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An Investigation of Secondary Teachers' Knowledge of Rate of Change in the Context of Teaching a Standards-Based Curriculum

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AN INVESTIGATION OF SECONDARY TEACHERS' KNOWLEDGE OF RATE OF CHANGE IN THE CONTEXT OF TEACHING A STANDARDS-BASED CURRICULUM

by

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CHAPTER I

INTRODUCTION

Statement of the Problem

Over the past 15 years interest in teacher knowledge has increased (Ball, 1988a, 1991; Even, 1993; Shulman, 1987). Recent calls for increasing students' opportunities to learn, and to enhance their proficiency, place a heavy demand on teachers' expertise. Reform efforts are calling for a change in the way teachers teach, with an emphasis placed on teaching for understanding and meaning rather than developing isolated procedures and skills (National Council of Teachers of Mathematics [NCTM], 1989, 1991, 2000; National Research Council [NRC], 1989). There is general consensus that, in order to do so, teachers themselves need to have a solid knowledge of the subject of mathematics. This need creates an interest in doing three key things: 1) defining and analyzing what subject matter knowledge for teaching means, 2) understanding how subject matter knowledge, in the form of mathematical content knowledge and pedagogical content knowledge, is applied in classroom teaching, and 3) exploring factors that contribute to the acquisition of subject matter knowledge.

Many of the reform efforts in mathematics education are based on an image of mathematics teaching that emphasizes student thinking as the springboard for mathematical discourse and meaning (Ball, 1991; NCTM, 1991). It is argued that such an approach requires teachers to understand well what they are teaching and be able to connect students with the content. That is, such an approach requires teachers to have

highly developed mathematical content knowledge and pedagogical content knowledge (Barnett, 1991; Fennema & Franke, 1992). Knowing how to teach one's subject matter includes knowing what concepts, ideas, and principles make up the content of the discipline, knowing how the discourse within a discipline relates to the teaching of the subject, and knowing how fundamental principles and ideas can be transformed into appropriate and useful representations that make this knowledge comprehensible to learners (Barnes, 1989; McDiarmid, Ball, & Anderson, 1989; Shulman, 1987).

Bolte (1995) argues that to be effective, a secondary mathematics teacher must not only possess a body of knowledge on a subject, but that the knowledge also be well integrated and organized. This integration and organization of knowledge aids a teacher in transforming his or her content knowledge into a form that is usable within the classroom. Expert-novice studies have indicated the importance of knowledge organization in cognitively complex domains. Expert teachers displayed more mathematical content and pedagogical content knowledge, but in addition, "their conceptual system, or cognitive schemata, for organizing and storing this knowledge are more elaborate, interconnected, and accessible than the novices' schemata" (Brown & Borko, 1992, p. 213).

There are a few kinds of experiences that most teachers have in learning and developing their knowledge. Courses and workshops offer teachers opportunities to revisit and relearn the content of the school curriculum. A RAND study (2002) points out the practical problem that lies herein: Because of weaknesses in the research base, these initiatives have been designed and implemented with scant evidence about their effectiveness. Despite some successful efforts to develop teachers' mathematical

knowledge through professional development (e.g., Lappen & Even, 1989; Schifter, 1993), participating teachers are not necessarily able to put new knowledge to use in their classrooms. Even when they do gain important mathematical insights, they may be no better able to understand their students' ideas, to ask strategic questions, or to analyze the mathematical tasks contained in their textbooks.[1]

Although common sense suggests that the best preparation for teaching mathematics might be an undergraduate degree in mathematics, a few complications are evident. First, the mathematics of the K-12 curriculum does not map well onto the curriculum of an undergraduate mathematics degree (Conference Board of the Mathematical Sciences [CBMS], 2001). That is, even if preservice teachers majored in mathematics as undergradutes, they might not have studied the mathematics in the K-12 curriculum since they were K-12 students themselves. Second, there is wide variation across institutions of higher education in what constitutes a major (e. g., at some colleges and universities, one does not "major" at all, but fulfills a "concentration"). There are studies that investigated the relationship between teachers' subject matter preparation and student achievement (e. g., Begle, 1979; Ferguson, 1991; Monk, 1994). In general the studies reached a similar conclusion – the amount of mathematics taken by teachers was not likely to produce significant effects on student performance. However, Begle (1979) did find some positive correlations between mathematics methods courses and student achievement. As in Begle's analysis, Monk (1994) uncovered significant effects of courses in undergraduate mathematics pedagogy. Together, these studies suggest measures such as undergraduate degrees or number of courses taken in mathematics

[1] See, for example, Borko et al., 1993

content may be poor indicators of what teachers actually know and how they can use that knowledge in teaching.

Recently, some have suggested that school curriculum materials themselves can provide another opportunity for teachers to learn and develop their knowledge (Ball & Cohen, 1996; Collopy, 1999). They argue that curriculum materials can support the development of teachers' understanding of mathematical content. For example, curriculum materials can provide a variety of ways of representing rate of change and its relationships to other mathematical topics. Also curriculum materials can contribute to the development of pedagogical content knowledge by suggesting ways to monitor students' understanding and providing examples of common ways students express particular mathematical ideas. To explore possible contributions made by curriculum materials to teacher learning, it is important to study teachers as they utilize particular curriculum materials. Because knowledge is open to change and development with experience (Fennema & Franke, 1992), it is important to conduct research in classrooms where teachers use their knowledge in practice and have the opportunity to continue to learn about mathematical ideas through their teaching.

To examine teachers' subject matter knowledge and the role it plays in classroom teaching, a mathematical context is necessary. For instance, the importance of the mathematics of change in all grade levels of school mathematics is well recognized (NCTM, 1989, 2000; Stewart, 1990). Understanding change is fundamental to understanding function, which plays a central role in mathematics, and to understanding many ideas presented in the news (NCTM, 2000). *Rate of change*, with accumulation, is one of two fundamental concepts of mathematical change (Noble, Nemirovsky, Wright,

& Tierney, 2001). Rate of change can be used to describe many situations in the real world, such as how fast a person runs down the street. It can be recognized in calculus class as the derivative. In algebra classes it is concerned with the slopes of lines or the consecutive differences or ratios in a table of numbers, which can show change over time for functions (Confrey & Smith, 1995). Rate of change can also be part of learning about rational numbers in presecondary school mathematics (Behr, Lesh, Post, & Silver, 1983; Harel & Confrey, 1994). Therefore, rate of change has the potential to tie together subjects such as algebra and geometry, representations such as tables and graphs, and contexts such as mathematical structure and mathematical modeling.

To benefit from the mathematical power of this concept, instead of just manipulating symbols and developing isolated skills, current reforms recommend that students be able to interpret real-world phenomena with a variety of contexts for exploring rate of change, represent and analyze mathematical relationships in various representational forms, translate between representational forms, and develop conceptual understanding of the related procedures and the ideas about what rate of change is (NCTM, 1989, 2000). All of these conceptual ideas are fundamental to doing mathematics meaningfully.

From the literature review, only few mathematical topics, such as fractions at the elementary level and functions at the secondary level, have been studied in much depth and many other mathematical topics remain unexplored or need to be studied in more depth. Teachers' knowledge of rate of change is one of those topics we know little about and, at the same time, research indicates that this is not an area that students typically understand with much depth, even after taking calculus courses in which the idea is

formalized (e. g., Smith, 2000). There exist a few studies on the knowledge in domains related to rate of change. Some include Stump's (1997) study interviewing preservice secondary teachers on the concept of slope, and a two-serial study by Thompson and Thompson (1994, 1996) investigating one middle school teachers' knowledge of speed as a rate. These studies have demonstrated that teachers' knowledge is often weak and lacks of connections among different aspects of slope and rate, respectively.

Purpose of the Study

The purposes of this study are 1) to provide an in-depth description of secondary school teachers' mathematical content and pedagogical content knowledge of rate of change, and 2) to compare and contrast differences, if any, between teachers who are experienced using a Standards-based curriculum versus those who are less experienced using those same materials. The study focuses on the following research questions:

1. What characterizes teachers' mathematical content knowledge and pedagogical content knowledge of rate of change in the context of teaching Standards-based curriculum materials?
2. What qualitative differences exist between experienced and less experienced teachers using Standards-based curriculum materials with respect to their knowledge of rate of change?

The concept map in Figure 1 elaborates the research question domain. Words in bold reflect the primary aim of this study: to characterize teachers' mathematical content knowledge and pedagogical content knowledge with respect to rate of change. Two-way arrows indicate inter-relationships between ideas. Solid arrows represent influences that are supported by the literature (though they are not restricted to the domain of rate of

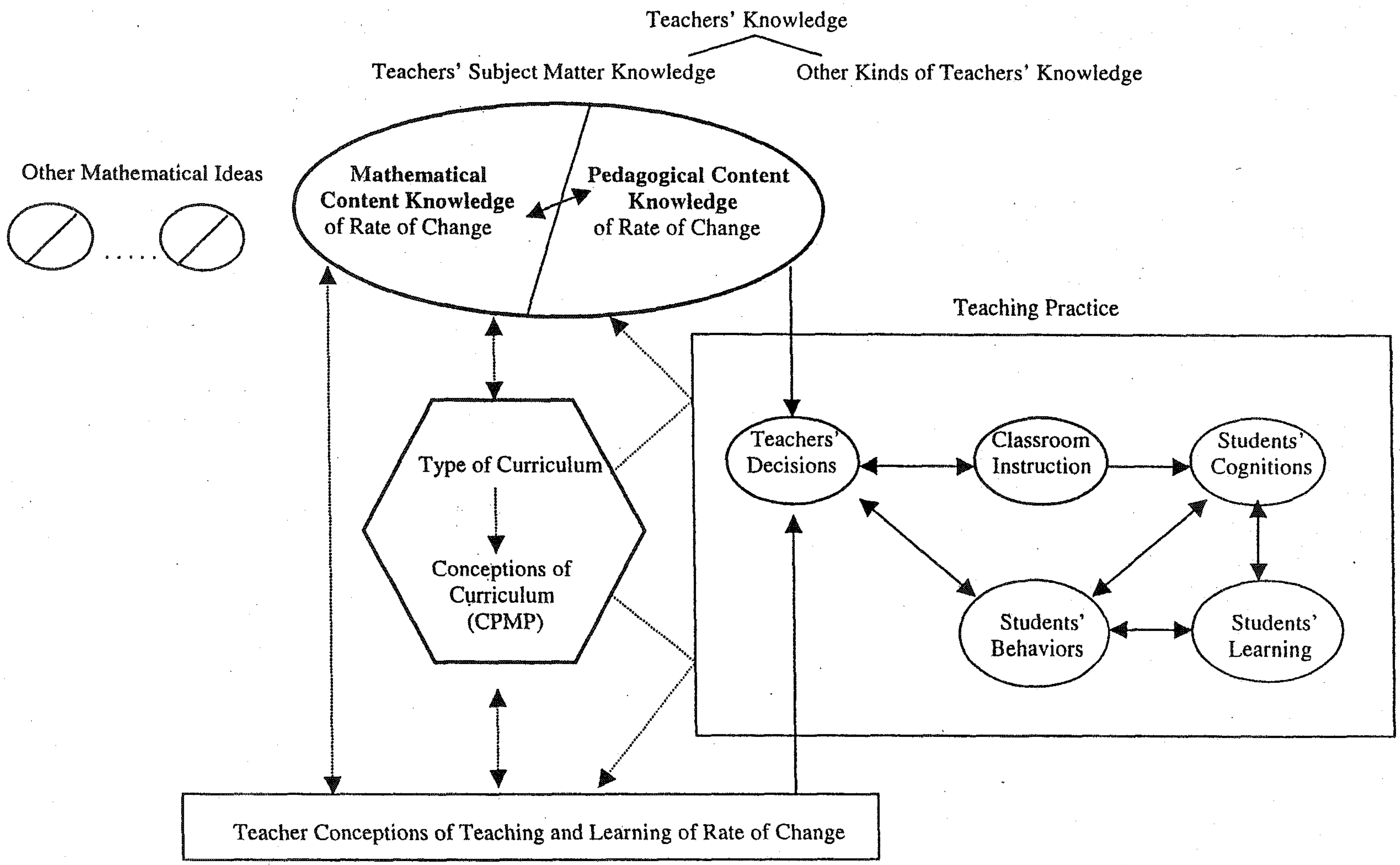

Figure 1. Concept map that displays the research domain
Note. Adapted from Cognitively Guided Instruction (CGI) Research Model (Fennema, Carpenter, & Peterson, 1989, p. 204).

change) and dotted arrows represent influences that were probed by the investigation of this study.

Significance of the Study

Currently there is a growing body of research concerning teacher knowledge, but as noted by Ball and her colleagues (2001), it is based overwhelmingly on studies of preservice elementary teachers. This study contributes to the research base on teacher knowledge in several ways. First, this study broadens the picture of mathematical content and pedagogical content knowledge possessed by experienced teachers as well as inexperienced teachers at secondary level. Research has shown that U. S. teachers' knowledge is generally weaker than their counterparts in other nations such as China and Japan (i.e., Ma, 1999). However, there has not been more recent research on teachers in the context of using reform curricula where content is developed differently. This information may then have the potential to inform teacher education in terms of characterizing more clearly teachers' knowledge of rate of change.

The second contribution of this study is a characterization of teachers' conceptions of teaching and learning mathematics of rate of change. For example, it may be useful to know what concept image of rate of change a teacher has when learning is conceived as a mastery of procedures. It will also be useful to know how a teacher makes connections among mathematical ideas when teaching is perceived as decomposing a complex problem into a sequence of basic skills. By examining teachers' conceptions of teaching and learning mathematics with respect to the concept of rate of change, this study may provide useful information concerning teachers' thinking about teaching and

learning the concept of the rate of change and how it relates to their thinking about the concept of rate of change within the secondary school curriculum.

The third contribution of this study comes from the contextual examination of teachers' mathematical content and pedagogical content knowledge base as manifested in their teaching in the classroom. It is not typical to examine teachers' content knowledge by watching them teach and discussing their teaching with them afterwards. However, Ball and her colleagues (2001) address the importance of looking at teachers' teaching to probe their knowledge, by saying that classrooms where teachers respond to students' responses, explain a mathematical procedure whose meaning is buried, or consider connections among ideas create conditions where teachers have to make explicit their understanding of the mathematical ideas and procedures. An examination of teachers' knowledge within the context of their classroom gives another tool to probe their knowledge and complements other data collection methods situated outside of the classroom environment.

Lastly, the fourth contribution of this study relates to the context of using Standards-based curriculum materials. Standards-based curricula such as the CPMP materials provide a comprehensive high school mathematics curriculum with integrated content strands. The CPMP curriculum introduces and develops mathematical concepts including change/rate of change in a systemic way across grades levels 9 – 12. However, studies that focus on the impact of Standards-based curricula, such as the CPMP materials, in developing particular mathematical ideas are only beginning to emerge (Huntley et al, 2000; Lloyd & Wilson, 1998; Senk & Thompson, 2003) and documenting teachers' knowledge of the concept of rate of change remains largely unexplored. This

study will contribute to the growing body of knowledge about the impact of Standards-based curricula on such topics. This may, in turn, be informative to curriculum developers at all grade levels and help mathematics educators better design teacher education programs and professional development programs.

CHAPTER II

LITERATURE REVIEW

Teacher Knowledge

Recent reform efforts put an emphasis on teaching for understanding (Hiebert, 1992; NCTM, 1989, 2000). Hiebert (1992) defines understanding in terms of the way an individual represents and structures information. A mathematical idea, procedure, or fact is understood if its mental representation is part of an internal network of representations. The degree or extent of understanding is determined by the number and the strength of the connections. A fundamental assumption of cognitive psychology is that these knowledge structures and mental representations of the world play a central role in an individual's perceptions, thought, and actions (Brown & Borko, 1992). In the case of teaching, teachers' thinking is directly influenced by their knowledge, both mathematical and pedagogical, and their thinking, in turn, contributes to their instructional behavior. If a main goal of teacher education programs is to help teachers implement programs of instruction that develop student understanding of mathematics, it is reasonable to expect that teachers must have, and continue to develop, a connected and extensive knowledge base to bring to their mathematics teaching.

Categories of Teacher Knowledge

The knowledge that teachers bring to their teaching and interaction with students to produce an effective learning environment encompasses a broad range of skills, actions, and behaviors. Shulman's (1987) conceptualization of the domain of teachers'

knowledge has served as a framework for much of the current research on teacher knowledge (Ball, 1990a; Ball & McDiamid, 1990; Borko et al., 1992; Ebert, 1993). This theoretical model allows one to examine teacher knowledge, the relationships among the domains of knowledge identified with the theory, and its effects on classroom practices and outcomes.

In the process of developing his model for outlining teacher knowledge, Shulman and his colleagues carefully observed teachers as they moved from being teacher education students to neophyte teachers.

> Their development from students to teachers, from a state of expertise as learners through a novitiate as teachers, exposes and highlights the complex bodies of knowledge and skill needed to function effectively as a teacher. The result is that error, success, and refinement—in a word, teacher-knowledge growth—are seen in high profile and in slow motion. The neophyte's stumble becomes the scholar's window. (Shulman, 1987, p. 4)

Comparing these accounts with observations of highly successful veteran teachers shows that the knowledge, understanding, and skill displayed haltingly among beginners are often demonstrated with ease by those more experienced. In examinations of their cases, Shulman and his colleagues repeatedly asked what teachers knew, or failed to know, that permitted them to teach in the manner that they did.

Making some basic assumptions on the capacity to teach provides (indirectly) the categories of knowledge that underlie the understanding necessary to promote comprehension among students. Shulman holds that to be put in a position to teach, one must believe that they comprehend critically a set of ideas both in content and purpose and that they can transform understanding, performance skills, or desired attitudes or values into pedagogical representations and actions. This transformation, "the process wherein one moves from personal comprehension to preparing for the comprehension of

others, is the essence of the act of pedagogical reasoning, of teaching as thinking, and of planning the performance of teaching" (Shulman, 1987, p. 16).

Shulman outlines seven categories of knowledge that underlie the understanding needed to promote comprehension among students: subject matter knowledge (mathematical content knowledge), general pedagogical knowledge, curriculum knowledge, pedagogical content knowledge, an understanding of learners and their characteristics, knowledge of educational contexts, and knowledge of educational ends, purposes, and values, and their philosophical and historical grounds. These categories have been built upon and refined by other scholars (e.g., Ball, 1991; Grossman, 1990). The two domains particularly relevant to this study are content knowledge and pedagogical content knowledge.

Mathematical content knowledge. In Shulman's view, the first domain, mathematical content knowledge, consists of two types of understandings, substantive knowledge and syntactic knowledge.[2] Substantive knowledge consists of the key facts, concepts, principles, and explanatory frameworks in a discipline, whereas syntactic knowledge consists of the rules of evidence and proof within that discipline. Ball (1988b) also works from a conceptual framework that focuses on these two aspects of subject matter knowledge—in her words, knowledge of mathematics and knowledge about mathematics. One qualitative dimension Ball uses to analyze teachers' substantive knowledge is connectedness. The measure of connectedness addresses the need to ensure integration and accessibility of knowledge. This is contrary to the tendency to

[2] The terms "substantive knowledge" and "syntactic knowledge" were originally described by Schwab (1961/1978).

compartmentalize school mathematics which reduces the discipline to one of applying procedures and regurgitating truths.

Syntactic knowledge consists of knowledge of the rules of evidence and proof accepted within the discipline. Balls' knowledge about mathematics parallels Shulman's dimension of syntactic knowledge, and includes an understanding of the nature of knowledge in the discipline. Teachers bring to their teaching a perspective of where mathematics comes from, how it changes, how truth is established, and what it means to know and do mathematics. To facilitate classroom discourse consistent with a vision of conceptually-oriented mathematics instruction of mathematical ideas (e.f., NCTM, 1989, 2000) and to evaluate students' mathematical arguments, teachers need to know about the sources and development of mathematical knowledge, the standards for evidence, argument, and proof.

Leinhardt and Smith (1985), in their exploration of the cognitive aspects of teaching, also identified two core areas of knowledge, subject matter knowledge and lesson structure. They would add to the description of subject matter domain by making explicit the need for understanding of classes of student errors as they relate to the structure of the discipline. Shulman (1987) separates out this strand of knowledge, recognizing it as a special area of knowledge unique to the needs of teachers. He calls this pedagogical content knowledge.

<u>Pedagogical content knowledge.</u> Shulman's second knowledge domain relevant to this study, pedagogical content knowledge, identifies the blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized and adapted to the diverse interests and abilities of learners, and presented for

instruction. Shulman proposed that teachers possess knowledge beyond that of content specialists. In addition to knowing the substantive and syntactic structures of a subject, teachers must be able to convey content to students. Shulman (1986) labeled the "ways of representing and formulating the subject that make it comprehensible to others" (p. 9) as pedagogical content knowledge. Grossman (1990) elaborated on Shulman's conceptualization of pedagogical content knowledge by proposing four components. The first component is the overarching purpose for teaching mathematics that guides the planning and enactment of instruction. Second, topic specific knowledge of students' common preconceptions and misconceptions and of how students learn helps teachers anticipate typical difficulties and monitor students' developing understandings. Third, teachers draw upon their knowledge of the organization of the mathematics curriculum and available materials for teaching specific topics. Fourth, teachers use knowledge of instructional strategies and representations to foster students' understanding of mathematical topics.

Research on Teacher Knowledge

As Shulman and others have argued, teachers' knowledge affects both the content and the processes of their instruction, influencing both what and how they teach (Carpenter, Fennema, Peterson, Chiang & Loef, 1989). Thus, it is important to know whether teachers have sufficient subject matter knowledge and pedagogical understanding of the subject matter they teach.

Existing studies associated with teacher knowledge provide evidence that both subject matter knowledge and pedagogical content knowledge of teachers—elementary and secondary, preservice and experienced—are often insufficient in quantity and

unsatisfactory in quality. Understandably, this is an essential reason why there have been so many reports and documents calling for improving teachers' knowledge (e.g., CBMS, 2001; NCTM, 1991, 2000). A review of some studies of teachers' knowledge reveals that an overwhelming majority of the studies have been conducted with preservice elementary teachers (Ball, 1990a; Graeber, Tirosh, & Glover, 1989; Mayberry, 1983; Simon, 1993) and only a few studies have focused on the knowledge of practicing teachers (e.g., Post, Harel, Behr, & Lesh, 1991; Stein, Baxter, & Leinhardt, 1990; Wilson, 1994). Moreover, most of the studies investigating teachers' knowledge have been limited to some specific domains of a particular subject topic, such as arithmetic operations and functions. Among the studies cited above, most were conducted in the area of arithmetic operations with the exception of Mayberry's work on geometry and Post and his colleagues' and Wilson's work on functions.

One of the reasons why research has focused on elementary teachers is possibly explained by a continued assumption that content knowledge is not a problem for secondary teachers, who, by virtue of specialized study in mathematics, "know their subject" (Ball, 2001). However, research on secondary teachers repeatedly reveals the fallacy of this assumption (e.g., Ball, 1988a, 1990b; Even, 1993; Wilson, 1994). Still the preponderance of this work is on preservice teachers. Only a few studies of practicing secondary teachers' knowledge exist (Lee, 1994; Lloyd & Wilson, 1998; Norman, 1992; Stump, 1997; Vinner & Dreyfus, 1989). Those studies mainly investigated the limit concept (Lee, 1994) and functions (Lloyd & Wilson, 1998, Norman, 1992; Vinner & Dreyfus, 1989) in regards to definitions, properties, and operation and translation

between different representational forms. Thus, many mathematical topics, including rate of change, have not been studied in any depth.

Teachers' Acquisition of Knowledge

In addressing the need for teachers to learn the subject matter knowledge and pedagogical content knowledge needed to guide conceptually-oriented teaching, it is important that teachers have ongoing support to learn about and improve them. It is also important to examine the effectiveness of the existing supports, and when those are found lacking, consider other ways to improve the supports.

University Preservice Programs

One of the prominent ways in which teachers acquire their knowledge is as students in university preservice programs. University preservice programs herein means teachers' educational experiences at their university before formally becoming school teachers. There is no question that a main purpose of teacher preparation programs is to provide preservice teachers with adequate knowledge and skills needed for teaching. However, the findings of studies investigating the extent to which university preservice programs influence the development of teachers' knowledge related to teaching for understanding is limited. For instance, Begle (1979) conducted analyses across studies of teacher effects, attempting to identify the trends in how teachers' mathematics study impacted their effectiveness with students. Begle's analysis produced a surprise: advanced mathematics course-taking produced positive main effects on students' achievement in only 10% of the cases, and –more startling—negative main effects in

8%.[3] His study produced another result, less-often reported, which is that the relationship between the number of credits in mathematics *methods* courses and student performance yielded the greatest positive main effects—23%. This result is notable, but still low.[4]

Consistent with Begle's claims, McDiarmid (1990) argued that, if preservice teachers did take college mathematics courses, the kinds of experiences they would encounter would not promote either the subject matter knowledge or pedagogical content knowledge that mathematics educators advocate. The university courses might be successful at increasing specific content knowledge, but were disappointing with respect to what teachers were able to *do* with the mathematics in their own practice.

Focusing on mathematics methods courses, McDiarmid examined a four-week field experience of a group of preservice teachers paired with unconventional teachers. The researcher noted that some students were willing to reconsider their understandings and beliefs in a short amount of time, but the changes may have been superficial and short-lived. Although admitting a 10-week mathematics methods course did have some impact on preservice teachers, Ball (1989) also raised the same concern as McDiarmid about the extent to which the impact would continue. Moreover, a broader question that has been touched upon earlier but remains open is whether the effects of the experiences at universities will be washed out by school teaching experiences (Zeichner & Tabachnick, 1981).

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[3] A positive main effect would mean that more credits of mathematics were associated with greater student performance; a negative main effect meant that more credits of mathematics were associated with lower levels of achievement.

[4] Monk (1994) found a similar conclusion to that reached by Begle—namely, that the number of mathematics courses makes a difference, but only up to a point.

Professional Development Programs

Professional development programs provide another avenue for practicing teachers to explore and learn pedagogical content and subject matter knowledge needed to enhance their conceptual teaching. However, many scholars (c.f., Little, 1993; Lord, 1994) note that most professional development is poorly suited to support teachers acquiring or even expanding such knowledge, although the results of some programs seem promising (e.g., Featherstone, Smith, Beasley, Corbin, & Shank, 1993; Lappan & Even, 1989; Schifter, 1993; Simon & Schifter, 1991). A limitation, however, is that such programs, while immensely helpful for participants and possibly for their colleagues, can deal with only a tiny fraction of teachers. In fact, the mathematical understandings that teachers need to know are too varied, complex, and context-dependent to be anticipated in one or even several courses, or once-in-a-while programs such as professional development (Russell, Schifter, Bastable, Yaffee, Lester, & Cohen, 1994).

In summary, a review of the literature on the effectiveness of teacher preparation programs at the university and through professional development programs reveals that these supports often fall short of their potential and are not sufficient to develop the knowledge necessary to teach for understanding.

Textbook Use and Curriculum Materials

More recently, the role of curriculum materials themselves and their potential to impact teacher learning has been a point of discussion (Ball & Cohen, 1996; Russell et al, 1994).

Traditional textbooks. As observed by Goodlad (1984), textbooks have been typically conceived of as providing a scope and sequence of learning objectives for

students with corresponding instructional and assessment activities. A heavy emphasis is placed on mechanics in the topics covered by teachers and quizzes emphasize short answers and recall of specific information. According to Goodlad, similar textbooks yield similar teaching. He also noted that teachers select some topics to teach because they have taught them traditionally and they quite often teach in the ways that they have been taught. This suggests that the opportunities for teachers to change their teaching practice are not cultivated in traditional textbooks with traditional approaches to mathematics and are not supported in conventional school cultures.

Potential support of curriculum materials. The publication of *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989) prompted scholars to start thinking about how curriculum materials could help teachers continue to develop the knowledge needed to teach mathematics in the vision of the Standards. Ball and Cohen (1996) urged designers to consider including information that would support teacher learning about student thinking, knowledge of mathematics, development of a classroom community, mathematical content, and reasoning about instructional designs. They went on to argue that such "educative" curriculum materials could embed features that contribute to successful learning opportunities for teachers, along with the usual scope and sequence of instructional activities for students.

The potential that curriculum materials have to be educative for teachers was also addressed by Russell and her colleagues (1994). From their study exploring the development of teachers' mathematical understanding while they teach, they argued that the process of teacher learning can be bolstered by materials that are closely linked to practice and offer ongoing support because curriculum materials are ubiquitous in

classrooms. That is, curriculum materials can support teachers' learning in ways that contribute to establishing contexts where teacher learning takes place. They asserted that curriculum materials seem well situated to contribute to teachers' opportunities to learn subject matter knowledge and pedagogical content knowledge.

Recent Reform in Mathematics Education

Recent recommendations by major mathematics education professional organizations, such as the National Council of Teachers of Mathematics (NCTM), the Mathematical Science Education Board (MSEB), and the National Research Council (NRC), call for fundamental changes in school mathematics curricula, instruction, and assessment (NCTM, 1989, 1991, 1995, 2000; MSEB and NRC, 1990). The proposed changes include design of curricula with a common core of broadly useful mathematics for all students, emphasis on student-centered instruction that engages students in exploration of mathematical facts and principles through collaborative work on authentic problems, and assessments of student learning through a variety of strategies that are embedded in regular classroom activity (Huntley, Rasmussen, Villarubi, Sangtong, & Fey, 2000).

In order to implement these recommended changes, teachers need to establish mathematics classrooms in which students engage actively and cooperatively in exploration and discussion, and use a variety of representations and technological tools to solve rich problems and reason mathematically (NCTM, 1991). In contrast with traditional classroom activities that emphasize repetition, practice, and memorization of skills and rules, reform-oriented instruction emphasizes student development of understanding and connections among important concepts and procedures, both within

mathematics and to the real world. Reform-oriented instruction can offer learning opportunities that do not typically occur in traditional classrooms by encouraging students to communicate their ideas and questions, agree and disagree among themselves, and negotiate the development of rich joint theories and ideas (Lloyd & Behm, 2002).

Reform-Oriented Curriculum Materials

Specific reform recommendations have resulted in, among other things, a flourish of curriculum development actively across grade levels (e.g., Investigations in Number, Data, and Space; Connected Mathematics Project; Core-Plus Mathematics Project; Mathematics in Context; etc. c.f. Senk & Thompson (2003)) during the decade since publication of the 1989 NCTM *Curriculum and Evaluation Standards for School Mathematics*. Although these materials have incorporated specific aspects of reform recommendations in diverse ways (emphasizing different themes or activities), the materials share certain qualities that distinguish them from traditional mathematics textbooks in at least two important ways.

First, reform curricula explicitly incorporate reform ideas about mathematics and pedagogy by emphasizing inquiry mathematics: student explorations of real-world mathematical situations and discussions related to problem-centered activities. Furthermore, the materials are formatted to support these mathematical and pedagogical differences. Traditional mathematics textbooks are typically divided into chapters outlining self-contained daily lessons for the teacher to present (often composed primarily of definitions and examples of the lesson's content focus) followed by numerous practice exercises for the students. In contrast, most reform-oriented curriculum materials have lessons which encourage students to investigate, experiment,

and debate mathematics. Mathematical activities in lessons pose large-scale problems and situations centered on particular mathematical themes and content areas.

A second substantive difference is that reform-oriented materials generally offer more extensive information for teachers. In addition to providing correct answers to problems, most of these new curriculum materials offer such things as details about different representations of content, historical information about mathematical and pedagogical ideas, examples of what students might believe or understand about particular activities and content, and/or potentially fruitful questions for eliciting discussion. The inclusion of these details has been motivated in part by the failure of the "teacher-proof" curriculum materials written in the 1950s and 1960s to facilitate substantial educational change (Lloyd & Behr, 2002). Such findings support the position that it is teachers, and not texts alone, that determine how the innovations envisioned by reformers and curriculum developers become implemented in classrooms.

The Curriculum of the Core-Plus Mathematics Project (CPMP)

One set of high school curriculum materials designed and developed to assist teachers' learning in the ways described above is the Core-Plus Mathematics Project. The Core-Plus Mathematics Project (CPMP), now published as *Contemporary Mathematics in Context: A Unified Approach* (Coxford, Fey, Hirsch, Schoen et al., 2003), was one of five reform high school curriculum projects funded by the National Science Foundation in 1992. As a response to the proposed reforms, the CPMP curriculum consists of a 3-year integrated mathematics program for all students plus a 4^{th}-year course continuing the preparation of students for college mathematics. CPMP's novel treatment of complex subject matter enacts many components of the recommendations of the NCTM

Curriculum and Evaluation Standards for School Mathematics, as well as research on teaching and learning, and thus challenges traditional approaches to content (Hirsch, 2001). The curriculum, built upon the theme of mathematics as sense-making, emphasizes the modeling of real-world situations as a way to explore mathematical concepts and incorporates numerous features of graphics calculators. Each year of the CPMP curriculum features topics in algebra and functions, statistics and probability, geometry and trigonometry, and discrete mathematics. The four major content strands are unified by common themes such as data, symbolic reasoning, representation, shape, and change; by common topics like graphical representations, curve fitting, and matrices; and by habits of mind like visual thinking, recursive thinking, and reasoning with multiple representations, and providing convincing arguments (Hirsch, Coxford, Fey, & Schoen, 1995).

In addition to the efforts to provide successful opportunities for students to learn, the curriculum materials embed opportunities to assist teachers' learning. To support the development of teachers' understanding of mathematical concepts, procedures, and representations, and the connections between them, the CPMP curriculum provides discussions of ideas central to a mathematical topic, a variety of ways of representing a mathematical topic and relationships to other mathematical topics. This information is provided through unit overviews and backgrounds of mathematical content. A teacher could draw on them as needed to prepare for and enact instruction during a unit involving the mathematical ideas. Also, the curriculum materials support the development of teachers' knowledge about the nature of mathematics as a discipline through discussions and concrete illustrations of mathematical thinking and arguments by making explicit

developers' conceptions of the nature of mathematics as compared to conventional beliefs about the nature of mathematics.

To provide opportunities for teachers to develop pedagogical content knowledge, it is important to have teachers understand how to develop students' conceptual understanding of mathematics, in addition to developing procedural skills (NCTM, 1991). CPMP directly addresses the purpose or approach that underlies the design of the materials and makes explicit arguments for the circumstances under which the curriculum's approach is beneficial and how its aims contrast with those of other approaches. Also, the curriculum materials point out likely difficulties and preconceptions that students bring to the study of particular topics as well as ideas and strategies that commonly arise during instruction. Tangentially, the curriculum materials suggest ways to monitor students' understanding and provide examples of common ways students express particular mathematical ideas.

Curriculum materials often provide information on the scope and sequence of topics they cover at a given grade level and across grade levels. Sometimes, however, they include more topics than can be covered in a school year (Freeman & Porter, 1989). The CPMP curriculum provides information that supports teachers' decisions about what topics to include and omit, including relationships between topics, and offers various plans for content coverage within and across grade levels.

Scholars suggest that curriculum materials can also support teacher learning by offering guidance on how to make use of the information they offer (Ball & Cohen, 1996; Russell, 1994). While curriculum materials in general provide representations of content embedded in activities and lesson plans, curriculum developers' reasons for choosing

representations or activities out of the universe of possibilities is rarely made explicit. Information about the merits of alternative instructional strategies and representations, which is what CPMP makes explicit, can support teachers' reasoning about the potential impact on their students.

Rate of Change

Mathematics educators have identified the mathematics of change as an important strand of school mathematics (NCTM, 2000; Stewart, 1990), not only because of its critical role historically and in present day mathematics, the sciences, and the social sciences, but also because the concepts of the mathematics of change are rooted in everyday experiences of people—young and old (Confrey & Smith, 1995; Kaput & Nemirovsky, 1995).

Rate of change is one of the fundamental concepts of the mathematics of change (Noble, Nemirovsky, Wright, & Tierney, 2001). It is also identified as a key concept to the understanding of functions in mathematics and to the understanding of many ideas related to everyday life [for example, when describing qualitative change (" I grew taller over the summer") and quantitative change ("I grew two inches in the last year")] (NCTM, 2000). NCTM (2000) recognizes it is an important goal in the development of algebraic thinking to be able to analyze change in various contexts by understanding relations and to model phenomena using appropriate tools. In particular, a conceptual understanding of rate of change is especially crucial for the study of calculus. Researchers claim that understanding of advanced topics such as calculus "develops from basic intuitions that children construct in their daily experiences with physical and symbolic change" and that calculus learning can be facilitated by earlier experiences that

allow children to study and represent situations involving change (e.g., Nemirovsky, 1993). Rate of change can also be part of learning about rational numbers in presecondary school mathematics (Behr, Lesh, Post, & Silver, 1983; Harel & Confrey, 1994). It can be recognized in algebra classes concerned with the slopes of lines or the consecutive differences or ratios in a table of numbers, which can show change over time for functions (Confrey & Smith, 1995). And, rate of change can be recognized in calculus class as the derivative.

<u>Student Knowledge of Rate of Change</u>

An important component of pedagogical content knowledge is knowing how students think within a specific mathematical domain. Within a given domain, teachers must understand what makes the learning of a specific topic difficult and know appropriate strategies to facilitate the remediation of students' misconceptions. It is important for teachers to understand what students know or don't know about the subject domain involving rate of change.

The concept of rate of change can be introduced to students from elementary through high school in a variety of contexts. Distance traveled per second is an example of a simple way to think about rate of change that many elementary students can grasp. Traditionally the concept of rate of change is typically introduced via *slope* in first-year algebra and it has been identified as a difficult concept even for high school and college students to understand (Nemirovsky & Rubin, 1992; Monk, 1992). Several student misconceptions have been associated with the concept of slope. For example, when a linear function is written in the form $y = mx + b$, it may be difficult for students to consider the slope as a ratio if m is an integer. Another confusion often exists as to

whether slope is computed as "x over y" or "y over x." Furthermore, students sometimes believe that the "order of the points" matters for computing slope $((y_2 - y_1)/(x_2 - x_1)$ or $(y_1 - y_2)/(x_1 - x_2))$ (Barr, 1980, 1981). There is also confusion between the roles of m and b when a function is written in the form $y = mx + b$ (Barr, 1980, 1981; Schoenfeld, Smith, & Arcavi, 1993). Orton (1984) presented several tasks involving rate of change to high school students and undergraduates. He identified several errors in students' reasoning. In their responses to a word problem involving the speed of a car, students were confused between average speed and constant speed. In a problem involving a difference table, students exhibited confusion between the existence of a number pattern in the table and the presence of a constant rate of change.

Several studies deal with knowledge of rate of change of precalculus students. Researchers at TERC (Technical Education Research Centers) used a device with a car moving on a track so that both the position and velocity of the car at each moment in time cold be graphed (Monk & Nermirovsky, 1994; Rubin & Nemirovsky, 1991). Students were shown a velocity graph and asked to predict a corresponding position graph. The main result from these studies was that students assumed that the position graph should resemble the velocity graph. If the velocity graph increased, these students believed the position graph did too, and vice versa. For these students any change in the velocity graph meant the same kind of change in the position graph. These studies also reported students' errors in interpreting the sign of the velocity. Some students in the studies associated the sign of the velocity with the speed of the car—positive velocity means going faster and negative velocity means going slower.

A conceptual understanding of rate of change is especially crucial for the study of calculus and physics. Calculus typically begins with the study of derivatives and rates of change, using slopes of tangent lines to develop these concepts. Physics assumes the ability of students to interpret slopes as a functional relationship between two quantities (McDermott, Rosenquist, & van Zee, 1987). Thus, in order to help students acquire a deep understanding of rate of change, it is important that secondary mathematics teachers have a robust mathematical understanding of rate of change and be able to utilize that knowledge when working with their students.

Teacher Knowledge of Rate of Change

Very few studies of teachers' knowledge related to rate of change exist. Stump (1997) examined both preservice and inservice secondary teachers' repertoires of representations of slope through paper-and-pencil surveys. This study showed that geometric (steepness) representations of slope were included in all teachers' responses and algebraic (as a parameter of a linear equation) and physical representations were mentioned more often that functional, trigonometric, or ratio representations. Differences were found between preservice and inservice teachers—inservice teachers made more references and had greater understanding of the trigonometric representations of slope.

Thompson and Thompson (1994, 1996) examined how one middle school teacher's mathematical knowledge was reflected in the language he used to teach the concept of speed to one student. The researchers found that the teacher's difficulty in speaking conceptually about rates had an effect on the student's understanding. While the student was oriented to how she was thinking about mathematical situations, the teacher was oriented to the calculations she employed. The teacher's use of calculational

language interfered with his intention to facilitate the student's conceptual grasp of a situation. For example, in discussing the situation of trying to travel to an airport 120 miles away in 3.5 hours, the teacher said "We multiply 3.5 hours by his speed to go 120 miles," instead of something more situationally attuned, such as "He's going to go at some constant speed for 3.5 hours, and at the end of 3.5 hours he should have traveled 120 miles."

NCTM Standards Related to Rate of Change

Consistent with other scholars who identify change as being fundamental to understanding mathematical ideas such as functions and how they relate to real-life phenomena, PSSM (2000) also recognizes the importance of being able to analyze change in various contexts by understanding relations and modeling phenomena using appropriate tools. It is asserted that ideas of change should receive a more explicit focus in the early grades and throughout all grade levels, and that high school students should have a strong basis for understanding the idea so they can learn about the idea of slope as representing a constant rate of change in linear functions.

Aspects of Knowledge of Rate of Change

Several approaches to the study of various aspects of teachers' content-specific and pedagogical knowledge related to rate of change are relevant to this study.

Concept images and definitions

Concept images and concept definitions are useful constructs for characterizing teachers' knowledge. A concept image is the total cognitive structure that is associated with a given concept. A concept definition is a set of words used to specify a given concept. Discussion of the concept image gives rise to an important distinction between

the accepted formal mathematical definition for a concept and the way that an individual thinks about that concept. Usually, when the concept name is seen or heard, what is evoked is not the formal definition, but the concept image (Tall & Vinner, 1981). A person can hold a concept image for slope that does not seem to correspond directly to a formal mathematical definition, as found in the investigation of teachers' understanding of functions by Vinner and Dreyfus (1989). However, this study did not attempt to distinguish between teachers' concept images and concept definition of rate of change. Rather, it considers concept images and concept definitions together as one component of teachers' knowledge.

Multiple representations and connections

Due to the complex nature and various uses of the idea of rate of change, including slope, situations embedding change lend themselves to a variety of representations such as equations, graphs, tables, and verbal descriptions. Multiple representations have increasingly found their way into the school curriculum as new technology facilitates more efficient constructions of them (PSSM, 2000). However, it has been noted that, in traditional courses, students are not provided with enough opportunities to develop different representations of mathematical ideas beyond those based on symbolic representations (Porzio, 1997). Porzio studied the impact of the instructional approaches used in three different differential calculus courses on students' understanding of the relationship between slope, rate of change, and the first derivative. The results indicated that whereas students in a traditional calculus emphasizing use of symbolic representations to present concepts and solve problems were more likely to form a disjoint or weakly-connected network of knowledge related to the concept of

slope (which may help explain their lack of understanding of the relationship between slope, rate of change, and the first derivative), students in the course emphasizing use of multiple representations were more likely to form strongly-connected knowledge related to the concept of slope.

As such, a regard for the multitude of ways that the same relationship can be represented is indispensable for teaching. A teacher whose concept image of slope highlights the associations between different representations is empowered to make the choice of which representations to employ (Dreyfus, 1991, p. 39). For teachers, the ability and inclination to work with the concept of slope in a variety of formats is necessary for providing students with opportunities to explore and construct connections between alternative representations of situations embedding constant rate of change. Depending on how a relationship is displayed, different interpretations can be made. For example, a graph can provide insight into both global and local features (Leinhardt, Zaslavsky, & Stein, 1990), whereas a table of values may illuminate more local features. When combined, the information gleaned from diverse representations contributes to a deeper, more comprehensive understanding of the underlying situations of change. The importance of multiple representations and connections among them is also compatible with the viewpoint elaborated by Thompson (1994), who posited that

> Our sense of "common referent" among tables, expressions, and graphs is just an expression of our sense, developed over many experiences, that we can move from one type of representational activity to another, keeping the current situation somehow intact. Put another way, the core concept of "function" is not represented by any of what are commonly called the multiple representations of function, but instead by our making connections among representational activities. (p. 39)

Mathematical modeling

A sense of connection among representations for the concept of a mathematical idea is enhanced through a variety of experiences with applied problem settings that allow students to describe, explain, manipulate, and predict a wide range of problem situations (Huntley, et al., 2000). The creation of a mathematical representation of a real-world or non-mathematical situation is often referred to as *modeling* (Dreyfus, 1991). Many current reform recommendations value the mathematical modeling of phenomena as one of the most powerful uses of mathematics, and emphasize modeling and contextualized problem-solving across the mathematics curriculum (NCTM, 1989, 2000). However, research indicates that most school problems being posed to students do not involve the students in creating, modifying or extending systems of representations for meaningful problem situations (e.g., Doerr, 1998). Even in solving typical textbook "word problems," students generally try to make meaning out of questions that are often simply a thin layer of words disguising an already carefully quantified situation.

In PSSM (2000) it is recommended that high school students should be able to develop, identify and find the best fitting model for real-world data by drawing on their own knowledge of ideas and methods that they have developed. They should also be able to explain why that model seems reasonable. Because a teacher makes choices of which problems to engage students in, a teacher's capacity to use and appreciate the importance of the concept in varying contexts is critical.

Summary

Since a major goal of this study is to characterize teachers' knowledge of rate of change, definitions of teachers' knowledge, rate of change, and the associated research

were included in the literature review. This study employs Shulman's conceptualization of teacher knowledge in terms of both content knowledge and pedagogical content knowledge. It addresses important aspects of knowledge of rate of change and is analyzed in terms of the following categories: a) concept definitions and images, b) multiple representations, c) connections, and d) mathematical modeling. Studies of students' knowledge of rate of change have noted some common misconceptions such as understanding of slope and misinterpretation of graphs. Studies of teachers' knowledge of rate of change remain largely unexplored.

Another important component in this study is the exploration of how curriculum materials effect teachers' acquisition of knowledge. Thus, some ways in which teachers acquire their knowledge and the nature of new Standards-based curricular were also included in the literature review.

CHAPTER III

METHODOLOGY

Overview

This study investigated secondary school mathematics teachers' knowledge of rate of change as evidenced by a survey, their work on mathematics tasks and their instructional practice while teaching rate of change. This included an investigation into the differences among teachers with different levels of experiences teaching a particular set of Standards-based curriculum materials that emphasize change as a central theme. The research questions upon which this study focused were:

1. What characterizes teachers' mathematical content knowledge and pedagogical content knowledge of rate of change in the context of teaching Standards-based curriculum materials?
2. What qualitative differences exist between experienced and less experienced teachers using Standards-based curriculum materials with respect to their knowledge of rate of change?

To gain a richer description of teacher knowledge in the context of teaching a Standards-based curriculum, thirteen teachers at various levels of experience with the curriculum were studied. These teachers were selected to provide a range of similar and contrasting levels of experiences to investigate. The selection of the teachers was based on the teachers' educational background and the number of years they had been teaching

and using the Standards-based curriculum developed by the Core-Plus Mathematics Project (CPMP).

Because of the exploratory nature of the study, qualitative data collection and analysis procedures were utilized. Data collection was divided into three phases. In phase I, a pre-survey assessed teachers' conceptions of teaching and learning rate of change. Although teachers' conceptions were not a primary focus of this study, the information gathered from this data helped gain additional insight into teachers' mathematical content and pedagogical content knowledge of rate of change. In phase II, teachers were individually interviewed using a set of mathematics problems to assess primarily their mathematical content knowledge of rate of change. In phase III, a subset of four teachers from the group of 13 was observed while they taught units where the focus on rate of change was central, to further investigate differences in their knowledge of rate of change.

Data analysis was an ongoing process. As the data were gathered, emerging themes and possible categories were noted, guided by the framework designed for this study. The analysis proceeded in two stages. Initially, individual teacher's data were examined and described. Then, a comparison was made among three groups of teachers categorized by their experience teaching the CPMP curriculum. In both stages, triangulation between various data sources was conducted to confirm emerging patterns, themes, and conclusions.

This chapter first presents a guiding framework for interpreting secondary mathematics teachers' knowledge of rate of change, then discusses the research procedures used in this investigation. Next, the teachers who participated in the

investigation and instruments used to collect data related to the two research questions are described. The final section describes the methods used to analyze the data.

A Guiding Framework for Secondary Mathematics Teachers' Knowledge of Rate of Change

A framework, consisting of a blocked list of questions, was designed to guide this investigation of secondary mathematics teachers' knowledge of the concept of rate of change. Because rate of change ideas are closely related to the concept of function, this framework was developed from a synthesis of various frameworks of teachers' knowledge of functions (e.g., Even, 1990; Norman, 1992; Wilson, 1994), and in part adapted from the framework used by Stump (1997) to investigate teachers' knowledge of slope. Also included in the synthesis were the recommendations outlined in the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989) and *Principles and Standards for School Mathematics* (NCTM, 2000). Because this study also investigated some aspects of teachers' pedagogical content knowledge of rate of change, the guiding framework included aspects of this type of knowledge. (See Figure 2)

This framework included five major components designed to interpret teachers' knowledge of rate of change: a) Concept image and concept definition, b) Multiple representations and connections, c) Linear connection, d) Mathematical modeling, and e) Interpretation of rate of change for teaching students. Given that there is some natural overlap to these components, the first four were designed to primarily assess mathematical content knowledge, and the last focused on pedagogical content knowledge. All five components of the framework were used to characterize participating

Concept Image and Concept Definition 1. What are the teachers' concept images and concept definitions of rate of change? As an algebraic concept? As a geometric concept? As a physical concept? As a trigonometric concept? As a characteristic of function? 2. What examples do they use to illustrate their concept images and concept definitions? 3. What are relations between their concept images and concept definitions?
Multiple Representations and Connections 1. Can the teachers use their concept of rate of change to recognize patterns of change between variables represented in tabular, graphic, symbolic, and verbal form? 2. Can they use tables, graphs, verbal descriptions, and symbolic rules to represent relationship between two variables involving rate of change? 3. Can they translate from one representation of rate of change to another? 4. Can they construct situations in which a rate of change is an essential component? 5. Can they choose an appropriate representation to explore different contexts that exhibit constant (or nearly constant) or non-constant rate of change?
Linear Connections 1. Can they discriminate between constant rate of change (linear) and non-constant rate of change (nonlinear) in situations involving relationships between two variables? 2. Can they use their understanding of rate of change to help to make sense out of situations involving change in which rate of change is not constant? • Can they describe how the rate of change of a function at various points is shown by the shape of the function's graph? • Can they estimate the rate of change of a function at any point on its graph? • Do they understand the similarities and differences in estimating the rate of change for linear and nonlinear functions, and how those methods are related to finding the slope of a line? • Can they interpret the derivative function to predict the behavior of its original function?
Mathematical Modeling 1. Can they apply their understanding of rate of change to model and solve problems in which linear or nonlinear relationships exist between variables of interest? That is, can they analyze data, represent key aspects of problems, interpret results, make predictions from data, and generalize a method that can be used and adapted to find solutions to problems in a range of contexts that exhibit linear (or nearly linear)or non-linear relationships?
Interpretation of Rate of Change for Teaching Students 1. How do teachers develop or plan to develop the concept of rate of change? 2. What contexts, problems, representations, analogies, illustrations, examples, explanations, demonstrations do they use or plan to use in teaching the concept of rate of change? 3. What concepts do they identify as prerequisite for students' understanding of rate of change? 4. (How) can they explain student difficulties with the concept of rate of change? 5. Do they emphasize concepts or procedures in teaching about rate of change?

Figure 2. Guiding framework for secondary teachers' knowledge of rate of change

teachers' knowledge in response to the research questions of this study. The framework was also used in comparing and contrasting the knowledge of the participating teachers based on their experience in teaching the CPMP curriculum.

The guiding framework evolved as the investigation progressed. In its original form, the framework was used to develop pilot questions for the research instruments and to modify them. As the questions were modified, so was the framework. In order to insure that key objectives were addressed, the framework was also reviewed by three mathematics educators. Figure 2 shows the final version of the guiding framework used in this study.

Research Procedures

This study investigated secondary school teachers' mathematical content knowledge and pedagogical content knowledge of rate of change as evidenced by a self-report survey, work on mathematical tasks and observation of instructional practice while teaching rate of change. This study also examined differences among teachers with different levels of experiences using the CPMP curriculum materials.

The process of participant selection began in July of 2003. This study required teachers with various levels of teaching experience with the CPMP materials, so the categories (shown in Figure 3) were used for selecting a potential pool of teachers. Initially, CPMP project staff and Western Michigan University mathematics educators recommended teachers for the study who they believed fell into the categories described in Figure 3. These teachers were contacted and the nature of the research project was explained. In addition to explaining the general intent of the study, the types of data to be collected and the time commitments were explained to the teachers. Seventeen teachers agreed to

participate after this initial contact. However, by the time the study began, 13 of those teachers, from seven different schools, agreed to participate: Two for category 1, three for category 2, one for category 4, two for category 6, one for category 7, and four for category 8. Although an attempt was made to include teachers from each category, no teachers were selected for category 3 and 5.

Category 1:	Teachers who were just beginning to teach Course 1 (preferably with no other CPMP experience)
Category 2:	Teachers who had at least two years experience teaching Course 1 (preferably with no other CPMP experience)
Category 3:	Teachers who had no more than one year (> 0) experience teaching Course 2 (preferably having also taught Course 1)
Category 4:	Teachers who had at least two years experience teaching Course 2 (preferably having also taught Course 1)
Category 5:	Teachers who had no more than one year (> 0) experience teaching Course 3 (preferably having also taught Course 1 and 2)
Category 6:	Teachers who had at least two years experience teaching Course 3 (preferably having also taught Course 1 and 2)
Category 7:	Teachers who had no more than one year (> 0) experience teaching Course 4 (preferably having also taught the other 3 courses)
Category 8:	Teachers who had at least two years experience teaching Course 4 (preferably having also taught the other 3 courses)

Figure 3. The categories used in selecting the participants

All 13 teachers were given (via e-mail or in person) the survey at the beginning of the study to be completed on their own time. They reported that it took them approximately 30 minutes to complete the survey. The modes they used to return their survey varied: via mail, e-mail or in person. In addition to the survey, all 13 teachers were also individually interviewed in October through December of 2003 using a set of mathematical problems designed to address their range of knowledge of rate of change.

All of the interviews were conducted at the teachers' schools, usually in their classrooms. During the interviews, teachers were asked to think aloud as they completed the problems and responses were audiotaped and later transcribed. The researcher did not deviate from the problems, however, teachers were often asked to clarify the meaning of their responses or probed more deeply about their understanding. The length of the interviews varied from one hour to two hours. Written work done by the teachers during the interview was collected for later analysis.

A subset of four teachers from the group of 13 was purposefully selected for classroom observations and interviews, with the potential to provide the greatest contrast in experience levels. The two least experienced teachers and the two most experienced teachers in teaching the CPMP curriculum were chosen. Observations focused on particular CPMP lessons/investigations where rate of change was a central theme. Because of the placement of the units within the school year, for the most experienced teachers was observed September through November of 2003. The least experienced teachers were observed December of 2003 through February of 2004. The same class was observed from four to six times for each teacher, and in the case of teachers teaching more than one section of the course to be observed, the classes were chosen on the basis of timing. Teachers were asked to provide the researcher with copies of supplemental assignments, quizzes and tests given during the observation period. Classes were audiotaped and later transcribed.

A semi-structured interview was conducted before and after each observation. The pre- and post-observation interviews were audiotaped and transcribed.

Participants

The thirteen teachers who completed the survey and the mathematical task-based interview were from six high schools and one middle school. The participants included 9 males and 4 females. All but two teachers had experience teaching the CPMP curriculum. The two teachers (in category 1) were first-year teachers who were beginning to teach Course 1. All but one teacher were currently teaching a CPMP course. The years of teaching experience for these teachers ranged from half a year to thirty seven years (including two first-year teachers), with an average of 12.4 years and a median of 9 years. Nine of the teachers held bachelor degrees in mathematics. Four held bachelor degrees in mathematics education. Four had received masters degrees in mathematics education and one was completing a masters degree in mathematics education at the time of the study. One had received a masters degree in mathematics. Six teachers had been exposed to the CPMP curriculum before they started using it in their classroom and eight teachers had been exposed to other reform-oriented curricula in their methods courses, conferences, or both. Five teachers who had relatively little teaching experience started their first and also current teaching job with the CPMP curriculum. All but two teachers had participated in some kinds of professional development programs related to the CPMP curriculum materials.

Background information for all 13 teachers is provided in Table 1 and more detailed information on the teachers' teaching experience relative to the CPMP curriculum is presented in Table 2. A coding system was used for teachers' names to indicate three levels of teaching experience with the CPMP curriculum: 1) HE for the

Table 1
Participant background information

Teacher Identifications[a]		Degree(s) & Major(s)		Number of years teaching	Number of years teaching CPMP	Exposed to CPMP before the first time teaching it	Exposed to other reform curricula	Participated in PD related to CPMP	Experience teaching calculus
		Bachelor Major/Minor	Master						
LE	L1A$_1$*	Math Ed/English	None	First year	First year	Yes	Yes	Yes	No
	L1A$_2$*	Math/Psychology	None	First year	First year	No	No	Yes	No
	L3A$_3$	Math Ed/ Family consume	None	2.5	2.5	Yes	Yes	Yes	No
ME	M1B	Math/Biology	None	4	4	No	No	No[d]	No
	M1C[b]	Math	Math	35	6	No	Yes	Yes	No
	M2C	Math	Math Ed	9	4	Yes	Yes	Yes	No
	M3C	Math	Math Ed[c]	20	6	Yes	No	Yes	No
	M3B	Math Ed/ Earth science	None	5	5	Yes	No	No[d]	No
	M4F	Math/Physics	None	13	5	No	No	Yes	No
HE	HF	Math	Math Ed	6	5	Yes	Yes	Yes	No
	HD*	Math/French	Math Ed	13.5	7	No	No	Yes	No
	HB	Math	Math Ed	15	7	No	Yes	Yes	Yes
	HG*	Math/Science	None	37	10	No	Yes	Yes	Yes

Note. [a] Keys for the teachers' coded identifications: The first letter indicates the group category in which the teacher was classified. The number in middle (used for LE and ME teachers) indicates the highest course in the CPMP course series that the teacher had experience of teaching. The letter at the end indicates the teacher's school identification (i.e., teachers of which coded identifications end in the same letter means they were at the same school.) [b] The teacher is a middle school teacher. [c] The teacher was completing the master degree at the time of the study. [d] The teacher indicated that he/she received interdepartmental development (f.g., working with colleague teachers on lesson planning or assessment). * Indicates classroom observation participant.

Table 2

<u>Participants' experience levels teaching the CPMP curriculum</u>

Teacher Identifications		Number of years teaching the particular CPMP course(s) before participation in the study				The CPMP course(s) taught during the study			
		Course 1	Course 2	Course 3	Course 4	Course 1	Course 2	Course 3	Course 4
LE	$L1A_1$*	First year				X	X		
	$L1A_2$*	First year				X	X		
	$L3A_3$	2		1		X		X	
ME	M1B	4				X			
	M1C	6				X			
	M2C	2	3				X	X	
	M3C	6	4	2			X	X	
	M3B	2	5	5			X	X	
	M4F	3	2		1				
HE	HF	3	3	3	2		X	X	
	HD*	6	4	4	2		X	X	X
	HB	4	5	6	4			X	
	HG*	4	6	6	4		X	X	X

<u>Note.</u> * Indicates classroom observation participant.

for highly experienced teachers who had taught all four CPMP courses each for at least two years, 2) ME for teachers having a moderate amount of experience teaching the CPMP curriculum—the teachers in this category had a mix of experience teaching the first three CPMP courses (with one having taught Course 4 once, but not Course 3), and 3) LE for the least experienced teachers who had not taught any CPMP courses in the past or those whose experience was limited mainly to Course 1 and 2 (with a single exception of one teaching Course 3 once). Thus, for example, teachers whose coded names begin with the letter "H" are HE teachers who had taught all four CPMP courses each for at least two years.

Observation Participants

From the group of thirteen teachers, classroom observations (with pre- and post-interviews) were conducted with four teachers. Selection of observation participants focused on generating cases where the teachers varied in level of teaching experience with the CPMP curriculum, in order to gain additional insight into how teachers' experience in teaching the CPMP curriculum might enhance their knowledge of rate of change. To accomplish this, two LE teachers, $L1A_1$ and $L1A_2$, and two HE teachers, HD and HG, were selected. Criteria used for selection of these four teachers were their willingness to be observed in their classroom and their teaching assignment to either Course 1 or Course 4 during the time of this study.

Ideas of mathematical change and rate of change are addressed in each course of the CPMP curriculum. The discussion of "change" begins in Course 1 and revolves around linear and exponential functions, Course 2 focuses on power functions and periodic change, and Course 3 deals with families of functions and sequential change

using recursion. In Course 4, the rate of change idea is more intensively discussed and formalized in ways that underpin Calculus.

Investigating teachers with the largest contrast in experience with the CPMP curriculum seemed to have the most potential for exploring possible differences in their knowledge. Therefore, the selected four teachers were those with the least ($L1A_1$ and $L1A_2$) and most (HD and HG) experience teaching the CPMP materials among the thirteen. In addition, teachers HD and HG had taught all four CPMP courses. The two LE teachers were first-year teachers and had no formal teaching experience using the CPMP materials or any other curriculum materials. These four teachers' background information is presented in Tables 1 and 2. More detail on these teachers' background information is presented in Chapter IV.

Instruments

The survey, the mathematical task-based interview and the classroom observation protocol were designed to probe teachers' mathematical content and pedagogical content knowledge of rate of change. Questions were selected to address the five components of the guiding framework for secondary mathematics teachers' knowledge of rate of change devised for this study: concept image and concept definition, multiple representations and connections, linear connections, mathematical modeling and interpretation of rate of change for teaching students. The complete forms of the survey, mathematical task-based interview and observation protocol appear in the appendices. A description of each instrument and its relation to the research question is presented in the sections that follow.

Survey

The initial data collection phase of the study was by a pre-survey of each of the participating teachers, administered for take-home completion. This survey consisted of three sections: Background information, Perceptions of teaching and learning rate of change, and Perceptions of textbooks and the CPMP curriculum.

Background information

The first section of this survey contained items concerning information about the educational and teaching history of each teacher and their teacher preparation program and teaching experiences, such as courses taught and types of professional development programs attended. Moreover, it also contained background on their school, such as how CPMP is used, and the support (preparation) teachers received for teaching CPMP. This particular section of the survey was taken from a survey designed and used by the Core-Plus Mathematics Project. Information from this section was used to help characterize individual teachers (presented in a previous section). See Appendix A: Background Information for a listing of the corresponding survey items.

Conceptions of teaching and learning of rate of change

This section of the survey consisted of two open-ended questions (Items 1 and 2) and 14 Likert scale items (Items 3 – 16). In order to gain additional insight into teachers' mathematical content and pedagogical content knowledge of rate of change, the two open-ended questions assessed teachers' thoughts about learning goals and prerequisites for students to understand rate of change. Teachers were asked to identify two or three of the most important ideas they want students to learn about rate of change (in Item 1) and

concepts students need to know in order to understand the idea of rate of change (in Item 2).

Items 3 – 16 were designed to identify important beliefs the teachers have about teaching and learning the idea of rate of change. Teachers were asked to indicate the extent of their agreement with each belief statement using a five-point Likert scale (SD = Strongly disagree (1), D = Disagree (2), N = Neutral (3), A = Agree (4), SA = Strongly agree (5)). Although teacher beliefs were not the primary focus of this study, researchers suggest that in order to understand teachers' thinking about a mathematical idea, either in planning for instruction or in teaching, it is important to understand their beliefs about the mathematical idea or their teaching as well as their knowledge (Cooney & Wilson, 1992). Teacher beliefs have the potential to affect the way teachers learn mathematics (which, in turn, affects what the teacher learns) and also the way that the teacher teaches mathematics. Thus, information on this section of the survey helped provide a clearer perspective on what characterized the teachers' knowledge of rate of change. For example, a teacher's reliance on an algebraic representation for teaching slope (e.g., "m" in an equation $y = mx + b$) may originate in her strong belief that students need to demonstrate a procedural understanding of slope before doing any investigation (see item 13 in this section of the survey). Moreover, the teacher's own procedural-based understanding of slope may contribute to the development of such a belief. And the teacher's understanding of slope may be restricted by the way the idea of slope is presented in the curriculum he/she uses.

The 14 Likert scale items were adapted from a survey used in an evaluation study of the NSF-funded TRUMPET (Targeting Reform of the Undergraduate Mathematics

Preparation of Elementary/Middle School Teachers) project (Ziebarth, 2002) to study the conceptions of probability and statistics held by elementary preservice teachers. They were altered to address the domain of rate of change. Items were pilot tested to ensure that each item was clear, distinct from other questions, and meaningful for the goal of the survey. See Appendix A: Conceptions of Teaching and Learning of Rate of Change for a listing of the corresponding survey items.

Conceptions of textbooks and CPMP

The third section of the survey contained five open-ended questions (Items 1 – 3, 16 and 17) and 12 Likert scale items (Items 4 – 15). Questions 1–3, shown below, were designed to assess teachers' beliefs about textbooks (or other curriculum materials):

1. What is the purpose of textbooks (or other curriculum materials)?
2. What makes a "good" textbook *for students* who are learning mathematics?
3. What makes a "good" textbook *for teachers* of mathematics?

Items 4–15 were designed to identify teachers' views about the CPMP curriculum with respect to rate of change– how well CPMP materials support students to learn, and teachers to teach and learn, rate of change. Response choices were indicated on a five-point Likert scale (SD = Strongly disagree (1), D = Disagree (2), N = Neutral (3), A = Agree (4), SA = Strongly agree (5)). For example, Item 8 asked teachers to indicate the extent to which CPMP investigations involving rate of change challenge *students* to think more deeply about those ideas. In Item 9, teachers were asked to indicate a level of their agreement with that same statement but with respect to themselves (i.e., the extent CPMP investigations involving rate of change challenge *you* [as a teacher] to think more deeply about ideas of slope and rate of change.). These Likert-scale items were developed from

an instrument used by Lloyd et al. (2002) to examine elementary preservice teachers' beliefs about textbooks. In addition, two open-ended questions (Items 16 and 17) asked for teachers' thoughts about the impact of the CPMP curriculum on their teaching and mathematical learning.

Responses on the items in this section of the survey provided information to explore possible contributions made by CPMP to enhancing teachers' own learning of the concept of rate of change. See Appendix A: Conceptions of Textbooks and CPMP for a listing of the corresponding survey items.

Mathematical Task-Based Interview

This interview (see Appendix B) consisted of eight questions assessing various aspects of mathematical understanding of rate of change as elaborated in four of the five components of the guiding framework. Questions 1 and 2, shown below, directly related to the first component of the framework. The questions were designed to assess teachers' concept of rate of change as shown in their descriptions of concept images and concept definitions of rate of change:

1. In the context of algebra and function, what does "rate of change" mean to you? How would you illustrate this idea?
2. How would you define rate of change? Can you give one or more examples to illustrate?

The remaining questions were selected in relation to the second, third and fourth components of the framework. Guided by the framework, 17 problems were originally chosen as a sample pool from various references, including the *Professional Standards for School Mathematics* (NCTM, 2000), an instrument developed by a comparative study to investigate effects of a Standards-based curriculum, in particular CPMP, on students'

understanding (Huntley et al., 2000) and Harvard Consortium's *Calculus* (Huges-Hallet et al., 1998). To establish content validity, the 17 problems were initially reviewed by one of the authors of CPMP and then reviewed by two other mathematics educators. In response to reviewers' comments on the richness of problems and the length of time for their completion in an interview setting, some questions were modified, combined or replaced as necessary.

Six problems (Problems 3 – 8) were then selected to examine the aspects of mathematical understanding of rate of change, identified in the framework as: a) use and interpretation of multiple representations and translation between representations, and connections among representations (primarily assessed in Problems 3, 5 and 8), b) connections between different types of rate of change (primarily assessed in Problems 6 and 7), and c) modeling of real world situations (primarily assessed in Problem 4). That instrument, together with the other two questions related to concept images and definitions, were pilot-tested with two high school teachers and three student teachers doing internships at a high school and changes were made as a result.

The first purpose of the pilot interviews was to determine the clarity of the questions and procedures. Some adjustments were made in how questions were displayed and ordered in response to the experience in the pilot interviews. The second purpose was to determine how long the interview would last with the number of questions proposed. Although there was some variation in time required to complete the interview, the results of the sample interviews confirmed the interview could reasonably be conducted in about 1.5 hours. The third, and perhaps most critical, purpose of the pilot interviews was to determine the potential of the rate of change questions to reflect differences in teachers'

rate of change knowledge. In the process, the sample interviews also provided an opportunity for the researcher to discover and anticipate possible approaches to the rate of change items. For example, in Problem 5 (taken from CPMP Course 4) of the pilot instrument, pilot-teachers were asked to use multiple representations to describe the rate of change of an exponential function $f(x) = ab^x$:

> Problem 5. Suppose *f* is an exponential function of *x* with rule in the form
> $f(x) = ab^x$
>
> (a) What is the rate of change of f ?
> (b) How is the rate of change of f shown in tables, graphs, and rules of the function?
> (c) What rule gives the rate of change of f at any point x?

One teacher (who had the most experience teaching Course 4 where rate of change ideas are intensively discussed) showed understanding of relating patterns in numerical results to the shape of a graph and making connections between the derivative function (which he mentioned as a general rule giving the rate of change of the given function *f* at any point) and the rule for finding the slope of a straight line. Another teacher (who has used CPMP for 7 years but not taught CPMP Course 4) was able to use a table and graph that she drew to explain instantaneous rate of change at various points, but was unable to discover ways to use such information to find a general rule of rate of change of the given function. She tried to remember its derivative function that she knew would give the answer and finally gave up, commenting about how you lose things if you don't teach them. One student teacher (doing his internship when interviewed) said the rate of change of $f(x) = ab^x$ is "b," which is the multiplicative factor that appeared in a table that is multiplied by each previous term to get the successive term. During the interview, he created a table for x = 1, 2 and 3, and said:

> The rate of change [of $f(x) = ab^x$] is b so it has the constant rate of change which is the slope of the line...but I know exponential functions can't be a line [drawing a graph of an increasing exponential function] . . . I mean straight line. Maybe, maybe it goes up by the same speed something.

This suggests that his understanding of rate of change is predominated by constant rate of change. He misunderstood the change in the table of the given exponential function f(x) as b which is a constant consistently multiplied to the next term. (He had to subtract a previous term from its next terms to find a change.) His response also highlights the misconception that "slope" is an equivalent term with "constant rate of change," rather than slope is one representation among many that could be used to represent constant rate of change. Piloting this instrument, thus, showed that these items certainly had the potential to uncover teachers' thinking and to highlight differences in teachers' rate of change knowledge. This problem was ultimately changed to a particular, but simpler, function, $f(x) = x^2$, for use in this study. Such decision was made because of concerns that teachers' *levels of understanding of function* may obstruct the exploration of rate of change (e.g., if they could not graph $f(x) = ab^x$, it would be a limiting factor of thinking about rate of change, perhaps causing more confusion.).

<u>Classroom Observations</u>

Since teachers' instructional behavior also reflects what teachers know about the mathematical domain being taught, it was reasonable to explore instructional tasks and classroom discourse as another way to depict what they know. Observation of teaching has not often been used in the past to capture teacher knowledge, because teaching using traditional textbooks did not easily lend itself to such scrutiny. In that environment, teachers typically present procedural exercises. Discourse is limited, usually *from* teacher *to* students. In such an environment, teachers' knowledge is minimally explicated through

the way they use their knowledge in practice. Thus, it was not realistic to expect to capture, through observation of teaching practice, the aspects of the teachers' knowledge that couldn't be captured by their work on a series of mathematics problems. Conversely, the context of Standards-based mathematics curricula creates conditions where teachers have to make explicit their understandings of mathematical ideas and procedures and use their understanding to probe and push students' thinking (Ball, et al, 2001). Thus, classroom observation became an important tool in this study to understand the mathematical understanding of teachers using Standards-based textbooks and to probe possible contributions made by curriculum materials to their knowledge.

The four observation teachers were observed teaching units in which ideas of mathematical change are central. (Information on the observed units is presented in the section "Results of Classroom Observations" of Chapter IV.) The classroom observation protocol (see Appendix D) was used to record field notes during observations to help supplement the audio recordings and form the basis for post-observation interviews. The observation protocol was developed based on the guiding framework devised for this study and partly adapted from one developed by Mendez (1998). The classroom observation protocol was designed to record observations in a systemic manner that was appropriate for the setting and participants in this study and that facilitated analysis of the observations. The observation protocol included descriptions of the physical setting and verbal portraits of the people involved, a summary of the lesson, and a record of the nature and substance of the teacher's actions and interactions with students or those among students. Substantial actions and interactions were observed according to the following aspects:

a) repertoire of aspects of rate of change utilized during the lesson (e. g., algebraic, geometric, or functional characteristic concept);
b) multiple representations (tabular, graphic, symbolic and verbal) and connections among them utilized during the lesson;
c) calculational and conceptual misunderstandings that surfaced during the lesson;
d) connections between different rates of change and with different mathematical or real world topics utilized during the lesson;
e) procedural and conceptual generalization of an idea involving rate of change utilized during the lesson, and;
f) important mathematical ideas about rate of change that are pursued or not pursued.

The representations that were utilized and elicited were noted, because they were an important indication of teachers' mathematical content and pedagogical content knowledge. For example, a teacher's ability to discuss instantaneous rate of change of a nonlinear function at one point by relating the derivative rule to the graph of the function shows a depth of understanding that mere memorization of the symbolic rule misses. Whether and how teachers made connections among representations of rate of change were also noted. Connections can also be made between different mathematical ideas, and the most complex of connections is a generalization which sorts examples into categories or recognizes patterns and gets at the structure of a broader theme within mathematics. Also, instances of when teachers decided to pursue or not to pursue ideas were recorded. For example, student confusions or student responses involving significant mathematical ideas that were or were not pursued in class were noted.

Finally, the observation protocol notes included the researcher's impressions and initial interpretations. They were of an introspective nature as the researcher began to reflect on what was happening in the classroom. None of the four teachers involved in the classroom observations used written lesson plans (preventing the collection of such artifacts), but student handouts, quizzes and exams were collected. These supporting documents, along with field notes and audiotapes of the classes, contributed to a characterization of teacher knowledge and teacher conceptions of rate of change.

Pre- and Post-Observation Interviews

Before each observation period began, a semi-structured interview was conducted to partly probe the participating teachers' mathematical and pedagogical understanding of the idea of rate of change with respect to the lesson being observed, as well as to form a basis for what to focus on during classroom observations. Thus, information gained from these pre-observation interviews, in part, addressed Research Question 1. The teachers were asked to indicate mathematical prerequisites required for students' understanding the idea of rate of change within the upcoming lesson, to tell what they wanted students to know from the lesson, and to identify difficulties that students may have with the idea discussed in the lesson.

As a follow-up to each observation, teachers were asked to reflect on the lesson by indicating whether they thought goals were reached and providing reasons for making any unplanned alterations. Some of the questions in this semi-structured interview followed-up and built upon the data previously gathered for each teacher. In this respect, the post-observation interview varied somewhat in its focus from teacher to teacher. Discussions revolved around what the individual teacher had done in class, what other

representations could have been incorporated, what connections could have been made and/or how these ideas could be generalized. Because it was assumed that behavior is purposeful and expressive of deeper values and beliefs (Marshall & Rossman, 1999), the rationale for teachers' decisions in probing students' thinking, noted during classroom observations, were discussed to gain an understanding of why such decisions were made. The rationale often explained what they understood about the mathematical ideas and students' thinking. Therefore, how the decisions were tied to their mathematical content knowledge and pedagogical content knowledge regarding that idea was probed. (See Appendices E and F for a listing of questions in the pre- and post-observation interview protocols).

Observation and post-observation interview protocols were pilot tested with two high school teachers and three preservice secondary teachers. For each of the high school pilot teachers, two classroom periods were observed and either audio- or video-taped. During this time, the focus of both teachers' classes (CPMP Course 1) was on studying and making connections between numeric (constant rate of change), graphic (constant slope), and symbolic ($y = mx + b$) characteristics of linear relationships.

During the time when the preservice teachers were observed, they were teaching small groups of students (Grade 8) in one middle school classroom setting, as part of a research project, MTEC.[5] They used a Standards-based middle school textbook, *Connected Mathematics Project* (Variables and Patterns), and the observed lessons were centered on exploring three ways (verbal description, table, and graph) of representing a

[5] MTEC (Mathematics Teacher Education Community) is a research project, directed by Van Zoest at Western Michigan University, that aims to form various communities of mathematics teachers focusing on improving teaching.

changing situation. Although the goal of this study is to investigate teachers' knowledge at the high school level, the mathematical content being discussed was relevant and the students were completing their 8th grade year.

Throughout the observation period, the researcher took field notes, using the observation protocol developed for this study, to check the appropriateness of if the predetermined categories for documenting complex actions and interactions, if there were some themes to be added, and if using this protocol would not intrude excessively on the ongoing flow of events. Also, it permitted the researcher to practice taking notes while observing and become familiar with the format of the protocol.

As a result of the pilot, the format was altered to make it more user-friendly. Some questions were made less specific on the new version. For example, in its original form, the representation question was specified in subcategories (unpacked, amplified, and single), but the three categories were combined in the final form, how many and what representations would be discussed was not clear until the end of the lesson. The predetermined big themes, described in the section on Classroom Observation, remained unchanged and no other themes were added.

During the pilot, interviews were conducted following each observation using the post-observation interview protocol developed for this study to check if the predetermined questions addressed all aspects outlined in the observation protocol. As a result of the pilot, some questions were made more specific to avoid general responses. For example, "What connection could be made?" in its original form was changed to "What connections could be made among the ideas discussed in class?, with different rates of change?, with different mathematical ideas?, or with real-world situations?"

Some questions were expanded to a series of questions to push teachers to reflect on their actions and interactions. For example, "Why did you decide to pursue that mathematical idea?" was expanded to two questions in "Why did you decide to follow-up the students' mathematical ideas? In what ways did the follow-up clarify your rationale?"

Data Analysis

The data gathered from the survey and the mathematical task-based interview were used to provide an in-depth description of CPMP teachers' mathematical content and pedagogical content knowledge and their conceptions of rate of change with respect to the CPMP curriculum. Responses were examined by participant, by question, and by relevant aspects of the concept of rate of change, with an emphasis on distinguishing general patterns or trends within the participants' responses individually and as a group. The guiding framework for interpreting secondary mathematics teachers' knowledge of rate of change was used to analyze the extent and cohesiveness of their knowledge. Because the purpose of the data collection was to provide a description of teachers' knowledge, and not to rank the extent or correctness of their knowledge, the analysis of the data was predominately qualitative in nature. Thus, the analysis of the data was an ongoing and iterative process. The analysis process began as the data was being gathered for each teacher. Initial insights about the data and speculative patterns were recorded in the margins of the researcher's journal. A quantitative measure, frequency, was used to provide general insight into similarities and differences in teachers' responses in terms of the number of specific responses or approaches.

The Survey Data

The results of the survey consisting of a mixture of Likert-scale items and open-ended questions were analyzed item by item and then emerging patterns were identified among all of the teachers as well as within the three groups of teachers (LE, ME and HE teachers). In particular, analysis of open-ended questions involved repeated scans of teachers' written responses in order to identify emerging issues. In some cases, emerging issues were related to categories originating from the guiding framework designed for this study, but some other issues suggested by the data created new categories. A tentative list of categories from the initial scan was revised and redefined during the subsequent reviews of the data. Data were then coded according to those categories and tabulated, as displayed in the tables in Chapter IV.

The Mathematical Task-Based Interview Data

The items of the mathematical task-based interview were classified based on the four categories of the guiding framework: concept image and concept definition, multiple representations and connections, the linear connection and mathematical modeling. Within each of the categories, teachers' responses were analyzed to identify response patterns, major themes and levels of complexity. The responses were grouped according to the following: preference on a particular representation, such as numerical vs. graphical representations; frequently demonstrated misconceptions, such as the signs of velocity; and procedurally- vs. conceptually-based understanding. Approaches teachers used to solve the problems, types of representations utilized, and errors (if any) were also coded.

The Observation and Follow-Up Interview Data

The results of the classroom observations and follow-up interviews were analyzed by individual teacher and then looked at across the range of teachers (LE and HE teachers) for patterns and themes. As mentioned earlier, the researcher's journal was used as a tool for clarifying assumptions and theoretical orientation. Ongoing speculation about emerging patterns was used as the basis for questions during the interviews with teachers and for generating themes to be followed up in subsequent observations. During analyses, efforts were made to search for regularities in the data and find events that represented distinct but substantial segments of data.

Data were coded initially using the existing categories of mathematical content and pedagogical content knowledge in the guiding framework, but new or more specified categories arising from the data, such as categories involving "graph as picture," misconception, and understanding of second derivative, were developed as well. The coding encompassed both single propositions and activity segments, and follow-up interview transcripts were used to clarify and/or look for explanations for teachers' instructional decisions in the coded propositions and activity segments. No attempt was made to code every proposition, and some propositions received more than one code. Then significant themes within and across the two groups of teachers were determined. At this level of analysis, the researcher referred to other data sources of individual teachers in an attempt to find consistencies and inconsistencies, permitting the researcher to further analyze the findings for particular teachers. Finally, significant statements and examples were incorporated into the report of findings for the purpose of comparison between the two groups of teachers. Decisions about which lesson vignettes to include for

support were determined by how representative the lesson was of what the teachers did and did not know and how the information related to the research questions being explored.

Summary

An overarching goal of this study is to characterize teachers' content and pedagogical content knowledge with respect to rate of change in the context of teaching the CPMP curriculum. A framework was designed to provide a comprehensive guide for analyzing different aspects of rate of change knowledge consisting of concept image and definition, multiple representations and connections, linear connection, mathematical modeling, and interpretation of rate of change for teaching students. Thirteen teachers with a wide variety of teaching experience with the CPMP materials were selected to provide a range of similar and contrasting levels of experiences to investigate. All teachers completed a survey documenting perceptions of teaching and learning, and individual interviews consisting of two questions (one relating concept image and the other relating concept definition) and a series of six mathematics tasks. Four teachers, two with the least amount of experience teaching the CPMP curriculum and two with extensive experience (having taught all CPMP courses), were selected for classroom observations. Data collected from all sources were primarily analyzed qualitatively and provided evidence to characterize teachers' knowledge of rate of change.

CHAPTER IV

RESULTS

This chapter presents results of this study of secondary school teachers' knowledge of the concept of rate of change. First, the results of the survey, completed by all 13 participating teachers, are presented to illustrate their conceptions of teaching and learning rate of change and their thoughts about textbooks and the CPMP curriculum materials. Next, the results of the mathematical task-based interview, conducted with each participant, are presented in relation to the components of the guiding framework for analyzing secondary teachers' knowledge of rate of change devised for this study: Concept Image and Concept Definition of Rate of Change, Multiple Representations and Connections, The Linear Connection, and Mathematical Modeling. Finally, the results of classroom observations and follow-up interviews for a subgroup of four teachers are reported to contrast the least and most experienced teachers using the CPMP materials. Although data were collected from individual teachers, in reporting the results, themes related to the teachers' knowledge base relative to rate of change are reported for the group rather than for individuals.

These data provide a broad but in-depth characterization of secondary mathematics teachers' mathematical content knowledge and pedagogical content knowledge of rate of change. This characterization also addresses the issue of how teachers' experience in teaching the CPMP curriculum might enhance their knowledge of rate of change.

Results of the Survey

All 13 teachers completed the survey. The survey consisted of three sections: background information, conceptions of teaching and learning rate of change, and conceptions of textbooks and the CPMP curriculum. Since the participating teachers' background information was reported earlier (see Chapter III), only results from the remaining two survey sections are reported below.

Conceptions of Teaching and Learning Rate of Change

This section of the survey consisted of two open-ended questions (Questions 1 and 2) assessing teachers' thoughts about learning goals and prerequisites for students to understand rate of change, and 14 Likert scale items (Items 3 – 16) identifying important beliefs that teachers had about teaching and learning rate of change.

Learning goals

With the intention of gaining insight into teachers' mathematical content knowledge of rate of change, the first open-ended question asked teachers to identify two or three of the most important ideas they want students to learn about rate of change. A wide variety of ideas were listed and many of the ideas were mentioned by more than one teacher. (See Table 3) There were no teachers who focused solely on computational procedures or formulas. Most teachers (11 out of 13) listed a goal that students be able to use and interpret multiple representations of rate of change. Of those eleven, nine of the teachers mentioned the ability to recognize patterns of change or rate of change shown in different representations and two teachers mentioned the ability to use multiple representations (e.g., tables, graphs and equations) to represent the relationship between two variables involving rate of change. Of those nine teachers, two teachers referred to

tables, graphs and equations ($L1A_1$, $L1A_2$), and two teachers referred to only tables and graphs (M3B, HD). The remaining five did not make reference to a particular representation, but made a more general statement such as "describing rate of change from various representations."

Ideas about the process of approximating the rate of change of a curve were listed by five teachers. The responses included "When a graph is curved, an estimate for rate of change is really the linear 'piece'. So it's like finding the slope of the line segment" (HD), and "How to find a rate of change at any given point as well as over an interval" (M3C). All HE teachers, no LE teachers, and only one ME teacher considered the idea of the linear connection as an important idea about rate of change. "Understanding of derivatives" was mentioned by three teachers, with two HE teachers further elaborating that it is "knowing derivative functions of different function families and graphing the derivative functions."

The importance of understanding functions was identified by three ME teachers as understanding of linear relationships, understanding of dependent and independent variables, and understanding of function families (e.g., linear, quadratic, exponential and trigonometric) (2, 1 and 1 incidents, respectively). Two teachers included "developing graphing calculator skills" as an important idea for students in learning rate of change. Other ideas, such as ratio and applying real-life situations, each received a single mention. Table 3 shows the ideas teachers listed and response rates by teacher experience category.

As illustrated in Table 3, differences were found in the LE and HE teachers' responses. While LE and HE teachers both considered the ability to recognize patterns of

Table 3
Teacher responses identifying key ideas important for students' learning rate of change

Identified ideas	Number of responses that include this idea			Teacher identifications
	HE (n = 4)	ME (n = 6)	LE (n = 3)	
Use and Interpretation of Multiple Representations				
1. Recognizing patterns of change represented in multiple representations	3	4	2	$L1A_1$, $L1A_2$, M1B, M2C, M3B, M4F, HB, HD, HG
2. Using tables, graphs and equations to represent rate of change	0	1	1	$L3A_3$, M1C
Linear Connection				
3. Discriminating non-constant rate of change	4	1	0	M3C, HB, HF, HD, HG
4. Knowing instantaneous vs. average rate of change	1	1	0	M3C, HF
5. Estimating rate of change for nonlinear functions is similar to finding the slope of a line	1	0	0	HD
Derivative				
6. Understanding of derivatives (with no details)	2	1	0	M3C, HD, HG
7. Knowing derivative functions of different function families	1	0	0	HD
8. Graphing derivative functions	1	0	0	HG
Functions				
9. Understanding of linear relationship	0	2	0	M1B, M2C
10. Knowledge of dependent and independent variables	0	1	0	M1C
11. Understanding of function families	0	1	0	M1C
Others				
12. Developing graphing calculator skills	0	1	1	$L3A_3$, M1C
13. Understanding the idea of ratio	0	1	0	M4F
14. Applying to real-world situations	0	0	1	$L1A_2$

Note. The column total number of responses is greater than the number of teachers responding since some teachers gave multiple responses. The row total number of responses is the number of teachers including the corresponding idea in their response.

change between two variables displayed in different representations to be one of the most important ideas about rate of change, no LE teachers related that idea to "non-constant rate of change," which is the idea mentioned by all HE teachers. Other ideas that were mentioned by LE teachers, but were not mentioned by any of the HE teachers, were using multiple representational forms to represent rate of change, applying real-life situations and graphing calculator use. All HE teachers, and some ME teachers, addressed ideas concerning the linear connection, derivative and functions. No LE teachers responded in these regards.

Prerequisites for students' understanding of rate of change

Question 2 focused on aspects of pedagogical content knowledge. Teachers were asked to identify concepts students need to know in order to understand the idea of rate of change. The most common responses involved experience with graphs and knowledge of dependent and independent variables (7 and 6 responses, respectively). "Experience with graphs" includes being able to make and interpret graphs and using graphing calculators as a way to make graphs. Knowledge of slope, solving equations, and function families were mentioned less often (5, 3 and 2 responses, respectively). "Knowledge of slope" includes being able to find the slope of a line (4 responses) and knowing that the slope indicates the constant rate of change of the line (1 response). Two LE teachers considered being able to make and read tables as prerequisite concepts for students' understanding as well as make and read graphs ($L1A_1$, $L1A_2$). Other concepts listed were arithmetic skills such as subtraction and fractions ($L1A_1$) and ratios (M4F). One teacher did not list any concept as a prerequisite, stating "not sure if there is any prerequisite concepts" (M3C).

Table 4 presents the prerequisite concepts listed by teachers and response rates by teacher experience category.

Table 4
Teacher responses identifying key prerequisite concepts needed for students' learning of rate of change

Identified Concepts	Number of responses that include this idea			Teacher Identifications
	HE (n = 4)	ME (n = 6)	LE (n = 3)	
1. Being able to make and interpret graphs	2	3	2	$L1A_1$, $L1A_2$, M1C, M2C, M3B, HG, HF
2. Knowledge of variables	1	3	2	$L1A_1$, $L1A_2$, M1B, M1C, M3B, HB
3. Knowledge of slope	2	1	1	$L3A_3$, M2C, HD, HG, HF
4. Solving equations	0	1	2	$L1A_2$, $L3A_3$, M1C
5. Being able to make and interpret tables	0	0	2	$L1A_1$, $L1A_2$
6. Knowledge of function families	2	0	0	HB, HD
7. Ratios	0	1	0	M4F
8. Arithmetic skills	0	0	1	$L1A_1$
9. Use of graphing calculators	0	1	0	M1C
10. No concepts identified	0	1	0	M3C

Note. The column total number of responses is greater than the number of teachers responding since some teachers gave multiple responses. The row total number of responses is the number of teachers including the corresponding idea in their response.

Beliefs about teaching and learning of rate of change

Items 3 through 16 asked teachers to indicate the extent of their agreement with each belief statement using a five-point Likert scale (SD = Strongly disagree (1), D = Disagree (2), N = Neutral (3), A = Agree (4), SA = Strongly agree (5)). For all teachers,

rate of change was perceived as an important mathematics topic. All teachers indicated that basic ideas of rate of change and slope are not too difficult to learn. But two teachers indicated agreement with the item "Introducing slope as a rate of change makes it more difficult to understand." (Teacher $L3A_3$ strongly agreed and teacher M1C agreed.) However, they were not asked to explicate what they considered "basic ideas," so there may have been variation in how they interpreted that statement. All teachers felt confident with their ability to teach concepts of rate of change. They also believed that they needed to be aware of how students learn mathematics and that it was important that they be able to interpret what students are doing as they work on mathematics problems. A majority of teachers (12 of 13) agreed or strongly agreed that learning rate of change in context and through different representations is important (with $L3A_3$ choosing "neutral" to the item), and that students should learn mathematics by being actively engaged in solving problems (with M4F choosing "neutral"). Also, most teachers (10 of 13) agreed or strongly agreed that working cooperatively is an effective way to learn mathematics (with $L3A_3$, M2C and M4F choosing "neutral"). All but one teacher ($L3A_3$) disagreed with the statement "Students need to demonstrate procedural understanding of rate of change before doing any investigation." Five teachers felt neutral or agreed with the statement "The best ways to teach how to solve complex problems is to decompose them into a sequence of basic skills." Three teachers (M2C, M3B, M4F) chose "neutral" and two teachers ($L3A_3$, M1B) chose "agree."

While there did not seem to be large differences in response patterns with respect to teachers' level of experience in using the CPMP curriculum, the response patterns of one teacher varied significantly from those of her peers on specific items. This teacher,

$L3A_3$, had a more procedural view of teaching and learning mathematics in general, and concepts of rate of change in particular, by strongly agreeing that students need to demonstrate procedural skills before doing any investigation. She also believed that decomposing problems into a series of basic skills and teaching for mastery of the skills is the best way of teaching. She saw "making sense" as important when doing mathematics, yet gave "using context and multiple representations in learning" less value than all of her peers.

Conceptions of Textbooks and the CPMP Curriculum

This section of the survey consisted of three open-ended questions (Questions 1 – 3) assessing teachers' beliefs about textbooks (or other curriculum materials), 12 Likert scale items (Items 4 – 15) identifying teachers' views about the CPMP curriculum with respect to rate of change, and two open-ended questions (Items 16 and 17) asking teachers' thoughts about the impact of the CPMP curriculum on their teaching and mathematical learning.

Beliefs about textbooks

When asked about the purpose of using textbooks or other curriculum materials, the most often mentioned responses included "acquiring mathematics content to learn" and "providing examples solved for students and problems to solve" (7 and 6 responses, respectively). The teachers made such comments as "[Textbooks provide materials] for students to review what they learn" (M1C) and "[Textbooks allow] students to have a common experience and common language to use in their discussions of the topics" (HB). These responses suggest that the teachers may principally view textbooks as materials that communicate information for students to learn, not necessarily for teachers.

Three ME teachers who mentioned ideas related to both teachers and students included a statement indicating that a purpose of textbooks is to present teachers information on what to teach and how (M2C, M3B, M4F).

Questions 2 and 3 asked teachers to state their thoughts about what makes a "good" textbook for students and teachers of mathematics. A variety of ideas were listed. Six ideas that were identified as key aspects needed for both students and teachers were that they contain examples (solved for students), problems requiring deep-level thinking but also focusing on skills, clear explanations of ideas, connections to different areas of mathematics and real-world situations, and helpful visual materials (e.g., pictures, diagrams).

Other textbook features considered to benefit students were "review/remedial materials" (M1C), "easy to understand steps for solving problems" (L3A_3) and "being colorful" (M3B).

Beneficial features for teachers included "resources for review/remediation" (M1C, HD), "being written in an easy to understand way" (L3A_3), and "being formatted in a way that is doable in class periods" (M3C). Three teachers (L1A_1, M2C, M3B) also mentioned the importance of having teacher notes involving the background or development of content, suggestions for presenting ideas, and areas where students may have difficulty and suggestions for how to address them. Table 5 reports textbook features listed by teachers and response rates by teacher experience category.

As illustrated in Table 5, teachers' statements about mathematics textbooks involved ways in which textbooks present mathematical topics. Many teachers believed that textbooks display mathematical knowledge that students should acquire through

examples, explanations, and visual aids. Moreover, the textbooks should display the knowledge with clarity. About half of the teachers (including 3 of 4 HE teachers) indicated "problems and/or questions requiring student thinking" as a key aspect that

Table 5
Teacher views about what mathematics textbooks should contain

Response Categories	Number of responses that include this idea					
Mathematics textbooks should be/contain:	For students			For teachers		
	HE (n = 4)	ME (n = 6)	LE (n = 3)	HE (n = 4)	ME (n = 6)	LE (n = 3)
1. Problems/questions engaging student thinking	3	2	1	3	2	0
2. Problems connecting to real-world situations and different mathematical topics	1	4	0	2	1	0
3. Clear explanations of the topics	2	2	1	1	2	1
4. Problems focusing on skills	1	2	0	0	0	1
5. Example problems (solved)	1	1	1	1	1	0
6. Helpful visual materials	0	0	2	0	0	1
7. Review/remedial materials	0	1	0	1	1	0
8. Written in an easy to understand ways	0	0	1	0	0	1
9. Formatted in a way that is doable in class periods	0	0	0	0	1	0
10. Teacher notes	0	0	0	0	2	1

Note. The column total number of responses is greater than the number of teachers responding since some teachers gave multiple responses. The row total (in each section of "For students" and "For teachers") number of responses is the number of teachers including the corresponding idea in their response.

makes a textbook good for students (6 responses) and teachers (5 responses). Only one LE teacher ($L1A_2$) considered "problems and/or questions requiring student thinking" to be beneficial for students and no LE teachers related it to teachers. The importance of clear explanations of mathematical topics and example problems solved for students were mentioned (5 and 3 responses in conjunction with students, and 4 and 2 responses in conjunction with teachers).

Views about the CPMP curriculum

In items 4 through 15 teachers were asked to focus specifically on the CPMP curriculum and indicate their level of agreement with each statement using a five-point Likert scale (SD = Strongly disagree (1), D = Disagree (2), N = Neutral (3), A = Agree (4), SA = Strongly agree (5)). The CPMP curriculum was perceived by most (11 out of 13) teachers as focusing on student thinking in general. All but one LE teacher ($L3A_3$) indicated that the materials challenge *students* to think more deeply about the idea of rate of change. Of the 12 teachers who agreed with this statement, nine also felt that the CPMP materials challenge *teachers themselves* as well as students. There were three teachers who viewed the materials as challenging for students but not for them as a teacher ($L3A_3$, M2C, M4F). Teacher HG chose "neutral" as to whether it challenges students or teachers. On the items concerning the slope idea presented in the CPMP materials, most teachers (11 out of 13) disagreed with the statement "introducing slope as a rate of change makes it more difficult to understand." Only teacher $L3A_3$ strongly agreed and M1C agreed. All teachers seemed to be comfortable with considering explaining slope as a rate of change, but six teachers (no HE teachers) indicated that approaching the slope idea as rate of change may not be easy for *students* to understand.

On the item concerning their perceptions about the problem-based nature of the CPMP materials, 9 out of 13 teachers believed that students better retain what they have learned through materials of this kind. The remaining four teachers chose "neutral".

The response patterns of two teachers differed from their peers on specific items. Teacher M1C felt more strongly than the others that introducing slope as a rate of change makes it more difficult to understand, in particular for students. But that teacher seemed to be comfortable with the approach of addressing slope in this way, agreeing that CPMP's explanations of slope as a rate of change are easy for teachers to follow. Interestingly, he did not believe that CPMP materials challenged him to think differently about slope and rate of change. One LE teacher, $L3A_3$, shared some of the same views as teacher M1C regarding CPMP's approach to the idea of slope. In addition, she did not think the CPMP materials do a good job of helping students learn the concept of slope. It is noteworthy that, in the survey section about teaching and learning rate of change, she showed very procedural views of teaching and learning mathematics in general and concepts of rate of change in particular, emphasizing learning procedural skills and teaching for mastery of the skills. Two ME teachers (M2C, M4F) also valued emphasizing procedural aspects in teaching and learning.

Influence of the CPMP curriculum

In the final two open-ended questions 16 and 17, teachers were asked to reflect on the extent to which they felt using the CPMP curriculum has changed their teaching and helped them expand their own mathematical understanding.

On teaching. When asked whether the CPMP curriculum has had an effect on their teaching, all seven teachers who had used a traditional textbook before using the

CPMP curriculum, answered "yes." Five teachers (all LE teachers, M4F and HF) indicated that this question was not applicable to them since they had begun their teaching career with the CPMP curriculum. All of them stated that the way they approach preparing for and teaching their class is different from the way they were taught. One teacher ($L3A_3$) answered "no" to this question. She recognized CPMP's distinct approach to teaching and she somewhat understood why it utilized this approach. However, she explained that she used an approach that was more comfortable to her. This was the same teacher who also expressed a procedural view of how students should be taught.

When asked how their teaching has changed since they started using the CPMP materials, nine teachers (all seven teachers who indicated the curriculum has changed their teaching and two LE teachers who answered "not applicable") responded. The teachers who responded noticed various things that they now pay more attention to or spend more time on as a result of using the CPMP materials. Their responses were categorized in terms of instructional format, teacher role and teacher focus (i.e., nature of teachers' questions and comments). As shown in Table 6, while LE teachers noted changes about instructional format (such as, spend more time on group work and less time on lecture) the most, HE teachers noted changes about a teacher's role (such as, pay more attention to listening to students' ideas and monitoring group work) the most.

<u>On their own learning of mathematics.</u> All but one teacher (M4F) indicated that there were times when using the CPMP materials that they had to review some mathematics again for themselves. When asked to recall what mathematical ideas those were in general or with respect to slope and rate of change, five teachers had trouble recalling and responded more generally. The following are some sample responses:

Table 6
Teacher responses on what has changed about their teaching

Response categories	Number of responses that include this idea		
	HE (n = 3[a])	ME (n = 4[a])	LE (n = 2[a])
Instructional format			
Spend more time on:			
1. Whole class discussion to build some idea or try to elicit opinions - not just "right answers"- from students	0	1	1
2. Presenting or sharing to have students communicate their findings, solutions, or strategies to the whole class	0	2	0
3. Group work	1	0	2
Spend less time on:			
4. Lecture/Demonstration to show how to solve a problem	1	2	2
Teacher role			
Pay more attention to:			
5. Posing questions and/or problems	2	2	1
6. Listening to students ideas, strategies, interpretations, and reasoning	2	1	1
7. Encouraging to consult with peers as they work	0	1	2
8. Monitoring group work	2	0	0
Teacher focus			
Emphasize more:			
9. Conceptual meaning of mathematical ideas and mathematical processes such as reasoning, justifying solutions	2	1	0
Emphasize less:			
10. Procedures (algorithms)	0	1	0
11. Correctness/accuracy	0	1	1

Note. The column total number of responses is greater than the number of teachers responding since some teachers gave multiple responses. The row total number of responses is the number of teachers including the corresponding idea in their response.
[a] Number of teachers who did not provide responses were subtracted from the total number of teachers in each category.

> [It] helped me think about a lot of the concepts in new and deeper ways. ($L1A_2$)

> I'm constantly learning new ideas in math because of CPMP. ($L1A_1$)
> I can safely say that I have confidence in my math skills as a result of teaching it [the CPMP curriculum]." (M1C)

> Ideas in the curriculum are sometimes presented in new and deeper ways from what I've known. (M2C)

> Connections between tables and graphs, connections to real world applications have increased my understanding. (HB)

Even teacher $L3A_3$, whose teaching style remained traditional, felt that using the CPMP curriculum impacted her mathematical understanding, commenting that "New connections are presented, often that had never been apparent to me before."

Other teachers made reference to "use and connections of multiple representations," and "connections between different mathematical ideas and to real-world situations" in their responses on how the curriculum helped them look differently at slope and rate of change. For example:

> CPMP did require me to look at slope differently. I learned it simply as "rise over run," and learned it within the confines of the concept from a classroom perspective. CPMP's approach to slope uses real-life examples and a piecemeal approach, which was both helpful for me and for the students. (M1B)

> Learning it [slope] in problems that could happen in every day life alone shed new light on the number that before was just a number to me. And derivative, when I read through the text it felt like I completely re-learned about it, though I had many calculus courses in college. (HD)

<u>Summary</u>

Rate of change was perceived by all teachers as an important and generally a not-too-difficult-to-learn mathematical topic. Many teachers viewed use and interpretation of multiple representations and the linear connection as key ideas in learning rate of change, although no LE teachers mentioned ideas concerning the linear connection. The ability to

make and interpret representations was the most often identified prerequisite concept for learning rate of change. Not surprisingly, almost all teachers felt that learning in context and through different representations is important. Working cooperatively and being engaged in solving problems were viewed as effective ways to learn mathematics. All teachers indicated they were confident with their ability to teach concepts of rate of change and slope, and most teachers (11 of 13) felt that approaching slope as a rate of change did not make it more difficult to understand.

When asked about the role of textbooks, the most often mentioned responses were "acquiring mathematics content to learn" and "providing examples solved for students and problems to solve." This indicates that the teachers may conceive textbooks mainly as materials that communicate information for students, not necessarily for teachers. "Problems engaging student thinking and connecting to other topics" and "clear explanations of topics" were most frequently identified as features that can benefit both students and teachers. However, features concerning the nature of problems/questions were not considered by any LE teachers.

With respect to the CPMP curriculum, seventy percent of teachers felt that the CPMP materials challenge teachers themselves, as well as students, to think more deeply about the idea of rate of change and also that students better retain what they have learned through problem-based materials such as CPMP. Almost all of the teachers indicated that the way they approach preparing for and teaching their classes is different from the way they were taught. All the teachers who had taught prior to using the CPMP materials noted changes in their teaching since they started using the CPMP materials, with regard to instructional format, teacher role and teacher focus. Most teachers indicated that using

the CPMP materials had an impact on their own learning of mathematical ideas to varying degrees.

Results of the Mathematical Task-Based Interview

All 13 teachers were individually interviewed using the Mathematical Task-Based Interview (shown in Appendix B). A total of eight interview items were designed to assess various aspects of mathematical understanding of rate of change as elaborated in four of the five components of the guiding framework. Questions 1 and 2 directly related to the first component of the framework, Concept Images and Definitions of Rate of Change. The other three components (Multiple Representations and Connections, The Linear Connection and Mathematical Modeling) were addressed in questions 3 through 8. A summary of teachers' responses to each problem precedes a discussion of general trends that emerged during the analysis of the interview data.

Concept Images and Concept Definitions

The first two interview questions (see below) requested a statement of teachers' concepts of rate of change and an illustration of their ideas that contributed to an understanding of their concept image and concept definition of rate of change:

1) In the context of algebra and function, what does "rate of change" mean to you? How would you illustrate this idea?
2) How would you define rate of change? Can you give one or more examples to illustrate?

Concept images

In terms of the concept image, all but one teacher (M3C) stated and illustrated their concept images of rate of change in more than one way and several themes were evident

in teachers' responses. Table 7 shows the distribution of teacher responses classified (in the most specific category) by similar descriptions and mathematical content area (type).

Table 7
Teachers' concept images of rate of change

Description of concept images of rate of change	Type[a]	# of Teachers (n = 13)	Teacher identifications
1. Slope of a line = $(y_2 - y_1)/(x_2 - x_1)$	Geometric	13	All
2. Rise over run	Geometric	2	$L1A_1$, $L3A_3$
3. For the line y = mx + b, m is the rate of change	Algebraic	4	$L3A_3$, M1B, M3B, HG
4. Constant increase/addition	Algebraic	3	$L1A_1$, M1B, M1C
5. How fast/much something changes over time	Functional	3	$L1A_2$, M2C, HF
6. Relationship between two variables	Functional	4	M1C, M3B HB, HD,
7. Derivative as the slope of the tangent line of the curve at a point	Functional	2	HB, HG
8. Derivative	Procedural	3	$L1A_2$, M2C, M3B
9. Comparison, a ratio	Ratio	1	M4F

Note. The column total number of teachers is greater than the total number of participating teachers since some teachers gave multiple responses. [a] Types were identified through direct references mentioned by teachers or interpreted by the interviewer (the researcher).

As illustrated in Table 7, regardless of level of experience with the CPMP materials, all of the 13 teachers mentioned "slope of a line" illustrated by $(y_2 - y_1)/(x_2 - x_1)$ as a way to conceptualize rate of change. Functional concepts were the second most frequently mentioned type of concept image. Of the nine teachers who mentioned a function concept, four teachers stated their concept images as relationships between two variables using the generic terms, one (or one variable) vs. the other (or the

other variable), while three teachers referred specifically to time as a variable. For example, one HE teacher (HF) said, "It's how much something changes over time. How much is water pouring out of a container over time, say per minute, or how much is a car traveling per hour." To illustrate this functional characteristic of their concept images many teachers drew a rough sketch of the graph of a function. Some sample responses were:

> A linear graph of some sort. As time passes the amount of pay changes in a job. It [the graph] will probably steadily increase. ($L1A_2$)
>
> A linear function. For example, the amount of weight or force that's applied and that's going to make a difference in a stretch or in a distance. (M1C)
>
> If you are looking at population of a city over time and asking them [students], as the years go by, what's happening to the population in this picture. And most of the time population is going to be an exponential growth. And so they'd see the curve increasing and hopefully seeing it not increasing straight because it's increasing at an increasing rate. (M3B)
>
> I usually see it [rate of change] as a curve. Maybe a ball going through the air. A parabolic function where the rate of change is increasing. (HG)

The graphical examples that less experienced teachers used, however, were often restricted to those with a linear relationship.

A total of five teachers mentioned the term "derivative" to illustrate their image of rate of change. However, they seemed to have different understandings of the idea of derivative. In responding to a follow-up question that probed their view of the derivative concept, three of these five teachers ($L1A_2$, M2C, M3B) failed to provide a rationale for their response. These teachers did not seem to understand the concept of derivative beyond a procedural rule for calculating one. Thus their responses were classified as "derivative" (in description) and "procedural" (in type) because they used terminology

without much understanding. For example, two of the teachers commented in the following manner:

> Derivative means rate of change. That's what I've heard from my calculus class in college. I don't remember it all but I do remember some easy ones. 2x is the derivative of x^2 and x becomes 1. But this is it. This is all I remember about derivative. I know derivative means rate of change, but don't know how I can use it to tell you how I think about rate of change. Then I need to go back to the "rise over run" thing. ($L1A_2$)

> In terms of rate of change, the first idea that comes to mind is actually the derivative of a function, which is more of a calculus idea. Why I brought that up is... well, I guess basically I was taught that rate of change is derivative, so I guess for me it was a fact that I remembered what I was taught in college. (M2C)

The remaining two teachers (HB, HG) demonstrated a more sophisticated understanding by stating that the derivative is a function of a point and the slope of the tangent line of the given function at that point. This suggests that their view of the derivative concept in relation to the rate of change concept is functional in nature, as compared to the other three teachers whose view of derivative was more procedural.

Algebraic types of description, referring to "the parameters in the equation $y = mx + b$" or "increment in a table of numerical data," were mentioned by seven teachers including one HE teachers. To exhibit their ideas, those seven teachers all described a linear model in which the rate of change is constant. Of all the teachers, one (M4F) made reference to the idea of ratio by describing his concept image as "a comparison, a ratio of two measurements."

Other possible concepts that can be used to illustrate one's image of the rate of change idea are physical (e.g., the steepness of an incline such as pitch of a roof or grade of a road) or trigonometric (the tangent of the angle of inclination). Neither physical nor trigonometric concepts were mentioned by teachers in this study.

As shown in Figure 4 and Table 8, while geometric concepts dominated LE and ME teachers' concept image of rate of change, every HE teacher made reference to a characteristic of function as well as geometric concepts to illustrate their image of rate of change. Algebraic concepts were used more often by LE and ME teachers than they were by HE teachers.

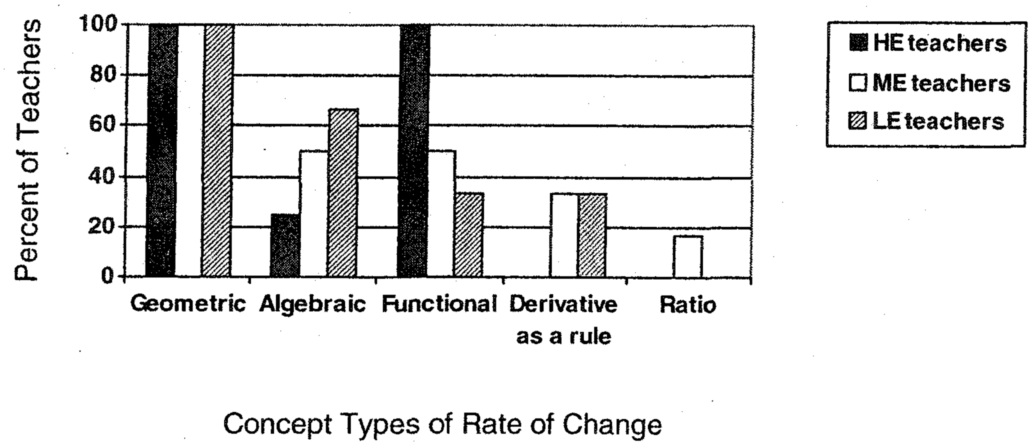


Figure 4. Percent of teachers who mentioned multiple approaches in describing the concept of rate of change in their concept images

Table 8
Distribution of teachers using multiple concepts to describe rate of change in their concept images

Types	HE teachers (n = 4)	ME teachers (n = 6)	LE teachers (n = 3)
	Number (%)	Number (%)	Number (%)
Geometric	4 (100)	6 (100)	3 (100)
Algebraic	1 (25)	3 (50)	2 (67)
Functional	4 (100)	3 (50)	1 (33)
Derivative as a rule	0 (0)	2 (33)	1 (33)
Ratio	0 (0)	1 (17)	0 (0)

Note. The column total number of responses is greater than the number of teachers responding since all but one ME teacher gave multiple responses.

Concept definition

Unlike the variation in teachers' concept image of rate of change, all but one teacher (M4F) defined the concept as "how fast/how much something changes over time," which is a "functional" definition of rate of change, as shown in Table 9. Their selection of this definition as their formal definition of rate of change may point to an influence of using the CPMP curriculum (which introduces the term "rate of change" in situations that exhibit a functional relationship). The teacher who did not use a functional approach defined rate of change as a ratio of two measurements, which was also the approach used to illustrate his concept image of rate of change.

Table 9
Teachers' concept definitions of rate of change

Descriptions of concept definitions of rate of change	Types[a]	# of teachers (n = 13)	Teacher identifications
1. How fast/how much something changes over time or over a certain period	Functional	12	All except M4F
2. A ratio	Ratio	1	M4F

Note. [a] Types were identified through direct references mentioned by teachers or interpreted by the interviewer (the researcher).

After giving a definition of rate of change, seven teachers also indicated that they may alter their definition slightly based on factors such as age of the students in their classroom and class appropriateness, as well as the demands of the task at hand (saying, for example, "it depends on what the student already knows and what problem they are working on"). However, they did not provide specific examples of these alterations. Some teachers seemed to defer to what was personally preferential. For example, teachers $L1A_2$ and M1B indicated that they always try to do an algebraic approach first unless a

graphical way seems more familiar or more applicable to the situation at hand. Teachers HD and HF expressed a preference for using a graphical approach to think about rate of change.

Multiple Representations and Connections

The other six interview problems (see Appendix C for a brief solution of each problem), in full or in part, required teachers to demonstrate their ability to use and make connections among various reprsentations (tables, graphs, symbolic rules, or verbal descriptions), to represent relationships between two variables involving rate of change, and interpret results from a given representational form. These questions were presented both in context and context-free situations, as well as in situations involving constant (or nearly constant) and non-constant rate of change.

When asked about the role of the two parameters typically used to describe a linear relationship between two variables involving a constant rate of change in different representational forms (i.e., mathematically, the coefficient of x and the constant term in a linear equation) (see Figure 5), all 13 teachers provided a correct response when discussing the effects of the parameters on numeric and graphic patterns.

3) Movies-To-Go has a membership plan for its customers. It costs $20 per year to become a member, and members pay $1.50 to rent any movie video. What is the role of the numbers 20 and 1.50 in a table, a graph and an equation relating annual cost *C* and the number of videos rented *V*?

Figure 5. Question 3 in the mathematical task-based interview

Regardless of their level of experience with CPMP, teachers were able to recognize patterns of change in the two variables of interest from the problem context and

show how the recognized patterns appear in tables, graphs and equations. Figure 5 shows a contextualized problem in which the rate of change is constant (Question 3).

The following two teacher responses were typical of the ways in which teachers talked about their solutions:

> In a table, the 20 would represent if you rent zero videos, then it would still cost you $20.00 for that month. So you would see that in the table when V is zero and C would be 20. In the graph you can find this [20] as the Y intercept. So when V = zero, then the C value would be 20. That's the starting point for both the table and the graph. In the C-V equation, you would see C= $1.50V + 20. The role of 1.50 is, so every time you...every time that the V column moves up by 1, the C column moves up by 1.5. So for instance, we have zero videos gives you $20. One video would give you a total cost of $21.50 and that continues in the table, $1.50 more every video. In the graph every one more V that you move right, the rise goes up another 1.5 and then in the equation that's the...it was C = $1.50V + 20 so the $1.50 would be the rate of change. (HF)

> 20 is the constant amount, so in the equation it would be like $20 plus how much it costs each video to rent. So 1.5 would be the rate of change because that's how much it costs to rent each video. $1.50 would be multiplied by the V, for the number of videos. The graph would be a line and it would start at, since $20 is the starting point on the Y axis, it would start at 20. The 1.50 is shown by how much it goes up from one video to the next. In the table form...It would start at zero videos like $20 and each time, it would go up $1.50 from one to the next. ($L1A_1$)

These examples show that the teachers demonstrated an ability to move among different representations in their responses. For example, teacher HF moved from equation to table and graph to equation. This suggests that teachers seemed to be confident in their ability to translate from one representation of a constant rate of change to another.

The order in which teachers used various representations in their solutions varied. That is, although the questions were written with a suggested order of table, graph and equation, some teachers began with the graph while others began with the equation. One teacher (M3B) began by commenting "... So in a table... well no, where do I want to

start? I guess in a graph…" and then began to work with a graph. Also, teachers often used their previous responses to answer the next part of the problem, rather than using the information originally given in the problem. Many teachers based what they knew about the role of the numbers 20 and 1.50 in a table to answer questions about what a graph would show. For example, after creating a table, M1B said "Okay, now when we're in a graph... um, I don't need to go back [to the given problem]. I can use what I got here [indicating his written work for that part of the table]."

Questions 5 and 8 emphasized interpretations of a graphical representation that exhibits non-constant rate of change or a mixture of constant and non-constant rates of change. These questions required teachers to recognize patterns of change represented in graphs and use the recognized patterns to construct another situation in which rate of change is an essential component. In question 5, shown in Figure 6, teachers were asked to give a story that describes patterns shown in a (time, distance) graph and then create a (time, velocity) graph matching the given graph.

5) Give a story for the following (time, distance) graph. Then sketch a (time, velocity) graph corresponding to the given graph.

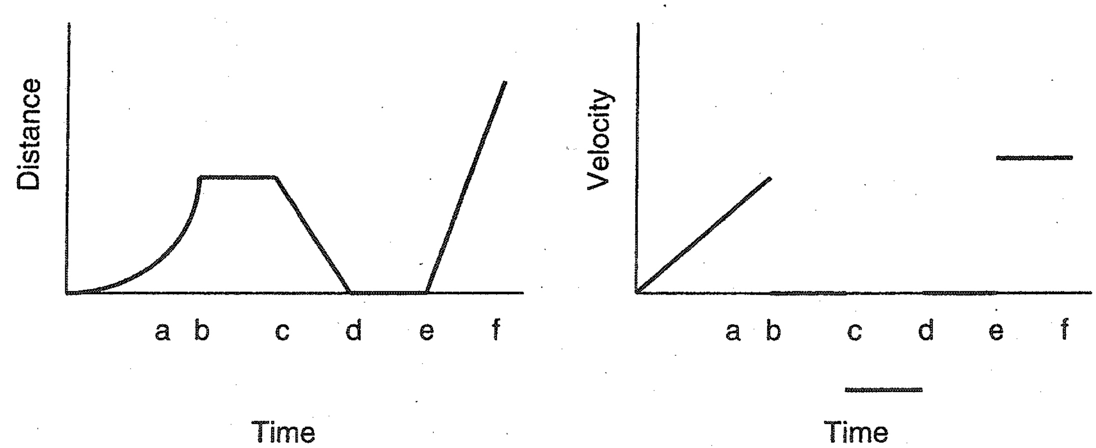


Figure 6. Question 5 in the mathematical task-based interview
Note. The velocity graph and letters a – f were added for the purpose of reporting results.

Teachers had fewer problems in creating a story for the given graph. Twelve teachers gave a reasonable story by recognizing changes in both rate and direction of motion. The following is representative of the teachers' stories:

> I'm going to start my run from my house [indicating the origin in the graph] and I've stretched out and everything and I'm feeling pretty good. So I began by going through a sprint. So my distance increases quite quickly because I'm running fast, I'm accelerating at that point [at time a]. [Indicating the segment (a, b)] so… as time goes by then my distance is increasing faster and faster and faster so, then I get tired so I stop. [Indicating the segment (b, c)] give me time to rest. [Indicating the segment (c, d)] when I feel refreshed, I turn around and I head back towards my house. This time just doing a steady jog at a constant rate. [Indicating the segment (d, e)] and then I stop and I have my stop watch set and I want to rest for the same amount of time that I rested on the other end [the segment (b, c)] and I'm ready to go. [Explaining the segment (e, f)] And I'm going to miraculously start at one speed and continue at that speed until I'm finished, which is going to be a faster speed. (M3C)

Only one teacher, M1B, had difficulty interpreting the segments showing decreasing distance (segment (c, d)) and zero distance (segment (d, e) only). That teacher was able to provide a correct interpretation of the segment (b, c) (which is another horizontal line shown in the distance graph), saying that " There's no distance being covered. The object is just standing now. . . . " He also demonstrated an understanding that horizontal segments have zero velocity later when asked to predict a velocity-time graph corresponding to the distance graph. However, "zero" distance did not seem to make sense to the teacher. He looked puzzled and said "How can the distance be zero? You walked some distance." Here he needed to understand the change of the distance in terms of direction of motion in reference to the origin from which the object started – increasing distance means moving away from the origin and decreasing distance means moving back towards the origin. In this context zero distance meant the object got to the point where it started and stayed there for a period of time.

When asked to sketch a (time, velocity) graph, seven teachers completed a graph correctly. All 13 teachers recognized changes in velocity in the following segments: from the origin to b (an increasing velocity), from b to c (zero velocity) and from d to e (zero velocity). They also provided a proper sketch of those corresponding segments in their (time, velocity) graph. However, some teachers made errors such as ignoring the sign of the velocity for the segment (c, d) (5 incidences) and assuming that the velocity graph resembles the distance graph (for some parts) (4 incidences). Of the five teachers who were unable to indicate that velocity is negative when indicating an opposite direction, three teachers ($L1A_2$, M1B, M3B) drew straight lines that went up to the right for the last segment (e, f) showing an increasing velocity for the segment. They often said something like, "they are walking or running faster," which is what they said about the segment when giving a story for the (time, distance) graph. It may be that their incorrect mental picture of the path of a person moving was distracting them from the distance/time information on the axes or that their use of the term "faster" caused them to misinterpret it as an increasing rate.

Teacher M1B was the only teacher who had difficulty interpreting the segment (c, d) showing decreasing distance. This teacher predicted the segment (c, d) in the distance graph as yielding a *positively* decreasing velocity. Also, this teacher was unable to interpret the segment (d, e) and left the segment blank in his velocity graph. The velocity graph that teacher M1B drew is presented in Figure 7. Of eight teachers who considered the sign of the velocity, only one teacher (M2C) made an error in predicting the segment (c, d) as yielding a *negatively* decreasing velocity. The velocity graph that teacher M2C drew is also presented Figure 7.

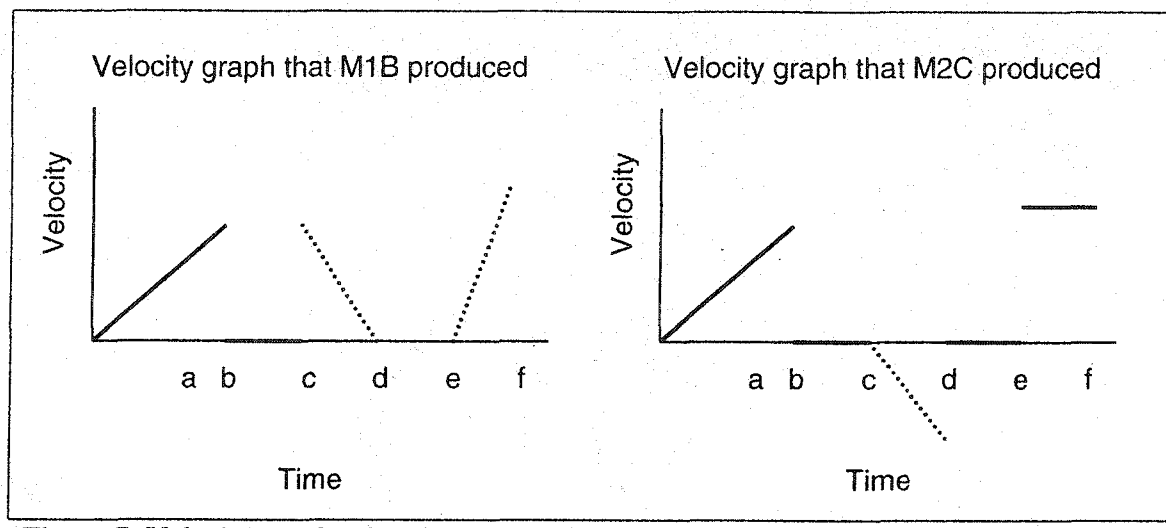


Figure 7. Velocity graphs showing reported errors
Note. The dotted lines shown indicate the teachers' misinterpretations of the given distance graph.

As shown in Table 10, no LE teachers were able to take into consideration the directional characteristic of velocity, but the responses of teachers $L1A_1$ and $L3A_3$ would have been appropriate with respect to speed instead of velocity. In contrast, every HE teacher gave a proper sketch of the (time, velocity) graph. Teacher HG provided two different graphs with respect to velocity vs. speed. The approaches that teachers used are summarized in Table 10.

Table 10
Teachers' approaches in sketching a (time, velocity) graph of Question 5

Approaches	Teacher Identifications
Considered the directional characteristic of velocity	
Appropriate	M1C, M3C, M4F, HB, HD, HG, HF
Made an error in predicting the segment (c, d) as yielding a negatively decreasing velocity	M2C
Ignored the directional characteristic of velocity	
Appropriate with respect to speed	$L1A_1$, $L3A_3$
Made an error in predicting the segment (c, d) as yielding a positively decreasing velocity	M1B
Made an error in predicting the segment (e, f) as yielding a positively increasing velocity	$L1A_2$, M1B, M3B

Question 8 focused on evaluating a derivative function from a graph to describe the behavior of the original function. Figure 8 shows the graph of the derivative function f′(x). A rough sketch of a possible original function f(x) is added for discussion purposes. It was not included on the interview question.

This problem appeared to be one of the most difficult items in the interview and teachers' responses varied. M1B was unable to provide a response to this item, saying "I don't have the wisdom to answer this problem and I have stayed far away from this stuff for a long time." The teacher felt "everything" was difficult about rate of change with respect to a nonlinear function. In fact, this teacher was not successful with any of the questions involving non-constant rate of change associated with the idea of derivative.

8) What can you tell about the behavior of the original function f(x) from the behavior of its derivative function f′(x) illustrated below? Be as complete as possible in your response.

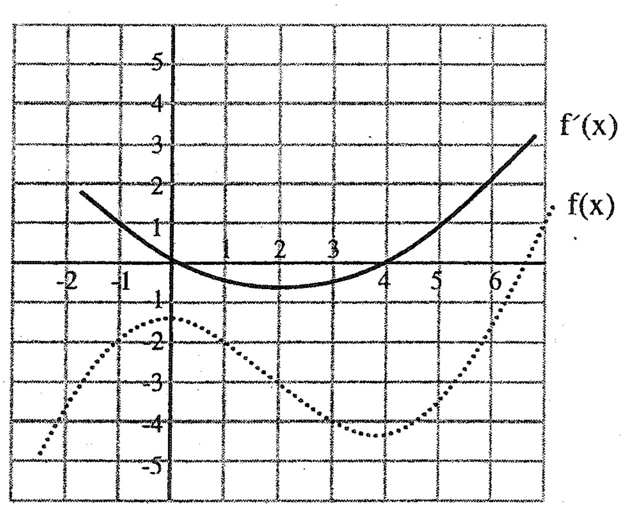


Figure 8. Question 8 in the mathematical task-based interview
Note. A sketch of a possible function f(x) were added for the purpose of reporting results.

All of the 12 teachers who attempted to answer this problem stated that the derivative function f′(x) gives the rate of change or the slope of f(x). Most teachers used the terms derivative and slope interchangeably in their responses. Eleven of the 12 respondents were able to verbally describe how the rate of change in f changes by checking to see whether the function values of f′ were increasing or decreasing, and positive or negative. One teacher who failed to give a verbal description of the changing aspect of the rate of change in f had difficulty thinking about the original function f from its derivative f′. This teacher wrestled with ideas and finally gave up and did not complete his response to this problem. He ended the problem by commenting:

> This is an opposite way of thinking, which I'm not used to. It would not have been too hard if it was f and asked for f′. I used to learn it gives acceleration, which must mean if I have time [for the x-axis]...[f′ is a] time-speed graph. I know derivative is rate of change of f, so it gives acceleration. I'm just thinking back to all the things I've ever learned about a derivative. But I don't remember much. ($L1A_1$)

Additionally, all 12 teachers were able to explain their thinking about the antiderivative of a quadratic function and referred to a "cubic function" to talk about the behavior of the original function f. Six teachers (all LE teachers and 3 ME teachers) began by inferring that the given graph was a quadratic function and the other six teachers (all HE teachers and 2 ME teachers) indicated that the original function would "act or look like" a cubic function, because the part of f′ showing looks like a quadratic function and f′ as a whole would be a higher powered function. Four teachers (M1C, M2C, HD, HF) attempted to calculate an antiderivative for the given graph f′ assuming that f′ or at least the part of f′ showing was a quadratic function. Of the four teachers, M1C and M2C demonstrated their knowledge of the symbolic rule for finding an antiderivative of a quadratic function, but used an improper equation for the graph f′(x).

The two HE teachers, HD and HF, correctly computed a possible equation of $f'(x)$ as $f'(x) = 1/6\ x^2 - 2/3\ x$, by approximating the location of the vertex point of $f'(x)$ as (2, -2/3). They then found a proper antiderivative of the quadratic equation in the form of $f(x) = 1/18\ x^3 - 1/3\ x^2 + C$. They both indicated that C could be any number. However, while teacher HF used the letter C, not assigning any specific value for the constant, teacher HD used zero for C to continue his work on the problem.

All 12 teachers, except one LE teacher ($L1A_1$), attempted to represent the function f(x) graphically, whether correct or incorrect. Of the other 11 teachers, seven teachers began by examining the function values of $f'(x)$ at various points along the x-axis and drew imaginary tangent lines that had the slope corresponding to the function values they tested. They then sketched a proper shape for f ignoring coordinates. The remaining four teachers began by drawing a graph of a cubic function, but then somewhat modified it based on their knowledge about rate of change in f. Three teachers ($L1A_2$, $L3A_3$, M3B) used a graph of a constantly increasing cubic function (shown in part (a) of Figure 9) and left their initial graphs unchanged—which is an incorrect shape for the graph of f. The fourth teacher (M3C) used a graph of a different cubic function (shown in part (b) of Figure 9)—which is a correct shape for the graph of f.

Only three teachers (M4F, HG, HF) correctly showed where the local maximum, minimum, and inflection point would appear on the graph of f(x) (i.e., at x = 0, 4 and 2, respectively). The most common error was made in positioning the inflection point at x = 2.

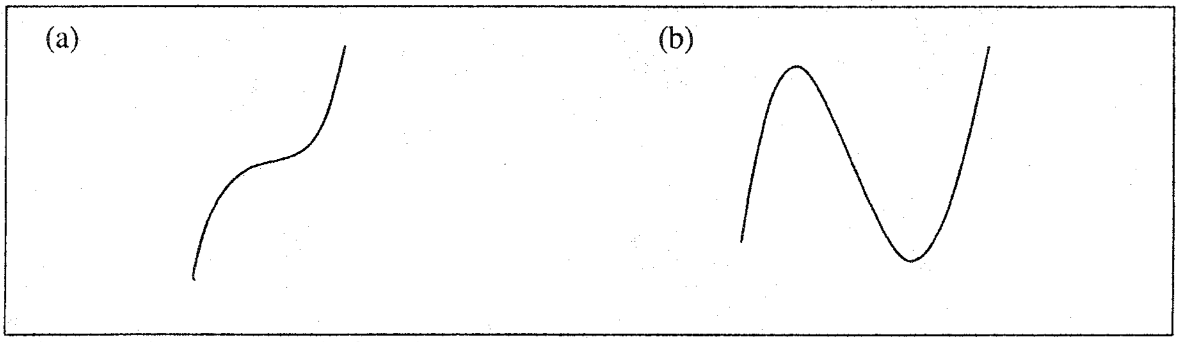


Figure 9. Shapes of cubic functions used to infer an original function f(x) from the given function $f'(x)$

Linear Connection

Questions 6 and 7 (shown in Figure 10) focused on the idea of linear approximation to find the rate of change in nonlinear functions. That is, finding a rate of change depends on treating a curve as if it were a series of very short line segments which approximate the curve. Making the segments as short as possible makes the estimation process more accurate. This is the process which is taken to its logical limit in

6) Suppose that a laboratory experiment uses fruit flies that double in number every five days. If the initial population contains 100 flies, the number at any time *t* days into the experiment will be modeled by the function with the rule $P(t) = 100(2^{0.2t})$.

   Use the function rule above to answer these questions as accurately possible.

   (a) What is the average growth rate of the population (flies per day) from day 0 to day 20?
   (b) What are estimated rates at which the fly population will be growing on day 10 and on day 20?
   (c) How are the growth rates calculated in parts (a) and (b) shown in the shape of a (t, P(t)) graph?

7) Suppose f is a quadratic function of *x* with rule in the form $f(x) = x^2$.

   (a) How is the rate of change of f shown in tables, graphs, and rules of the function?
   (b) What rule gives the rate of change of f at any point x?

Figure 10. Questions 6 and 7 in the mathematical task-based interview

finding derivative functions. Overall, all teachers were successful in discriminating between constant rate of change (linear) and non-constant rate of change (nonlinear) in situations involving relationships between two variables. However, many teachers had difficulty making sense out of situations involving changes in which rate of change was not constant.

Question 6 was a contextualized problem that investigated the exponential growth of the fly population. . Among LE and ME teachers, five expressed some hesitancy and uncertainty about responding to this question. Excerpts from teachers expressing hesitancy included the following:

I'm not familiar with this type of question. ($L1A_1$, M1B)

Should I divide or subtract? (M1C)

In part (a) where teachers were asked to estimate the average rate of change, the common strategy used by teachers was to figure out or approximate the function values (representing the fruit fly population) at each of the given days, subtract the two function values, and then divide by the difference in the days to find the average growth rate of the population of flies per day (i.e., $(P(20) - P(0)) / 20$). Teachers $L1A_1$ and M1B forgot to divide by the difference in the days, leaving the solution as the overall change in the population over the period of time. Many teachers did not seem to understand instantaneous rate of change could be estimated by an average rate of change for a time period that is very small.

In part (b) where teachers were asked to estimate rates of the fly population at particular points, day 10 and day 20, only six teachers (all four HE and two ME teachers) provided an appropriate solution. These teachers used a similar method for finding the

estimates in that they took two nearby points (or one nearby point and the particular point) and divided the difference in their function values by the difference in their t-values. For example, to estimate the rate of change at day 10, four teachers (HB, HD, HF and M2C) used methods in the form of $(P(10 + h) - P(10)) / h$ and two teachers (HG and M3C) used $(P(10 + h) - P(10 - h)) / 2h$.

Of the remaining seven teachers who were unable to give an adequate answer, four teachers (M4F, M3B, M1B and $L1A_2$) indicated that the rate at day 20 would be greater than the rate at day 10, noting the pattern of change of the exponential growth of the fruit fly population. However, these teachers could not go beyond this interpretation. Teacher M3B said she could find an exact number for the estimates if she knew the derivative of the function P(t), commenting ,"[The] rate at day 10 is the slope of the tangent line. I need the equation of that line [to find the slope]. But I don't remember what's the derivative of that function [P(t)]." Teachers M1B and $L1A_1$ showed difficulty understanding what rate of change at particular point meant and said, "I need two points to find rate of change at that point. Change is something occurring over an interval which has two points. Do we know two points?" Two teachers ($L3A_3$nd M1C) calculated the function values (P(10) and P(20)) as the rate and did not realize that those values represented the population.

In part (c) when asked how the estimated rates (calculated in part (a) and (b)) appeared in a graph of the function P(t), five of the six teachers (who gave appropriate answers for both part (a) and (b)) provided adequate answers and demonstrated an understanding of how the methods of estimating non-constant rate of change are related to finding the slope of a line. These five teachers (HB, HD, HG, M3C and M2C) said the

average growth rate in part (a) was the slope of the secant line over the interval [0, 20] and each of the rates estimated in part (b) was the slope of the line over the interval containing the particular point. Teachers HF (who was one of those teachers who gave adequate answers to both part (a) and (b)) and M4F (who was one of the teachers who did not give an adequate answer to part (b)) were able to interpret how the average growth rate would be shown graphically but were unable to do so for the rates at particular points. For the six remaining teachers who did not give an adequate answer to part (b), two teachers ($L1A_1$ and M1C) could only verbally describe the pattern of change as continuously increasing and four teachers ($L1A_2$, $L3A_3$, M1B and M3B) could not provide an answer.

Question 7 was a context-free problem that addressed the same idea. Teachers demonstrated whether or not they had a flexible understanding of a graphical representation of rate of change in a nonlinear case by examining a simple nonlinear quadratic function ($f(x) = x^2$) and answering questions about how the rate of change of x^2 is shown in tables, graphs, and rules. Less experienced teachers (LE and ME teachers) had more difficulty using graphs than tables or equations, and compared to HE teachers, to represent non-constant rate of change.

All 13 teachers were able to discover a numerical pattern in the table of (x, f(x)) and identified the rate of change of f in ways that showed how f(x) values changed when x values changed. An x-increment of 1 was initially used by all teachers. However, $L1A_2$, and M1B were unable to provide a graphical representation of the rate of change of the function and eventually gave up. In addition, six other teachers gave unconvincing responses to this question ($L1A_1$, $L3A_3$, M4F, M1C, M2C, M3B). Of the six teachers,

three teachers ($L1A_1$, $L3A_3$, M3B) made sketches of a parabola representing $f(x) = x^2$ and a straight line of $f'(x) = 2x$ over the parabola, indicating that the straight line represented the rate of change of the function. However, they were unable to justify their responses when asked how the line of $f'(x) = 2x$ was related to the way they explained the rate of change of $f(x) = x^2$ as shown in a table (changes in two consecutive y-values over changes in two consecutive x-values with the increment of 1). These three teachers drew several secant lines over the intervals they used to show the rate of change in a table as well as a straight line representing $f'(x) = 2x$. These teachers looked somewhat puzzled and were unable to connect their interpretations across two different representations for the rate of change of $f(x) = x^2$. The following is representative of these teachers' responses:

> The rate of change is 2x and 2x is this line [indicating the line they sketched over x^2]. I mean... graphically. But these numbers [the values that were produced as rates of change in different x-intervals] are the slope of the [secant] lines here [indicating the drawing]. I'm not making sense about how the 2x line comes into play. (M4F)

Every HE teacher and one ME teacher (M3C) also mentioned both 2x and secant lines to illustrate how the rate of change of $f(x) = x^2$ can be graphically represented. They explained with confidence how the secant lines were related to $f'(x) = 2x$. In fact, most of the LE and ME teachers had difficulty relating their interpretations of the numerical pattern that they identified as the rate of change of $f(x) = x^2$ to its derivative function $f'(x) = 2x$, while three of the four HE teachers demonstrated a solid understanding of the connection. ME teachers' responses included the following:

> It [The rate of change] is 2x + 1. 2x by itself is just the derivative of x squared, then you need to get into the calculus of it. (M2C)

> It [2x] is being doubled but it [2x +1] is not. I don't know... The only thing I can see is it's the difference [second differences in y values] between my differences [first differences in y values]... I think this [2x] is the right one [derivative function of x^2]. The exponent goes to the front. (M3B)

These teachers' comments indicate that they may possess a rule-based understanding of the derivative concept that may be more procedural than conceptual. This seems to be consistent with their understanding of derivative as shown in their concept images. For example, these teachers mentioned the term derivative as an illustrative example of rate of change, but did not incorporate it into the explanations of their other examples.

Mathematical Modeling

This component of the guiding framework dealt with the ability to apply understanding of rate of change to model and solve problems in which linear or non-linear relationships exist between variables of interest. Question 4 in Figure 11 was a contextualized problem that provided a situation involving a nearly linear relationship represented in a table.

4) The junior class at Centennial High School planned to make a class sweatshirt with every student's name on it. They tested several possible selling prices and got the following data relating price P and probable sales S. What would the predicted number of shirts sold be when the price per shirt is set at $5?

Price per Shirt P in dollars	10	20	30	40	50
Number of Shirts Sold S	200	150	90	50	5

Figure 11. Question 4 in the mathematical task-based interview

This problem required teachers to analyze the data, interpret results and make predictions from the given data to estimate a possible answer to the problem. The results of teachers' approaches to this problem are found in Table 11.

Ten out of 13 teachers began by using the table to find patterns in change from the table in the number of shirts sold (that is, 50, 60, 40, 45 as price per shirt increased) and indicated that the rate of change was not constant. The remaining three teachers (M1C, M3B, HG) started entering the data values into a graphing calculator to find a regression line that best fit the given data and to find an equation. They did not numerically figure out whether the relationship was a constant rate of change or non-constant rate of change. (M1C and HG also searched for patterns in the table.) They showed their familiarity with this kind of problem. For example, the following is an excerpt from the responses made by one ME teacher who has taught Course 1 for six years:

> This is a typical problem in Unit 3 of Course 1 where they [students] draw a modeling line that they believe is a good fit for the trend in the data, or use their graphing calculator to make a scatter plot of the sample data. I'll do a regression and then [while entering the plots into his calculator] I don't know what regression I would do. I'll have a better idea once I see the plots. (M1C)

Table 11
Teachers' approaches in responding question 4

Approaches that teachers started by:	Teacher Identifications
1. Find a regression model using graphing calculators	M1C, M3B, HG
2. Find patterns of change in the table	$L1A_1$, $L1A_2$, $L3A_3$, M1B, M2C, M3C, M4F, M1B, M2C, HB, HD, HF
Approaches that teachers continued by:	**Teacher Identifications**
1. Trace the graph on the calculator	M3B
2. Find an average of the differences in the number of shirts sold	$L1A_1$, $L1A_2$, $L3A_3$, M3C, M4F, HB, HG, HD
3. Make a rough sketch using the ordered pairs shown in the table	M1B, M1C, M2C, HF

All but one teacher (M3B, who traced the graph on the calculator) searched for patterns in the table at some point. Of those teachers, eight maintained an algebraic approach to continue the problem. That is, they found an average of all of the differences by either estimating or actually figuring out the differences in the table. An estimated average of 50 was the most often used (6 teachers) and 47 was used by one other teacher. One teacher ($L3A_3$) calculated the exact average of the four numbers, which is 48.75, and rounded it down to 48 in order to ease the calculation for finding half of the average. Every LE teacher used this approach to solve the problem. The remaining four teachers used a graph to explore the problem further. They created a rough scatter plot of the five ordered pairs for the price per shirt and number of shirts sold. These teachers drew a line of best fit over their plots and used their graphs to estimate the number of shirts when the shirt price is $5.

Once these twelve teachers found the rate of change, their approaches to finding the number of shirts differed. Nine teachers used proportional reasoning, two teachers ($L1A_2$ and M4F) created an equation, and one teacher (HF) extended his graph to find the number of shirts when the price was at $5. The teachers who used proportional reasoning figured that they would need to add half the rate of change to determine the number of shirts sold at $5. The teachers who created an equation used a particular point (such as, $10, 200 shirts) to determine values of the parameters in their equation.

Two ME teachers (M1C and M3C) brought up the concern about the reality of the situation. For example, teacher M3C, who figured an average rate of 50 and 225 (200 + 25) to be the number of shirts when the price goes down to $5, commented "I would take

into consideration the fact that the cheaper price may get even more people who buy shirts. And so it may be greater than 225."

Summary

Teachers demonstrated a similar understanding in areas involving constant rate of change (and nearly constant rate of change) and varied understanding in areas involving non-constant rate of change. Teachers demonstrated a solid understanding of multiple representations in contexts involving a linear relationship and a weak understanding in contexts involving a nonlinear relationship. Teachers were successful in constructing representations—tables, graphs, and equations—for a constant rate of change and talking about the roles and effects of the parameters (that are typically used to describe a linear model) in tables, graphs and equations. They could move flexibly among different representations. In situations involving non-constant rates of change, teachers' levels of understanding of multiple representations differed. Most teachers showed flexibility in moving from graphs to words, whether in a context or not. Teachers used such words as "same," "increasing," "decreasing," "positive," and "negative" to describe phenomena involving change.

Generally, teachers were less flexible in moving from graphs of one phenomenon to graphs involving a related phenomenon and less experienced teachers had more difficulty than their peers. In particular, teachers showed less flexibility in moving from graphs of rate of change to graphs of accumulation (such as, sketching a graph of an original function from a graph of its derivative), but were more capable moving in the opposite direction (such as, sketching a velocity/time graph from a distance/time graph). Teachers often confused the slope of a tangent line of the derivative function with the rate

of change of the original function at a particular point. Misconceptions, such as interpreting graphs as a picture of an event and confusion about the sign of the velocity, were demonstrated by some LE and ME teachers.

HE teachers showed strength in making the connections between the concepts of average rate of change and instantaneous rate of change in non-linear functions. Generally, teachers were better able to discuss average rate of change and instantaneous rate of change as well as their connection when the problem was contextualized (such as the fruit fly population situation).

Estimating a rate of change at a specific day (i.e., finding an instantaneous rate of change) was difficult for LE teachers and some ME teachers. Some teachers understood that the slope of a derivative function at a given point was a way to find instantaneous rate of change at the point, but they were unable to infer the symbolic rule for the derivative nor connect it with the method for finding the slope of a line. This was mostly evident in a context-free problem where most teachers were unsuccessful in connecting a numerical pattern of change in a quadratic function and the function values of its derivative. However, teachers were able to indicate that the function values of its derivative would give an instantaneous rate of change at various points.

Finally, teachers were able to use their understanding of rate of change to analyze data showing a non-linear relationship (although nearly linear) to model and solve a problem. While teachers' approaches varied, there appeared to be no difference among LE, ME and HE teachers' understanding.

Results of Classroom Observations

This section presents results of classroom observations and pre- and post-interviews for the selected four teachers: two LE teachers, $L1A_1$ and $L1A_2$, having the least amount of experience among the 13 participants using the CPMP curriculum, and two HE teachers, HD and HG, having the most extensive amount of experience using the curriculum. For ease in reporting, the teachers' coded names were replaced by pseudonyms, with no attempt to match their gender: Bob ($L1A_1$), Sally ($L1A_2$), Ian (HD) and Misook (HG). These teachers' background information, such as educational and teaching histories, were briefly described in Chapter III and are reported with more detail in this section.

One of the LE teachers, Bob, was observed while he was teaching two CPMP units in Course 1: *Patterns of Change* and *Linear Models* and the other LE teacher, Sally, was observed teaching the Course 1 unit, *Linear Models*. The *Patterns of Change* unit emphasizes informal exploration of functional relationships using contextualized problems and offers students opportunities to understand patterns between variables through the exploration of multiple representations. The *Linear Models* unit follows the *Patterns of Change* unit and emphasizes connections among numeric, graphic, and symbolic representations of linear functions. HE teachers, Ian and Misook, were observed while they were teaching the Course 4 unit, *Rates of Change*, which emphasizes understanding of average rate of change and instantaneous rate of change. Each teacher was observed at least four times and interviewed before and after each observation. The results are presented in the following section and are analyzed according to the various themes that emerged.

The LE Teachers: Bob and Sally

Both Bob and Sally were in their first year of teaching at the time of the study. Both taught in the same high school. Thus, neither of them had any prior teaching experience using the CPMP materials nor much experience with other textbooks, except for the texts used during their internships. Their school was a CPMP curriculum pilot site in 1993 and all courses of the curriculum were being used in the school at the time of this study. The CPMP curriculum is used with all 9^{th}-grade students. The majority take Course 1 with a small subgroup accelerated to Course 2 based on teacher recommendation. Courses 2 – 4 are offered in grades 10 - 12 along with the University of Chicago School Mathematics Project (UCSMP) curriculum. Sophomores choose either curriculum, and once chosen, typically stay in that curriculum throughout high school.

Bob and Sally's classrooms were organized in a similar manner. Their desks were in the front. The student desks in Bob's classroom were arranged in double rows so that pairs of students sat and worked together. Students in Sally's classroom sat at two-person tables in three rows. In each of the classrooms, students were allowed to make a bigger group to work with if they preferred, and there was enough room for teachers to walk among the students. During the observation segment of the study, both teachers taught Courses 1 and 2 of the CPMP curriculum and one algebra class for 10^{th} graders. For both, one of their Course 1 classes was observed four times. Each of the observed classes contained approximately 30 ninth-grade students. Bob characterized his observed class as better than to his other Course 1 class: "They [the students] have good listening skills, and are always with you. They have improved on organization and neatness." He noted that there were some strong students who tended to be quite vocal. He said that he had

"tended to get off track more" in the class and worried a bit about the more verbal students stifling those that were quieter. Sally described her observed class as typical.

Bob entered this first year of teaching directly from college. He had an undergraduate major in mathematics education and a minor in English. He took "typical college mathematics" courses including linear and modern algebra, modern geometry, calculus, and graph theory. He also completed three mathematics methods courses in teaching secondary school mathematics. He was exposed to the CPMP curriculum and other Standards-based curricula in the methods courses. He had a week-long workshop on Course 1 before starting his teaching job. The workshop was organized by the Core-Plus Mathematics Project and was designed to introduce teachers to Core-Plus Mathematics, its instructional model, and its assessment philosophy. During this study, he did not participate in any additional professional development.

Sally was also a first year graduate from college. She had an undergraduate major in mathematics and a minor in psychology. She took similar mathematics courses as those taken by Bob. Sally also took a course entitled "Math for Secondary Education," but did not take any mathematics methods courses. Her first experience with the CPMP curriculum was participation in the same week-long workshop that Bob had attended and, like him, she received no additional professional development.

Bob and Sally both reported that they performed well in most of the math classes they took in college. They both felt that "higher-level" calculus classes were the most difficult. Both of them took three semesters of calculus and referred to Calculus II and III as the "higher-level" calculus classes. Bob said "I did good in most of the math classes I took, but in Calculus II and III, [I did] just enough to pass. The teacher showed us a

bunch of formulas to have memorized, but I didn't fully understand how they were related."

Course 1 Unit 2: *Patterns of Change*

In this unit, students study a variety of types of change inherent in real situations and develop the idea of a dependence relationship between two variables. This unit develops students' ability to recognize important patterns of change and represent the recognized patterns of change in different ways—using tables, graphs, rules and verbal descriptions. This unit precedes the *Linear Models* unit.

Course 1 Unit 3: *Linear Models*

While the *Patterns of Change* unit provides students with a broad picture of patterns of change, this unit focuses on linear functions as a specific type of change among the variety of types of change they were exposed to in the previous unit. The principle goal of this unit is to develop "linearity sense" by studying linear and nonlinear (mostly, nearly linear) relationships, and understand the connection among multiple representational forms, such as numeric (constant rate of change), graphic (constant slope), and symbolic ($y = a + bx$). Throughout the lessons, students are to become proficient at recognizing and representing patterns of change in various representational forms. Thus, the idea of change is central to the unit.

What is slope?: Concepts and procedures

During the observation of Unit 3, Lesson 2, Bob's students were working on writing rules that represent a given linear model. During a brief pre-interview, Bob was asked where he anticipated the students having difficulty. He began by saying " I don't know for sure, because this is the first time teaching it." He went on to speculate,

however, that students "would have trouble finding the slope the most. [They] do not follow the formula $\Delta y/\Delta x$ to find slope, rate of change here. They don't understand it. They switch the order [to $\Delta x/\Delta y$] or they make computation errors." This response suggests that he may be focused more on procedural knowledge. Although the term "rate of change" was cited with slope in his response, it was evident later from his teaching that he used "rate of change" not because he conceptualized slope as a measure of rate of change, but because he tried to be consistent with CPMP's approach.

At one point in the observation, when a pair of students asked Bob how to find the slope, he began by asking, "Can you describe in your own words what slope is?" One student, Mary, answered, "Relation in the change between x and y coordinates," while another student, Heather, answered, "Given two points on a graph, slope is that 'delta' thing, $(x_1 - x_2) / (y_1 - y_2)$." He responded by saying "You [Heather] were almost right but not entirely. Flip that [the fraction mentioned by Heather] over. Then the fraction gives you the slope." In a follow-up interview, he said, "It was interesting to see Mary's memory of slope to be more conceptual and Heather's memory of slope to be related to the formula (even if it was an incorrect answer)." Bob then recalled that he had had discussions about the differences between conceptual and procedural knowledge in a methods course he had taken in college. It appeared that Bob was attempting to incorporate what he had learned into his knowledge for teaching. However, he seemed to express a tension or unresolved ideas about the role of conceptual and procedural knowledge when discussing his students' performance:

> The area in which both of the students were weak was working with equations concerning an algebraic representation of slope. They both had trouble coming up with an equation for a line that was already graphed. This is something that I found very surprising, as well as encouraging. In the methods courses we

> discussed the need to focus more on conceptual understanding, and not so much on memorizing formulas. Well, these students exhibited these characteristics almost to a fault. They were very good on the conceptual questions, and not so good with algebraic representations and formulas.

Despite his initial propensity toward a procedural aspect of slope, he also seemed to recognize that some students were not getting it. He then explained in the interview that he wanted to try teaching with a focus on the development of conceptual understanding next time. He seemed to make this decision almost based on the assumption that the researcher would like to see more of a conceptual approach in his class, rather than an honest change of mind. As a result, in the beginning of the next class, he spent the first few minutes "philosophizing" about slope, rambling a bit, while setting the stage for a discussion about the "meaning" of slope. The students did not seem to understand where he was coming from and this was clearly not a typical kind of discussion held in this class. One of the students finally became impatient with his verbal treatise, and interrupted him by saying, "I have a question. What exactly is slope? Tell us how to find one."

Although Bob used the term "rate of change" in the discussion of the meaning of slope (which is how the term "slope" is introduced in the CPMP textbooks), he eventually defined slope as "vertical change / horizontal change," and referred to a graph of the line, shown in the textbook, passing through particular points ((2,4) and (4,5) in this case). He emphasized that the slope was a fraction, 1/2, up 1, over 2. This is consistent with his geometric concept image of rate of change as rise over run. Later, while students were working at their seats, one student, Natalie, was having difficulty understanding how the two fractions (-2)/3 and 2/(-3) could both represent the same slope.

When asked in a follow-up interview to describe her difficulty, he said: "They think you are describing a movement as opposed to you describing a number, a measure of steepness." Bob seemed to use the term "movement " to refer to the way Natalie was visualizing "rise" and "run," where you move from one point on the graph to another point. Natalie is interpreting the slope (-2)/3 by thinking about the start point, going *down* by 2 units, then going to the *right* by 3 units, whereas the slope 2/(-3) is interpreted as going *up* by 2 units, then going to the *left* by 3 units. However, she does not yet realize that these two different phenomena represent the same slope.

Emphasis on the procedural aspect of slope was also evident in Sally's teaching. For example, her class was working on an activity in Unit 3, Lesson 2 involving the development of understanding of situations where the appropriate linear model is a line with a negative slope. In that activity, the content was relating spring length in a mattress to weight on the spring. Sally discovered that numerous students used an inconsistent order to find the changes in length and weight, that is, they subtracted y_1 from y_2, but then subtracted x_2 from x_1. Upon recognizing this error, Sally said the following to the students:

> When you have two points in front of you, you always do the second one minus the first one—the second y value minus the first y value. And you do the same thing with the x's—the second x minus the first x. And you know the y value is on the top and the x value is on the bottom.

After this, she gave the class two points (1, 3) and (3, 2) and had them find the slope of a line given the two points of the line. While she walked around the class, she told students to relate to the "rule" (as described in the cited quotation above).

During the follow-up interview, she showed disappointment in the CPMP materials, because of the absence of discussion of the slope equation

$(y_2 - y_1) / (x_2 - x_1)$, when ordered pairs are lined up as (x_1, y_1) and (x_2, y_2). She questioned CPMP's treatment of slope as a measure of rate of change. She went on to add more detail, displaying her static view of the nature of mathematics:

> There is one point where they [the CPMP textbook] show how to find the rate of change in the linear model in the very beginning of the first investigation. They use [the term] rate of change a lot more than slope. But what they used is the slope equation anyhow, though they did not say it. There must be a reason why they did not show that formula [the slope equation], maybe for later math. But I think if they [the students] had the formula, which is "universally true," and know where it is so they can go back to see when they are unsure which one [needs to be subtracted], it will definitely help.

When asked why the subtracted numbers should come from the first ordered pair, (x_1, y_1), she even more strongly emphasized her notion of "writing an idea as a rule that is universally true":

> Yes, it doesn't need to be always (x_1, y_1). You can reverse the order [i.e., the subtracted numbers can come from the point (x_2, y_2).]. But this way their chance of making errors will be less once they remember it. You know there are so many things you need to remember to be successful [in math]. When they have powerful rules that they know always work it will less stress them out and they will be more willing to study. It may not be one of their [the CPMP textbooks] goals, but perhaps that is where it needs to end up practically. In here [the CPMP textbooks] it doesn't look like there are any rules and formulas. When an idea is universally true, make it look like a formula so students know that's something they need to remember until at least the end of the semester.

These responses reveal that Sally was unclear about the focus of the materials and how the concept of slope was developed throughout the materials, so she had to rely on her own knowledge of slope and goals for student learning, which shaped her translation of the curriculum into practice.

Later in the interview, she showed her own discomfort about differentiating between "concepts" and "procedures," referring to the week-long workshop on CPMP that she had attended a few weeks before she started her teaching job:

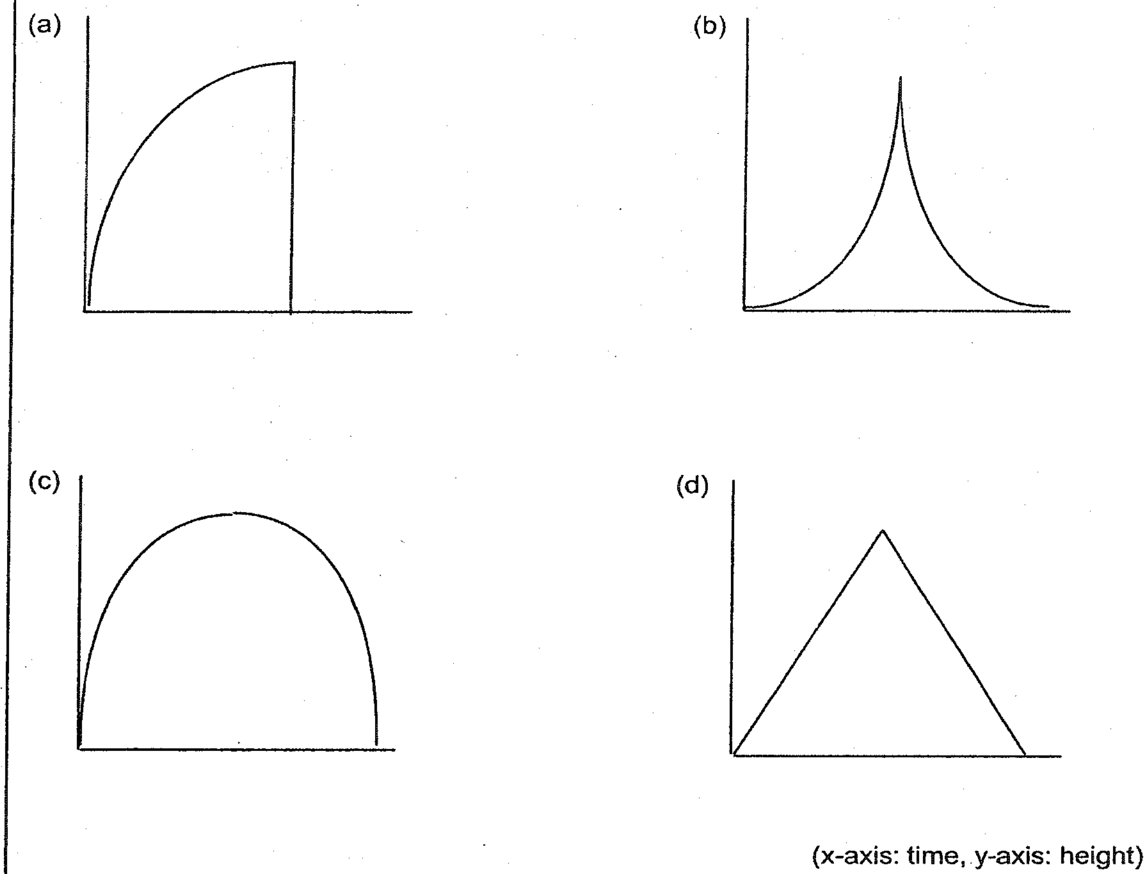
> In the workshop the lead teacher said this curriculum emphasizes more mathematical concepts than facts and procedures. What is a concept? and what is a procedure? To me, you know the concept, whatever you call concept, "through" the procedure and that [procedure] is the place where the concept is applied. You just can't separate them. They are associated but this association is not automatic for most students.

Her discomfort talking about concepts and procedures separately may be related to her inappropriate way of using the two terms interchangeably. In one of the interviews before an observation, she was asked to describe the learning goal of the upcoming lesson, and explained "The concept of slope—rise over run, positive and negative slope, horizontal and parallel lines." Although she referred to the "concept," of slope, her response seemed to indicate a focus on developing a series of procedures.

<u>Interpreting graphs: Message of a graph vs. picture of the phenomenon</u>

Bob's class was discussing a situation involving projectile motion (a ball thrown straight up into the air) in preparation for an investigation (Unit 2, Lesson 4) studying patterns of change using tables and graphs. During the discussion of the situation, students were asked to determine which of a given set of graphs best matches the pattern of (time, height) data describing the ball's flight. The graphs and their descriptive information are presented in Figure 12. (The descriptive information was not provided in the text.)

One student, John, said he believed the third sketch (c) best described a ball's flight when thrown straight up into the air. Several students chimed in that they agreed with John. Bob did not ask John to explain how he knew (c) was the best fit, but did ask investigation after he saw no hands up, Melissa put her hand up and said "I'll say the answer's wrong but I don't know what the real correct answer is." Rather than pursuing if there was anyone who disagreed. Just as he was about to move on to the main this, Bob



Information on the graphs:

- Graph (a) shows a rise to a maximum height with gradually slowing speed, then an instantaneous fall to ground.
- Graph (b) shows an accelerating rise, then a very rapid descent at first, gradually slowing as it approaches the ground.
- Graph (c) shows a rapid rise at first, with a gradual slowing as the ball nears its peak, then the descent begins slowly and gradually picks up speed.
- Graph (d) shows constant speed during ascent, an instantaneous stop at the peak, and a constant rate during descent.

<u>Figure 12.</u> Graphs provided in a problem determining which of the graphs best describe a ball's flight thrown straight up into the air

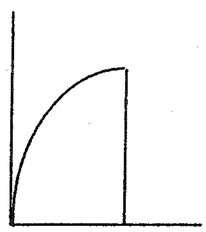
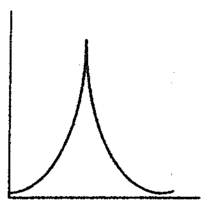
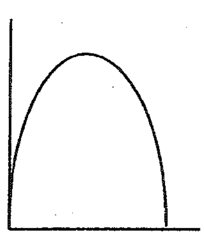
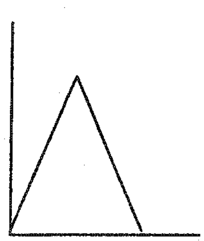
decided to have the class watch how an actual ball would fly. He asked John to come up front and stand over at the opposite end of the board. He tossed the ball to John, and had John toss it back to him, intending for the ball to travel parabolically (even though the ball was to be thrown *straight up* into the air). After the demonstration, Bob asked the whole class, "What did *the ball* do?" [italics added] and students drew a parabola up in the air, which is the shape of the path they saw the ball traveled. Bob confirmed the students' responses by drawing a parabola on the board and labeling the axes, implying he accepted their responses as reasonable. No further discussion was facilitated and the class moved on to a new page. The way he handled this activity with his students suggests his fragile understanding of rate of change.

In a follow-up interview, Bob was asked to reexamine the graphs and explain why each of them was or was not a reasonable sketch describing the ball's flight. Bob's explanations corresponding to each of the graphs are presented in Table 12.

Bob described the graphs as actual paths of projectile motion, rather than as changes in the ball's height over time. His main concern in determining which graph was more reasonable was to check whether he could possibly throw a ball in a similar way to any of the sketches. Thus, to Bob, graphs (a), (b) and (d) did not seem to be reasonable as they did not match any path of a thrown ball. In interpreting the graphs, he failed to recognize how speed was represented or whether there was any change in speed. Even though he did mention the word "velocity" when he talked about graph (c) (shown in italics in Table 12), he was describing the "ball" (i.e., the path of the ball that was pictured in that graph), not the "changes in velocity." The velocity of the ball is actually decreasing until it reaches the peak. He was asked to be more specific about this response

Table 12

Bob's rationales about reasonableness of the graphs describing the flight of a ball thrown straight up into the air

Graphs	Bob's choice	Bob's explanations
(a) 	Unreasonable	The first half [of the graph] is right, but it [the vertical line] can't be this way. Time would have passed at least a bit while the ball changed direction. It's like you ran into a wall and fell down instantaneously.
(b) 	Unreasonable	Well, this is more like a…how do I say that…okay, what I don't think it is…, it's because it comes up here [its peek] and stops and it comes back down. It [the peak] can't be that sharp.
(c) 	Reasonable	That [the graph] shows it's…the time goes by and the height goes up. There is a spot where the *velocity is no longer going up* any more and it stops and it goes back down in the same pattern as it went up. This is exactly what I showed to them [the students] in class.
(d) 	Unreasonable	This one shows that, um…it kind of goes up in a straight line over time and then it stops and then it's back down in the same straight line over time. It's pretty hard to throw a ball like this. [Each of the two steep lines] should be more gradual.

and the following is the dialogue that occurred. (R is the researcher):

Bob: When the ball reaches the top, which is the maximum height the ball can go up to, it has to stop and change the direction. So, right here [the peak of the graph (c)] [the] ball does not move theoretically… I mean the height from the ground, though it's less likely noticeable. So, the velocity will be zero at that moment because it's not moving.

R: Ok. Then what can you tell about velocity before the ball reached the top?

Bob: Before? It drops cause it's fighting through gravity, It slows down. That's why it's a parabola, not a straight line. [A] straight line has a constant rate of change.

R: You said velocity is going up.

Bob: Did I? If I did, I want to take that back. The ball is going up, not the velocity.

In this dialogue he changed his language. This exchange might make him think about the situation once again and perhaps be able to realize his misconception. However, it is not clear whether he came to a true understanding of what was going on or just said what he assumed was supposed to be.

A similar misconception about graphs was apparent with Sally. The class had been working on an investigation (in Unit 3, Lesson 2) focusing on the connection between slope, constant rate of change and parameters in a linear function of the form y = a + bx. Although little time remained in the class period, Sally decided to have a whole-class discussion to close the investigation. She drew two linear graphs with the same y-intercept on the chalkboard on the same pair of axes (time, distance) with no scales shown. (See Figure 13) One had a positive slope and the other was a horizontal line. The following are the last few exchanges of the class discussion:

Sally: Look at these two lines. I just made these up. Do these lines have a constant rate of change?

Students: Yes.

Sally: What does constant rate of change mean?

Students: Keep going… doesn't change.

Sally: Yes, it stays the same. [Pointing to the horizontal line] What's the slope of this line, the rate of change?

Student 1: Zero.
Sally: How can you tell?
Student 2: It's flat.
Sally: Right. [Walking on the floor of the classroom] I'm not going up and down. I just walk on the floor.

[Class ended.]

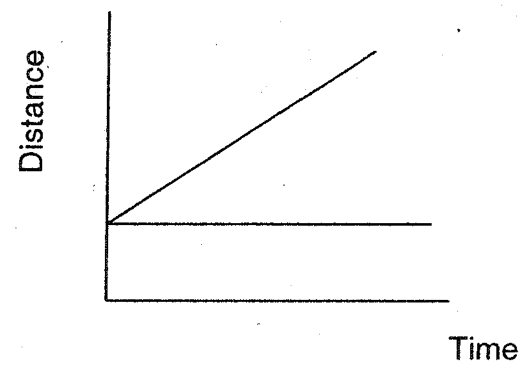


Figure 13. Reproduction of Sally's display of two constant rate of change graphs on the chalkboard

Sally's last comment is not appropriate in this context of relating distance to a period of time. The flat line indicates no distance is gained as time passes, meaning that the object being examined is not moving. However, with her example of walking on the flat floor, there would be an increase in distance. In other words, when walking on the floor, the rate of change would be constant if you walk at a steady rate. However, the rate of change could not be zero as long as you walk at any rate. Sally's last comment itself might be appropriate if the graph was showing the steepness of the path you walked. Then the graph would be horizontal as long as you are on a flat floor no matter how fast or how far you walk. However, this interpretation does not make sense in the context of a time-distance relation, which is the way that the graph was shown. It is noteworthy that during a follow-up interview (cited in an earlier section) she preferred interpreting slope

as rise over run instead of as a measure of rate of change. Her reluctance to approach slope as a rate of change may have resulted in her fragile understanding of rate of change.

During the mathematical task-based interview, Bob and Sally were very successful with problems involving a linear (or nearly linear) pattern, but much less accomplished in recognizing and representing rate of change in a non-linear situation, especially those involving graphical representations. In an interview problem that involved investigating a (time, distance) graph, their mental picture of the path of a person moving seemed to distract them from the distance/time information on the axes and further from inferring the velocity/time information from the distance/time graph. This difficulty in separating the interpretation of a graph from the picture of the phenomenon being investigated surfaced again in their teaching.

<u>Role of different representations</u>

Although both Bob and Sally's concept images and problem solving processes were dominated by numerical displays, they both appreciated the centrality of the multiple representations in the CPMP units they were using. They valued the variety of tables, graphs, rules, and verbal descriptions in the activities, because of the different information that each representation provides. For example, as Bob explained in a follow-up interview:

> Some people might be able to see the relationship with an equation and see that what this really says is that there is some continuous increasing amount that it is going up by. But I think that by making a table, then it can maybe help them see that too.

Although tables and rules were viewed as the most helpful to Bob personally, he recognized that investigating additional representations offered increased opportunities

for students to understand a relationship, because various features are more easily retrieved from certain representations.

The fact that both teachers valued a variety of representations was evident in their classrooms. For example, Unit 3 involved activities focusing on inferring the pattern of a table or rules from the slope and location of a line and vice versa. In the case of equations, both the *NOW-NEXT* equations and explicit equations are used. As students worked on the activities relating slope to *NOW-NEXT* equations and explicit equations, both teachers were faced with questions of the type "Why do we have to do this?" In response, Bob emphasized the importance of understanding it in a couple of different ways. He repeatedly asked students the question, "Which one is more useful?" to lead them to the notion that the *NOW-NEXT* equation only gives answers for the next one, whereas the explicit equation allows calculations for any value of the independent variable, or what one student described as "jumping up to 2," for instance.

Similarly, Sally communicated her appreciation for the different utilities of the two types of rules during a follow-up interview:

> The *NOW-NEXT* . . . they might be able to use that idea to generate answers to problems. As far as the other one, the more explicit formula is certainly a more general rule to be able to get any number down the line in a sequence without necessarily having the term just prior to it.

Additionally she suggested the following reason for students to have competency in using both explicit and *NOW-NEXT* equations:

> Sometimes a pattern might be kind of difficult to explain explicitly because the formula equation might be kind of difficult but, yet, you might be able to say, "Well, this is what's happening, you know, this is what happened to get the next term and I can see that." So, I think both of them are valid.

Thus the cited comments in follow-up interviews and classroom episodes indicate their high regard for the role of different representations (albeit different kinds of symbolic representations in the cited examples) for the divergent perspectives and uses they supply.

Links between representations

In addition to their appreciation of the variety of perspectives provided by multiple representations, Bob emphasized the links between different displays of the same relationship. Bob repeatedly communicated to students that the "equation, table, and graph all show the same thing." His demonstrations of connections included attention to the varied appearance of a particular feature of a relationship in the different representations. For example, when students asked him for help with describing the meanings of the numbers in a (weight on spring, spring length in a mattress) equation $L = 10 + (-0.1W)$, Bob often related the students' equation, table, and graph to the role of the equation's numbers in the situation, as reflected in the following discussion:

Tess: What does the minus 0.1 stand for?
Bob: Well, what was happening to the graph?
Tess: It was decreasing.
Bob: Was it decreasing at the same rate each time?
Tess: I don't know.
Bob: Well, let's check and see. Calculate the difference between the table values. What are the increments?
Tess: The increment is 2.
Bob: 2 what?
Tess: Well, the length [of the spring] is going down by 2 centimeters as weight increases by 20 kilograms.
Bob: So, what would be the increment, actually decrement, as each kilogram of weight is added?
Tess: [You] need to divide that by 20. [She calculated on her calculator.] I got 0.1.
Bob: What does that mean?
Tess: The spring [length] is going down by 0.1 centimeters per kilogram.

Thus, Bob and the student made connections between the number - 0.1 in the rule, the decreasing nature of the graph, and the equal decrements of 0.1 in the table. These details of the three representations contributed to the conclusion that the number - 0.1 reflects the constant rate of change in the relationship between weight on the spring and the spring length.

Although Bob and Sally communicated similar sentiments about the importance of multiple representations, on several occasions they gave preference to certain formats during class discussions of particular problems. In these cases, students reminded them of the usefulness of the neglected representations. For example, the following transcription is of Bob's interaction with two students as they describe a relationship between two variables represented in both graphical (given in the activity) and tabular (created by the students) forms, using an equation:

> Jamal: What do we do here?
>
> Bob: They want you to come up with a rule, an explanation of how you would figure it out. So is there a way, whatever you think, of how this pattern is occurring on the graph?
>
> Joe: [Turned around in his seat in front of Jamal] Is it as the weight increases by 1, the [coil spring] length increases by 2?
>
> Bob: What do you think about that? [Looking at Jamal]
>
> Jamal: I agree with him.
>
> Bob: Is that kind of the pattern that's occurring there? [Points to graph]
>
> Jamal: Yeah.
>
> Joe: You don't have to look at this necessarily [Points to graph]. You can look over here. [Points to the table]
>
> Bob: Right. Some people like to see a graph, some people can figure it out from a table.

Joe's recognition of the pattern in the table directly challenged Bob's graphical tendency here.

Similar student reminders occurred when Sally relied more heavily on "tables," which she conceded was a personal preference, in her whole-class discussion of

numerous problems during the unit. For example, when Sally stood at the board discussing the table and then the equation for a problem, Alisha excitedly pointed out the graph's display of the same relationship:

> Sally: How could you use your table? . . . . I want to know for 6 weeks, so I've got to go down… to 6. There's 6 and it's 142 dollars.
>
> Alisha: You can see it on your graph too! Hit graph and hit trace. Then use your right arrow.

After Alisha articulated how to use a calculator graph to think about the question, Sally attended to graphical features in the remainder of this problem discussion. These types of student reactions to Sally's occasional over-reliance on a particular representation may have contributed to her less biased treatment of graphs, tables, and equations in the classroom. Also, this may suggest the ability of the curriculum to empower students and contribute to the development of a learning community where neither the teacher nor the text are the sole authority.

<u>The HE Teachers: Ian and Misook</u>

During the time of this study, Ian was in his fourteenth year of teaching at a suburban public high school. He had taught various courses, such as algebra, geometry, trigonometry and statistics, using various textbooks, but he had never taught a calculus course. He started using the CPMP materials when his school became a field-test site for the curriculum in 1995. All CPMP courses are now being offered to students and they choose between the CPMP courses or the UCSMP curriculum. Ian had experience teaching every CPMP course, with six years teaching Course 1 (the most) and two years teaching Course 4-only the first few units- (the least). During the observational segment of the study, he was teaching Course 2, 3 and 4 (the first few units) and a statistics course using a non-CPMP textbook.

Ian had an undergraduate major in mathematics at a technical university and earned a master's degree in mathematics education to get a teaching license from a state university. The mathematics courses he had taken included calculus, probability and statistics, linear algebra, finite math, and graph theory. Courses related to calculus and probability and statistics appeared to be the major parts of his work in mathematics. As part of his M.Ed. work, he took two methods courses: a general education methods course and a course on teaching middle school mathematics. He thought that the mathematics courses had the most impact on his teaching in that they prepared him for the content he would teach. He said: "The math courses were so good and filled with content that when I was exposed to the reform curriculum [the CPMP curriculum], the material wasn't new and I could focus on the pedagogy!"

Before Ian started his teaching job, he had not been exposed to the CPMP curriculum nor any other reform-oriented curricula. But, since he started teaching the curriculum, he participated in four week-long summer workshops, provided annually by the curriculum project on all four CPMP courses. When he completed teaching all the courses, he became a lead teacher and began facilitating workshops for other teachers. He also participated in a regional leadership team on reflective teaching.

Ian's classroom was adjacent to the central area for mathematics faculty desks in his department. Student desks were arranged in double rows so that pairs of students sat and worked together. Sometimes Ian also directed students to push desks together in order to make a bigger group. Approximately 23 students were in the observed Course 4 class, all juniors and seniors. Ian described them as a strong class, "[They] may be the second best I've had over recent years." He remarked on the unusual strength of the girls

in the class: "It's normally true in higher math classes that boys speak out more, but girls in this class, they are bright."

Misook was the most experienced of all 13 participating teachers, and thus, of the four teachers being observed as well. She taught for over 35 years, 10 of them using the CPMP curriculum as well as teaching from other textbooks. Like Ian's school, hers was a CPMP field-test site and she started using the CPMP materials when they were introduced in her school in 1994. Each CPMP course is now offered and students choose between the CPMP curriculum and the UCSMP curriculum. She has experience teaching all four CPMP courses, having at least four years with each course. At the time of this study, she was teaching CPMP course 2, 3 and 4 and a calculus class.

Misook had an undergraduate major in mathematics and minors in science and physical education. The mathematics courses she had taken included courses on calculus, algebra, geometry and some computer-related courses. As part of her minor, she took a general education methods course and a course on techniques in teaching physical education. She answered "no thought" to an item on the survey asking which of the university education programs she thought had had the most impact on her teaching.

She has been quite active professionally. She has participated in many workshops and professional development programs in general and also specifically related to CPMP curriculum materials as a lead teacher as well as a participant. She appraised her professional development experience as "good" with a comment that professional development always opens a person's eyes to new approaches. She has supervised many student teachers over the years and served as a mentor-teacher for her colleagues who were new to the CPMP curriculum. She has presented talks on teaching calculus and the

CPMP curriculum at a national conference and also at state mathematics teacher meetings. She said "calculus" is her favorite subject to teach. In fact, she has periodically served as an instructor of a college pre-calculus course for about five years.

Her desk was in the back corner of the classroom with an overhead projector in the middle at the front. Student desks were arranged in groups of three or four. The class observed contained approximately 20 students, juniors and seniors. Misook described her students as "kind of average" and said "There are always some who are brighter than some others."

Course 4 Unit 1: *Rates of Change*

The important ideas of this unit are rate of change and total change in a quantity, two key topics in calculus. In the lessons that contribute to understanding of rate of change, the goal is about estimating average and instantaneous rate of change in different contexts. Related to this goal, students are to develop the ability to estimate and represent rate of change for a variety of variables using tables, graphs and symbolic rules, and understand the similarities and differences in estimating the rate of change in linear and nonlinear functions.

What is derivative?: Rate of change in a function

Misook's class was working on an investigation of CPMP Course 4, Unit 1, Lesson 2 where students were to reason about derivative functions for linear and quadratic functions using a numerical slope-estimation technique with the aid of graphing calculators. The students first worked on an activity where they developed a symbolic rule for the derivative function of $f(x) = x^2$ by studying a table of estimates for the derivative at several different points. They then worked in groups to develop symbolic

rules for the derivative of several different linear and quadratic functions, including $f(x) = x^2 - 4$. While some students worked quickly and started on $f(x) = x^2 - 4$ by making a table of estimates for its derivative (which was what the problem asked the students to do), Misook stopped the class and asked the whole class to make a guess about what the derivative was going to be. The following is the whole class discussion:

Students:	We haven't started yet. [They yelled.]
Misook:	That's okay. Just make a guess. Don't touch your calculator. Just think about it. [She waited a little but saw many blank faces, so she decided to provide a clue.] Well, you've just worked on x^2 and our function is $x^2 - 4$. What happens?
Student 1:	To x squared minus 4?
Misook:	Yeah.
Student 1:	It's decreasing by 4.
Misook:	Decreasing by 4. Everybody buy that? If we were to graph this [$x^2 - 4$], this would be...
Student 2:	It moves down.
Misook:	Yes, there's some sort of a shift. [She drew two graphs on the same pair of axes.] Now, look, it doesn't change the graph as much as it changes the location of it. Everyone buying that? . . . What's going to be the derivative here, Jacob? [He had raised his hand.]
Jacob:	It would be linear?
Misook:	It would be linear. Okay. Everybody... Is everybody happy with what he just said? Jacob, can you say why you thought so?
Jacob:	It's... maybe, x squared and x squared minus 4 are both quadratic. And [the derivative of] x squared was linear...
Misook:	Do you mean, [the derivative] stays the same? What about linear?
Jacob:	[Inaudible]
Misook:	[Drawing a tangent line on the graph of x^2] what is this?
Students:	The slope?
Misook:	It's the slope, it's the rate of change. [Pointing to the graph of $x^2 - 4$] does the shape change at all?
Students:	No.
Misook:	What happens here to the derivative? [She pointed to the position on the graph of $x^2 - 4$ at the same x-value used when she drew the tangent line on x^2.]
Students:	It stays the same, same slope.
Misook:	So, what would be the equation?
Amber:	Oh, shouldn't it be the same? I mean 2x [they found this rule in the activity that preceded this problem.]
Misook:	Why the hell did that happen?
Amber:	'Cause it doesn't change the rate. It's just shifting down.

After this discussion, she had them check their conjecture by creating the table as they were asked to do in the problem. Then, she put more problems on the board, such as $x^2 + 4$ and $x^2 - 10$, and asked the students to think about the derivative functions. The students explained that they thought 2x would be the derivative of the two functions, and referenced the idea of "shifting down or up." In a follow-up interview, when asked about the rationale for this action, she said:

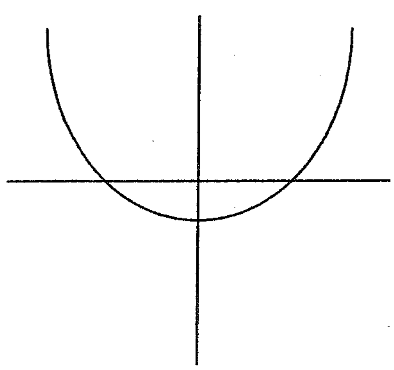
> We would love to have our college students say that, wouldn't we? That's exactly what you...because you looked at this [pointing to the drawings of the two graphs used in class], you're starting to see some things. It is more important that it make sense to them why some functions have the same derivative. Derivative is rate of change and the derivative won't change unless the rate of change changes. . . . [I wanted to] remind them, the constant is a shift. It just moves it up or down. The only thing that changes is the location of the graph, not the rate of change . . . the rate of change is the same in every case.

Misook's instructional actions and comments suggest that she conceptualizes derivatives and slope as rate of change and wanted to develop it that way with her students. She offered the students additional opportunities to conjecture about the derivative functions. This may assist students in developing a more meaningful sense of the symbolic rules of derivative functions, such as understanding why a constant in a function becomes zero in its derivative function.

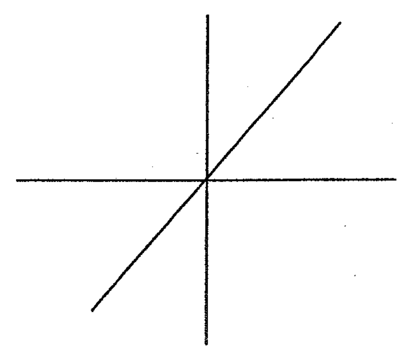
<u>Emphases on students' creation of multiple representations</u>

In an investigation of the same Unit 1, Lesson 2 determining derivative graphs by studying the shape of graphs, Ian gave the students worksheets to sketch a graph of the derivative function for each of the given functions with no scales shown. The worksheet contained different shapes (such as straight lines and curves) of graphs, without their function rules. For the curve shown in Figure 14, it seemed the class assumed (although

implicit) the graphs were of a quadratic function. Ian found that to do this activity numerous students relied on their memory about the shape of the estimate of the derivative function. For example, with the graph of the parabola shown in Figure 14, students began by inferring that the graph was $f(x) = x^2 - a$ (with 1 being the most preferred value for a). They then computed $f'(x) = 2x$ as its derivative and drew a sketch of $y = 2x$. (Also shown in Figure 13) Students also used their graphing calculators to graph the derivative function by producing a table of estimates of the rate of change for (for instance) $f(x) = x^2 - 1$, using a numerical method of approximating rates of change, $f'(x) = (f(x + 0.1) - f(x - 0.1)) / 0.2$.



Sketch the graph of the derivative of the function whose graph is shown



A representative of students' drawings to complete the task shown at left

Figure 14. A derivative sketching task of the worksheet Ian used in his CPMP Course 4 class

Ian stopped the class and acknowledged that everybody got a line that looked similar to a graph of 2x for the graph of the derivative of the parabolic graph. He asked the students to think about how they would know their sketches were correct assuming they did not know the symbolic rule to find the derivative. Students then started to work

by examining slopes for the function at several different points along the x-axis. The following is typical of how students explained their thinking about this:

> Well, I'm trying to do the derivative of this graph. It's [the slope is] a greater negative here [pointing to the negative part of the x-axis of Figure 14 (the graph on the left)], and then it [the slope] slowly becomes more positive [meaning less negative] near 0. As we go to quadrant I over here, from quadrant II, it [the slope] goes from 0 to more and more positive values. So, I knew the line [graph of the derivative] had to go from quadrant III to quadrant I and on up [as seen in Figure 14 (the graph on the right)].

During observations prior to this investigation, it seemed that most of the students understood that the slope of the graph of a function is the y-value of the derivative. But, by encouraging them to present their ideas using a different representation from the one they chose to use, students benefited from being able to recognize an idea embedded in the different representation. This was an additional opportunity to reinforce their understanding of the connections between the slopes of the tangent lines to the graph of a function and the behavior of its derivative. This perhaps prompted the students to become more fluent with the idea.

A similar occurrence was seen in Misook's teaching in regards to developing an informal understanding of an important mathematical idea that is typically introduced in calculus class. During one of the lessons observed, Misook had the students complete a worksheet (which she made up), where they had to sketch a graph of the derivative for each of the graphs shown without the symbolic rules. Misook's worksheet contained the same kinds of graphs that were on the worksheet Ian used, such as graphs of linear, quadratic and cubic functions. In addition, Misook included a saw tooth graph as shown in Figure 15.

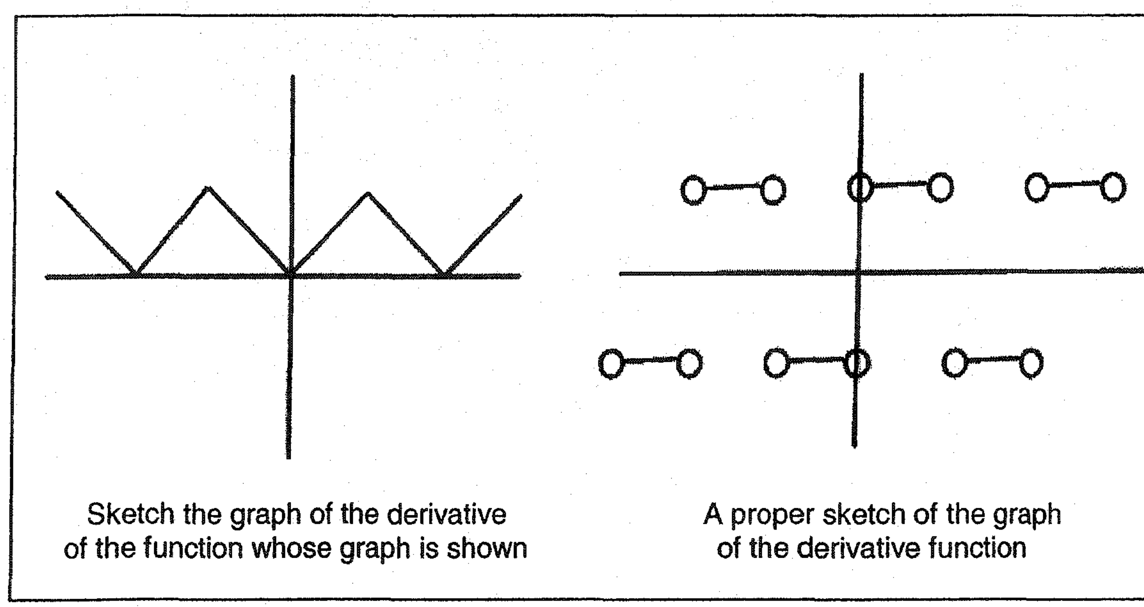


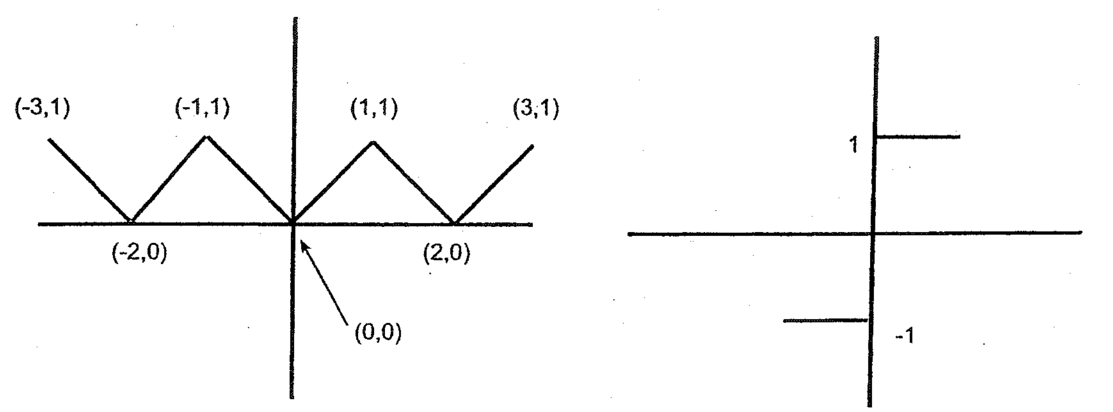
Figure 15. A derivative sketching task of the worksheet Misook used in her CPMP Course 4 class

When moving among the groups of students as they worked, Misook saw one student, Kurt, looking somewhat puzzled, staring at his drawing. It was inferred from his work that he assigned values for ordered pairs on the graph of the function for the task, as shown in Figure 16. He then used the ordered pairs to calculate numerical values for the slopes of the graph on the right-hand side and the left-hand side of the y-axis, which he determined to be 1 and -1, and then used these slopes to draw the derivatives as shown in Figure 16. Kurt's performance indicates that he preferred an algebraic way of solving the problem.

Kurt did not continue the process with the other parts of the graph, but instead asked Misook, "What is the derivative at $x = 0$?" Misook encouraged him to think more on his own and left to go to another group, saying she would be back to him to discuss it further. When she returned to him, he posed a different question "What is the equation of this graph? Once I know that, I think I can estimate it [the derivative]." He still

maintained an algebraic approach to the task. She then asked him to "draw a tangent line at x = 0." He then began to work on her suggestion, talking to himself:

> I don't know... you could...[paused] Well, you could draw as many [tangent lines] as you want, 'cause this is just one point... *one point that you can get close to.* So, maybe, something like this? [He drew the sketch shown in Figure 17 and looked to Misook to try to read her expression.] I think...if we took a tangent line, on this corner, we could actually have all types of tangent lines going around that corner. So, I guess we can't say what it [the derivative] is [at x = 0], because there's too much change going on around this corner. It would be true for every corner here . . . .



Kurt's allocation of values for the graph in Figure 15

Kurt's unfinalized drawing of the derivative of the graph in Figure 15

Figure 16. Kurt's intermediate work for the derivative sketching task in Misook's class

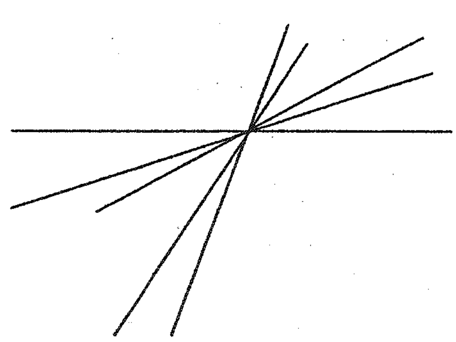


Figure 17. Kurt's sketch of tangent lines for the derivative sketching task

This discussion foreshadowed an important idea in calculus; that is, the idea that there is a function which fails to have a derivative at certain points and that such a point is a sharp corner. Kurt was getting the idea, albeit in the initial stages. In a follow-up interview, Misook was asked about the rationale for her prompt question "draw a tangent line at $x = 0$," and she explained:

> It's [the derivative is] undefined at the corner. But they can't know that, because they haven't learned it yet. They will know when they get to calculus. But I didn't want to completely shut him off on the question... but I do want him to at least get a feel for the idea that the derivative doesn't exist at a sharp corner . . .

That the function of Figure 15 is not differentiable at $x = 0$ (actually, at each sharp point on the graph) can be proved using a mathematical approach. Assuming the middle two parts of the graph behave like the function $f(x) = |x|$, when trying to compute the derivative of $f(x) = |x|$ at $x = 0$, the left limit is -1 and the right limit is 1. Since the limits are different on each side of zero, the derivative at $x = 0$ does not exist. One can also think about this visually in that near $x = 0$, even close-up views of the graph look the same, so this is a corner which can not be straightened out even by zooming in. While Kurt's thinking was not at this formal level, his reasoning was solid. And Misook's willingness to encourage this exploration demonstrates her understanding of differentiability and linear approximation.

Misook's capacity to comfortably utilize graphical representations to assess characteristics of problem situations was clear during the mathematical task-based interview. This cited example illustrates that she was able to use her proficient graphical knowledge to create an environment that helped the student view a mathematical idea from a different perspective and therefore reveal a new mathematical idea that might not have otherwise emerged.

Interpretation of the second derivative

In the mathematical task-based interview, problems requiring understanding of the derivative appeared to be the most challenging for teachers. Even in the case of HE teachers, who demonstrated more robust understanding overall, some of their understanding of the derivative proved to be limited. The results from classroom observations added additional evidence to confirm these findings.

As an introduction to an investigation on the linear connection in Course 4, Unit 1, Lesson 2, Misook had the students work on exploring methods for using the rule of a given function to find the rule of its derivative function. Misook put a function, $f(x) = 2x^2 - 3x - 5$, on the chalkboard and asked the students to find the rate of change at $x = 3$, using an interval of 1, 0.5 and 0.1 on either side of $x = 3$. She told the students the three values would all be different. Her goal was to help make explicit the idea that an estimate could be more precise by choosing a smaller interval containing the given point. Previously students studied three different methods for finding the rate of change at some specified x-value: using a segment just to the left of a point, using a segment just to the right, and using a segment that straddles the given point. By this time, the students had settled on the third method, as intended in the CPMP materials. In this particular activity, the students found the rate of change at the three different intervals and were puzzled by the fact that all the estimates were the same (all came out to be 9). The findings contradicted their anticipated results, which were that the values would have been different (actually, more accurate) as different (actually, smaller) intervals were used. When they asked why, Misook said "in this particular case it came out to be 9 again." It was not clear what she meant by "this particular case"—whether it was the issue of being

a quadratic function (which is the case here) or the issue of selection of some of the numbers used in the problem (i.e., numbers in the parameters of the equation, numbers used as the end points of the given interval segments, or the value of the point of interest). When probed about what she meant by "this particular case," she explained:

> I wrote the wrong number down. It just happens. . . . [She said she went over this problem previously] I had a different one rigged up so that you would get different values. I had it so that where it comes around... [did not complete the sentence]. This was a nice symmetric... And so if you choose any point on that [the right-hand part, from the y-axis, of the graph], it will always be 9. It would always be 9 because I went symmetric . . . . What the one that I wanted to choose... [did not complete the sentence] . . . . I don't know what it was. What I had chosen originally, that the vertex was between there and so my first value [estimate at an interval of 1 on either side] would've been a lot different. I must've, when I practiced this. And believe it or not, I do practice these...[She attempted to calculate by hand the x-value of the vertex point of $f(x) = 2x^2 - 3x - 5$ and came up with an incorrect answer of 3/2. The correct x-value of the vertex point is 3/4] I must've chosen two, and so that one you would have gotten a funny answer on both sides. You wouldn't have gotten 9 on all cases. But I made a mistake. When I practiced this, I must've chosen 2. When I practiced this yesterday, because I wanted my first interval . . . . My first interval to be like... past the vertex . . . .

In this response, Misook indicated that the three estimates would have been all different if she had used x = 2 instead of x = 3 to investigate the rate of change, reasoning that the vertex being within an interval would yield an inconsistent estimate at the interval compared to estimates at other intervals. She reasoned that estimates would be always the same as long as each of the two end points of an interval used to estimate the rate of change for a given point were located in the same part of the graph—either the right-hand side or left-hand side. This is actually an incorrect idea. In fact, the incident is due to a feature of quadratic functions, which have a constant change in differences of the y-values (the second derivative of a quadratic function is constant). Thus, since the changes in differences in y-values for constant increases in x-values are all the same, the

length of an interval does not make a difference in estimates for the rate of change as long as the two end points of an interval are kept the same distance from the point in question.

Interpreting graphs: Message of a graph vs. picture of the phenomenon

Like the two LE teachers, Ian showed a tendency to misinterpret graphs as pictures of the phenomenon being graphed, rather than the correct interpretation of the relationships between the two variables of interest.

During Course 4, Unit 1, Lesson 1 that explored how the information in a table or graph for different variables can be used to estimate and explain the rate of change of those variables at specific points, students examined a situation relating distance traveled by a trapeze flyer along an arc from the takeoff point to the time of the swing from one side to the other and back again.

In one of the interviews before an observation, Ian gave "Being able to describe and estimate rates of change using tables and graphs" as the goals of this lesson and this lesson was not difficult for him to prepare for teaching. Toward the end of class, Ian had a whole-class discussion to sum up the investigation of the situation using the graph (as shown in Figure 18). Ian ended the investigation with an exchange about what a negative velocity would tell about the graph:

Ian:	What does a negative number for velocity indicate about the graph?
Student 1:	Backwards?
Ian:	What would "backward" mean in terms of the trapeze person?
Student 2:	You're falling
Ian:	You are falling because you're going down. Going up would be positive, going down would be negative [he drew a going-up line and then a falling down line into the air]

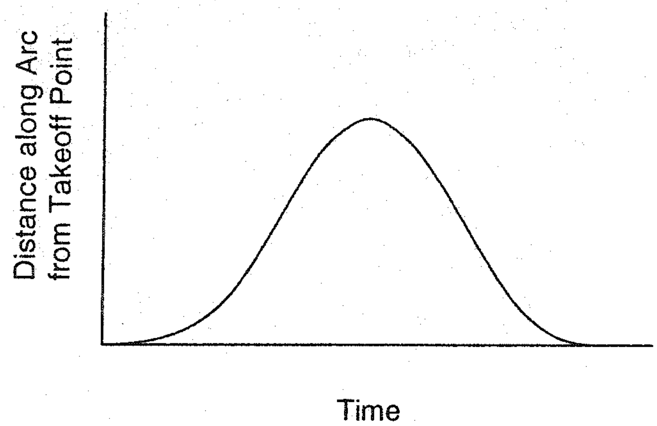


Figure 18. A graph of the (time, distance) relationship of a complete swing from one side to the other and back to the start

As seen in this dialogue, Ian confused the (time, distance) graph with the actual motion of a trapeze flyer. A feature of this situation, which is that the peak of the graph is at the end of a single arc, not in mid-arc, may cause confusion. So, in a follow-up interview he was asked to be more precise about what he was referring to as a "negative number." He began by relating the change in distance over the change in time to the motion of the swing. He then indicated that the second half of the graph was where the velocity is negative, because it was where the trapeze flyer was coming back to where he started from the other platform. He also commented that he was not really thinking carefully about the inflection points (where a decreasing rate starts increasing and an increasing rate is decreasing), saying that it was in these that the trapeze flyer started accelerating or decelerating. His response suggests that when given time to think carefully about the context, he was able to catch his misconception. However, the misconception apparent in his teaching suggests his graphical knowledge about rate of change may have room for additional improvement.

Summary

During pre- and post-interviews, Both Bob and Sally often pointed out their lack of familiarity with the materials, their emphases, and where lessons were headed. They repeatedly claimed that a year's experience would make them much more effective in their teaching of the CPMP materials. Ian and Misook indicated that because they were fairly comfortable with the mathematical content (they attributed that comfort to experience with the CPMP materials), they focused more on implementing the material to its full potential.

Observations provided examples of both what the teachers know about rate of change as well as how they used that knowledge to develop their students' understanding. Bob and Sally showed their inclination towards non-graphical representations (e.g., tables and symbolic rules) in problem situations. But, at the same time, both teachers acknowledged the power of varied forms of representation to help students gain multiple perspectives on problem situations. They also emphasized the links among different representations of the same relationship. But their somewhat procedurally-based understanding of slope hampered their abilities to deal with students' difficulties and resulted in *missed opportunities* for students to understand the concept more fully.

Ian and Misook also valued "having multiple ways to represent mathematical concepts." However, in addition to stressing the importance of using multiple representations, both Ian and Misook also focused on student-generated representations as ways of enhancing their learning or developing new mathematical ideas. They provided tasks that encouraged students to approach problems from different perspectives. Their flexible understanding and graphical proficiencies seemed to

empower them to create such opportunities. It seemed that their solid understanding resulted in a more facile use of that knowledge in their teaching. However, some confusion did surface as well. One of the HE teachers, Ian, as well as both of the LE teachers, interpreted a graph as a picture of an event instead of as a depiction of a relationship between two variables. This confusion did not surface during the task-based interview. The idea of derivative proved to be challenging for HE teachers as well.

CHAPTER V

SUMMARY, DISCUSSION AND RECOMMENDATIONS

Teachers' subject matter knowledge is enjoying a renaissance in research on teaching and teacher education. From a range of perspectives and with a variety of approaches, researchers are focusing increasingly on the subject matter knowledge of teachers and its role in teaching. In research on the teaching of mathematics, researchers have investigated inservice and preservice teachers' understanding (or misunderstanding) of specific topics (Ball, 1988a; Even, 1993; Knuth, 2002; Norman, 1992; Post, Harel, Behr, & Lesh, 1991; Wilson, 1994). These studies often claim that teachers possess insufficient understanding of the mathematical topics they teach which often results in inadequate opportunities for teachers to develop the requisite mathematical knowledge or the ability to use it in practice (e.g., when teachers use curriculum materials and teaching methods that emphasize skills and procedures without allowing students to develop the conceptual understanding underlying those procedures). In addition, there has been increasingly more interest in research on teachers' knowledge due to the availability of new curricula that require a deeper knowledge across a broad range of mathematics topics in order to teach them well.

In 1992, the National Science Foundation (NSF) responded to the need for curriculum materials to help high school mathematics students as well as teachers by funding curriculum development projects based on the recommendations in the NCTM Standards documents (NCTM, 1989, 1991). In particular, five high school-level

curriculum projects resulted from NSF funding: a) Applications/Reform in Secondary Education (ARISE), b) Core-Plus Mathematics Project (CPMP), c) Interactive Mathematics Program (IMP), d) Math Connections, and e) Systemic Initiative for Montana Mathematics and Sciences (SIMMS). Since these curriculum materials reflect the recommendations in the NCTM Standards documents, the term "Standards-based" is used here to describe them. One important aspect of the NCTM Standards recommendations, which is also reflected in the NSF-funded high school curriculum projects, is that "making sense" of mathematics is an essential goal and, in particular, that understanding of algebra is more than manipulation with a set of symbolic rules. The Core-Plus Mathematics (CPMP) curriculum takes such an approach by emphasizing the development of algebraic ideas, such as relationships among quantities, multiple representations, and analysis of change (e.g., rate of change), in addition to developing symbolic reasoning and manipulation skills.

Although much research has demonstrated that teachers have a weak understanding of important mathematical concepts, there has been little research on teachers' knowledge in the context of using these Standards-based curricula where content is developed differently. The Third International Mathematics and Science Study (1995) along with the work of other scholars (Ball, 1996; Russell et al, 1994) suggests the need for research to reflect the qualitatively different nature of theses new textbooks in terms of the educational opportunities they provide for teaching practice, teacher learning and student learning. In terms of content, change/rate of change has been advocated by many (e.g., Nemirovsky, 1993; Noble et al, 2001) as an important mathematical concept. It has its basis in everyday experiences like motion and growth and is a fundamental

organizing idea for relationships between variables (Confrey & Smith, 1995; Monk & Nemirovsky, 1992). The rate of change concept is fundamental to the study of much theoretical and applied mathematics. Thus, students can approach change at many levels, from elementary arithmetic to advanced calculus, and across many disciplines.

In CPMP, the ideas of change/rate of change are addressed in each grade level course while continually growing in sophistication. The discussion of "change" begins in Course 1 and revolves around linear and exponential functions, while Course 2 focuses on power functions and periodic change and Course 3 deals with families of functions and sequential and discrete aspects of change using recursion. In Course 4, rate of change ideas are more intensively discussed and formalized in ways that underpin Calculus. The concepts of change/rate of change in each course is developed through investigation of real-world phenomena within a variety of contexts, representing and analyzing mathematical relationships in various representational forms, translating between representational forms, and making connections with other mathematical ideas and to other disciplines. Problem situations or contexts engage students in meaningful and challenging discussions. Students launch into discussions about why, for example, graphs based in real-life contexts are changing, how they are changing, and how actions and inputs affect the resulting graphs. These dialogues are valuable, for they encourage students to rethink their own understanding, to consider multiple representations, and to look at a problem from various points of view. This approach has the potential to enhance teachers', as well as students', learning of rate of change.

What Does It Mean to Know Rate of Change?

In an attempt to analyze the knowledge necessary to understand such a broad concept as rate of change, a guiding framework was designed and used for this study to begin articulating what it means to know rate of change. It was developed with three major foci in mind: concept image and concept definition of rate of change, mathematical understanding of rate of change, and certain aspects of pedagogical understanding of rate of change. In investigating the mathematical understanding of rate of change, three contexts were identified and elaborated upon that seemed to capture a range of rate of change knowledge: use and interpretation of multiple representations, linear connections, and modeling and solving problems that exhibit a rate of change. The mathematical understanding of rate of change across these contexts is a continuum beginning with basic concepts involving simple rate of change ideas to deeper and more-connected understandings. A basic understanding of rate of change is often limited to working within a single context and would include the ability to:

a) construct representations for constant rate of change and non-constant rate of change;
b) analyze representations by recognizing patterns of change within a given context, and;
c) translate between one representation of rate of change to another.

At the other end of the continuum a more-connected and complex understanding of rate of change would include the ability to:

a) make connections between representations of change across contexts (for example, connect velocity graphs (graphs of change) and position graphs (graphs of accumulation));

b) use knowledge of rate of change to interpret and make sense of new representations and situations involving change (for example, think of situations or contexts that a given graph could represent), and;

c) conjecture and construct a system of rules that govern the behavior of situations (for, example, generalize a rule that gives the rates of change of a nonlinear function at various points).

Findings and Discussion

This is a descriptive study with the intent of providing a rich characterization of teachers' knowledge of rate of change. The nature of teachers' understanding and qualitative differences among teachers were investigated by collecting data through a survey, an interview using mathematics tasks, and classroom observations and interviews with a subset of four teachers. The participating teachers were grouped as: a) LE teachers for the least experienced teachers who had not taught any CPMP courses or those whose experience was limited mainly to Course 1 and 2; b) ME teachers for teachers having a moderate amount of experience teaching CPMP, mainly with Course 1, 2 and 3 each for at least 2 years; and, c) HE teachers for the highly experienced teachers who had taught all four CPMP courses each for at least 2 years.

The results from all data sources provided evidence to characterize teachers' knowledge of rate of change with respect to each of the components in the framework and identify differences in the characterized knowledge in relation to teachers' level of CPMP teaching experience. A summary of key findings associated with the rate of change knowledge and relevant discussion follow.

Concept Image and Concept Definition of Rate of Change

All teachers included a description of slope of a straight line in their image of rate of change, which is a geometric concept as a property of a graph of a line. Few teachers thought of rate of change as a derivative or ratio. Slightly more than half of the teachers used functional language to illustrate their image. It was a surprising finding that the functional representation of rate of change was missing from the descriptions of many teachers' concept images, because a functional interpretation is inherent in the term rate of change (e.g., change as a function of time). Physical or trigonometric representations of rate of change were virtually non-existent in teachers' concept images. This suggests that many teachers' mental images of rate of change are restricted to linear situations in that they likely think of rate of change as an attribute of slope, rather than slope as one representation among many that could be used to represent rate of change. This may be due to limitations in their own learning experience where some teachers might be less comfortable explicitly talking about rate of change, compared to talking about slope, since the term "rate of change" has only more recently become prominent in school mathematics curricula. Although teachers might be capable of making connections between various representations of rate of change, few have actually incorporated these representations into their concept images.

There did not seem to be a direct relationship between the representation used in their concept images and those used in their concept definitions. Unlike the various representations shown in their concept images, the use of functional language was much more prevalent in teachers' concept definitions.

Comparison among LE, ME and HE teachers

In regards to their concept images, all HE teachers made reference to features of function as well as geometric concepts (such as rise over run). However, only three (50%) ME teachers and one (33%) LE teacher mentioned a functional characteristic to illustrate their concept image of rate of change. While these ME and LE teachers were more likely to refer to the parameters in a linear equation (such as "m" in the equation $y = mx + b$), HE teachers incorporated the ideas of derivative and/or slope of a tangent line utilizing nonlinear functional situations. However, all but one ME teacher used a functional approach to define rate of change. It is puzzling that although most teachers preferred a description involving function characteristics for their formal definition of rate of change, many of them did not utilize a functional approach for an informal image of rate of change. This may suggest that their mental images of mathematical concepts are confined by contexts with which they are more familiar. Their formal definitions seemed to be more linked to how the concept is presented in the textbooks they have used, since the concept of rate of change is introduced and developed in various situations that exhibit a functional relationship in the CPMP materials.

Mathematical Understanding of Rate of Change

Teachers demonstrated a similar understanding in situations involving constant rate of change (and nearly constant rate of change) and varied understanding in situations involving non-constant rate of change. In situations involving constant rate of change, teachers demonstrated flexibility in their thinking about and ability to describe change/rate of change using a variety of types of representations—tables, graphs, equations and verbal descriptions. Teachers were able to move flexibly among different

representations (i.e., using the information obtained from a table to create a graph and making a connection between the information provided by the table and the graph). In the CPMP curriculum, use of multiple representations is consistently required in many problems using the prompts: make a table, make a graph, write a rule, and write *NOW-NEXT* equation. Based on this study, it appears that such an approach helped teachers understand the importance of using different representations and view rate of change in multiple ways.

In situations involving non-constant rates of change, teachers' levels of understanding of multiple representations differed. All teachers were able to distinguish between constant and non-constant rates of change and construct representations to recognize patterns of change. It was common for teachers to begin to describe the patterns with words first. Most teachers demonstrated the ability to move between various representations. These teachers were the most flexible in moving from graphs to words and the least flexible in moving from one type of graph to another. In particular, the teachers showed better flexibility in moving from graphs of accumulation to graphs of rate of change (such as, sketching a velocity graph from a given distance graph), compared to moving in the opposite direction.

Inferring graphs of rate of change from graphs of accumulated quantities proved to be difficult for most teachers (i.e. predicting a graph of the original function from its derivative function without reference to its symbolic rule). Teachers often confused the slope of the tangent line of the derivative function as the rate of change of the original function. In fact, in relation to the concept of derivative, making connections between average rate of change and instantaneous rate of change appeared to be the most

challenging area to understand. Even teachers who demonstrated understanding of the slope of the tangent line at a given point as a way to find instantaneous rate of change at that point were often unsuccessful in using their understanding of finding average rate of change to estimate an instantaneous rate of change and to make sense of the symbolic rule for the derivative. The fact that only four teachers (HB, HD, HG and M3C) were able to use their knowledge of derivative to make sense of a situation where it was necessary to go beyond manipulative facility and utilize conceptual understanding, suggests many teachers held a very procedural understanding of derivative. This confirmed the idea that, as shown in their concept image, three teachers (two ME and one LE) did not seem to understand the concept of derivative other than knowing a procedure for calculating one. These three teachers mentioned the term derivative as an illustrative example of rate of change, but did not provide any rationale for using the term other than demonstrating their knowledge of the symbolic rules involved.

Context played an important role with regard to the teachers' ability to explore rate of change. More teachers were able to interpret situations involving non-constant change when they were embedded in a context-rich setting. For example, teachers were more accomplished in discussing average rate of change and instantaneous rate of change and their connection in a contextualized problem such as the fruit fly population problem. There was only one teacher (M1B) who "gave up" when completing the task-based interview, due to the teacher's insufficient understanding of mathematics. The teacher consistently had difficulty discussing rate of change in nonlinear situations, and in fact, was not successful with any of the problems involving non-constant rates of change that he attempted to solve. This may also relate to the fact that the teacher has taught only

CPMP Course 1 for several years, and deeper treatment of non-constant situations are explored in the later CPMP courses. It was a surprising result, given this teacher's mathematics background and experience teaching high school mathematics for four years. This may suggest that there have been limited opportunities for this teacher to develop his understanding of situations involving non-constant rates of change.

Comparison among LE, ME and HE teachers

HE teachers demonstrated a strength at working with contexts involving non-constant rate of change and the concept of derivative, not exhibited by the LE or ME groups. These teachers approached problems in a sense-making way using their understanding of rate of change in linear relationships, reflecting a strong conceptual understanding of rate of change ideas and the ability to distinguish between constant rate of change and non-constant rate of change. Teachers who had more experience using the CPMP curriculum were more apt to use graphs to describe non-constant rates of change, while LE teachers demonstrated a strong tendency to analyze information on tasks using tables and equations. In particular, HE teachers and some ME teachers showed a strength in describing derivative using graphical representations as well as other representations. These teachers were better able to recognize similar and contrasting characteristics of different types of representations across contexts.

LE teachers and some ME teachers demonstrated a weakness in understanding instantaneous rate of change. They did not seem to recognize that finding a rate of change depends on treating the curve as if it were a series of very short line segments that approximate the curve. For them, it was necessary to have two points or know the symbolic rule for the derivative function to find instantaneous rates of change. Although

HE teachers demonstrated a deeper and more well-connected understanding of ideas involving the concept of derivative than most of the other teachers, one of the observed HE teachers did not demonstrate the recognition of the effect of a second derivative function on the rate of change in its original function. This suggests that HE teachers may also exhibit gaps in their understanding of rate of change and that certain aspects of derivative remain challenging.

Among the participants in this study, two of the HE teachers (one who was observed and the other who was not observed) had taught a calculus course. Since the concept of rate of change is central to the study of calculus and it is reasonable to think that teaching an advanced course would impact knowledge, influence a person's perspective, one may argue that the strengths demonstrated by HE teachers over their peers could be attributed to experiences teaching calculus. However, the two teachers' understanding of rate of change was more similar to those of HE teachers as a whole. Thus, calculus teaching experience did not seem to be an influential factor in teacher's understanding of rate of change.

An error in viewing a graph as a picture of an event instead of depicting a relationship between two variables was demonstrated by some LE and ME teachers on the task-based interview and one HE teacher during classroom teaching. This HE teacher's error did not surface when he was interviewed. This may suggest that this teacher's graphical ideas of rate of change is not thorough and, more generally, that teaching demands more mathematical flexibility in order to use such knowledge in practice. For example, when teachers hold class discussions and they make decisions about which (whose) ideas to pursue, teachers need to readily unpack their own

knowledge to understand and appraise mathematical ideas being discussed. Also, teachers face many factors (e.g., classroom management issues) that could distract them from focusing on the mathematics under discussion. Thus, teaching requires teachers to develop and acquire mathematical knowledge in usable and useful ways.

Some differences demonstrated among teachers' understanding may be due to their levels of experience using the CPMP curriculum, and in particular, with CPMP Course 4. Although different aspects of the rate of change idea are developed throughout each course, in Course 4 the rate of change idea is intensively discussed and formalized in ways that underpin calculus. The possible positive impact of the curriculum materials on teachers' mathematical understanding is supported by the results of the survey. A majority of the teachers reported that by using the CPMP materials they were given opportunities to rethink some mathematical topics from new perspectives as they worked through the texts. When asked to recall what mathematical ideas those were, some teachers indicated ideas related to rate of change, such as the meaning of slope and derivative. Some teachers also indicated that the CPMP materials challenged them to think differently about the idea of rate of change.

With respect to slope, there were two teachers ($L3A_3$ and M1C) who seemed to be skeptical about CPMP's approach of introducing slope as a rate of change. This belief may be a result of their experience teaching rate of change thus far in the courses they have taught (Courses 1 and 3 for $L3A_3$ and only Course 1 for M1C). In the CPMP courses, the term "slope" is not used until students are well-acquainted with rate of change. Understanding slope of a line as a rate of change is crucial to developing the functional concepts underlying the meaning of slope and making sense out of situations

involving change in which rate of change is not constant. Since those teachers had not taught Course 4, they might not have had the opportunity to see why slope is presented in such a way in the CPMP materials. This position was also supported by voluntary comments written by two teachers on their survey: "You see how useful it is when you get to Course 4" (HG, HB). This suggests that the development of knowledge of rate of change may be enhanced by the use of curriculum materials that contain rich connections among different rates of change through various representations, such as those in the CPMP curriculum.

However, some of the misconceptions displayed by HE teachers also suggest that additional work may be required. For example, one HE teacher who has taught all CPMP courses showed difficulty interpreting a graph as a relationship between two variables. Instead, this teacher tended to see a graph as a picture of an event. And another HE teacher's understanding of derivative was fragmented and seemed held as separate bits of knowledge. These same weaknesses have been noted by others (Monk & Nemirovsky, 1992; Porzio, 1997), indicating that these ideas may require more time to develop, regardless of how the concepts are developed in a particular curriculum.

Pedagogical Content Knowledge

Evidence of four aspects of pedagogical content knowledge emerged through teachers' self-report surveys and classroom observations: knowledge of development of ideas involving rate of change, knowledge of the prerequisites for student understanding of rate of change, knowledge of students' difficulties with ideas involving rate of change, and knowledge of representations for teaching ideas involving rate of change. As previously discussed, some teachers' understandings of rate of change were more

procedural than conceptual. However, no teachers identified computational procedures or formulas as the *only* "key" ideas for learning about ideas involving rate of change. A majority of teachers mentioned the ability to use and interpret multiple representations to recognize and/or represent rate of change as a learning goal. This result may be due to their experience using curriculum materials that value learning through different representations. Teachers reported the importance of a wide variety of other learning goals including understanding of functions (e.g., knowledge of dependent and independent variables and function families), ideas related to finding a rate of change in nonlinear functions (e.g., understanding average rate of change vs. instantaneous rate of change) and the concept of derivative (e.g., finding derivative functions).

The most common ideas teachers identified as key prerequisites needed for students to understand rate of change was experience with graphs (being able to make and interpret graphs) and knowledge of variables. Other concepts included experience with tables, knowledge of slope, and being able to solve equations. One teacher did not mention any particular prerequisite concepts.

When considering students' difficulties with understanding ideas involving rate of change, all teachers thought that basic ideas of rate of change and slope were not too difficult to learn, and most teachers indicated that introducing slope as a rate of change did not make it more difficult to understand.

In terms of teachers' knowledge of instructional representations for teaching ideas involving rate of change, teachers seemed to acknowledge the importance of concepts over procedures. However, results of the observations revealed LE teachers' propensity towards developing the procedural aspects of rate of change. In the case of teaching

change involving slope, it seemed that LE teachers were more likely affected by their concept image rather than their concept definitions. These teachers seemed to isolate the study of slope from the study of rate of change. There was little evidence that these teachers emphasized the connection between slope as a geometric concept and slope as rate of change. Thus, these teachers' students likely missed an opportunity to make this important connection while they were forming their conception of rate of change and slope.

Limitations

There are several limitations in this study, mainly due to the sample of teachers available for this study. The most prominent limitation may have been the inability to control for other factors (besides curriculum) which might contribute to teachers' understanding, such as educational background, professional development experiences, opportunities to learn from students, years of teaching, experience teaching Calculus, etc. The varying degrees of these pre-experiences may have given rise to prominent knowledge differences among teachers. This study did not attempt to even out the teachers' pre-experiences, but rather used them to categorize the characterized knowledge of teachers. An attempt was made to gain some initial understanding of curriculum impact, but no attempt was made to establish causal relationship between teachers' knowledge of rate of change and curriculum materials used.

This study was also limited by the sample size of teachers. Although this study employed some of the methods of quantitative research, larger sample sizes would have allowed for more robust statistical treatment of the data and possibly relationships could have been more firmly established between variables representing the various aspects of

teachers' knowledge of rate of change. However, much of this study was qualitative in nature, and whereas qualitative research typically focuses intently on only a few subjects, this study was conducted with a relatively large number of participants in this regard. Thus, a limited amount of time was spent with each teacher. A more complete picture of teachers knowledge might have been developed with further interviews.

This study employed classroom observation as a means to further contribute to the portrait of teachers' knowledge of rate of change, but because of both time constraints and limited resources, only four teachers were observed and they were chosen to heighten contrasts in experience level. A larger sample using teachers with a wide variety of levels of experience with the CPMP curriculum would have been useful and perhaps provided a more complete picture.

With regard to the interview tasks, some of the elements of the framework have been assessed more thoroughly than others in this study. For example, gaps in teachers' knowledge in using graphs to estimate an instantaneous rate of change were apparent, but their knowledge in using tables to estimate an instantaneous rate of change was not investigated. In addition, some elements of the framework deserve more careful attention than the time allowed in this study. For example, multiple questions are needed to fully assess teachers' ability to apply their understanding of rate of change to model and solve problems in which linear or nonlinear relationships exist.

The results of this study suggest that some elements of the framework should be elaborated upon more fully to be of more practical use. For example, one of the original elements in the framework was "Can the teachers use their concept of rate of change to recognize patterns of change between variables represented in tabular, graphic, symbolic,

and verbal form?" Based on the findings from this study, this element should be expanded to account for specific differences in the ways teachers use each representation. The suggested elaboration is shown in Figure 19.

Original element:

Can the teachers use their concept of rate of change to recognize patterns of change between variables represented in tabular, graphic, symbolic, and verbal form?

Elaborated element:

Using tables

- Can teachers use the values in a table to describe where a function is increasing /decreasing and use differences to describe where it is increasing /decreasing at faster, slower, and steady rate?
- Can teachers use tables of values, size and direction of differences to describe the amount and type of change in a function over an interval?
- Can teachers use average rate of change over intervals before and after a given point to estimate the instantaneous rate of change at that point?

Using graphs

- Can teachers use the shape and trend of a graph to describe where a function is increasing /decreasing and where it is increasing /decreasing at faster, slower, and constant rates?
- Can teachers use vertical rises and falls or visible slopes of secant lines to describe intervals where a function is increasing /decreasing and use differences to describe where it is increasing /decreasing at faster, slower, and constant rates?
- Can teachers use "eye-ball" slopes of tangent lines to a graph at a particular point to estimate the instantaneous rate of change at that point?

Using equations

- Can teachers use algebraic properties of equations to describe where a function is increasing /decreasing and where it is increasing /decreasing at faster, slower, and constant rates?
- Can teachers use f' to describe where a function is increasing /decreasing and where it is increasing /decreasing at faster, slower, and constant rates?
- Can teachers compute two f values to find the average rate of change over an interval?
- Can teachers use an equation to obtain more accurate estimates of instantaneous rate of change by computing average rate of change over intervals of decreasing width?
- Can teachers use f' to exactly compute the instantaneous rate of change at a particular point?

Figure 19. An elaborated element of the framework

Comparisons to Other Research

One finding from this study was that teachers who demonstrated a better understanding of rate of change overall were more flexible using multiple representations. Those teachers showed a consistent and strong inclination to use graphical representations to work on tasks, rather than numerical or symbolic representations. The literature (e.g., Huntley et al, 2000; NCTM, 2000, Porzio, 1997) clearly indicates that having multiple ways to represent mathematical concepts is beneficial as ways of learning with understanding. For example, graphical representations convey mathematical information visually, whereas expressions represented symbolically can often be easier to manipulate, analyze, or transform. Porzio (1997) investigated rate of change knowledge of students in different differential calculus courses and found that students in the course that emphasized use of multiple representations to present concepts and solve problems were more likely to form a strongly-connected understanding of the relationship between slope, rate of change and the first derivative. He claimed that the differences between the students from the different courses may be related to the amount of time students spent working on problems designed to help them make connections between different representations of the concept of rate of change. This may partially account for the differences among LE, ME and HE teachers in this study, since multiple representations are one of the features emphasized in the CPMP curriculum.

Although one representational scheme is not superior to another, generally, the ability to visualize in mathematics is thought to be advantageous and the role of visual thinking is fundamental to understanding calculus (Vinner, 1989). However, studies have consistently shown that students' understanding is typically algebraic and not visual (Lee,

1994; White & Mitchelmore, 1996) and teachers' graphical ideas are often shallow (Berenson & Carter, 1994). These observations were similar to the findings in this study. Overall, teachers in this study were less accomplished with tasks involving graphical representations. However, teachers who had relatively more experience with CPMP were much better at utilizing graphical representations (with some expressing a preference for using them).

Stump (1997) studied preservice and inservice secondary teachers' knowledge of slope and found that inservice teachers made more references to and had more connected understandings of different representations of slope. She did a similar study (2000) with high school precalculus students and found that students were more likely to view slope as a measure of steepness than as a measure of rate of change. In this study, teachers who had more experience teaching CPMP displayed a wider repertoire of representations for slope and demonstrated a well-developed understanding of slope. Some teachers who had less experience teaching CPMP demonstrated discomfort and reluctance in approaching slope as rate of change. This suggests that the kind of understanding of slope that Stump refers to may be developed as a result of using a curriculum, such as CPMP, that approaches slope conceptually.

There were two areas that proved to be the most difficult for teachers in this study. One was connecting the concepts of average rate of change and instantaneous rate of change. This difficulty was observed in a study Thompson (1994) conducted with advanced undergraduate and beginning graduate mathematics students working on a problem requiring understanding of the relationship between derivative and integral as expressed in the Fundamental Theorem of Calculus. His students struggled with the

concept of average rate of change as expressed by $(f(x + h) - f(x))/h$. When they computed it, they did not know how to interpret it. They also did not know what significance there was in computing average rate of change for decreasing values of h.

Another difficulty for teachers was understanding the calculus derivative functions. Generally, teachers had difficulty making sense of symbolic rules and graphs of derivatives and connecting the rules and graphs with estimated rate of change at particular points. Aspinwall, Shaw and Presmeg (1997) found a similar finding in their single case study with one college calculus student. They developed non-routine problems to have the student determine the graphs of derivatives given only graphs of functions and also routine problems to determine the student's procedural knowledge about derivative formulas. The results showed that although the student was able to calculate derivatives in various situations utilizing the product, quotient and chain rules, he showed difficulty in discriminating graphically between a function and its derivative and making connections between the symbolic and graphical representation of the derivative of $f(x) = x^2$.

Such difficulties cited in the above two studies were also exhibited by teachers in this study. In particular, those difficulties showed that building deep understanding requires making solid connections among related mathematical ideas (even after taking calculus). Those difficulties may, in part, be due to how most textbooks and teachers, as well as teachers of teachers, often present the concepts to students. For example, in many traditional curricula, students are often asked to compute or manipulate symbols before they have a chance to make sense of why such mechanics are valuable. They may spend

considerable time doing calculations and have few opportunities to actually think conceptually and contextually about what they are doing.

Some common errors/struggles were observed in this present study: a) position/velocity (speed) confusion, b) sign of velocity and c) interpreting graph as a picture. These errors have been noted in other studies. Patrick and Alba Thompson (1994, 1996) conducted interviews with one middle school student around situations involving constant rate and gave a detailed account of the student as she came to understand constant rate of change. Initially she thought about speed as distance or how far an object moves in one unit of time. Over several sessions, her understanding of rate was extended as she came to see that distance and time changed together so that the ratio of changes in distance to corresponding changes in time were all the same constant as was the ratio of the total accumulated distance to the total accumulated time after an increment of each. Also, several studies conducted by TERC found that students were often confused by position/velocity graphs (Monk & Nemirovsky, 1994; Rubin & Nemirovsky, 1991). A main result of these studies was that precalculus students often assumed that a position graph should resemble the velocity graph. If the velocity graph increased, these students believed that the position graph did as well. When a velocity graph changes from positive to negative, the position function changes from increasing to decreasing. For these students, any change in the velocity graph meant the same kind of change in the position graph. In an interview with one of these students, Nemirovsky (1994) reported her discovery of errors in interpreting the sign of the velocity and coming to a new understanding of what that sign means. That students' alternative conceptions (in the above studies) were similar to those of the teachers in this present study may further

support a widely accepted research finding that teachers' content knowledge does influence what students learn. This may indicate that both teachers and curriculum materials they use need to find ways to address such points of confusion.

Implications for Use and Further Study

Rate of change is an important concept both in secondary level mathematics and everyday life. This study provided a characterization of secondary mathematics teachers' knowledge of rate of change and identified some strengths and weaknesses of their understanding with an intent to explore the impact of curriculum materials that contribute to their characterized knowledge.

Teachers in this study all demonstrated a good understanding of rate of change in linear relationships. This understanding occurred in various representations. Further research could investigate the factors that contributed to teachers' understanding and flexibility in viewing linear relationships exhibiting constant rate of change. Teachers in this study showed flexibility in describing rates of change using multiple representations. They were the least flexible with graphical representations. Further studies could investigate the factors that inhibit growth in graphical reasoning.

Teachers who had more experience the CPMP curriculum demonstrated a strength in dealing with situations involving non-constant rates of change. This finding could be attributed to the fact that these teachers also had more teaching experience in general. Therefore, further research could explore what characterizes understanding across a broader range of teacher backgrounds, especially those who have a substantial amount of teaching experience with more traditional curriculum materials.

The concept of derivative proved to be difficult to understand, even for teachers having a great deal of experience with the CPMP materials, although such teachers demonstrated a more solid understanding than their less experienced peers. Research is needed to explore what might make the concept of derivative difficult to understand over a much broader range of experience levels. Studies of this nature could include: which calculus courses teachers have taken and how these courses impacted their understanding of derivative as well as other concepts relative to rate of change.

The evidence that many teachers lack conceptual understanding of mathematics suggests that teachers may complete their formal teacher preparation programs with deficiency in their knowledge base. According to Anderson (1989), many students are not aware of a) the structure of knowledge in mathematics, b) the function or purpose of knowledge in mathematics, and c) the development of knowledge in mathematics. Instead, they are only learning declarative knowledge ("knowing that" vs. "knowing how" or "knowing why"), and they are unable to use that knowledge productively. The results from this study show that LE teachers in particular (who were recent graduates from college) did not demonstrate a deep understanding of rate of change and had a propensity toward developing the procedural aspects of rate of change with their students. Thus, in order to promote the kind of mathematical understanding necessary for teaching, teacher preparation programs may need to: a) approach content differently in order to help preservice teachers develop a conceptual understanding of the content they will be expected to teach and the ability to apply knowledge in new situations, and b) provide preservice teachers with opportunities to interact with Standards-based curriculum materials where topics are developed more conceptually.

In terms of the methodology used in this study, the use of classroom observation to capture teachers' knowledge was an important feature and can contribute to the way mathematics educators conceptualize research on teacher knowledge. The observations often provided enhanced information that went beyond what was garnered from the mathematical task-based interviews. More research is needed to understand what might help or hamper teachers' ability to use their knowledge in teaching practice and how teachers' practice could be used as a site for assessing their knowledge. Teaching (particularly in the context of using Standards-based materials) creates conditions where teachers have to make explicit their understanding of mathematical ideas and procedures, be able to understand and appraise the way their students think about such procedures, and make decisions about whether an idea is mathematically significant and worth taking up. While using interviews are useful to probe teachers' knowledge, watching teachers' practice will help to probe the mathematical nature and demands of teaching. This approach also adds insight into understanding how teachers' knowledge needs to be held as well as what teachers need to know for teaching.

In relation to students' learning, further studies could investigate: What characterizes students' knowledge of rate of change in the context of using a Standards-based curriculum? and, how is teachers' knowledge of rate of change related to their students' knowledge of rate of change?

Summary

In order to teach for understanding, teachers must have extensive and deep knowledge of important mathematical concepts. This study has examined the knowledge of secondary school teachers in relation to a fundamental mathematical concept, rate of

change in the context of teaching a Standards-based high school curriculum, the CPMP curriculum. The results of this study confirm the complexity of the concept of rate of change and how it continues to be a challenge for teachers to understand.

Studies that focus on the impact of Standards-based curriculum in developing particular mathematical ideas are only beginning to emerge (Huntley et al, 2000; Lloyd & Wilson, 1998). This study does not claim that teaching experience with Standards-based curriculum resulted in the differences in teacher knowledge of rate of change shown in this study. Rather, this study hoped to detect the role that curriculum materials might play in offering learning opportunities beyond those of typical teacher education experiences and the findings of this study suggest that Standards-based curriculum materials may support teachers as they learn ideas involving rate of change as they teach them.

This study is a step toward understanding and categorizing teachers' knowledge of rate of change and provides one glance at the impact of Standards-based curriculum materials on teacher learning. More studies will need to be done to investigate factors that aid or impede the development of an understanding of the concept of rate of change with methods other than those used in this study. Such studies will give mathematics educators a better picture of how the teaching and learning of rate of change can be improved.

APPENDICES

APPENDIX A

Survey

Survey

School: ____________________________ Name: _________________________

Age: __20-24 __25-29 __30-34 __35-39 __40-44 __45-49 __50-54 __55-60 __60-

Sex: Male ____ Female ____

** For items not applicable to you, write N/A.

Background Information
Part A: Teacher Preparation Program

1. What degree(s) do you have? What major? Minor? From where? When?
   Check and/or describe in each category.

	Bachelor		Master	Doctorate
Mathematics	Major _____	Minor _____		
Math Education	Major _____	Minor _____		
Other (Then, describe)	Major _____ ( )	Minor _____ ( )		
Where				
When				

2. What mathematics content courses have you taken?

3. Briefly describe the mathematics methods courses you have taken.

4. Before you started your teaching job, had you been exposed to the CPMP curriculum?
   ______ Yes ______ No
   If yes, describe in what way you became exposed to CPMP materials (e.g., methods course(s), internship, conference/workshop, etc.)

5. If you have been exposed to a reform curriculum other than CPMP materials, then check all that apply.
   ____ ARISE ____ IMP ____ MATH Connections _____ SIMMS

Describe in what way you became exposed to the materials you checked (e.g., methods course(s), internship, conference/workshop, etc.) and level of exposure.

6. Of the education programs you completed at the university, which do you think have or will have (the most) impact on your teaching (e.g., math courses, methods courses, internship, etc) and in what ways (e.g., understanding of mathematical content, pedagogical approach, etc)?

Part B: Teaching Experience

1. How many years have you taught mathematics?

2. (Non-CPMP use) If you have used a non-CPMP textbook, indicate the number of years you taught each course and when you last taught them.

	Years taught	Textbook used*	Last taught
First-year Algebra	________	________	________
Second-year Algebra	________	________	________
Geometry	________	________	________
A course involving Trigonometry	________	________	________
Calculus	________	________	________
Statistics	________	________	________
Other ( )	________	________	________

* If you do not remember the name of textbook you used, please indicate whether the materials were traditional or reform programs.

3. (CPMP use) If you have used CPMP textbooks, indicate the number of years you teach each course the courses you are currently teaching.

	How many years taught	Course(s) you are currently teaching
Course 1	________	________
Course 2	________	________
Course 3	________	________
Course 4	________	________

4. Have you participated in any workshops, inservices, or other forms of professional development programs in general and/or specifically related to CPMP curriculum materials?
   ______ Yes   ______ No
   If yes, briefly describe your previous experience (what it(they) was(were), why you went, overall appraisal of the experience, etc.).

   In what ways has your professional development experience had an impact on your teaching?

Part C: School Climate

1. Check all of the following that apply to your school:
   _____ Urban   _____ Suburban   _____ Rural
   _____ Public   _____ Public Charter   _____ Independent (non-public)

2. Approximate number of students in school: _________
   Total students enrolled in classes using CPMP textbooks: __________

3. Number of teachers now teaching CPMP textbooks: __________
   Total number of math teachers in your school: __________

4. Date (year) of first use: _________

5. Have you made an official adoption? ______ Yes   ______ No

6. CPMP textbooks now being used:
   ___ 1A ___ 1B ___ 2A ___ 2B ___ 3A ___ 3B ___ 4A ___ 4B

7. CPMP classes are usually
   _____ homogeneously grouped by ability
   _____ heterogeneously grouped
   _____ Other. Please describe: ____

8. CPMP curriculum is used with
   _____ ONLY accelerated students
   _____ all students EXCEPT accelerated students
   _____ all students
   _____ those students who choose CPMP textbooks rather than a parallel track
   _____ all EXCEPT special needs students
   _____ only students not successful in earlier math classes
   _____ Other. Please describe:____

9. What type of preparation did most teachers receive to teach the CPMP curriculum? Please check all that apply.
_____ Five or more full-day sessions
_____ Two to four full-day sessions
_____ Less than two days of sessions
_____ No professional development was given
_____ All teachers teaching CPMP had the same preparation
_____ Some teachers had more preparation, and help the others
(These teachers are given time for helping _____ Yes _____ No)
_____ All professional development was completed before teaching CPMP curriculum
_____ Professional development was spread out while teaching CPMP curriculum
_____ Other. (Please describe: ______________________________________)

Conceptions of Teaching and Learning of Rate of Change

1. What are two or three of the most important ideas you want students to learn about rate of change?

2. What prerequisite concepts do students need to understand in order to understand the idea of rate of change?

(3 – 16) For each of the following statements, please circle the response that most accurately represents your feelings. SD = strongly disagree (1), D = disagree (2), N = neutral (3), A = agree (4), SA = strongly agree (5).

	SD	D	N	A	SA
3. It is important for teachers to be aware of how students learn mathematics.	1	2	3	4	5
4. It is important for teachers to be aware of how students learn rate of change.	1	2	3	4	5
5. Rate of change is an important mathematics topic.	1	2	3	4	5

6. Basic ideas of slope are too difficult to learn.	1	2	3	4	5
7. Basic ideas of rate of change are too difficult to learn.	1	2	3	4	5
8. Learning rate of change in contexts is important.	1	2	3	4	5
9. Learning rate of change through different representations is important.	1	2	3	4	5
10. When working mathematics problems, it is important that what you are doing makes sense to you.	1	2	3	4	5
11. I am confident of my ability to teach concepts of rate of change.	1	2	3	4	5
12. Students need to demonstrate procedural understanding of slope before doing any investigation.	1	2	3	4	5
13. The best ways to teach how to solve complex problems is to decompose them into a sequence of basic skills and then teach for mastery of the skills one at a time.	1	2	3	4	5
14. Attempting to introduce slope as a rate of change makes it more difficult to understand.	1	2	3	4	5
15. Students should learn mathematics by being actively engaged in solving problems.	1	2	3	4	5
16. Working cooperatively (in groups, pairs, etc.) is an effective way for students to learn mathematics.	1	2	3	4	5

Conceptions of Textbooks and the CPMP Curriculum with respect to Rate of Change

(1 – 3) First, think about what makes a good mathematics textbook *in general*. Please write your thought.

1. What is the purpose of textbooks (or other curriculum materials)?

2. What makes a "good" textbook *for students* who are learning mathematics?

3. What makes a "good" textbook *for teachers* of mathematics?

** **Please** do not go back to items 1-3 at all when completing the rest of this section.

(4 – 15) Now, think about *CPMP* textbook with respect to slope and rate of change. For each of the following statements, please circle the response that most accurately represents your thoughts. SD = strongly disagree (1), D = disagree (2), N = neutral (3), A = agree (4), SA = strongly agree (5). Take "you" to mean yourself as a math teacher.

	SD	D	N	A	SA
4. CPMP does a good job of helping students learn the concept of rate of change	1	2	3	4	5
5. CPMP explanations of slope as a rate of change are easy for *students* to follow.	1	2	3	4	5
6. Attempting to introduce slope as a rate of change makes it more difficult to understand.	1	2	3	4	5
7. CPMP explanations of slope as a rate of change are easy for *you* to follow.	1	2	3	4	5
8. CPMP investigations challenge students to think more deeply about ideas of slope and rate of change.	1	2	3	4	5

9. CPMP investigations challenge *you* to think more deeply about ideas of slope and rate of change.	1	2	3	4	5
10. CPMP activities focus on student thinking.	1	2	3	4	5
11. CPMP questions in activities focus on student thinking.	1	2	3	4	5
12. CPMP activities and questions help you think differently about *learning* mathematics.	1	2	3	4	5
13. CPMP activities and questions help you think differently about *teaching* mathematics.	1	2	3	4	5
14. CPMP does a good job of helping students attempt to make sense of mathematical rules and relationships.	1	2	3	4	5
15. I believe students better retain what they have learned through problem-based material such as CPMP.	1	2	3	4	5

* Teachers sometimes say that curriculum materials have a significant effect on teacher learning and teaching. Has that been true for you?

16. Is there anything you've noticed about your teaching that has changed since you've started using the CPMP materials?

17. Have you noticed that presentation of content in CPMP make you think through the mathematics again for yourself in new ways compared to the past? For example, seeing new aspects of slope, expanding your own understanding of rate of change, etc.

APPENDIX B

Mathematical Task-Based Interview

Mathematical Task-Based Interview

Teacher: ______________________ Date: ______________________

1. In the context of algebra and functions, what does "rate of change" mean to you? How would you illustrate this idea?

2. How would you define rate of change? Can you give one or more examples to illustrate?

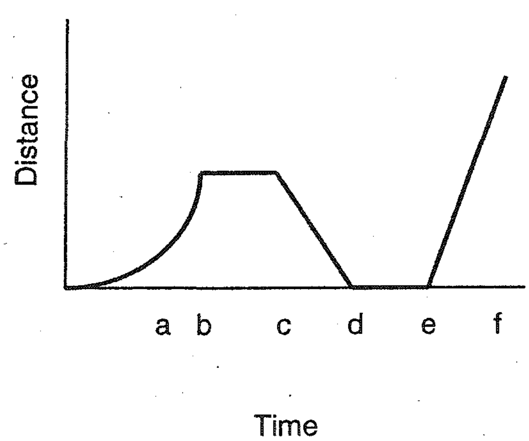
3. Movies-To-Go has a membership plan for its customers. It costs \$20 per year to become a member, and members pay \$1.50 to rent any movie video. What is the role of the numbers 20 and 1.50 in a table, a graph and an equation relating annual cost C and the number of videos rented V?

	Table	Graph	C-V Equation
20			
1.50			

4. The junior class at Centennial High School planned to make a class sweatshirt with every student's name on it. They tested several possible selling prices and got the following data relating price P and probable sales S. What would the predicted number of shirts sold be when the price per shirt is set at $5?

Price per Shirt P in dollars	10	20	30	40	50
Number of Shirts Sold S	200	150	90	50	5

5. Give a story for the following (time, distance) graph. Then sketch a (time, velocity) graph corresponding to the given graph.



6. Suppose that a laboratory experiment uses fruit flies that double in number every five days. If the initial population contains 100 flies, the number at any time t days into the experiment will be modeled by the function with the rule $P(t) = 100(2^{0.2t})$.

   Use the function rule above to answer these questions as accurately as possible.

   (a) What is the average growth rate of the population (flies per day) from day 0 to day 20?

   (b) What are the estimated rates at which the fly population will be growing on day 10 and on day 20?

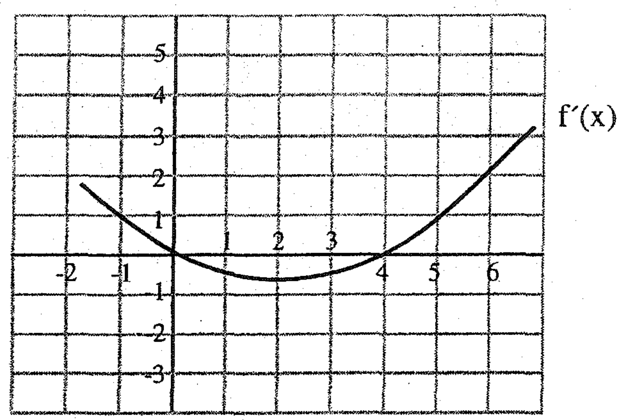
   (c) How are the growth rates calculated in parts (a) and (b) shown in the shape of a (t, P(t)) graph?

7. Suppose f is a quadratic function of x with rule in the form $f(x) = x^2$.

   (a) How is the rate of change of f shown in tables, graphs, and rules of the function?

   (b) What rule gives the rate of change of f at any point x?

8. What can you tell about the behavior of the original function f(x) from the behavior of its derivative function f′(x) illustrated below? Be as complete as possible in your response.



APPENDIX C

Brief Solutions of Mathematical Task-Based Interview Questions 3 – 8

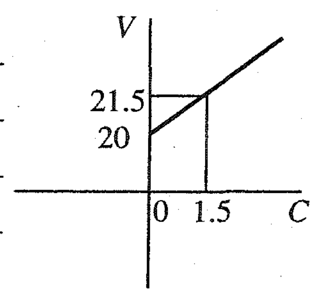
Brief Solutions of Mathematical Task-Based Interview Questions 3 – 8

3. Movies-To-Go has a membership plan for its customers. It costs $20 per year to become a member, and members pay $1.50 to rent any movie video. What is the role of the numbers 20 and 1.50 in a table, a graph and an equation relating annual cost C and the number of videos rented V?

Table

# of videos (V)	Cost (C)
0	20
1	21.5
2	23
3	24.5

Graph



C-V Equation

$V = 20 + 1.5C$

Rate of change in cost C is 1. 5 (slope) and initial value for cost C is 20 (y-intercept).

4. The junior class at Centennial High School planned to make a class sweatshirt with every student's name on it. They tested several possible selling prices and got the following data relating price P and probable sales S. What would the predicted number of shirts sold be when the price per shirt is set at $5?

Price per Shirt P in dollars	10	20	30	40	50
Number of Shirts Sold S	200	150	90	50	5

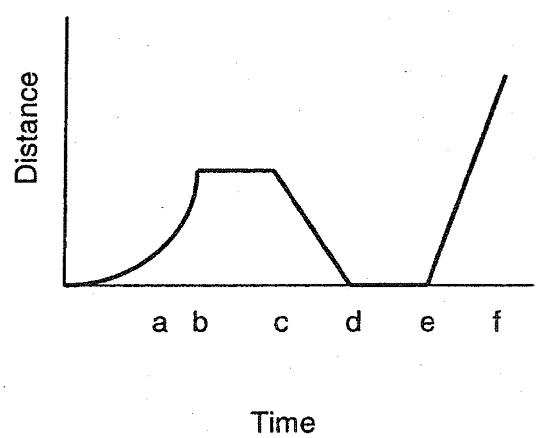
A possible method 1) Find the difference in number of shirts and an average of all of the differences.
A possible method 2) Use a graphics calculator to find a regression model.

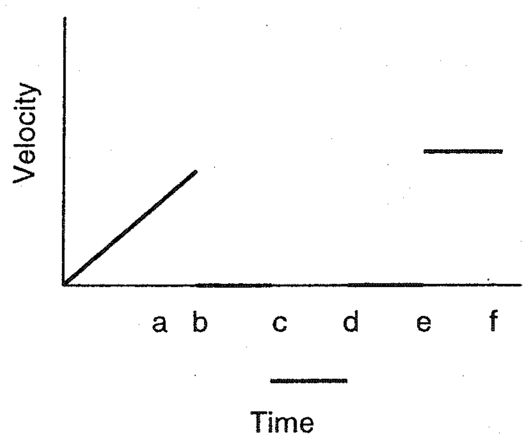
5. Give a story for the following (time, distance) graph. Then sketch a (time, velocity) graph corresponding to the given graph.

A possible story for the (time, distance) graph.

I'm going to start my run from my house and began by going through a sprint. So my distance increases quickly because I'm running fast, I'm accelerating at point a. [For a – b] As time goes by then my distance is increasing faster and faster and faster so, then I get tired so I stop. [For b – c] I gave myself time to rest. [For c – d] I turn around and I head back towards my house. This time just doing a steady jog at a constant rate. [For d – e] Then I stop and want to rest. [For e – f] And I start off again at a constant speed.

A (time, velocity) graph





6. Suppose that a laboratory experiment uses fruit flies that double in number every five days. If the initial population contains 100 flies, the number at any time t days into the experiment will be modeled by the function with the rule $P(t) = 100(2^{0.2t})$.

Use the function rule above to answer these questions as accurately possible.

(a) What is the average growth rate of the population (flies per day) from day 0 to day 20?

$$\frac{100(2^{0.2(20)}) - 100(2^{0.2(0)})}{20} = 75 \text{ flies/day}$$

(b) What are the estimated rates at which the fly population will be growing on day 10 and on day 20?

Responses may vary, depending on the estimation strategy. However, estimates for the growth rate should be close to the following.

Day 10: $$\frac{100(2^{0.2(10.1)}) - 100(2^{0.2(9.9)})}{10.1 - 9.9} \approx 55 \text{ flies/day}$$

Day 20: $$\frac{100(2^{0.2(20.1)}) - 100(2^{0.2(19.9)})}{20.1 - 19.9} \approx 222 \text{ flies/day}$$

(c) How are the growth rates calculated in parts (a) and (b) shown in the shape of a (t, P(t)) graph?

The growth rate in part (a) is the slope of the secant line over the interval [0, 20] For the growth rates in part (b), for example, the growth rate at day 10 is the slope of secant line over the interval defined by two points very close to (10, f(10)); the slope of the tangent line at t = 10.

7. Suppose f is a quadratic function of x with rule in the form $f(x) = x^2$.

(a) How is the rate of change of f shown in tables, graphs, and rules of the function?

In tables: Should make reference to the difference in the function values; the ratio of $(f(x_1) - f(x_0)) / (x_1 - x_0)$

In graphs: Should make reference to ideas that if the graph is increasing, then the rate of change is positive and if the graph is decreasing, then the rate of change is negative; the steeper the graph is, the greater (in absolute value) the rate of change; slope of secant line over the interval defined by two points; slope of tangent line at a given point.

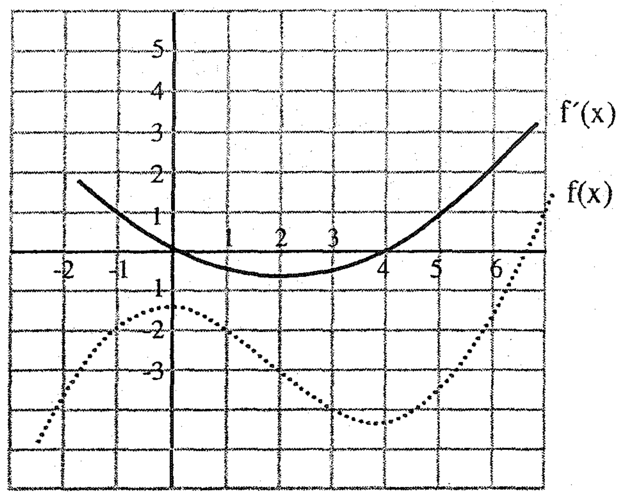
Using the function rule: Should make reference to calculating average rate of change $(f(x_1) - f(x_0)) / (x_1 - x_0)$; instantaneous rate of change (its derivative) $2x$

(b) What rule gives the rate of change of f at any point x?

Responses may vary, depending on choices of points. However, an answer should be something like:

$$\frac{f(x + 0.1) - f(x - 0.1)}{0.2} \quad \text{or} \quad 2x$$

8. What can you tell about the behavior of the original function f(x) from the behavior of its derivative function f′(x) illustrated below? Be as complete as possible in your response.



The original function f starts going up at positively decreasing rates up to the peak. It begins going down at increasing rates (rates are negative) over [1, 2]. The function is still going down but at decreasing rates (rates are negative) over [2, 4]. An inflection point occurs at
$x = 2$. After $x = 4$, it starts going up at positively increasing rates.

In the case that f′(x) is inferred as a quadratic function, a possible equation of f′(x) is a form of $f'(x) = ax(x-4)$ and its antiderivative is $f(x) = a/3\ x^3 - 2ax^2 + C$ (a and C are constants). For example, when approximating the location of the vertex point of f′(x) as (2, -2/3), the symbolic rule of f′(x) is $1/6\ x^2 - 2/3\ x$, and the symbolic rule of f(x) is $1/18\ x^3 - 1/3\ x^2 + C$.

APPENDIX D

Classroom Observation Protocol

Classroom Observation Protocol

Teacher: ______________________ CPMP Course: #_____ N= _____

Date: _____________ Scheduled time for class: From ______ to ______

Time instruction actually began: ______ Explain if different from scheduled time.

Time instruction actually ended: ______ Explain if different from scheduled time.

Unit: ______ Lesson: _______ Part of Lesson: ______________________

Sketch of Classroom:

Resources used:

Technological tools Curriculum materials other than CPMP text

Manipulatives Supplemental paper-and-pencil materials

Teacher-made materials Other:

Summary of Lesson:

Repertoire of Aspects of Rate of Change

Algebraic Geometric Physical Trigonometric Function

Characteristic

Representations and Links among them

Representations

Focus on one More than one All possible

Tabular

Graphic

Symbolic

Verbal

Links

None Partly Entirely

Misunderstandings

Calculational Conceptual

Initially showed, then corrected

Remained incorrect until end of the lesson

Connections

Contexts or statements used to make connection between different rates of change

Examples or statements used to make connection with different mathematical or real world topic

Generalization

Generalization beyond particular examples or categorization or recognition of a pattern

Procedural

Conceptual

Limited to one particular situation

Important Mathematical ideas about rate of change that were pursued

Important Mathematical ideas about rate of change that were not pursued

APPENDIX E

Pre-Observation Interview Protocol

Pre-Observation Interview Protocol

1. What is the mathematical goal of the lesson?

   What is the most important thing you want your students to learn from this upcoming lesson?

   Are there other major things you want your students to learn?

2. What do you think students need to know to learn the idea of rate of change within this lesson?

3. What aspects of the mathematics do you think the students will struggle with the most?

   How do you anticipate handling these?

4. Was this lesson challenging for you to prepare for teaching?
   (For experienced teachers, ask to recall how it was first time they taught.)

   If so, in what ways?

5. How did you use CPMP Teacher's guide as you planned to teach this lesson?
   If you don't use it now, how did you do first time you taught the lesson?

   What was most challenging?
   What information was most useful? Least useful?
   (For instance, structure of lesson, format of activities, math content covered, explanations of math content, question asked, assessment, etc.)

6. Did you make any alterations on lesson plan from the text?

   If so, what is the purpose of the alterations you made?

APPENDIX F

Post-Observation Interview Protocol

Post-Observation Interview Protocol

1. How do you think the lesson went and to what extent was the main goal achieved?

   What did this lesson tell you about what your students are learning and still need to learn regarding the ideas in the lesson?

2. Were any alterations made during the lesson from the ones you had planned?

   If any, why did you have to make changes? Did you like the way the lesson went in class or do you wish you could have stayed on the original plan? Why or Why not?

3. What student mathematical utterances were you surprised by or did not expect? Why?

(In the rest of questions, some may be discussed according to what came to the researcher's consideration during observation.)

4. What other representations would you like to see?
   How could the representations you mentioned have been incorporated with already-discussed ones in class to make connections between them?

5. What connections could have been made among the ideas discussed in class?, with different rates of change?, with different mathematical ideas?, or with real-world situations?

6. How could you generalize these ideas?

7. Why did you decide to follow-up the students' mathematical ideas?
   In what ways did the follow-up clarify your rationale?

8. Why did you not follow up the students' ideas?
   If you wish you could, then how would you do?

9. Can you recognize there were times where you made miscalculations and/or showed misunderstandings? Why did you think so?

APPENDIX G

Human Subjects Institutional Review Board Approval

WESTERN MICHIGAN UNIVERSITY

Human Subjects Institutional Review Board

Centennial
1903-2003 Celebration

Date: August 11, 2003

To: Kate Kline, Principal Investigator
Steven Ziebarth, Co-Principal Investigator
Jihwa Noh, Student Investigator for dissertation

From: Mary Lagerwey, Chair Mary Lagerwey

Re: HSIRB Project Number: 03-07-04

This letter will serve as confirmation that your research project entitled "An Investigation of Secondary School Mathematics Teachers' Knowledge of Rate of Change in the Context of Teaching a Standards-Based Curriculum" has been **approved** under the **expedited** category of review by the Human Subjects Institutional Review Board. The conditions and duration of this approval are specified in the Policies of Western Michigan University. You may now begin to implement the research as described in the application.

Please note that you may **only** conduct this research exactly in the form it was approved. You must seek specific board approval for any changes in this project. You must also seek reapproval if the project extends beyond the termination date noted below. In addition if there are any unanticipated adverse reactions or unanticipated events associated with the conduct of this research, you should immediately suspend the project and contact the Chair of the HSIRB for consultation.

The Board wishes you success in the pursuit of your research goals.

Approval Termination: August 11, 2004

Walwood Hall, Kalamazoo, MI 49008-5456
PHONE: (269) 387-8293 FAX: (269) 387-8276

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