Capacitated and Uncapacitated Facilities Location Allocation Problem with Sensitive Prices, Stochastic Demands, and Inventory

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CAPACITATED AND UNCAPACITATED FACILITIES LOCATION-ALLOCATION PROBLEM WITH SENSITIVE PRICES, STOCHASTIC DEMANDS, AND INVENTORY

by

Yaser Al-Alawi

A Dissertation
Submitted to the
Faculty of The Graduate College
in partial fulfillment of the
requirements for the
Degree of Doctor of Philosophy
Department of Industrial and Manufacturing Engineering

Dr. Abdolazim Houshyar and Bob White

Western Michigan University
Kalamazoo, Michigan
December 2005
This study developed a quantitative model that solves a facility location-allocation (LA) problem that maximizes the net profits generated from expected sales. The model was formulated to take into consideration demand uncertainty, sensitive prices, and existence of inventory for capacitated and uncapacitated facilities. Four new models for the LA problem with stochastic demands and inventory were developed and studied. The four models were combinations of capacitated and uncapacitated facilities and sensitive and insensitive prices.

A new method was proposed for achieving optimality, and an approximation procedure was developed that could find near-optimal solutions for problems that could not be solved optimally. Additionally, an existing Two-Phase Method was modified to solve this problem, and a heuristic algorithm was developed to improve the solution of the modified Two-Phase Method for capacitated facilities.

Inventory plays a critical role in either maximizing profit or minimizing cost, and it also has an impact on shipping arrangements. However, to the author’s best knowledge, the effect of inventory on profit-maximization LA problems has never been investigated in the literature. Therefore, one of the objectives of the proposed research was to study the effect of inventory on a profit-maximization LA problem.
Inventory impact was found to be significant, which significantly changed the expected profit level and shipping arrangements.

All methods were capable of optimizing small LA problems with uncapacitated facilities. The proposed optimization method was shown to have the ability of solving small LA problems with capacitated facilities as well. However, for large problems the Approximation Method was the only technique capable of solving the uncapacitated facilities' problems. Furthermore, the Approximation Method always had the best available solution for capacitated facilities with large problems.

Hence, the study proposed a way to optimally maximize expected profit by locating a number of new facilities that serve a specific set of customers with uncertain demands, sensitive prices, and an existence of inventory. Also, the study provided an efficient method capable of solving very large LA problems.
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Yaser Al-Alawi
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CHAPTER 1
INTRODUCTION

Choosing a location is one of the most strategic decisions an enterprise can make. By locating and building new facilities, a business incurs the one-time cost of opening the facilities, a cost that can be very high. Moreover, closing existing facilities and locating new ones might not be feasible because of this cost and the difficulty of finding suitable available locations. Thus, the location decision is not a trial and error process.

Most of the research conducted addressed the location problem for cost minimization because of the vast applications it has in the supply chain and because of the availability of models. However, the end of the supply chain involves transporting commodities to customers, which is associated with sales. Therefore, profit maximization models are the most appropriate for end of supply chain applications, even if the profit maximization location problem requires a higher complexity of formulations.

Previous researchers proved the significant effect of inventory policies on the cost minimization location problem, even though, inventory is a secondary decision. Hence, this study considered the impact of inventory on the profit maximization location problem in an effort to find better methods of solving the problem with the existence of inventory.
Section 1.1 of this chapter defines terminology that will be used extensively throughout the succeeding chapters. Section 1.2 presents general background information about the research problem. Section 1.3 lists the research objectives, and Section 1.4 describes the study's significance. Research assumptions and limitations are presented in Section 1.5.

1.1. Definitions

This study uses terminology that is common in the literature. Precise definitions, therefore, are essential to avoid any misunderstanding and to expedite the research process. Five subsections are presented here, one for each definition.

1.1.1. Location-Allocation Problem

Location-allocation (LA) problem is an optimization procedure to locate number of new facilities and allocate them to service a specific number of customers.

1.1.2. Capacitated Facilities and Uncapacitated Facilities

Uncapacitated facilities are those without capacity constraints in their shipping and production areas. These facilities are able to serve customers optimally at convenient demand centers because the facilities can easily transfer goods and services.
1.1.3. Stochastic Demands

The requests of customers at demand centers are variable. Therefore, demands are of a probabilistic nature that can follow certain probability density distribution functions. In contrast, demand is deterministic when it can take only a single specific value.

1.1.4. Sensitive Prices

Elasticity of demand as an economic concept describes the relationship between price and demand. This inversely proportional relationship shows that there will be a drop in demand when the price increases and vice versa. In other words, the price is a function of the expected demand.

1.1.5. Inventory

Excess inventory occurs when the number of units shipped to the demand center is more than what is demanded. This condition is a direct consequence of the stochastic demands, where the shipping order is based on expected demand rather than actual demand.

1.1.6. Expected Net Profit

The expected net profit at each demand location is the difference between the revenue generated from selling units and the total cost. The total cost is composed of
the production cost at the production plant, the transportation cost, and the cost of inventory at the demand center.

1.2. Problem Background

The LA problem is to locate a set of new facilities to serve certain customers at demand locations. LA problem for the private sector can be modeled as a transportation-type plant-location problem. The transportation-type problems are a combination of five factors:

- Facilities (Uncapacitated or Capacitated)
- Demand (Deterministic or Stochastic)
- Price (Insensitive or Sensitive)
- Inventory (Incorporated or Not incorporated)

All eight combinations of the first three factors have been addressed in the literature. In general there are two ways of formulating the LA problems: 1) minimizing the total cost or 2) maximizing the total profit.

1.3. Objective

The objective of this study was to develop a quantitative algorithm that solves the transportation-facility LA problem by maximizing the expected net profits generated from sales. The algorithm was formulated to help capacitated/uncapacitated facilities with stochastic demands, price sensitivity, and usage of inventory. Heuristic
and approximation procedures were developed to solve the problems that could not be solved optimally.

1.4. Significance of the Research

Location-allocation problems can play a crucial role in many aspects of business practices at both strategic and tactical levels. Examples of such problems are supply-chain, material-handling, and location issues.

An example of a company with supply-chain LA problem is a grocery retailer. Consider a retailer that sells groceries and has many branches in a city. The demand for groceries is variable and uncertain and depends on the willingness of customers to buy products within a specific period of time. However, certain amounts of groceries must be shipped to the branches so that consumers can start purchasing them. Even though the demand for groceries could be forecasted based on previous periods, the forecasted sales level is not necessarily the most profitable setup for the entire firm because shipping and handling costs change and supply and storage facilities have limited capacities.

Therefore, allocating the shipping or storage facilities to satisfy the demand at branches is crucial to maximizing the total profit of the retailer. Selecting the shipment size is essential too. The more units shipped, the higher the cost of production and shipping, and the units might not even be sold. However, shipping fewer units than needed could lead to lost sales and send customers to a competitor; this is known as the opportunity cost. One way to look at this supply-chain problem is
as a complex transportation issue in which the cost is a function of units produced and the profit is a function of the expected sales.

1.5. Assumptions and Limitations

Certain assumptions were made in order to simplify and accelerate the solution process of the research problem. This also helped to keep the focus on the main issues of the scope of the research. The assumptions, however, are also limitations of the model. Therefore, removing or altering these assumptions can lead to future work. The following assumptions were made for the LA model:

- A single type of unit was produced, shipped, and sold (single commodity).
- A single-period inventory policy.
- Unlimited shipping batch sizes.
- Unlimited inventory capacity.
- Satisfying all demand centers was not obligatory. (Demand was forecasted with a continuous probability density function.)
CHAPTER 2
LITERATURE REVIEW

Cooper (1963) introduced the location-allocation (LA) problem in the literature. Since then, researchers have continued to investigate the problem from different perspectives. One type of problem, the fixed charge facility location problem, is considered the foundation of many location models used for supply chain design. Balinski (1965) formulated the fixed charge location problem for a set of candidate facility locations and a set of customer locations with known demands. In Balinski's formulation, a fixed cost would be charged if a candidate facility were opened. Demands were satisfied by shipments from the opened facility, with a known unit cost between the customer and the candidate facility. The objective was to minimize the total cost of locating the facility and shipping the demand requirements. The problem formulation is presented in Formulation 2.1.

Formulation 2.1 – Fixed Charge Location Problem

Minimize \[ \sum_{vi} \sum_{vj} d_{ij} c_{ij} x_{ij} + \sum_{vi} f_i y_i \] 

Subject to

\[ \sum_{vi} x_{ij} = 1 \quad \forall j \in J \] [2]

\[ x_{ij} - y_i \leq 0 \quad \forall i \in I \cup J \] [3]

\[ y_i \in \{0, 1\} \quad \forall i \in I \] [4]
$X_{ij} \geq 0 \quad \forall i \in I; \forall j \in J$ [5]

Where:

$I$ Set of candidate facility locations, indexed by $I$

$J$ Set of customer locations, indexed by $j$

$f_i$ Fixed cost of locating a facility at candidate site $i \in I$

$c_{ij}$ Shipping cost per unit shipping from candidate facility $i$ to customer $j$

$d_j$ Demand at customer location $j \in J$

Decision Variables:

$X_{ij}$ Fraction of the demand at customer location $j$ that is served by a facility at $i$

$Y_i \begin{cases} 
1 & \text{if candidate facility is located at site } i \\
0 & \text{otherwise} 
\end{cases}$

Equation [1] of the formulation, the objective function, minimizes the total cost of shipping and the total cost of locating the facility. Demand requirements at location $j$ are satisfied by a demand constraint in Equation [2]. Equation [3] ensures that no shipment will be made if a candidate facility is not located in location $i$. Equations [4] and [5] represent the decision variables of the problem. Formulation 2.1 assumed uncapacitated shipping facilities and deterministic demands.

The capacitated facilities case was addressed later on by incorporating the supply and shipping capacity ($A_i$) of the candidate facilities. This led to the replacement of $d_jX_{ij}$ with the $x_{ij}$ decision variable that represents the number of units shipped from facility $i$ to customer $j$ (Efroymson and Ray, 1966). Thus, Formulation 2.1 is modified as follows (Formulation 2.2):
Formulation 2.2 – Fixed Charge Location Problem for Capacitated Facilities

Minimize  \[ \sum_{i_j} \sum_{j_l} c_{ij} x_{ij} + \sum_{i_l} f_i y_i \]  \[ \text{[1]} \]

Subject to:

\[ \sum_{j_l} x_{ij} = d_j \quad \forall j \in J \]  \[ \text{[2]} \]

\[ \sum_{i_j} x_{ij} \leq A_i y_i \quad \forall i \in I ; \forall j \in J \]  \[ \text{[3]} \]

\[ Y_i \in \{0,1\} \quad \forall i \in I \]  \[ \text{[4]} \]

\[ x_{ij} \geq 0 \quad \forall i \in I ; \forall j \in J \]  \[ \text{[5]} \]

Geoffrin and Graves (1974) extended Formulation 2.2 to include multiple commodities. Several heuristics have been introduced in the literature to tackle the complexity of the LA problem, especially for large problems. Few researchers addressed rectilinear distance instead of Euclidean; for instance, Liu, Kao, and Wang (1994) used simulated annealing to solve a minisum location-allocation problem. The model solved for uncapacitated facilities with rectilinear distances. Zhou (2000) integrated neural network, genetic algorithms, and stochastic simulation to solve the LA problem of uncapacitated facilities, stochastic demands, and facilities coordinates. The model minimized both the total fixed cost and the variable cost, which was the cost of transportation determined by a known weight, multiplied by the Euclidean distance between the shipping and receiving facilities.
Liu (1997) investigated the theoretical framework of dependent chance programming, dependent chance multi-objective programming, and dependent chance goal programming. These models were useful when there were multiple stochastic inputs and multiple tasks whose reliability levels were to be optimized. While they did not solve the LA problem, the models provided a very good infrastructure for solving it. A hybrid intelligent algorithm that integrated network simplex algorithm, stochastic simulation, and genetic algorithm solved the cost minimization LA problem for capacitated facilities with stochastic demands (Zhou & Liu, 2003). Francis, McGinnis, and White (1992) addressed various types of LA problems through a real-life example that illustrated the complexity of LA problems and ways to solve them. The researchers presented a case that solved the assignment of digging oil wells in the sea to different platforms. The objective function was to minimize the total cost of digging. For the same case study, they presented different models that were of deterministic nature. The concepts of capacitated facilities and uncapacitated facilities were addressed as well. An optimal solution was hard to achieve, so different heuristics were developed and implemented.

Drezner and Wesolowsky (1999) formulated LA problem to deal with changing costs as a result of economies of scale. The model was classified as a capacitated facilities, cost-sensitive, and deterministic demands model. The problem was to find p locations for p facilities servicing a given set of demand points. Each demand point obtained service from the closest facility, and the objective was to minimize the sum of weighted Euclidean distances between the demand points and
their servicing facility. Costs per customer decreased because they were shared by a larger number of customers. The decreases were reflected in the prices charged to customers. Customers base their selection of a facility on the total cost: the cost of transportation (proportional to the distance to the facility) and the mill price charged by the facility. Thus, customers might choose to obtain service from a capacitated facility, rather than from the nearest facility. In another paper, Drezner and Wesolowsky (1999) also investigated demand-dependent cost. While in their previous work only the fixed cost was divided by the number of units demanded, in this study the total cost depended on the number of units demanded. Berman, Drezner, and Wesolowsky (2003) investigated the LA problem where distance depends on the reliability of the service facility. The objective was to find the location of the facility that maximized the expectation for satisfactory service. Finally, Drezner and Drezner (2003) solved the LA problem for a retail facility where market conditions were expected to change during the planning phase.

Stochastic programming and robust optimization are the main approaches to decision-making under uncertainty. Stochastic programming minimizes the expected cost by creating discrete scenarios with a given probability of occurrence for the uncertain parameters. Robust optimization minimizes the worst case or the regret that is the difference in the objective function between a given scenario and the optimal solution (Daskin et al, 2003). Examples of stochastic programming are the work of Weaver and Church (1983) and Mirchandani, Oudjit, and Wang (1985). The latter
model solved multiple scenarios of the p-median problem. The model was translated in terms of the LA format as follows in Formulation 2.3:

Formulation 2.3 – Stochastic Programming LA Fixed Charge Location Problem

Minimize  \[ \sum_{s} \sum_{j} \sum_{i} q_s d_{js} c_{is} X_{is} + \sum_{i} f_i y_i \]  \[ \text{[1]} \]

Subject to

\[ \sum_{i} X_{is} = 1 \quad \forall j \in J; \forall s \in S \]  \[ \text{[2]} \]
\[ X_{is} - y_i \leq 0 \quad \forall i \in I; \forall j \in J; \forall s \in S \]  \[ \text{[3]} \]
\[ y_i \in \{0,1\} \quad \forall i \in I \]  \[ \text{[4]} \]
\[ X_{is} \geq 0 \quad \forall i \in I; \forall j \in J; \forall s \in S \]  \[ \text{[5]} \]

Where:

* Set of candidate facility locations, indexed by *I*
* Set of customer locations indexed by *j*
* Set of scenarios
* Probability of scenario *s* occurring
* Fixed cost of locating a facility at candidate site *i* \(i \in I\)
* Shipping cost per unit shipping from candidate facility *i* to customer *j* for scenario *s*
* Demand at customer location *j* \(j \in J\) when scenario *s* occurs
Decision Variables:

\[ X_{ij} \]  
\[ Y_i \] 
- Fraction of the demand at customer location \( j \) that is served by a facility at \( j \) when scenario \( s \) occurs
- 1 if candidate facility is located at site \( i \)
- 0 otherwise

Most of the research conducted in the area of LA focuses on minimizing total cost; very few researches solve the LA problem for maximizing profit. Logendarn and Terrell (1988), however, modeled the LA problem to maximize the total expected profit for uncapacitated facilities with stochastic demand and sensitive prices. They studied two models of LA along with their optimization formulations. The first LA model was of uncapacitated facilities, stochastic demands, and insensitive prices; the other LA model was of capacitated facilities, stochastic demands, and sensitive prices. A heuristic algorithm was proposed to solve large problems that could not be solved optimally. The same authors extended the focus to solve for capacitated plants (Logendran and Terrell, 1991). It was found that, while Branch and Bound was sufficient for solving small and medium-sized problems, it was insufficient for solving large problems, and so a heuristic was used. In the models of Logendran and Terrell, neither distance nor inventory was incorporated. They relied on the cost matrix between facilities, under the assumption of fixed plant locations. Thus, the coordinates of the locations of the facilities were not incorporated; the from/to matrix of distances was used instead. The algorithm was solved in two phases to maximize the total expected profit. The first phase obtained the candidate shipments. The
second phase assigned these shipments from shipping plants to demand centers. Both phases are presented as follows:

Formulation 2.4 – Two-Phase Algorithm

Phase I:

\[
UB_j \int_{x_j} f(q_j) dq_j = \frac{r_q}{p_j}
\]  \hspace{1cm} [1]

\[
L(x_{ij}) = p_j \mu_q - p_j \int_{x_j} f(q_j) dq_j - r_q x_{ij}
\]  \hspace{1cm} [2]

Phase II:

Maximize \hspace{1cm} OB_T = \sum_{j=1}^{n} \sum_{i=1}^{m} z_{ij} L(x_{ij}) - \sum_{i=1}^{m} f_i y_i
\]  \hspace{1cm} [3]

Subject to

\[
\sum_{i=1}^{m} z_{ij} = 1 \hspace{1cm} ; j = 1,2,...,n
\]  \hspace{1cm} [4]

\[
\sum_{j=1}^{n} x_{ij} z_{ij} \leq A_i y_i \hspace{1cm} ; i = 1,2,...,m
\]  \hspace{1cm} [5]

\[
y_i, z_{ij} \in \{0,1\}
\]  \hspace{1cm} [6]

Where:

\(A_i\) \hspace{1cm} the capacity of candidate SP_i (equals infinity for uncapacitated facilities)

\(f_i\) \hspace{1cm} the fixed cost of candidate SP_i

\(x_{ij}\) \hspace{1cm} the optimal shipment size produced in SP_i and shipped to DC_j

\(L(x_{ij})\) \hspace{1cm} the expected profit for shipping \(x\) units from SP_i to DC_j

The decision variables:
\[ z_{ij} = \begin{cases} 
1 & \text{if production plant } i \text{ supplies demand center } j \\
0 & \text{otherwise} 
\end{cases} \]

\[ y_i = \begin{cases} 
1 & \text{if candidate facility is located at site } i \\
0 & \text{otherwise} 
\end{cases} \]

Inventory can play a critical role in either maximizing profit or minimizing cost, and it has an impact on shipping arrangements. However, the effect of inventory on the LA problem has not been investigated widely in the literature. In a PhD dissertation, Shen (2000) studied a Blood Bank in Chicago, but it was a case of special circumstances with few applications to the LA problem. The case was of known fixed locations for suppliers/retailers. The objective was to determine the optimal number of distribution centers that act as connecting facilities between suppliers and retailers. Also, the study aimed to optimize the retailers assigned to each distribution center, as well as to optimize the ordering policies at the distribution centers. Shen and Coullard (2001) and Shen, Coullard and Daskin (2002) modified the study to incorporate reorder size, reorder interval, and safety stock. These modifications were an extension of the traditional uncapacitated fixed charge facility location problem. Thus, Formulation 2.1 was modified to incorporate the inventory policy as follows in Formulation 2.5. The approach requires the demand for all retailers in each scenario to have a variance identically proportional to the mean demand. This restriction was later removed (Shu et al., 2004).
Formulation 2.5 – Fixed Charge Location Problem with Inventory Policy

Minimize  

\[ \sum_{i \in I} \sum_{j \in J} \mu_j c_{ij} X_{ij} + \sum_{i \in I} f_i Y_i + \sum_{i \in I} \rho_i \sqrt{\sum_{j \in J} \mu_j X_{ij}} + \sum_{i \in I} \omega_i \sqrt{\sum_{j \in J} \sigma^2_{ij} X_{ij}} \]  

Subject to  

\[ \sum_{j \in J} X_{ij} = 1 \quad \forall i \in I \]  

\[ X_{ij} - Y_i \leq 0 \quad \forall i \in I \; ; \; \forall j \in J \]  

\[ Y_i \in \{0,1\} \quad \forall i \in I \]  

\[ X_{ij} \geq 0 \quad \forall i \in I \; ; \; \forall j \in J \]  

Where:

- \( \mu_j \)  mean of the demand per unit time at customer \( j \)
- \( \sigma^2_j \)  variance of the demand per unit time at customer \( j \)
- \( c_{ij} \)  the annualized cost of supplying and shipping a unit from facility \( i \) to a customer \( j \)
- \( \rho_i \)  the fixed order cost plus the fixed transportation cost per shipment from the supplier to facility \( i \) plus the working inventory holding cost
- \( \omega_i \)  the lead time of shipments and safety stock holding cost the supplier to facility \( i \)

The objective function consists of four terms. The first term represents the total shipping cost and variable cost. The second term represents the total fixed cost of locating the facility in the candidate sites. The third term represents the working
inventory costs that include all fixed costs of shipments. The last term represents the safety stock.

The available LA models of the profit maximization do not integrate all of the parameters into one formulation. Thus, the solutions of these models do not guarantee optimality. The impact of inventory was not studied on these types of LA problems.
CHAPTER 3
METHODOLOGY

This chapter is divided into six sections. Section 3.1 describes the research problem of the dissertation in detail. Section 3.2 lists the notation that will be used in the subsequent sections and chapters. Section 3.3 describes about the data set ranges. Section 3.4 lists the steps and techniques used to solve the problem. (Individual techniques will be explained extensively in separate chapters or sections following this chapter.) Section 3.5 describes the proposed models and explains how their creation led to the development of the final desired model. Section 3.6 is concerned with the apparatus and requirements used for this research.

3.1. Problem Statement

The problem was a location-allocation problem where there were \( n \) demand centers (DC) and \( m \) shipping plants (SP). Each DC was exposed to stochastic demands that were forecasted and fitted to follow a certain continuous probability density function (pdf). The SP candidate locations were known prior to the assignments. \( SP_i \) was charged a one-time fixed cost once it was opened. Also, it was restricted with a known capacity. The capacity of the SP attributed to the capacity of the supply warehouse or production capacity restriction during the assignment period. Shipments were made from \( SP_i \) to \( DC_j \) without capacity limitation. Inventory
capacities of DC’s were unlimited. Thus, the desired level of demand was achieved irrespective of any capacity limitation other than that of the SP capacity.

A variable cost was charged per unit shipped from SP\textsubscript{i} to DC\textsubscript{j}. This variable cost varied for each combination of i and j. This was due to such factors as the distance between SP\textsubscript{i} and DC\textsubscript{j} (per-unit distance cost). For example, the fuel cost per mile from SP\textsubscript{2} to DC\textsubscript{3} might have been different from the fuel cost from SP\textsubscript{2} to DC\textsubscript{4}. All costs associated with this transportation problem were fixed and predetermined based on actual or forecasted values.

Sales were realized at the DCs only. Each DC charged a price per unit sold based on the price-demand relationship. Thus, the selling price varied from one DC to another. The price and demand relationship is also known as price sensitivity, which relates the value of the selling price to the demand level. Because the shipments were made prior to the sales, the selling price was determined based on the forecasted demand mean.

In this specific application of the LA problem, not all demands needed to be satisfied, nor did all shipping plants need to be used. The decision to open or close a certain SP or DC was based on the location-allocation solution. Therefore, the aim was to solve the LA problem in order to maximize the total expected profit that could be generated from sales under the circumstances mentioned earlier.
3.2. Notation

The following is the notation that will be used throughout the remainder of this dissertation. These variables are common among all chapters and sections.

\( SP_i \) = Candidate shipping plant \( i \)

\( DC_j \) = Candidate demand center \( j \)

\( m \) = The number of candidate plants

\( n \) = The number of candidate demand centers

\( f_i \) = Fixed cost of opening and locating a plant in location \( i \)

\( A_i \) = The capacity of candidate SP\( i \)

\( r_{ij} \) = Total variable cost per unit of product produced at SP\( i \) and shipped to DC\( j \)

\( p_j \) = Unit selling price received at DC\( j \)

\( h_j \) = Holding cost charged per unit stored at DC\( j \)

\( S_j \) = Stock out cost or backordering cost per unit short at DC\( j \)

\( x_{ij} \) = Number of units shipped from SP\( i \) to DC\( j \).

\( x_{ij}^* \) = Number of units shipped from SP\( i \) to DC\( j \) when a single SP\( i \) is allowed to satisfy all demand requirements at DC\( j \)

\( L(x_{ij}^*) \) = The expected net profit for selling \( x_{ij}^* \) units shipped from SP\( i \) and sold at DC\( j \)

\( q_j \) = Demand random variable at DC\( j \)

\( f(q_j) \) = Probability density function describing the random demand \( q_j \) at
DC$_j$

$F(q_j)$ = Cumulative density function describing the random demand $q_j$ at DC$_j$

$\mu_j$ = Mean of the forecasted demand distribution at DC$_j$

$\sigma_j$ = The standard deviation of the demand distribution at demand DC$_j$

$LB_j$ = Lower bound on the demand distribution at DC$_j$

$UB_j$ = Upper bound on the demand distribution at DC$_j$

$P^{*}_{rij} = F(Q^*)$ = Critical probability at which no shortage will occur when a single SP$_i$ is allowed to satisfy all demand requirements at DC$_j$ with $Q^*$ units

$M$ = Infinity (or big number)

$OB_w$ = The total net profit generated using method W

Table 3.1 summarizes the location-allocation problem notation and presents the problem in a transportation-problem format. The table shows that $x_{ij}$ units were produced and shipped from SP$_i$ to DC$_j$ at a cost of $r_{ij}$ per unit of $x_{ij}$. The sum of $x_{ij}$ for each SP$_i$ was within the SP$_i$ capacity ($A_i$). The demand at DC$_j$ was associated with demand pdf that was satisfied by the sum of $x_{ij}$. All shipments were optimized in order to maximize the profit gained from selling the sum of $x_{ij}$ at every DC$_j$ minus the shipping cost, SP fixed cost (if it was used), per-unit stock out cost and per-unit holding cost at each DC$_j$. 

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Table 3.1 – LA Problem Notation Summary

<table>
<thead>
<tr>
<th>j</th>
<th>DC_j</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( A_1 )</td>
<td>( f_1 )</td>
</tr>
<tr>
<td>2</td>
<td>( A_2 )</td>
<td>( f_2 )</td>
</tr>
<tr>
<td>..</td>
<td>..</td>
<td>..</td>
</tr>
<tr>
<td>m</td>
<td>( A_m )</td>
<td>( f_m )</td>
</tr>
</tbody>
</table>

3.3. Data for Test Runs

The LA problem was solved using the following data for the sake of illustration. This allowed the comparison of the performance of the methods with the model differences. The independent parameters are the fixed cost \( f_i \) of \( \text{SP}_i \), the
capacity ($A_i$) of SP, the variable cost ($r_{ij}$), the holding cost ($h_j$), the stock out cost ($s_j$), the insensitive price ($p_j$), and the mean and standard deviation of demand pdf ($\mu_j, \sigma_j$). The values of these parameters were generated randomly from known probability distributions as shown in Table 3.2.

### Table 3.2 – Test Data

<table>
<thead>
<tr>
<th>Independent Variable</th>
<th>Distribution</th>
<th>Limits</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_i$</td>
<td>Uniform</td>
<td>[200, 300]</td>
</tr>
<tr>
<td>$A_i$</td>
<td>Type (a)</td>
<td>Uniform [300, 350]</td>
</tr>
<tr>
<td></td>
<td>Type (b)</td>
<td>Uniform [100, 250]</td>
</tr>
<tr>
<td>$r_{ij}$</td>
<td>Uniform</td>
<td>[0.5, 2.5]</td>
</tr>
<tr>
<td>$h_j$</td>
<td>Uniform</td>
<td>[0.5, 1.5]</td>
</tr>
<tr>
<td>$s_j$</td>
<td>Uniform</td>
<td>[0.5, 1.0]</td>
</tr>
<tr>
<td>$\mu_j$</td>
<td>Uniform</td>
<td>[100, 200]</td>
</tr>
<tr>
<td>$\sigma_j$</td>
<td>Uniform</td>
<td>[5, 25]</td>
</tr>
<tr>
<td>$p_j$</td>
<td>Price insensitive</td>
<td>Uniform [3, 6]</td>
</tr>
<tr>
<td></td>
<td>Price sensitive</td>
<td>$p_j = 600/\mu_j$</td>
</tr>
</tbody>
</table>

3.4. Approach

The problem was approached in four stages. The first stage was to modify a method called the Two-Phase method (Logendran and Terrell, 1988, 1991), which is
found in the literature. The solution that used the Two-Phase method was considered a lower bound on the optimal solution of the LA problem. The second stage was to solve the LA problem with a new method called the Combined Method. The Combined Method was capable of finding a global optimal solution. If the method terminated before guaranteeing the global optimal solution was not achieved, then the nonlinear part of the Combined Method was approximated linearly. Thus, the third stage was the linear approximation of the Combined Method. The purpose of this approximation was twofold: to solve large problems efficiently when the non-linear Combined Method terminated at local optimal solution and to shorten the time used to solve this complex problem. If the approximation tended to be inefficient under any kind of circumstance, then a heuristic was employed. Therefore, the final stage was to develop a heuristic that improved the solution of the Two-Phase Method. The stages of solving the problem were as follows:

1. Two-Phase Method:

   Phase one determined the candidate shipment size from SP\textsubscript{i} to DC\textsubscript{j}. This phase ignored the impact of the capacity and fixed cost of the SP. Also, it assumed that only a single SP would satisfy the demand at each DC.

   Phase two was a purely binary assignment problem. This phase located SP\textsubscript{i} and allocated the shipments from SP\textsubscript{i} to DC\textsubscript{j}. Therefore, the LA problem was not completed simultaneously and thus optimality is not guaranteed.
2. Combined Method:

This method combined the phases of the Two-Phase Method into one formulation. The constraints were modified so that the problem was globally optimized. These modifications imitated the real-world application and did not violate any logical concept.

3. Approximation Method:

This method linearly approximated the nonlinear decision variables of the Combined Method in the objective function.

4. Heuristic Method:

The heuristic employed considered the solution of the Two-Phase Method as the lower bound. Thus, the solution of the Two-Phase Method was the starting point from which the heuristic improved the solution.

Figure 3.1 summarizes the entire methodology hierarchically, from creating a single LA problem scenario to finding the best possible solution. Thus, the start was to input the LA problem parameters. Then the Two-Phase Method and the Combined Method were concurrently applied. If the solution of the Combined Method was greater than that of the Two-Phase Method, the best solution was achieved. However, if the solution of the Combined Method was less than the solution of the Two-Phase Method, the Combined Method was approximated linearly if possible. If the linear approximation solution was greater than the solution of the Two-Phase Method, a heuristic was developed that attempted to improve the solution of the Two-Phase
method. The reasoning behind this methodology will be described more in details in the subsequent chapters.

![Flow Chart of Solution Approach](image)

Figure 3.1 – Flow Chart of Solution Approach
3.5. Stages of Creating the Required Model

The desired model was the one that adopted the price sensitivity, cost sensitivity, stochastic demands, and existence of inventory for both capacitated facilities and uncapacitated facilities. Such a model was achieved in steps, and its reliability was assured through continuous checks. The purpose of the checks was to validate and verify the applicability of the model before it became too complex to be analyzed. Therefore, the final model required the development of several preliminary models, from the simplest form of the problem to the final desired model. The final model had five dimensions:

1. Demand: deterministic or stochastic.
2. Price-demand relationship: price insensitive or price sensitive.
3. Capacity of shipping facilities: uncapacitated facilities or capacitated facilities.
4. Inventory: existed or did not exist.

Twelve combinations were created and analyzed, and each combination was called a model. The selected models are addressed in Table 3.3. The twelve models of interest were grouped in three bundles; each bundle consists of four models. The first bundle, models one to four, had deterministic demands. The second bundle, models five to eight, addressed the impact of price sensitivity. The last bundle, models nine to twelve, addressed the impact of inventory. The first bundle of models was approached efficiently in the literature. Thus, models one to four were not studied in this research. The second bundle of models was approached inefficiently in literature. So models...
five to eight were approached again in this study for two main reasons. The first was to benchmark this study's algorithm against what was approached in the literature. The second was to study the impact of inventory on the LA problem by comparing the new developed models, nine to twelve, with the similar ones without the inventory.

Hence, model one was a combination of deterministic demands, insensitive price, insensitive cost, uncapsacitated facilities, and no inventory. For the deterministic models, inventory was meaningless as inventory in this research was used to overcome the uncertainty of demand. The inventory played a crucial role in maximizing the profit when the demand was stochastic rather than deterministic. Inventory existed when the number of shipped units exceeded the actual units demanded. Table 3.3 lists all models that were addressed in this research with their dimensions and combinations and whether they were achieved previously in the literature or not. Moreover, the table indicates the chapters that contain examples of the models. Models five and six were solved using the Two-Phase Method (see examples 4.1a and 4.1b). The same models were solved again using the Combined Method (see examples 5.1a and 5.1b). Models seven and eight were solved (see examples 6.1a and 6.1b). Models nine and ten were solved using the Two-Phase Method (see example 7.1). The same models were solved again using the Combined Method (see example 7.2).
Table 3.3 – Models and their Combinations

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<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
</tr>
<tr>
<td></td>
<td>Uncapacitated</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
</tr>
<tr>
<td>Inventory</td>
<td>Exists</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>√</td>
</tr>
</tbody>
</table>

3.6. System Requirements

The LA problem is considered to be an operations-research problem. It was either formulated as a linear program (LP) or a non-linear program (NLP). Thus, efficient LP/NLP software was essential for reliable results. In addition, using a programming language made the application of the heuristic method robust. Another purpose of this programming language was to prepare the input file of the LA problem to be fed into the LP/NLP software. The programming language used in this study was Visual Basic version six 6.0, and the LP/NLP software used was GAMS. GAMS is a reliable commercial operations research software that is available with
different solvers. GAMS model can be constructed as a text file using a text editor or
GAMS-IDE. GAMS-IDE is capable of locating the syntax errors and their sources,
monitoring the completion, and monitoring the execution of models. The free solvers
available on the GAMS website are limited in their ability to solve large problems.
However, the full version of GAMS includes solvers capable of solving problems
with a large numbers of integer variables. It can be purchased from the GAMS
website. However, the full version still suffers from the limitations of long
computation time and the inability to optimize complex nonlinear problems. The
barrier of long computation time was overcome through the use of the NEOS server,
which is an excellent resource for solving large problems with several LP/NLP
solvers since parallel machines are used to speed up the process. Therefore, the trail
version of the GAMS solvers were used to solve small size problems and NEOS
server was used to solve medium and large size problems.

This study formulated the LA problem as mixed integer programming (MIP)
and mixed integer nonlinear programming (MINLP). GAMS solved the MIP
formulation with a CPLEX solver that is available on the NEOS server through the
XpressMP solver type. GAMS solved the MINLP formulation with an SBB solver
that integrated Branch and Bound (BB) and nonlinear programming (NLP) solvers.
The nonlinear terms in the MINLP formulations of this study were found to be
quadratic, which led to the use of mixed integer quadratic programming (MIQCP) as
a solver format with an SBB solver. GAMS had a limitation of 1,000 resource's usage
and 10,000 iteration counts for SBB and CPLEX solvers.
CHAPTER 4
TWO-PHASE METHOD

The work of Logendran and Terrell (1987 and 1991) was the inspiration for this research. Logendran and Terrell modeled an LA problem of uncapacitated/capacitated facilities that had price-sensitive stochastic demands. Logendran and Terrell’s model did not use any type of distance between the facilities or their location coordinates. It assumed that the plants’ candidate locations were fixed prior to the assignment. Thus, it allocated certain locations (shipping plants) to certain customers (demand centers). Moreover, the original formulation of the method did not incorporate inventory in the formulation.

Some modifications were applied to this model in order to make it more realistic for the scope of the current study. One such modification was the incorporation of inventory and stock-out costs in the formulation. Another modification was the alteration of the implemented concepts of selling price and demand probability density function (pdf). Logendran and Terrell treated the selling price in the same sense as variable cost, which is the price that is dependent on the link between each SP_i and DC_j. Thus, they assumed sales could take place along the shipping route rather than at the end of the shipping route, which is the demand center. Therefore, they used p_{ij} for the price charged for units shipped from SP_i to DC_j instead of p_j, which would imply that the same price was charged for all units sold at DC_j. This notion of pricing and sales did not fit the scope of the current study, in
which sales exist only at the final destination, DC\textsubscript{j}, not along the shipping route between SP\textsubscript{i} and DC\textsubscript{j}. Also, Logendran and Terrell presented demand as if it were realized for each link between SP\textsubscript{i} and DC\textsubscript{j}. Thus, each parameter of the demand pdf, such as the mean ($\mu_{ij}$), was attributed to the source (i) and destination (j) of the shipment. This was not realistic for the current study for two main reasons. First, demand was realized only at the end of the supply chain, DC\textsubscript{j}, not along the shipping route between SP\textsubscript{i} and DC\textsubscript{j}. Second, the forecasted demand was independent of the source of shipment, especially for new routes that had never been utilized for any shipment. If the current study had used Logendran and Terrell’s method of presenting demand, $\mu_{ij}$ would have had to be changed to $\mu_i$, and the other demand pdf parameters would have had to be changed as well.

The base model of Logendran and Terrell’s method was solved in two phases. The first phase was responsible for determining the number of units needed for shipment from SP\textsubscript{i} to DC\textsubscript{j}. Knowing the optimal numbers of needed/supplied units, the second phase solved the LA problem as a pure binary programming problem. The objective of the second phase was to allocate the demand centers to the supply facilities so that the total profit of the assignments would be maximized.

4.1. Phase I

In general, Logendran and Terrell calculated profit at each demand center by selling ($x_{ij}$) units in DC\textsubscript{j} that were shipped from SP\textsubscript{i} and subtracting the cost of
transporting \((x_{ij})\) units between these two facilities. Thus, the expected profit \((L(x_{ij}))\) from the sale of \((x_{ij})\) units was calculated by (derivation presented in Appendix B):

Net profit at each demand center

\[
L(x_{ij}) = p_{ij} \cdot \mu_{q_{ij}} - p_{ij} \int_{x_{ij}}^{q_{ij}} f(q_{ij})dq_{ij} - r_{ij}x_{ij}
\]

\[4.1\]

Elmaghraby (1960) established the optimality conditions for a stochastic transportation problem with continuous demand distributions, and he used the same conditions to determine the optimal shipment quantities \((x^*_{ij})\). Therefore, equation 4.2 was solved to determine \(x^*_{ij}\). The right-hand side of equation 4.2 represents the critical probability at which no shortage will occur when a single \(SP_i\) is allowed to satisfy all demand requirements at \(DC_j\) with \(x^*_{ij}\) units. Thus, the left-hand side represents the demand cumulative pdf that is integrated from the lower bound of the demand pdf to \(x^*_{ij}\). The derivation of equation 4.2 is presented in Appendix A. Figure 1 illustrates the relationship between the critical probability and \(x^*_{ij}\) based on Equation 4.2. For the sake of illustration, a normal distribution is utilized in Figure 4.1 with the shaded area representing the critical probability associated with the point of \(x^*_{ij}\).
Consequently, the optimal profit for the shipment of $x_{ij}^*$ units from $SP_i$ to $DC_j$ is given in the following formula:

$$L(x_{ij}^*) = p_j \mu_{q_j} - p_j \int_{x_i^*}^{UB_{q_j}} (q_j - x_{ij}^*) f(q_j) dq_j - r_q x_{ij}^*$$ \[4.3\]

$x_{ij}^* \geq 0$

A simplified form of Equation 4.3 is as follows (derivation presented in Appendix C):

$$L(x_{ij}^*) = p_j \int_{LB_{q_j}}^{UB_{q_j}} q_j f(q_j) dq_j$$ \[4.4\]
4.2. Phase II

Phase II optimizes the allocation of the demand centers to the production facilities in order to maximize the total profit. Thus, the input parameters for this phase are:

- $A_i$ (the capacity of candidate SP$_i$)
- $f_i$ (the fixed cost of candidate SP$_i$)
- $x^*_y$ (the optimal shipment size produced in SP$_i$ and shipped to DC$_j$)
- $L(x^*_y)$ (the expected profit for shipping $x^*_y$ units from SP$_i$ to DC$_j$)

The decision variables of the IP assignment problem were:

- $z_{ij} = \begin{cases} 1 & \text{If production plant } i \text{ supplies demand center } j \\ 0 & \text{Otherwise} \end{cases}$
- $y_i = \begin{cases} 1 & \text{If plant } i \text{ is open} \\ 0 & \text{If plant } i \text{ is closed} \end{cases}$

The modified version of Phase II of Legondran and Terrell (1988, 1991) Method for capacitated facilities was formulated in Formulation 4.1 as follows:

Formulation 4.1 – Phase II

Maximize

$$OB_T = \sum_{j=1}^{n} \sum_{i=1}^{m} z_{ij} L(x^*_y) - \sum_{i=1}^{m} f_i y_i$$

Subject to

$$\sum_{i=1}^{m} z_{ij} \leq 1 \quad ; \quad j = 1,2,\ldots,n$$
\[ \sum_{j=1}^{n} x_{ij}^* z_{ij} \leq A_i y_i \quad ; i = 1,2, \ldots, m \]  
\[ y_i, z_{ij} \in \{0,1\} \]  

4.2.1. Objective Function

The first term of the first equation of the above formulation \( \left( \sum_{j=1}^{n} \sum_{i=1}^{m} z_{ij} L(x_{ij}^*) \right) \) addresses the total profit generated from selling \( x_{ij}^* \) units in \( DC_j \) minus the total variable cost. The variable cost is the cost of producing \( x_{ij}^* \) units in \( SP_i \) and shipping them to \( DC_j \).

The second term of the objective function \( \left( \sum_{i=1}^{m} f_i y_i \right) \) addresses the total fixed cost of opening and locating \( SP \) in location \( i \).

4.2.2. Constraints

The second equation of the above formulation dictates that each \( DC_j \) may not receive shipments from more than a single \( SP_i \). According to Logendran and Terrell (1987), this restriction ensures that the price per unit is common for all units shipped to the same \( DC_j \). After implementing the modifications that were mentioned at the beginning of this chapter, this rationale for limiting the shipments received from a single shipping facility was no longer valid. However, the constraint not to duplicate shipments or send more than needed at \( DC_j \) was still valid. In addition, the shipments \( (x_{ij}^*) \) were predetermined in Phase I, and no partial value of this shipment could be
made. Finally, the second phase allowed the model to be optimized without forcing it to satisfy the demand level $D_{C_j}$ if it was not a profitable assignment.

Equation 3 of Formulation 4.1 assures that the total number of shipped units from each production facility is within the capacity of $SP_i$. The base formulation for the uncapacitated facilities is the same as Formulation 4.1 except that $A_i$ will be equal to infinity ($M$).

4.3. Example 4.1

The Two-Phase Method was implemented to determine the expected total net profit when there are two shipping plants and three demand centers, and demands are uniformly distributed at all demand centers. The upper and lower limits of the demand parameters for the uniform pdf for $D_{C_1}$, $D_{C_2}$, and $D_{C_3}$ were (90, 170), (144, 209), and (62, 147), respectively. The selling price per unit at all $D_{C_j}$ was $4. The first shipping plant had a fixed cost of $270 and a capacity of 233 units. The second shipping plant had a fixed cost of $250 and a capacity of 172 units. The costs per unit produced at $SP_1$ and shipped to $D_{C_j}$ were $2.05, $2.13, and $1.33 respectively. Values of $r_{2j}$ per unit were $0.53, $1.92, and $2.23 respectively. The problem was reworked with unlimited supply capacities (uncapacitated facilities).
Solution:

a. Capacitated Facilities (Model 6):

In order to apply the Two-Phase Method, Phase I was solved first, and then the outcomes were used to solve the second phase.

Phase I:

For uniform distribution and problem input parameters, the critical probabilities were calculated by Equation 4.2 and summarized in Table 4.1. The values of critical probabilities were used to determine the \( (x^*_{ij}) \) and \( L(x^*_{ij}) \) values.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>DC₁</th>
<th>DC₂</th>
<th>DC₃</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_i )</td>
<td>90</td>
<td>144</td>
<td>62</td>
</tr>
<tr>
<td>( b_j )</td>
<td>170</td>
<td>209</td>
<td>147</td>
</tr>
<tr>
<td>( p_j )</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>( r_{ij} )</td>
<td>SP₁</td>
<td>2.05</td>
<td>2.13</td>
</tr>
<tr>
<td></td>
<td>SP₂</td>
<td>0.53</td>
<td>1.92</td>
</tr>
<tr>
<td>( p_{ij}^{c*} )</td>
<td>SP₁</td>
<td>0.488</td>
<td>0.468</td>
</tr>
<tr>
<td></td>
<td>SP₂</td>
<td>0.868</td>
<td>0.520</td>
</tr>
</tbody>
</table>

Table 4.1 – Example 4.1 Critical Probabilities
Then \( x_{ij}^* \) and \( L(x_{ij}^*) \) were calculated according to Equations 4.2 and 4.4. The uniform demand pdf parameters were implemented in Equations 4.2 and 4.4. The following are Equations 4.2 and 4.4 after the modifications were made:

\[
x_{ij}^* = a_j + \frac{p_j - r_{ij}}{p_j} (b_j - a_j)
\]

\[
L(x_{ij}^*) = p_j \frac{x_{ij}^* - a_j^2}{2(b_j - a_j)}
\]

Thus, \( x_{ij}^* \) and \( L(x_{ij}^*) \) values are summarized in the following table:

<table>
<thead>
<tr>
<th>SP$_i$</th>
<th>( x_{ij}^* )</th>
<th>L(( x_{ij}^* ))</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>DC$_1$</td>
<td>DC$_2$</td>
</tr>
<tr>
<td>1</td>
<td>129.000</td>
<td>174.388</td>
</tr>
<tr>
<td>2</td>
<td>159.400</td>
<td>177.800</td>
</tr>
</tbody>
</table>

Phase II:

Formulation 4.1 was applied with the given data of \( (x_{ij}^*, L(x_{ij}^*), A_b, \text{ and } f) \) and presented as follows:

Maximize

\[
OB_T = 213.525z_{11} + 297.692z_{12} + 241.285z_{13} + 432.709z_{21} + 334.672z_{22} + 143.027z_{23} - 270y_1 - 250y_2
\]

S.T.
\[ z_{11} + z_{21} \leq 1 \]
\[ z_{12} + z_{22} \leq 1 \]
\[ z_{13} + z_{23} \leq 1 \]
\[ 129z_{11} + 174.388z_{12} + 118.7381z_{13} \leq 233y_1 \]
\[ 159.4z_{21} + 177.8z_{22} + 99.613z_{23} \leq 172y_2 \]
\[ y_i, z_j \in \{0, 1\} \]

The above LA problem was solved as pure binary LP and optimized with \( OB_T \) equal to (210.401) and \((z_{11} = z_{21} = y_1 = y_2 = 1)\). This meant the total net profit was expected to be $210,401 for shipping 174.388 units from SP_1 to DC_2 and 159.4 units from SP_2 to DC_1. The solution shows that no shipments were made to DC_2. The corresponding LA problem was formulated and summarized in GAMS format in Appendix D.

b. Uncapacitated Facilities (Model 5):

Example 4.1 was reworked with unlimited supply capacities. The effect of supply capacities existed only in Phase II. Thus, Phase I calculations remained unchanged, and Phase II was resolved with unlimited capacities as follows:

\[ \text{Maximize} \]
\[ OB_T = 213.525z_{11} + 297.692z_{12} + 241.285z_{13} + 432.709z_{21} + 334.672z_{22} + 143.027z_{23} \]
\[ -270y_1 - 250y_2 \]
\[ \text{S.T.} \]
\[ z_{11} + z_{21} \leq 1 \]
\[ z_{12} + z_{22} \leq 1 \]
\[ z_{13} + z_{23} \leq 1 \]
\[ 129z_{11} + 174.388z_{12} + 118.7381z_{13} \leq My_1 \]
\[ 159.4z_{21} + 177.8z_{22} + 99.613z_{23} \leq My_2 \]
\[ y_i, z_i \in \{0,1\} \]

The above LA problem was solved as pure binary LP and optimized with \( OB_T \) equal to 660.409 and \( (z_{21} = z_{22} = z_{23} = y_2 = 1) \). This meant the total net profit was expected to be $660,409 for shipping 159.4, 177.8, and 99.613 units from SP2 to DC1, DC2, and DC3 respectively. Thus, the first shipping plant was not used, and all the demand centers were satisfied with the optimal shipments. The problem formulated in GAMS format is in Appendix E.

4.4. The Generation of the Parameters

The input parameters of the test problems were: \( A_i, f_i, \mu_{q_j}, \sigma_{q_j}, r_{iy} \) and the pdf of the \( q_j \). Although there are variables Phase I (\( p_j, x^*_j \)), the actual decision variables are the results from Phase II (\( z_{ij}, y_i \)). The following flow chart summarizes these variables and addresses the Two-Phase Method’s implementation steps. The implementation steps start with the creation of LA problem parameters from the independent variables’ range limits. Then they proceed with generating values for these independent from the given distributions for the entire problem; that is, they generate \( A_i \) and \( f_i \) for all \( i \) (1 to m). Also for all \( j \) (1 to n), they generate \( h_{iy}, \mu_{p_j} \) and \( \sigma_{q_j} \). Finally, they generate \( r_{ij} \) for each combination of \( i \) and \( j \). The next step is the calculation of \( p_j \) values whether the prices are sensitive or insensitive.

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The Two-Phase Method was applied in the proceeding two operations. The first operation was the calculation of phase-one variables. These variables were the critical probability for each demand pdf, the $x^*_j$ candidate shipments' sizes, and the associated candidate profits $L(x^*_j)$. The phase-two LA formulation parameters were complete at this stage. Thus, the final step was solving the LA formulation.
Two-Phase Method

Input parameters:
{n, m, the distribution function with its parameters \( \mu_p \), \( f_a \), and \( r_q \)}

From each designated distribution generate:
{\( f_a, A_l \)} values for each SP,
{\( \mu_p, \mu_q \)} values for each DC, and
{\( r_{pq} \)} values for each combination of i & j.

Phase I: calculate:
The critical probability for each \( q \) probability distribution.
{\( x^*_{ij} \)} The independent optimal possible shipments.
{\( L(x^*_{ij}) \)} The expected profit from shipping \( x^*_{ij} \).

Phase II: LA problem,
Optimizing \( y_i \), \( z_{ij} \),
Let \( OB = OB_t \)

Figure 4.2 – Implementation Steps of Test Problems for the Two-Phase Method
CHAPTER 5

COMBINED METHOD

This chapter is the core of the current study because it presents the development of a new method of solving the LA problem optimally. The method is referred to as the Combined Method because it merges the phases of the Two-Phase Method into a single phase.

The chapter is divided into three sections. The first section describes the concept of the Combined Method, and the second section presents its general formulation. Section three discusses the development of a customized formulation for uniformly distributed demands and includes a numerical example.

5.1. Concept

The concept of the Combined Method was inspired by the Two-Phase Method's weaknesses, which prevent the solution from guaranteeing optimality. Optimality is not achieved in the Two-Phase Method because the candidate shipment sizes are determined prior to the LA optimization. Therefore, candidate shipment size is not affected by the SP's capacity and fixed cost. Moreover, in the Two-Phase Method, the DCj is limited to receiving shipments from one SPi only. This leads to underutilization of the capacity of SPi, which can prevent it from becoming a good candidate.

Hence, in order to solve the model for optimality, the phases of the Two-Phase Method were merged into one formulation. The developed formulation
optimized the total net profit generated from expected sales at all assigned demand centers because the shipment sizes and SPi assignments were optimized.

5.2. General Formulation

In order to optimize the solution to the LA problem, the number of units produced at SPi and shipped to DCj \( (x_{ij}) \) was added to the decision variable sets. As shipments were assigned when \( x_{ij} \) was a decision variable, the \( z_{ij} \) variable was eliminated. Another new set of decision variables \( (z_j) \) was needed to open or close DCj. The \( z_j \) variable is binary and equals one when DCj is satisfied by any shipment from SPi. Therefore, there were three sets of decision variables for the Combined formulation:

\[
x_{ij} = \text{positive real number representing shipment size from SPi to DCj}
\]

\[
y_i = \begin{cases} 
1 & \text{SPi is open} \\
0 & \text{SPi is closed}
\end{cases}
\]

\[
z_j = \begin{cases} 
1 & \text{DCj receives a shipment} \\
0 & \text{Otherwise}
\end{cases}
\]

In this chapter the total shipments to DCj are the sum of all \( x_{ij} \) for each specific DCj. This sum was referred to as \( D_j \). Thus, the sum of all shipments \( (D_j) \) received at DCj was responsible for satisfying the demand requirements at that specific DCj. However, the actual demand level \( (q_j) \) at DCj was known only after the \( x_{ij} \) were received from all the SPi. Thus, the actual demand might be greater than, less than, or equal to the available units at DCj. The following table addresses the situations and determines the associated expected revenue at each DCj.
Table 5.1 – Expected Revenue at DC$_j$

<table>
<thead>
<tr>
<th>Actual Demand ($q_j$)</th>
<th>$q_j \leq D_j$</th>
<th>$q_j &gt; D_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Quantity Sold</strong></td>
<td>$q_j$</td>
<td>$D_j$</td>
</tr>
<tr>
<td><strong>Expected Revenue</strong></td>
<td>$E(q_j, p_j</td>
<td>q_j \leq D_j \cdot P(q_j \leq D_j) - D_j \cdot P(q_j &gt; D_j)$</td>
</tr>
</tbody>
</table>

$$= p_j \int_{q_j}^{D_j} f(q_j) dq_j$$

$$= p_j D_j \int_{D_j}^{UB_j} f(q_j) dq_j$$

The total profit generated from expected sales of $x_j$ units produced at SP$_i$ and shipped to DC$_j$ was determined as follows:

Expected Total Profits = Expected Total Revenue – Total Variable Cost – Total Fixed Cost

Total Revenue = Sum of Expected Revenue from Sales at all DC$_j$

$$= \sum_j p_j \left[ \int_{LB_j}^{D_j} q_j f(q_j) dq_j + D_j \int_{D_j}^{UB_j} f(q_j) dq_j \right]$$

Total Variable Cost = Sum of all Units Produced and Shipped

$$= \sum_i \sum_j r_{ij} x_{ij}$$

Total Fixed Cost = Sum of all Opened SP$_i$ Fixed Costs

$$= \sum_i f_i y_i$$

where:
The general formulation of the Combined Method is presented in Formulation 5.1. The formulation is composed of the objective function, two sets of constraints, and three sets of decision variables. The objective function presents the total expected profits. The first set of constraints is the demand constraint, which ensure that no shipments will be made to DC$j$ if DC$j$ is going to be closed. The second set of constraints ensures that fewer units are produced and shipped from SP$i$ than the SP$i$ supply capacity. The decision variables are addressed in Equations 4 and 5 of the formulation.

Formulation 5.1 — General Formulation of the Combined Method

Maximize

\[
OB_C = \sum_{j=1}^{n} p_j \int_{q_j}^{LB_j} q_j f(q_j) dq_j + \sum_{j=1}^{n} p_j D_j \int_{q_j}^{UB_j} f(q_j) dq_j - \sum_{j=1}^{n} \sum_{i=1}^{m} r_{ij} x_{ij} - \sum_{i=1}^{m} f_i y_i
\]  

[1]

Subject to

Demand: \[ \sum_{i=1}^{m} x_{ij} \leq M z_j \quad j = 1,2,\ldots,n \]  

[2]

Supply: \[ \sum_{j=1}^{n} x_{ij} \leq A_i y_i \quad i = 1,2,\ldots,m \]  

[3]

\forall x_{ij} \geq 0 \quad \forall y_i, z_j \in \{0,1\} \quad \text{[4] and [5]}

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The differences between the Two-Phase formulation and the Combined formulation are constraint sets, problem orientation, and the number and type of variables. In the Two-Phase Method all \((mn+m)\) variables were binary, while in the Combined Method the number of variables increased to \((mn + m+n)\). However, the number of binary variables was reduced to \((m+n)\). The remaining \((mn)\) variables were positive real numbers associated with \(x_{ij}\). Merging the two phases changed the problem from a pure binary integer problem (BIP) to a mixed nonlinear binary problem (MINLP). The non-linearity of variables appears clearly in the objective function Equation 1 of Formulation 5.1. The MINLP is one of the most difficult types of problem to solve, and commercial solvers available on the market are limited in the ability to solve these problems to optimality efficiently.

5.3. Uniformly Distributed Demands

In this section, the demand probability density function (pdf) follows the uniform probability distribution at all \(DC_j\). The nonlinear objective function was derived by substituting the \(F(q_j)\) values with the demand's uniform pdf.

5.3.1. Derivation of the Objective Function

In order to derive the nonlinear part of the objective function, the per-unit profit function \((L(x_{ij}))\) was re-derived after incorporating the uniform pdf equation. Therefore, the objective function was simplified to the following form:

For \(q_j \sim U[a_j, b_j]\)
\[ f(q_j) = \frac{1}{b_j - a_j} \]

Revenue at DC\(_j\) = \( p_j \left[ \int_{LB_i}^{UB_i} q_j f(q_j) dq_j + D_j \int_{b_j}^{U_b} f(q_j) dq_j \right] \)

\[ = p_j \int_{LB_i}^{UB_i} \frac{1}{b_j - a_j} dq_j + D_j \int_{b_j}^{U_b} \frac{1}{b_j - a_j} dq_j \]

\[ = p_j \left[ \frac{q_j^2}{2(b_j - a_j)} \right]_{b_j}^{U_b} + p_j D_j \left[ \frac{q_j}{b_j - a_j} \right]_{b_j}^{U_b} \]

\[ = p_j \left[ \frac{D_j^2 - a_j^2}{2(b_j - a_j)} \right] + p_j D_j \left[ \frac{b_j - D_j}{b_j - a_j} \right] \]

\[ = \frac{p_j}{2(b_j - a_j)} [D_j^2 - a_j^2 + 2D_j b_j - 2D_j^2] \]

\[ = \frac{p_j}{2(b_j - a_j)} [2D_j b_j - a_j^2 - D_j^2] \]

\[ = \frac{p_j}{2(b_j - a_j)} [2b_j (\sum_{i=1}^{m} x_{ij}) - a_j^2 - (\sum_{i=1}^{m} x_{ij})^2] \]

5.3.2. Final Formulation

The final formulation of the LA problem with uniformly distributed demands was shown in Formulation 5.2. This LA formulation was solved as MINLP because of the existence of a nonlinear term in the objective function. The nonlinear term \( (\sum_{i=1}^{m} x_{ij})^2 \) is associated with the total shipments received at DC\(_j\).
Formulation 5.2 – Combined Method of Uniformly Distributed Demands

Maximize

\[
OB_c = \sum_{j=1}^{n} \frac{p_j}{2(b_j - a_j)} \left[ 2b_j \left( \sum_{i=1}^{m} x_{ij} \right) - a_j^2 z_j - \left( \sum_{i=1}^{m} x_{ij} \right)^2 \right] - \sum_{j=1}^{n} \sum_{i=1}^{m} r_{ij} x_{ij} - \sum_{i=1}^{m} f_i y_i
\]  \[1\]

Subject to

Demand: \[ \sum_{i=1}^{m} x_{ij} \leq M z_j \quad j = 1, 2, \ldots, n \] \[2\]

Supply: \[ \sum_{j=1}^{n} x_{ij} \leq A_i y_i \quad i = 1, 2, \ldots, m \] \[3\]

\[ \forall x_{ij} \geq 0 \] \[4\]

\[ y_i, z_j \in \{0,1\} \] \[5\]

5.3.3. Example 5.1

Example 4.1 was re-solved using the Combined Method and the results were compared with the Two-Phase Method solution.

Solution:

a. Capacitated Facilities (Model 6):

The problem input variables are summarized in the following table. The LA of the Combined Method is as follows:
Table 5.2 – Input Variables of Example 5.1

<table>
<thead>
<tr>
<th>j \ r_j</th>
<th>DC_j</th>
<th>A_i</th>
<th>f_i</th>
</tr>
</thead>
<tbody>
<tr>
<td>i</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>SP_i</td>
<td>1</td>
<td>209</td>
<td>170</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>172</td>
<td>250</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Demand pdf</th>
<th>a_j</th>
<th>b_j</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>90</td>
<td>144</td>
</tr>
<tr>
<td></td>
<td>170</td>
<td>209</td>
</tr>
<tr>
<td>p_j</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>\frac{p_j}{2(b_j-a_j)}</td>
<td>0.0250</td>
<td>0.0308</td>
</tr>
</tbody>
</table>

Maximize

\[ OB_C = 0.025 \left[ 340(x_{11} + x_{12}) - 8100z_1 - (x_{11} + x_{12})^2 \right] + 0.0308 \left[ 418(x_{12} + x_{22}) - 20736z_2 - (x_{21} + x_{22})^2 \right] + 0.0235 \left[ 294(x_{23} + x_{33}) - 38446z_3 - (x_{23} + x_{33})^2 \right] - 2.058x_{11} - 2.13x_{12} - 1.33x_{13} - 0.53x_{21} - 1.92x_{22} - 2.23x_{23} - 270y_1 - 250y_2 \]

S.T.

\[ x_{11} + x_{21} \leq M z_1 \]
\[ x_{12} + x_{22} \leq M z_2 \]
\[ x_{13} + x_{23} \leq M z_3 \]
\[ x_{ij} + x_{kl} + x_{lm} \leq 233y_i \]
\[ x_{21} + x_{22} + x_{23} \leq 172y_2 \]
\[ x_j \square 0 \]
\[ y_i, z_j \in \{0,1\} \]

The above LA problem was optimized as an MINLP with \( OB_C \) equal to (438.451) and \( (z_I = z_2 = z_3 = y_1 = y_2 = 1) \). The optimized shipments were:

- \( x_{i2} = 130.274 \)
- \( x_{i3} = 102.726 \)
- \( x_{22} = 140.130 \)
- \( x_{22} = 31.870 \)

This meant that the expected total net profit was $438,451, which was generated from the optimized \( x_{ij} \) shipments to \( DC_j \) by opening the two \( SP_i \). (Note that it was different from the Two-Phase Method solution, which had \( x_{i2} = 174.388, x_{22} = 159.400, \) and \( OB_T = $210.401 \) ) All \( DC_j \) were satisfied by the sum of the shipments.

- \( D_1 = x_{i1} + x_{21} = 0 + 140.130 = 140.13 \) (units)
- \( D_2 = x_{i2} + x_{22} = 130.274 + 31.870 = 162.144 \) (units)
- \( D_3 = x_{i3} + x_{23} = 102.726 + 0 = 102.726 \) (units)

The GAMS format of the problem formulation is in Appendix F.

b. Uncapacitated Facilities (Model 5):

The same formulation was reworked with unlimited supply capacities. The effect of supply capacities existed in the supply constraints only. Here is the same formulation solved with the following supply constraints:
\[ x_{11} + x_{12} + x_{13} \leq M y_1 \]
\[ x_{21} + x_{22} + x_{23} \leq M y_2 \]
Thus, the LA problem was optimized with \( OB_C \) equal to 660.409 and \( z_1 = z_2 = z_3 = y_2 = 1 \). The optimal shipment arrangements were:
\[ x_{21} = 159.4 \]
\[ x_{22} = 177.8 \]
\[ x_{23} = 99.613 \]
This meant the total net profit was expected to be $660,409 for shipping 159.4, 177.8, and 99.613 units from SP_2 to DC_1, DC_2, and DC_3 respectively. Thus, the first shipping plant was not used, and all the demand centers were satisfied by the optimal shipments. The GAMS format of the problem formulation is in Appendix G.

Table 5.3 summarizes the total net profit for both parts of Examples 5.1 and 4.1. In the case of uncapacitated facilities, the net profit was the same for both methods. However, in the case of capacitated facilities, the value of the objective function was $210,401 when the Two-Phase Method was used, and it was $438,451

<table>
<thead>
<tr>
<th>Example</th>
<th>Method</th>
<th>Capacitated Facilities</th>
<th>Uncapacitated Facilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1</td>
<td>Two-Phase</td>
<td>$210.401</td>
<td>$660.409</td>
</tr>
<tr>
<td>5.1</td>
<td>Combined</td>
<td>$438.451</td>
<td>$660.409</td>
</tr>
</tbody>
</table>
when the Combined Method was implemented. Thus, the solution of the Combined Method was 108.39% better than the solution of the Two-Phase Method. The only disadvantage of the Combined Method is that the MINLP can get stuck in the optimal solutions if it is used to solve a large problem with many commercially available solvers.
CHAPTER 6
SENSITIVE PRICES

This chapter presents the incorporation of a sensitive prices factor in the LA problem. Section 6.1 of this chapter explains the concept of sensitive prices. Section 6.2 solves Model 7 and Model 8 with the Two-Phase Method. Section 6.3 solves the same models using the Combined Method.

6.1. Concept

The elasticity of demand concept was used to maintain the profitability level that resulted from sales at DC\textsubscript{j}. Therefore, the profitability level remained constant regardless of the sales level. Another purpose of implementing this notion was to make the product more attractive to customers, which yield a more realistic expectation of sales. Accordingly, the price for a product received at DC\textsubscript{j} varied depending upon the expected sales. In the case of retailers, discrete batches of sales took place at DC\textsubscript{j} before all sales were realized for a specific period. Thus, price per unit was determined at DC\textsubscript{j} before the actual sales took place. The price level had an impact on the revenue terms of the objective function. As part of the proposed optimization formulations, the value of price should be determined prior to solving the LA problem.

The sensitive price was determined by the following equation, where the constant represents the desired profitability level (Logendran and Terrell 1991).
Selecting a reasonable value for the profitability constant was crucial for imitating the real situation. However, determining the level of profitability (a constant) was more of a strategic tool, and this was outside the scope of this research.

\[ p_j = \frac{\text{Constant}}{\mu_j} = \frac{600}{\mu_j} \]  \hspace{1cm} [6.1]

Equation 6.1 was applied in the LA formulation for all models with sensitive prices in the following sections and chapters. The following sections present examples of Models 7 and 8. Both models incorporated sensitive price factors.

6.2. Example 6.1 (Two-Phase Method: Model 8 and Model 7)

Example 4.1 was re-solved using the price-demand relationship defined in Equation 6.1, with a profitability constant of 600. The results were compared with the Two-Phase Method solution.

Solution:

a. Capacitated Facilities (Model 8):

The LA model for this problem consisted of stochastic demands, sensitive prices, and capacitated facilities. Thus, the problem was categorized as Model 8 according to the definition in Chapter 3. The LA problem was solved using the Two-Phase Method. Phase I was solved first, and then the outcomes were used to solve the second phase.
Phase I:

For uniform distribution and problem input parameters, the critical probabilities were calculated by Equation 4.2 and summarized in Table 6.1. The values of critical probabilities were used to determine the $x^*_y$ and $L(x^*_y)$ values.

Table 6.1 – Example 6.1 Critical Probabilities

<table>
<thead>
<tr>
<th>Parameter</th>
<th>DC1</th>
<th>DC2</th>
<th>DC3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_j$ (units)</td>
<td>90</td>
<td>144</td>
<td>62</td>
</tr>
<tr>
<td>$b_j$ (units)</td>
<td>170</td>
<td>209</td>
<td>147</td>
</tr>
<tr>
<td>$\mu_j = \frac{a_j + b_j}{2}$ (units)</td>
<td>130</td>
<td>176.5</td>
<td>104.5</td>
</tr>
<tr>
<td>$p_j = \frac{600}{\mu_j}$ ($/unit)$</td>
<td>4.615</td>
<td>3.399</td>
<td>5.742</td>
</tr>
<tr>
<td>$r_{ij}$ ($/unit)$</td>
<td>SP1</td>
<td>2.05</td>
<td>2.13</td>
</tr>
<tr>
<td>$\sigma_j$</td>
<td>SP2</td>
<td>0.53</td>
<td>1.92</td>
</tr>
<tr>
<td>$P_{ij}^\sigma = \frac{p_j - r_{ij}}{p_j}$</td>
<td>SP1</td>
<td>0.556</td>
<td>0.373</td>
</tr>
<tr>
<td></td>
<td>SP2</td>
<td>0.885</td>
<td>0.435</td>
</tr>
</tbody>
</table>

Then $x^*_y$ and $L(x^*_y)$ were calculated according to Equations 4.2 and 4.4. The uniform demand pdf parameters were implemented in these equations as well. The following are Equations 4.2 and 4.4 after the modifications were made:

$$x^*_y = a_j + p_{ij}^\sigma (b_j - a_j)$$
The $x^*$ and $L(x^*)$ values are summarized in the following table:

### Table 6.2 – Example 6.1 Values of Phase I

<table>
<thead>
<tr>
<th>SP₁</th>
<th>$x^*$</th>
<th>$L(x^*)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>DC₁</td>
<td>DC₂</td>
</tr>
<tr>
<td>1</td>
<td>134.467</td>
<td>168.273</td>
</tr>
<tr>
<td>2</td>
<td>160.813</td>
<td>172.288</td>
</tr>
</tbody>
</table>

Phase II:

Formulation 4.1 was applied with the given data of $x^*$, $L(x^*)$, $A_i$, and $f_i$ and presented as follows:

**Maximize**

$$OB_T = 287.922z_{11} + 198.205z_{12} + 417.584z_{13} + 512.334z_{21} + 233.964z_{22} + 309z_{23} - 270y_1 - 250y_2$$

**S.T.**

$$z_{11} + z_{21} \leq 1$$

$$z_{12} + z_{22} \leq 1$$

$$z_{13} + z_{23} \leq 1$$

$$134.467z_{11} + 168.273z_{12} + 127.310z_{13} \leq 233y_1$$

$$160.813z_{21} + 172.288z_{22} + 113.987z_{23} \leq 172y_2$$

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The previous LA problem was solved as pure binary LP and optimized with an \( OB_T \) equal to 409.919 and \( z_{13} = z_{21} = y_1 = y_2 = 1 \). This meant the total net profit was expected to be $409,919 for shipping 127.31 units from SP_1 to DC_3 and 160.813 units from SP_2 to DC_1. The solution shows that no shipments were made to DC_1. The corresponding LA problem was formulated and summarized in GAMS format in Appendix H.

b. Uncapacitated Facilities (Model 7):

The LA model for this problem consisted of stochastic demands, sensitive prices, and uncapacitated facilities. Thus, the problem was categorized as Model 7 according to the definition in Chapter 3. Example 4.1 was reworked with unlimited supply capacities. The effect of supply capacities existed only in Phase II. Thus, Phase I calculations remained unchanged, and Phase II was resolved with unlimited capacities as follows:

\[
\text{Maximize} \quad OB_T = 287.922z_{11} + 198.205z_{12} + 417.584z_{13} + 512.334z_{21} + 233.964z_{22} + 309z_{23} - 270y_1 - 250y_2 \\
\text{S.T.} \\
z_{11} + z_{21} \leq 1 \\
z_{12} + z_{22} \leq 1 \\
z_{13} + z_{23} \leq 1
\]
\[ 134.467z_{11} + 168.273z_{12} + 127.31z_{13} \leq My_1 \]
\[ 160.813z_{21} + 172.288z_{22} + 113.987z_{23} \leq My_2 \]
\[ y_i, z_q \in \{0,1\} \]

The above LA problem was solved as pure binary LP and optimized with an 
\( OB_T \) equal to 805.299 and \( (z_{21} = z_{22} = z_{23} = y_2 = 1) \). This meant the total net profit was
expected to be $805.299 for shipping 160.813, 172.288, and 113.987 units from SP_2
to DC_1, DC_2, and DC_3, respectively. Thus, the first shipping plant was not used and
all the demand centers were satisfied with the optimal shipments. The problem was
formulated in GAMS format in Appendix I.

6.3. Example 6.2 (Combined Method: Model 8 and Model 7)

Example 5.1 was re-solved using the price-demand relationship defined in
Equation 6.1, with a profitability constant of 600. The results were compared with the
solution of Example 6.1.

Solution:

a. Capacitated Facilities: (Model 8)

The input variables and calculated variables were presented in Table 6.3.
These variables were substituted to Formulation 5.2.
Table 6.3 - Input Variables of Example 6.2

<table>
<thead>
<tr>
<th>j</th>
<th>DC_1</th>
<th>DC_2</th>
<th>DC_3</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.05</td>
<td>1.13</td>
<td>1.74</td>
<td>233</td>
</tr>
<tr>
<td>2</td>
<td>0.36</td>
<td>0.92</td>
<td>2.03</td>
<td>172</td>
</tr>
</tbody>
</table>

Maximize

\[ OB_C = 0.0288 \left[ 340(x_{11} + x_{12}) - 8100z_1 - (x_{11} + x_{12})^2 \right] \]
\[ + 0.0261 \left[ 418(x_{12} + x_{22}) - 20736z_2 - (x_{12} + x_{22})^2 \right] \]
\[ + 0.0338 \left[ 294(x_{23} + x_{33}) - 38446z_3 - (x_{23} + x_{33})^2 \right] \]
\[ - 2.058x_{11} - 2.13x_{12} - 1.33x_{13} - 0.53x_{21} - 1.92x_{22} - 2.23x_{23} \]
\[ - 270y_1 - 250y_2 \]

S.T.

\[ x_{11} + x_{21} \leq Mz_1 \]
\[ x_{12} + x_{22} \leq M z_2 \\
x_{13} + x_{23} \leq M z_3 \\
x_{11} + x_{12} + x_{13} \leq 233 y_1 \\
x_{21} + x_{22} + x_{23} \leq 172 y_2 \\
x_{ij} \leq 0 \\
y_i, z_j \in \{0, 1\} \\
\]

The previous LA problem was optimized as a MINLP, with an \( OB_C \) equal to 588.608 and \( (z_1 = z_2 = z_3 = y_1 = y_2 = 1) \). The optimized shipments were:

\begin{align*}
    x_{12} &= 119.482 \\
x_{13} &= 113.518 \\
x_{21} &= 141.024 \\
x_{22} &= 30.976
\end{align*}

This meant that the expected total net profit was $588,608, generated from the optimized \( x_{ij} \) shipments to DC\( j \) by opening the two SP\( j \). All DC\( j \) were satisfied with the sum of the shipments:

\begin{align*}
    D_1 &= x_{11} + x_{21} = 0 + 141.024 = 141.024 \text{ (units)} \\
    D_2 &= x_{12} + x_{22} = 119.482 + 30.976 = 150.458 \text{ (units)} \\
    D_3 &= x_{13} + x_{23} = 113.518 + 0 = 113.518 \text{ (units)}
\end{align*}

The GAMS format of the problem is in Appendix J.
b. Uncapacitated Facilities: (Model 7)

The same formulation was reworked with unlimited supply capacities. The effect of supply capacities existed in the supply constraints only. Therefore, the same formulation was solved with the following supply constraints:

\[
\begin{align*}
    x_{11} + x_{12} + x_{13} & \leq My_1 \\
    x_{21} + x_{22} + x_{23} & \leq My_2
\end{align*}
\]

Thus, the LA problem was optimized with an \( OB_C \) equal to 805.299 and \( z_i = z_2 = z_3 = y_2 = 1 \). The optimal shipment arrangements were:

\[
\begin{align*}
    x_{21} &= 160.813 \\
    x_{22} &= 172.288 \\
    x_{23} &= 113.987
\end{align*}
\]

This meant the total net profit was expected to be $805.299 for shipping 160.813, 172.288, and 113.987 units from SP\(_2\) to DC\(_1\), DC\(_2\), and DC\(_3\), respectively. Thus, the first shipping plant was not utilized and all the demand centers were satisfied with the optimal shipments. The GAMS format of the problem is available in Appendix K.

Table 6.4 summarizes the values of the total net profit for both parts of Examples 6.1 and 6.2. In the case of uncapacitated facilities, the net profit generated was the same for both methods. However, in the case of capacitated facilities, the value of the objective function was $409.919 using the Two-Phase Method, while it was $588.608 when the Combined Method was implemented. Thus, the solution of the Combined Method was 43.59% better than the solution of the Two-Phase Method.
This indicates the superiority of the Combined Method over the Two-Phase Method in the case of capacitated facilities.

Table 6.4 – Comparison between Examples 6.1 and 6.2

<table>
<thead>
<tr>
<th>Example</th>
<th>Method</th>
<th>Capacitated Facilities</th>
<th>Uncapacitated Facilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.1</td>
<td>Two-Phase</td>
<td>$409.919</td>
<td>$805.299</td>
</tr>
<tr>
<td>6.2</td>
<td>Combined</td>
<td>$588.608</td>
<td>$805.299</td>
</tr>
</tbody>
</table>
CHAPTER 7
INVENTORY AND STOCK-OUT

This chapter presents the incorporation of inventory and stock-out costs in the Two-Phase Method and Combined Method. It presents the final formulation of both Methods to solve the LA problem of Models 9, 10, 11, and 12.

Hence, the chapter is divided into two sections. The first section describes the modification of the Two-Phase Method to incorporate the inventory and stock-out costs. Section two describes the modification of the Combined Method to incorporate the inventory and stock-out costs. Part (a) and part (b) of example 7.1 and example 7.2 present Model 9 and Model 10, respectively.

7.1. Two-Phase Method with Inventory and Stock-out Costs

The actual demand level \((q_j)\) at DC\(_j\) is uncertain; it might be greater than the available units \((x^*y_j)\) shipped to DC\(_j\), less than \(x^*y_j\), or equal to \(x^*y_j\). Stock out occurs in the first case when the actual demand is greater than what is available for sale. This situation can incur an opportunity cost or a backordering cost. In this case the inventory level is zero. The second case, which occurs when the actual demand is equal to the available units for sale, is very rare. In this case no inventory or stock out is experienced. The third situation is when the actual demand is less than the available units. This will cause a partial sale of \(x^*y_j\) units, and the remainder of \(x^*y_j\) will be
stocked or sold with a salvage value. The three cases are presented in the following table:

<table>
<thead>
<tr>
<th>Case</th>
<th>Level of inventory</th>
<th>Level of stock-out</th>
</tr>
</thead>
<tbody>
<tr>
<td>(q_j &gt; x_j^{*})</td>
<td>None</td>
<td>(q_j - x_j^{*})</td>
</tr>
<tr>
<td>(q_j = x_j^{*})</td>
<td>None</td>
<td>None</td>
</tr>
<tr>
<td>(q_j &lt; x_j^{*})</td>
<td>(x_j^{*} - q_j)</td>
<td>None</td>
</tr>
</tbody>
</table>

Table 7.1 – Possibilities of Inventory and Stock-out Levels

Hence, the base model was modified so that the extra units that were unsold were considered inventory and included as part of the optimization model. In addition, the modified model included the stock-out cost for the units short of sale. This took place in the cost part of Formulation 4.1. Therefore, Equation 4.3 was modified to include the inventory cost and stock-out cost.

7.1.1. Phase I (with Inventory and Stock-out Costs)

When the size of the shipment is known and the actual number of demanded units at \(DC_j\) is less than what was estimated \((q_j < x_j^{*})\), the unsold units \((x_j^{*} - q_j)\) are going to be stocked with probability of \(P(q_j < x_j^{*})\). Thus, the holding cost was represented by the term \((E(x_j^{*} - q_j)H_j|q_j < x_j^{*})P(q_j < x_j^{*})\). On the other hand, the cost of shipping was still \((r_{ij}x_j^{*})\) because all the \((x_j^{*})\) units will be shipped to demand center \(j\) before the actual level of demand was known. Tables 7.2 and 7.3 summarize...
sales based on shipped units and actual demand level. They also determine the size of inventory and its expected cost.

Table 7.2 – Sales Based on Demand and Quantities Shipped

<table>
<thead>
<tr>
<th>Demand Level</th>
<th>FORMULA 1</th>
<th>FORMULA 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Revenue</td>
<td>( E(q_j x_j^*</td>
<td>q_j &lt; x_j^*) )</td>
</tr>
<tr>
<td>Shipping</td>
<td>( x_j^* r_j x_j^* )</td>
<td>( x_j^* r_j x_j^* )</td>
</tr>
<tr>
<td>Inventory</td>
<td>( E((q_j - x_j^*) s_j</td>
<td>q_j &gt; x_j^*) )</td>
</tr>
<tr>
<td>Stock out</td>
<td>( - )</td>
<td>( E((q_j - x_j^*) s_j</td>
</tr>
</tbody>
</table>

Table 7.3 – Formulas of Sales Based on Demand and Quantities Shipped

<table>
<thead>
<tr>
<th>Demand Level</th>
<th>FORMULA 1</th>
<th>FORMULA 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Revenue</td>
<td>( p_j \int_{q_j}^{q_j^*} f(q_j) dq_j )</td>
<td>( p_j x_j^* \int_{x_j}^{x_j^*} f(q_j) dq_j )</td>
</tr>
<tr>
<td>Shipping</td>
<td>( r_j x_j^* \int_{q_j}^{q_j^*} f(q_j) dq_j )</td>
<td>( r_j x_j^* \int_{x_j}^{x_j^*} f(q_j) dq_j )</td>
</tr>
<tr>
<td>Inventory</td>
<td>( h_j \int_{q_j}^{q_j^*} f(q_j) dq_j )</td>
<td>( - )</td>
</tr>
<tr>
<td>Stock out</td>
<td>( - )</td>
<td>( s_j \int_{q_j}^{q_j^*} f(q_j) dq_j )</td>
</tr>
</tbody>
</table>

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Using the formulas in Table 7.3 led to the determination of the net profit per unit \( x^* \) shipped from SPi to DCj. Thus, the net profit per unit equals the expected revenue minus the expected cost. The expected cost was the sum of expected shipping and production cost, inventory cost, and stock-out cost. The final form of the profit formula was derived as follows:

\[
\text{Profit} = \text{Revenue} - (\text{cost of shipping and production} + \text{cost of inventory} + \text{stock-out cost})
\]

\[
\text{Revenue} = \int_{LB_j}^{UB_j} p_j f(q_j) dq_j + p_j x^*_j \int f(q_j) dq_j
\]

\[
= \int_{LB_j}^{UB_j} p_j f(q_j) dq_j + p_j x^*_j (1 - F(x^*_j))
\]

\[
\text{Cost of shipping and production} = r_j x^*_j \int f(q_j) dq_j + r_j x^*_j \int f(q_j) dq_j = r_j x^*_j
\]

\[
\text{Cost of inventory} = h_j \int_{LB_j}^{UB_j} (x^*_j - q_j) f(q_j) dq_j = h_j x^*_j \int_{LB_j}^{UB_j} f(q_j) dq_j - h_j \int_{LB_j}^{UB_j} q_j f(q_j) dq_j
\]

\[
= h_j x^*_j F(x^*_j) - h_j \int_{LB_j}^{UB_j} q_j f(q_j) dq_j
\]

\[
\text{Cost of stock out} = s_j \int_{LB_j}^{UB_j} (q_j - x^*_j) f(q_j) dq_j = s_j \int_{LB_j}^{UB_j} q_j f(q_j) dq_j - s_j x^*_j \int_{LB_j}^{UB_j} f(q_j) dq_j
\]

\[
= s_j (\mu_j - \int_{LB_j}^{UB_j} q_j f(q_j) dq_j) - s_j x^*_j [1 - F(x^*_j)]
\]
Table 7.4 – Expected Profit Terms

<table>
<thead>
<tr>
<th>Terms</th>
<th>$d$ or $q$</th>
<th>$P_j$</th>
<th>$-p_j x^*_q$</th>
<th>$+p_j x^*_q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Revenue</td>
<td>$P_j$</td>
<td>$-p_j x^*_q$</td>
<td>$+p_j x^*_q$</td>
<td></td>
</tr>
<tr>
<td>Shipping</td>
<td></td>
<td></td>
<td></td>
<td>$r_q x^*_q$</td>
</tr>
<tr>
<td>Inventory</td>
<td>$-h_j$</td>
<td>$h_j x^*_q$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Stock out</td>
<td>$-s_j$</td>
<td>$s_j x^*_q$</td>
<td>$s_j \mu_j - s_j x^*_q$</td>
<td></td>
</tr>
<tr>
<td>Profit</td>
<td>$p_j + h_j + s_j$</td>
<td>$-(p_j + h_j + s_j)x^*_q$</td>
<td>$(p_j - r_q + s_j)x^*_q - s_j \mu_j$</td>
<td></td>
</tr>
</tbody>
</table>

Table 7.4 segregates the terms of the profit function into three general categories. It lists the coefficients of the revenue and costs in terms of the profit function in the common category. Therefore, the total expected profit is determined as follows:

$$L(x^*_q) = (p_j + h_j + s_j) \int_{L_{j}}^{x^*_q} q_j f(q_j) dq_j + [p_j - r_q + s_j - (p_j + h_j + s_j)F(x^*_q)]x^*_q - s_j \mu_j [7.1]$$

Hence, the expected profit from shipping $x^*$ units from SP$_i$ to DC$_j$ with incorporation of inventory and stock-out costs was determined after the critical probability was substituted in Equation 7.1.
Critical Probability Determination: (Models with inventory and stock-out costs)

The critical probability \( P_{crij} \) is defined as the cumulative probability at which no shortage will occur when a single SP\(_i\) is allowed to satisfy all demand requirements at DC\(_j\) with \( x^*_j \) units. Therefore, the stock-out cost has no effect on the determination of the critical probability formula. According to the above definition, critical probability was determined by equating the values of expected gain and expected loss from shipping an extra unit from SP\(_i\) to DC\(_j\). The expected gain equals the price per unit minus the cost per unit of production and shipping multiplied by the probability of the extra unit produced and shipped to be sold. On the other hand, the expected loss equals the probability of not selling the extra unit produced at SP\(_i\) and shipped to DC\(_j\) multiplied by the costs incurred from this process. The costs incurred are the per-unit costs of holding, production, and shipping. The critical probability was derived as follows:

\[
\text{Expected gain from shipping an extra unit} = \text{expected loss from shipping an extra unit}
\]

\[
(p_j - r_{ij})(1 - F(x^*_j)) = (r_j + h_j)F(x^*_j)
\]

\[
(p_j - r_{ij}) = (p_j - r_{ij})F(x^*_j) + (r_j + h_j)F(x^*_j)
\]

\[
(p_j - r_{ij}) = (p_j - r_{ij} + r_j + h_j)F(x^*_j)
\]

\[
F(x^*_j) = \frac{p_j - r_{ij}}{p_j + h_j}
\]
For equation $P_{ij}$, the critical probability is given by $P_{ij} = F(x^*_i) = \frac{p_j - r_j}{p_j + h_j}$.

\[\therefore P_{ij} = F(x^*_i) = \frac{p_j - r_j}{p_j + h_j}\]  

[7.2]

Equation 7.2 of the critical probability indicates that holding cost per unit ($h_j$) is inversely proportional to the critical-probability value. Thus, the higher the holding cost, the lower the value of the critical probability. This leads to an inverse relationship between holding cost and the optimal shipment size ($x^*_i$). Figure 7.1 indicates the relationship of the critical probability, holding cost, and optimal shipment size. The higher the holding cost, the lower the optimal shipment size.

The first term of Equation 7.1 shows that the holding and stock-out costs are directly proportional to the net profit, which is irrational. The remaining terms of the equation indicate the counter effect of this notion. Moreover, the total net profit does not increase if the holding cost ($h_j$) and/or stock-out cost ($s_j$) increases. This is due to...
the effect of these costs on the critical probability (Equation 7.2). The critical probability formula shows that the higher the holding cost, the lower the shipment size. This decreases the profit expected for selling a smaller number of units.

Satisfying demand at all DCj was not obligatory for the current study. Thus, the demand constraint of Formulation 4.1 was relaxed to an inequality to allow the model to work with the limited capacities of the plants. This was effective for cases in which the summation of the plants’ capacities was less than the expected demand. This may keep some of the demand centers from receiving any shipments. However, it also allows the model to maximize the total net profit without launching new facilities, when capacity is still available.

7.1.2. The Final Formulation (Two-Phase Method with Inventory and Stock-out Costs)

Equations 7.1 and 7.2 summarize Phase I, which adopt inventory and stock-out costs. Keep in mind the changes occur only in the determination of Phase-I outcomes. The formulation of Phase-II was unaffected by the modifications because its role is to assign the candidate shipments and profits that are determined in Phase-I. However, the values of the critical probability in Equation 4.6, Chapter 4, might become negative if the associated shipping and production costs are greater than the selling price. The negative critical probability is not rational because it represents the area under the pdf curve, which never goes below zero. Therefore, the optimal shipment size associated with the negative critical probability is less than the lower bound of the associated demand pdf. This leads to a negative sign of the net profit.
$L(x^*_{ij})$, which means this specific shipment causes a net loss rather than a net profit. In reality, this will never occur since the demand will never dip below the lower bound, which defines the associated pdf.

The final formulation of the Two-Phase Method incorporating inventory and stock-out costs is as follows.

Formulation 7.1 – Two-Phase Method with Inventory and Stock-out Costs

Phase-I:

$$F(x^*_{ij}) = \frac{p_j - r_y}{p_j + h_j}$$  \[1\]

$$L(x^*_{ij}) = (p_j + h_j + s_j) \int_{LB_j}^{x^*_{ij}} q_j f(q_j) dq_j + [p_j - r_y + s_j - (p_j + h_j + s_j)F(x^*_{ij})]x^*_{ij} - s_j \mu_j$$  \[2\]

Phase-II:

Maximize

$$OB_T = \sum_{j=1}^{n} \sum_{i=1}^{m} z_{ij} L(x^*_{ij}) - \sum_{i=1}^{m} f_i y_i$$  \[3\]

Subject to

$$\sum_{i=1}^{m} z_{ij} \leq 1 \quad ; j = 1,2,\ldots,n$$  \[4\]

$$\sum_{j=1}^{n} x^*_{ij} z_{ij} \leq A_i y_i \quad ; i = 1,2,\ldots,m$$  \[5\]

$$y_i, z_{ij} \in \{0,1\}$$  \[6\]
7.1.3. Example 7.1 (Two-Phase Method with Inventory and Stock-out Costs)

Example 4.1 was reworked with holding and stock-out costs, and the results were compared. The holding cost equaled $1.00 per unit stored at any \( DC_j \), and the stock-out cost equaled $0.50 per unit demanded and not available at any \( DC_j \).

Solution:

a. Capacitated Facilities (Model 10):

For uniform distribution and problem input parameters, the critical probabilities were calculated by Equation 4.6, Chapter 4, and summarized in Table 4.8. The values of critical probabilities were used to determine the \( x^*_{ij} \) and \( L(x^*_{ij}) \) values.

Table 7.5 – Example 7.1 Critical Probabilities

<table>
<thead>
<tr>
<th>Parameter</th>
<th>( DC_1 )</th>
<th>( DC_2 )</th>
<th>( DC_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_j )</td>
<td>90</td>
<td>144</td>
<td>62</td>
</tr>
<tr>
<td>( b_j )</td>
<td>170</td>
<td>209</td>
<td>147</td>
</tr>
<tr>
<td>( p_j )</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>( h_j )</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( s_j )</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>( r_{ij} )</td>
<td>( SP_1 ) 2.05</td>
<td>2.13</td>
<td>1.33</td>
</tr>
<tr>
<td></td>
<td>( SP_2 ) 0.53</td>
<td>1.92</td>
<td>2.23</td>
</tr>
<tr>
<td>( p_{ij}^{\alpha} = \frac{p_j - r_{ij}}{p_j + h_j} )</td>
<td>( SP_1 ) 0.390</td>
<td>0.374</td>
<td>0.534</td>
</tr>
<tr>
<td></td>
<td>( SP_2 ) 0.694</td>
<td>0.416</td>
<td>0.354</td>
</tr>
</tbody>
</table>
Then \( x_{ij}^* \) and \( L(x_{ij}^*) \) were calculated from Equations 1 and 2 of Formulation 7.1 after substituting the values for the uniform pdf and parameters, as shown in the equations below. The \( x_{ij}^* \) and \( L(x_{ij}^*) \) values calculated using the following equations are summarized in Table 7.6.

\[
\begin{align*}
\therefore F(x_{ij}^*) &= \int_{a_{ij}}^{x_{ij}^*} f(q_j) dq_j = \frac{1}{b_j - a_j} \int_{a_{ij}}^{x_{ij}^*} dq_j = \frac{x_{ij}^* - a_j}{b_j - a_j} \\
\therefore P_{ij}^{\gamma} &= \frac{P_j - r_{ij}}{P_j + h_j} \\
\therefore P_{ij}^{\gamma} &= F(x_{ij}^*) \\
\therefore x_{ij}^* &= a_j + P_{ij}^{\gamma} (b_j - a_j) \quad [7.3a] \\
\int_{a_{ij}}^{x_{ij}^*} q_j f(q_j) dq_j &= \frac{1}{b_j - a_j} \int_{a_{ij}}^{x_{ij}^*} q_j dq_j = \frac{1}{2(b_j - a_j)} [q_j^2]_{a_{ij}}^{x_{ij}^*} = \frac{x_{ij}^* - a_j^2}{2(b_j - a_j)} \quad [7.3b]
\end{align*}
\]

Equations 7.3a and 7.3b were substituted in Equation 7.1:

\[
L(x_{ij}^*) = (p_j + h_j + s_j) \frac{x_{ij}^* - a_j^2}{2(b_j - a_j)} + [p_j - r_{ij} + s_j - (p_j + h_j + s_j) P_{ij}^{\gamma}] x_{ij}^* - s_j \mu_j \quad [7.4]
\]

<table>
<thead>
<tr>
<th>SP1</th>
<th>( x_{ij}^* )</th>
<th>( L(x_{ij}^*) )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>DC1</td>
<td>DC2</td>
</tr>
<tr>
<td>1</td>
<td>121.20</td>
<td>168.31</td>
</tr>
<tr>
<td>2</td>
<td>145.52</td>
<td>171.04</td>
</tr>
</tbody>
</table>

Table 7.6 – Phase I Outcome Parameters for Example 7.1

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Phase II:

Formulation 7.1 was applied with the given data for $x^*_y$, $L(x^*_y)$, $A_i$, and $f_i$. In the case of unlimited supply capacities, the values were set to infinity ($A_i = M$). Substituting Phase II parameters in Formulation 7.1 gave the following assignment problem:

\[
\text{Maximize} \\
OB_T = 198.478z_{11} + 285.642z_{12} + 221.521z_{13} + 406.754z_{21} + 322.099z_{22} + 127.502z_{23} - 270y_1 - 250y_2
\]

\[
S.T.
\]

\[
z_{11} + z_{21} \leq 1
\]

\[
z_{12} + z_{22} \leq 1
\]

\[
z_{13} + z_{23} \leq 1
\]

\[
121.20z_{11} + 168.31z_{12} + 107.39z_{13} \leq 233y_1
\]

\[
145.52z_{21} + 171.04z_{22} + 92.09z_{23} \leq 172y_2
\]

\[
y, z \in \{0,1\}
\]

The above LA problem was solved as a pure binary LP and optimized with $OB_T$ equal to (222.099) and $(z_{11} = z_{13} = z_{22} = y_2 = 1)$. This meant the total net profit was expected to be $222,099 for shipping 121.20 units from SP_1 to DC_1, 107.39 units from SP_2 to DC_3, and 171.04 units from SP_2 to DC_2. Thus, both shipping plants were opened and all the demand centers were satisfied by the optimal shipments. The problem formulated in GAMS format appears in Appendix L.
b. Uncapacitated Facilities (Model 9):

Part (a) of the example was reworked with unlimited supply capacities. The effect of supply capacities existed only in Phase II. Thus, Phase-I calculations held unchanged, and Phase II was resolved with the unlimited capacities as follows:

Maximize

\[
OB_T = 198.478z_{11} + 285.642z_{12} + 221.521z_{13} + 406.754z_{21} + 322.099z_{22} + 127.502z_{23} \\
- 270y_1 - 250y_2
\]

S.T.

\[
\begin{align*}
z_{11} + z_{21} & \leq 1 \\
z_{12} + z_{22} & \leq 1 \\
z_{13} + z_{23} & \leq 1
\end{align*}
\]

\[
121.20z_{11} + 168.31z_{12} + 107.39z_{13} \leq My_1
\]

\[
145.52z_{21} + 171.04z_{22} + 92.09z_{23} \leq My_2
\]

\[y_1, z_{ij} \in \{0,1\}\]

The above LA problem was solved as pure binary LP and optimized with \(OB_T\) equal to (606.356) and \((z_{21} = z_{22} = z_{23} = y_2 = 1)\). This meant the total net profit was expected to be $606,356 for shipping 145.52 units, 171.04 units, and 92.09 units from SP2 to DC1, DC2, and DC3 respectively. Thus, the first shipping plant was not used, and all the demand centers were satisfied by the optimal shipments. The problem formulated in GAMS format appears in Appendix M.

The following table summarizes the values of the total net profit for both parts of Examples 4.1 and 7.1. The net profit generated in the uncapacitated facilities case

77
decreased when inventory and stock-out costs were employed. This was logical because the model charged costs for expected stocked units and expected shortage. According to the results listed in the table, this logic was violated for the capacitated facilities case. In cases with inventory, the value of the critical probability decreased, which in turn decreased the size of $x_i^*$. Therefore, the optimization assignment (Phase II) was more flexible when small sizes of shipments were assigned rather than large ones, which maximized profit. This was a limitation of the Two-Phase Method.

<table>
<thead>
<tr>
<th>Example</th>
<th>Inventory &amp; Stock-out costs</th>
<th>Capacitated Facilities</th>
<th>Uncapacitated Facilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1</td>
<td>None</td>
<td>$210.401</td>
<td>$660.409</td>
</tr>
<tr>
<td>7.1</td>
<td>√</td>
<td>$222.099</td>
<td>$606.356</td>
</tr>
</tbody>
</table>

7.2. Combined Method with Inventory and Stock-out Costs

The previous section addressed the situations in which inventory and stock out occur for the Two-Phase Method. Even though this notion stays the same, the formulas derived in for the Combined Method have to be updated because the number of units shipped to DC$_j$ has changed. In section 7.1 the total shipments to DC$_j$ were equal to $x_i^*$. In this Section the total shipments to DC$_j$ were $D_j$. Therefore, at DC$_j$, the actual demand level ($q_j$) might be greater than the available sum of shipments ($D_j$) or less than or equal to $D_j$. Stock out was experienced in the first case, when the actual
demand was greater than what was available for sale. This situation still incurred an opportunity cost or backordering cost. Inventory existed when the actual demand was less than or equal to the available units; this situation caused either a full or a partial sale of $D_j$ units. If there was any remainder of $D_j$, those units were stocked or sold with a salvage value.

Hence, the base model was modified so that the unsold units were considered inventory and included as part of the optimization model. Tables 7.8 and 7.9 address the two cases of the relationship between the actual demand level and the total shipped units at DC$_j$. These tables present the expected sales level, revenue, and cost for each DC$_j$. The total cost of $D_j$ was the sum of shipping and production, inventory, and stock-out costs. The cost of inventory was determined from the expected stocked units. Moreover, the cost of stock out was determined from the expected shortage units.

<table>
<thead>
<tr>
<th>Table 7.8 – Expected Levels of Sales, Revenue, and Costs</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Quantity Sold</strong></td>
</tr>
<tr>
<td><strong>Revenue</strong></td>
</tr>
<tr>
<td><strong>Cost</strong></td>
</tr>
<tr>
<td><strong>Inventory</strong></td>
</tr>
<tr>
<td><strong>Stock out</strong></td>
</tr>
</tbody>
</table>
### Table 7.9 – Formulas of Expected Levels of Sales, Revenue, and Costs

<table>
<thead>
<tr>
<th>$q_j$</th>
<th>$D_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Quantity Sold</strong></td>
<td></td>
</tr>
<tr>
<td>Revenue</td>
<td>$p_j \int_{LB_j}^{UB_j} q_j f(q_j) dq_j$</td>
</tr>
<tr>
<td>Shipping</td>
<td>$r_i D_j \int_{LB_j}^{UB_j} f(q_j) dq_j$</td>
</tr>
<tr>
<td>Cost</td>
<td></td>
</tr>
<tr>
<td>Inventory</td>
<td>$h_j \int_{LB_j}^{UB_j} (D_j - q_j) f(q_j) dq_j$</td>
</tr>
<tr>
<td>Stock out</td>
<td>$c_j \int_{LB_j}^{UB_j} (q_j - D_j) f(q_j) dq_j$</td>
</tr>
</tbody>
</table>

7.2.1. General Formulation (with Inventory and Stock-out Costs)

Using the formulas in Table 7.9 led to the determination of the expected net profit generated at $DC_j$. Thus, the net profit per unit was equal to the expected revenue minus the expected cost. As mentioned earlier, the total expected cost was the sum of expected shipping and production, inventory, and stock-out costs for $D_j$ units. The total expected net profit generated from the LA system was derived as follows using the Combined Method ($OB_C$):

$$OB_C = Total \ Revenue - Total \ Variable \ Cost - Total \ Fixed \ Cost - Total \ Inventory$$

$$Cost - Total \ Stock-out \ Cost$$

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Total Revenue = Sum of Expected Sales at all DC_j

\[ \text{Total Revenue} = \sum_{j} p_j \left[ \int_{L_{B_j}}^{D_j} q_j f(q_j) dq_j + D_j \int_{D_j}^{U_{B_j}} f(q_j) dq_j \right] \]

\[ = \sum_{j} p_j \left[ \int_{L_{B_j}}^{D_j} q_j f(q_j) dq_j + D_j (1 - F(D_j)) \right] \]

Total Variable Cost = \sum_{j=1}^{n} \sum_{i=1}^{m} r_{ij} x_{ij} \left\{ \int_{L_{B_j}}^{D_j} f(q_j) dq_j + \int_{D_j}^{U_{B_j}} f(q_j) dq_j \right\} = \sum_{j=1}^{n} \sum_{i=1}^{m} r_{ij} x_{ij}

Total Fixed Cost = \sum_{i} f_i y_i

Total Inventory Cost = \sum_{j=1}^{n} h_j \int_{L_{B_j}}^{D_j} (D_j - q_j) f(q_j) dq_j

\[ = \sum_{j=1}^{n} h_j \left[ D_j \int_{L_{B_j}}^{D_j} f(q_j) dq_j - \int_{L_{B_j}}^{D_j} q_j f(q_j) dq_j \right] \]

\[ = \sum_{j=1}^{n} h_j [D_j F(D_j) - \int_{L_{B_j}}^{D_j} q_j f(q_j) dq_j] \]

Total Stock-out Cost = \sum_{j=1}^{n} s_j \int_{D_j}^{U_{B_j}} (q_j - D_j) f(q_j) dq_j

\[ = \sum_{j=1}^{n} s_j \left[ \int_{D_j}^{U_{B_j}} q_j f(q_j) dq_j - D_j \int_{D_j}^{U_{B_j}} f(q_j) dq_j \right] \]

\[ = \sum_{j=1}^{n} s_j [ \int_{D_j}^{U_{B_j}} q_j f(q_j) dq_j - D_j (1 - F(D_j))] \]
The above equations show that the terms of revenue, inventory cost, and stock-out cost have common coefficients. Table 7.10 separates these terms of profit function to highlight their common coefficients. Therefore, the following steps determine the total expected profit according to these coefficients. The formula for total expected profit was simplified in Equation 7.5.

Table 7.10 – Coefficients of Revenue, Inventory, and Stock-out

<table>
<thead>
<tr>
<th>Terms</th>
<th>( \int_{LB_j}^{D_j} q_j f(q_j) dq_j )</th>
<th>( F(D_j) )</th>
<th>Others</th>
</tr>
</thead>
<tbody>
<tr>
<td>Revenue</td>
<td>( p_j )</td>
<td>(- p_j D_j)</td>
<td>(+ p_j D_j)</td>
</tr>
<tr>
<td>Cost</td>
<td>( h_j )</td>
<td>(- h_j D_j)</td>
<td>_</td>
</tr>
<tr>
<td>Stock-out</td>
<td>( s_j )</td>
<td>(- s_j D_j)</td>
<td>( s_j D_j - s_j \mu_j )</td>
</tr>
<tr>
<td>Profit</td>
<td>( p_j + h_j + s_j )</td>
<td>(- (p_j + h_j + s_j) D_j)</td>
<td>((p_j + s_j) D_j - s_j \mu_j)</td>
</tr>
</tbody>
</table>
Thus,

\[
OB_C = \sum_j (p_j + h_j + s_j) \int_{LB_j}^{D_j} q_j f(q_j) dq_j - (p_j + h_j + s_j) D_j F(D_j) + (p_j + s_j) D_j - s_j \mu_j
\]

\[-\sum_i \sum_m r_{ij} x_{ij} - \sum_i f_i y_i \]  

[7.5]

The general formulation of the Combined Method is presented in Formulation 7.1 for LA problems with inventory and stock-out costs.

Formulation 7.2 – General Formulation of the Combined Method with Inventory and Stock-out Costs

Maximize

\[
OB_C = \sum_j (p_j + h_j + s_j) \int_{LB_j}^{D_j} q_j f(q_j) dq_j - (p_j + h_j + s_j) D_j F(D_j) + (p_j + s_j) D_j - s_j \mu_j
\]

\[-\sum_i \sum_m r_{ij} x_{ij} - \sum_i f_i y_i \]  

[1]

Where:

\[
D_j = \sum_{i=1}^m x_{ij}
\]

Subject to

Demand: \[ \sum_{i=1}^m x_{ij} \leq Mx_j \quad j = 1, 2, \ldots, n \]  

[2]

Supply: \[ \sum_{j=1}^n x_{ij} \leq A_i y_i \quad i = 1, 2, \ldots, m \]  

[3]
\[ \forall x_j \geq 0 \]  \hspace{1cm} \text{[4]} \\
\[ y_j, z_j \in \{0, 1\} \]  \hspace{1cm} \text{[5]} 

### 7.2.2. Uniformly Distributed Demands

To illustrate the implementation of the Combined Method for LA models that incorporate inventory and stock-out costs, the uniform probability distribution was assumed for all demands at DC\(j\). The uniform pdf was included in Formulation 7.2, which appeared in two terms of the objective function. The first term was the cumulative probability function \((F(D_j))\). The other term was \[ \int_{L_{B_j}}^{D_j} q_j f(q_j) dq_j \]. These two terms were integrated into the uniform pdf as follows:

\[ q_j \sim U[a_j, b_j] \]

\[ f(q_j) = \frac{1}{b_j - a_j} \]

\[ F(D_j) = \int_{L_{B_j}}^{D_j} f(q_j) dq_j \]

\[ = \int_{a_j}^{b_j} \frac{1}{b_j - a_j} dq_j = \frac{q_j}{b_j - a_j} \bigg|_{a_j}^{b_j} \]

\[ \therefore F(D_j) = \frac{D_j - a_j}{b_j - a_j} \]  \hspace{1cm} \text{[7.6]}
Hence, Equations 7.6 and 7.7 were substituted in Equation 7.5. The result was Equation 7.8.

\[
OB_c = \sum_j (p_j + h_j + s_j) \frac{D_j^2 - a_j^2}{2(b_j - a_j)} - \frac{(p_j + h_j + s_j)D_j}{b_j - a_j} + (p_j + s_j)D_j - s_j \mu_j
- \sum_i r_{ij} x_{ij} - \sum_i f_i y_i
\]  

Substituting Equation 7.8 as the objective function in Formulation 7.2 resulted in Formulation 7.3. However, Equation 7.8 had many common coefficients, so it was simplified before being included in the final formulation. In addition, Equation 7.8 has some constant terms that were associated with the DC\(_j\) demand parameters. Thus, they were multiplied by \(z_j\) to turn off these constants when DC\(_j\) is not assigned any shipments.

\[
OB_c = \sum_{j=1}^{n} \frac{p_j + h_j + s_j}{2(b_j - a_j)} D_j^2 - \sum_{j=1}^{n} \frac{p_j + h_j + s_j}{2(b_j - a_j)} a_j^2 z_j - \sum_{j=1}^{n} \frac{(p_j + h_j + s_j)}{b_j - a_j} D_j^2
+ \sum_{j=1}^{n} \frac{p_j + h_j + s_j}{b_j - a_j} a_j D_j + \sum_{j=1}^{n} (p_j + s_j) D_j - \sum_{j=1}^{n} s_j \mu_j z_j - \sum_i r_{ij} x_{ij} - \sum_i f_i y_i
\]

\[
= \sum_{j=1}^{n} \frac{p_j + h_j + s_j}{b_j - a_j} a_j D_j - \sum_{j=1}^{n} \frac{p_j + h_j + s_j}{2(b_j - a_j)} a_j^2 z_j - \sum_{j=1}^{n} \frac{p_j + h_j + s_j}{2(b_j - a_j)} D_j^2
\]
\[
+ \sum_{j=1}^{n}(p_j + s_j)D_j - \sum_{j=1}^{n}s_j\mu_jz_j - \sum_{j}^{n}\sum_{i}^{m}r_{xy}x_{iy} - \sum_{i}^{m}f_iy_i
\]
\[
= \sum_{j=1}^{n}\left[ \frac{a_j(p_j + h_j + s_j)}{b_j - a_j} + p_j + s_j \right]D_j - \sum_{j}^{n}\left[ \frac{a_j^2(p_j + h_j + s_j)}{2(b_j - a_j)} + s_j\mu_j \right]z_j
\]
\[
- \sum_{j=1}^{n}\left[ \frac{p_j + h_j + s_j}{2(b_j - a_j)} \right]D_j - \sum_{j}^{n}\sum_{i}^{m}r_{xy}x_{iy} - \sum_{i}^{m}f_iy_i
\]

\[
\therefore OB_c = \sum_{j=1}^{n}\sum_{i=1}^{m}K_{1j}x_{iy} - \sum_{j}^{n}K_{2j}z_j - \sum_{j=1}^{n}K_{3j}(\sum_{i=1}^{m}x_{iy})^2 - \sum_{j}^{n}\sum_{i}^{m}r_{xy}x_{iy} - \sum_{i}^{m}f_iy_i \quad [7.9]
\]

Where:

\[
K_{1j} = \frac{a_j(p_j + h_j + s_j)}{b_j - a_j} + p_j + s_j \quad [7.10]
\]

\[
K_{2j} = \frac{a_j^2(p_j + h_j + s_j)}{2(b_j - a_j)} + s_j\mu_j \quad [7.11]
\]

\[
K_{3j} = \frac{p_j + h_j + s_j}{2(b_j - a_j)} \quad [7.12]
\]

Equations 7.9 to 7.12 were each substituted as the objective function in Formulation 7.2, which became Formulation 7.3. These substitutions made Formulation 7.3 able to use the Combined Method to solve LA problems of uniformly distributed demands and inventory and stock-out costs.

Formulation 7.3 - Combined Method with Inventory and Stock-out Costs and Uniformly Distributed Demands

Maximize
\[ OB_c = \sum_{j=1}^{n} \sum_{i=1}^{m} K_1 x_{ij} - \sum_{j=1}^{n} K_2 z_j - \sum_{j=1}^{n} K_3 (\sum_{i=1}^{m} x_{ij})^2 - \sum_{j=1}^{n} \sum_{i=1}^{m} r_{ij} x_{ij} - \sum_{i=1}^{m} f_i y_i \]  

Where:

\[ K_{1j} = \frac{a_j (p_j + h_j + s_j)}{b_j - a_j} + p_j + s_j \]  

\[ K_{2j} = \frac{a_j^2 (p_j + h_j + s_j)}{2(b_j - a_j)} + s_j \mu_j \]  

\[ K_{3j} = \frac{p_j + h_j + s_j}{2(b_j - a_j)} \]  

Subject to

Demand \[ \sum_{j=1}^{n} x_{ij} \leq Mz_j \quad ; j = 1, 2, \ldots, n \]  

Supply \[ \sum_{j=1}^{n} x_{ij} \leq A_i y_i \quad ; i = 1, 2, \ldots, m \]  

\[ x_{ij} \in \{0, 1\} \]  

\[ y_i, z_j \in \{0, 1\} \]  

7.2.3. Example 7.2 (Combined Method with Inventory and Stock-out Costs)

Example 7.1 was reworked using the Combined Method and the results were compared.

Solution:

a. Capacitated Facilities (Model 10):
The three constants \((K_{lj}, K_{2j}, \text{and } K_{3j})\) of the objective function were calculated with Equations 1a, 1b, and 1c and presented in the following table. Then the LA problem of the example was solved with Formulation 7.3.

Table 7.11 – Example 5.2 \(OB_{C}\) Constants

<table>
<thead>
<tr>
<th>Parameter</th>
<th>(DC_1)</th>
<th>(DC_2)</th>
<th>(DC_3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a_j)</td>
<td>90</td>
<td>144</td>
<td>62</td>
</tr>
<tr>
<td>(b_j)</td>
<td>170</td>
<td>209</td>
<td>147</td>
</tr>
<tr>
<td>(p_j)</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>(h_j)</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(s_j)</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>(K_{1j})</td>
<td>10.688</td>
<td>16.685</td>
<td>8.512</td>
</tr>
<tr>
<td>(K_{2j})</td>
<td>343.438</td>
<td>965.542</td>
<td>176.615</td>
</tr>
<tr>
<td>(K_{3j})</td>
<td>0.034</td>
<td>0.042</td>
<td>0.032</td>
</tr>
<tr>
<td>(r_{ij})</td>
<td>(SP_1)</td>
<td>2.05</td>
<td>2.13</td>
</tr>
<tr>
<td></td>
<td>(SP_2)</td>
<td>0.53</td>
<td>1.92</td>
</tr>
</tbody>
</table>

Maximize

\[
OB_{C} = 10.688(x_{11} + x_{21}) + 16.685(x_{12} + x_{22}) + 8.512(x_{13} + x_{23})
- 343.438z_{1} - 965.542z_{2} - 176.615z_{3}
- 0.034(x_{11} + x_{21})^2 - 0.042(x_{12} + x_{22})^2 - 0.032(x_{13} + x_{23})^2
- 2.058x_{11} - 2.13x_{12} - 1.33x_{13} - 0.53x_{21} - 1.92x_{22} - 2.23x_{23}
\]
\[-270y_1 - 250y_2\]

\[S.T.\]
\[x_{11} + x_{21} \leq M z_1\]
\[x_{12} + x_{22} \leq M z_2\]
\[x_{13} + x_{23} \leq M z_3\]
\[x_{11} + x_{12} + x_{13} \leq 233y_1\]
\[x_{21} + x_{22} + x_{23} \leq 172y_2\]
\[x_{ij} \in \{0,1\}\]
\[y_1, z_j \in \{0,1\}\]

The above LA problem is optimized as an MINLP with \(OB_C\) equal to 394.345 and \((z_1 = z_2 = z_3 = y_1 = y_2 = 1)\). The optimized shipments are:

\[x_{12} = 130.395\]
\[x_{13} = 102.605\]
\[x_{21} = 136.798\]
\[x_{22} = 35.202\]

This means that the expected total net profit is $394.345, which is generated from the optimized \(x_{ij}\) shipments to DC\(_j\) by opening the two SP\(_i\). All DC\(_j\) are satisfied by the sum of the shipments:

\[D_1 = x_{11} + x_{21} = 0 + 136.798 = 136.798 \text{ (units)}\]
\[D_2 = x_{12} + x_{22} = 130.395 + 35.202 = 165.597 \text{ (units)}\]
\[D_3 = x_{13} + x_{23} = 102.605 + 0 = 102.605 \text{ (units)}\]

The GAMS format of the problem formulation is in Appendix N.
b. Uncapacitated Facilities:

The same formulation was reworked again with unlimited supply capacities. The effect of supply capacities existed in the supply constraints only. Thus, the above formulation was solved with the following supply constraints:

\[ x_{11} + x_{12} + x_{13} \leq M y_1 \]
\[ x_{21} + x_{22} + x_{23} \leq M y_2 \]

Therefore, the LA problem is optimized with \( OB_C \) equal to 607.837 and \( z_1 = z_2 = z_3 = y_2 = 1 \). The optimal shipment arrangements are:

\[ x_{21} = 147.745 \]
\[ x_{22} = 174.491 \]
\[ x_{23} = 97.082 \]

This means the total net profit is expected to be $607.837 for shipping 147.745, 174.491, and 97.082 units from SP_2 to DC_1, DC_2, and DC_3 respectively. Thus, the first shipping plant is not used, and all the demand centers are satisfied by the optimal shipments. The GAMS format of the problem formulation is in Appendix O.

Table 7.12 summarizes the total expected net profit for both parts of Examples 7.1 and 7.2. For both uncapacitated and capacitated facilities, the net profit generated by the Combined Method is greater than that generated by the Two-Phase Method. However, in the case of capacitated facilities the difference between the Combined Method and the Two-Phase Method is remarkable. The value of the objective function was $222.099 when the Two-Phase Method was used, but it was $394.345 when the Combined Method was implemented. The solution of the Combined
Method was 77.55% better than the solution of the Two-Phase Method for this specific example.

Formulation 7.2 still consists of a nonlinear term that makes the model solvable as an MINLP. The performance of the MINLP decreases when the problem size increases, because the MINLP tends to get stuck in the local optima solutions. Thus, the performance of the Combined Method is dependent on the performance of the MINLP solver chosen.

Table 7.12 – Comparison between Examples 7.1 and 7.2

<table>
<thead>
<tr>
<th>Example</th>
<th>Method</th>
<th>Capacitated Facilities</th>
<th>Uncapacitated Facilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.1</td>
<td>Two-Phase</td>
<td>$222.099</td>
<td>$606.356</td>
</tr>
<tr>
<td>7.2</td>
<td>Combined</td>
<td>$394.345</td>
<td>$607.837</td>
</tr>
</tbody>
</table>
Section 8.1 of this chapter determines a tight demand upper bound. Section 8.2 presents a newly developed method for solving the LA problem using a linear approximation. Section 8.3 presents a heuristic algorithm to improve the solution of the Two-Phase Method for capacitated facilities models.

8.1. Demand Upper Bounds Determination

For small- and medium-sized problems that were expected to be solved optimally, the demand upper bound was set to infinity ($M$) in the Combined Method formulation. The reason for this was because the optimal cutoff point on demand was part of the objective function. Thus, the global optimal solution of the LA determined the optimal cutoff point value automatically. However, setting a tighter upper bound expedited the solution search, especially for the methods of approximation that were efficient at solving large problems.

Hence, this section aimed to set a tight upper bound on the demand pdf that should not violate any constraint or change the solution of the Combined Method formulation. In the Two-Phase Method, the cutoff point was determined by the critical probability associated with the candidate-shipment link. This setup was violated, as more than one SP$_i$ was allowed to send shipments to the same DC$_j$. Therefore, a new parameter was introduced to set a tighter upper bound ($d_j$) on the
demand pdf at each DC$_j$. The $d_j$ upper bound might not always be achieved, depending on the optimal solution. Consequently, the total available shipment ($D_j$) realized at DC$_j$ is less than or equal to the $d_j$ level. This relationship was presented in the following figure.

![Figure 8.1 - Demand Upper Bound and Available Shipments at DC$_j$ Relationship](image)

The pdf of demand was assumed to be a continuous function that could be limited with real positive upper and lower bounds. Setting the $d_j$ assured that the total shipments to DC$_j$ would not exceed the expected sales. Critical probability provided optimality under the single source shipment condition because it was derived to determine the cut-off point on demand pdf in the case of a single source shipment. The derivation of the critical probability was based on the idea of equating the expectation of gain from selling an extra unit and the expectation of loss of not selling the extra unit.
The determination of \( d_j \) was associated with the SP_1's capacity, the number of SP_i shipping units to DC_j, and the critical probability for every participating SP_i to DC_j.

Proposition 8.1. For any demand pdf at DC_j, irrespective of the SP_i capacity, a \( d_j \) demand upper bound of total shipments to DC_j is given by:

\[
d_j = \max_{\forall i} \{ x_i^* \} \quad \text{[5.1]} \]

Proof:

As mentioned in chapter 4, Elmaghraby (1960) established the criterion for finding the optimal cutoff point on a pdf by applying the critical probability. The impact of the critical probability was inversely related to the \( x_i^* \). In other words, the higher the ratio of the critical probability, the lower the value of \( x_i^* \). This is shown clearly in the following formula and Figure 8.2.

\[
P_{ij}^{cr} = F(x_i^*) = \int_{-\infty}^{x_i^*} f(q_j) dq_j = \frac{p_j - r_{ij}}{p_j}
\]

Figure 8.2 - Impact of Critical Probability on \( d_j \)
The SPi was categorized in uncapacitated facilities and capacitated facilities. The uncapacitated SPi was a facility with infinite supply and shipping capacity. On the other hand, the capacitated SPi was a facility with a limited supply and shipping capacity. The number of SPi was categorized in single SPi and multiple SPi. Thus, there were four combinations of SPi capacity and number of SPi. Each combination affected the value of $d_j$ differently.

8.1.1. Uncapacitated Single SPi

A single SPi with unlimited capacity accommodates all demand requirements at DCj so that $x_{ij}$ equals $x^*_ij$. Therefore, $d_j$ equals $x^*_ij$ as shown in the following figure.

![Figure 8.3 - $d_j$ with Single Uncapacitated SP](image)

8.1.2. Uncapacitated Multiple SPi

A multiple SPi with unlimited capacity accommodates all demand requirements at DCj so that the total $x_{ij}$ received equals one of the $x^*_ij$ candidates. The SPi of minimum fixed cost and maximum critical probability supplies the DCj with $x_{ij}$
units. Thus, $d_j$ was not achieved, as the $SP_i$ fixed cost played a role in the total optimization. However, $d_j$ still equals the maximum $x_{ij}^*$, as shown in the following figure.

<table>
<thead>
<tr>
<th>$SP_i$</th>
<th>$\sum_{i=1}^{m} x_{ij} \leq d_j z_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$r_{1j}$</td>
</tr>
<tr>
<td>2</td>
<td>$r_{2j}$</td>
</tr>
<tr>
<td>3</td>
<td>$r_{3j}$</td>
</tr>
<tr>
<td>DC$_j$</td>
<td>$p_j$</td>
</tr>
<tr>
<td>$x_{ij}$</td>
<td>$x_{ij}^<em>$ $x_{ij}^</em>$ $x_{ij}^<em>$ $x_{ij}^</em>$ $LB_j$ $UB_j$</td>
</tr>
<tr>
<td>$\Lambda = \infty$</td>
<td>$P_{2j} &gt; P_{3j} &gt; P_{1j}$</td>
</tr>
</tbody>
</table>

Figure 8.4 – $d_j$ with Multiple Uncapacitated SP

8.1.3. Capacitated Single $SP_i$

A single $SP_i$ with limited capacity cannot accommodate the demand requirement at DC$_j$. (The capacity of $SP_i$ might be less than or equal $x_{ij}^*$). Therefore, two possibilities of shipment size $x_{ij}$ are determined in the following table. The $d_j$ level represents the upper bound, which means it is $x_{ij}^*$. Thus, $x_{ij}$ is less than or equal to $x_{ij}^*$, and so it is $d_j$. The following figure addresses the $d_j$ level for the single capacitated $SP_i$.  

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Table 8.1 - Possible Values of $x_{ij}$

<table>
<thead>
<tr>
<th>$A_i &lt; x^*_{ij}$</th>
<th>$A_i \geq x^*_{ij}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_{ij} = A_i$</td>
<td>$x_{ij} = x^*_{ij}$</td>
</tr>
</tbody>
</table>

8.1.4. Capacitated Multiple SP$_i$

The most complex situation exists when there are more than one SP$_i$ with limited capacities allowed to fulfill the required demand at DC$_j$. The required level of demand can be achieved with shipments from more than just a single SP$_i$. Determining the values of each $x_{ij}$ is not possible without solving the entire LA, since $x_{ij}$ is part of the optimization process. The maximum value $d_j$ could withstand was when only a single SP$_i$, with the maximum critical probability, supplied the entire needed demand at DC$_j$. This was identical to a single uncapacitated SP$_i$ case. Thus, $x_{ij}$, $x^*_{ij}$, and $d_j$ were equal. Therefore, determining the tightest valid upper bound was the maximum $x^*_{ij}$ for all possible SP$_i$ candidates. The following figure shows the determination of $d_j$ when three capacitated SP$_i$ are available.

Figure 8.5 – $d_j$ Level at Single Capacitate SP Case
8.1.5. Example 8.1

Re-solve Example 5.1 with upper bounds on demands.

Solution

a. Capacitated Facilities: (Model 6)

The input parameters and calculated variables were presented in Table 8.1.
Table 8.2 – Input Parameters and Demand Upper Bounds of Example 8.1

<table>
<thead>
<tr>
<th></th>
<th>DC_1</th>
<th>DC_2</th>
<th>DC_3</th>
<th>A_i</th>
<th>f_i</th>
</tr>
</thead>
<tbody>
<tr>
<td>i</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>j</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>r_{ij}</td>
<td>SP_1</td>
<td>2.05</td>
<td>2.13</td>
<td>1.33</td>
<td>233</td>
</tr>
<tr>
<td></td>
<td>SP_2</td>
<td>0.53</td>
<td>1.92</td>
<td>2.23</td>
<td>172</td>
</tr>
<tr>
<td>Demand</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>pdf</td>
<td>a_j</td>
<td>90</td>
<td>144</td>
<td>62</td>
<td></td>
</tr>
<tr>
<td></td>
<td>b_j</td>
<td>170</td>
<td>209</td>
<td>147</td>
<td></td>
</tr>
<tr>
<td>Supply</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>p_j</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ p_{ij}^* = \frac{p_j - r_{ij}}{p_j} \]

\[ x_{ij}^* = \text{SP}_1 \]

\[ d_j = \max \{ x_{ij}^* \} \]

\[ \frac{p_j}{2(b_j - a_j)} = 0.0250, 0.0308, 0.0235 \]

Maximize

\[ \text{OB}_C = 0.025 \left[ 340(x_{11} + x_{12}) - 8100z_1 - (x_{11} + x_{12})^2 \right] \]

\[ + 0.0308 \left[ 418(x_{12} + x_{22}) - 20736z_2 - (x_{12} + x_{22})^2 \right] \]

\[ + 0.0235 \left[ 294(x_{23} + x_{23}) - 38446z_3 - (x_{23} + x_{23})^2 \right] \]

\[ - 2.058x_{11} - 2.13x_{12} - 1.33x_{13} - 0.53x_{21} - 1.92x_{22} - 2.23x_{23} \]

\[ - 270y_1 - 250y_2 \]

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The previous LA problem was optimized as a MINLP with the exact optimal solution to that in Example 5.1.

b. Uncapacitated Facilities: (Model 7)

The same formulation was reworked with unlimited supply capacities. The previous LA problem was optimized as a MINLP with the exact optimal solution to that in Example 5.1. Therefore, setting a tight upper bound on demand pdf does not affect the solution of the Combined Method for either the capacitated facilities or the uncapatitated facilities.

8.2. Linear Approximation

The Combined Method resulted in a MINLP formulation that was capable of optimizing the LA problem. However, the optimality was not guaranteed for large
problems due to the capabilities of much commercial software. Therefore, the MINLP formulation was simplified for large problems with a Linear Approximation Method.

The Linear Approximation Method approximates linearly the nonlinear terms existing in the Combined Method formulation. The Linear Approximation Method is an optimization-based approximation method. The Combined Method general form is recalled from Chapter 5, which consists of two nonlinear terms in the objective function for the case of the uniform pdf for the demands. Both terms were associated with the \( x_{ij} \) decision variable. The \( D_j \) represented the sum of all shipments to \( DC_j \). The lowest value of the objective function occurred when the negative \( D_j \) terms were at their maximum. A tight upper bound \( d_j \) was enforced on the demand pdf, as determined in the previous section, and was found to have no effect on the MINLP solution. Since \( D_j \) cannot exceed the demand upper bound \( d_j \), \( D_j \) was approximated to \( d_j \) for the negative terms of the objective function. This penalized the objective function by the worst-case scenario for these terms. The negative terms in the formulation were transformed into constants associated only with the \( DC_j \) and then they were multiplied by the decision variable \( z_j \) to stop charging the objective function with their values if the \( DC_j \) was not assigned any shipments. Therefore, all the terms were linear, and the Combined Method formulation was now an MIP rather than an MINLP. The general Combined Method formulation with the approximation is presented in Formulation 8.1. Formulation 8.2 represents the Combined Method formulation with the Linear Approximation for the uniformly distributed demands.
Two examples are presented in the following subsection to address the applicability of the method.

Formulation 8.1 – General Formulation of Combined Method with Approximation

Maximize

\[ OB_A = \sum_{j=1}^{n} p_j \int_{LB_j}^{D_j} q_j f(q_j) dq_j + \sum_{j=1}^{n} p_j D_j - \sum_{j=1}^{n} p_j d_j F(d_j) z_j - \sum_{j=1}^{n} \sum_{i=1}^{m} r_j x_{ji} - \sum_{i=1}^{m} f_i y_i \] \[ 1 \]

Subject to

Demand:
\[ \sum_{j=1}^{m} x_{ji} \leq d_j z_j \quad j = 1, 2, \ldots, n \] \[ 2 \]

Supply:
\[ \sum_{j=1}^{n} x_{ij} \leq A_i y_i \quad i = 1, 2, \ldots, m \] \[ 3 \]
\[ \forall x_{ij} \geq 0 \] \[ 4 \]
\[ y_i, z_j \in \{0, 1\} \] \[ 5 \]

Where:
\[ d_j = \max_{\forall i} \{x^*_i\} \quad j = 1, 2, \ldots, n \] \[ 6 \]

Formulation 8.2 – Uniformly Distributed Demands with Linear Approximation

Maximize

\[ OB_A = \sum_{j=1}^{n} \frac{p_j}{2(b_j - a_j)} [2b_j (\sum_{i=1}^{m} x_{ji}) - a_j^2 z_j - d_j^2 z_j] - \sum_{j=1}^{n} \sum_{i=1}^{m} r_j x_{ji} - \sum_{i=1}^{m} f_i y_i \] \[ 1 \]

Subject to

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Demand: \[ \sum_{i=1}^{n} x_{ij} \leq d_j z_j \quad j=1,2,...,n \]  \hspace{1cm} [2]

Supply: \[ \sum_{j=1}^{n} x_{ij} \leq a_i y_i \quad i=1,2,...,m \]  \hspace{1cm} [3]

\[ \forall x_{ij} \geq 0 \]  \hspace{1cm} [4]

\[ y_i, z_j \in \{0,1\} \]  \hspace{1cm} [5]

Where: \[ d_j = \max_{\forall i} \{x_{ij}^*\} \quad j=1,2,...,n \]  \hspace{1cm} [6]

8.2.2. Example 8.2

Apply Formulation 8.2 to solve Example 5.1.

Solution:

a. Capacitated Facilities: (Model 6)

The input variables and calculated variables that were presented in Table 8.2 were substituted in Formulation 8.2 as follows:
Table 8.3 – Input Variables and Demand Upper Bounds of Example 8.2

<table>
<thead>
<tr>
<th>j</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>DCj</td>
</tr>
<tr>
<td>i</td>
<td>1</td>
</tr>
<tr>
<td>r_ij</td>
<td>SP_1</td>
</tr>
<tr>
<td></td>
<td>SP_2</td>
</tr>
<tr>
<td>Dema nd pdf</td>
<td>a_j</td>
</tr>
<tr>
<td></td>
<td>b_j</td>
</tr>
<tr>
<td>p_j</td>
<td>4</td>
</tr>
<tr>
<td>p_j^* = p_j - r_ij</td>
<td>SP_1</td>
</tr>
<tr>
<td></td>
<td>SP_2</td>
</tr>
<tr>
<td>x_j^*</td>
<td>SP_1</td>
</tr>
<tr>
<td></td>
<td>SP_2</td>
</tr>
<tr>
<td>d_j = max_{ui} (x_j^*)</td>
<td>159.400</td>
</tr>
<tr>
<td>p_j</td>
<td>0.0250</td>
</tr>
<tr>
<td>\frac{p_j}{2(b_j-a_j)}</td>
<td>0.0250</td>
</tr>
</tbody>
</table>

Maximize

\[ OB_A = 0.025 \left[ 340(x_{11} + x_{12}) - 8100z_1 - (159.4)^2 z_1 \right] \]
\[ + 0.0308 \left[ 418(x_{12} + x_{22}) - 20736z_2 - (177.8)^2 z_2 \right] \]
\[ + 0.0235 \left[ 294(x_{23} + x_{33}) - 38446z_3 - (118.738)^2 z_3 \right] \]
\[ - 2.058x_{11} - 2.13x_{12} - 1.33x_{13} - 0.53x_{21} - 1.92x_{22} - 2.23x_{23} \]
\[ - 270y_1 - 250y_2 \]

S.T.

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\[ x_{11} + x_{21} \leq 159.4z_j \]
\[ x_{12} + x_{22} \leq 177.8z_j \]
\[ x_{13} + x_{23} \leq 118.738z_j \]
\[ x_{11} + x_{12} + x_{13} \leq 233y_1 \]
\[ x_{21} + x_{22} + x_{23} \leq 172y_2 \]
\[ x_i \geq 0 \]
\[ y_i, z_i \in \{0,1\} \]

The previous LA problem was optimized as a MINLP with an \( OB_A \) equal to 212.689 and \( z_1 = z_2 = y_1 = y_2 = 1 \). The optimized shipments were:

\[ x_{12} = 165.2 \]
\[ x_{21} = 159.4 \]
\[ x_{22} = 12.6 \]

This meant the total net profit was expected to be $212,689, generated from the optimized \( x_i \) shipments to DC\(_j\) by opening the two SP\(_j\). All DC\(_j\) are satisfied with the sum of the shipments:

\[ D_1 = x_{11} + x_{21} = 0 + 159.4 = 159.4 \text{ (units)} \]
\[ D_2 = x_{12} + x_{22} = 165.2 + 12.6 = 177.8 \text{ (units)} \]
\[ D_3 = x_{13} + x_{23} = 0 + 0 = 0 \text{ (units)} \]

b. Uncapacitated Facilities: (Model 5)

The same formulation was reworked with unlimited supply capacities. Thus, the LA problem was optimized with an \( OB_A \) equal to 651.802 and \( z_1 = z_2 = z_3 = y_2 = 1 \).

The optimal shipment arrangements were:
This meant the total net profit was expected to be $651,802 for shipping 159.4, 177.8, and 118.737 units from SP₂ to DC₁, DC₂, and DC₃, respectively. Thus, the first shipping plant was not utilized, and all the demand centers were satisfied with the optimal shipments.

Table 8.4 – Comparison between Examples 4.1, 5.1, and 8.2

<table>
<thead>
<tr>
<th>Example</th>
<th>Method</th>
<th>Capacitated Facilities</th>
<th>Uncapacitated Facilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1</td>
<td>Two Phase</td>
<td>$210.401</td>
<td>$660.409</td>
</tr>
<tr>
<td>5.1</td>
<td>Combined</td>
<td>$438.451</td>
<td>$660.409</td>
</tr>
<tr>
<td>8.2</td>
<td>Linear Approximation</td>
<td>$212.689</td>
<td>$651.802</td>
</tr>
</tbody>
</table>

Table 8.4 summarizes the values of the total net profit for both parts of Examples 4.1, 5.1, and 8.2. In the case of capacitated facilities, the Linear Approximation Method performed better than the Two-Phase Method. However, in the uncapacitated facilities case, the Linear Approximation Method could not beat the Two-Phase Method as the Two-Phase solution was the optimal solution.
8.2.3. Example 8.3

Apply the Linear Approximation Method and Two-Phase Method to solve Model 5 and Model 6 with 200 SP_i and 200 DC_j.

Solution:

The LA solution of the Linear Approximation Method and the Two-Phase Method was summarized in the following table (Table 8.5) for Model 5 and Model 6. The table indicates that, for Model 5, the Linear Approximation Method solution was 6.35% better than the Two-Phase Method solution. The percentage increased drastically, to 115%, for Model 6. Thus, the superiority of the Linear Approximation Method over the Two-Phase Method increased with limited SP_i capacities. Moreover, the increase of the problem's size had greater effect on the Two-Phase Method performance than on the Linear Approximation Method. This was justifiable because the number of binary decision variables in the Two-Phase was greater than in the Linear Approximation Method by n(m-1).

Table 8.5 – Example 8.3 Solution Summary

<table>
<thead>
<tr>
<th>Method</th>
<th>Model 5 (uncapacitated facilities)</th>
<th>Model 6 (capacitated facilities)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Two-Phase</td>
<td>102,000</td>
<td>51,381</td>
</tr>
<tr>
<td>Linear Approximation</td>
<td>110,000</td>
<td>109,000</td>
</tr>
<tr>
<td>Difference (%)</td>
<td>6.35</td>
<td>115.02</td>
</tr>
</tbody>
</table>
8.3. Heuristic

In the previous examples it was noted that the Two-Phase Method performed poorly for the capacitated facilities case but approached an optimal solution for all uncapatitated facilities problems. Therefore, a heuristic was developed to improve the Two-Phase Method solution for LA models with capacitated facilities. The heuristic procedure attempted to assign the unused supply and shipping capacities of SP; to unsatisfied DCj in order to maximize the total profit. It searched for the SP; candidates first, and then located the most profitable candidate shipment route to the unsatisfied DCj. The developed heuristic algorithm is provided in Subsection 8.3.1, followed by an example in Subsection 8.3.2.

8.3.1. Algorithm

Step 1: Solve the LA problem with the Two-Phase Method for a starting point:

Two-Phase optimal solution: \{y_b, z_{ij}, and OB_T\}

Let: k = 0

\[ OB_{Bu} = OB_T \]

For i = 1, 2, ..., m:

\[ \Delta_i = A_i y_i - \sum_{j=1}^{n} x_{ij} z_{ij} > 0 \]

Step 2: k = k + 1

If k = m+1, go to Step 8

Else, If \( y_k = 0 \), repeat Step 2

Else, For i = k

\[ k \Delta_i = \Delta_i - \sum_{j=1}^{n} x_{ij} z_{ij} > 0 \]

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Else, repeat Step 2

Step 3: If all $\sum_{i=1}^{m} z_{ij} = 1$, go to Step 2

Maximum $p^c_{ik} = p^c_{jk}$

If $p^c_{ik} < 0$, go to Step 2

Step 4: $x_{ij} = \Delta_i$

Step 5: For the corresponding $p^c_{ik}$:

$$L(x_{ij}) = (p_j + h_j + s_j) \int_{0}^{x_{ij}} q_j f(q_j) dq_j + [p_j - r_j + s_j - (p_j + h_j + s_j)F(x_{ij})]x_{ij} - s_j \mu_j$$

If $L(x_{ij}) < 0$ Set $x_{ij} = 0$ and go to Step 2

Step 6: For the corresponding $p^c_{ik}$:

$z_{ij} = 1$

Step 7: $OB_{Hk} = OB_{H(k-1)} + L(x_{ij})$

Return to Step 2.

Step 8: Stop

8.3.2. Example 8.4

Apply the heuristic algorithm to improve the solution of Example 4.1 for the capacitated facilities case.

Solution:

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Table 8.6 -- Solution Summary of Example 4.1A

<table>
<thead>
<tr>
<th>j</th>
<th>DCj</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>SP1</td>
<td>129.000</td>
<td>174.388</td>
</tr>
<tr>
<td>SP2</td>
<td>159.400</td>
<td>177.800</td>
</tr>
</tbody>
</table>

Table 8.4 presented the summary of the Two-Phase Method solution for Example 4.1a. The table indicated DC3 was not satisfied with any shipment. It also showed that the unused supply capacities were 58.612 and 12.6 units for SP1 and SP2, respectively. The heuristic algorithm was applied to improve the Two-Phase solution ($210.4$) and presented as follows:

**Iteration 1:**

Step 1: Two-Phase optimal solution: \{y_1 = y_2 = 1, z_{12} = z_{21} = 1, and OB_T = $210.4$\}

Let: k = 0
\( OB_{H0} = OB_T = 210.4 \)

\[
A_1 = A_1 y_1 - \sum_{j=1}^{n} x_{1j} z_{1j} = 233(1) - 129(0) - 174.388(1) - 118.738(0) = 58.612
\]

\[
A_2 = A_2 y_2 - \sum_{j=1}^{n} x_{2j} z_{2j} = 172(1) - 159.4(1) - 177.8(0) - 99.613(0) = 12.6
\]

Step 2: \( k = k + 1 = 0 + 1 = 1 \)

\( y_1 > 0 \)

\( ^{1}A_1 = A_1 - (0+1+0)(0) = 58.612 \)

Step 3:

\( z_{11} + z_{21} = 1 = 1 \)

\( z_{12} + z_{22} = 1 + 0 = 1 \)

\( z_{13} + z_{23} = 0 + 0 = 0 \)

Maximum \( P_{y}^{cr} = P_{y}^{cr} \)

Maximum \( \{ P_{13}^{cr}, P_{23}^{cr} \} = \) Maximum \( \{ 0.667, 0.442 \} = 0.677 > 0 \)

\( P_{13}^{cr} = \max P_{y}^{cr} = P_{13}^{cr} \)

Step 4: \( x_{13} = ^{1}A_1 = 58.612 \)

Step 5: For the corresponding \( P_{13}^{cr} \):

\( s_3 = h_3 = 0 \) (given)

\( p_3 = 4 \) (given)

\( [a_3, b_3] = [62, 147] \) (given)

\[
L(x_y) = p_j \int_{x_y}^{x_y} q_j f(x_j) dq_j + [p_j - r_j - p_j F(x_y)]x_y
\]

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\[ L(x_{13}) = p_3 \int_{x_{13}}^{x_4} q_3 f(q_3) dq_3 + [p_3 - r_{13} - p_3 F(x_{13})] x_{13} \]

\[ L(x_{13}) = 4\left( \frac{x_{13}^2 - 62^2}{2(147 - 62)} \right) + 4 - 1.33 - 4(\frac{x_{13} - 62}{147 - 62}) x_{13} \]

\[ L(x_{13}) = L(58.612) = 156.22 > 0 \]

Step 6: For the corresponding \( p_{13}^{\alpha_1} \):

\[ z_{13} = 1 \]

Step 7: \( OB_{II} = OB_{II0} + L(x_{13}) = 210.4 + 188.14 = 398.54 \)

Return to Step 2.

**Iteration 2:**

Step 2: \( k = 1 + 1 = 2 \)

\[ y_2 = 1 \]

\[ ^2A_2 = A_2 - (1 + 0 + 0)(0) = 12.6 > 0 \]

Step 3:

\[ z_{11} + z_{21} = 0 + 1 = 1 \]

\[ z_{12} + z_{22} = 1 + 0 = 1 \]

\[ z_{13} + z_{23} = 1 + 0 = 1 \]

Since, all \( \sum_{i=1}^{m} z_{ij} = 1 \) go to Step 2

**Iteration 3:**

Step 2: \( k = 2 + 1 = 3 > m = 2 \)

Then go to Step 8
Step 8: Stop

The final solution was $398.54, generated from shipping $x_{12}^*, x_{13}^*$, and $x_{21}^*$ of 174.388, 58.612, and 159.4, respectively. The heuristic improved the Two-Phase Method solution by 72.25%, which was 16.33% less than the optimal solution of the Combined Method. It was noticed that SP2 still had 12.6 units remaining of unused supply and shipping capacity. However, the algorithm fathomed this option as all DCj were satisfied with shipments.
CHAPTER 9
COMPARATIVE ANALYSIS

This chapter benchmarks the proposed methods against similar algorithms available in the literature. This chapter is divided into four sections. The first section is an overview of the available models and methods in the literature. The second section presents a statistical analysis of uncapacitated facilities models for all methods. Section three presents a statistical analysis of capacitated facilities models for all methods. Section four compares the advantages and disadvantages of all methods.

9.1. Overview

Logendran and Terrell (1988, 1991) developed the Two-Phase Method that was implemented using two solving techniques. The first technique, a Branch and Bound (BB), was limited by LA problem size due to the limited capability of software at the time of the research. The second technique was a heuristic found to be efficient for small problems. The efficiency of the heuristic could not be evaluated for large problems because of the unavailability of the BB solution. Numerical results for both techniques were presented for uncapacitated facilities, stochastic demands, and sensitive prices (Model 7) in the 1988 article. Logendran and Terrell (1991) also presented numerical results using both techniques for capacitated facilities, stochastic demands, and sensitive prices (Model 8). Table 9.1 presents the numerical results of
the Logendran and Terrell (1988, 1991) study for uniformly distributed demands. The final values (FV) of the objective function of both techniques were presented for small and large problems.

Model 7 and Model 8 were solved with the substitute of the LA problem parameters drawn from a specified data set. For $m SP_i$ and $n DC_j$, the data set for both models was provided in both articles but not the exact values of parameters. A single scenario of six different sizes of small problems and four sizes of large problems was solved for Models 7 and 8; but only two scenarios of an 8x10 (mxn) problem were solved. Thus, the FV values did not present an average value for each tested problem size. Therefore, the values provided in Table 9.1 were not used for the benchmarking process.

This study proposed three new methods: the Combined Method, the Linear Approximation Method, and the Heuristic Method. The first two methods were compared against the modified Two-Phase Method presented in Chapter 4. The Heuristic Method was not included in the comparative analysis as it was based on improving the solution of the Two-Phase Method and was limited only to models of uncapacitated facilities.
Table 9.1 – Results Summary of Logendran and Terrell (1988,1991)

<table>
<thead>
<tr>
<th>Problem Size</th>
<th>Uncapacitated (M7)</th>
<th>Capacitated (M8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type</td>
<td>FV&lt;sub&gt;HEUR&lt;/sub&gt;</td>
<td>FV&lt;sub&gt;BB&lt;/sub&gt;</td>
</tr>
<tr>
<td>Small</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2x2</td>
<td>547.34</td>
<td>547.34</td>
</tr>
<tr>
<td>2x3</td>
<td>1073.66</td>
<td>1073.66</td>
</tr>
<tr>
<td>3x4</td>
<td>1280.65</td>
<td>1280.65</td>
</tr>
<tr>
<td>4x5</td>
<td>1805.88</td>
<td>1805.88</td>
</tr>
<tr>
<td>5x6</td>
<td>2134.26</td>
<td>2134.26</td>
</tr>
<tr>
<td>6x8</td>
<td>3174.64</td>
<td>3174.64</td>
</tr>
<tr>
<td>8x10</td>
<td>3741.19</td>
<td>3837.11</td>
</tr>
<tr>
<td>Large</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10x15</td>
<td>6715.97</td>
<td>6715.97</td>
</tr>
<tr>
<td>15x20</td>
<td>9101.64</td>
<td>9194.06</td>
</tr>
<tr>
<td>20x25</td>
<td>11601.83</td>
<td>11601.83</td>
</tr>
<tr>
<td>30x40</td>
<td>18883.10</td>
<td>—</td>
</tr>
</tbody>
</table>

9.2. Uncapacitated Facilities

Model 7 was used as an example of uncapacitated facilities models for the sake of illustration. Two groups of problem sizes were studied individually in the following subsections. The first group consisted of the small problems and encompassed five sizes (2x3, 2x4, 3x4, 4x5, and 5x5). The second group consisted of the large problems in three sizes (10x10, 100x100, and 200x200). Four scenarios for
each problem size were solved using the Combined Method, the Linear Approximation Method, and the Two-Phase Method.

9.2.1. M7 Small

The Combined Method performance decreased with the increase of problem size. Figure 9.1 presents the objective function average value of five small problems for the three methods. The three methods performed identically for this group of LA problems except for the problem size 5x5. The Combined Method performance deteriorated for the 5x5 problem due to the increased size of the problem, which also increased the number of nonlinear terms in the objective function.

![Method Performance for Small Problems (Model 7)](image)

Figure 9.1 – Method Performance for Uncapacitated Facilities Small Problems
9.2.2. M7 Large

The Combined Method formulation could not solve this group of problems due to the large number of nonlinear variables that increased the complexity of MINLP. Figure 9.2 presents the superiority of the Linear Approximation Method over the other methods for the uncapacitated facilities model of large size problems. The Two-Phase Method was supposed to optimize the LA problem of this group, as it was uncapacitated. However, due to the large number of binary variables in the Two-Phase Method, the number of iterations increased and exceeded the software limitation on iterations (10,000). The Linear Approximation Method was faster, simpler, and required fewer computations than the Two-Phase Method.

![Method Performance for Large Problems](image)

Figure 9.2 – Method Performance for Uncapacitated Facilities Large Problems
9.3. Capacitated Facilities

Logendran and Terrell (1991) used capacities that were uniformly distributed between [300, 350] units. In the same study they used demand means that were uniformly distributed between [100, 200]. This ensured the SPi capacities would always be sufficient to satisfy the demand requirements, especially since they tested problems with the number of SP almost equal to the number of DC. Thus, the SPi behaved like uncapacitated facilities, which made the capacitated facilities case not achieved appropriately in the 1991 article. Therefore, two sets of SPi capacities were applied in this chapter. The first set was what Logendran and Terrell (1991) used, Uniform [300, 350] units, which was referred to as type (a). The other set, Uniform [100, 250] units, was referred to as type (b).

Subsection 9.3.1 addresses the statistical analysis for capacitated facilities and small problems with the use of type (a) SPi capacities (M8a). Subsection 9.3.2 addresses the analysis for capacitated facilities and small problems with type (b) SPi capacities (M8b). Subsections three and four do the same for large problems.

9.3.1. M8a Small

Figure 9.1 presents the objective function average value of five small problems for the three methods. The three methods performed closely for this group of LA problems except for sizes 2x4 and 4x5. This might have been due to the difference between the number of SPi and the number of DCj, which made the SPi capacities incapable of satisfying the demand requirements of all DCj. According to
Figure 9.2, the least difference between the methods' performances was in problem size 2x3. On the other hand, the most difference occurred in problem size 2x4. Thus, the two sizes were analyzed as follows. Problem size 3x4 was not considered with the least difference as all methods performed equally.

Figure 9.3 – Method Performance for Capacitated Facilities Small Problems

In all problem sizes, the Combined Method always had the highest OB value, followed by the Linear Approximation Method and, finally, the Two-Phase Method. A randomized block design analysis of variance (ANOVA) was implemented with the methods as the Treatments and run as the Blocks. Table 9.2 and Table 9.3 show that there were no statistical differences between the three methods, with a significance level of 0.05 for problem sizes 2x3 and 2x4, respectively.

Figures 9.4 and 9.5 present the main effect plots of the three methods for problem sizes 2x3 and 2x4, respectively. In both figures the Two-Phase Method
performed worse than the other two methods. The Two-Phase Method also had a negative effect on the scenario run compared to a considerably positive effect on the scenario by the Combined Method.

Table 9.2 – ANOVA for M8a (2x3)

<table>
<thead>
<tr>
<th>Source</th>
<th>DF</th>
<th>SS</th>
<th>MS</th>
<th>F</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>Method</td>
<td>2</td>
<td>14388</td>
<td>7194</td>
<td>1.36</td>
<td>0.326</td>
</tr>
<tr>
<td>Runs</td>
<td>3</td>
<td>25709</td>
<td>8570</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Error</td>
<td>6</td>
<td>31782</td>
<td>5297</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>11</td>
<td>71880</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 9.3 – ANOVA for M8a (2x4)

<table>
<thead>
<tr>
<th>Source</th>
<th>DF</th>
<th>SS</th>
<th>MS</th>
<th>F</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>Method</td>
<td>2</td>
<td>179782</td>
<td>89891</td>
<td>10.46</td>
<td>0.011</td>
</tr>
<tr>
<td>Runs</td>
<td>3</td>
<td>91663</td>
<td>30554</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Error</td>
<td>6</td>
<td>51554</td>
<td>8592</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>11</td>
<td>322998</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

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9.3.2. M8b Small

The same analysis for the same scenarios and problem sizes was held with SP capacities drawn from the type (b) data set. Here, the case was more realistic than the
previous subsection because the SP$_1$ capacities were more limited and more dispersed. Figure 9.6 addresses the impact of the limited SP$_1$ capacity on the methods’ performances. The Combined Method performed the best, while the Two-Phase performed the worst. Tables 9.4 and 9.5 indicate a significant statistical difference between the three methods, with a significance level of 0.05 for sizes 2x3 and 2x4, respectively.

**Method Performance for Small problems**

*(Model 8b)*

<table>
<thead>
<tr>
<th>Problem Size (mxn)</th>
<th>Combined</th>
<th>Two-Phase</th>
<th>Linear Approx</th>
</tr>
</thead>
<tbody>
<tr>
<td>2x3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2x4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3x4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4x5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5x5</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 9.6 – Method Performance for Capacitated Facilities Type (a) Small Problems
Table 9.4 – ANOVA for M8b (2x3)

<table>
<thead>
<tr>
<th>Source</th>
<th>DF</th>
<th>SS</th>
<th>MS</th>
<th>F</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>Method</td>
<td>2</td>
<td>127777</td>
<td>63889</td>
<td>13.29</td>
<td>0.006</td>
</tr>
<tr>
<td>Runs</td>
<td>3</td>
<td>41219</td>
<td>13740</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Error</td>
<td>6</td>
<td>28853</td>
<td>4809</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>11</td>
<td>197849</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 9.5 – ANOVA for M8b (2x4)

<table>
<thead>
<tr>
<th>Source</th>
<th>DF</th>
<th>SS</th>
<th>MS</th>
<th>F</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>Method</td>
<td>2</td>
<td>130606</td>
<td>65303</td>
<td>6.07</td>
<td>0.036</td>
</tr>
<tr>
<td>Runs</td>
<td>3</td>
<td>53280</td>
<td>17760</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Error</td>
<td>6</td>
<td>64497</td>
<td>10750</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>11</td>
<td>248383</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figures 9.7 and 9.8 present the main effect plots of the three methods for problem sizes 2x3 and 2x4, respectively. In both figures the Combined Method had the largest mean and the Two-Phase Method the smallest.
9.3.3. M8a Large

Figure 9.9 presents the objective function average value of five scenarios of large problems for two methods. The two methods are the Linear Approximation
Method and Two-Phase Method. The Combined Method was omitted from the analysis of the large problems because the available solver could not return any reasonable values for these problems for the given time and number of iterations. The remaining methods, the Linear Approximation and Two-Phase Methods, performed closely. However, the Two-Phase Method’s performance decreased with the increase in problem size. This was due to reaching the limit on the number of iterations set by the NEOS server.

![Method Performance for Large Problems (Model 8a)](image)

**Figure 9.9 – Method Performance for Capacitated Facilities Type (a) Large Problems**

9.3.4. M8b Large

Figure 9.10 presents the objective function average value of five scenarios of large problems for the two methods. The Linear Approximation Method’s performance consistently increased with increasing problem size. The Two-Phase Method performed closely for the 10x10 and 100x100 problem sizes. However, the

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Two-Phase Method's performance decreased drastically with increasing problem size. This was due to reaching the limit on the number of iterations set by the NEOS server and tight SP1 capacities.

![Method Performance for Large Problems](image)

Figure 9.10 – Method Performance for Capacitated Facilities Type (b) Large Problems

9.4. Comparisons between All Methods

Table 9.6 addresses the three methods of interest and the criteria for comparison. In other words, the table summarizes the advantages and disadvantages of each method.
Table 9.6 – Method Comparisons

<table>
<thead>
<tr>
<th>Criterion</th>
<th>Combined</th>
<th>Linear Approximation</th>
<th>Two-Phase (Phase-II)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of constraints</td>
<td>$m + n$</td>
<td>$m + n$</td>
<td>$m + n$</td>
</tr>
<tr>
<td>Number of variables</td>
<td>$m + n + mn$</td>
<td>$m + n + mn$</td>
<td>$m + mn$</td>
</tr>
<tr>
<td>Number of binary variables</td>
<td>$m + n$</td>
<td>$m + n$</td>
<td>$m + mn$</td>
</tr>
<tr>
<td>Formulated as</td>
<td>MINLP</td>
<td>MIP</td>
<td>BIP</td>
</tr>
<tr>
<td>Small uncapacitated</td>
<td>Optimal Solution</td>
<td>Optimal Solution</td>
<td>Optimal Solution</td>
</tr>
<tr>
<td>Large uncapacitated</td>
<td>—</td>
<td>Optimal Solution</td>
<td>Good Solution</td>
</tr>
<tr>
<td>Small capacitated</td>
<td>Optimal Solution</td>
<td>Good Solution</td>
<td>Lower Bound</td>
</tr>
<tr>
<td>Large capacitated</td>
<td>—</td>
<td>Best available Solution</td>
<td>Lower Bound</td>
</tr>
<tr>
<td>Computation time</td>
<td>Longest</td>
<td>Shortest</td>
<td>Long</td>
</tr>
<tr>
<td>Exceed NEOS iterations limit</td>
<td>—</td>
<td>Never</td>
<td>$mxn &gt; 10x10$</td>
</tr>
<tr>
<td>Multiple sourcing</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>One formulation</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
</tr>
</tbody>
</table>
CHAPTER 10
SENSITIVITY ANALYSIS

This chapter studied the impact of every parameter of the Combined Method formulation on the objective function value. Therefore, the chapter is divided into nine sections. The first eight sections individually analyze each of the parameters, and the last section summarizes the impact of all of the parameters. Most of the parameters had a consistent trend of impact for models 5 to 12. Thus, a sample model (Model 12) of size 2x3 is presented for illustration unless the studied parameter had a different impact on the various models. Four different LA scenarios for each level of the parameter were studied, which made the randomized block design analysis of variance (ANOVA) the appropriate testing procedure. The Treatments were the parameters' levels and the Blocks were the runs.

10.1. Fixed Cost

The SP₁ fixed cost ($f_i$) was studied by generating four different LA scenarios for Models 5, 6, 7, 8, 9, 10, 11, and 12 at three constant values of $f_i$. The three values were $200, $250, and $300. Fixed cost was inversely proportional to the value of the objective function ($OB_c$) (i.e. the higher the fixed cost, the lower the objective function). The same pattern of relationship was common for all eight models. Figure 10.1 presents Model 12 to illustrate the inverse relationship between $f_i$ and the $OB_c$. Table 10.1 indicates that the values of the $OB_c$ at the three levels of $f_i$ are significantly
different from each other. Thus, the effect of the fixed cost was determined to be significant:

![Impact of Fixed Cost](image)

**Figure 10.1 – Impact of Fixed Cost**

**Table 10.1 – ANOVA for Fixed Cost**

<table>
<thead>
<tr>
<th>Source</th>
<th>DF</th>
<th>SS</th>
<th>MS</th>
<th>F</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fixed Cost</td>
<td>2</td>
<td>74693.4</td>
<td>37346.7</td>
<td>608.12</td>
<td>0.000</td>
</tr>
<tr>
<td>Runs</td>
<td>3</td>
<td>343843.6</td>
<td>114614.5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Error</td>
<td>6</td>
<td>368.5</td>
<td>61.4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>11</td>
<td>418905.5</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
10.2. Variable Cost

The variable cost \((r_y)\) of supplying a unit at \(SP_i\) and shipping it to \(DC_j\) was studied at three levels: $0.50, $1.50, and $2.50 per unit. The variable cost was inversely proportional to the value of the \(OB_C\) (i.e. the higher the variable cost, the lower the objective function). The same pattern of relationship was common for all eight models. Figure 10.2 presents Model 12 to illustrate the inverse relationship between the \(r_y\) and the \(OB_C\). Table 10.2 indicates that the values of the \(OB_C\) at the three levels of \(r_y\) are significantly different from each other. Thus, the effect of the variable cost was determined to be significant.

![Impact of Variable Cost](image)

Figure 10.2 – Impact of Variable Cost
10.3. Inventory Cost

The inventory cost \((h_j)\) was studied by generating four different LA scenarios of the inventory models (Models 9, 10, 11, and 12) at two levels: $0.75 and $1.50 per unit. The third level of zero holding cost ($0.00) represented the non-inventory models (Models 5, 6, 7, and 8). Inventory cost was inversely proportional to the value of the \(OB_C\) (i.e. the higher the inventory cost, the lower the objective function). The same pattern of relationship was common for all models. Figure 10.3 addresses the inverse relationship between the \(h_j\) and the \(OB_C\). Table 10.3 indicates that the values of the \(OB_C\) at the three levels of \(h_j\) are significantly different (0.01 level of significance) from each other.
Impact of Inventory Cost

Figure 10.3 – Impact of Inventory Cost

Table 10.3 – ANOVA for Inventory Cost

<table>
<thead>
<tr>
<th>Source</th>
<th>DF</th>
<th>SS</th>
<th>MS</th>
<th>F</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inventory</td>
<td>2</td>
<td>300.1</td>
<td>150.1</td>
<td>11.33</td>
<td>0.009</td>
</tr>
<tr>
<td>Runs</td>
<td>3</td>
<td>143169.5</td>
<td>47723.2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Error</td>
<td>6</td>
<td>79.5</td>
<td>13.2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>11</td>
<td>143549.1</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

10.4. Stock-out Cost

The stock-out cost \( (s_f) \) was studied at three levels: $0.00, $0.50, and $1.00 per unit. The first level, zero ($0.00) stock-out cost, represented the non-inventory models (Models 5, 6, 7, and 8). Stock-out cost was inversely proportional to the value
of the $OB_C$ (i.e. the higher the stock-out cost, the lower the objective function). The same pattern of relationship was common for all models. Figure 10.4 addresses the inverse relationship between the $s_j$ and the $OB_C$. Table 10.4 indicates that the values of the $OB_C$ at the three levels of $s_j$ are not significantly different (0.1 level of significance) from each other. However, the $s_j$ was expected to be significantly different for a wider range of values, as the pattern of its impact was consistently downward.

![Impact of Stock-out Cost](image)

Figure 10.4 – Impact of Stock-out Cost
Table 10.4 – ANOVA for Stock-out Cost

<table>
<thead>
<tr>
<th>Source</th>
<th>DF</th>
<th>SS</th>
<th>MS</th>
<th>F</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stock-out</td>
<td>2</td>
<td>2873</td>
<td>1437</td>
<td>3.06</td>
<td>0.121</td>
</tr>
<tr>
<td>Runs</td>
<td>3</td>
<td>154526</td>
<td>51509</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Error</td>
<td>6</td>
<td>2817</td>
<td>470</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>11</td>
<td>160217</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

10.5. SP1 Capacity

The SP1 capacity ($A_i$) was studied at three levels: 100, 150, and 200 units for capacitated facilities models (Models 6, 8, 10, and 12). The capacity was directly proportional to the value of the $OB_C$ (i.e. the larger the capacity, the higher the objective function). The same pattern of relationship was common for all models. Figure 10.5 addresses the relationship between the $A_i$ and the $OB_C$. However, the impact of the capacity was dependent on the demand requirements and the profitability of sales of the extra units. For instance, scenarios one and three in Figure 10.5 did not increase the $OB_C$ value when capacity increased from 150 units to 200 units because the demand requirements for these scenarios were almost optimized, according the maximum profit. Thus, the demands were satisfied, and the extra units of capacity were almost unused. Table 10.5 indicates that the values of the $OB_C$ at the three levels of $A_i$ are significantly different (0.05 level of significance) from each other.
10.6. Price and Profitability

For models of insensitive prices (Models 5, 6, 9, and 10), price had a constant value drawn from a uniform distribution $[3, 6]$. On the other hand, for models with sensitive prices (Models 7, 8, 11, and 12), the price was equal to a function of the profitability constant and demand mean. Thus, subsection 10.6.1 addresses the impact...
of price for models of insensitive prices, and subsection 10.6.2 addresses the impact of the profitability constant for models of sensitive prices.

10.6.1. Price

The price was studied by generating four different LA scenarios of every model of insensitive price (Models 5, 6, 9, and 10) at three levels of $3.00, $4.50, and $6.00 per unit. Price was directly proportional to the value of the $OB_C$ (i.e. the higher the price, the higher the objective function). The same pattern of relationship was common for all four models. Figure 10.6 presents Model 12 to illustrate the positive relationship. Table 10.6 indicates that the values of the $OB_C$ at the three levels of price are significantly different from each other. Thus, the effect of the price was of significant impact on the value of the total net profit for insensitive price LA models.

![Impact of Price](image)

Figure 10.6 – Impact of Price
10.6.2. Profitability Constant

The Profitability Constant was studied by generating four different LA scenarios of every model of sensitive pricing (Models 7, 8, 11, and 12) at three profitability constant values of $600, $700, and $800. The Profitability Constant was directly proportional to the value of the $OB_c$ (i.e. the higher the Profitability Constant, the higher the objective function). The same pattern of relationship was common for all four models. Figure 10.7 presents Model 12 to illustrate the positive relationship. Table 10.7 indicates that the values of the $OB_c$ at the three levels of Profitability Constant are significantly different from each other. Thus, the effect of the Profitability Constant was significant.
Figure 10.7 – Impact of Profitability Constant

Table 10.7 – ANOVA for Profitability Constant

<table>
<thead>
<tr>
<th>Source</th>
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<th>SS</th>
<th>MS</th>
<th>F</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>Profitability</td>
<td>2</td>
<td>406632</td>
<td>203316</td>
<td>167.20</td>
<td>0.000</td>
</tr>
<tr>
<td>Runs</td>
<td>3</td>
<td>212303</td>
<td>70768</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Error</td>
<td>6</td>
<td>7296</td>
<td>1216</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>11</td>
<td>626232</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
10.7. Demand Mean

The demand mean ($\mu_j$) was studied at three levels of 100, 150, and 200 units for all models. The demand mean had a different impact on the price sensitive models than on the models of insensitive prices. In the insensitive price models, the demand mean had a positive impact on the $OB_C$ (i.e. the higher the demand mean, the higher the objective function). Conversely, the demand mean had a negative impact on the $OB_C$ for models of sensitive prices. It was suspected that limited capacities might play a critical role for this unexpected trend of mean impact. However, Figure 10.8 proved the opposite. The limited capacities were not the reason for the trend. The same relationship was addressed in the Two-Phase and Linear Approximation Methods. Thus, this trend was not a cause or function of the Combined Method formulation or capacitated facilities models.

Hence, the one plausible reason for this negative relationship between the demand mean and $OB_C$ in price sensitive models was the elasticity of demand formula. The elasticity of demand formula is:

$$p_j = \frac{\text{Profitability Constant}}{\mu_j}$$

It was expected that, according to the elasticity of demand concept, the level of $OB_C$ would stay constant for different demand mean levels. However, it turned out that the impact of the formula on the LA problems was negative instead of the supposed constant.
10.8. Demand Standard Deviation

The demand standard deviation ($\sigma_j$) was studied at three levels of 5, 15, and 25 units. It was found that the standard deviation had a more consistent impact on the uncapacitated facilities models than on the capacitated facilities models. However, the general trend of impact was inversely proportional to the value of the $OB_C$ (i.e. the higher the standard deviation, the lower the objective function). Figure 10.9 addresses the inverse relationship between the demand standard deviation and the $OB_C$ for Model 12. Table 10.8 indicates that the values of the $OB_C$ at the three levels of $\sigma_j$ are significantly different from each other at every level of significance.
Table 10.8 – ANOVA for Demand Standard Deviation

<table>
<thead>
<tr>
<th>Source of Variation</th>
<th>DF</th>
<th>SS</th>
<th>MS</th>
<th>F</th>
<th>P-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard Deviation</td>
<td>2</td>
<td>826304.890</td>
<td>413152.44</td>
<td>164.80</td>
<td>0.00</td>
</tr>
<tr>
<td>Runs</td>
<td>4</td>
<td>75306.739</td>
<td>18826.68</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Error</td>
<td>8</td>
<td>20055.568</td>
<td>2506.95</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>14</td>
<td>921667.197</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

10.9. Summary

Table 10.9 summarizes the impact of each studied parameter on the value of the objective function. The table states the $OB_C$ value with a unit increase in the value of a specific parameter. However, evaluating the value of the $OB_C$ according to the parameters' increase would not be sufficient, as many parameters interact and affect
each other. Equation 10.1 states a hypothetical regression model of the $OB_C$, where $\beta_i$ represents the slope associated with each parameter. In other words, the slope is the rate of unit change of the objective function for a unit change of the parameter. The parameters' rates were roughly determined from the average value of the average slopes of all the parameter charts. The slopes were listed in descending order in Table 10.9 as follows. As the interaction terms were not studied they were chopped off, and Equation 10.1 simplified into Equation 10.2. Thus, a unit change in any parameter would lead to a $\beta_i$ change in the $OB_C$ value. However, this is a hypothetical situation, as every change in any parameter has an effect on the rest of the parameter and may change the basic solution. Therefore, solving the Combined Method formulation is the best way to determine the new value of the $OB_C$.

$$OB_C = \beta_0 + \beta_1 f_i + \beta_2 r_q + \beta_3 h_j + \beta_4 s_j + \beta_5 A_i + \beta_6 p_j I + \beta_7 Const_j (1 - I) + \beta_8 \mu_j I$$

$$+ \beta_9 \mu_j (1 - I) + \beta_{10} \sigma_j + \text{Interactions}$$  \hspace{1cm} [10.1]

Where

$$I = \begin{cases} 1 & \text{Insensitive Prices} \\ 0 & \text{Sensitive Prices} \end{cases}$$

$$OB_C = \beta_0 + \beta_1 f_i + \beta_2 r_q + \beta_3 h_j + \beta_4 s_j + \beta_5 A_i + \beta_6 p_j I + \beta_7 Const_j (1 - I) + \beta_8 \mu_j I$$

$$OB_C = \beta_0 + \beta_1 f_i + \beta_2 r_q + \beta_3 h_j + \beta_4 s_j + \beta_5 A_i + \beta_6 p_j I + \beta_7 Const_j (1 - I) + \beta_8 \mu_j I$$  \hspace{1cm} [10.2]

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Table 10.10 summarizes the percentage change in the \( OB_C \) value for one percent change in a specific parameter. The absolute values of the parameters' impacts were listed in descending order to address the most influential parameter of the model. The \( SP_i \) capacity was found to have the highest impact on the value of the objective function. The least impact on the value of the objective function was associated with the inventory cost. However, the inventory cost was determined to have a significant, but not minor, impact on the objective function. Thus, a one percent increase in inventory cost would yield a 0.02 percent reduction in the objective function value. The values listed in Tables 10.9 and 10.10 are credible for the tested LA scenarios only, and they were determined to roughly compare the impact of parameters on the \( OB_C \) value.

Table 10.9 – Parameter Impact Rate

<table>
<thead>
<tr>
<th>1 unit change in Parameter</th>
<th>Units change in ( OB_C )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Price</td>
<td>902.52</td>
</tr>
<tr>
<td>Capacity</td>
<td>41</td>
</tr>
<tr>
<td>Demand Mean of (Insensitive Price) models</td>
<td>34.51</td>
</tr>
<tr>
<td>Profitability Constant</td>
<td>0.98</td>
</tr>
<tr>
<td>Fixed Cost</td>
<td>-0.70</td>
</tr>
<tr>
<td>Demand Standard Deviation</td>
<td>-3.70</td>
</tr>
</tbody>
</table>

Continued...
### Table 10.9 continued…

| Demand Mean of (Insensitive Price) models | -4.61 |
| Holding Cost | -8.20 |
| Stock-out Cost | -70.00 |
| Variable Cost | -195.00 |

### Table 10.10 – Percentage Impact of the Parameters

<table>
<thead>
<tr>
<th>1% Change of Parameter</th>
<th>% Change $OB_C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Capacity</td>
<td>24.02</td>
</tr>
<tr>
<td>Demand Mean of (Insensitive Price) models</td>
<td>7.92</td>
</tr>
<tr>
<td>Price</td>
<td>5.79</td>
</tr>
<tr>
<td>Profitability Constant</td>
<td>1.05</td>
</tr>
<tr>
<td>Variable Cost</td>
<td>-0.97</td>
</tr>
<tr>
<td>Demand Mean of (Insensitive Price) models</td>
<td>-0.46</td>
</tr>
<tr>
<td>Fixed Cost</td>
<td>-0.24</td>
</tr>
<tr>
<td>Stock-out Cost</td>
<td>-0.08</td>
</tr>
<tr>
<td>Demand Standard Deviation</td>
<td>-0.07</td>
</tr>
<tr>
<td>Holding Cost</td>
<td>-0.02</td>
</tr>
</tbody>
</table>
CHAPTER 11
CONCLUSION AND FUTURE DIRECTIONS

The purpose of this study was to develop a quantitative algorithm that solves the facility location-allocation (LA) problem in order to maximize the net profits generated from expected sales. The proposed formulation was supposed to take into consideration the multiple dimensions of the LA problem, which are uncapacitated/capacitated facilities, stochastic demands, insensitive/sensitive prices, inventory cost, and stock-out cost.

Four new models of the LA problem with stochastic demands, inventory cost, and stock-out cost were developed and studied. The four new models were combinations of capacitated/uncapacitated facilities and sensitive/insensitive pricing. Three new methods were developed to solve the LA problem. Moreover, an existing Two-Phase Method was modified to incorporate inventory and to cope with the scope of the study. The newly developed methods were: the Combined Method and the Linear Approximation Method. These methods were benchmarked against the modified Two-Phase Method.

The number of binary variables in the Linear Approximation Method and the Combined Method was \((m(n-1))\) less than in the Two-Phase Method. Moreover, the Linear Approximation Method and the Combined Method allowed multiple sourcing of shipments, while the Two-Phase Method was restricted to single sourcing. The Combined Method was capable of optimizing small LA problems only. Due to the
complexity of MINLP formulation, the Combined Method could not return any realistic solutions for large problems. The Linear Approximation Method was able to optimize the LA models of uncapacitated facilities regardless of the problem's size. Additionally, the Linear Approximation Method yielded a good solution for small problems of capacitated facilities LA models. However, for large problem capacitated facilities LA models, the Linear Approximation Method gave the best solution among all other methods. In fact, the Linear Approximation Method could solve more than ten times the size of the Two-Phase Method before it reached the NEOS server's 10,000 iterations limit. The modified Two-Phase Method optimized only the small uncapacitated facilities LA models and gave a lower bound (LB) on the rest of the problem types. The Heuristic was developed to improve the solution of the Two-Phase Method for capacitated facilities LA models only. Thus, the Heuristic algorithm was limited to assigning the leftover capacities of already opened facilities rather than opening new facilities. Therefore, the Heuristic Method was not investigated widely in this study. Table 11.1 summarizes the solving capabilities of the methods as follows.

Inventory and stock-out costs had a critical impact on the LA basic solution in terms of the objective function's value and shipping arrangements. The inventory impact was found to be significant, which sharply lowered the expected profit level and shipping arrangements. However, the inventory cost parameter had the least impact on the value of the total expected profit. On the other hand, shipping plant
capacity was found to be the most influential parameter, followed by the demand mean and price for LA models of insensitive price.

Table 11.1 – Solving Capabilities of All Methods

<table>
<thead>
<tr>
<th>Method</th>
<th>Uncapacitated Facilities</th>
<th>Capacitated Facilities</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Small Size</td>
<td>Large Size</td>
</tr>
<tr>
<td>Combined</td>
<td>Optimal</td>
<td>N/A</td>
</tr>
<tr>
<td>Linear Approximation</td>
<td>Optimal</td>
<td>Optimal</td>
</tr>
<tr>
<td>Two-Phase</td>
<td>Optimal</td>
<td>LB</td>
</tr>
<tr>
<td>Heuristic</td>
<td>-----</td>
<td>-----</td>
</tr>
</tbody>
</table>

The Combined Method was limited to small problems for the SBB solver's capabilities. The Linear Approximation Method was tested only on uniformly distributed demands. Thus, implementing the Linear Approximation Method on a different demand pdf would be a logical direction for future work. Studying the impact of economies of scale on the LA problem would be another direction for future study. And finally, the proposed Heuristic Method needs to be improved (to open a closed SP,) and statistically tested (to be compared against other available methods).

The proposed methods would fit reality if the LA problem was formulated for multi-commodity supply chain and multi-period allocations. Moreover, testing the
method's performance with actual data sets drawn from real-life practice would enhance the evaluation of the actual performance of the proposed methods.
REFERENCES


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Retrieved from: www.elsevier.com/orms


APPENDIX A

Derivation of Critical Probability
Derivation of Critical Probability

Expected gain from shipping an extra unit = expected loss from an extra unit

\[(p_j - r_y)[1 - F(q_j)] = r_y F(q_j)\]

\[(p_j - r_y) - (p_j - r_y) F(q_j) = r_y F(q_j)\]

\[p_j - r_y = (p_j - r_y) F(q_j) + r_y F(q_j)\]

\[p_j - r_y = (p_j - r_y + r_y) F(q_j)\]

\[p_j - r_y = p_j F(q_j)\]

\[F(q_j) = \frac{p_j - r_y}{p_j}\]

\[P_{ij}^c = F(q_j) = \frac{p_j - r_y}{p_j}\]
APPENDIX B

Derivation of Phase-I Formula
Derivation of Phase-I Formula

Net Profit per unit = Revenue per unit – cost per unit

Cost per unit = \( r_{ij} x_{ij} \)

Revenue per unit = \( E(q_j p_j | q_j < x_q) P(q_j < x_q) + x_{ij} p_j P(q_j \geq x_q) \)

\[
\begin{align*}
\text{Revenue per unit} &= p_j \int_{\text{LB}_j}^{\text{UB}_j} q_j f(q_j) dq_j + p_j x_{ij} \int_{x_q}^{\text{UB}_j} f(q_j) dq_j \\
&= p_j \int_{\text{LB}_j}^{\text{UB}_j} q_j f(q_j) dq_j + p_j x_{ij} \int_{x_q}^{\text{UB}_j} f(q_j) dq_j - p_j \int_{x_q}^{\text{UB}_j} q_j f(q_j) dq_j
\end{align*}
\]

\[
\Rightarrow \mu_j = \int_{\text{LB}_j}^{\text{UB}_j} q_j f(q_j) dq_j = \int_{\text{LB}_j}^{\text{UB}_j} q_j f(q_j) dq_j + \int_{x_q}^{\text{UB}_j} q_j f(q_j) dq_j
\]

\[
\therefore \text{Revenue per unit} = p_j \mu_j - p_j \int_{x_q}^{\text{UB}_j} q_j f(q_j) dq_j + p_j x_{ij} \int_{x_q}^{\text{UB}_j} f(q_j) dq_j \\
= p_j \mu_j - p_j \int_{x_q}^{\text{UB}_j} (q_j - x_q) f(q_j) dq_j
\]

\[
L(x_q) = p_j \mu_q - p_j \int_{x_q}^{\text{UB}_q} (q_j - x_q) f(q_j) dq_j - r_q x_q
\]

**Hint:** This formula is derived and used in Legondran (1991).
APPENDIX C

Simplified Formula of Phase-I
Simplified Formula of Phase-I

\[
\therefore \int_{x_y}^{UB_{x_j}} f(q_j) dq_j = \frac{r_y}{p_j}
\]

\[
L(x_y^*) = p_j \mu_{x_y} - p_j \int_{x_y}^{UB_{x_y}} (q_j - x_y^*) f(q_j) dq_j - r_y x_y^*
\]

\[
= p_j \int_{LB_j}^{x_y} q_j f(q_j) dq_j + p_j x_y^* \int_{x_y}^{UB_{x_y}} f(q_j) dq_j - p_j \int_{x_y}^{UB_{x_y}} q_j f(q_j) dq_j - r_y x_y^*
\]

\[
= p_j \int_{LB_j}^{x_y} q_j f(q_j) dq_j + p_j x_y^* \left( \frac{r_y}{p_j} \right) - r_y x_y^*
\]

\[
= p_j \int_{LB_j}^{x_y} q_j f(q_j) dq_j
\]

\[
\therefore L(x_y^*) = p_j \int_{LB_j}^{x_y} q_j f(q_j) dq_j
\]
APPENDIX D

GAMS File for Example 4.1A
SETS
   I   plants   /SP1, SP2/
   J   demand centers   /DC1, DC2, DC3/ ;

PARAMETERS
   A(I)   capacity of SPi
          / SP1  233
          SP2  172 /
   F(I)   fixed cost of SPi
          / SPI  270
          SP2  250 /
   MU(J)   Mean of the demands at DCj
          / DC1  130
          DC2  176.5
          DC3  104.5 /
   SD(J)   standard deviation of the demands at DCj
          / DC1  23.094
          DC2  18.764
          DC3  24.537 / ;

TABLE   R(i,j) the variable cost per unit shipped from i to j
          DC1  DC2  DC3
   SPI  2.05  2.13  1.33
   SP2  0.53  1.92  2.23 ;

PARAMETERS   AU(j)   lower limit of the uniform demands at DCj ;
   AU(j) = MU(j) - SD(J)*SQRT(3) ;

PARAMETERS   BU(j)   upper limit of the uniform demands at DCj ;
   BU(j) = MU(j) + SD(J)*SQRT(3) ;

PARAMETERS   P(j)   price per unit sold at DCj ;
   P(j) = 4 ;

PARAMETERS   K(j)   first constant ;
   K(j) = 0.5*P(j)/(BU(J)-AU(J)) ;

PARAMETERS   XS(i,j)   Optimal shipment size based on independent assignment
from i to j ;

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PARAMETERS \( X_U(j) \) upper bound at demand \( j \);
\[
X_U(j) = \text{smax}(I, X_S(I, J)) \;
\]

PARAMETERS \( L(i, j) \) profits
\[
L(i, j) = K(j) * (SQR(X_S(i, j)) - SQR(AU(j))) \;
\]

VARIABLES
\( Y(I) \) equals to 1 if the plant is utilized
\( Z(I, J) \) equals to 1 if there is a shipment from plant \( i \) to demand \( j \)
\( OB \) total transportation profits;

BINARY VARIABLES
\( Y(I) \)
\( Z(I, J) \);

EQUATIONS
\( \text{PROFIT} \) objective function
\( \text{SUPPLY}(I) \) observe supply limit at plant \( i \)
\( \text{DEMAND}(J) \) limit demand \( j \) to receive from only 1 plant;

\( \text{PROFIT} \) .. \( OB = \text{E=} \sum(I, J, L(I, J) * Z(I, J)) - \sum(I, F(I) * Y(I)) \);
\( \text{SUPPLY}(I) \) .. \( \sum(J, X_S(I, J) * Z(I, J) = L= A(I) * Y(I)) \);
\( \text{DEMAND}(J) \) .. \( \sum(I, Z(I, J)) = L= 1 \);

MODEL EXAMPLE41a /ALL/;
SOLVE EXAMPLE41a USING MIP MAXIMIZING OB;
APPENDIX E

GAMS File for Example 4.1B
SETS
  I   plants /SPI, SP2/
  J   demand centers /DC1, DC2, DC3/ ;

PARAMETERS
  A(I)  capacity of SPi
        / SP1 10000
        SP2 10000 /
  F(I)  fixed cost of SPi
        / SP1 270
        SP2 250 /
  MU(J) Mean of the demands at DCj
        / DC1 130
        DC2 176.5
        DC3 104.5 /
  SD(J) standard deviation of the demands at DCj
        / DC1 23.094
        DC2 18.764
        DC3 24.537 / ;

TABLE  R(i,j) the variable cost per unit shipped from i to j
        DC1  DC2  DC3
   SPI 2.05  2.13  1.33
   SP2 0.53  1.92  2.23 ;

PARAMETERS  AU(J) lower limit of the uniform demands at DCj ;
    AU(J) = MU(J)- SD(J)*SQRT(3) ;

PARAMETERS  BU(J) upper limit of the uniform demands at DCj ;
    BU(J) = MU(J)+ SD(J)*SQRT(3) ;

PARAMETERS  P(j) price per unit sold at DCj ;
    P(j) = 4 ;

PARAMETERS  K(J) first constant ;
    K(J) = 0.5*P(j)/(BU(J)-AU(J)) ;

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PARAMETERS XS(i,j) Optimal shipment size based on independent assignment from i to j ;
XS(i,j) = BU(j)- R(i,j)/P(j)*(BU(J)-AU(j)) ;

PARAMETERS XU(j) upper bound at demand j ;
XU(j) = smax((I),XS(I,J)) ;

PARAMETERS L(i,j) profits ;
L(i,j) = K(j)*(SQR(XS(i,j))-SQR(AU(j))) ;

VARIABLES
Y(I) equals to 1 if the plant is utilized
Z(I,J) equals to 1 if there is a shipment from plant i to demand j
OB total transportation profits ;

BINARY VARIABLES
Y(I)
Z(I,J) ;

EQUATIONS
PROFIT objective function
SUPPLY(I) observe supply limit at plant i
DEMAND(J) limit demand j to receive from only 1 plant ;

PROFIT .. OB =E= SUM((I,J), L(I,J)*Z(I,J))-SUM((I),F(I)*Y(I)) ;
SUPPLY(I) .. SUM((J), XS(I,J)*Z(I,J)) =L= A(I)*Y(I) ;
DEMAND(J) .. SUM((I), Z(I,J)) =L= 1 ;

MODEL EXAMPLE41b /ALL/ ;
SOLVE EXAMPLE41b USING MIP MAXIMIZING OB ;
APPENDIX F

GAMS File for Example 5.1A
SETS
   I   plants /SP1, SP2/
   J   demand centers /DC1, DC2, DC3/ ;

PARAMETERS
   A(I)  capacity of SPi
         /SP1  233
         SP2  172 /
   F(I)  fixed cost of SPi
         /SP1  270
         SP2  250 /
   MU(J) Mean of the demands at DCj
         /DC1  130
         DC2  176.5
         DC3  104.5 /
   SD(J) standard deviation of the demands at DCj
         /DC1  23.094
         DC2  18.764
         DC3  24.537 / ;

TABLE  R(i,j) the variable cost per unit shipped from i to j
        DC1  DC2  DC3
   SP1  2.05  2.13  1.33
   SP2  0.53  1.92  2.23 ;

PARAMETERS  AU(j) lower limit of the uniform demands at DCj
             AU(j) = MU(j) - SD(J)*SQRT(3) ;

PARAMETERS  BU(j) upper limit of the uniform demands at DCj
             BU(j) = MU(j) + SD(J)*SQRT(3) ;

PARAMETERS  P(j) price per unit sold at DC j
             P(j) = 4 ;

PARAMETERS  K(j) constant
             K(j) = 0.5*P(j)/(BU(J)-AU(J)) ;
PARAMETERS  
XS(i,j) Optimal shipment size based on independent assignment from i to j 
XS(i,j) = BU(j) - R(i,j)/P(j)*(BU(j)-AU(j)) ;

PARAMETERS  
XU(j) upper bound at demand j 
XU(j) = smax((I),XS(I,J)) 

VARIABLES  
Y(I) equals to 1 if the plant is utilized  
Z(J) equals 1 if there is a shipment from plant i to demand j  
X(I,J) number of optimal units shipped from plant i to demand j  
OB total transportation profits ;

POSITIVE VARIABLES  
X(I,J) ;

BINARY VARIABLES  
Y(I)  
Z(J) ;

EQUATIONS  
objective function  
SUPPLY(I) observe supply limit at plant i  
DEMAND(J) observe demand upper bound limit at demand j ;

PROFIT..  
OB =E= SUM((J), K(J)*2*BU(j)*SUM((i),X(I,J)))  
- SUM((J), K(J)*SQR(SUM((i),X(I,J))))  
- SUM((J), K(J)*Z(J)*SQR(AU(j)))  
- SUM((I,J), R(I,J)*X(I,J))-SUM((I),F(I)*Y(I)) ;

SUPPLY(I)..  
SUM((J), X(I,J)) =L= A(I)*Y(I) 

DEMAND(J)..  
SUM((I), X(I,J)) =L= XU(j)*Z(J) ;

MODEL EXAMPLE51a /ALL/ ;
SOLVE EXAMPLE51a USING MIQCP MAXIMIZING OB ;

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APPENDIX G

GAMS File for Example 5.1B
SETS
I plants /SP1, SP2/
J demand centers /DC1, DC2, DC3/ ;

PARAMETERS
A(I) capacity of SPi
/ SP1 10000
SP2 10000 /

F(I) fixed cost of SPi
/ SP1 270
SP2 250 /

MU(J) Mean of the demands at DCj
/ DC1 130
DC2 176.5
DC3 104.5 /

SD(J) standard deviation of the demands at DCj
/ DC1 23.094
DC2 18.764
DC3 24.537 / ;

TABLE R(i,j) the variable cost per unit shipped from i to j
DC1 DC2 DC3
SP1 2.05 2.13 1.33
SP2 0.53 1.92 2.23 ;

PARAMETERS AU(j) lower limit of the uniform demands at DCj ;
AU(j) = MU(j)- SD(J)*SQRT(3) ;

PARAMETERS BU(j) upper limit of the uniform demands at DCj ;
BU(j) = MU(j)+ SD(J)*SQRT(3) ;

PARAMETERS P(j) price per unit sold at DC j ;
P(j) = 4 ;

PARAMETERS K(j) constant ;
K(j) = 0.5*P(j)/(BU(J)-AU(J)) ;

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PARAMETERS XS(i,j) Optimal shipment size based on independent assignment from i to j;
XS(i,j) = BU(j)- R(i,j)/P(j)*(BU(j)-AU(J));

PARAMETERS XU(j) upper bound at demand j;
XU(j) = smax((I),XS(I,J))

VARIABLES
Y(I) equals to 1 if the plant is utilized
Z(J) equals 1 if there is a shipment from plant i to demand j
X(I,J) number of optimal units shipped from plant i to demand j
OB total transportation profits;

POSITIVE VARIABLES
X(I,J);

BINARY VARIABLES
Y(I)
Z(J);

EQUATIONS
PROFIT objective function
SUPPLY(I) observe supply limit at plant i
DEMAND(J) observe demand upper bound limit at demand j;

PROFIT.. OB =E= SUM((J), K(J)*2*BU(j)*SUM((i),X(I,J)))
- SUM((I), K(J)*SQR(SUM((i),X(I,J))))
- SUM((I,J), K(J)*Z(J)*SQR(AU(j)))
- SUM((I,J), R(I,J)*X(I,J))-SUM((I),F(I)*Y(I));
SUPPLY(I).. SUM((I), X(I,J)) =L= A(I)*Y(I);
DEMAND(J).. SUM((I), X(I,J)) =L= XU(J)*Z(J);

MODEL EXAMPLE51b /ALL/;
SOLVE EXAMPLE51b USING MIQCP MAXIMIZING OB;
APPENDIX H

GAMS File for Example 6.1A
SETS
  I  plants /SPI, SP2/
  J  demand centers /DC1, DC2, DC3/ ;

PARAMETERS
  A(I)  capacity of SPI
       /SPI 233
       SP2 172 /
  F(I)  fixed cost of SPI
       /SPI 270
       SP2 250 /
  MU(J)  Mean of the demands at DCj
         /DC1 130
         DC2 176.5
         DC3 104.5 /
  SD(J)  standard deviation of the demands at DCj
         /DC1 23.094
         DC2 18.764
         DC3 24.537 /

TABLE  R(i,j)  the variable cost per unit shipped from i to j
       DC1 DC2 DC3
  SPI  2.05 2.13 1.33
  SP2  0.53 1.92 2.23 ;

PARAMETERS  AU(J)  lower limit of the uniform demands of DCJ ;
             AU(J) = MU(J)- SD(J)*SQRT(3) ;

PARAMETERS  BU(J)  upper limit of the uniform demands of DCJ ;
             BU(J) = MU(J)+ SD(J)*SQRT(3) ;

PARAMETERS  P(J)  price per unit sold at DCJ ;
             P(J) = 600/MU(J) ;

PARAMETERS  K1(J)  first constant ;
             K1(J) = -0.5*P(J)/(BU(J)-AU(J)) ;
PARAMETERS XS(i,j) Optimal shipment size based on independent assignment from i to j;
XS(i,j) = BU(j)- R(i,j)/P(j)*(BU(J)-AU(J));

PARAMETERS XU(j) upper bound at demand j;
XU(j) = smax((I),XS(I,J));

PARAMETERS L(i,j) profits;
L(i,j) = K1(j)*(SQR(AU(j))-SQR(XS(i,j)));

VARIABLES Y(I) equals to 1 if the plant is utilized
Z(I,J) equals to 1 if there is a shipment from plant i to demand j
OB total transportation profits;

BINARY VARIABLES Y(I)
Z(I,J);

EQUATIONS PROFIT objective function
SUPPLY(I) observe supply limit at plant i
DEMAND(J) limit demand j to receive from only 1 plant;

PROFIT .. OB =E= SUM((I,J), L(I,J)*Z(I,J))-SUM((I),F(I)*Y(I));
SUPPLY(I) .. SUM((J), XS(I,J)*Z(I,J)) =L= A(I)*Y(I);
DEMAND(J) .. SUM((I), Z(I,J)) =L= 1;

MODEL eg61a /ALL/;
SOLVE eg61a USING MIP MAXIMIZING OB;
APPENDIX I

GAMS File for Example 6.1B
SETS
  I  plants  /SPI, SP2/
  J  demand centers  /DC1, DC2, DC3/ ;

PARAMETERS
  A(I)  capacity of SPi
       / SPI  10000
       SP2  10000 /
  F(I)  fixed cost of SPi
       / SPI  270
       SP2  250 /
  MU(J)  Mean of the demands at DCj
         / DC1  130
         DC2  176.5
         DC3  104.5 /
  SD(J)  standard deviation of the demands at DCj
         / DC1  23.094
         DC2  18.764
         DC3  24.537 / ;

TABLE  R(i,j)  the variable cost per unit shipped from i to j

  DC1  DC2  DC3
SPI   2.05  2.13  1.33
SP2   0.53  1.92  2.23 ;

PARAMETERS  AU(j)  lower limit of the uniform demands of DCj ;
  AU(j) = MU(j) - SD(J)*SQRT(3) ;

PARAMETERS  BU(j)  upper limit of the uniform demands of DCj ;
  BU(j) = MU(j) + SD(J)*SQRT(3) ;

PARAMETERS  P(j)  price per unit sold at DCj ;
  P(j) = 600/MU(j) ;

PARAMETERS  K1(j)  first constant ;
  K1(j) = -0.5*P(j)/(BU(J)-AU(J)) ;
PARAMETERS  XS(i,j) Optimal shipment size based on independent assignment from i to j ;
XS(i,j) = BU(j)- R(i,j)/P(j)*(BU(J)-AU(J)) ;

PARAMETERS  XU(j) upper bound at demand j ;
XU(j) = smax((i),XS(I,J)) ;

PARAMETERS  L(i,j) profits ;
L(i,j) = K1(j)*SQR(AU(j))-SQR(XS(i,j)) ;

VARIABLES
  Y(I) equals to 1 if the plant is utilized
  Z(I,J) equals to 1 if there is a shipment from plant i to demand j
  OB total transportation profits ;

BINARY VARIABLES
  Y(I)
  Z(I,J) ;

EQUATIONS
  PROFIT objective function
  SUPPLY(I) observe supply limit at plant i
  DEMAND(J) limit demand j to receive from only 1 plant ;

PROFIT ..  OB =E= SUM((I,J), L(I,J)*Z(I,J))-SUM((I),F(I)*Y(I)) ;
SUPPLY(I) ..  SUM((J), XS(I,J)*Z(I,J)) =L= A(I)*Y(I) ;
DEMAND(J) ..  SUM((I), Z(I,J)) =L= 1 ;

MODEL  eg61a /ALL/ ;
SOLVE  eg61a USING MIP MAXIMIZING OB ;
APPENDIX J

GAMS File for Example 6.2A
SETS
   I  plants /SP1, SP2/
   J  demand centers /DC1, DC2, DC3/ ;

PARAMETERS
   A(I) capacity of SPi
       / SP1 233
           SP2 172 /
   F(I) fixed cost of SPi
       / SP1 270
           SP2 250 /
   MU(J) Mean of the demands at DCj
       / DC1 130
           DC2 176.5
           DC3 104.5 /
   SD(J) standard deviation of the demands at DCj
       / DC1 23.094
           DC2 18.764
           DC3 24.537 / ;

TABLE  R(i,j) the variable cost per unit shipped from i to j
       DC1  DC2  DC3
   SP1  2.05  2.13  1.33
   SP2  0.53  1.92  2.23 ;

PARAMETERS  AU(j) lower limit of the uniform demands of demand center j ;
             AU(j) = MU(j)- SIG(J)*SQRT(3) ;

PARAMETERS  BU(j) lower limit of the uniform demands of demand center j ;
             BU(j) = MU(j)+ SIG(J)*SQRT(3) ;

PARAMETERS  P(j) price per unit sold at demand center j ;
             P(j) = 600/MU(J) ;

PARAMETERS  K(j) constant ;
             K(j) = 0.5*P(j)/(BU(J)-AU(J)) ;

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PARAMETERS

- **XS(i,j)** Optimal shipment size based on independent assignment from \(i\) to \(j\):
  
  \[ XS(i,j) = BU(j)- R(i,j)/P(j)*(BU(j)-AU(j)) \]

- **XU(j)** upper bound at demand \(j\):
  
  \[ XU(j) = \text{smax}(i), XS(i,j)) \]

VARIABLES

- **Y(I)** equals to 1 if the plant is utilized
- **Z(J)** equals 1 if there is a shipment from plant \(i\) to demand \(j\)
- **X(I,J)** number of optimal units shipped from plant \(i\) to demand \(j\)
- **OB** total transportation profits

POSITIVE VARIABLES

- **X(I,J)**

BINARY VARIABLES

- **Y(I)**
- **Z(J)**

EQUATIONS

- **PROFIT** objective function
- **SUPPLY(I)** observe supply limit at plant \(i\)
- **DEMAND(J)** observe demand upper bound limit at demand \(j\)

**PROFIT..**

\[
OB = \text{SUM}(J), K(J)*2*BU(j)*\text{SUM}(i), X(I,J)) - \text{SUM}(J), K(J)*\text{SQR}(\text{SUM}(i), X(I,J))) \\
- \text{SUM}(J), K(J)*Z(J)*\text{SQR}(AU(j))) \\
- \text{SUM}(J), R(I,J)*X(I,J)) - \text{SUM}(I), F(I)*Y(I))
\]

**SUPPLY(I)..**

\[ \text{SUM}(I,J), X(I,J)) = L = A(I)*Y(I) \]

**DEMAND(J)..**

\[ \text{SUM}(I,J), X(I,J)) = L = 10000*Z(J) \]

MODEL eg62a /ALL/;

SOLVE eg62a USING MIQCP MAXIMIZING OB;
APPENDIX K

GAMS File for Example 6.2B
SETS
   I   plants /SP1, SP2/ 
   J   demand centers /DC1, DC2, DC3/ 

PARAMETERS
   A(I)   capacity of plant i 
          / SP1 10000 
          SP2 10000 / 
   F(I)   fixed cost of SPi 
          / SP1 270 
          SP2 250 / 
   MU(J)  Mean of the demands at DCj 
          / DC1 130 
          DC2 176.5 
          DC3 104.5 / 
   SD(J)  standard deviation of the demands at DCj 
          / DC1 23.094 
          DC2 18.764 
          DC3 24.537 / 

TABLE R(i,j) the variable cost per unit shipped from i to j

<table>
<thead>
<tr>
<th></th>
<th>DC1</th>
<th>DC2</th>
<th>DC3</th>
</tr>
</thead>
<tbody>
<tr>
<td>SPI</td>
<td>2.05</td>
<td>2.13</td>
<td>1.33</td>
</tr>
<tr>
<td>SP2</td>
<td>0.53</td>
<td>1.92</td>
<td>2.23</td>
</tr>
</tbody>
</table>

PARAMETERS
   AU(j)  lower limit of the uniform demands of demand center j ;
   AU(j) = MU(j) - SIG(J)*SQRT(3) ;

PARAMETERS
   BU(j)  lower limit of the uniform demands of demand center j ;
   BU(j) = MU(j) + SIG(J)*SQRT(3) ;

PARAMETERS
   P(j)   price per unit sold at demand center j ;
   P(j) = 600/MU(J) ;

PARAMETERS
   K(j)   constant ;
   K(j) = 0.5*P(j)/(BU(J)-AU(J)) ;
PARAMETERS XS(i,j) Optimal shipment size based on independent assignment from i to j;
XS(i,j) = BU(j) - R(i,j)/P(j)*(AU(J) - BU(J));

PARAMETERS XU(j) upper bound at demand j;
XU(j) = smax((I),XS(I,J));

VARIABLES
Y(I) equals to 1 if the plant is utilized
Z(J) equals 1 if there is a shipment from plant i to demand j
X(I,J) number of optimal units shipped from plant i to demand j
OB total transportation profits;

POSITIVE VARIABLES
X(I,J);

BINARY VARIABLES
Y(I)
Z(J);

EQUATIONS
PROFIT objective function
SUPPLY(I) observe supply limit at plant i
DEMAND(J) observe demand upper bound limit at demand j;

PROFIT.. OB =E= SUM((J), K(J)*2*BU(j)*SUM((i),X(I,J)))
- SUM((J), K(J)*SQR(SUM((i),X(I,J))))
- SUM((J), K(J)*Z(J)*SQR(AU(j)))
- SUM((I,J), R(I,J)*X(I,J))-SUM((I),F(I)*Y(I));
SUPPLY(I).. SUM((J), X(I,J)) =L= A(I)*Y(I);
DEMAND(J).. SUM((I), X(I,J)) =L= 10000*Z(J);

MODEL eg62b /ALL/;
SOLVE eg62b USING MIQCP MAXIMIZING OB;
APPENDIX L

GAMS File for Example 7.1A
SETS
    I plants /SP1, SP2/;
    J demand centers /DC1, DC2, DC3/;

PARAMETERS
    A(I) capacity of SPi
        / SP1 233
        SP2 172 /
    F(I) fixed cost of SPi
        / SP1 270
        SP2 250 /
    MU(J) Mean of the demands at DCj
        / DC1 130
        DC2 176.5
        DC3 104.5 /
    SD(J) standard deviation of the demands at DCj
        / DC1 23.094
        DC2 18.764
        DC3 24.537 /;
    h(J) holding cost per unit stored at DCj
        / DC1 1
        DC2 1
        DC3 1 /;
    s(J) stock out cost per unit short at DCj
        / DC1 0.5
        DC2 0.5
        DC3 0.5 /;

TABLE R(i, j) the variable cost per unit shipped from i to j

<table>
<thead>
<tr>
<th></th>
<th>DC1</th>
<th>DC2</th>
<th>DC3</th>
</tr>
</thead>
<tbody>
<tr>
<td>SP1</td>
<td>2.05</td>
<td>2.13</td>
<td>1.33</td>
</tr>
<tr>
<td>SP2</td>
<td>0.53</td>
<td>1.92</td>
<td>2.23</td>
</tr>
</tbody>
</table>

PARAMETERS
    AU(J) lower limit of the uniform demands at DCj
    AU(J) = MU(J) - SD(J)*SQRT(3) ;
PARAMETERS BU(j) upper limit of the uniform demands at DCj
BU(j) = MU(j)+ SD(J)*SQRT(3)
PARAMETERS P(j) price per unit sold at DCj
P(j) = 4
PARAMETERS K(j) first constant
K(j) = 0.5*P(j)/(BU(J)-AU(J))
PARAMETERS XS(i,j) Optimal shipment size based on independent assignment from i to j
XS(i,j) = BU(j)- R(i,j)/P(j)*(BU(J)-AU(J))
PARAMETERS XU(j) upper bound at demand j
XU(j) = smax((l),XS(I,J))
PARAMETERS L(i,j) profits
L(i,j) = K(j)*(SQR(XS(i,j))-SQR(AU(j)))
VARIABLES
Y(I) equals to 1 if the plant is utilized
Z(I,J) equals to 1 if there is a shipment from plant i to demand j
OB total transportation profits
BINARY VARIABLES
Y(I)
Z(I,J)
EQUATIONS
PROFIT objective function
SUPPLY(I) observe supply limit at plant i
DEMAND(J) limit demand j to receive from only 1 plant
PROFIT .. OB =E= SUM((I,J), L(I,J)*Z(I,J))-SUM((I),F(I)*Y(I))
SUPPLY(I) .. SUM((J), XS(I,J)*Z(I,J)) =L= A(I)*Y(I)
DEMAND(J) .. SUM((I), Z(I,J)) =L= 1
MODEL EXAMPLE71a /ALL/ ;
SOLVE EXAMPLE71a USING MIP MAXIMIZING OB ;
APPENDIX M

GAMS File for Example 7.1B
SETS
  I  plants /SPI, SP2/  
  J  demand centers /DC1, DC2, DC3/ ;

PARAMETERS
  A(I)  capacity of SPI 
        /SPI 10000
        SP2 10000 /
  F(I)  fixed cost of SPI 
        /SPI 270
        SP2 250 /
  MU(J)  Mean of the demands at DCj 
         / DC1 130
         DC2 176.5
         DC3 104.5 /
  SD(J)  standard deviation of the demands at DCj 
         / DC1 23.094
         DC2 18.764
         DC3 24.537 ;
  h(J)  holding cost per unit stored at DCj 
         / DC1 1
         DC2 1
         DC3 1 /
  s(J)  stock out cost per unit short at DCj 
         / DC1 0.5
         DC2 0.5
         DC3 0.5 ;

TABLE  R(i,j)  the variable cost per unit shipped from i to j 
          DC1   DC2   DC3
    SPI  2.05  2.13  1.33
    SP2  0.53  1.92  2.23 ;

PARAMETERS  AU(j)  lower limit of the uniform demands at DCj ;
             AU(j) = MU(j) - SD(J)*SQRT(3) ;
PARAMETERS  
BU(j) upper limit of the uniform demands at DCj ;  
BU(j) = MU(j) + SD(J)*SQRT(3) ;

PARAMETERS  
P(j) price per unit sold at DCj ;  
P(j) = 4 ;

PARAMETERS  
K(j) first constant ;  
K(j) = 0.5*P(j)/(BU(J)-AU(J)) ;

PARAMETERS  
XS(i,j) Optimal shipment size based on independent assignment  
from i to j ;  
XS(i,j) = BU(j) - R(i,j)/P(j)*(BU(J)-AU(J)) ;

PARAMETERS  
XU(j) upper bound at demand j ;  
XU(j) = smax((I),XS(I,J)) ;

PARAMETERS  
L(i,j) profits ;  
L(i,j) = K(j)*(SQR(XS(i,j))-SQR(AU(j))) ;

VARIABLES  
Y(I) equals to 1 if the plant is utilized  
Z(I,J) equals to 1 if there is a shipment from plant i to demand j  
OB total transportation profits ;

BINARY VARIABLES  
Y(I)  
Z(I,J) ;

EQUATIONS  
PROFIT objective function  
SUPPLY(I) observe supply limit at plant i  
DEMAND(J) limit demand j to receive from only 1 plant ;

PROFIT .. OB =E= SUM((I,J), L(I,J)*Z(I,J))-SUM((I),F(I)*Y(I)) ;
SUPPLY(I) .. SUM((J), XS(I,J)*Z(I,J)) =L= A(I)*Y(I) ;
DEMAND(J) .. SUM((I), Z(I,J)) =L= 1 ;

MODEL EXAMPLE71b /ALL/ ;
SOLVE EXAMPLE71b USING MIP MAXIMIZING OB ;
APPENDIX N

GAMS File for Example 7.2A
SETS
   I plants /SP1, SP2/ 
   J demand centers /DC1, DC2, DC3/ ;

PARAMETERS
   A(I) capacity of SPi
      / SP1 233 
      SP2 172 / 
   F(I) fixed cost of SPi
      / SP1 270 
      SP2 250 / 
   MU(J) Mean of the demands at DCj
      / DC1 130 
      DC2 176.5 
      DC3 104.5 / 
   SD(J) standard deviation of the demands at DCj
      / DC1 23.094 
      DC2 18.764 
      DC3 24.537 / ;
   h(J) holding cost per unit stored at DCj
      / DC1 1 
      DC2 1 
      DC3 1 / ;
   s(J) stock out cost per unit short at DCj
      / DC1 0.5 
      DC2 0.5 
      DC3 0.5 / ;

TABLE  R(i,j) the variable cost per unit shipped from i to j
      DC1  DC2  DC3 
   SP1  2.05  2.13  1.33 
   SP2  0.53  1.92  2.23 ;

PARAMETERS  AU(j) lower limit of the uniform demands at DCj 
   AU(j) = MU(j) - SD(J)*SQRT(3) ;

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PARAMETERS  BU(j) upper limit of the uniform demands at DCj ;
BU(j) = MU(j) + SD(J)*SQRT(3) ;

PARAMETERS  P(j) price per unit sold at DC j ;
P(j) = 4 ;

PARAMETERS  K(j) constant ;
K(j) = 0.5*P(j)/(BU(J)-AU(J)) ;

PARAMETERS  XS(i,j) Optimal shipment size based on independent assignment from i to j ;
XS(i,j) = BU(j) - R(i,j)/P(j)*(BU(J)-AU(J)) ;

PARAMETERS  XU(j) upper bound at demand j ;
XU(j) = smax(I),XS(I,J)) ;

VARIABLES
Y(I) equals to 1 if the plant is utilized
Z(J) equals 1 if there is a shipment from plant i to demand j
X(I,J) number of optimal units shipped from plant i to demand j
OB total transportation profits ;

POSITIVE VARIABLES
X(I,J) ;

BINARY VARIABLES
Y(I)
Z(J) ;

EQUATIONS
PROFIT objective function
SUPPLY(I) observe supply limit at plant i
DEMAND(J) observe demand upper bound limit at demand j ;

PROFIT.. OB =E= SUM((J), K(J)*2*BU(j)*SUM((i),X(I,J)))
- SUM((J), K(J)*SQR(SUM((i),X(I,J))))
- SUM((J), K(J)*Z(J)*SQR(AU(j)))
- SUM((I,J), R(I,J)*X(I,J)-SUM((I),F(I)*Y(I))) ;
SUPPLY(I).. SUM((J), X(I,J)) =L= A(I)*Y(I) ;
DEMAND(J).. SUM((I), X(I,J)) =L= XU(j)*Z(J) ;

MODEL EXAMPLE72a /ALL/ ;
SOLVE EXAMPLE72a USING MIQCP MAXIMIZING OB ;
APPENDIX O

GAMS File for Example 7.2B
SETS
   I plants /SP1, SP2/
   J demand centers /DC1, DC2, DC3/ ;

PARAMETERS
   A(I) capacity of SPi
       / SP1 10000
       SP2 10000 /
   F(I) fixed cost of SPi
       / SP1 270
       SP2 250 /
   MU(J) Mean of the demands at DCj
       / DC1 130
       DC2 176.5
       DC3 104.5 /
   SD(J) standard deviation of the demands at DCj
       / DC1 23.094
       DC2 18.764
       DC3 24.537 / ;
   h(J) holding cost per unit stored at DCj
       / DC1 1
       DC2 1
       DC3 1 /
   s(J) stock out cost per unit short at DCj
       / DC1 0.5
       DC2 0.5
       DC3 0.5 / ;

TABLE  R(i,j) the variable cost per unit shipped from i to j
       DC1   DC2   DC3
       SP1   2.05  2.13  1.33
       SP2   0.53  1.92  2.23 ;

PARAMETERS  AU(j) lower limit of the uniform demands at DCj ;
   AU(j) = MU(j) - SD(J)*SQRT(3) ;
PARAMETERS BU(j) upper limit of the uniform demands at DCj;
BU(j) = MU(j) + SD(J)*SQRT(3);

PARAMETERS P(j) price per unit sold at DC j;
P(j) = 4;

PARAMETERS K(j) constant;
K(j) = 0.5*P(j)/(BU(J)-AU(J));

PARAMETERS XS(i,j) Optimal shipment size based on independent assignment from i to j;
XS(i,j) = BU(j) - R(i,j)/P(j)*(BU(J)-AU(J));

PARAMETERS XU(j) upper bound at demand j;
XU(j) = smax((I),XS(I,J));

VARIABLES
Y(I) equals to 1 if the plant is utilized
Z(J) equals 1 if there is a shipment from plant i to demand j
X(I,J) number of optimal units shipped from plant i to demand j
OB total transportation profits;

POSITIVE VARIABLES
X(I,J);

BINARY VARIABLES
Y(I)
Z(J);

EQUATIONS
PROFIT objective function
SUPPLY(I) observe supply limit at plant i
DEMAND(J) observe demand upper bound limit at demand j;

PROFIT.. OB =E= SUM((J), K(J)*2*BU(J)*SUM((i),X(I,J)))
- SUM((J), K(J)*SQR(SUM((i),X(I,J))))
- SUM((J), K(J)*Z(J)*SQR(AU(J)))
- SUM((I,J), R(I,J)*X(I,J))-SUM((I),F(I)*Y(I));
SUPPLY(I).. SUM((J), X(I,J)) =L= A(I)*Y(I);
DEMAND(J).. SUM((I), X(I,J)) =L= XU(J)*Z(J);

MODEL EXAMPLE72b /ALL/;
SOLVE EXAMPLE72b USING MIQCP MAXIMIZING OB;