e^{-s|z|^2} dx_i dy_i = \left( \frac{s}{\pi} \right)^n \prod_{i=1}^n \frac{s}{s} e^{\frac{t_i|z|^2}{4s}} = e^{\frac{t|z|^2}{4s}}.
\]

Lemma 2.3 shows that, for a real number \( t \) and for fixed \( a \in \mathbb{C}^n \), the function \( f(z) = e^{t(z,a)} \) belongs to the Fock space \( \mathcal{F}_p \), \( 0 < p < \infty \). However, a more general result holds.

**Lemma 2.4.** Let \( w \in \mathbb{C}^n \) and let \( b \in \mathbb{C} \). Let \( a \in \mathbb{C}^n \) be such that \( |a| = \max_{i=1, \ldots, n} |a_i| < \frac{s}{p} \) and let \( P \) be a polynomial in \( \mathbb{C}^n \). Then the function

\[
f(z) = P(z)e^{a_1 z_1^2 + \cdots + a_n z_n^2 + b(z,w)}
\]

belongs to \( \mathcal{F}_p \), \( 0 < p < \infty \).

**Proof.** Let \( \epsilon \) be a real number satisfying \( 0 < \epsilon < \frac{s}{2p}(s - p|a|) \). For large \( |z| \) we get that

\[
|P(z)| \leq e^{\epsilon |z|^2} \quad \text{and} \quad |b(z,w)| \leq \epsilon |z|^2.
\]

It follows that, for large \( |z| \), we have

\[
|f(z)|^p \leq |P(z)|^p e^{pRe(a_1 z_1^2 + \cdots + a_n z_n^2 + b(z,w))} \leq e^{p|z|^2} e^{p(|a||z|^2 + |b(z,w)|)} = e^{(2p + p|a|)|z|^2}.
\]

Therefore we get that for sufficiently large \( R \) and \( |z| > R \), we have

\[
||f||_{p,a}^p = \int_{|z|<R} |f(z)|^p dv_\nu(z) + \int_{|z|>R} |f(z)|^p dv_\nu(z)
\]

\[
\leq \int_{|z|<R} |f(z)|^p dv_\nu(z) + \left( \frac{s}{\pi} \right)^n \int_{|z|>R} e^{-|z|^2(s - 2p + p|a|)} dv(z) < \infty,
\]

\[
\square
\]
and the result follows. □

Remark 2.5. The inequality $|a| < \frac{r}{p}$ in the preceding lemma is sharp. To see this we note that if $f(z) = e^{\frac{z}{p}z^2}$, then a calculation shows that $\|f\|_p^p = \frac{r}{p} \int e^{-2n\beta} dx_1 dy_1 = \infty$.

Next, we turn our attention to the kernel function of Fock space. Using (2.2) and (2.5) we get that, for $w \in \mathbb{C}^n$, the normalized reproducing kernel at $w$ is the function:

$$k_w(z) = \frac{K_w(z)}{||K_w||} = e^{s(z,w) - \frac{1}{2} |w|^2}.$$

The next lemma shows the asymptotic behavior of $k_w$.

Lemma 2.6. $k_w \to 0$ weakly in $L^2 (\mathbb{C}^n, dv_z)$ as $|w| \to \infty$.

Proof. We use the fact that the bounded functions with compact support are dense in $L^2 (\mathbb{C}^n, dv_z)$. Let $F$ be a bounded linear functional on $L^2 (\mathbb{C}^n, dv_z)$. We need to show that $F(k_w) \to 0$ as $|w| \to \infty$. Since $F$ is bounded there exists $g \in L^2 (\mathbb{C}^n, dv_z)$ such that $F(f) = \langle f, g \rangle$. On the other hand, for $\epsilon > 0$ there exists $\phi$, a bounded function with compact support, such that $\|g - \phi\|_2 < \epsilon$. We have,

$$|F(k_w)| = |\langle k_w, g \rangle| \leq |\langle k_w, g - \phi \rangle| + |\langle k_w, \phi \rangle| < \epsilon + |\langle k_w, \phi \rangle|.$$

Using Lemma 2.3 we get that

$$|\langle k_w, \phi \rangle| = \left| \frac{1}{C_n} \int k_w(z) \overline{\phi(z)} dv_z(z) \right| \leq ||\phi||_2 \int \left| k_w(z) \right| dv_z(z)$$

$$||\phi||_2 e^{-\frac{1}{2} |w|^2} \int_{\mathbb{C}^n} e^{s(z,w)} dv_z(z) = ||\phi||_2 e^{-\frac{1}{2} |w|^2} e^{\frac{1}{2} |w|^2} = ||\phi||_2 e^{-\frac{1}{4} |w|^2}.$$

It follows that $|F(k_w)| \leq \epsilon + ||\phi||_2 e^{-\frac{1}{2} |w|^2}$ Letting $|w| \to \infty$ we get that $|F(k_w)| \leq \epsilon$, and since $\epsilon$ was arbitrary the result follows. □
The next lemma will be used in the proof of Vukotic's inequality.

**Lemma 2.7.** Let $0 < p < \infty$ and let $a \in \mathbb{C}^n$. If $f \in F^p_s$, let $V_{a,p,s}f(z) = e^{\frac{(2s-1)a}{p}}f(z - a)$.

Then $V_{a,p,s}$ maps $F^p_s$ onto itself with inverse $e^{\frac{-2a}{p}}V_{-a,p,s}$.

**Proof.** Straightforward calculations show that $V_{a,p,s}$ is invertible and its inverse is given by $e^{\frac{-2(s-1)a^2}{p}}V_{-a,p,s}$. On the other hand,

$$||V_{a,p,s}f||_{s,p}^p = \int_{\mathbb{C}^n}|V_{a,p,s}f(z)|^p dv_s(z) = \int_{\mathbb{C}^n}|e^{(2s-1)a}||f(z - a)|^p dv_s(z).$$

We change variables, $z - a = y$, and we get that

$$||V_{a,p,s}f||_{s,p}^p = \int_{\mathbb{C}^n}|e^{(2s-1)a^2}||f(y)|^p \left(\frac{s}{\pi}\right)^n e^{-s|y|^2} dv(y)$$

$$= \int_{\mathbb{C}^n}e^{(s-1)|a|^2}e^{(2s-1)a^2}||f(y)||^p \left(\frac{s}{\pi}\right)^n e^{-s|y|^2} e^{-2s\Re(y,a)} dv(y)$$

$$= e^{(s-1)|a|^2} \int_{\mathbb{C}^n}||f(y)||^p dv(y) = e^{(s-1)|a|^2}||f||_{s,p}^p.$$

It follows that

$$||V_{a,p,s}f||_{s,p} = e^{\frac{(s-1)|a|^2}{p}}||f||_{s,p}. \quad (2.8)$$

**Remark 2.8.** Equation (2.8) shows that the operator $V_{a,p,s}$ is a constant multiple of an isometry of $F^p_s$. In particular, when $s = 1$, we get that $V_{a,p,1}$ is an isometry of $F^p_1$ with inverse $V_{-a,p,1}$, (cf., [2, Proposition 2]).
A crucial result we will use in the proof of the reproducing formula is related to the density of polynomials in Fock spaces. More precisely, Wojtaszczyk and Garling proved that the polynomials are dense in $F^p$. We state below their result adapted to the Fock space $F^p$.

**Lemma 2.9.** For any $0 < p < \infty$ and $s > 0$ the polynomials are dense in $F^p$.

**Proof.** See Proposition 5 of [2]. □

**Corollary 2.10.** If $p \geq 1$ then the Fock space $F^p$ is a separable Banach space.

**Proof.** The set of polynomials whose coefficients are complex numbers with real and imaginary parts rational numbers is a countable dense subset of $F^p$. It is worth mentioning that when $0 < p < 1$ the space $F^p$ is a separable metric space. □

### 2.3 Vukotić’s inequality

In this section we prove that point evaluation functionals are bounded on Fock spaces. Also, we obtain, as a consequence of Lemma 2.11, some basic properties of Fock spaces including the *nested property*, estimation of the Taylor coefficients and the *end behavior* of functions in $F^p$. The next lemma shows how fast a function in $F^p$ can grow when $|z|$ is large.

**Lemma 2.11.** *(Vukotić’s inequality).* Let $0 < p < \infty$ and let $z \in \mathbb{C}^n$. Then for all $f \in F^p$ we have,

$$|f(z)| \leq e^{\frac{s}{2}|z|^2} \|f\|_{s,p}.$$ 

**The first proof.** We follow an idea of Zhu, (cf., [14, Theorem 2]). If $t > 0$, we note that the function $F(u) = |f(z + u)e^{-t(|z|^2 + 2\langle u, z \rangle)}|^p$ is plurisubharmonic in $\mathbb{C}^n$. It follows that,

$$|F(0)| \leq \int_{S_n} |F(r\xi)|d\sigma(\xi).$$
This implies that,

\[
|f(z)|^pe^{-tp|z|^2} \leq \int_{S_n} |f(z + r\xi)|^p e^{-tp(|z|^2 + 2r\xi \cdot z)} \, d\sigma(\xi)
\]

= \int_{S_n} |f(z + r\xi)|^p e^{-tp|z + r\xi|^2} \, d\sigma(\xi).

Thus,

\[
|f(z)|^pe^{-tp|z|^2} e^{-tpr^2} \leq \int_{S_n} |f(z + r\xi)|^p e^{-tp|z + r\xi|^2} \, d\sigma(\xi). \tag{2.9}
\]

Multiplying (2.9) by \(2nr^{2n-1}\) and integrating from 0 to \(\infty\) we get that,

\[
2n \int_0^\infty r^{2n-1} e^{-tpr^2} \, dr |f(z)|^pe^{-tp|z|^2} \leq 2n \int_0^\infty r^{2n-1} \, dr \int_{S_n} |f(z + r\xi)|^p e^{-tp|z + r\xi|^2} \, d\sigma(\xi).
\]

Using Lemma 1.1 and (1.2) we get that,

\[
|f(z)|^p e^{-tp|z|^2} \left(2n \int_0^\infty r^{2n-1} e^{-tpr^2} \, dr\right) \leq \frac{n!}{\pi^n} \int |f(z + u)|^p e^{-tp|u + z|^2} \, dv(u).
\]

Since \(2n \int_0^\infty r^{2n-1} e^{-tpr^2} \, dr = \frac{\Gamma(n+1)}{(tp)^n}\), it follows, based on the substitution \(z + u = y\), that

\[
|f(z)|^p e^{-tp|z|^2} \left(\frac{tp}{\pi}\right)^n \int |f(y)|^p e^{-tp|y|^2} \, dv(y) = e^{tp|z|^2} \int |f(y)|^p \, dv_h(y).
\]

Finally if \(tp = s\), we get that \(|f(z)| \leq e^{\frac{s}{p}|z|^2} \|f\|_{s,p}\). \qed
The second proof of the lemma is based on an application of Lemma 2.7.

The second proof. Let \( f \in F_2^p \). Since \( |f(z)|^p \) is plurisubharmonic we have

\[
|f(0)|^p \leq \int_{S_n} |f(r\xi)|^p d\sigma(\xi).
\]

(2.10)

Multiplying both sides of (2.10) by \( 2n_s e^{n-1} e^{-s^2} \) and integrating from 0 to \( \infty \) we get that,

\[
2n_s \int_0^\infty 2n_r e^{-s^2} |f(0)|^p dr \leq 2n \int_0^\infty |f(r\xi)|^p e^{-s^2} |s|^n d\sigma(\xi)
\]

\[
\left( \frac{s}{\pi} \right)^n \int_{\mathbb{C}^n} |f(z)|^p e^{-|s|^2} \frac{n_s}{n!} dV(z) = \left( \frac{s}{\pi} \right)^n \int_{\mathbb{C}^n} |f(z)|^p e^{-|s|^2} dV(z) = \int_{\mathbb{C}^n} |f(z)|^p dV_a(z).
\]

A calculation shows that the left hand side of the preceding inequality equals \( |f(0)|^p \), and hence we get that

\[
|f(0)| \leq \|f\|_{s,p}.
\]

(2.11)

This proves the desired result when \( z = 0 \).

We fix \( a \in \mathbb{C}^n \) and let \( f \in F_2^p \). We note that \( (V_{-a,p,s}f)(0) = e^{-\frac{|a|^2}{q}} f(a) \). Combining (2.8) and (2.11) we get that,

\[
\left| e^{-\frac{|a|^2}{q}} f(a) \right| = |(V_{-a,p,s}f)(0)| \leq \|V_{-a,p,s}f\|_{s,p} = e^{-\frac{(q-1)|a|^2}{q}} \|f\|_{s,p},
\]

and the lemma is proved. \( \square \)

**Lemma 2.12.** Let \( a \in \mathbb{C}^n \) and let \( F_a : F_2^p \to \mathbb{C} \) be given by \( F_a(f) = f(a) \). Then

\[
\|F_a\|_{F_2^p \to \mathbb{C}} = e^{-\frac{|a|^2}{q}}.
\]

18
Proof. If $a = 0$ we get, based on (2.11), that $||F_0||_{F^p \to \mathbb{C}} \leq 1$. On the other hand, since $F_0 1 = 1$, we get that $||F_0||_{F^p \to \mathbb{C}} \geq 1$ and hence $||F_0||_{F^p \to \mathbb{C}} = 1$. Let $a \in \mathbb{C}^n$. We have

$$F_0 (V_{-a,p,s} f) = V_{-a,p,s} f(0) = e^{-\frac{\|a\|^2}{p}} f(a) = e^{-\frac{\|a\|^2}{p}} F_a(f).$$

It follows that

$$F_a(f) = e^{-\frac{\|a\|^2}{p}} F_0 (V_{-a,p,s} f).$$

Using (2.8) we get that

$$|F_a(f)| = e^{-\frac{\|a\|^2}{p}} |F_0 (V_{-a,p,s} f)|$$

$$\frac{|F_0 (V_{-a,p,s} f)|}{||f||_{s,p}} = e^{-\frac{\|a\|^2}{p}} |F_0||_{F^p \to \mathbb{C}} = e^{-\frac{\|a\|^2}{p}},$$

Since $V_{-a,p,s} f \in F^p_\infty$ when $f \in F^p_\infty$, (cf., Lemma 2.7), we get that

$$||F_0||_{F^p \to \mathbb{C}} = e^{-\frac{\|a\|^2}{p}} |F_0||_{F^p \to \mathbb{C}} = e^{-\frac{\|a\|^2}{p}},$$

and the lemma is proved. \hfill \Box

Remark 2.13. It is worth mentioning that an inequality similar to that of Lemma 2.11 was used by Vukotić to calculate the norm of point evaluation functionals on Bergman space $A^p_\infty$, (cf., [12, Corollary]).

Lemma 2.11 shows that if $f \in F^p_\infty$ and $z \in \mathbb{C}^n$ then $f(z) e^{-\frac{\|z\|^2}{p}}$ is bounded by $||f||_{s,p}$. However, approximating a function in $F^p_\infty$ by polynomials, we get that a stronger result holds.

Corollary 2.14. Let $f \in F^p_\infty$. Then,

$$\lim_{|z| \to \infty} f(z) e^{-\frac{\|z\|^2}{p}} = 0.$$  

Proof. Let $\epsilon > 0$ and let $f \in F^p_\infty$. Since the polynomials are dense in $F^p_\infty$, there is a polynomial $P$ such that $||f - P||_{s,p} < \epsilon$. We have

$$|f(z) e^{-\frac{\|z\|^2}{p}}| \leq |f(z) - P(z) e^{-\frac{\|z\|^2}{p}}| + |P(z)| e^{-\frac{\|z\|^2}{p}}.$$
Using Lemma 2.11 we get, since \( f - P \in F^p_s \), that
\[
|f(z)e^{-\frac{1}{p}|z|^2}| \leq e + |P(z)|e^{-\frac{1}{p}|z|^2}. 
\]
It follows that \( \lim_{|z| \to \infty} |f(z)e^{-\frac{1}{p}|z|^2}| \leq e \), and since \( e \) was arbitrary the result follows. \( \square \)

The next lemma, which deals with the nested property of Fock spaces, shows that \( F^p_s \) decreases when \( p \) increases.

**Lemma 2.15.** If \( 0 < p < q < \infty \) then \( F^p_s \subseteq F^q_s \).

**Proof.** Let \( f \in F^q_s \). Using Lemma 2.11, we get that
\[
|f(z)| \leq e^{\frac{p}{q}|z|^2} \|f\|_{s,q}. 
\]
Then,
\[
\int_{\mathbb{C}^n} |f(z)|^p |dz| \leq \int_{\mathbb{C}^n} e^{\frac{2p}{q}|z|^2} \|f\|_{s,q}^p |dz| < \infty.
\]

To prove that the inclusion is strict, we use the following approach. If \( g \) is an analytic function in \( \mathbb{C} \), then \( f(z) = g(z_1)z_2 \cdots z_n \), is a holomorphic function in \( \mathbb{C}^n \). A calculation shows that,
\[
\|f\|_{p,q}^p = \|g\|_{s,p}^p \prod_{j=2}^n |z_j|^p |dz_j|, \quad (2.12)
\]
hence \( f \in F^p_s (\mathbb{C}^n) \) if and only if \( g \in F^p_s (\mathbb{C}) \). Let \( 0 < p < q < \infty \) and let \( \lambda \) be a real number such that \( \frac{p}{q} < \lambda < \frac{q}{p} \). If \( g(z) = e^{\lambda z^2} \), we prove that \( g \in F^p_s (\mathbb{C}) \) and \( g \notin F^q_s (\mathbb{C}) \).

We have,
\[
\|g\|_{s,p}^p = \frac{s}{\pi} \int_{\mathbb{C}} e^{\lambda |z|^2} |e^{-s|z|^2} dA(z) = \frac{s}{\pi} \int_{\mathbb{R}} e^{(\lambda p-s)x^2} dx \int_{\mathbb{R}} e^{-(\lambda p+s)y^2} dy < \infty,
\]
and
\[
\|g\|_{s,q}^q = \frac{s}{\pi} \int_{\mathbb{C}} e^{\lambda q |z|^2} |e^{-s|z|^2} dA(z) = \frac{s}{\pi} \int_{\mathbb{R}} e^{(\lambda q-s)x^2} dx \int_{\mathbb{R}} e^{-(\lambda q+s)y^2} dy = \infty.
\]

20

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Thus, equation (2.12) implies that $f \in P^p_r(C^n)$ and $f \notin P^p_s(C^n)$.

Let $p \geq 1$ and let $a \in C^n$. Let $f \in P^p_r$ and let $g$ be the function given by $g(z) = f(z + a)$. It is natural to investigate the membership of $g$ in $P^p_s$. Unfortunately, as the next lemma shows, the translation function $g$ need not be in $P^p_s$.

**Lemma 2.16.** Let $p \geq 1$ and let $a \in C^n$, $a \neq 0$. There exists $f \in P^p_r$ such that $g \notin P^p_s$.

**Proof.** Assume to the contrary that $g \in P^p_s$ for all $f \in P^p_r$. Let $A : P^p_r \to P^p_s$ be the operator given by $A(f) = f$. It is straightforward to show that $A$ is a linear operator. We prove that $A$ is a continuous operator. If $(f_n)_{n \in \mathbb{N}}$ is a sequence of functions in $P^p_r$ which converges in $P^p_s$ to a function $f$ and $A(f_n)$ converges in $P^p_s$ to a function $g$, we prove that $A(f) = g$. Using Lemma 2.11 we get that,

$$|f_n(z) - f(z)| \leq e^p|z|^p \|f_n - f\|_{s,p},$$

from which it follows that, for $z \in C^n$, we have $f_n(z) \to f(z)$. On the other hand, an application of the same lemma shows that,

$$|Af_n(z) - g(z)| \leq e^p|z|^p \|Af_n - g\|_{s,p},$$

hence, for $z \in C^n$, we get that $Af_n(z) \to g(z)$. This implies, since $Af_n(z) = f_n(z + a)$, that $f_n(z + a) \to g(z)$. It follows, since $f_n(z + a) \to f(z + a)$, that $g(z) = f(z + a)$. An application of the Closed Graph Theorem, (cf., [7, Proposition 12]), shows that $A$ is a continuous operator.

Let $n$ be a natural number and let $f_n(z) = e^{n(z,a)}$. Using Lemma 2.3 we have

$$\int_{C^n} |f_n(z)|^p dv(z) = \int_{C^n} |e^{n(z,a)}|^p dv(z) = e^{np||a||^2} < \infty,$$

hence $f_n \in P^p_r$. On the other hand,

$$Af_n(z) = f_n(z + a) = e^{n(z+a,a)} = e^{n(z,a)}e^{n||a||^2} = f_n(z)e^{n||a||^2}.$$
Thus,

\[ \frac{|A f_n|_{s,p}}{|f_n|_{s,p}} = e^{n|a|^2} \to \infty, \quad n \to \infty, \]

which contradicts the boundedness of \( A \). \( \square \)

**Remark 2.17.** Let \( p \geq 1, \ b \in \mathbb{C}^n \) and let \( g(z) = f(z)e^{s(x,b)} \), where \( f \) is a holomorphic function in \( \mathbb{C}^n \). Since

\[ \int_{\mathbb{C}^n} |g(z)|^p dv_\alpha(z) = e^{s^2|b|^2} \int_{\mathbb{C}^n} \left| f \left( \frac{pb}{2} + y \right) \right|^p dv_\alpha(y), \]

Lemma 2.16 also shows that, in general, if \( f \in F^p_s \) then \( g \notin F^p_s \). However, an application of Hölder's inequality shows that if \( f \in F^p_s, \ p > 1 \), then \( g \in F^p_s \).

If \( f \) is a holomorphic function in \( \mathbb{C}^n \) then the Taylor series of \( f \) can be written as:

\[ f(z) = \sum_{m} a_m z^m = \sum_{m} \frac{\partial^m f(0)}{m!} z^m. \]

Wojtaszczyk and Garling, (cf., [2, Lemma 1]), obtained estimates for the Taylor coefficients of functions which belong to \( F^p_s \). The next lemma gives their result adapted to \( F^p_s \) spaces. We include it here for the sake of completeness.

**Lemma 2.18.** If \( f \in F^p_s \) then,

\[ |a_m| \leq \|f\|_{s,p} \prod_{j=1}^{n} \left( \frac{2\pi s}{pm_j} \right)^{\frac{m_j}{2}}. \]

**Proof.** Let \( r = (r_1, r_2, \ldots, r_n) \in R^+_n \) and let \( T^n_r(0) \) be the polytorus with center 0 and polyradius \( r \), that is,

\[ T^n_r(0) = \{ z \in \mathbb{C}^n : |z_j| = r_j, \ \text{for all} \ \ j = 1, 2, \ldots, n \}. \]
By Cauchy Integral Formula, (cf., [10, Theorem 1.3.3]), we get that,

\[ a_m = \frac{\partial^m f(0)}{m!} = \frac{1}{(2\pi i)^n} \int_{\gamma(0)} \frac{f(w)}{w^{m+1}} dw \]

\[ = \frac{1}{(2\pi i)^n} \int_{|w_1|=r_1} \cdots \int_{|w_n|=r_n} \frac{f(w)}{w_1^{m_1+1} \cdots w_n^{m_n+1}} dw_1 \cdots dw_n. \]

It follows, based on Lemma 2.11, that

\[ |a_m| \leq \|f\|_{s,p} \prod_{j=1}^n \int_{|w_j|=r_j} e^{\frac{s}{r_j} |w_j|^2} |dw_j| = \|f\|_{s,p} \prod_{j=1}^n \int_0^{2\pi} e^{\frac{s}{r_j} r_j^2} d\theta_j = \|f\|_{s,p} \prod_{j=1}^n \frac{e^{\frac{s}{r_j} r_j^2}}{r_j}. \]

Thus,

\[ |a_m| \leq \|f\|_{s,p} \prod_{j=1}^n \frac{e^{\frac{s}{r_j} r_j^2}}{r_j}. \]

If we set \( r_j = \sqrt{\frac{pm_j}{2s}} \), we get that

\[ |a_m| \leq \|f\|_{s,p} \prod_{j=1}^n \left( \frac{2es}{pm_j} \right)^{m_j}, \]

and the lemma is proved. \( \square \)

### 2.4 The reproducing formula

In this section we prove the reproducing formula. More precisely, we show that formula (2.6) holds for all \( f \in F_p^p \), \( p > 1 \). The following result, also known as the reproducing formula, gives an integral representation for functions in \( F_p^p \) when \( p > 1 \).
Lemma 2.19. Let $1 < p < \infty$ and let $f \in F^p_d$. Then,

$$f(z) = \int_{\mathbb{C}^n} f(w)e^{s(z,w)} dv_s(w), \quad \text{for all } z \in \mathbb{C}^n. \quad (2.13)$$

Proof. We follow an idea of [11] adapted to our case. First, we prove that formula (2.13) holds for monomials and hence for polynomials. An application of Lemma 3.4, with $n = 1$, shows that

$$\int_{\mathbb{C}} e^{bzw} w^k dA_s(w) = \frac{b^k z^k}{s^k}. \quad (2.14)$$

Letting $b \to s$ we get, based on Lebesque Convergence Theorem, that

$$\int_{\mathbb{C}} e^{szw} w^k dA_s(w) = z^k. \quad (2.14)$$

Let $f(z) = z_1^{m_1}z_2^{m_2} \cdots z_n^{m_n}$ be a monomial. Using (2.14) we get that,

$$\int_{\mathbb{C}^n} f(w)e^{s(z,w)} dv_s(w) = \prod_{j=1}^n \int_{\mathbb{C}} w_j^{m_j} e^{sz_jw_j} dA_s(w_j) = \prod_{j=1}^n z_j^{m_j} = f(z).$$

Thus, if $z \in \mathbb{C}^n$ and $P$ is a polynomial, we have that

$$P(z) = \int_{\mathbb{C}^n} P(w)e^{s(z,w)} dv_s(w). \quad (2.15)$$

Let $p > 1$ and let $f \in F^p_d$. Using Lemma 2.9 we get that for $\epsilon > 0$ there is a polynomial $P$ such that $||f - P||_{s,p} \leq \epsilon$. An application of Lemma 2.11 shows that

$$|f(z) - P(z)| \leq e^{s|z|^2}||f - P||_{s,p} \leq e^{s|z|^2}\epsilon. \quad (2.16)$$
On the other hand, a calculation based on (2.15) and (2.16) shows that

\[
\begin{align*}
|f(z) - \int_{C^n} f(w)e^{s(z,w)}dv_{s}(w)| & \leq |f(z) - P(z)| + |P(z) - \int_{C^n} f(w)e^{s(z,w)}dv_{s}(w)| \\
& = |f(z) - P(z)| + \left| \int_{C^n} P(w)e^{s(z,w)}dv_{s}(w) - \int_{C^n} f(w)e^{s(z,w)}dv_{s}(w) \right| \\
& \leq e^{\frac{1}{2}|z|^2} + \int_{C^n} |P(w) - f(w)||e^{s(z,w)}|dv_{s}(w) \leq e^{\frac{1}{2}|z|^2} + \epsilon + \epsilon \int_{C^n} e^{\frac{1}{2}|w|^2} |e^{s(z,w)}|dv_{s}(w).
\end{align*}
\]

Using Lemma 2.3 we get that,

\[
\begin{align*}
\int_{C^n} e^{\frac{1}{2}|w|^2} |e^{s(z,w)}|dv_{s}(w) &= \int_{C^n} \left( \frac{s}{\pi} \right)^n e^{-s\left(1-\frac{1}{p}\right)|w|^2} |e^{s(z,w)}|dv(w) \\
& = \left( \frac{p}{p-1} \right)^n \int_{C^n} |e^{s(z,w)}|dv_{s(1-\frac{1}{p})}(w) = \left( \frac{p}{p-1} \right)^n \frac{e^{\frac{1}{2}|z|^2}}{e^{\epsilon(p-1)}}.
\end{align*}
\]

It follows that,

\[
\begin{align*}
|f(z) - \int_{C^n} f(w)e^{s(z,w)}dv_{s}(w)| & \leq \left( e^{\frac{1}{2}|z|^2} + \left( \frac{p}{p-1} \right)^n \frac{e^{\frac{1}{2}|z|^2}}{e^{\epsilon(p-1)}} \right) \epsilon,
\end{align*}
\]

and since \( \epsilon \) is arbitrary the result follows.
Chapter 3

3.1 Bergman type integral operators on $L^p(\mathbb{C}^n, dv_s)$

In this chapter we study the boundedness of a general class of integral operators, induced by the kernel functions of Fock spaces. Recall that the Bergman projection operator $P_s : L^2(\mathbb{C}^n, dv_s) \to F^2_s$ is an integral operator defined by:

$$P_s f(z) = \int_{\mathbb{C}^n} f(w) e^{s(z,w)} dv_s(w), \quad f \in L^2(\mathbb{C}^n, dv_s).$$

In 1987 Janson, Peetre, and Rochberg (cf., [4, Corollary 9.1]), showed that, surprisingly, $P_s$ is a bounded operator on $L^p(\mathbb{C}^n, dv_s)$ if and only if $p = 2$. Thus, it is natural to investigate the boundedness of a more general class of integral operators. More precisely, for $a$, $b$, $c$ real parameters we consider the operators $S_{a,b,c}$ and $T_{a,b,c}$ on $L^p(\mathbb{C}^n, dv_s)$ defined by

$$S_{a,b,c} f(z) = \int_{\mathbb{C}^n} e^{a|z|^2 + b(z,w) + c|w|^2} f(w) dv(w),$$

and

$$T_{a,b,c} f(z) = \int_{\mathbb{C}^n} \left| e^{a|z|^2 + b(z,w) + c|w|^2} f(w) dv(w).$$

The main results of this chapter are the following theorems.

**Theorem 3.1.** Let $b \neq 0$, $a < \frac{8}{p}$, and $p \geq 1$. Then the following conditions are equivalent:
1) $T_{a,b,c}$ is bounded on $L^p(\mathbb{C}^n, dv_z)$.
2) $S_{a,b,c}$ is bounded on $L^p(\mathbb{C}^n, dv_z)$.
3) $b^2 p^2 + 4(c + a)p(s - ap) + 4(s - ap)^2 \leq 0$.

The next theorem deals with the case $b = 0$.

**Theorem 3.2.** Let $a < \frac{s}{p}$.

A) Let $p > 1$ and let $q = \frac{p}{p - 1}$ be its conjugate index. Then the following conditions are equivalent:
1) $S_{a,0,c}$ is bounded on $L^p(\mathbb{C}^n, dv_z)$.
2) $pc + s < 0$.

Furthermore,

$$\|S_{a,0,c}\|_{L^p(\mathbb{C}^n, dv_z) \to L^p(\mathbb{C}^n, dv_z)} = \frac{\pi^n}{(s + q(c + s))^{\frac{n}{2}}(s - ap)^{\frac{n}{2}}}.$$ 

B) Let $p = 1$. Then the following conditions are equivalent:
1) $S_{a,0,c}$ is bounded on $L^1(\mathbb{C}^n, dv_z)$.
2) $c + s \leq 0$.

Furthermore,

$$\|S_{a,0,c}\|_{L^1(\mathbb{C}^n, dv_z) \to L^1(\mathbb{C}^n, dv_z)} = \frac{\pi^n}{(s - a)^n}.$$ 

The special case $a = 0$ turns out to be very useful for studying the boundedness of $S_{a,b,c}$ and $T_{a,b,c}$ on $L^p(\mathbb{C}^n, dv_z)$. In this case, we denote the corresponding integral operators by $S_{b,c}$ and $T_{b,c}$. Thus, we have that

$$S_{b,c}f(z) = \int_{\mathbb{C}^n} e^{b(z,w) + c|w|^2} f(w) dv(w) \quad \text{and} \quad T_{b,c}f(z) = \int_{\mathbb{C}^n} e^{b(z,w) + c|w|^2} f(w) dv(w).$$ 

(3.1)
The operators $T_{a,b,c}$ and $S_{a,b,c}$ were introduced in an earlier version of [1] by Kehe Zhu. It is worth mentioning that a similar class of integral operators induced by the kernel function of Bergman spaces on the unit ball $B_n$ have been studied in [6]. We base the proofs of Theorems 3.1 and 3.2 on the study of the operators $S_{b,c}$ and $T_{b,c}$. The next lemma gives a way for calculating norms of certain monomials.

**Lemma 3.3.** Let $m = (m_1, m_2, \ldots, m_n)$ be an $n$-tuple of nonnegative integers. If $s > 0$ and $p > 0$ then,

$$
\int_{\mathbb{C}^n} |z|^m |p| dv_{s}(z) = \prod_{k=1}^{n} \frac{\Gamma \left( \frac{m_k}{2} + 1 \right)}{s^2}.
$$

In particular,

$$
\int_{\mathbb{C}^n} \left| z^{m_1} \right|^p dv_{s}(z) = \frac{\Gamma \left( \frac{m_1}{2} + 1 \right)}{s^{m_1}}.
$$

**Proof.** First we note that,

$$
\int_0^{\infty} r^{2n + |m|p - 1} e^{-sr^2} dr = \frac{\Gamma \left( \frac{n + |m|p}{2} \right)}{2s^{n + |m|p/2}}. \quad (3.2)
$$

A calculation based on (1.2) and Lemma 1.1 shows that,

$$
\int_{\mathbb{C}^n} |z|^m |p| dv_{s}(z) = \frac{s^n}{n!} \int_{\mathbb{C}^n} |z|^m |p| e^{-s|z|^2} dV(z) = \frac{s^n}{n!} \int_{0}^{\infty} r^{2n + |m|p - 1} e^{-sr^2} dr \int_{S_n} |z|^m |p| d\sigma(\xi).
$$

Thus, using Lemma 1.3 and (3.2) the result follows. \(\Box\)

The next lemma is crucial in our analysis and it will be used in the proof of Lemma 3.10.
Lemma 3.4. Let $b$ be a real number, let $k$ be a nonnegative integer, and let $s > 0$. The following integral formula holds:

\[
\int_{\mathbb{C}^n} e^{b(z,w)} w_1^k \, dv_s(w) = \frac{b^k}{s^k}.
\]

The first proof. Let $J = \int e^{b(z,w)} e^{-|w|^2} \, dxdy$. If $w = x + iy$ and $z \in \mathbb{C}$, then

\[
J = \int_R e^{b(z,x)} e^{-sx^2} \, dx \int_R e^{-by^2} e^{-sy^2} \, dy
\]

\[
= \left( e^{\frac{s^2}{4x^2}} \int_R e^{-\left( \sqrt{sx} - \frac{bx}{2\sqrt{s}} \right)^2} \, dx \right) \left( e^{-\frac{s^2}{4y^2}} \int_R e^{-\left( \sqrt{sy} + \frac{by}{2\sqrt{s}} \right)^2} \, dy \right).
\]

If we make the obvious substitutions in the integrals above we get that $J = \frac{\pi}{s}$, i.e.,

\[
\int_{\mathbb{R}^2} e^{b(z,w)} e^{-|w|^2} \, dxdy = \frac{\pi}{s}. \tag{3.3}
\]

Therefore

\[
F(b) = \int_{\mathbb{C}^n} e^{b(z,w)} w_1^k \, dv_s(w) = \left( \frac{s}{\pi} \right)^n \int_{\mathbb{R}^2} e^{b(z,w_1)} w_1^k e^{-|w_1|^2} \, dx_1 \, dy_1 \prod_{j=2}^n \int_{\mathbb{R}^2} e^{b(z,w_j)} e^{-|w_j|^2} \, dx_j \, dy_j
\]

\[
= \frac{s}{\pi} \int_{\mathbb{R}^2} e^{b(z,w_1)} w_1^k e^{-|w_1|^2} \, dx_1 \, dy_1.
\]

Differentiating formula (3.3) $k$ times with respect to $s$, we obtain that

\[
\int_{\mathbb{R}^2} e^{b(z,w)} |w|^2 e^{-|w|^2} \, dxdy = \frac{\pi k!}{s^{k+1}}. \tag{3.4}
\]
Differentiating $F(b)$ $k$ times with respect to $b$, and using (3.4) we obtain that

$$F^{(k)}(b) = \frac{z_1^k S}{\pi} \int_{\mathbb{R}^2} e^{b z_1} |w_1|^{2k} e^{-s|w_1|^2} dx_1 dy_1 = \frac{k! z_1^k}{s^k}.$$  

This implies that $F(b)$ is a polynomial of degree $k$. Furthermore, if $0 < j < k - 1$,

$$F^{(j)}(0) = \frac{z_1^j S}{\pi} \int_{\mathbb{R}^2} |w_1|^{2j} e^{-s|w_1|^2} dx_1 dy_1 = \frac{z_1^j S}{\pi} \int_{0}^{\infty} r^{k+j+1} e^{-sr^2} dr \int_{0}^{2\pi} e^{(k-j)\theta} d\theta = 0.$$

Thus $F(b) = \frac{z_1^k}{s^k}$, and the lemma is proved. \(\square\)

*The second proof.* A different proof of Lemma 3.4 is a simple application of the reproducing formula as follows:

$$\int_{\mathbb{C}^n} e^{b(z,w)} w_1^k dw_1 w_2 dw_3 = \int_{\mathbb{C}^n} e^{s(bz,\overline{w})} w_1^k dw_1 w_2 dw_3 = \left( \frac{b z_1}{s} \right)^k.$$

\(\square\)

*Remark 3.5.* For $k = 0$, we get as a consequence of Lemma 3.4 that

$$\int_{\mathbb{C}^n} e^{b(z,w)} dw_2(w) = 1. \quad (3.5)$$

In other words, for fixed $z \in \mathbb{C}$, $b \in \mathbb{R}$, and $s > 0$, the measure $d\mu(w) = e^{b(z,w)} dw_2(w)$ is a probability measure.

A useful tool for studying the boundedness of integral operators on $L^p$ spaces is the *Schur’s test* (cf., [13, Theorem 2.9]), which will be used in the proof of Lemma 3.11.
Lemma 3.6. (Schur’s Test). Suppose $H(x, y)$ is a positive kernel and
\[ Tf(x) = \int_X H(x, y)f(y)d\mu(y) \]
is the associated integral operator. Let $1 < p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. If there exists a positive function $h$ on $X$ and positive constants $C_1$ and $C_2$ such that
\[ \int_X H(x, y)h^q(y)d\mu(y) \leq C_1 h^q(x), \quad x \in X, \]
and
\[ \int_X H(x, y)h^p(x)d\mu(x) \leq C_2 h^p(y), \quad y \in X, \]
then the operator $T$ is bounded on $L^p(X, d\mu)$. Moreover, the norm of $T$ on $L^p(X, d\mu)$ does not exceed $C_1^{1/q}C_2^{1/p}$.

The next result which is well-known in the theory of integral operators deals with the adjoint of a bounded operator. We are going to use it several times later on.

Lemma 3.7. Let $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. If an integral operator
\[ Tf(x) = \int_X K(x, y)f(y)d\mu(y) \]
is bounded on $L^p(X, d\mu)$ then its adjoint
\[ T^* : L^q(X, d\mu) \to L^q(X, d\mu) \]
is an integral operator given by
\[ T^* f(x) = \int_X \overline{K(x, y)}f(y)d\mu(y). \]
Proof. The proof of this result can be found in [3]. □

Recall that, for $b, c$ real numbers, and $s > 0$,

$$S_{b,c}f(z) = \int_{\mathbb{C}^n} e^{b(z,w) + c|w|^2} f(w) dv(w)$$

and

$$T_{b,c}f(z) = \int_{\mathbb{C}^n} \left| e^{b(z,w) + c|w|^2} \right| f(w) dv(w)$$

are operators acting on $L^p(\mathbb{C}^n, dv_s)$. Clearly,

$$S_{b,c}f(z) = \left( \frac{\pi}{8} \right)^n \int_{\mathbb{C}^n} e^{b(z,w) + (c+s)|w|^2} f(w) dv_s(w),$$

and

$$T_{b,c}f(z) = \left( \frac{\pi}{8} \right)^n \int_{\mathbb{C}^n} \left| e^{b(z,w) + (c+s)|w|^2} \right| f(w) dv_s(w).$$

It follows, based on Lemma 3.7, that the adjoints of $T_{b,c}$ and $S_{b,c}$ with respect to the integral pairing

$$\langle f, g \rangle_s = \int_{\mathbb{C}^n} f(z)\overline{g(z)} dv_s(z)$$

are given respectively by:

$$S_{b,c}^*f(z) = \left( \frac{\pi}{8} \right)^n e^{(c+s)|z|^2} \int_{\mathbb{C}^n} e^{b(z,w)} f(w) dv_s(w), \quad (3.6)$$

and

$$T_{b,c}^*f(z) = \left( \frac{\pi}{8} \right)^n e^{(s+c)|z|^2} \int_{\mathbb{C}^n} |e^{b(z,w)}| f(w) dv_s(w). \quad (3.7)$$
The case $b = 0$ needs a special treatment. In this situation we have that

$$S_c f(z) = T_c f(z) = \left( \frac{\pi}{s} \right)^n \int e^{(c+s)|w|^2} f(w) d\nu_s(w), \quad (3.8)$$

and

$$S_c^* f(z) = T_c^* f(z) = \left( \frac{\pi}{s} \right)^n e^{(c+s)|z|^2} \int f(w) d\nu_s(w). \quad (3.9)$$

### 3.2 Necessary conditions for the boundedness of $S_{b,c}$

In this section we prove some technical lemmas which are going to be used in the study of the boundedness of the operators $S_{b,c}$ and $T_{b,c}$ defined by (3.1). First we will establish some necessary conditions for the boundedness of $S_{b,c}$.

**Lemma 3.8.** Let $p > 1$. If $S_{b,c}$ is bounded on $L^p(\mathbb{C}^n, d\nu_s)$, then $pc + s < 0$.

**Proof.** Let $x$ be a real number such that $x > -\frac{s}{p}$ and $x > c$, and let $f(z) = e^{-x|z|^2}$.

A calculation shows that $||f||_{L^p} = \left( \frac{s}{px + s} \right)^n$. On the other hand, we have in view of (3.5) that

$$S_{b,c} f(z) = \left( \frac{\pi}{x - c} \right)^n \int e^{b(z,w)} d\nu_{x-c}(w) = \left( \frac{\pi}{x - c} \right)^n.$$

Now, the boundedness of $S_{b,c}$ implies that there is a constant $C > 0$ such that

$$\left( \frac{\pi}{x - c} \right)^{np} \leq C \left( \frac{s}{px + s} \right)^n$$

and, hence,

$$\left( \frac{x - c}{s} \right)^n \leq C \left( \frac{(x - c)p}{px + s} \right)^n. \quad (3.10)$$
We note that if \(-s/p = c\), then (3.10) becomes \((\pi^n/s)^n \leq C \left(\frac{(s/p + s)^{p-1}}{p^p}\right)^n\), and letting \(x\) converge to \(-s/p\) in the last inequality, we get that \((\pi^n/s)^n \leq 0\) which is definitely a contradiction. On the other hand, if \(-s/p < c\), then letting \(x\) converge to \(c\) in (3.10), we get that \((\pi^n/s)^n \leq 0\), which is false. Thus, we must have that \(c < -s/p\), which implies that \(pc + s < 0\). \(\square\)

Remark 3.9. It is worth mentioning that for \(p = 1\), it follows from the proof of Lemma 3.8 that if \(S_{b,c}\) is bounded on \(L^1(C^n, dv_s)\) then \(c + s \leq 0\).

The next lemma gives another necessary condition for the boundedness of the operator \(S_{b,c}\) in the case when \(b \neq 0\).

**Lemma 3.10.** Let \(b \neq 0\), \(s > 0\), and \(p \geq 1\). If the operator \(S_{b,c}\) is bounded on \(L^p(C^n, dv_s)\), then \(b^2p^2 + 4pcs + 4s^2 \leq 0\).

**Proof.** First we consider the case \(p = 1\). Let \(z \in C^n\), and define \(f_z(w) = \frac{e^{b(w,z)}}{|e^{b(w,z)}|}\). Obviously \(\|f_z\|_\infty = 1\), for all \(z \in C^n\). On the other hand, using (3.6) and Lemma 2.3 we get that,

\[
S_{b,c} f_z(z) = \left(\frac{\pi}{\theta}\right)^n e^{(c+s)|z|^2} \int_{C^n} |e^{b(z,w)}| dv_s(w) = \left(\frac{\pi}{\theta}\right)^n e^{(c+s+b^2/4)|z|^2}.
\]

Since \(S_{b,c}^*\) is bounded on \(L^\infty(C^n)\), there exists a positive constant \(C\) such that

\[
\left(\frac{\pi}{\theta}\right)^n e^{(c+s+b^2/4)|z|^2} = |S_{b,c}^* f_z(z)| \leq \|S_{b,c}^* f_z\|_\infty \leq C \|f_z\|_\infty = C,
\]

for all \(z \in C^n\). This implies that \(b^2 + 4sc + 4s^2 \leq 0\).

Next we consider the case \(p > 1\). Since \(S_{b,c}\) is bounded on \(L^p(C^n, dv_s)\), Lemma 3.8 implies that \(pc + s < 0\). Let \(x\) be a real number such that \(x > -\frac{b}{p} > c\), and let \(f(z) = e^{-x|z|^2/2k}\), where \(k\) is a positive integer. Then, a calculation based on Lemma 3.3

---

34

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
shows that \( ||f||^p_{s,p} = \left( \frac{s}{px + s} \right)^n \frac{\Gamma\left( \frac{k}{2} + 1 \right)}{(px + s)^{\frac{k}{2}}} \). On the other hand, using Lemma 3.4, we have that

\[ S_{b,c} f(z) = \left( \frac{\pi}{x - c} \right)^n \int e^{b(x,w)u^k} d\mu_{x-c}(w) = \left( \frac{\pi}{x - c} \right)^n \frac{b^k z^k}{(x - c)^k}. \]

A calculation based on Lemma 3.3 shows that

\[ \|S_{b,c} f\|^p_{s,p} = \frac{n^{np} |b|^k p}{(x - c)(n + kp)} \int |z|^k d\mu_s(z) = \frac{n^{np} |b|^k p}{(x - c)(n + kp)} \frac{\Gamma\left( \frac{k}{2} + 1 \right)}{(px + s)^{\frac{k}{2}}}. \]

The boundedness of \( S_{b,c} \) implies that there exists a positive constant \( C \) such that

\[ \frac{n^{np} |b|^k p}{(x - c)(n + kp)} \frac{\Gamma\left( \frac{k}{2} + 1 \right)}{(px + s)^{\frac{k}{2}}} \leq C \left( \frac{s}{px + s} \right)^n \frac{\Gamma\left( \frac{k}{2} + 1 \right)}{(px + s)^{\frac{k}{2}}} \cdot \]

This is equivalent to

\[ \left( \frac{\pi}{x - c} \right)^n \frac{px + s}{s} \left( \frac{b^2(px + s)}{(x - c)^2 s} \right)^{\frac{k}{2}} \leq C. \quad (3.11) \]

Letting \( k \to \infty \) in (3.11), we obtain that \( \frac{b^2(px + s)}{(x - c)^2 s} \leq 1 \), and hence that

\[ x^2 s - x(2cs + b^2 p) + c^2 s - b^2 s \geq 0. \quad (3.12) \]

The discriminant of this inequality is \( \Delta = b^2(b^2 p^2 + 4cps + 4s^2) \). We claim that \( \Delta \leq 0 \). Suppose to the contrary that \( \Delta > 0 \). This implies that the quadratic equation associated with inequality (3.12) has two real roots given by

\[ x_1 = \frac{2cs + b^2 p - |b| \sqrt{b^2 p^2 + 4cps + 4s^2}}{2s} \quad \text{and} \quad x_2 = \frac{2cs + b^2 p + |b| \sqrt{b^2 p^2 + 4cps + 4s^2}}{2s}. \]
Thus, for the inequality (3.12) to be valid for all $x > -\frac{s}{p}$, the largest root of the quadratic equation, namely $x_2$, cannot exceed $-\frac{s}{p}$. In other words we must have

$$\frac{2cs + b^2 p + |b|\sqrt{b^2 p^2 + 4cps + 4s^2}}{2s} \leq -\frac{s}{p}.$$ 

This implies that $2s^2 + 2cps + b^2 p^2 + p|b|\sqrt{b^2 p^2 + 4cps + 4s^2} \leq 0$. Therefore we get that 

$$\left(\sqrt{4s^2 + 4cps + b^2 p^2 + |b|p}\right)^2 \leq 0.$$ 

It follows that $\sqrt{4s^2 + 4cps + b^2 p^2 + |b|p} = 0$, which is definitely a contradiction. Thus, we must have that $b^2 p^2 + 4cps + 4s^2 \leq 0$. □

### 3.3 Sufficient conditions for the boundedness of $T_{b,c}$

The next lemma gives a sufficient condition for the boundedness of the operator $T_{b,c}$ on $L^p(C^n, dv_s)$.

**Lemma 3.11.** Let $p \geq 1$ and $b \neq 0$. If $b^2 p^2 + 4cps + 4s^2 \leq 0$, then the operator $T_{b,c}$ is bounded on $L^p(C^n, dv_s)$.

**Proof.** First we consider the case $p = 1$. In this case we have that $b^2 + 4cs + 4s^2 \leq 0$. We have

$$|T_{b,c}f(z)| \leq \left(\frac{\pi}{s}\right)^n \int_{C^n} |e^{b(z,w)+(c+s)|w|^2}||f(w)||dv_s(w).$$

Thus, by Tonelli’s theorem and by Lemma 2.3, we obtain that

$$\int_{C^n} |T_{b,c}f(z)||dv_s(z) \leq \left(\frac{\pi}{s}\right)^n \int_{C^n} |f(w)||e^{c+(c+s)|w|^2}dv_s(w) \int_{C^n} |e^{b(z,w)}|dv_s(z)$$

$$= \left(\frac{\pi}{s}\right)^n \int_{C^n} |f(w)||e^{c+(c+s)b^2|w|^2}dv_s(w)$$

$$\leq \left(\frac{\pi}{s}\right)^n \int_{C^n} |f(w)||dv_s(w).$$
Consequently, $T_{b,c}$ is bounded on $L^1(\mathbb{C}^n, dv_s)$.

Now we consider the case $p > 1$. Since $b^2p^2 + 4cps + 4s^2 < 0$, it follows that $p c + s \leq -\frac{b^2}{4s} < 0$, and $s - (c + s)q = -\frac{s}{p}(s + pc) \geq \frac{aqb^2}{4s} > 0$, where $q = \frac{p}{p-1}$. We are going to use Lemma 3.6 with $H(z, w) = (\pi/s)^n |e^{b(z,w)} + (c+s)|w|^2|$, and $h(z) = e^{-\lambda|z|^2}$, where $\lambda = \frac{s(q-1)}{p} = -\frac{s}{pq}$. Since $cp + s < 0$, we have that $\lambda > \frac{s}{q}$, and an application of Lemma 2.3 shows that, for $z \in \mathbb{C}^n$,

$$\int_{\mathbb{C}^n} H(z, w)h^q(w)dv_s(w) = \left(\frac{\pi}{\lambda q - c}\right)^n \int_{\mathbb{C}^n} |e^{b(z,w)}|d\nu_{\lambda q - c}(w) = \left(\frac{\pi}{\lambda q - c}\right)^n e^{\frac{\lambda^2|z|^2}{4(\lambda q - c)}}$$

$$\leq \left(\frac{\pi}{\lambda q - c}\right)^n e^{-\lambda|z|^2} = \left(\frac{\pi}{\lambda q - c}\right)^n h^q(z).$$

On the other hand, $\lambda > -\frac{s}{p}$, and another application of Lemma 2.3 shows that, for $w \in \mathbb{C}^n$,

$$\int_{\mathbb{C}^n} H(z, w)h^p(z)dv_s(z) = \left(\frac{\pi}{s}\right)^n e^{(c+s)|w|^2} \int_{\mathbb{C}^n} |e^{b(z,w)}|d\nu_{\lambda p + s}(z)$$

$$= \left(\frac{\pi}{\lambda p + s}\right)^n e^{(c+s)|w|^2} \int_{\mathbb{C}^n} |e^{b(z,w)}|d\nu_{\lambda p + s}(z) \leq \left(\frac{\pi}{\lambda p + s}\right)^n e^{(c+s+\frac{\lambda^2}{4(\lambda p + s)})|w|^2}$$

$$\leq \left(\frac{\pi}{\lambda p + s}\right)^n e^{-\lambda|w|^2} = \left(\frac{\pi}{\lambda p + s}\right)^n h^p(w),$$

and the lemma is proved. \qed

### 3.4 Proofs of the main results

In this section we characterize the boundedness of the operators $S_{b,c}$ and $T_{b,c}$ on $L^p(\mathbb{C}^n, dv_s)$, and we prove the main results of the chapter which are Theorem 3.17 and Theorem 3.19 below.
Theorem 3.12. Let $b \neq 0$, $s > 0$ and $p \geq 1$. The following conditions are equivalent:

1) $T_{b,s}$ is bounded on $L^p(C^n, dv_s)$.
2) $S_{b,s}$ is bounded on $L^p(C^n, dv_s)$.
3) $b^2p^2 + 4cps + 4s^2 \leq 0$.

Proof. It is straightforward to verify that 1) implies 2). The implication 2) $\Rightarrow$ 3) is Lemma 3.10 and 3) $\Rightarrow$ 1) is Lemma 3.11.

The next theorem deals with the case when $b = 0$.

Theorem 3.13. Let $s > 0$, and let $c$ be a real number.

a) When $p > 1$, $S_c$ is bounded on $L^p(C^n, dv_s)$ if and only if $pc + s < 0$.

b) When $p = 1$, $S_c$ is bounded on $L^1(C^n, dv_s)$ if and only if $c + s \leq 0$.

Proof. a) If $S_c$ is bounded on $L^p(C^n, dv_s)$ the implication follows from Lemma 3.8. In the other direction, if $f \in L^p(C^n, dv_s)$, then

$$|S_c f(z)| \leq \left(\frac{s}{\pi}\right)^n \int_{C^n} e^{(c+s)|w|^2} |f(w)| dv_s(w) \leq \left(\frac{s}{\pi}\right)^n \|f\|_{s,p} \left(\int_{C^n} e^{q(c+s)|w|^2} dv_s(w)\right)^{1/q}.$$ 

Since $s - q(c+s) = -\frac{q}{p}(s + pc) > 0$, we obtain that

$$|S_c f(z)| \leq \left(\frac{s}{\pi}\right)^n \left(\frac{s}{s - q(c+s)}\right)^{n/q} \|f\|_{s,p},$$

and the result follows from the fact that $dv_s$ is a probability measure.

b) When $p = 1$, Remark 3.9 shows that if $S_c$ is bounded on $L^1(C^n, dv_s)$ then $c + s \leq 0$.

On the other hand if $c + s \leq 0$, then we get that

$$|S_c f(z)| \leq \left(\frac{s}{\pi}\right)^n \int_{C^n} e^{(c+s)|w|^2} |f(w)| dv_s(w) \leq \left(\frac{s}{\pi}\right)^n \int_{C^n} |f(w)| dv_s(w) = \left(\frac{s}{\pi}\right)^n \|f\|_{s,1}.$$
Integrating the above inequality completes the proof.

An interesting case is obtained when \( c = 0 \), for which the corresponding integral operators are

\[
S_b f(z) = \int_{C^n} e^{b(z,w)} f(w) dv(w) \quad \text{and} \quad T_b f(z) = \int_{C^n} |e^{b(z,w)}| f(w) dv(w).
\]

As a consequence of Theorems 3.12 and 3.13, we obtain the following corollary.

**Corollary 3.14.** Let \( p \geq 1 \) and let \( s > 0 \). The integral operators \( S_b \) and \( T_b \) are unbounded on \( L^p(C^n,dv_\lambda) \) for any real value of \( b \).

Another case worth mentioning is when \( b = t \) and \( c = -t \), where \( t \) is a positive real number. In this case the corresponding integral operators denoted by \( T_t \) and \( S_t \) are given by

\[
S_t f(z) = \left( \frac{\pi}{t} \right)^n \int_{C^n} e^{t(z,w)} f(w) dv_t(w) \quad \text{and} \quad T_t f(z) = \left( \frac{\pi}{t} \right)^n \int_{C^n} |e^{t(z,w)}| f(w) dv_t(w).
\]

(3.13)

As a consequence of Theorem 3.12 we recover Theorem 11 of [1].

**Corollary 3.15.** Let \( t > 0 \), \( s > 0 \), and \( p \geq 1 \). The following conditions are equivalent:

1) \( T_t \) is bounded on \( L^p(C^n,dv_\lambda) \).

2) \( S_t \) is bounded on \( L^p(C^n,dv_\lambda) \).

3) \( pt = 2s \).

**Remark 3.16.** We get, as a consequence of Corollary 3.15, that the densely defined operator \( S_t : L^p(C^n,dv_\lambda) \to L^p(C^n,dv_\lambda) \) is bounded on \( L^p(C^n,dv_\lambda) \) if and only if \( p = 2 \). It follows that the Bergman projection \( P_t = \left( \frac{1}{t} \right)^n S_t \) is a bounded operator on \( L^p(C^n,dv_\lambda) \) if and only if \( p = 2 \). This is quite different than the situation in the Bergman spaces. Namely, (cf., [13, Theorem 2.11]), the Bergman operator \( P_\alpha \) is bounded on \( L^p(B_n,dv_\alpha) \) if and only if \( p > 1 \).
Now we are ready to prove the main results of this chapter which are simple applications of Theorems 3.12 and 3.13. For the sake of completeness we restate the theorems we are going to prove here.

**Theorem 3.17.** Let $b \neq 0$, $a < \frac{d}{p}$, and $p \geq 1$. Then the following conditions are equivalent:

1) $T_{a,b,c}$ is bounded on $L^p(\mathbb{C}^n, dv_s)$.

2) $S_{a,b,c}$ is bounded on $L^p(\mathbb{C}^n, dv_s)$.

3) $b^2 p^2 + 4(c + a)p(s - ap) + 4(s - ap)^2 \leq 0$.

**Proof.** We will prove that 2) and 3) are equivalent and mention that the proof that 1) and 3) are equivalent is analogous. Since $S_{a,b,c}$ is bounded on $L^p(\mathbb{C}^n, dv_s)$, there is a positive constant $C$, such that, for all $f \in L^p(\mathbb{C}^n, dv_s)$,

$$\int_{\mathbb{C}^n} \left| \int_{\mathbb{C}^n} e^{b|z|^2 + b(z,w) + c|w|^2} f(w)dv(w) \right|^p dv_s(z) \leq C \int_{\mathbb{C}^n} |f(z)|^p dv_s(z),$$

or

$$\int_{\mathbb{C}^n} \left| \int_{\mathbb{C}^n} e^{b(z,w) + c|w|^2} f(w)dv(w) \right|^p dv_{s-ap}(z) \leq C \int_{\mathbb{C}^n} |f(z)|^p dv_{s-ap}(z).$$

Let $g(z) = f(z)e^{-a|z|^2}$. Thus, since $f \in L^p(\mathbb{C}^n, dv_s)$ if and only if $g \in L^p(\mathbb{C}^n, dv_{s-ap})$, we obtain that

$$\int_{\mathbb{C}^n} \left| \int_{\mathbb{C}^n} e^{b(z,w)+(c+a)|w|^2} g(w)dv(w) \right|^p dv_{s-ap}(z) \leq C \int_{\mathbb{C}^n} |g(z)|^p dv_{s-ap}(z).$$

This is equivalent to the fact that the operator $S_{b,c+a}$ is bounded on $L^p(\mathbb{C}^n, dv_{s-ap})$. The desired result follows from Theorem 3.12. □

**Remark 3.18.** We note that condition $a < \frac{d}{p}$ is necessary for the boundedness of the operator $S_{a,b,c}$. Otherwise, if $a \geq \frac{d}{p}$, the integral operator $S_{a,b,c}$ is unbounded on the
space $L^p(\mathbb{C}^n, dv_x)$. To see this, we let $\lambda$ be a positive real number such that $\lambda > -c - s/p$, and let $f(z) = e^{-\lambda |z|^2}$. Then, a calculation shows that $||f||_{L^p}^p = \left(\frac{s}{s + p + c\lambda}\right)^n$. On the other hand, based on (3.5), we have that

$$S_{a,b,c}f(z) = \left(\frac{\pi}{\lambda}\right)^n e^{s|z|^2} \int_{\mathbb{C}^n} e^{b(z,w)} dw \lambda(w) = \left(\frac{\pi}{\lambda}\right)^n e^{s|z|^2}.
$$

This implies that

$$||S_{a,b,c}||_{L^p}^p = \left(\frac{\pi}{\lambda}\right)^n \left(\frac{s}{\pi}\right)^n \int_{\mathbb{C}^n} e^{(ap-s)|z|^2} dv(z) = \infty.
$$

The next theorem deals with the case $b = 0$.

**Theorem 3.19.** Let $a < \frac{s}{p}$.

A) Let $p > 1$ and let $q = p/(p - 1)$ be its conjugate index. Then the following conditions are equivalent:

1) $S_{a,0,c}$ is bounded on $L^p(\mathbb{C}^n, dv_x)$.

2) $pc + s < 0$.

Furthermore,

$$||S_{a,0,c}||_{L^p(\mathbb{C}^n, dv_x) \to L^p(\mathbb{C}^n, dv_x)} = \frac{\pi^n}{(s - q(c + s))^{\frac{n}{q}} (s - ap)^{\frac{n}{p}}}.$$

B) Let $p = 1$. Then the following conditions are equivalent:

1) $S_{a,0,c}$ is bounded on $L^1(\mathbb{C}^n, dv_x)$.

2) $c + s \leq 0$.

Furthermore,

$$||S_{a,0,c}||_{L^1(\mathbb{C}^n, dv_x) \to L^1(\mathbb{C}^n, dv_x)} = \frac{\pi^n}{(s - a)^n}.$$
Proof. The proof that the two conditions of the theorem are equivalent follow by using a similar argument as in the proof of Theorem 3.17 and by using Theorem 3.13. We will concentrate on calculating the norm of $S_{a,0,c}$ when $p > 1$ and $p = 1$.

A) When $p > 1$, we have that

$$|S_{a,0,c}f(z)| \leq \left(\frac{\pi}{s}\right)^n e^{q|z|^2} \int_{\mathbb{C}^n} e^{(c+s)|w|^2} |f(w)| \, dv_s(w).$$

An application of Hölder's inequality shows that

$$|S_{a,0,c}f(z)| \leq \left(\frac{\pi}{s}\right)^n e^{q|z|^2} \|f\|_{s,p} \left(\frac{s}{s-q(c+s)}\right)^{\frac{n}{q}}.$$

This implies that

$$\|S_{a,0,c}f\|_{s,p} \leq \frac{\pi^n}{(s-q(c+s))^{\frac{n}{q}}} \left(\frac{s}{s-ap}\right)^{\frac{n}{q}} \|f\|_{s,p}.$$

Let $\lambda < \frac{s}{p} < -c$ and let $f(z) = e^{\lambda|z|^2}$. A calculation shows that

$$\|f\|_{s,p} = \left(\frac{s}{s-\lambda p}\right)^{\frac{n}{p}}.$$

Another calculation shows that

$$S_{a,0,c}f(z) = e^{q|z|^2} \left(\frac{\pi}{-c-\lambda}\right)^n$$

and

$$\|S_{a,0,c}f\|_{s,p} = \left(\frac{s}{s-ap}\right)^{\frac{n}{p}} \left(\frac{\pi}{-c-\lambda}\right)^n.$$

It follows that,

$$\|S_{a,0,c}\|_{L^p(C^n, dv_s)} \to L^p(C^n, dv_s) \geq \frac{\|S_{a,0,c}f\|_{s,p}}{\|f\|_{s,p}} = \left(\frac{s-\lambda p}{s-ap}\right)^{\frac{n}{p}} \left(\frac{\pi}{-c-\lambda}\right)^n.$$
We claim that there is $\lambda \in (-\infty, \frac{s}{p})$ such that:

$$\left(\frac{s - \lambda p}{s - ap}\right)^\frac{n}{p} \left(\frac{\pi}{-c - \lambda}\right)^n = \frac{\pi^n}{(s - q(c + s))^{\frac{n}{p}} (s - ap)^{\frac{n}{p}}}.$$ 

Equivalently,

$$\frac{s - \lambda p}{(-c - \lambda)^p} = \left(\frac{1}{s - q(c + s)}\right)^\frac{p}{q}.$$

Let $\lambda = (c + s)(q - 1) = \frac{c + s}{p - 1} < \frac{s}{p}$. It is straightforward to check that

$$s - \lambda p = -c - \lambda = s - q(c + s) = \frac{-cp - s}{p - 1},$$

and hence the desired result follows.

B) We have, since $c + s \leq 0$, that

$$|S_0,0,cf(z)| \leq \left(\frac{\pi}{s}\right)^n e^{\alpha|z|^2} \int_{C^n} e^{(c+s)|w|^2} |f(w)| dv_\alpha(w) \leq \left(\frac{\pi}{s}\right)^n e^{\alpha|z|^2} \int_{C^n} |f(w)| dv_\alpha(w)$$

$$= \left(\frac{\pi}{s}\right)^n e^{\alpha|z|^2} \|f\|_{s,1}.$$ 

It follows that $\|S_0,0,cf\|_{s,1} \leq \left(\frac{\pi}{s-a}\right)^n \|f\|_{s,1}$.

Let $\lambda < s \leq -c$ and let $f(z) = e^{\lambda|z|^2}$. A calculation shows that $\|f\|_{s,1} = \left(\frac{s}{s-\lambda}\right)^n$.

On the other hand,

$$S_{a,0,c}f(z) = e^{\alpha|z|^2} \left(\frac{\pi}{-c - \lambda}\right)^n \quad \text{and} \quad \|S_{a,0,c}f\|_{s,1} = \left(\frac{\pi}{-c - \lambda}\right)^n \left(\frac{s}{s-a}\right)^n.$$
It follows that,
\[
||S_{a,0,c}||_{L^1(C^n, dv)} \leq \frac{||S_{a,0,c}f||_{s,1}}{||f||_{s,1}} = \left( \frac{\pi}{s-\alpha} \right)^n \left( \frac{s-\lambda}{-\ell-\lambda} \right)^n.
\]
Letting \( \lambda \to -\infty \) we get that the desired result follows. \( \square \)

A special case worth mentioning is when \( a = -\alpha, b = a + \beta, \) and \( c = -\beta, \) where \( \alpha \) and \( \beta \) are real positive numbers. As a consequence of Theorem 3.1 we recover Corollary 17 of [1].

**Corollary 3.20.** Let \( \alpha > 0, \beta > 0, s > 0, \) and \( p \geq 1. \) The following conditions are equivalent:

1) \( T_{-\alpha, \alpha + \beta, -\beta} \) is bounded on \( L^p(C^n, dv_3). \)
2) \( S_{-\alpha, \alpha + \beta, -\beta} \) is bounded on \( L^p(C^n, dv_3). \)
3) \( (\alpha + \beta)p = 2(s + p\alpha). \)

### 3.5 Applications

In this section, based on the main results, we show that, in general, contrary to the situation in Bergman and Besov spaces, the Fock spaces cannot be characterized by membership of partial derivatives in \( L^p \) spaces. First, we introduce another Fock space whose weight measure depends on \( p. \) For \( 0 < p < \infty \) let \( \mathcal{L}_p^p \) be the set of measurable functions on \( C^n \) such that \( f(z)e^{-\frac{1}{2}|z|^2} \in L^p(\mathbb{C}, dv). \) The subspace of \( \mathcal{L}_p^p \) which consists of holomorphic functions is denoted by \( \mathcal{F}_p^p. \) Thus,

\[
\mathcal{F}_p^p = \left\{ f : f \in H(C^n), \int_{\mathbb{C}^n} |f(z)e^{-\frac{1}{2}|z|^2}|^p dv(z) < \infty \right\},
\]

where \( H(C^n) \) is the set of holomorphic functions in \( C^n. \) If \( p = \infty \) then

\[
\mathcal{F}_\infty^\infty = \{ f : f \in H(C^n), ||f||_{\infty, t} < \infty \},
\]
where $||f||_{\infty,t} = \text{esssup}_{z \in \mathbb{C}^n} |f(z)e^{-t/2|z|^2}|$. This space, $\mathcal{F}^p_t$, also known as the Fock space, has been studied in [4] and more recently, the one dimensional case, in [11]. We note that $F^p_t = \mathcal{F}^p_t$ if and only if $p = 2$. Let $P_t = (t/\pi)^n S_t$, where $S_t$ is as in (3.13), and let $E^p_t = P_t(L^p(\mathbb{C}^n, d\nu_t))$. The next theorem shows that the Bergman projection $P_t$ is a bounded operator on $L^p_t$ for all $1 \leq p \leq \infty$, (cf., [4, Theorem 7.1]).

**Theorem 3.21.** The operator $P_t$ defined by

$$P_t f(z) = \int_{\mathbb{C}^n} e^{t(z,w)} f(w) d\nu_t(w)$$

is bounded on $L^p_t$ for all $1 \leq p \leq \infty$.

**Proof.** Let $1 \leq p < \infty$. First we note that $L^p_t = L^p(\mathbb{C}^n, d\nu_{tp/2})$ and $P_t = (\frac{1}{t})^n S_{0,t,-t}$. Since $t^2 p^2 + 4(-t + 0)p t^2 + 4(\frac{p}{t})^2 = 0$ we get, based on Theorem 3.1, that $P_t$ is bounded on $L^p_t$ for all $1 \leq p < \infty$. If $p = \infty$ a calculation based on Lemma 2.3 shows that $|P_t f(z)| \leq \left( \frac{1}{\pi} \right)^n \int_{\mathbb{C}^n} |f(w)e^{-\frac{1}{2}|w|^2}|e^{t(z,w)}|e^{-\frac{1}{2}|w|^2} d\nu_t(w)$

$$\leq 2^n ||f||_{\infty,t} \int_{\mathbb{C}^n} e^{t(z,w)} d\nu_{tp/2}(w) = 2^n ||f||_{\infty,t} e^{\frac{1}{2}|z|^2}.$$ 

It follows that $||P_t f||_{\infty,t} \leq 2^n ||f||_{\infty,t}$ and the theorem is proved. \[Q.E.D.\]

**Remark 3.22.** When $1 \leq p < \infty$ the norm of $P_t$ satisfies the inequality $||P_t||_{L^p_t - L^q_t} \leq 2^n$, (cf, [1, Main Theorem]). When $p = \infty$ we prove that $||P_t||_{L^p_{\infty} - L^q_{\infty}} = 2^n$. Let $f(z) = e^{t/2|z|^2}$. We have that $f \in L^p_{\infty}$ and $||f||_{\infty,t} = 1$. On the other hand, a calculation based on (3.5), shows that

$$P_t f(z) = \int_{\mathbb{C}^n} e^{t(z,w)} e^{t/2|w|^2} d\nu_t(w) = 2^n \int_{\mathbb{C}^n} e^{t(z,w)} d\nu_{tp/2}(w) = 2^n.$$ 

It follows that $||P_t f||_{\infty,t} = \text{esssup}_{z \in \mathbb{C}^n} |2^n e^{-t/2|z|^2}| = 2^n$, hence $||P_t||_{L^p_{\infty} - L^q_{\infty}} \geq 2^n$ which combined with a previous calculation shows that the norm equality holds.
Recall the reproducing formula of Fock space $\mathcal{F}_t^p$.

**Lemma 3.23.** Let $t > 0$ and $0 < p < \infty$. If $f \in \mathcal{F}_t^p$, then for all $z \in \mathbb{C}^n$,

$$f(z) = \int_{\mathbb{C}^n} f(w) e^{t(z,w)} dv_t(w).$$

**Proof.** This formula, proved for the one dimensional case in [11, Theorem 3.7], can be proved for polynomials and then extended to $\mathcal{F}_t^p$ using the density of polynomials. □

The first application of our results is to show that the space $E_t^q$ is a Fock space, in other words, the image of $L^p(\mathbb{C}^n, dv_t)$ under the Bergman projection is a Fock space.

**Theorem 3.24.** Let $1 < p < \infty$ and let $q = \frac{p}{p-1}$ be its conjugate index. Then $E_t^q = \mathcal{F}_t^{q/p}$. \\

**Proof.** To show that $E_t^q \subset \mathcal{F}_t^{q/p}$, we let $f \in E_t^q$. Then $f(z) = \int_{\mathbb{C}^n} g(w) e^{t(z,w)} dv_t(w)$, for some $g \in L^q(\mathbb{C}^n, dv_t)$. Consequently,

$$\int_{\mathbb{C}^n} |f(z) e^{-\frac{q}{4} |z|^2}|^q dv(z) = \left( \frac{t}{\pi} \right)^{n(q-1)} \int_{\mathbb{C}^n} \left| \int_{\mathbb{C}^n} e^{t(q-tp/4) |z|^2 + t(z,w) - t|w|^2} g(w) dv(w) \right|^q dv_t(z)$$

$$\leq \left( \frac{t}{\pi} \right)^{n(q-1)} \int_{\mathbb{C}^n} |S_{t/q-tp/4, t-t} g(z)|^q dv_t(z) \leq \left( \frac{t}{\pi} \right)^{n(q-1)} \|S_{t/q-tp/4, t-t}\|_{l^q}^q |g|_{l^q}^q,$$

where the boundedness of $S_{t/q-tp/4, t-t}$ on $L^q(\mathbb{C}^n, dv_t)$ follows from Theorem 3.1. Thus $E_t^q \subset \mathcal{F}_t^{q/p}$.

To prove the reverse inclusion, let $f \in \mathcal{F}_t^{q/p}$ and note that

$$\int_{\mathbb{C}^n} |f(z)|^q e^{-\frac{q}{4} |z|^2} dv(z) < \infty. \quad (3.14)$$
Let \( g(z) = f(\frac{2}{p})e^{(1-2/p)|z|^2} \). A straightforward calculation based on (3.14) shows that \( g \in L^q(\mathbb{C}^n, d\nu_t) \). Furthermore, another calculation shows that \( P_t((\frac{2}{p})^n g)(z) = \int_{\mathbb{C}^n} f(w) e^{tp/2(z,w)} d\nu_{tp/2}(w) \). An application of Lemma 3.23 shows that the last expression equals \( f(z) \), for \( f \in F^q \). Thus, \( f \in E^q_t \), and the inclusion is proved. \( \square \)

Remark 3.25. Theorem 3.24 shows that the Bergman projection operator \( P_t \) maps \( L^q(\mathbb{C}^n, d\nu_t) \) onto \( F^q_{tp/2} \), and thus, we have recovered the first part of Corollary 9.1 of [4]. However our proof, which is different than that in [4], is based on the main results of this chapter. Theorem 3.24 also shows that \( E^q_t \), the dual space of \( F^p_t \), is different from \( F^q_t \) unless \( p = 2 \).

Since \( E^p_t \) consists of holomorphic functions it is possible to look for a characterization of functions in \( E^p_t \) in terms of their partial derivatives. Recall that, if \( m = (m_1, m_2, \cdots , m_n) \) is a multi-index of nonnegative integers and \( f \) is a holomorphic function on \( \mathbb{C}^n \), we use notation \( \partial^m f = \frac{\partial^{m_1} f}{\partial z_1^{m_1}} \frac{\partial^{m_2} f}{\partial z_2^{m_2}} \cdots \frac{\partial^{m_n} f}{\partial z_n^{m_n}} \). The following theorem, which gives a necessary condition for membership of a holomorphic function \( f \) in \( E^p_t \), is another application of our results.

Theorem 3.26. Let \( k \in (0,1) \), \( p > \frac{1}{1-k} \), \( t > 0 \), let \( l \) be a real number such that

\[
l \leq \frac{-p^2 + 4p(1-k) - 4}{4p(p(1-k) - 1)} t,
\]

and let \( m \) be a fixed multi-index of nonnegative integers. If \( f \in E^p_t \) then \( e^{l|z|^2} \partial^m f \in L^p(\mathbb{C}^n, d\nu_t) \).

Proof. We let \( f \in E^p_t = P_t(L^p(\mathbb{C}^n, d\nu_t)) \). Thus \( f(z) = \int_{\mathbb{C}^n} e^{l(z,w)} g(w) d\nu_t(w) \), for some \( g \in L^p(\mathbb{C}^n, d\nu_t) \). Differentiating we get that,

\[
e^{l|z|^2} \partial^m f(z) = t^{m_1} e^{l|z|^2} \int_{\mathbb{C}^n} e^{l(z,w)} \overline{g(w)} \partial^m d\nu_t(w).
\]
A calculation shows that \( d\nu_t(w) = \frac{e^{-tk|w|^2}}{(1-k)^n} d\nu((1-k))_. \) This implies that,

\[
e^{s|z|^2} \partial g(z) = \frac{t |m|}{(1-k)^n} e^{s|z|^2} \int_{C^n} e^{t(z,w)} g(w) \|w\|^m e^{-tk|w|^2} d\nu((1-k))(w).
\]

Thus,

\[
e^{s|z|^2} \partial g(z) \leq \frac{t |m|}{(1-k)^n} e^{s|z|^2} \int_{C^n} |e^{t(z,w)}|g(w)||w|^m e^{-tk|w|^2} d\nu((1-k))(w)
\]

\[
\leq \frac{t |m|}{(1-k)^n} C e^{s|z|^2} \int_{C^n} |e^{t(z,w)}|g(w)|d\nu((1-k))(w),
\]

where \( C = \sup_{w \in C^n} |w|^m e^{-tk|w|^2}. \) It follows that,

\[
\int_{C^n} e^{p|z|^2} |\partial g(z)|^p d\nu_t(z) \leq \frac{t |m|}{(1-k)^n} C^p \int_{C^n} \left( \int_{C^n} e^{p|z|^2} e^{t(z,w)}|g(w)|d\nu((1-k))(w) \right)^p d\nu_t(z)
\]

\[
= \frac{t |m|+n}{\pi^n} \int_{C^n} \left( \int_{C^n} e^{p|z|^2+t(z,w)-t(1-k)|w|^2} |g(w)| d\nu_t(w) \right)^p d\nu_t(z)
\]

\[
= \frac{t |m|+n}{\pi^n} C^p \int_{C^n} (T_{t,t,(1-k)} h(z))^p d\nu_t(z),
\]

(3.15)

where \( h(z) = |g(z)|. \) Since \( l \leq \frac{-p^2 + 4p(1-k) - 4}{4p(p(1-k) - 1)}, \) we get that

\[
t^2 p^2 + 4(-t(1-k) + l)p(t - lp) + 4(t - lp)^2 \leq 0,
\]
whence Theorem 3.1 implies that the operator $T_{1,t,-(1-k)}$ is bounded on $L^p(C^n, du_t$). Thus,

$$\int_{C^n} (T_{1,t,-(1-k)} h(z))^p \, du_t(z) \leq ||T_{1,-t,-(1-k)}||_{L^p(C^n, du_t)}^p ||\vartheta||_{L^p}^p.$$  \hspace{1cm} (3.16)

Combining (3.15) and (3.16) we get that the theorem is proved. \hfill \Box

As a consequence of Theorem 3.26 we obtain the following corollary.

**Corollary 3.27.** Let $k$ and $p$ be as in Theorem 3.26, let $T = \frac{4p(p(1-k)-1)}{p^2-4p(1-k)+4}$, and let $m$ be a multi-index of nonnegative integers with $|m| = N$. If $f \in E_T^p$, then $e^{-N|z|^2} \vartheta^m f \in L^p(C^n, du_T).$

**Proof.** Since $p > \frac{1}{1-k} > 1$ we get by Lemma 2.19 that $F_T^p \subseteq E_T^p$. When $f \in E_T^p$, the result follows from Theorem 3.26 with $l = -N$ and $t = T$. \hfill \Box

Unfortunately, as the following example shows, the necessary condition in Theorem 3.26 is not sufficient.

**Example 3.28.** Let $f(z) = e^{\frac{q}{4}z^2}$, $q = \frac{p}{p-1}$, let $N$ be an integer such that $N \geq \frac{p^2-4p(1-k)+4}{4p(1-k)-1}t$, and let $m$ be a fixed positive integer. We will show that $e^{-N|z|^2} f^{(m)} \in L^p(C, dA_t)$ and $f \notin E_T^p$. A calculation shows that

$$\int_{C} |f(z)|^p e^{-\frac{q}{4}|z|^2} dA(z) = \int_{R^2} e^{-\frac{q}{4}y^2} dxdy = \infty,$$

thus, based on Theorem 3.24, we get that $f \notin E_T^p$. On the other hand, we have that $f^{(m)}(z) = e^{\frac{q}{4}z^2} P(z)$, for some polynomial $P$. Let $\epsilon$ be a real number such that $0 < \epsilon < \frac{q}{4}$.
We note that,

\[
\frac{p^2 qt}{4(p-1)(p-k-1)}, \quad \int \left| e^{-N|z|^2} f^{(m)}(z) \right|^p dA(z) = \frac{t}{\pi} \int \left| e^{-N|z|^2 + \frac{cp}{4} z^2} dz \left( e^{-t|z|^2} \right)^p dA(z)
\]

\[
\leq \frac{M^p t}{\pi} \int e^{- \frac{q^2 t}{4}} dx \int e^{- \frac{q^2 t}{4}} dy < \infty,
\]

since \( \epsilon < \frac{p^2 qt}{4(p-1)(p-k-1)} < N + \frac{t}{p} - \frac{q^2}{4} \), and \( M = \sup_{z \in \mathbb{C}} |P(z)| e^{-\epsilon |z|^2} < \infty. \)

**Remark 3.29.** Corollary 3.27 and Example 3.28 show that, in general, the Fock spaces \( F^p \) and \( F_F^p \) cannot be characterized in terms of membership of partial derivatives in \( L^p \) spaces since the condition in question is only necessary. This is quite different from the situation in the Bergman space (cf., [13, Theorem 2.17]) and in the Besov space (cf., [6, Theorem 13]).

### 3.6 Open problems and remarks

In this section we state some problems that we plan to study in the near future. In Section 4, we proved that the integral operators \( S_{a,b,c} \) and \( T_{a,b,c} \) are bounded on \( L^p(C^n, dv) \) provided that the conditions of Theorem 3.1 and Theorem 3.2 are satisfied and, we have also calculated the norms of these operators acting on \( L^p(C^n, dv) \) when \( b = 0 \). Thus, it is natural to ask for the calculation of the norms of \( S_{a,b,c} \) and \( T_{a,b,c} \) when \( b \neq 0 \). We mention that if \( t > 0, a = 0, b = t, c = -t, \) and \( pt = 2s \), it was proved in [1] that:

\[
\|T_{0,t,-t}\|_{L^p(C^n, dv) \to L^p(C^n, dv)} = 2^n.
\]

**Problem 1.** Let \( b \neq 0 \). Find,

\[
\|S_{a,b,c}\|_{L^p(C^n, dv) \to L^p(C^n, dv)} \quad \text{and} \quad \|T_{a,b,c}\|_{L^p(C^n, dv) \to L^p(C^n, dv)}.
\]
The next problem is related to the study of the compactness of $S_{a,b,c}$ and $T_{a,b,c}$ on $L^p(\mathbb{C}^n, dv_\lambda)$. A special case is worth mentioning. Recall that, if $p = 2$, the Bergman projection $P_t : L^2(\mathbb{C}^n, dv_t) \rightarrow F_t^2$, given by

$$P_t f(z) = \int_{\mathbb{C}^n} f(w) e^{i(z,w)} dv_t(w)$$

is not a compact operator. To see this, we have, by Lemma 2.6, that $k_w$ converges weakly to 0 in $L^2(\mathbb{C}^n, dv_t)$. On the other hand, since $k_w$ is a holomorphic function, we get that $P_t(k_w) = k_w$. Thus,

$$||P_t(k_w)||_t^2 = ||k_w||_t^2 = 1,$$

hence $P_t$ fails to be compact, (cf., [15, Theorem 1.3.4]).

Problem 2. Study the compactness of $T_{a,b,c}$ and $S_{a,b,c}$ on $L^p(\mathbb{C}^n, dv_\lambda)$.

The third problem is about proving the reproducing formula of Fock space $F_p^p$ when $p \in (0,1]$. It is natural to ask, since the polynomials are dense in $F_p^p$, whether the reproducing formula holds for all $f \in F_p^p$, $0 < p \leq 1$.

Problem 3. Let $p \in (0,1]$. Prove the reproducing formula for $f \in F_p^p$. 

51

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Bibliography


