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Learning to Construct Proofs in a First Course on Mathematical Proof

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LEARNING TO CONSTRUCT PROOFS IN A FIRST COURSE ON MATHEMATICAL PROOF

by

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ON MATHEMATICAL PROOF

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This study examined the conceptions of proof that undergraduate students have upon entry to a transition course on mathematical proof, how they develop skill in planning and reporting proofs, obstacles encountered, and effects of instruction on their performance in solidifying schema in proof-planning and proof-reporting.

The subjects were sophomores and juniors (n=16) in a transition course at a large midwestern university. The course was taught by one of the co-authors of the text, "Mathematical Proofs" (Chartrand, Polimeni, and Zhang, 1999, in press). Assessment of learning to construct proofs was through quizzes and a final exam developed by the professor with input from the researcher. These written assessments were augmented by case studies of six students.

A pretest and initial interviews provided baseline measures of the students' understandings. Subsequent assessments revealed how each student constructed direct proofs, proofs by contrapositive, proofs by contradiction, and proofs by mathematical induction. Half the students demonstrated that they understood the statement to be proved, and recognized definitions of the terms involved. Ten of the

16 students also showed correctly that negative results could be established by a counterexample.

The study confirmed obstacles previously identified in the literature: starting direct proofs and proofs by contrapositive, using definitions, and using universal and existential quantifiers. In addition, other obstacles were prominent: choosing mathematical notation and representations, forming induction assumptions for proofs by complete induction, and constructing proofs by contradiction.

Students' proof-constructions demonstrated habits that appropriated the presentations in the textbook and classroom. They gave clear statements of the starting assumptions, the proof strategy, and the framework of proofs by mathematical induction. The statements of starting assumptions for proofs by contradiction and the induction assumption for complete induction, however, were not successfully emulated. The study included a formulation of schema for constructing proofs that distinguished between proof concepts and mathematical concepts.

The study concluded by noting limitations of the research, suggesting avenues for further related research, and making recommendations for practice.

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CHAPTER ONE

BACKGROUND OF THE PROBLEM

Mathematics is distinguished from the other quantitative sciences by its dependence upon deductive reasoning. Observations and experiments have their place in discovering patterns of shape and number, but proofs by deductive reasoning are the accepted means by which results are publicly established in the mathematics community (Baylis, 1983; Steen, 1989). The synergy of geometric and algebraic thinking that occurs throughout undergraduate mathematics is connected in the context of mathematical reasoning. From applications of the Pythagorean Theorem to the investigations of general topology and graph theory, reasoning and proofs show how mathematical concepts work together. Higher-level undergraduate mathematics employs results that are frequently the consequences of complex chains of reasoning that upper-level students must be able to follow (Kleiner, 1991; Selden and Selden, 1987a, 1994; Van Dormolen, 1977). Whereas lower-level courses, such as calculus, linear algebra, and differential equations, are algorithmic and process-driven, upper level courses are more abstract and proof-intensive.

Proofs that are written in textbooks for undergraduate mathematics students serve to explain why a mathematical result is true and to convince them that the result is a valid consequence of mathematical reasoning (Hersh, 1993; Davis, 1985). For such students, understanding why a proof is necessary in the first place and

understanding how the proof involves their previous knowledge of mathematics are essential building blocks in their appreciation of the role of proofs. Knowing the kinds of assumptions that are needed to start a proof provides an intellectual foothold, and the confidence of knowing a finite set of initial assumptions gives undergraduate mathematics students a place from which to begin constructing a proof. Even before a proof is completed, they should understand that choices have been made about the solution path, and that there may be other successful ways to construct a proof. Doing proofs includes coordinating other activities such as building understandings of connections between mathematical constructs, recalling definitions, using logical inferences, and judging the validity of proofs (de Villiers, 1999; Selden and Selden, 1987a; Harel, 1987, 1997; Hart, 1992). Appreciating these features of proofs increases the awareness of and facility with the connections between and among mathematical concepts.

Unfortunately, undergraduate mathematics students hit a wall when they encounter proofs in upper-level mathematics courses. They commonly have difficulty distinguishing between an inductive argument from examples and a deductive argument from abstract principles (Martin and Harel, 1989). An inductive argument only verifies a result for the examples that are treated, but is not a proof in general. Undergraduate mathematics students have difficulty understanding how to apply a proof to new circumstances to which it applies (Selden and Selden, 1987). Poincaré observed that, "There is nothing mysterious in the fact that every one is not capable of discovery. That every one should not able to retain a demonstration he has once learnt

is still comprehensible. But what does seem most surprising, when we consider it, is that any one should be unable to understand a mathematical argument at the very moment it is stated to him" (quoted in Hart 1994, 49). Sometimes the difficulties are simply remembering the definitions of recently-introduced mathematical objects; sometimes the symbols used are confusing or unusual to the student (Moore, 1991; 1995; Harel, 1997, 1999); and sometimes students are defeated by having the freedom to choose a representation for the mathematical objects. A frequent complaint of mathematics students is that they do not know how to start a proof (Moore, 1990, 1991, 1994; Hart, 1994; Selden and Selden, 1995). The construction of a valid proof in mathematics, therefore, is seen to require a variety of skills and concepts, successfully coordinated. Instead of blaming students for their difficulties, however, we should consider the context in which they are asked to produce proofs.

The Curriculum Gap

There is a great discontinuity in the undergraduate mathematics curriculum from the algorithmic and example-driven lower level courses to the abstract and proof-intensive upper-level courses (Dubinsky, Elterman, & Gong, 1988; Robert and Schwarzenberger, 1991). The lower-level courses referred to here are principally the studies of single- and multiple-variable calculus, differential equations, and sophomore-level linear algebra. The upper-level courses are primarily advanced calculus and abstract algebra; however, topology, complex analysis, graph theory, and

all other upper-level mathematics courses use proofs as their means of exposition. Although to many people, mathematics means calculating with numbers or solving equations, to the mathematician who is engaged in research, writing, or teaching, it is constructing and validating proofs that characterize mathematical activity.

This curriculum gap between lower-level and upper-level undergraduate mathematics was created in two stages. The first stage was a tendency in the twentieth century to move abstract mathematics from graduate to undergraduate studies. In analysis and algebra, the development of abstract notions of limits, function spaces, groups, rings, and fields came in the late 1900s with the work of Weierstrass, Peano, Hilbert, and E. Noether. This way of doing advanced calculus and modern algebra from an abstract viewpoint filtered rapidly from mathematics research (1920s) to graduate courses (1930s) to undergraduate courses (1950s) (Kleiner, 1991; Bell, 1992).

The second stage consisted of two contradictory movements. The tendency to introduce abstractions at earlier places in the curriculum continued in the 1970s with the introduction of sophomore-level linear algebra courses. The purposes of such courses were to introduce the language and concepts of vectors and matrices at an earlier point in students' mathematical experience and to introduce proofs in the axiomatic setting of vector spaces. At the same time, and in the opposite direction, the place of proofs in calculus began to wane throughout the 1970s and 1980s, a trend which accelerated in the 1990s with the reform calculus movement. Proofs using the epsilon-delta limits require algebraic manipulations with inequalities as well as an

understanding of the conditional statement being proved, but this kind of proof has been either de-emphasized or eliminated altogether in current calculus textbooks. Topics such as the Mean Value Theorem are frequently explained by a diagram but left unproved.

Another change in the undergraduate curriculum since the 1970s has been the compression of what was four semesters of calculus, multivariable calculus, and differential equations into three semesters. This had the effect of introducing sophomore-level linear algebra into the four semesters of core mathematics for the mathematics major. The reason for this compression is that the number of required courses for a major is bounded by a complex balance of schedules, finances, and course offerings at most colleges and universities.

The Bridge

Several curricular approaches have attempted to address the students' difficulties with this curriculum discontinuity. Precalculus textbooks with emphases on sets, abstract functions, and proofs (Kreshner and Wilcox, 1950; Oliphant, et al., 1965; Zwier and Nyhoff, 1969) included proofs with abstract sets and functions and detailed discussions of epsilon-delta proofs of limits as a prelude to calculus. For the students who arrived at college with the proper prerequisites to take calculus, however, these books were regarded just as reference books or as supplementary reading. Both students and professors disregarded reading outside the new, large calculus textbooks.

These books, however, had the unintended effect of isolating the study of proofs from the rest of the mathematics curriculum. The difficulties that students had in transferring the notations, concepts, and proof-strategies to courses in calculus and linear algebra persisted to their upper-level courses. A continuous strand emphasizing inferences and proof throughout the undergraduate mathematics curriculum would reinforce the ideas begun in these precalculus books.

Secondly, some textbooks for upper-level courses in abstract algebra (Gilbert and Gilbert, 1999) and advanced calculus (Dangello and Seyfried, 2000) included sections on the mechanics and logic of proofs in either an appendix or an introductory chapter. Like the first approach, this had the unfortunate consequence of making the material of proofs appear to be just one topic in one course, instead of integrating the necessary language and logic of proofs in and through all courses of the undergraduate mathematics curriculum.

Thirdly, bridging the curricular gap is the focus of the approach on reasoning across the mathematics curriculum as a theme for the undergraduate mathematics program. The principle feature is to integrate logic concepts and vocabulary early and to review them often. Since textbook authors do not write complete undergraduate curricula in the way high school textbooks are often developed, this has not been done. Individual textbooks include paeans to the beauty or strength of proofs in establishing mathematical results, but the beauty and strength is not apprehended by students who are looking at mathematics as truth delivered by a professor or textbook. There are examples of high school mathematics curricula now that demonstrate the integration of

strands regarding inference and proof. These examples are influencing the current deliberations about the undergraduate mathematics curriculum, for example, by the Mathematical Association of America's Committee on the Undergraduate Program in Mathematics (Mathematics Association of America [MAA], 2000).

Fourthly, intermediate courses to address the curricular gap in the undergraduate mathematics curriculum have been designed as a prerequisite to the advanced calculus and abstract algebra courses. Such courses, often called transition courses, bridge courses, or foundations of higher mathematics courses, are intended to prepare students to construct proofs of abstract results of the sort that they will encounter in advanced calculus and abstract algebra. The course typically confines its mathematical content to a subset of the following topics: sets, functions, divisibility of integers, algebra, calculus, linear algebra, construction of the real number field, and cardinality. Each design for such a course, and each textbook proposed for such a course, has to deal with the balance between emphasizing the mathematical logic needed for mathematical proofs, and the amount of content that will be new to students. Material on sets, abstract functions, and divisibility of integers is not familiar to students at this stage. Making a place for such a course in the mathematics program is subject to the same objection that was mentioned earlier: It isolates the topic of proofs, instead of integrating it with the rest of the curriculum for mathematics majors. It does look forward to the upper-level courses, but is still disjoint from the lower-level part of the program.

Transition courses approach the problems that students have with proofs in a different way. Some focus on logic and its vocabulary and implications for building proofs as the most important need (Solow, 1989; Velleman, 1990). Others treat number theory or topology as situations in which students could learn the structures of proofs (Cuprillari, 1989; Schumacher, 1996). Some see the emphasis on sets and functions to be essential (Smith, et al., 1997). Others use the context of calculus and linear algebra of the lower-level courses to reinforce the notions of limits, upper bounds, and conditional statements (Exner, 1996). All of these have the purpose of helping students prepare for the proof expectations of advanced calculus, abstract algebra, and other advanced mathematics courses. Education research can assess the effectiveness of transition courses in assisting students to overcome the difficulties mentioned, and therefore, the pertinent research will be discussed next.

Research on Proofs in Undergraduate Mathematics Education

In order to identify problems in the enterprise of teaching and learning, studies are first done to determine student errors and to compare their performance with those who are considered experts in the field. Studies of these kinds have been done with students in abstract algebra (Selden and Selden, 1987a, 1997a; Hart, 1994) and transition courses (Moore, 1990, 1991, 1994). Their emphasis on error-detection is an important first step to finding more effective ways to characterize teaching and learning proofs. They found that students have difficulties choosing helpful

representations, deciding what starting assumptions to use, and knowing how to use definitions within the proofs. In particular, Selden and Selden (1999, 6) have objected that the courses that purport to prepare students for advanced mathematics do not address the kinds of conceptual problems the students actually have in understanding proofs. These authors emphasize the need for instruction in understanding logical forms of propositions, and specifically mention the following understandings as ones that transition courses frequently fail to address: substitution (a universal statement may take any particular realization), interpreting the logical structure of informally written statements, applying theorems and definitions to situations in proofs, understanding the language of proofs, and recognizing logical structures in the context of mathematics.

Others have looked at the effect that a transition course has on proof-performance (Moore, 1990, 1991, 1994). The main findings of the study were that even after a transition course, students continue to have difficulties in starting proofs, in choosing representations, and in coordinating definitions within the proofs. Moore's recommendation is that earlier courses should give attention to the logic and symbolism that students will encounter in proofs in the transition course and advanced courses. In other words, he is proposing that the ability of students to successfully construct proofs would be encouraged by a conscious strand knitting together the themes of inference and proof into the content of undergraduate mathematics.

Besides research on transition courses *per se*, a different type of research has been to categorize the kinds of thinking that students adopt when considering proofs

(Marshall, 1995; Harel and Sowder, 1998; Sowder and Harel, 1999). Harel and Sowder (1998, 245) studied lower-level undergraduate mathematics courses and found several common schemes that students have in understanding proofs. External conviction proof schemes include accepting proofs because of their form, because of the authority of a textbook or professor, or because of the particular symbols the proof may use. Empirical proof schemes include the use of inductive reasoning and accepting proofs because of the convincing power of a diagram. Analytical proof schemes include proofs by transformations as well as reasoning in an axiomatic context. There are subcategories of many of these, and as a whole, the study shows that an empirical collation of how students look at proofs provides a helpful classification of the ways that students understand the role of proofs.

Harel and Sowder (1998, 246) also point out a danger in any program that emphasizes mathematical reasoning, namely, that students may see the symbolic proofs as somehow more "acceptable" than verbal or geometric proofs. The standards for acceptable proofs should be clearly stated, and succinctly illustrated. For example, the dictum, "do not divide by zero" is seen by students as a command from previous instructors or textbooks; a command to be obeyed because the authority says so. (*ibid*, 235). The proof of the mathematical statement, "There is no number x such that $x = 2/0$" is not difficult, but it requires temporarily accepting a hypothesis that is intended to be contradicted. This is a thought that might be a cognitive obstacle to many students. Further, students who are not familiar with axiomatic proofs regard them as either proving the obvious, or a tedious way to prove a known result

(Schoenfeld, 1985). E.g., the proof that $0a = 0$ for any number a is viewed by students as a rule they memorized in school, or so obvious that it needs no proof. The use of symbols can also be a sensitive area, since often textbooks and instructors use the same symbols to mean different things. With scalar multiplication of vectors, for example, in the statement $0\mathbf{v} = \mathbf{0}$, the symbol "0" is used to mean the real number zero in the first instance, but the same symbol means the zero vector in the second instance. If the vector $\mathbf{v}$ is an n-tuple, then beginners are satisfied with a proof by examining coordinates, but then they will miss the power of the argument using the axiomatic definition of a vector space. The value of the proof from axioms is that it is valid for n-tuple vectors for numbers, for matrices, for functions, i.e., for any objects that form a vector space with operations of vector addition and scalar multiplication.

Like Moore in the study discussed above, Harel and Sowder (1998) also make recommendations that address the curriculum gap in the undergraduate mathematics curriculum. After making a particular study of how mathematical induction is taught, they prescribe a general instructional treatment for the undergraduate mathematics curriculum. Their three principles for teaching are the duality between ways of student thinking and ways of student understanding, creating a necessity for new concepts, and structuring problems to lead students through repeated occasions to use similar reasoning skills. This again is a suggestion for a strand of inference and proof throughout the undergraduate mathematics curriculum.

Prior to the introduction of transition courses, students were expected to learn the necessary logic and organization of proofs from their experiences in advanced

calculus and modern algebra. It is no surprise that mathematics education research done since then has found students' difficulties with constructing mathematical proofs are easy to detect (Selden and Selden, 1987a; Hart, 1994; Dubinsky, et al., 1994; Leron and Dubinsky, 1995; Zaskis and Dubinsky, 1996). But detecting errors in student work is just the first step in research into the teaching and learning of any mathematical concept. There are many more issues surrounding teaching proof concepts and their acquisition by learners, some of which are identified next.

Issues in the Teaching of Proof

Since the predominant way of treating the problem of the gap in the undergraduate mathematics curriculum has been to introduce a transition course, questions about its efficacy are natural.

1. Do students show better understanding of how to construct proofs in advanced courses as a consequence of taking a transition course?

2. Are students able to show better understandings of how to form contrapositive statements as a result of taking a transition course?

3. In what ways do students develop skill in planning and reporting proofs?

4. What are the obstacles to students beginning and concluding proofs?

5. Does practice with simple proofs aid students' understandings of complex arguments?

6. Do students understand proofs by mathematical induction better as a result of taking a transition course?

7. What impact does instruction in proof strategies have on the proof-planning performance of mathematics students?

THE RESEARCH QUESTIONS

A transition course to advanced mathematics is an appropriate course in which to study students' understandings of proof-construction since they have a background in mathematical reasoning from their calculus and elementary linear algebra courses. They have seen many examples of direct proofs – calculations (e.g., the product rule for derivatives; the determinant of an orthogonal matrix); constructions (e.g., the chain rule; the construction of an inverse matrix); and proofs by contrapositive (e.g., if a sequence of nonnegative numbers converges, then the limit must be nonnegative). In order to study the characteristics of student understandings of proofs, this study will address these specific research questions:

1. What conceptions of proof do students have upon entry to a course on mathematical proofs?
2. How do students develop skill in planning and reporting proofs?
3. What are the obstacles to students beginning proofs?
4. What are the obstacles to students completing proofs?
5. How does instruction in a course on mathematical proofs affect the ability of students to understand what they are trying to prove?
6. How does instruction in proof strategies improve the performance of students in solidifying schema in proof-planning and proof-reporting?

There is no standardized instrument for measuring these dimensions of understanding, so the investigator developed the assessment tasks and the interview questions and protocols. Although the issue of proof-construction is independent of the mathematical subject matter of the course in mathematical proofs, that content provided the environment for the assessment tasks. Some of the tasks were adapted from the similar work done in pilot studies in courses in sophomore linear algebra and abstract algebra.

The Design of this Study

In order to investigate how and how well students learn to construct proofs in a transition course, it is necessary to assess their understandings of proofs early and often, and to obtain their comments on how this understanding progresses. This study will show their understandings of proof structure, proof ingredients, and proof logic. The assessments were a pretest, class quizzes, clinical interviews, and a final exam. The transition course emphasized a small set of proof strategies, namely direct proof, proof by contrapositive, proof by contradiction, and proof by mathematical induction. The course applied these strategies to the context of sets, divisibility of integers, relations, and abstract functions.

A transition course addresses the curricular gap from formulas, factoring, and step-by-step processes in lower-level undergraduate mathematics courses to a concern about axiomatic systems, proofs and relations among mathematical objects in upper-

level courses. It follows that there should be an assessment of student understandings of proof at this hinge. The research questions above show the scope and bounds of those understandings, and these determined the direction of the interactive clinical interviews. Designing the interviews required knowledge of the educational research that has been done concerning the teaching and learning of proofs, which is the topic of the next chapter.

Terminology

An analysis of some common textbooks for courses in Advanced Calculus (e.g., Fulks, 1969; Binmore, 1982) and Abstract Algebra (e.g., Gallian, 1998; Gilbert and Gilbert, 1999; Fraleigh, 1999) shows that 90% of the proofs are direct proofs, the remainder being evenly divided between proofs by mathematical induction and proofs by contradiction. For this reason, this study will concentrate on students' understandings of the construction of direct proofs of the sort that they will encounter in undergraduate mathematics courses.

There are several variations of the common classification of proofs into direct proofs and indirect proofs (O'Daffer and Thornquist, 1993; Galbraith, 1995; Solow, 1990; Chartrand, et al., 1999). The term "indirect proof" is used by some to mean proof by contrapositive or proof by contradiction; others reserve the term "indirect proof" for proof by contradiction only; but others reserve the term "indirect proof" for

proof by contrapositive only. Textbook authors frequently assume that their readers understand the assumptions, intentions, and conclusions of a proof by contradiction.

The vocabulary of proof concepts uses many words that are not part of college students' usage. In particular, the words "converse", "contrapositive", "counterexample", and "contradiction" all begin with the same syllable and are frequently confused. The term "contradiction" in particular, if it has meaning for students at all, carries the meaning, "That's wrong."

There are three methods of mathematical induction, going by various names. All three are based on the fact that any set of positive integers has a least element. One of these methods is called ordinary mathematical induction, or the first principle of mathematical induction, or strong mathematical induction. A second method is called complete induction, or the second principle of mathematical induction, or weak induction. In fact, those two methods are equivalent. The last method of mathematical induction is called the method of minimum counterexample. This is a direct application of the greatest lower bound property for positive integers, but the proof itself is a proof by contradiction. The method of minimum counterexample is also equivalent to the first two principles of mathematical induction.

The choice of the form in which a proof is written down is not restricted to one accepted format: It is entwined with the current standards of rigor for proofs (Kleiner, 1991). The textbooks of advanced courses write most proofs in paragraph form, with the justification for each step given in apposition. For this reason, one of the objectives of this transition course is to expose students to reading and writing

paragraph proofs. A common alternative way to report a proof is called the list form, or two-column form, or tabular form. This used to be very common in high school geometry, although that is not generally the case now. The results in geometry whose proofs were written in the list form were the same as those in Euclid's Elements, which, ironically, was written with paragraph proofs. There are other possible formats for reporting proofs, such as diagrams or flowcharts, but they were not employed in this course and they were not generated by the students.

The term "conceptual schema" describes how individuals organize their thoughts about a particular concept. Each individual may have different modes of thinking, as well as different categories into which they classify knowledge new to them. At the same time, such modes of thinking are invisible to others, particularly to the researcher. This project is directed towards understanding the conceptual schema that undergraduate mathematics students have towards constructing proofs. The ingredients of constructing proofs are not simple recipes with lists of contents; they include also the processes of combining the mathematical and logical concepts together. These processes themselves are then items in the contents in a self-referential way that reminds one of the paradoxes of set theory. The collection of ideas included here – the mathematical concepts, the logical machinery, the processes of constructing proofs – all become part of the conceptual schema (Hart, 1994, 62) that describe what it means for an individual to "understand making proofs." In the

same way, Dubinsky (1991) says that a schema is a "coherent collection of objects and processes" with which individuals frame their ways of thinking.

The Proposed Study

In the Winter Term (January-to-April), 2000, the research study was done in a transitions course in Mathematical Proofs for mathematics majors who were sophomores or juniors at a fairly large midwestern university. The students had completed two courses in calculus and one course in elementary linear algebra; and some had completed courses in differential equations or multivariable calculus. The course topics included: sets, logic, elementary number theory, relations, functions, direct proofs, proofs by contrapositive, proofs by contradiction, and mathematical induction. The theme of how to start proofs was evident throughout the course. The course text was a preliminary version of a textbook on mathematical proofs (Chartrand, et al., 1999). The investigator developed the research instruments. The written assessments included a pre-test to set a level of the students' understanding of judging the validity of mathematical proofs and results; and quizzes and a final exam. Clinical interviews explored how the students understood the starting assumptions and the construction of proofs.

These research instruments paid attention to the obstacles to the students' understandings that have been identified in the research literature. Moore (1991, 1994) has classified some of these, according to the difficulties students had with

constructing proofs. Selden and Selden (1995, 1999) have identified many of the obstacles in logic related to "unpacking" the symbols, quantifiers, and inferences that one must make to understand advanced undergraduate mathematics. Chapter Two will review the research into mathematics students' understandings of these forms of proofs.

CHAPTER TWO

RESEARCH ON LEARNING AND TEACHING PROOF

The activity of constructing proofs involves the coordination of several ideas: logical inference (Selden and Selden, 1995, 1996) understandings of the roles of definitions (Rin, 1982; Vinner, 1991), examples (Mason and Pimm, 1984; Zaslavsky and Peled, 1996), and prior theorems in context (Hazzan and Leron, 1996). "Mathematical Proving is a process that uses definitions, postulates, previously proven statements, and deductive reasoning to produce a sequence of true statements providing a valid argument that a statement to be proved is true." (O'Daffer and Thornquist, 1993, 49) This process of constructing proofs is, then, itself a problem-solving activity. But the difficulties that students have in constructing proofs are severe enough that Hart (1994, 53) concludes, "Proof continues to be a problem-solving task at which most students fail." The present study focuses on the proof-production aspect of the definition of proof above.

Research on the learning and teaching of proof has focused on the following areas: (a) expert/novice comparison studies, (b) studies of student errors, (c) studies involving curricular issues, and (d) studies of schema related to proof. This chapter will treat each of these areas in turn.

Expert/Novice Comparison Studies

The research which employs the contrast between experts in a certain field and novices attempts to bring out the differences in how these people think and perform. It operates with the tacit assumption that if the instructor knows the shape of the experts' understandings about the field, then this will translate into better ways to teach to novices (Schoenfeld, 1980). Schoenfeld used this line of research into problem-solving to find that experts had more fruitful connections among mathematical concepts, better control of the use of definitions and theorems to apply, and a willingness to abandon unpromising paths. These results provide both a starting point for the present research, and a source of sounding points against which the learning states of undergraduate mathematics majors may be investigated.

Hart (1994) used tasks within the context of an abstract algebra course in which students had completed the study of elementary group theory. His study consisted of 29 students from three abstract algebra courses: (1) a junior-level introduction to abstract algebra, (2) an advanced undergraduate abstract algebra course, and (3) a first-year graduate abstract algebra course. The participants in the study were ten students from each of the first two courses, and nine from the graduate course. The objective of this study was "to analyze the processes, errors, and self-assessment that college students... exhibit...." (*ibid*, 54) The principle instrument was a test requiring written construction of six proofs from elementary group theory; a second instrument was a written self-assessment by the students concerning their

perceptions of the difficulty of the test and their performance strategies. There were no interviews; however, the students were instructed to write down what they were thinking as they attempted each proof. The self-assessment was done immediately after each proof had been attempted. The test questions and the questionnaire had been refined by two sample pilot experiences. The students' conceptual understanding of group theory was classified into four categories. Conceptual analysis revealed four different levels of understandings about proof-construction. One immediate finding was that the number of undergraduate mathematics courses the students had taken was unrelated to this conceptual category. The findings include that on difficult proofs, the lower-performing students incorrectly rated themselves as being successful; the higher performing students had more accurate perceptions of their work. The experts worked more often directly from the hypotheses towards the conclusion; but the novices were more likely to begin with the conclusion and work backwards. Other processes, such as reformulating a problem and choosing notation, were more prevalent among the experts. In many cases, there was no clear pattern for strategies like abandoning unpromising solution paths – both novices and experts either persisted in their initial strategy or showed willingness to change. The most common errors were: (a) confusing the group operations, (b) incorrect deductions, and (c) assuming the result. The first is a concept usage problem. All three of these error types are noted in all the other research on proof cited in this chapter.

These errors and processes were interpreted in terms of the students' conceptual schema (the mental conceptual map that a student constructs and uses to

apprehend objects of knowledge), concluding that the confusion in group operations is based on their over reliance on familiar concrete representations of group elements as numbers. This was manifest in the task to prove that the identity element of a group is unique. Many students were sidetracked at the start by over familiarity with the number zero with respect to addition. They too readily assumed commutativity and frequently assumed the uniqueness they were supposed to prove.

On the other hand, the graduate students avoided these errors due to their experience and maintained flexibility in their choices of proof strategies. They were willing to abandon unfruitful solution paths as soon as they perceived that a path might not lead them all the way to a successful conclusion. Furthermore, these experts made use of properties and definitions of mathematical objects in appropriate ways within their proofs, in striking contrast to the novices.

Hart concluded that rather than trying to teach novices the behavior exhibited by experts, we need to find a way to teach novices so that they acquire their own stable and powerful conceptual schema (*ibid*, 62). These schemas would coordinate the issues of starting proofs, using definitions, confusing the group operations, avoiding errors in deductions, and choosing useful representations of the mathematical objects. This study also noted the lack of a theory of proof-understanding at the college level. Hart concluded that there is still a need for more qualitative, cognitive-based research in order to know the reasons why students fail to construct proofs successfully.

Weber (2000) is currently conducting a study in which four undergraduates studying abstract algebra are compared to four graduate students who are specializing in algebraic fields of study. For a baseline, each student was given two proof-tasks in elementary group theory, and the challenge for the researcher was to discern differences in their performance. Both groups did extremely well at these tasks, which were, however, at a higher conceptual level than the tasks in Hart's study discussed above. Whereas Hart assessed students' understandings of concepts like uniqueness of inverse, and identifying the inverse with respect to a nonstandard group operation, Weber gave students tasks about group homomorphisms and about normal subgroups that were most easily completed in terms of the corresponding quotient groups.

Even the novices in this study had no difficulty with beginning their proofs in appropriate ways. But of the four more challenging problems, only six of twenty undergraduates' proofs were valid proofs, whereas 19 of the 20 graduate students' proof were valid. Weber's findings to date emphasize the need for strategic knowledge about definitions and mathematical concepts and when to use them within proofs. The term "strategic knowledge" comes from Schoenfeld's work on the kinds of knowledge needed for successful problem-solving. Weber, like Hart, views proof-construction as a problem-solving task. His findings indicate that to prove a theorem, the student needs: (1) Understanding of mathematical reasoning and of mathematical proof; (2) Knowledge of, and the ability to apply, the theorems of a knowledge domain; and (3) Strategic knowledge to judiciously choose which theorems to apply. (Weber, 2000)

Studies of Student Errors

Moore (1990, 1991, 1994) studied the proofs that undergraduate mathematics students constructed in a transitions course, finding that beginners often do not know how to start a proof. The students were eight mathematics majors, six mathematics education majors, and two graduate students. Interviews were conducted with three of the mathematics majors and two of the mathematics education majors, chosen for their representativeness of the class in terms of their previous mathematics courses and their grades in those courses. The students in the interviews indicated that they often did not know how to start a proof and did not know how to connect the hypothesis with the conclusion. Moore pursued this line of inquiry by assessing the students' uses of definitions within the proofs. He quoted students' difficulties with the concept of one-to-one function and their inability to use the definition in the course of constructing a proof. The root of their problem was that the definition is suitable for demonstrating specific examples of one-to-one functions, but the negation of the definition is what is usually needed in proofs about abstract functions: "...learning to translate a definition into symbolic form in which quantifiers are explicit – for example,

$$f \text{ is one-to-one if and only if } \forall x,\ \forall y\ (f(x) = f(y) \Rightarrow\ x = y)$$

– helped them see the logical structure of a proof based on the definition and facilitated their use of the definition." (Moore, 1991, 215). He found that getting

started in constructing proofs, errors in concept understanding, and problems with mathematical language and proof structures were what gave students the most difficulty. Moore's conclusions about cognitive obstacles to completing proofs by such students were used as an ingredient in the formation of questions for the interviews of the present study.

Moore observed the classes, led preceptorial sessions, tutored some students individually, and did the interviews. He used the entire class as a quantitative base and selected a subset of five interviewees to sharpen the focus on what the student difficulties were. The findings were several: Mathematics students in a transition course have (a) difficulty starting proofs; (b) difficulty with using mathematical language, notation, and definitions within proofs; and (c) a simplistic notion of proof as a procedure, not an explanation or a discovery method.

His prescription for improvement included: all mathematics courses should give more direct attention to proofs and reasoning, allowing students wherever possible to give reasons for their inferences, and to organize their schema coherently, in contrast the to the professor, for whom multiple definitions of concepts, well-developed concept images, and understanding of concept usage characterized his thinking about the course material. In a comment on curriculum, he concluded that "Until proof is integrated throughout the school and university mathematics curricula in the United States, I believe the abrupt transition to proof will continue to be a source of frustration for undergraduate students and teachers." (Moore, 1994, 264).

Selden and Selden (1987) found that students make errors of reasoning and application in abstract algebra proofs, but there are common features among these errors . Several classes of junior mathematics majors in abstract algebra submitted their proofs in homeworks and tests, which Selden and Selden analyzed for common traits. They classified misconceptions and other errors in simple group theory proofs as errors of generalization, use of theorems, notation and symbols, nature of proofs, and quantifiers (*ibid*, 468). The types of errors identified included: (a) confusing the converse with the proposition to be proved, (b) assigning a variable name to a quantity and assuming that it therefore existed, (c) not recognizing alternate representations, (d) extrapolating from arithmetic properties of numbers, (e) making circular arguments, (f) using the same symbol for different objects within a proof, and (g) tacitly changing the hypothesis to trivialize the theorem.

The authors' suggestions for improving the situation for the students who plan to take such an abstract algebra course include: lower-level courses should eschew the static view of mathematics as procedures and facts, and emphasize the creating and validating of algorithms as the most valuable feature of mathematical thinking (*ibid*, 469). They also point out that the kinds of reasoning errors that they have classified in this empirical study and the logic topics that are necessary to understand calculus proofs, are not the kinds of things studied in most transition courses.

Pursuing the difficulties that mathematics majors have with compound logical statements, Selden and Selden (1995) collated student work on tests from six transitions courses that they had taught to junior and senior mathematics students.

There were contributions from 61 students from the years 1986-1993. The prerequisite for the transitions course was one year of calculus. The students were 48 juniors and seniors, and 3 masters degree candidates in mathematics for secondary school teaching. There were 26 mathematics majors, 27 mathematics education majors, and the remaining 8 were majors in mathematically-related fields. The transition course was characterized as "designed to ease the transition from lower division, more computational, mathematics courses to upper division, more abstract, mathematics courses such as modern algebra and advanced calculus" (*ibid*, 135). The data were from three tests and five final examinations. The questions cited required explanations of multiply-quantified statements such as the definition of limit of a function, and explication of common statements from calculus. In one class of fourteen students, 126 responses to such tasks were mostly incorrect – the three correct responses all came from one student.

Recognizing several points of view about the functions of proofs, the authors chose to focus on the ability to clarify the logical structure of mathematics statement and their ability to use such structures in the construction and validation of proofs (*ibid*, 124). The authors found that students (a) could not reliably translate a compound logical statement into its constituent parts, (b) generally were not able to supply unwritten assumptions about universal quantifiers, and (c) were hindered in their attempts at validating proofs by these and other problems with logic and syntax. This work raised questions about how to refine the definition of validating a proof, especially concerning what skills are needed and what processes are expected.

Based on these studies, Selden and Selden (1996) recommend that transition courses should give attention to the following:

Substitution (Also Called the Universal Law of *modus ponens* or Universal Law of Detachment)

When the quantifier "for all x" occurs, then one logical consequence is that the statement is valid for a particular x. Many students confuse this with algebraic substitution, but it is different. When they have made an algebraic substitution, or a replacement of a symbol in a logical implication, students frequently will claim they have done "substitution." For example, if (P and ($P \Rightarrow Q$)), then they will conclude Q, and say the reason is "substitution." There may be a problem with symbol sense here – The students may be confounding the implication symbol "$\Rightarrow$" with an equals sign.

The need for thinking in terms of making a universal statement particular occurs in problems such as these:

a. Prove that if the dot-product of vectors, $\mathbf{u} \cdot \mathbf{v}$ is the zero number for all vectors $\mathbf{v}$, then $\mathbf{u}$ is the zero vector.

b. Let a, b, c be real numbers. Prove that if the quantity $a > 0$ and the quadratic $ax^2 + bx + c > 0$ for all real x, then $b^2 - 4ac < 0$.

Interpreting the Logical Structure of Statements

Students' abilities to properly judge the validity of proofs was hindered by their lack of understanding of unstated quantifiers and of the meanings of compound quantifiers. Mathematical writing in journal and textbooks offer a trimmed version of what is meant, as when they might say, "Show that the square of an even number is even." When what is meant, and assumed that the reader will understand is, "Show that for every even integer, its square is even." The unstated universal quantifier gives a completely different meaning than an existential quantifier, and the reader is expected to supply the correct interpretation. Similarly, the authors found that students frequently did not read $P \Rightarrow (Q \text{ or } R)$ correctly, and therefore could not formulate a beginning for a proof-strategy.

Applying Theorems and Definitions Within Proofs

Even when a completed proof had been verified by the students, they were unable to see that it could then be used to make conclusions about particular situations. The authors noted that definitions were seldom invoked due to students not having the mental connections that invite them to widen their vision. They also noted the conflict between the common-language definitions and mathematical definitions – in common language, definitions are not prescriptions for exactly what objects satisfy the definition, due to careless use of words. In mathematics, however,

a definition should specify exactly what is, and what is not, the object under consideration.

Understanding the Language of Proofs

There are several conventions of usage among mathematicians that are used in writing proofs. Even in textbooks, there are common assumptions made about the reader's ability to fill in missing quantifiers and understand incompletely described situations. A principle example is the word "Let", which is used in at least three different senses by mathematicians. If a property is true for all objects of a set, then we might say, "Let x be one of the elements with that property." This is particularlization, mentioned above. In another sense, "Let $\delta = \varepsilon/2$" or "let $f(x) = 1/x$" might occur in the way of defining an object *in situ*. A third sense is the hypothetical use of "let", for example, when proving the uniqueness of the identity element in a group: "Let u and v be identity elements of the group." Students will often mistake this latter use as conferring existence, and neglect to show that such objects do indeed exist. (Selden and Selden, 1987, 461)

Recognizing Logical Structures

Students have difficulties with "unpacking the logic" of mathematical statements. The authors observe that textbooks for transition courses assume too

easily that students can read the statements of theorems with compound hypotheses and correctly construct their negations. This particular difficulty is the topic of an interpretive paper (Selden and Selden, 1995).

In later studies, Selden and Selden (1997b, 1998b) asked eight students in a transition course to think aloud about whether elementary proofs in number theory were valid. They found that the students focused on individual steps of the proofs, and not on the structure. All of the eight students were mathematics majors, seven of them in the secondary education teaching program. Proofs written by students in previous transition courses were presented to the participants for analysis, and each participant was asked to judge the validity of four proofs on four separate occasions. On the first occasion, these mathematics students made correct judgements about the validity of the proofs constructed by other students in only 50% of the opportunities. On the fourth occasion, their correct judgements constituted 81% of the opportunities. Selden and Selden see this as evidence that instruction is effective, and recommend the use of instruments like theirs in transition courses.

On the basis of their research, Selden and Selden recommend that prerequisite courses to the transitions course should give more opportunity for students to develop proofs and discuss validity issues. These should be introduced early in college mathematics, or earlier, and refreshed often. Students should be asked to give reasons for their mathematical arguments regularly, and should be challenged to explain why their proposed proof is indeed a proof of the desired theorem, and not of some other theorem. (1995, 142). These recommendations are commensurable with the

curriculum discussions at this time to make mathematical reasoning more central in the school mathematics curriculum (NCTM, 2000) and in the undergraduate mathematics curriculum (Mathematical Association of America [MAA], 2000).

In a study of how preservice elementary teachers in mathematics programs understand proofs, Martin and Harel (1989) investigated how such students responded to purported proofs by inductive reasoning versus proofs by deductive reasoning. These 101 students were college sophomores, enrolled in a required course for preparation for elementary school teachers. The research report did not specify what the prior mathematical experience of the participants was, nor what the prerequisites for the course were. The participants were presented with purported proofs prepared by the researchers, and asked to judge whether the "proofs" were valid. The proofs presented concerned divisibility properties of integers. There were no interviews to confirm or explain why students wrote, or failed to write, what they did.

Martin and Harel found that the students would frequently accept an explanation by inductive reasoning, often involving only one example. But at the same time, they would identify the instructors' proposed deductive proof as acceptable, too. The researchers concluded that the students did not distinguish the proofs from their proof styles. That students would accept a mathematical proposition as true with the evidence of just one example is interpreted as meaning that they extrapolate to the approval by an authority of teacher or textbook. This study by Martin and Harel shows the dangers of assuming too much about students' reasoning concerning proof.

There is some convergence of the findings of these studies on student errors, or at least, some limit points to their conclusions. They indicate that the primary obstacles to students and their attempts to construct proofs are: (a) knowing how to begin a proof, (b) knowing how to use definitions in proofs (concept-usage), and (c) knowing basic logic as it is used in mathematical proofs. The present research will delve further into these topics as part of finding how students themselves think about the process of constructing proofs.

Studies Involving Curricular Issues

The program of mathematics courses in an undergraduate major underwent continuous change in the twentieth century (Mathematical Association of America [MAA], 2000). The role of the transition course for college sophomores and juniors has been defended as a necessary prerequisite for abstract algebra and advanced calculus. There has been little research to assess how well transition courses actually perform this function. The available research indicates that a one-course treatment does not solve all the students' problems in reading and producing proofs. The most common recommendation from education research is that the study and appreciation of proof should be integrated throughout the high school and college curriculum (Moore, 1990, 1991, 1994; Selden and Selden, 1987, 1995, 1997, 1998, 1996; Harel and Sowder, 1998). Many implementations of the course use it to introduce mathematical content that is otherwise not a part of the current curriculum, such as:

abstract functions, set theory, equivalence relations, elementary number theory, congruences, and various choices of additional topics selected by the textbook authors. Typical additional topics include: cardinality, construction of the real number system, unique factorization of polynomials, and/or limits. The Mathematical Association of America's Committee on the Undergraduate Program in Mathematics (CUPM) is preparing a report on these and other curricular issues at the present time (Mathematical Association of America [MAA], 2000).

Another approach to the teaching of proof in the undergraduate mathematics curriculum involves the Moore Method. This method would hold the students responsible for producing proofs based on definitions and propositions provided by the instructor. This Socratic method is rooted in the work of R. L. Moore with graduate topology courses at the University of Texas in 1918-1942. There have been many variations on Moore's approach (Jones, 1977; Reisel, 1982; Chalice, 1995), including providing more direction to undergraduates in such courses. In a "modified Moore course," the professor will present some proofs as exemplars for the students to study, and to learn about the process of proof-construction.

Proof Schema

Schema is the term for the coordinated networks in which the mind organizes repeated similar experiences.

> These are complex networks of concepts, rules, and strategies, not isolated facts or algorithms. Having information stored in this way helps an individual cope with new experiences. Such schemata develop over long periods of time and by continual exposure to related contextual events. ...New experiences either use one's existing schema (called assimilation), or force a change in a particular schema (called accommodation). (Romberg, 1991, 62)

Whereas the schema for a single concept has been the target of many research reports, the schema for a multi-faceted process like proof-construction has not. Ingredients like starting assumptions, the role of definitions, the understanding of prior proved results, and the importance of knowing the goal of the proof have already been mentioned. The deeper one's appreciation and knowledge of mathematical connections, the more readily one can see further into the possible paths one could take in pursuing a successful proof. Open subjects for research the formation of schema, particularly over the four years of a student's college career; the stability of schema when faced with new concepts to assimilate; and the transfer of knowledge in recognizing old concepts in new situations. Harel's and Sowder's research (Harel and Sowder, 1998; Sowder and Harel, 1999) into the production, understanding, and appreciation of proofs provide valuable contributions to our approximate knowledge of students' proof schema. The authors maintain that their empirical classification is a way to make sense of the diverse understandings about proof that students hold. Their research was done in six mathematics courses, whose audiences and numbers of students are detailed in Table 2.1.

Table 1

Harel and Sowder (1998)

Course	Number	Audience
Elementary Number Theory	22	College junior and senior math majors
College Geometry	25	College junior and senior math majors
Linear Algebra #1	23	College sophomore math majors
Linear Algebra #2	27	College sophomore math majors
Linear Algebra #3	20	College junior and senior math majors
Euclidean Geometry and Calculus	1	Precocious junior high school student
Total:	118	

Among their findings is the fact that students who hold empirical understandings of proofs, tend to do so in two very distinct ways. Either their first and firmest thought is to provide the evidence of (one or) several examples, claiming a completed proof; or they use a visualization or a heuristic, and claim that as their proof. These understandings would seem to be the most fragile, but in fact, they are the most resistant to change.

Currently, Harel and Sowder are conducting a planned program to determine how students view proofs, as part of classifying the students' schema about proofs. The researchers have used written assessments and clinical interviews in courses in precalculus, calculus, linear algebra, and differential equations. This empirical study

(Sowder and Harel, 1999) into the kinds of schemes that students use to frame their concepts of proofs has found the following three categories:

1. External Conviction Schema are revealed when a student relies on the form of a proof or the authority of a professor or of a textbook. This causes one of the most difficult challenges to teaching, since it is one of the most difficult student habits to correct.

2. Empirical Schema of understanding proofs are revealed when a student justifies a mathematical result by citing examples only, or by pointing at a graph. A concomitant danger of such inductive reasoning is that some students will not provide more than one example unless it is required.

3. Analytical Schema are revealed when a student exhibits transformational understandings, whereby they would declare that a proof is valid because it is a transformation of a proof that has already been accepted. Also included as Analytical Schema are the recognition and acceptance of proofs that are axiomatic in nature, whether they are a derivation from axioms, or a construction and implementation of an appropriate axiomatic system.

This provides some understanding of how students are viewing proof, and therefore, some insight into why some teaching methods do not work. But more must be done to compare teaching philosophies and pedagogical stances to see if there are things to do and things to avoid in teaching proof concepts.

Summary of the Research

Expert/novice research has demonstrated that mathematics students in advanced courses make both logical and syntactical errors in constructing and verifying proofs (Hart, 1994; Selden and Selden, 1987, 1995, 1996, 1997ab, 1998). Research into the types of errors that students make in constructing proofs is summarized in Moore (1990, 1991, 1994) and Selden and Selden (1987, 1994, 1997ab, 1998). This has provided a platform for a wider view into curricular issues, which has been addressed by Moore, Selden and Selden, and Harel and Sowder (1998; Sowder and Harel, 1999). Such research has then turned to an examination of the kinds of proof schemas that students are operating from mentally.

In particular, this present study was guided into its present form by the findings in these research studies. The details of the research style and the level of investigation pursued will be described in the next chapter, to which we now turn.

CHAPTER THREE

METHODOLOGY

The Purpose of This Study

As previously stated in Chapters One and Two, there is an important juncture in the undergraduate mathematics curriculum at which students must learn to construct and appreciate proofs as a methodology of advanced mathematics. This study focuses on this juncture, and seeks to investigate how advanced undergraduates learn to construct proofs during a semester course on mathematical proof. The research questions focus on this learning and the students' responses to instruction:

1. What conceptions of proof do students have upon entry to a course on mathematical proof?
2. How do students develop skill in planning and reporting proofs?
3. What are the obstacles to students beginning proofs?
4. What are the obstacles to students completing proofs?
5. How does instruction in a course on mathematical proof affect the ability of students to understand what they are trying to prove?
6. How does instruction in proof strategies improve the performance of students in solidifying schema in proof-planning and proof-reporting?

The Context of This Study

This research study was conducted during the 2000 Winter semester (January-to-April) in a course in Mathematical Proof for mathematics majors who were sophomores or juniors at a large midwestern university. The students had completed courses in calculus and elementary linear algebra; and some had completed courses in differential equations or multivariable calculus. Although most of the 22 students in this course were sophomore or junior mathematics majors, some were mathematics minors with majors in science, computer science, and communication. The university offers four mathematics major programs. This transition course is required in the general mathematics major and the secondary teaching of mathematics major. The statistics major program also requires this course, but no upper-level courses in that program have this transition course as a prerequisite. The transition course is an elective in the applied mathematics major.

The course text was a preliminary version of a textbook on mathematical proof (Chartrand, et al., 1999). The course topics included sets, logic, elementary number theory, functions, and relations. The explicit techniques for starting proofs were direct proof, proof by contrapositive, proof by contradiction, and mathematical induction. Selecting appropriate strategies to begin writing proofs and articulating mathematical reasoning were central themes of the course. The instructor was one of the co-authors of the text and had previously taught the course. The written assessments were a written pretest, biweekly quizzes, and a final exam. The interactive assessments were

taped interviews with six students. The researcher developed the pretest and selected research-oriented questions for the quizzes and for the final exam; the course instructor developed the quizzes and final exam. The researcher structured the interviews to repeatedly ask students to articulate their thinking about the issues of starting proofs, justifying reasoning, and concluding proofs.

The class met for 40 fifty-minute periods, followed by a two-hour Final Exam period. The instructor taught principally by lecture/discussion, using the blackboard throughout. There was an emphasis on good mathematical writing and speaking. The instructor reinforced the textbook's terminology and syntax. For example, she clearly stated and wrote each result to be proved in correct mathematical language that was usually identical to that in the textbook. She wrote at the beginning of each proof either a motivational statement or a statement of purpose about the proof-technique. The proofs were concluded with an explicit statement that the desired goal of the proof had been achieved.

In contrast to written assessments, however, clinical interviews offer much richer opportunities to expose how an individual is thinking. The interviewer can probe the situations when the subject's knowledge is incomplete or insecure, can detect the level of confidence the subject presents, and can plan sequences of tasks that reveal the extent of the subject's understanding of mathematics. The research discussed in Chapter Two demonstrated that interviews also provide occasions for surprises to the researcher about the understandings that students may hold. In an interactive interview, these surprises can be "teachable moments" for the researcher,

who then may formulate questions and request clarification. An important reason to include the interviews for this research project is to search for evidence that confirms or denies the conclusions from the written assessments. This kind of confirmation is called triangulation of the findings. (Miles and Hubermann, 1994, 41).

Comparing the performance of the interview subjects with the written work of the entire class showed on the one hand that the subset of interview subjects was representative of the entire class. These points are important for fairly making final judgments about how the results of this study may generalize to other populations of undergraduate mathematics majors.

From general information about the students ages, previous mathematical experience, and previous grades, six students were selected to participate in clinical interviews, some from each of the high, middle, and lower levels of mathematical abilities. All but one of the six students completed all five interviews, and one student completed three of the interviews. A balance of genders was achieved with three males and three females (Table 3.1). The interviews were conducted on campus, not at the same time as the class meetings. The researcher audiotaped and transcribed each interview.

The Research Paradigm for This Study

The nature of the research problem determined the research questions in Chapter One, and these in turn determined the research design. The design included

both a quantitative component and a qualitative component, because some research questions invited responses from the entire class while others required knowing how individuals were thinking about the process of constructing proofs. This bimodal way of conducting such research is common now in mathematics education research, as evidenced by the research cited in Chapter Two.

Table 2

Backgrounds of the Interview Students

Student	Mathematics Grades	Gender	Class	Major, Minor
S1	B	Male	Sophomore	Communications Major Mathematics Minor
S2	A	Female	Sophomore	Mathematics Major Physics Major
S3	A	Male	Senior	Math/Statistics Major Astronomy Minor
S4	A	Female	Sophomore	Mathematics Major Computer Science Minor
S5	B	Male	Sophomore	Mathematics Major Communications Minor
S6	C	Female	Sophomore	Mathematics Major Earth Sciences Minor

It is part of the human nature of thinking and communicating that students' understandings and their expressions of them will not be completely clear to a

researcher or instructor. This is distinct from the philosophy that some students may hold that mathematics is only concerned with right and wrong answers; this is an effort to accurately reflect what they understand from what they reveal in their own words about their thinking about the process of constructing mathematical proofs. For the researcher, finding useful ways to investigate that thinking is the challenge.

The written assessments were a natural way to use the course-embedded quizzes and tests as one source of information; the interviews were a way to both confirm that information and to extend it. Both assessment experiences spanned the entire semester, providing measurements through time to observe whether and how the students' understandings solidified.

The Quantitative Study

Although this research used the same quizzes and final exam that were employed for evaluation by the instructor, the purposes of the research were different from evaluations for grades. The students' responses were analyzed for what they revealed about how the students understood the starting assumptions of proofs and the construction of mathematical proofs. Of the 22 students who finished the course, complete data for this research was available for n=16 students. The data came from the pretest, two quizzes, and the final exam. Data was unavailable for the remaining four quizzes.

The Qualitative Study

The interview tasks repeatedly addressed the issue of choosing starting assumptions for proofs so that the students had ample opportunity to explain their thinking. There was only one student who did not complete the interview tasks. The tasks were written to involve the students in thinking about the course material and objectives concerning the construction of proofs. The research questions about the impact of instruction on how undergraduate mathematics students think about constructing proofs also require the inclusion of the variables of time and frequency of assessment. These features were recorded with the data throughout the semester, and included in the coding for analysis. The interview tasks elicited the occasions when students had incomplete notions about definitions and processes in constructing proofs. When a student did not know how to reply, the interview venue made it possible to persist in obtaining a response that showed how or what was the thinking. This was the special advantage of the interview process over the written assessments. The coordination of these two modes of information gathering through time provided a feedback loop. Responses on the written assessments shaped the interview protocols, which then informed the choice of questions on the succeeding interviews, and all of these were background for the analysis of the final exam data. The opportunities to pursue a line of questioning are what characterizes clinical interviews, in contract to structured interviews.

Analysis

The distinctions among the research questions led to some differences in how they were treated. The first research question, for example, asked about the students' understandings at the beginning of the course, and so depended entirely on the pretest. The data from the n=16 students who participated in all of the written assessments were examined in regard to the issues important for this study: How well did they recognize the consequences of a logical implication? Could they begin and essentially complete a direct proof? Could they explain a proof by contradiction? These attributes were examined and formed a foundational set of tentative results for the subsequent assessments. These tentative results of the quantitative material were systematically compared to the interview material to confirm the findings and detect specific comments in the interviews which would shed light on reasons for the students' responses. Throughout the analysis, the element of time of the semester was maintained in the coding of the data so that it could be included.

Research questions 2, 3, and 4 concerned how students expressed their understandings of the workings of proofs. These were investigated by the use of written assessments for the entire class, throughout the semester; and by the interviews with the select six students. The data for the entire class was examined first, to form observations about the trends displayed in their work. To validate these observations, the same points were compared to the corresponding items in the interview protocols.

The questions in the interviews were designed to invite the students to explain their thinking about just those attributes of constructing proofs.

Research Questions 5 and 6 differed in an important way. They addressed the influence of instruction on the ways that the students changed in their abilities to make proofs during the semester. Finding answers to these questions required closer attention to the element of time during the semester.

As stated in the research questions, the interview tasks required the students to assess their starting assumptions in their proofs, defend their choices of steps in proofs, and explain how they knew when their proof was complete. The complete interview protocols are in Appendix D. For example, various ways of questioning assessed their choice of a proofs initial assumption, as illustrated in Figures 1, 2, and 3.

How would you begin a proof of the fact that the square of an even integer is an even integer?

Figure 1. Sample of a Direct Proof Task in Pretest Item 4

If *a, b* are real numbers and *ab* is nonzero, then *a* is nonzero.

Figure 2. Sample of a Proof by Contrapositive Task from Interview 2

If x is a nonzero fraction and y is an irrational number, then xy is an irrational number.

Figure 3. Sample of a Proof by Contradiction Task from Interview 2

Based on the research by Moore (1990, 1991, 1994), discussed in Chapter Two, the questions for the quizzes, final exam, and the interviews were designed to compare students' understandings of how to start proofs, how to use definitions within proofs, and how to explain the logic of the proof. Following Selden & Selden (1987, 1995, 1997ab, 1999), the interviews for this research were intended to shed further light on the shortcomings that these authors identified as parts of the problem of constructing proofs. The protocols and question statements in this study were influenced by the previous several researchers about how wording of a task can impose obstacles for the students.

The questions on the pretest were designed to assess what students knew about how to start a proof and judge the validity of a proof, understanding how to apply a result, and understanding concepts such as contrapositive and converse. See Appendix A for the actual items. By gathering data from the entire class, the progress of the class during the semester could be tracked. By comparing the entire-class data to the sample of six interviewees, the validity of the sample for the interviews was shown. The details of this argument will be in Chapter Four.

Pilot Studies

Pilot studies for this research were done with courses taught by the researcher in elementary linear algebra (Spring 1997, Spring 1998), abstract algebra (Fall 1997), and advanced calculus (Fall 1998). The populations for the pilot studies were sophomore, junior, and senior undergraduate mathematics majors at a small suburban university. The pilot studies served two purposes. The initial purpose was to explore what questions were appropriate for potentially productive modes of investigating the research questions. The second purpose of the pilot studies was to try different wording of questions to see if the instruments were inadvertently introducing obstacles. Questions about the validity of mathematical arguments, in particular, were revised on the basis of these preliminary attempts. As an example, a task in the pilot study interviews read, "Show that if a and b are positive real numbers, then $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$." The wording was revised to set the task in an inquiry setting, as in Figure 4.

Parameters of the Present Study

In summary, the present research was conducted in a course for transition to advanced mathematics with n=16 students involved in written assessments and six students participating in a sequence of five interviews. The interviews were spread

over five weeks in the middle of the semester, and repeatedly assessed the students' understandings of how to commence proofs, how to incorporate the use of definitions within proofs, and how to explain the logic of the proofs as they were built.

Prove or Disprove:

If a and b are positive real numbers, then $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$

a. How would you start on this?
b. Do you think the equation is true for all a and b?
c. Decide whether to prove or disprove the Statement.
[Comment: At this point, students were invited to restate the proposition as an inequality ($\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$)].
d. To attempt a proof of the revised statement, would you like to try a Direct
e. Proof, a Proof by Contrapositive, or a Proof by Contradiction?
e. What would your starting assumption be in such a proof?
f. What would your next step be?
g. What would your goal be for the end of the proof?
h. What is in the way of your completing the proof now?

Figure 4. Example of a Revised Task in Interview 3, Item 4

Class Observations

The researcher observed 11 class meetings, in order to compare the instructor's way of teaching proofs with the approach in the textbook. This was important for establishing an initial contact with the students, and for constructing appropriate

questions for the quizzes and interviews. For example, the implementation of this particular course did not use examples from the prerequisite courses in calculus and linear algebra, so the questions designed for the quizzes and interviews deliberately avoided these topics. In addition, the class observations revealed that the instructor was staying very close to the textbook in the use of explanatory phrases within the proofs such as declaring whether a proof was a direct proof, a proof by contradiction, or a proof by mathematical induction, and other such comments.

The researcher sat in the back of the class and took personal notes on the style and presentation of the instructor, as well as informal comments on the frequency and nature of student participation. The instructor did not interfere with students' exploratory suggestions; neither did she refrain from making intentional mistakes to provoke classroom discourse.

Summary

One of the special advantages of naturalistic inquiry is to encounter the subjects of the study in their own environment. In this research, that culture is meeting students in the environment of a class in which they are currently investing time and mental energy. The thoughts that they reveal are the thoughts that are fresh to them because of their engagement with the course material. The interview situation serves to bring out these thoughts and discover the ways of thinking that the students are experiencing as they face the course material.

There are many variables which impact how students learn, as well as how they express what they have learned. The importance of this research is in measuring how they approach the construction of proofs, and how their understanding changes through the semester. There are other variables, both quantitative (e.g., the amount of time spent studying) and qualitative (e.g., students' attitudes) that were not studied. The results of the analysis described are reported in the following chapter.

CHAPTER FOUR

ANALYSIS

Introduction

A transition course to advanced mathematics is an appropriate course in which to study students' understandings of proof-construction, since the students have a background in mathematical reasoning from their calculus and elementary linear algebra courses. They have seen examples of direct proofs that calculate, construct, or derive other mathematical constructs; and they likely have seen a few proofs by contrapositive and proofs by contradiction. The transition course that was the focus of this research consisted of 22 students, including six students who participated in the interviews. Four students did not take the pretest, and two others did not take the final exam, so the quantitative analysis only used data from the n=16 students for whom complete written assessments were available. All of the tasks from the interview protocols and written assessments are in the Appendices. This study addressed the research questions of Chapter One using methodology and instruments informed by the research discussed in Chapter Two according to the methodology discussed in Chapter Three.

The Research Questions

Research Question 1: What Conceptions of Proof do Students Have Upon Entry to a Course on Mathematical Proof?

Answers to this question derive from student responses to the pretest. The pretest assessed understandings of conditional statements, converses, direct proofs, and proofs by contradiction. The six items on the pretest provide information on several features of understandings about proofs. The performance of all 16 students in the study was analyzed to provide a landscape of responses to the questions. The first pretest item in shown in Figure 5.

> 1. Here is a known fact: If $a < 0$, then the quadratic function $y = f(x) = ax^2 + bx + c$ has a maximum y-value.
>
> Given that the quadratic function $g(x) = -4.9x^2 + 9.8x - 32.1$, what can you conclude from the known fact?

Figure 5. Pretest Item 1

Thirteen of the 16 students were able to apply the simple conditional statement. Eight of the 13 correct responses said specifically that g had a maximum because $a = -4.9$ and the known fact. Although students might have responded according to their familiarity with quadratic functions from high school mathematics and college calculus, in fact none of them wrote a derivative down, and only one drew

a graph. In order to further examine their understanding of implications, a second item of the pretest involved a similar conditional statement which did not apply. (Figure 6)

2. Here is a known fact:
If the differentiable function $f(x)$ has a relative maximum at $x = a$, then $f'(a) = 0$.

Given that $g(x) = x^3 - 3x + 7$ has $g'(1) = 0$, what can you conclude from the known fact?

Figure 6. Pretest Item 2

Only two respondents correctly stated that no conclusion could be made from the given information, which is a measure of understanding when a conditional statement is applicable. Nine students made the incorrect inference that g has a maximum at $x=1$, indicating that they were not distinguishing the implication from its converse. The remainder tried to treat this item as a problem to be solved, two of them using a graph or calculus to show that $g(x)$ actually has a minimum at $x=1$, and two of them verifying the hypothesis that $g'(1) = 0$. These approaches were not germane to the question of when a conditional statement applies to a mathematical situation. Finally, there were two students who wrote unrelated comments. Two responses had incidental arithmetic mistakes in calculating $g(1) = 5$. One of the two responses that was deemed correct actually reveals an obstacle in understanding the unstated universal quantifier, "for every function $f(x)$", in the given information. The

student wrote, "No conclusions, the functions $f(x)$ and $g(x)$ are not related." This serves as a powerful warning about relying on written assessments to reveal student understandings. The items on both written assessments and interviews demanded careful wording in order to elicit responses showing how students were thinking.

After these first items regarding conditional statements, pretest Item 3 (Figure 7) presented a complete proof of an elementary number theory fact, and asked for responses concerning its validity.

3. You have probably noticed that when you add two odd integers, the sum always seems to be an even integer. In mathematics, it is commonplace for observations of patterns like this to lead to conjectures and then to attempted proofs. The above statement leads to this conjecture:

If a and b are odd integers, then $a + b$ (the sum) is an even integer.

One student's proof looked like this:

If a and b are odd integers, then a and b can be written $a = 2m + 1$ and $b = 2n + 1$, where m and n are other integers.
If $a = 2m + 1$ and $b = 2n + 1$, then $a + b = 2n + 1 + 2m + 1$.
If $a+b = 2m + 1 + 2n + 1$ then $a+b = 2m + 2n + 2$.
If $a+b = 2m + 2n + 2$ then $a+b = 2(m + n + 1)$.
If $a+b = 2(m + n + 1)$ then $a+b$ is an even integer.

a. Look at the first statement. What does she assume?
What additional assumptions, facts, or algebraic properties did she use within her argument?
b. Does the student's argument prove the conjecture? Describe the features of the argument that support your position.

Figure 7. Pretest Item 3

The given proof made the starting assumption for a direct proof that a and b are odd integers. Seven students said the starting assumption was that $a = 2n+1$, $b = 2m+1$ for some integers n, m. This is using a definition instead of the starting assumption. This suggests that they took the first line of the proof to be an implication, instead of the definition that it is. Students often come to college mathematics with an inductive definition of an odd number as a positive integer from the set $\{1, 3, 5, 7, 9, \ldots\}$. They will assent to writing an odd integer as $2n+1$, but they do not recognize that as a definition. Four students said that writing an odd number a as $a = 2m+1$ was an assumption, instead of a definition. One student showed that he was thinking of positive integers only, and this restricted his ability to understand the proof.

The second part of this item asks for a judgement of the validity of the direct proof by algebra which correctly shows that the sum of two odd integers is an even integer. Two students incorrectly said that the proof was invalid, for example, because even numbers were not considered in the hypotheses. Nine of the students who affirmed that the proof was valid, gave as a reason that the last line of the proof was in agreement with what was to be proved. In saying this, they revealed a possible prejudice for the form of the proof, and did not elaborate on the validity of the steps between the first and last. Two students wrote nothing. One student confessed, "I understand all of the calculations she made, but didn't understand how it proves it."

The pretest also included questions providing the students with opportunities to interpret the converse of a conditional statement (Items 4,5). When asked on the

pretest to start a proof that the square of an even integer is even (Item 4, Figure 8), 10 students made a correct starting assumption for a direct proof, but six made an assumption that would suit a proof of the converse statement.

4. How would you begin a proof of the fact that the square of an even integer is an even integer?

Figure 8. Pretest Item 4

Item 4 was similar in context and difficulty to the direct proof of Item 3, but required the students to start the proof. Seven students did so successfully, but the remaining nine did not. For two of them, an obstacle was the lack of understanding the proposition as a conditional statement: "If n is an even integer, then n^2 is an even integer." The reason the question was not stated that way was to see how the students would handle the freedom of choosing notation for themselves. On this issue of choosing notation, 11 chose appropriate notation, two chose inappropriate notation for the item, and three wrote either nothing or sentences without notation.

The purposes of pretest Item 5 (Figure 9) were to detect whether students knew when to look for a counterexample, and whether they maintained the distinction between the proposition and its converse. Ten students correctly indicated that they knew what a counterexample was and how to pursue finding one for this situation. Two students confused the proposition with its converse, one student correctly

described how a proof by contrapositive would commence, and three students wrote nothing. Two students showed initiative by producing counterexamples with negative values of n. For example, one student chose $n = -7$, for which $n^2 - n + 41 = 96$; and another student used incorrect reasoning to deduce that $n = -40$, for which $n^2 - n + 41 = 1681 = 41^2$.

5. Consider the expression $n^2 - n + 41$. A student claimed that if n is an integer, then $n^2 - n + 41$ is a prime number. How could you prove that this claim is not true?

Figure 9. Pretest Item 5

Although these students had not previously studied number theory, an item (Figure 10) involving divisibility was included to see how they handled an abstract definition.

6. Here is a definition: An integer m is a <u>factor</u> of the integer n if and only if there is a integer k such that $n = mk$. Prove that if a is a factor of b, and if b is a factor of c, then a is a factor of $a + c$.

Figure 10. Pretest Item 6

Thirteen of the 16 students were able to successfully start a proof, but only eight brought it to a valid conclusion. This problem brought several other issues to the surface concerning definition usage, symbol sense, quantifier understanding, and algebra. For example, eight of the 16 students interpreted the hypotheses "Let a be a factor of b and b be a factor of c" as the statements "$b = ka$ for some integer k and $c = kb$ for some integer k" with the same choice of symbol k in each case.

Five students solved $b = ka$ for a, writing $a = b/k$, a common step for students to do prior to studying number theory or abstract algebra. This suggests that they are unaware that they have changed the domain of discourse from the integers to the rational numbers, and what difference that might make. Changing the emphasis from divisibility to fractions was a complication which caused distractions from completing the proof, although only one committed a fatal error of algebra, writing "$b(x + 1/x) = b(1)$", apparently confusing addition with multiplication.

Seven students correctly used the definition of factor: six who made the error of using the same variable k twice, one who assumed that all multiples were multiples of two. One student changed the item by using "multiple" instead of "factor" throughout. Six students used fractions in their algebra unnecessarily, of whom two committed algebra mistakes. There were two miscellaneous algebra errors, and two students who wrote nothing at all. Three students misinterpreted the problem for its converse.

This discussion of student performance on the pretest establishes a baseline of what to expect them to build upon in a semester course that concentrates on proof-

construction. The students exhibited an understanding of the format that a proof should have, and they also demonstrated an understanding of the starting assumption of direct proofs. But they demonstrated that they did not have secure understandings of the concepts of applying conditional statements, discerning the converse of an implication, and framing the starting assumption for a direct proof. These concepts are the logical underpinnings of proofs. The students' responses on the pretest demonstrated that 13 of them could apply a conditional statement in an appropriate manner (*modus ponens*), but only two of them noticed that the converse was not the same implication (Item 2). The ability to form converse statements is necessary in order to formulate the statement of a contrapositive proposition. This is important so that one can make an early assessment of whether to attempt a direct proof or not. The pretest did not assess proofs which were not direct.

Collateral concepts required for successful proof construction which were not assessed on this pretest are knowing the role of definitions, the use of quantifiers, forming negations of statements, choosing appropriate notation, and using previously proved properties and theorems. The pretest showed that students do not automatically understand that a conditional statement does not apply when its hypotheses are not present. (Item 2, Figure 6, page 57).

Research Question 2: How do Students Develop Skill in Planning and Reporting Proofs?

This question inquires into the process of learning to plan and write proofs. A detailed analysis of six students' encounters with constructing proofs in interview situations provides insight into how this process happens. The interview tasks first provide situations in which the students reveal their concept images and their understandings of them. Then, to see how those thoughts worked out in the construction of proofs, the final exams of the interview participants will be compared to what they revealed in the interviews. For coding, the n=16 students of the entire class were designated S1 through S16, with the first six being the interview participants.

The interviews commenced half way through the semester and continued weekly for five weeks with five of the six students. Student S4 ceased coming to the interview sessions after Interview 3. What follows is a discussion of the six interview students through their interview experiences and their performances on the final exam. The interviews provided opportunities to assess the evolving impact of instruction on proofs, and in particular, students' growing understanding of converse, proof strategies, and the role of definitions within proofs.

The final exam was given in a two hour time block at the conclusion of the semester. The instructor wrote the exam, with input from the researcher. The complete exam is reproduced in Appendix C; individual items will be cited in the following discussions. The final exam included a rich variety of proof contexts that the

students had studied throughout the semester. It included opportunities for assessing understanding of all four proof techniques, the role of definitions in the context of proofs, and balanced treatment of sets, functions, divisibility of integers, and congruences.

Student S1's Performance

Interview 1 commenced with an open-ended refresher task (Figure 11).

1. In your own words, how would you describe the converse of an implication?

Figure 11. Interview 1, Task 1

Student S1 was unable to recall the meaning of the converse of an implication in this first interview. He said that it is like, "saying something is even or saying something is odd would be the converse", which is an example of the concept of negation of a statement. After working on the truth table of logically equivalent statements provided in Task 2 (Figure 12), he was still unable to articulate the distinction between proof by contrapositive and proof by contradiction. For the starting assumption of a proof by contradiction, he wrote, "Q is false", then crossed out "false" and wrote "Q is true." A correct response would be, "Assume P is true and Q is false."

2.
a. Fill in the Truth Table:

P	Q	$P \Rightarrow Q$ Implication	$Q \Rightarrow P$ Converse	$\sim P \Rightarrow \sim Q$ inverse	$\sim Q \Rightarrow \sim P$ contrapositive
T	T				
T	F				
F	T				
F	F				

b. Which of the four items above (the implication, its converse, its inverse, its contrapositive) are logically equivalent?
c. If you were starting a direct proof of $P \Rightarrow Q$, what would the starting assumption be?
If you were starting a proof by contrapositive of $P \Rightarrow Q$, what would the starting assumption be?
If you were starting a proof by contradiction of $P \Rightarrow Q$, what would the starting assumption be?

Figure 12. Interview 1, Task 2

In order to discuss the concepts of converse, inverse, and contrapositive within a familiar context, students were asked to choose one theorem from a list of ten. The complete list is in the Appendix; here the two choices that the six interview students made are reproduced (Figure 13).

After choosing to discuss the result from linear algebra, S1 stumbled at forming the negation of "det(A) is nonzero", and further erred in forming the negation of "A has an inverse matrix", saying "If **A** has an inverse matrix not equal to $\mathbf{A}^{-1}$." Even after the interviewer stated the correct negation, S1 repeated the incorrect statement. When S1 attempted to state the contrapositive of the proposition in Task

3, he was unable to complete the statement. The interviewer prompted him to write the given statement in terms of P and Q in order to form the contrapositive statement $\sim Q \Rightarrow \sim P$, but this, too, was unsuccessful. Three times in different ways, S1 attempted to say the starting assumption for a proof by contrapositive would be $\sim P$.

3.
a. You have seen implications proved in Algebra, Geometry, Trigonometry, Calculus, Linear Algebra, and Number Theory, such as the following. Choose one of them for us to discuss.
* If $b^2 - 4ac > 0$, then the quadratic equation $ax^2 + bx + c = 0$ has two real roots (Quadratic Formula).
* If the n x n matrix **A** has det(**A**) not zero, then **A** has an inverse matrix $\mathbf{A}^{-1}$
b. For the implication that you chose from the list above, state the converse, the inverse, and the contrapositive. Which of those four are true and which are false?
c. How would you start a Proof by Contrapositive of the implication you chose above? Just talk about the beginnings of the proof, and why you would make that beginning.

Figure 13. Interview 1, Task 3

Some of S1's same weaknesses were still present a week later, at Interview 2. The pretest had included proofs to read and proofs to construct involving integers and divisibility; and the textbook and lectures had included examples of such proofs to illustrate the proof techniques of the course. To assess how this had been assimilated, the second interview commenced with a question focused on the definition of an even integer (Figure 14).

1.
a. How would you give a definition of an even integer?
Write it down.
Now consider this implication: **If the integer *n* is a multiple of 4, then *n* is even.**
b. Write down the converse, the inverse, and the contrapositive of the above statement. Then for each of the four implications, tell which of them is true and which of them is false.
c. If you were to prove the implication,

If the integer *n* is a multiple of 4, then *n* is even

what would you write for a starting assumption?
d. What would you try to do next?
e. What would you ultimately want to show in such a proof?

Figure 14. Interview 2, Task 1

S1 gave a correct concept definition of an even integer, including the appropriate quantifier: "x is an even integer if $x = 2a$ for some a in **Z**." Building on this, he correctly stated the converse, inverse, and contrapositive of the implication that "if n is a multiple of four, then n is even." He correctly supplied what the goal of the proof was, but did not offer to write a complete proof as it was not requested.

Pursuing the theme of definitions and their use within proofs, the next task (Figure 15) treated the product of a rational number and an irrational number.

Student S1 was very hesitant defining a rational number, explaining that he thought he knew the textbook definition but that he had never used it to prove anything. He was successful in recalling the definition. When reading the proof through, S1 admitted that he thought it was a direct proof until he saw the keyword, "contradiction" near the end of the proof. Thinking out loud, S1 struggled to

understand why the quantity y is rational when it is not written as a ratio of integers, but managed to think through the contradiction.

2.

a. Write down your definition of a "rational number."

b. Here is a Result and a Proposed Proof. Is the proof correct? Why or why not?

Result: If x is a nonzero fraction and y is an irrational number, then xy is an irrational number.

Proposed Proof:

Assume x is a nonzero fraction and y is an irrational number and xy is a fraction. Then $y = (xy)/x$ is a fraction. This is a contradiction. Therefore, xy is irrational.

c. What was the strategy for this proof - - - Direct Proof? Proof by Contrapositive? Proof by Contradiction?

d. Explain what the "contradiction" is in the proof. Does the contradiction mean that the proof is wrong?

e. Would the proof still be valid if the restriction that x is nonzero were removed? Discuss why.

Figure 15. Interview 2, Task 2

In contrast to his hesitation and confusion on earlier tasks, he immediately responded to Task 3 (Figure 16), declaring that it was easier to work with the contrapositive.

The proposition may in fact be proved by any of the three proof methods of the course. A direct proof by cases, for example, would examine the four possibilities where a and b are positive, negative, or zero. He then proceeded to correctly state the

starting assumption for a proof by contrapositive, and complete the proof without hesitation or mistakes. He concluded the task by correctly stating the converse implication, declared it to be false, and gave justification for his answer.

3.

Result: Let a and b be real numbers. If ab is nonzero, then a is nonzero.

a. Discuss how you would begin a proof of this result.
b. What would be your next step?
c. How would you know when your proof would be done?
d. Do you think now that the result is true?
    State the converse of the Result. Is the converse true?
    State the contrapositive of the Result. Is the contrapositive true?

Figure 16. Interview 2, Task 3

The last task of Interview 2 was a matching question (Figure 17), asking for the student to identify correspondences between starting assumptions and proof strategies.

S1 responded correctly to all parts except the proof by contradiction, "'Assume P and $\sim Q$', that's.... I don't know about that one." It was characteristic of S1 that the compound hypothesis of the proof by contradiction was a hindrance to his recall and application of this proof strategy.

S1's remarks to this point indicate that he does not know the definitions of the terms describing implications although they have been integral to the course to this

date. He does know the definitions of mathematical terms, but admitted that he has not used them in proofs.

4. Suppose you are trying to prove that $P \Rightarrow Q$ for some statements P and Q.
a. Match the following assumptions with the kind of proof it leads to

Assume P	direct proof
Assume Q	proof by contradiction
Assume $\sim P$	proof by contrapositive
Assume $\sim Q$	nonsense (not helpful)
Assume P and $\sim Q$	

b. If you started a direct proof by assuming what you said above, what would be your ultimate goal that you wished to show?
c. If you started a proof by contrapositive by assuming what you said above, what would be your ultimate goal that you wished to show?
d. If you started a proof by contradiction by assuming what you said above, what would be your ultimate goal that you wished to show?

Figure 17. Interview 2, Task 4

The subsequent interviews were more specifically about the role of definitions and the role of previously proved results within proofs. As the same time, the tasks were inviting the students to explain how they think about process of constructing proofs (Figure 18).

Task 1 was to remind the student of the context of sets and functions from the coursework. S1 required hints and coaxing to construct the functions and to calculate the composite functions. For Task 2, he correctly wrote, "f is one-to-one if $f(a) \neq f(b)$, when $a \neq b$." But a few sentences later, he said, "If I was [*sic*] not using

mathematical terms, I would just say each input would have a unique output." This, however, is the definition of function, not of a function being one-to-one. These two tasks were preparation, leading the students to prove the following result using the definition of one-to-one function (Task 3, Figure 19)

1. Let $A = \{r,s,t,u\}$ be a set with four elements.

a. Give an example of a function $g : A \to A$ which is one-to-one, but not onto.
   Give an example of a function $h : A \to A$ which is onto, but not one-to-one.

[The students were instructed to define the functions by drawing arrows in the diagram below.]

g		h	
r	r	r	r
s	s	s	s
t	t	t	t
u	u	u	u

b. Find the composition functions $g \circ h$ and $h \circ g$:

$g \circ h$		$h \circ g$	
r	r	r	r
s	s	s	s
t	t	t	t
u	u	u	u

- -

2. Write down your definition of <u>one-to-one</u> function (same as the word <u>injective</u>.)

- -

Figure 18. Interview 3, Tasks 1,2

He tentatively chose his proof strategy in Part a to be direct proof, and he correctly stated the starting assumption (Part b). He made a perceptive remark soon

afterwards: "I was trying to figure out how I could tie it [producing a next step of the proof, Part e] in with what the definition that I wrote up there. Try to see if that's one-to-one." This shows that he has an idea of the importance of the role of definitions in these proofs, even when his concept definitions are weak.

3. Suppose $f: A \to B$ and $g: B \to C$ are functions.
Now suppose that you are required to prove the

Result: If $g \circ f: A \to C$ is one-to-one, then f is one-to-one.

a. What are your choices for a proof-strategy, and which one would you choose?
b. Given the proof-strategy that you just chose, what would your starting assumption be?
c. Review the definition of one-to-one that you wrote above.
d. How would you start writing the proof?
e. What would your next step be?
f. What are you trying to get to as a goal?
g. What is in the way of your completing the proof now?

Figure 19. Interview 3, Task 3

To probe further into students' thinking about converse and contrapositive implications, the next task (Figure 20) concerned a false implication that students were invited to rewrite as a true statement. Then they were asked to prove the revised statement.

S1 observed without hesitation that the implication as stated is false, and provided a counterexample. It is not clear whether his stated need for an example

expressed the understanding of the role of a counterexample, or a desire to find successful examples from which to extrapolate by inductive reasoning.

4. Prove or Disprove:

   Result: If *a* and *b* are positive real numbers, then $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$

a. How would you start on this?
b. Do you think the equation is true for all a and b?
c. Decide whether to prove or disprove the Result.
   [Interviewer at this point invited the students to state a corrected proposition: If a and b are positive real numbers, then $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$.]
d. Would you like to try a Direct Proof, a Proof by Contrapositive, or a Proof by Contradiction?
e. What would your starting assumption be in such a proof?
f. What would your next step be?
g. What would your goal be for the end of the proof?
h. What is in the way of your completing the proof now?

Figure 20. Interview 3, Task 4

After Task 4, Part c, S1 restated the proposition as an inequality, but was unable to proceed. After being prompted to square both sides of the purported equality, he otherwise completed the remainder of the proof successfully. This confirms the results of the pretest and shows that the inability to complete a direct proof is much more difficult for some students than others.

The role of definitions in proofs surfaced in the remaining interviews as well. Interview 4 began with situations requiring knowledge of the definitions of functions,

surjectivity, and composite functions (Figure 21). These tasks were parallel in form to the first few tasks of Interview 3 (Figures 18-19, pages 72-73).

1. Let $A = \{a,b,c\}$ be a set with three elements, and $B = \{r,s,t,u\}$ be a set with four elements.

a. Give an example of a function $g : A \to B$ which is one-to-one, but not onto.
Give an example of a function $h : B \to A$ which is onto, but not one-to-one.

	g		h	
a	r	r		a
b	s	s		b
c	t	t		c
	u	u		

b. Find the composition functions $g \circ h$ and $h \circ g$:

$g \circ h$ $\qquad\qquad$ $h \circ g$

[Space is provided for students to write the appropriate domains and ranges]

- -

2. How would you define an <u>onto</u> function (same as the word <u>surjective</u>)?

Figure 21. Interview 4, Tasks 1,2

When asked if he knew the definition of $g \circ h$, he said, "Ah.... No." However, he showed by what he wrote that he knew it well. Here it may have been the symbols that prompted him, since he knew what to do once he had written $g(h(a))$.

The next task (Figure 23) is dual to the corresponding task of the preceding interview (Interview 3, Task 3, Figure 19, page 73).

3. Suppose $f: A \to B$ and $g: B \to C$ are functions.
   Now suppose that you are required to prove the
   Result: If $g \circ f: A \to C$ is onto, then g is onto.

a. What are your choices for a proof-strategy, and which one would you choose?
b. Given the proof-strategy that you just chose, what would your starting assumption be?
c. Review the definition of one-to-one that you wrote above.
d. How would you start writing the proof?
e. What would your next step be?
f. What are you trying to get to as a goal?
g. What is in the way of your completing the proof now?

Figure 22. Interview 4, Task 3

He both wrote and said, "every element b in B has a corresponding element x." Here he correctly used a universal quantifier, but neglected to say to which set the element x belonged, and what the word "corresponding" meant in this situation. After repeated questioning, he demonstrated that he possessed the correct concepts, but did not know how to frame it in terms of the notation $g(f(x))$. He could say that every element in C came from a corresponding element from the set A via the function $g \circ f$, but he did not know the definition of $g \circ f$. He understood that to complete the proof he had to show that every element in C came from an element of the set B via the function g, but he could not devise a way to find such elements. If he had observed that if $g(f(x)) = b$ and $f(x)$ is the element of B that corresponds to b via the function g, the proof would have been finished. At the conclusion of the interview, the researcher showed him the rest of the proof.

In order to ascertain how the students were conceiving the process of constructing proofs at this point of the semester, they were asked in Task 4 (Figure 23) to arrange slips of paper with the aspects of the proof process into categories, i.e., prepare a small conceptual map.

At this point of the semester, students will have built some conceptions about the process of constructing a proof. The following task asks them to consider the various concepts involved, all at once.

4. Here are cards with some of the ingredients of proofs that you have seen this semester:

    [Prepared cards with the following items, one to a card. Students were asked to arrange the cards in any kind of linear, branching, or circular pattern that would show how they think of the process of proving.]

    Understand Definitions of Terms
    Choose Strategy
        Direct Proof
        Proof by Contrapositive
        Proof by Contradiction
    Choose Starting Assumption
    Algebra
    Mathematical Background Knowledge
    Number properties (divisibility, primes, factors)
    Prior Theorems from the course
    Check your progress towards the goal of the proof
    Think through a proof plan
    Possibly restate in your own words what it is you are to prove.

    [Prompt: You may write others on this paper.]
    [Prompt: Identify the hypothesis P and conclusion Q]

Figure 23. Interview 4, Task 4

Student S1 arranged the process in the following manner and described his organization orally:

Choose a Proof Strategy	Math Knowledge
Direct Proof	Understanding of Terms
Proof by Contrapositive	Prior Theorems from the course
Proof by Contradiction	Algebra
Think through a proof plan	Number properties

If necessary, Identify the hypothesis (P) and consequent (Q)
Choose Starting Assumption
Possibly restate in your own words what it is you are to prove.
Check your progress against the goal of the proof

His performance in this activity shows that S1 has understood what parameters, terminology, and forms of proofs are. This supports that he knows what the process of proving is about, even though he has difficulty performing individual parts. As he faces subsequent tasks, the changes in his ability to construct proofs can be traced.

The final task of Interview 4 began with directed tasks (Figure 24) concerning inverse functions, but the purpose was to see whether the students understood how to apply previously proved results. They had proved in homework and in interviews results about when a composite function is one-to-one or onto. The textbook had modeled a similar proof for them (Chartrand, et al., 104). Here they were invited to signify how their understanding and vision had expanded.

Student S1 correctly found the inverse function asked for, and calculated the composition of the given function with its inverse. He was not sure why the function given by $y = x$ is one-to-one or onto. Part d then asked a reason why

$$(\ (g \circ h)\ (x) = x \text{ for all } x\ ) \Rightarrow g \text{ is onto}$$

He was unable to provide a reason, although it was the very same result discussed in Task 3 of this same interview.

5.
a. Does the function $g(x) = 2/(x - 1), x > 1$, have an inverse function? Explain why.
b. Find the inverse function.
c. If the inverse function is $h(x)$, then show that
   $g \circ h(x) = x$ for almost all x; and
   $h \circ g(x) = x$ for almost all x.
d. Is $g \circ h$ injective? surjective? Explain why.
e. Is $h \circ g$ injective? surjective? Explain why.
f. Since $g \circ h$ and $h \circ g$ are "nice," what can you conclude about the functions g and h ?

Figure 24. Interview 4, Task 5

The final interview, Interview 5, revealed some persistent habits of thought in the way S1 approached reading proofs. In Task 1 (Figure 25), he looked for the keywords, "assume to the contrary", but not finding them, he could not decide what proof strategy had been employed – direct proof, proof by contrapositive, or proof by contradiction. After talking through the proof out loud, he confidently said that it was a valid, direct proof.

After he was satisfied with the proof, the researcher asked him to rewrite the proof in list form. This he did easily. As an opinion, he expressed a preference for the list form.

His search for keywords rewarded him in Task 2, as well (Figure 26). He identified the provided proof as a proof by contradiction "'Cause [pointing at the

printed proof] 'this is a contradiction'." He declared the proof valid without hesitation.

1. Someone wrote the following proof of a mathematical assertion.

Assume **A** is an *nxn* matrix and $\mathbf{A}^2 = \mathbf{I}$, where **I** is the *nxn* identity matrix. Then $\det(\mathbf{A}^2) = \det(\mathbf{AA}) = \det(\mathbf{A})\det(\mathbf{A}) = \det(\mathbf{I}) = 1$, so $(\det(\mathbf{A}))^2 = 1$. Therefore, $\det(\mathbf{A}) = \pm 1$.

a. What result is established by this proof?
b. What proof strategy was used?
c. What was the starting assumption?
c. Explain what the person was doing in the second sentence of the proof. What do you suppose was their reason for doing these steps?
e. Is the proof valid or not? Why?

Figure 25. Interview 5, Task 1

Student S1 identified the starting assumptions because they were the first step, and he understood the purpose of the algebra steps. He said that they were purposeful, and led to the contradiction as it is stated in the proof. He had no hesitation in proclaiming the proof valid.

In the same way that the previous interview had asked students to arrange slips of paper with the steps in the process of constructing proofs, Task 3 (Figure 27) asked them to repeat that analysis in their own words.

2. Here is another proof of another mathematical assertion.

Assume that a and b are any odd integers, and that $a^2 + b^2$ is a multiple of 4.
Then $a = 2k+1$ for some integer k,
$b = 2m + 1$ for some integer m, and
$a^2 + b^2 = 4n$ for some integer n.
It follows that

$$4n = a^2 + b^2 = (2k+1)^2 + (2m+1)^2 = 4k^2 + 4k + 1 + 4m^2 + 4m + 1$$
$$4n = 4(k^2 + k + m^2 + m) + 2.$$

This may be written

$4(n - k^2 - k - m^2 - m) = 2$, which says that 2 is a multiple of 4.

This is a contradiction. Therefore, $a^2 + b^2$ is not a multiple of 4.

a. What result is established by this proof?
b. What was the proof strategy?
c. What was the starting assumption?
d. Explain what the person was doing in the sentence of the proof which starts with "It follows that...." What do you suppose was the reason for doing these steps?
e. Is the proof valid or not? Why?

Figure 26. Interview 5, Task 2

3. Describe the process of constructing a proof.

Figure 27. Interview 5, Task 3

Although S1 did not provide as much detail as the directed activity with the slips of paper, he said,

> First, I'd look at the statements – what P and Q say. Decide what method that would be the easiest to use. Then, depending on what the

> choice was, I would make the starting assumption, and then I would take those assumptions, and work with them, until I got the desired result, or the conclusion.

He made additional comments about the necessity of using other mathematical knowledge and results when he was asked about ingredients in the first two proofs discussed in this interview.

Task 4 (Figure 28) moved away from giving a complete proof to discuss, and only provided some strong hints to the proofs. The task was to discuss possible proofs that $6^n \equiv 1 \pmod 5$ for all positive integers n. This was approached three ways: two proofs using congruence arithmetic and algebra, and one proof by mathematical induction. The class had studied congruence arithmetic several weeks before, and completed homework and a quiz on the material.

S1 was unable to calculate 6 (mod 5), 36 (mod 5), 216 (mod 5), and 1296 (mod 5) until prompted to think in terms of exchanging pennies for nickels and keeping the remaining pennies. This was another indication of S1's weakness concerning recall of definitions mentioned earlier. The first proof that $6^n \equiv 1 \pmod 5$ involved raising both sides of $6 \equiv 1 \pmod 5$ to the nth power; the second proof involved using the binomial theorem to expand $6^n = (1 + 5)^n$, and observing that each term except the first is divisible by 5. S1 required help seeing that addition mod 5 provides the solution. The prompts are detailed in the following quotation from the interview.

4. Let's find out what we can about the expression 6^n (mod 5), where n is a positive integer.

a. First, calculate some examples:

6^1 (mod 5) = 6 (mod 5) = _____ (mod 5)
6^2 (mod 5) = 36 (mod 5) = _____ (mod 5)
6^3 (mod 5) = 216 (mod 5) = _____ (mod 5)
6^4 (mod 5) = 1296 (mod 5) = _____ (mod 5)

b. Now, calculate 6^n (mod 5) = _____
Can you explain how you obtained your answer?

c. Next, let's look at 6^n in a different way, writing it as $(1 + 5)^n$, and using the binomial expansion from Algebra:

$$(1 + 5)^n = 1 + n(5)^1 + (n(n\text{-}1)/2)(5)^2 + \ldots + (5)^n$$

Then calculate that expression modulo 5.
Can you explain how you obtained your answer?

d. What have we established? Have we proved it?

e. So now you have two direct proofs from algebra which show that $6^n = 1$ (mod 5).
Now prove it by Mathematical Induction.

e. Was your proof of the inductive step a Direct Proof, a Proof by Contrapositive, or a Proof by Contradiction?

g. Did any of the proofs that $6^n = 1$ (mod 5) use the fact that 5 is a prime number?
Can you generalize the result?

h. Which of the above three proofs will work to prove your generalization? Explain.
If you are not sure, write them out in terms of your variable.

Figure 28. Interview 5, Task 4

R: ...What have you established at this point?

S1: That 6^n (mod 5) is equal to,... Well, it'd be kind of obvious. I was going to say that $6^n = (1 + 5)^n$ (mod 5).

R: True. But you simplified that.

S1: Oh, yes. 6^n (mod 5) is equal to that... series?

R: Well, we found that most of the terms were zero, is that right?

S1: Yes.

R: So, on our first step, we did 6^n (mod 5) for n a small positive integer. We kept getting ones. Then with arithmetic mod n, we found what they were right away. With the binomial expansion, we found we could show it, and we got the same answer. So what was the answer?

S1: I can't follow the connections here.

R: ...In (b), for example, with the binomial expansion, you said the answer, but you didn't write it down. What was this equal to?

S1: 6^n (mod 5)

R: After we looked at the individual terms?

S1: Oh, one.

The purpose of the prompts was to direct S1 to think in terms of arithmetic mod 5 instead of divisibility. However, the obstacle to him was actually the ability to state the result in terms of integers mod 5. In the next task, S1 was able to provide the proof by mathematical induction with no prompting. In the proof of the inductive step, he used correct arguments involving divisibility by 5, and the interviewer encouraged him to look back at his completed work and interpret it in terms of arithmetic mod 5, again pointing out the proofs by congruence arithmetic and by the binomial expansion.

The final part on this last interview was about generalization of the result to $(a+1)^n = 1 \bmod a$, but S1 did not respond to prompting. One explicit intent of this item was to direct the students to look back at the several proofs that 6^n (mod 5) = 1, and consider a question about it that invited them to compare them. A second purpose was to invite them to make a generalization of the result. S1 talked about the various

proofs and how they did not use the fact that five is a prime number, but did not see how to change the statement of the result to other numbers. His attempt to do so was, "$6^n \pmod{x} = 1$, where x is any prime number."

In summary, the interview sequence confirmed what the pretest exposed about S1's strengths and weaknesses. He knew the form that a proof should take and could construct elementary direct proofs. He understood the meaning of truth tables and knew the symbolic form of the starting assumptions for proofs by contrapositive and proofs by contradiction. However, he displayed misunderstandings of the converse of a conditional statement that signified that his performance was not consistent on constructing proofs that are not direct. He rarely used existential or universal quantifiers, and yet he understood that to disprove a universal assertion only required one counterexample. He could construct simple proofs by mathematical induction without assistance. He admitted that definitions are important for understanding the statements of propositions as well as for proceeding within a proof; however, he did not know the key definitions from this course in mathematical proof. When S1 was constructing a proof, he focused on how to proceed from one step to the next, neglecting to include in his planning the consideration of related theorems and the goal of the proof.

The effects of this course in learning about proof were measured by the cumulative final exam, which provided opportunities for students to show that they have mastered the course material and can exercise higher-order thinking skills such as synthesis and classification to address that material. The exam also is where the

students can show that they have overcome weaknesses displayed in their work during the semester. For the purposes of this research, the final exam was consulted for support and confirmation of the tentative findings, as well as any possible findings in a contradictory direction.

S1's Performance on the Final Exam. The direct proofs that S1 wrote on the final exam were adequate in their starting assumptions and structures, but his continued failure to use quantifiers to indicate specifically what he meant left them incomplete. Item 1 (Figure 29), for example, was to show that a given linear polynomial was one-to-one and onto.

1. A function $f : \mathbf{R} \to \mathbf{R}$ is defined by $f(x) = 7x - 1$.
(a) Prove that f is one-to-one.
(b) Prove that f is onto.

Figure 29. Final Exam, Item 1

In his proof that $f(x) = 7x - 1$ is onto, S1 first wrote the objective of the proof, which made it appear to be a step in the proof. The algebra steps were correct in the body of the proof, but he eschewed quantifiers and did not provide a statement of what he had been proved. His neglect of quantifiers at this point is a step in the proof process that he has not taken.

Item 2 required a proof about set equality. Student S1 had said in Interview 1 that proofs about sets were new to him

> S1: ...It just came out of left field at us, I think. It seemed like before, everything was kind of systematic, like we knew where we're going and what we were doing. Whereas we came out in Chapter Four with the 'Proofs of Sets' [*sic*, 'Proofs Involving Sets'], it was just kind of find your own way.... There wasn't a set way to doing it.

2. Let A, B, and C be sets. Prove that $(A \cap B) \times C = (A \times C) \cap (B \times C)$.

Figure 30. Final Exam, Item 2

S1's proof (Item 2) that the intersection of sets distributes over the cartesian product was a direct proof of a biconditional statement, and was written well. He began with a statement of what he intended to show first, namely, "First we show that $(A \cap B) \times C \subset (A \times C) \cap (B \times C)$." Next, he chose appropriate and helpful notation: "Let $(x,y) \in (A \cap B) \times C$, so $x \in (A \cap B)$ and $y \in C$." After making the correct inference from $x \in (A \cap B)$, he did make a slight notational error by saying, "Therefore $x \in (A \times C)$" instead of $(x,y) \in (A \times C)$. Besides that, he correctly inferred the conclusion of the first half of his proof. The second half of proving the reverse subset inclusion proceeded similarly, with the same notational error mentioned before. There was no final statement of what had been proved.

Item 3 (Figure 31) required the consideration of two cases, and S1 neglected the second case, leaving his proof incomplete. His initial statements showed marked

improvement over his confusion shown in the interviews concerning how to decide to pursue a direct proof by cases. He had claimed that he could read a proof involving cases, but had difficulty deciding for himself when and how to do that. On this item, he wrote: "Assume to the contrary that 1000 can be written as the sum of three integers, an even number of which are even. Then, $x + y + z = 1000$ where $x = 2a + 1$, $y = 2b$, $z = 2c$ for some $a, b, c \in \mathbf{Z}$." He then continued to a successful conclusion that this assumption that one of the terms was odd and the other two were even leads to a contradiction. He did not make a statement of what the attainment of a contradiction portended. There was no mention of the other possible case, that of 1000 being the sum of three integers, none of which is even. This may have been a simple oversight, or he may have thought that anyone would know that the sum of three odd integers is odd, or he may have overlooked this case because he did not consider the zero number of terms to be an even number of terms.

3. Prove that 1000 cannot be written as the sum of three integers, an even number of which are even.

Figure 31. Final Exam, Item 3

Item 4 (Figure 32) required direct proofs of the three properties of an equivalence relation given in terms of integers mod 6.

4. A relation R is defined on $\mathbf{Z}$ by xRy if $x^2 = y^2$ (mod 6)
(a) Prove that R is an equivalence relation.
(b) Determine the distinct equivalence classes.

Figure 32. Final Exam, Item 4

S1 wrote the proofs in clear and correct style, although he used the divisibility characterization of equality mod 6 instead of the arithmetic properties that had been proved in the class and textbook. His list of the equivalence classes of this equivalence relation was incorrect, but it seemed that he was distracted by the condition "mod 6" and forgot that the equivalence relation was defined on the set of all integers.

Item 5 (Figure 33) was another proof of a biconditional, this time from elementary number theory. S1 wrote a valid proof.

5. Prove the following:
Result: Let $x \in \mathbf{Z}$. Then x^3 is even if and only if $5x^2$ is even.

Figure 33. Final Exam, Item 5

To assist the student in organizing their work on this problem, the instructor wrote the two lemmas in Figure 34 on the blackboard; they were not printed on the Exam. All students wrote the lemmas down, and all except four students successfully proved them. S1 constructed valid proofs by contrapositive for each of the lemmas,

and then wrote a valid proof of the required result. In effect, the converses of the lemmas were proved within the proof of the main result. This was the same procedure that most of the students followed.

Lemma 1: If x^3 is even then x is even. Lemma 2: If $5x^2$ is even then x is even.

Figure 34. Final Exam, Item 5 Lemmas

Items 6 (Figure 35), 7 (Figure 36), and 8 (Figure 37), were varieties of mathematical induction proofs, from which the induction assumption is of interest. The last two of these problems designated which form of mathematical induction should be used, but the first problem left the choice of method to the students' discernment.

6. Prove that $2^n > n^2 + n$ for every integer $n \geq 5$

Figure 35. Final Exam, Item 6

Item 6, for example, specified ordinary mathematical induction to be used to prove that $2^n > n^2 + n$ for all positive integers n. S1 started the proof in an appropriate way, including a correct induction assumption and a statement of intent about what

needed to be proved in the induction step. But she was then unable to write the inequalities of the induction step to complete the proof.

7. A sequence $x_1, x_2, x_3, \ldots$ is defined recursively by $x_1 = 1, x_2 = 4$, and
$$x_n = -x_{n-1} + 2x_{n-2} + 6n - 7 \text{ for all } n \geq 3.$$
Use the strong form of induction to prove that $x_n = n^2$ for all positive integers n.

Figure 36. Final Exam, Item 7

A proof by complete induction was required in Item 7, but here S1 merely verified two initial calculations, and did not state the induction assumption. Item 8 required a proof by the method of minimum counterexample, and S1 started to produce such a proof. He began to state the necessary assumption, "Then there must exists [*sic*] an m such that 6 divides $(m^3 + 5m)$", but left out the essential word "smallest" to which the eventual contradiction builds. As an affirming observation, notice that his statement did include a correct existential quantifier.

8. Use the method of minimum counterexample to prove that $6 \mid (n^3 + 5n)$ for every positive integer n.
[Recall that $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$].

Figure 37. Final Exam, Item 8

There were five "Prove or disprove" parts to item 9 (Figure 38). The parts were very different from one another in character, and required judgement about whether to seek a counterexample, or whether to assay building a proof by one of the three proof strategies.

9. Prove or disprove.

(a) Let $x,y \in \mathbf{Z}$. If $6x + 7y$ is even, then y is even.

(b) In $\mathbf{Z}_6$, if $[a] + [b] = [5]$, then $a + b = 5$.

(c) Every even integer is the sum of two unequal odd integers.

(d) For every two sets A and B, $(A \cup B) - B = A$.

(e) If a set S of real numbers contains a least element, then S is well-ordered.

Figure 38. Final Exam, Item 9

S1 wrote his response to Item 9, Part a as a proof by contrapositive, assuming that y was odd; and then considered cases according to whether x was even or odd. In each case, he used correct algebra to conclude that 6x + 7y would have to be odd. Item 9, Part b involved the understanding of integers mod 6 versus integers, and S1 left this blank. Although one cannot infer much from a problem that a student leaves blank, recall that S1 also misunderstood mod n equivalence classes on Item 4 of this final exam.

Item 9, Part c proposed that "Every even integer is the sum of two unequal odd integers." S1 wrote, "The statement is false. Zero is an even integer and it must be the sum of two equal integers." The statement, however, is true, although there are at least two reasons that may explain S1's reasoning. If he is thinking that $0 = n - n$ and the two equal integers he refers to are n and n, then he is neglecting that the problem specified the sum, and not difference, of these numbers. A second reason he may have given his response, is that he may have restricted the problem to nonnegative integers, and thought that $0 = n + m$ will only have the solution $n = m = 0$, in which case he would be neglecting the condition that the integers must be non-equal.

Item 9, Part d required a proof of a set equality. S1 gave a correct proof of one inclusion, but he merely asserted the reverse. In fact, the reverse inclusion is not true. The form of his proof was correct, with appropriate first steps and logical statements throughout, except for the one misstep. In spite of the error, this suggests that his misgivings concerning his understanding of proofs about sets previously quoted (page 87) has been replaced by a confident writing of such a proof.

Item 9, Part e assessed students' understandings of the definition of well-ordered sets and required them to construct a counterexample to a given statement. Student S1 wrote nothing.

Item 10 (Figure 39) was a matching question about proof strategies, somewhat like questions in the textbook and in Interview 2, Task 4. Eight identifications were required, and S1 erred only on the two, including the one that concerned proof by contradiction. This weakness was noted earlier in the discussion of the interviews. He

was unable to say what the starting assumption was for a proof by contradiction in Interview 1, Tasks 2, 3. He did not recognize the starting assumption within a given proof by contradiction in Interview 2, Task 2. He was unable to construct a proof by contradiction in Interview 3, Part c of Task 4.

10. A proof of $P \Rightarrow Q$ is to be given. If the first step of the proof is given below, then which of the following is true:
    (1) a direct proof is being used.
    (2) a proof by contrapositive is being used.
    (3) a proof by contradiction is being used.
    (4) an error has been made.

(a) Assume that Q is true _____.
(b) Assume that P is true _____.
(c) Assume that $P \Rightarrow Q$ is true _____.
(d) Assume that Q is false _____.
(e) Assume that P is false _____.
(f) Assume that $P \Rightarrow Q$ is false _____.
(g) Assume that P is true and Q is false _____.
(h) Assume that P is false and Q is true _____.

Figure 39. Final Exam, Item 10

Significance of S1's Performance. Student S1 talked candidly about the material of the course with which he was confident, as well as material about which he was confused. He never overcame his failure to learn the starting assumptions for proof by contradiction, and he continued to have difficulty with arithmetic with congruences. Justifying set equalities requires proving a biconditional statement, and he handled the form well, in spite of an error within the proof on the final exam. S1

showed that he understood elementary direct proofs well, and on the final exam he began to use quantifiers. His earlier weakness of not knowing essential definitions was overcome in the final exam, with the exception of knowledge of equivalence classes.

Student S2's Performance

Student S2 was a diligent mathematics student with a distinctive communication style. In interview situations, she was usually silent, and then would write complete, correct work. She was able to articulate her thinking whenever asked, and was cognizant of her own preliminary planning of proofs. In the first interview, she did not hesitate in defining the converse, inverse, and contrapositive of an implication (Task 1, Figure 11, page 65). She confidently identified the three proof strategies (Task 2, Figure 12, page 66); and she correctly gave the converse, contrapositive, and inverse statements of the quadratic formula statement (Task 3, Figure 13, page 67). Her work establishes a very high mark for her initial understandings in the course. This made it interesting to measure her progress.

The second interview included more concentration on the strategy of proof by contradiction, and she treated it without difficulty. She defined rational numbers correctly (Task 2, Figure 15, page 69) and her discussion of the proof by contradiction that the product of a nonzero rational number and an irrational number is irrational demonstrated that she understood the proof. She gave a complete explanation of why it was a proof by contradiction, and provided an explanation of the contradiction within the proof.

Task 3 of Interview 2 (Figure 16, page 70) requested a proof that

ab nonzero $\Rightarrow$ *a is nonzero for real numbers a and b*

This may be accomplished by any of the three proof strategies. S2 wrote a complete proof by contradiction, without hesitation. She showed that she understood the form of the starting assumption, (P and $\sim Q$) in this proof, and how to use this assumption to advance the proof. This continues to indicate a high level of her understanding of proof processes.

S2 made a slight error in defining a function from the set $\{r,s,t,u\}$ to itself in Task 1 of Interview 3 (Figure 18, page 72). Her error was made when concentrating on constructing a function which was one-to-one, and she temporarily overlooked that she had made a change that made her diagram no longer a function. But she proceeded to define injective functions correctly, and produced a good outline of a proof that if $g \circ f$ is injective, then f is injective (Interview 3, Task 3, Figure 19, page 73). However, she hesitated after writing her proof, and the interviewer inquired why. She indicated that the definition of injective, as applied to the composite function $g \circ f$, was not secure in her mind. The interviewer prompted her to rewrite her proof in paragraph form to see if that would change her perspective. She did so without hesitation, and was satisfied.

When considering Item 4 of Interview 3 (Figure 20, page 74) concerning whether $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$, she immediately provided a counterexample, and explained why that proved that the purported equality was not true. After restating the conjecture as an inequality, she declared she would construct a proof by

contrapositive, after which she squared both sides and pursued the algebra to the completion of the proof without difficulty. She showed a good understanding of proof and algebra here.

In contrast to the initial tasks of Interview 3, in Interview 4 (Tasks 1,2; Figure 21, page 75) she did not reveal any hesitation or errors in defining discrete functions (Task 1) or in the definition of surjective functions (Task 2). She confidently wrote a complete and valid proof that

$$g \circ f \text{ is surjective} \Rightarrow g \text{ is surjective}$$

(Task 3, Figure 22, page 76), without the hesitation that she had in the previous interview's corresponding question. Again, she rewrote her proof in paragraph form without any difficulty.

Task 4 required her to arrange slips of paper with generic steps in constructing a proof (Figure 23, page 77). She arranged them to her satisfaction, and then read them back to the interviewer. The arrangement showed that she comprehended the notion of how a proof is organized, and confirmed that the proofs she had been discussing and constructing in the interviews was consonant with her proof conception.

1. Identify the hypothesis P and consequent Q
2. Possibly restate in your own words what it is you are to prove
3. Understand definitions of terms
4. Think through a proof plan
5. Choose a proof strategy
6. Choose the starting assumption
7. Mathematical Background Knowledge
    7a. Algebra
    7b. Prior theorems from the course
    7c. Math Knowledge

7d. Number properties

8. Check your progress towards the goal of the proof

Task 5 on inverse functions (Figure 24, page 79) demonstrated that she could do the necessary algebra to find the inverse function. She was reluctant, however, to initiate a sentence about inverse functions, preferring to out-wait the interviewer until he asked a question.

On the final interview, student S2 discussed (Interview 5, Task 1, Figure 25, page 80) the direct proof that

$$A^2 = I \Rightarrow \det(A) = \pm 1$$

without any problem; and she rewrote the proof in list form easily.

She discussed the given proof (Item 2, Figure 26, page 81) that

If a, b are odd integers, then $a^2 + b^2$ is not divisible by 4

with confidence. Item 3 (Figure 27, page 81) asked the students to describe the process of constructing a proof. S2 said she did not know what was asked. She was presumably expecting a more directive task. The intention of the task was to give the students a free response opportunity to repeat the task from the previous interview when they were supplied with slips of paper with the suggested ingredients for constructing proofs and asked to arrange them.

Task 4 (Figure 28, page 83) showed that S2 had the same weakness for inductive reasoning that the other interview students demonstrated - - - when asked how she had calculated that $6^n = (1 \bmod 5)$, she admitted it was because of the pattern of 1's in the previous part of the question. The shift from generating a conjecture in Part a to generating a proof in Part b was not made clear. The intent was to lead

students to see that a proof by congruence arithmetic was available to provide a direct proof. The proof using the binomial expansion, she did quickly and expertly. The proof by mathematical induction she did correctly and without hesitation, using the definition of 6 mod 5 in terms of divisibility, instead of using arithmetic mod 5.

She did not see the generalization to $(a+1)^n = 1 \bmod a$. Perhaps the wording of the task was sending a conflicting signal.

Throughout the interviews, student S2 consistently showed her competence with the concepts of constructing proofs that she had begun with the pretest. Although she did not verbalize her thinking processes unless asked, she wrote complete and correct mathematical sentences at every opportunity. She required very little prompting to advance her proofs; in fact, the prompting was simply to invite her to share what she was thinking. She quickly corrected the few errors that she made, and the only task left incomplete was the invitation to generalize $6^n = 1 \pmod 5$ (Interview 5, Task 4, Figure 28, page 83).

The Research Question was How do students develop skill in planning and reporting proofs? In S2's case, she was producing valid proofs with no errors from the very commencement of the course. She demonstrated command of the material and assimilated new concepts easily and diligently. The ways that she developed these skills appear to be disciplined study, and attitudes of inquisitiveness and persistence toward mathematics. She approached unfamiliar situations with the same confidence that she faced familiar ones.

S2's Performance on the Final Exam. Items 1-4 of S2's final exam were not delivered with the rest of her exam, and the loss of the data is regrettable. For Item 5 (Figure 33, page 89), she constructed valid proofs by contrapositive to show that

If x is an integer, then x^3 is even $\Leftrightarrow$ x is even $\Leftrightarrow$ $5x^2$ is even

The three proofs by induction (Figures 35-37, page 90-91) were done without errors, with correct statements of the induction assumptions with the appropriate quantifiers in each proof. None of the other students in the class constructed complete and valid proofs of these three proofs involving mathematical induction methods.

The five proofs requested in Item 9 (Figure 38, page 92) were not all successfully completed, but are so different in quality from one another that comments on each are worthwhile. Like several choices S2 made earlier in the final exam, Part a of Item 9 (Figure 38, page 92) was also written as a proof by contrapositive, and was valid. Part b of Item 9 (Figure 38, page 92) was to prove or disprove that

If $[a] + [b] = [5]$ in the integers mod 6, then $a + b = 5$ in the integers

The result is false, but S2 claimed to have proved it using a proof by contrapositive. Her error was to deduce from "$[a]$ is an element of $\mathbf{Z}_6$," that "a is an integer [in $\mathbf{Z}$] between 0 and 5." This is an error of coordination of the two number systems in proximity, and is common among students encountering congruences for the first time. The proof that S2 offered for Part c of Item 9 (every even integer is the sum of two unequal odd integers; Figure 38, page 92) was almost correct: She wrote $2x = (2x+1) + (-1)$, which demonstrates that every even integer is the sum of two odd integers, but the terms are unequal only if $x \neq -1$. She did not notice this exception.

Item 9, Part d was a proposed set equality (Figure 38, page 92), which S2 disposed of with a correct counterexample. Her example used finite sets, but no diagram, as was common with this class of students. Item 9, Part e concerned well-ordered sets (Figure 38, page 92), and she gave an appropriate counterexample.

The final item, Item 10 (Figure 39, page 94), of the final exam was fill-in-the-blank about the starting assumptions of the principal proof strategies, and S2 did all of them without error.

Significance of S2's Performance. Student S2 was using correct terms and definitions from the pretest on, and could construct proofs with any of the three strategies. She was equally at ease with proofs of elementary number theory, sets, and sequences. Her only weakness with content was with congruences, and her errors were of the sort that she should recognize and overcome the next time she deals with these concepts.

In all of her work, S2 showed that she is diligent and studious, quick to see mathematical connections, and expert at constructing proofs at the level of this transition course. As with her performance in the interview situations, S2 committed almost no errors. Her attention to details and quantifiers made her work stand out from the rest of the class.

Student S3's Performance

S3 was a capable, but not exceptional mathematics student. He was open and communicative about his perceived mathematical strengths and weaknesses. The specific weaknesses were clearly identified at the start by the pretest and the first two interviews This made it relatively easy to observe how he progressed in his understandings of proof construction through the semester.

Student S3 demonstrated by his commentary and his work that he was having difficulty with proofs by contradiction. S3 was unsure of the meaning of converse in Interview 1 (Task 1, Figure 11, page 65), and expressed insecurity about the validity of conditional statements (Task 2, Figure 12, page 66).

Student S3 chose the quadratic formula statement in Task 3 (Figure 13, page 67). He correctly stated its converse, contrapositive, and inverse statements, and, upon consideration, decided that all four statements were true. However, "I'm not positive. It's my assumption," he said.

In Interview 2, he gave a correct definition of even integer (Task 1, Figure 14, page 68), a correct verbal statement of the contrapositive of the given statement, and a valid proof of it. He admitted that he had forgotten "what the converse is", but determined it by eliminating the contrapositive statement and the inverse statement, which he knew.

When presented with a proof by contradiction in Interview 2 (Task 2, Figure 15, page 69), S3's initial reaction was that the proof had a contradiction and that

meant that the proof was wrong. He later expanded on this to correctly observe, "The contradiction is – You know that something's false... It doesn't mean the proof is wrong." However, the very next task in Interview 2 was to construct a proof that *ab* nonzero implies *a* is nonzero (Task 3, Figure 16, page 70). After several sincere attempts to construct a direct proof by cases, and several prompts to direct his thinking towards an alternate proof strategy, he was unable to formulate the contrapositive of the implication.

Task 4 (Figure 17, page 71) asked the student to match the starting assumptions to the proof strategies. Student S3 did not hesitate in correctly matching the three principal strategies to their starting assumptions. He saw for himself that to "Assume Q" would be an appropriate starting assumption for proving the converse of $P \Rightarrow Q$, but not for proving $P \Rightarrow Q$, because it would mean assuming what was to be proved. He was not as confident about why "Assume $\sim P$" was "not helpful." After questioning, he saw that "Assuming $\sim P$" led to an immediate contradiction.

Interview 3 began with tasks to construct a function between finite sets with specified properties that it should be injective but not surjective, and another function with the opposite attributes (Figure 15, page 69). He encountered difficulty balancing the separate demands of the definitions for function, for injective, and for surjective, but was successful in showing that he did understand the definitions. S3 confidently calculated the two composite functions of Task 1, Parts a and b. He also used the existential quantifier correctly when defining injective function in Task 2. He was able to articulate what the goal of the proof was in Task 3 (Figure 19, page 73), although

he did not understand how to use the definition of injective function applied to the function $g \circ f$. However, he did produce a diagram which convinced him that the result could be true; and observed that this was not a proof, just a plausibility argument.

S3 was much more confident with the Interview 3 task to prove or disprove $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$ (Task 4, Figure 20, page 74). He immediately assumed universal quantifiers (for all real numbers a and b,...) and declared "Well, I would start by saying that the statement is false.... I would give an example, $\sqrt{1} + \sqrt{3}$ is not equal to $\sqrt{4}$." He was not able to restate the proposition without assistance. The researcher said, "You've proved that the equation is not true for all real numbers. But are there some numbers for which it is true?" and "Would using the word "solve" make any difference?" and "Solve for a." With this prompting, he was able to restate the proposition as "If $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$, then either a=0 or b=0", and prove it successfully, although he said it was a "proof by contradiction? [*sic*]" His proof, which was correct and valid, was a proof by contrapositive.

Interview 4 repeated opening tasks similar to the previous interview. Task 1 (Figure 21, page 75) asked the student for two functions between finite sets with specified injective or surjective properties. Student S3 defined functions that assigned the first point of the domain to the first point of the range, the second point of the domain to the second points of the range, etc; and then attended to adjusting these functions to have the desired properties. His choices of these assignments made his calculations of the two composite functions extremely easy. They also were valuable to him in reviewing the concepts of defining functions, recognizing injective functions

and surjective functions, and calculating composite functions. For Task 2 (Figure 21, page 75), he gave a correct verbal definition of surjective function, but was dissatisfied that he could not give a symbolic definition. This was partly because he had no symbolism for the existential quantifier, and so a definition with symbols would still have more words than symbols in it. S3's desire to learn was hindered by his incomplete understandings of definitions. It shows that he has some awareness of his learning style, but not enough to direct him to learn definitions carefully.

Task 3 invited S3 to construct a proof that if the composite of two functions is onto, then the first function is onto (Task 3, Figure 22, page 76). He was able to correctly state the beginning assumption for a direct proof, and discussed the context enough to show that he understood it heuristically, but the notation and symbols defeated his attempts to construct a proof. At the researcher's prompting, he began to rewrite his proof in paragraph form. When he stopped, and was asked, "How far are you from completing the proof?" He responded "pretty far," even though he had but one step to go. At his final attempt, he produced a circular proof that g surjective implies g is surjective. He was not connecting the hypotheses of the statement to the situation in the proof, for example, by understanding how to apply the definition of onto function to $g \circ f$.

In his organization of the ingredients of a proof (Task 4, Figure 23, page 77), S3 revealed the importance he gives to advance planning, and to the role of correlative mathematical knowledge. He identified "Prior theorems from the course" perhaps

sooner than they would occur in actual proof making. His list of features in proof making was:

1. Understand definitions of terms
2. Think through a proof plan
3. Possibly restate in your own words what it is you are to prove
4. Prior theorems from the course
5. Choose a proof strategy

6a. Choose the starting assumption

6b. Identify the hypothesis P and consequent Q

7. Mathematical Background Knowledge
   7a. Number properties
   7b. Math Knowledge
   7c. Algebra
8. Check your progress towards the goal of the proof

The task concerning inverse functions (Task 5, Figure 24, page 79) required some algebra, but principally addressed the notion of how to calculate the inverse function. S3 did this easily, and also computed the required composite functions without difficulty. He observed that the composite of the given function with its inverse function was the identity function, and similarly for the other composite; and he related those facts to the earlier results from Task 3 and Interview 2, Task 3 successfully.

S3 was alert to the need for prior knowledge about determinants in the two-line proof that initiated Interview 5 (Task 1, Figure 25, page 80). He validated the steps of the proof, and rewrote the proof in a list form as requested in the interview protocol. When writing the proof in this format, he asked questions and verified the details of each step. This process was therefore part of his acceptance of the proof. Comparing both forms of the proof, he preferred the list form.

The proof given for discussion in Task 2 (Figure 26, page 81) was in list form, and student S3 said he would much prefer it in this form than in paragraph form. The proof strategy was proof by contradiction, which he identified by the keyword, "contradiction" near the end of the proof. He correctly identified the starting assumption, and pointed out that the expression, "assume to the contrary that $a^2 + b^2 = 4n$ for some integer n" might have been used.

Task 3 (Figure 27, page 81) repeated Task 4 of Interview 4, but without providing the ingredients for the student. S3 wrote for his description of the process of constructing a proof, "Assume that P is true. We want to show from this that Q is true. We use algebra and/or math identities to show that Q has to be true if P is assumed to [be true]." By presuming the proof strategy of direct proof, he omitted the ingredient of choosing the proof technique.

Task 4 invited the student to discuss three proofs that $6^n = 1 \pmod 5$ for all positive integers n (Figure 28, page 83). Two of the proofs involved algebra, and one involved mathematical induction. S3 judged the result to be true, based on inductive examples. Calculating modulo 5 was not familiar to him, and he repeatedly expressed the desire to calculate with specific values for n. The binomial expansion was written out in the interview protocol, but S3 claimed he was unfamiliar with it. He only completed that particular direct proof with assistance. He was able to confidently construct the proof by mathematical induction without help, including a correct statement of the induction assumption.

The interviews have revealed S3's strengths and substantial weaknesses. The term "potentially good mathematics student" describes S3, for he is familiar with the terms and processes of constructing proofs and interpreting contradictions; however, his schema are disconnected and incomplete. He can understand and construct simple direct proofs, but not proofs that involve compound hypotheses or contradictions. He uses quantifiers in his definitions, but not always the correct ones. S3's failure to identify the converse of a statement hindered his progress in the first two interviews especially, but he did not find that an obstacle thereafter. He was willing to explore heuristic arguments to test his confidence in propositions, but this did not always help him to construct proofs. Although his view of how to proceed when in the middle of a proof was often very local, he was one of only two interview students to see that the result about the composite functions from tasks 3 of both Interview 3 and Interview 4 could be applied in Interview 4, Task 5.

The obstacle that S3 found in proofs by contrapositive and proofs by contradiction appear to be grounded in the difficulty of coordinating two logical statements simultaneously, and not with difficulty with negations of statements. He relied on keywords to identify the mode of a given proof, and he began to see that keywords are not always supplied for him in published mathematical proofs. These were steps in his self-awareness and reflect how he changed in his understandings of proof construction.

A strength of S3's thinking that should be encouraged is his readiness to construct conjectures and examples to test the truth of propositions. He generated

simple examples on his own, as well as devising numbers and functions when requested in the interviews. Although his examples were sometimes the simplest possible, as in the functions between finite sets he made up in the first items of Interviews 3,4, he responded immediately to suggestions that he could make less trivial examples. This trait was useful for him when deciding whether to seek a counterexample to a conjecture. Indeed, he said that the chapter on "Prove or Disprove" [Chapter Six] was his favorite.

S3's Performance on the Final Exam. On item 1 of the final exam (Figure 29, page 86), S3 proved that a given function is one-to-one and onto. His proof included appropriate quantifiers, although the use of mathematical language was not fluent. His proof of the distributive property of the intersection and cartesian product of sets was correct and valid (Item 2, Figure 30, page 87). S3 provided a complete proof in Item 3 (Figure 31, page 88), even treating the case overlooked by many of his classmates.

The equivalence relation proof (Item 4, Figure 32, page 89), however, revealed some weaknesses in his understanding of implications. Several are shown in his proof of the reflexive property for the defined relation,

$$xRy \text{ if } x^2 \equiv y^2 \pmod 6$$

> Proof: First, we must show that R is reflexive. Assume that xRx, thus $x^2 \equiv x^2 \pmod 6$. This implies [*sic*] that $x^2 \mid (6 + x^2)$; by subtracting x^2 from both sides we obtain $0 \mid 6$. Since zero can be divided by anything, R is reflexive.

The errors were: (1) assuming what was to be proved, (2) interpreting $x^2 \equiv x^2$ (mod 6) to mean $x^2 \mid (6 + x^2)$ instead of $6 \mid (x^2 - x^2)$, (3) subtracting from a statement of divisibility is not valid, and (4) saying $0 \mid 6$ does not say that 6 divides 0. In the context of this transition course, these errors indicate (1) failure to understand direct proofs, (2) failure to understand the definition of congruence, (3) failure to understand divisibility properties of integers, (4) failure to use the symbol for divisibility properly, and (5) failure to understand divisibility properties of zero. Although these errors are fatal to this particular proof, they do not by themselves demonstrate that this student has failed to appropriate the course concepts entirely. They do show, however, that the monitoring of his understanding of these concepts by the conventional means of homework and quizzes did not accomplish their desired purposes. Since he was able to successfully construct direct proofs elsewhere on the final exam, it appears that only his first error is related to the concepts of proof. The other errors are failures to respect the concept definitions of divisibility and congruence. His attempt to prove symmetry of the relation R was invalid because of the same incorrect definition of congruence. He stated the transitivity property, but did not attempt to prove it. It appears that he could not foresee where the incorrect congruence definition would lead in such a proof, but it may be that he did not have time to pursue this proof. Compared to his earlier work, he has overcome his incomplete knowledge of the definition of equivalence relation at this point. He knows what kinds of algebra steps would be productive. It is only his incorrect definition of divisibility that kept him from advancing his proof.

The next item on the final exam was a proposition from number theory (Item 5, Figures 33-34, page 89-90). Student S3 proved each of the two lemmas as one-way, conditional statements; in each situation using a proof by contrapositive. His proofs were models of well-written proofs, for they included a statement of purpose, a precise assumption which was appropriate to a proof by contrapositive, and complete and accurate algebra. His proof of the main result was started properly, and proceeded well. However, it was continued on the back of the page, and that was not available in the copy provided.

The final exam included three proofs by induction, one by ordinary mathematical induction, one by complete induction, and one by the method of minimum counterexample. S3 had no difficulty writing a correct proof by mathematical induction (Figure 35, page 90), but his proof included two weaknesses: he used the universal qualifier "all" instead of the existential "some"; and he did not complete the algebra steps at the end of the proof of the inductive step. This would be judged an incorrect proof that can be easily corrected. The proof by complete induction (Figure 36, page 91) concerned a recursively-defined sequence. S3 had again produced the form of the proof, but the proof of the inductive step was not finished. He did not understand what was to be proved, in the sense of proving a direct formula for the recursively-defined sequence. The proof by the method of minimum counterexample (Figure 37, page 91) was not done according to the directions. S3 started a proof by complete induction, but did not advance beyond the induction assumption. His performance on the interviews showed weakness in

concept definitions of various concepts. Here they are serious obstacles to his proof making.

Item 9 (Figure 38, page 92) on the final exam included five situations to prove or disprove. S3 scratched out his preliminary attempts at a proof for item 9, Part a, which was suggestive of the ideas needed to construct the proof. He wrote, "see back" to indicate that another attempt was on the back of the page, but this information was not on the copy from which this analysis is done. For Item 9, Part b, he made a clear statement of his answer, but he was wrong. As with the interview opportunities to show his understanding of congruence arithmetic, he shows important misunderstandings. This is one topic that he has not learned in this course.

On Item 9, Part c, "Prove or disprove: Every even integer is the sum of two unequal odd integers", S3 replaced the universal quantifier with an existential quantifier, and claimed that the result was false because 2 = 1+1 and the odd numbers 1 are not unequal. This shows a misunderstanding of what was to be proved.

To show that the proposed equality of sets in Part d of Item 9 was incorrect, S3 provided a counterexample with a finite set. His counterexample was correct, but he calculated the difference of the two sets incorrectly. He did not do Part e of Item 9, writing only the incorrect assessment, "true,...."

Item 10 (Figure 39, page 94) assessed recognition of the proof strategies, and S3 did well on those, even scoring his first success with matching the proof by contradiction with its starting assumption.

Significance of S3's Performance. In summary, S3 began with strengths in recognizing the proof strategies, even though there were difficulties understanding proofs by contradiction. He was able to construct direct proofs, until definitions and using prior theorems required more coordination of knowledge than just listing or calculating. These problems were not overcome in the final exam. As with the interview opportunities to show understanding of congruence arithmetic, he reveals important misunderstandings. This is one topic that he has not learned in this course. He did attempt to use quantifiers where needed, but often used them incorrectly even on the final exam.

Student S4's Performance

S4 was the only student who did not complete all five interviews. She was discouraged at her performance on Interviews 1-3, and did not return for the last two interviews. Her comments are valuable, however, for the insight they offer, since she was an "A" student who had significant difficulty with definitions and mathematical connections. The comparison with her final Exam will show how her learning progressed in its understandings of proof conceptions.

The first interview with S4 was not conducted until the twelfth week of the semester, due to communication difficulties. The first question of the first interview (Task 1, Figure 11, page 65) was, "Tell me about your conception of the converse of an implication," to which she responded, "I would say P does not imply Q." This shows a lack of recall of a definition that has been part of the course since the first day

of the course. She was able to complete the truth tables (Interview 1, Task 2, Figure 12, page 66) without difficulty, and constructed columns for $\sim P$ and for $\sim Q$ on her own initiative. However, on Task 2, Part e, she could not start a proof by contradiction, although she said that she knew that $\sim Q$ was involved.

For Task 3 (Figure 13, page 67), she chose to discuss the implication "If $\mathbf{A}^{-1}$ exists, then det(**A**) is nonzero." She was able to correctly state the converse, inverse, and contrapositive of the implication, and said that the converse, inverse, and the original implication are all true, in distinction to the contrapositive implication, which she claimed was false. This is inconsistent with her earlier correct observation that the original statement is logically equivalent to the contrapositive. In Task 3, Part c, she described how to start a proof by contrapositive, without hesitation or difficulty.

Student S4 defined even integer correctly, but without quantifiers in Interview 2, Task 1 (Figure 14, page 68); and proceeded to state correctly the converse, inverse, and contrapositive of the implication "If n is a multiple of 4, then n is even." She talked out loud about how to start and complete a direct proof of this result, making comments that showed that she had good understandings of direct proofs. Task 2 (Figure 15, page 69) caused two difficulties for her. The definition of a rational number mistakenly was qualified to have numerator and denominator real numbers instead of being restricted to integers. The other difficulty was identifying the starting assumption of a proof by contradiction. She was treating the compound assumption (P and $\sim Q$) as a single item, and could not relate it to the conclusion of the proof. Upon being prompted to draw ovals around the hypothesis P and the conclusion Q,

she still did not observe that the negation of Q was one of the starting assumptions. Task 2, Part e asked if the proof would still hold if the expression "x is nonzero" were changed to "x is a real number", and she replied that, "If x is zero, it'd be easier", which did not answer the question.

Another proof involving contradiction (Task 3, Figure 16, page 70) asked for a judgement whether "ab nonzero $\Rightarrow$ a is nonzero" were a true statement. She immediately said, "Assume to the contrary, that a=0." But then she was confused about the form of the proof: "Like $\sim P \Rightarrow \sim Q$, could you do that?" She correctly stated the converse. She correctly stated the contrapositive, but then claimed, "I don't know if that's the contrapositive, but I know what I wrote down is true!" She had written the correct contrapositive proposition: $a = 0$ implies $ab = 0$.

Her confusion about the starting assumption of a proof by contradiction was confirmed by the matching question (Interview 2, Task 4, Figure 17, page 71), when she observed, " '(P and $\sim Q$)', that's 'not helpful' ." After she was prompted to expand the statement to, "Assume P and, to the contrary, $\sim Q$", then she identified it as the starting assumption of a proof by contradiction. However, she still could not describe what the final stage of such a proof would involve, namely a contradiction.

Interview 3 concerned functions and the property of being one-to-one (Tasks 1-3, Figures 18-19, pages 72-73). S4 was confused by the distinction between a function being one-to-one, and the function being onto. Furthermore, she did not know the definition of the composite function $g \circ f$. When asked to define for a function the property of being one-to-one, she responded the "horizontal line test."

When asked to expand on her statement, she wrote, "There can be only one corresponding letter in the range to one letter in the domain." This response has some appropriate words, but the concept she defined is that of function, not of injective. She continued, "I would say $g \circ f : A \rightarrow C$ is one-to-one and try to prove f is one-to-one." Her proposal would be for a direct proof, but the next question in the protocol asks, "What eventually would you be wanting to show?" S4 replied, "Prove any contradiction, probably that f is one-to-one." She may have been conflating two statements here, and not meaning that they were synonymous. The first part of her answer is a response to the question, "What is the goal of a proof by contradiction?"; and the second part of her answer is a response to the question, "What is the goal of this particular proof?" The interview protocol pursued this, asking, "What is in the way of your completing the proof?" She responded, "Knowledge – lack of." In particular, she clearly saw that a lack of knowledge of the definitions of the one-to-one property of functions and the application of this definition to the function $g \circ f$ were debilitating her efforts to construct a proof involving these concepts. She was unable to complete the proof that

$$g \circ f \textit{ is one-to-one} \Rightarrow f \textit{ is one-to-one.}$$

Task 4 (Figure 20, page 74) presented another proof which was not a direct proof. She immediately declared that $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$ is false, and provided an appropriate counterexample (let $a = 1$ and $b = 1$). She approached the restated problem $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$ by suggesting that she could square both sides. No prompting was necessary. She completed the algebra with one error which did not

affect the remainder of the proof: The coefficient "2" was omitted when squaring the binomial. Task 4, Part g asked about the goal of the proof, and she said, "Well, I know.... It would be [the] problem $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$." However, she was surprised to learn that the proof was complete.

Student S4 was clearly disappointed in her exposed lack of understanding about definitions and about proofs by contradiction. She did not return for the remaining Interviews 4, 5. She did not return messages requesting her response.

S4's performance on the interviews indicates that the growth of understanding constructing proofs is not always ideal. Since student S4 was bright and articulate, quick to state her opinions, and openly friendly, the interview sessions promised to be valuable for assessing her mathematical thinking. From the very first question, though, it was revealed that her mode of studying was to learn definitions and theorems in context, struggle through them on homework, and then finalize her concept definitions when studying for tests and quizzes. She had not mastered the initial concepts concerning logical statements and implications, did not know how to choose among the three proof strategies of the course, and did not know the details of the definitions of mathematical properties. She did not use quantifiers when specifying variables, relying on her mental images to remember what the domains of her variables were.

When faced with a result to prove, she knew that there was a choice of proof strategy to be made, but she did not seem to know how to narrow the choices to a feasible set of options. She was open to starting abstract proofs without waiting to construct examples, which was both a strength and a weakness. It was a strength

because she did not waste time before starting to formulate a plan for constructing a proof, although she was unable to complete her proofs due to weaknesses in understanding how to apply mathematical definitions and concepts. Her approach was a weakness, however, in that she did not formulate examples to help her understand what it was she was to prove, and why it should be true.

S4's Performance on the Final Exam. S4 constructed direct proofs that a given affine function was one-to-one and onto with correct manipulations of symbols, but no quantifiers. Others use the same sort of personal abbreviations that shows that they are not paying attention to the details of the definitions. Her set theory proof that intersection distributes over cartesian product (Item 2, Figure 30, page 87) demonstrated choices of representation that propelled the proof easily. Her writing was clear and concise. Similarly, in Item 3 (Figure 31, page 88), she wrote a clear and complete direct proof by cases that had no omissions.

The proof that a given relation is an equivalence relation was well written as a direct proof using the divisibility definition of congruence (Item 4, Figure 32, page 89). However, she omitted the proof of the part that the relation is symmetric. She wrote that symmetry had to be shown, and she wrote the appropriate starting step, but then did not write the two steps to complete the proof. As with other instances when students leave work unwritten, it is not clear whether this was the last problem she worked on and whether she ran out of time. She failed to find the equivalence classes for this equivalence relation, also, merely writing [0], which may also be an indication of running out of time.

For Item 5 (Figure 33, page 89), S4 constructed flawless proofs by contrapositive to show that

If x is an integer, then x^3 *is even* $\Leftrightarrow$ *x is even* $\Leftrightarrow$ $5x^2$ *is even*

Proofs by contrapositive were the strategies of choice, and they were properly done.

Proofs by mathematical induction were not developed by S4 in the Interviews because she stopped early, but on the final exam there were the three opportunities to construct proofs using three methods. S4 devised a direct proof by mathematical induction that $2^n > n^2 + n$ for all positive integers n (Figure 34, page 90). Her statement of the induction assumption was just right, and she completed the induction step with inequalities done without erasures from start to finish. The complete induction proof (Item 7, Figure 35, page 90) was not done, past the correct statement of the induction assumption. Again, the drawback of written assessments is that there is no opportunity to inquire about the reasons this was left at that point. In Item 8 (Figure 36, page 91), she started the proof by minimum counterexample appropriately, but made an error of forming the negation of a statement: The negation of "6 divides $(n^3 + 5n)$ for every positive integer n" is not "6 does not divide $(n^3 + 5n)$ for every [*sic*] positive integer n." She did verify the first few values of $n = 1, 2, 3, 4, 5$; and she defined the minimum integer for which 6 does not divide $(n^3 + 5n)$. But she failed to use the proofs assumption to pursue what happens for the integer preceding that minimum integer.

Item 9 (Figure 38, page 92) provided five items to prove or disprove. S4 gave a direct proof for Item 9, Part a concerning even integers, with a clear statement of her

conclusion. In Item 9, Part b, she misunderstood the definition of the equivalence classes [*a*] and [*b*] mod 6, saying, "In $\mathbf{Z}_6$, *a* and *b* have limited possibilities. They can only be 0, 1, 2, 3, 4, or 5." This hampered her ability to answer the problem correctly.

Item 9, Part c is a striking instance of the failure to distinguish the converse of an implication from the implication itself. Instead of proving "Every even integer is the sum of two unequal odd integers", she proved that the sum of two unequal odd integers is even, which is a true, but obvious statement.

The proposed false set equality in Item 9, Part d was correctly treated, for she gave an appropriate counterexample with finite sets. For the question on well-ordering, she said, "the statement is false", and gave an appropriate counterexample, but without explanation.

The recall items about proof strategies in Item 10 (Figure 39, page 94) were correctly answered, even the ones concerning proofs by contradiction.

<u>Significance of S4's Performance.</u> S4 said at the first interview that she was embarrassed not to know about converses and other basic terms needed for doing proofs, and emphasized that she was getting "A's" on her work. She is a diligent student, but did not finalize her concept images until studying for quizzes and exams. At the Interview, she said, "Oh, I should have studied for this!" She had no difficulty at all on the final exam with proofs in elementary number theory, congruences, and functions; but she did have some concept images that needed strengthening, particularly regarding set theory, equivalence classes, and quantifiers. She did produce

some quantifiers on the final exam, but not in all instances where they should have been employed. She described her way of learning in terms of how she did the homework problems, with many strikeovers and false starts and conjectures about symbols. Her undisciplined learning style obstructed her genuine interest in mathematics. By the end of the course she had purged her direct proofs of their deficiencies, but the construction of proofs that required more synthesis of concepts were still at a rudimentary level.

Describing her progress in learning to construct proofs must note that she was unable to complete any early tasks. Even recalling definitions appeared to be *terra incognito* for her. S4 was not a failing student, however. Her *forte* was written assessments. She had her way of last minute studying that assisted her to produce correct concept images and viable beginnings to proofs.

Student S5's Performance

S5 was confident from the beginning with the ideas of converse and contrapositive of a mathematical statement (Interview 1, Task 1; Figure 11, page 65). His response to the task with truth tables was immediate and correct in Task 2 (Figure 11, page 65). He did not need to construct auxiliary columns for $\sim P$ and for $\sim Q$, as the researcher was prepared to suggest. For Task 3 (Figure 13, page 67), he chose the quadratic formula statement, and immediately gave accurate statements of its converse, inverse, and contrapositive.

S5 correctly proved on Interview 2, Task 1 (Figure 14, page 68) that if n is a positive integer and n is a multiple of four, then n is even, although his choices of notation and algebra were clumsy. However, when asked to pick an irrational number in preparation for Task 2 (Figure 15, page 69), he did not hesitate: "$\sqrt{-2}$", he said, although this is not a real number. On this item, his knowledge of proof by contradiction was faulty until he was prompted that it sometimes included the phrase, "assume to the contrary." At these keywords, he affirmed that now he knew what to do. But he did not. He still concluded that the starting assumption was "assume to the contrary that P is false," even though he had said twice before that this was impossible as a proof strategy. In discussing the validation of the proof by contradiction in this item, he only identified the proof strategy because of the keyword "contradiction" near the end. He identified the contradiction in this proof correctly, but then said that since there was a contradiction, the proof is wrong. He had been incorrect so often in the last two tasks, that the researcher did not pursue the remaining tasks of this interview (Tasks 3,4, Figures 17-18, pages 71-72) for fear of discouraging him.

When S5 constructed functions in Task 1 of the third interview (Figure 18, page 72), he made the function g the identity function, so that it was a trivial example that did not satisfy the conditions asked for. The second function he also made as simple as possible. He was unable to discuss the composite functions (Task 2, Figure 18, page 72), because he said he was familiar with functions given by formulas in algebra, trigonometry, and calculus classes; but not with functions on finite sets. He

said he was very unhappy with abstract functions not given by formulas, and that the notation and symbols for functions were confusing him. The notation $f: A \to B$ for functions had been used in the textbook and in class. Task 3 (Figure 19, page 73) requested a proof that

$$g \circ f \text{ is one-to-one } \Rightarrow f \text{ is one-to-one}$$

S5 never got past the obstacle that the notation $f: A \to B$ caused him. Several times he said that if he only had a formula for $f(x)$, then he could proceed. He was unable to even choose a proof strategy until forced to choose. He never understood that he had to use the definition of one-to-one function as applied to the function $g \circ f$.

On the following problem (Task 4, Figure 20, page 74), to prove or disprove $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$, S5 immediately said that it was untrue. When questioned further, he said that he had always been told that it was false. He was comfortable with this appeal to authority. When prompted to supply numbers for an example, he gave the correct example with $a = 4$, $b = 4$. Actually, any nonzero numbers will suffice. Upon the researcher's suggestion to square both sides of the proposed equality, he easily completed his proof and identified it correctly as a proof by contrapositive.

Interview 4 began with tasks that were parallel to those of the previous interview. Tasks 1 and 2 (Figure 21, page 75) asked for a function from a three-point set to a four-point set, and another function from a four-point set to a three-point set; one function to be injective but not surjective, and the other with the opposite properties. The first function defined by S5 satisfied the desired conditions, but the second did not. When asked to define surjective function, he wrote, "Given a function

f, find a and b such that $f(a) = b$." He read this over twice, and commented that it did not seem right, but he thought that it had the right idea. What he has said, however, simply says that the function f is defined at one element, and the omitted universal quantifier for b renders his definition useless for the following proof.

He had no sense of what proof strategy to use, and when requested to choose one of the three strategies used in this course, he picked proof by contrapositive. He was unable to make any progress on the proof for there were two obstructions. He did not know how to prove the result that g was onto, and he did not know how to use the hypothesis that $g \circ f$ was onto. Both of these can be traced to his inability to give a correct definition of surjective function in the previous question. The coordination of the hypothesis of the proof and the abstract definition of surjective function defeated him. He could not decipher what it meant for $g \circ f$ to be onto.

S5 ranked the ingredients for proofs (Task 4, Figure 23, page 77) as follows:

1. Mathematical Background Knowledge
2. Identify the hypothesis P and consequent Q
3. Understand definitions of terms
4. Choose a proof strategy
5. Choose the starting assumption
6. Prior theorems from the course
7. Algebra, Number properties
8. Check your progress towards the goal of the proof
9. Draw a box to signify the end of the proof, as in the textbook.

He omitted two items:

1. Think through a proof plan
2. Possibly restate in your own words what it is you are to prove

This choice of ordering the ingredients showed that his view of the activity of proving is closely aligned with his view of solving equations. The first three topics he chose were his own weaknesses, as the interviews have shown.

The last task of Interview 4 (Task 4, Figure 24, page 79) explored inverse functions and the connection they have with the properties of being injective or surjective. S5 was much more comfortable with functions given by formulas, as provided here, than with the abstract functions of Tasks 1 and 2 of the same interview. He did not have obstacles to calculating the inverse function, or calculating the composite functions that were requested. When he saw that the composite of the given function with its inverse function was the identity function, he immediately saw that this composite function was injective and surjective. He was not using general properties of the identity function, however, just the specific algebraic properties of the equation $(g \circ h)(x) = x$ that he had proved. In thinking this way, he was not relating the current problem situation to the previous results from Task 3 of both Interviews 3 and 4.

Interview 5 commenced with Task 1 showing a two-sentence direct proof of a property of matrices (Figure 28, page 83). S5 recognized the proof as a direct proof, observed that the property $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$ was used in the proof, and appeared to have a good concept of the flow of the proof. But then he described the inference from the hypothesis $\mathbf{A}^2 = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix:

> And since det(**A**) is just **I** and the determinant of another **A** is just **I**, it's the same thing as det($\mathbf{I}^2$), or just det(**I**), which would be one. So they make a general assumption that det(**A**) is equal to 1, therefore, det(**A**) = ± 1.

This is, unfortunately, nonsense. The researcher then asked S5 whether the given proof was valid, and S5 said, "No." When asked what was wrong with it, he continued, "It seems to me that the determinant was always positive." So, in this case, an incorrect factual error hindered his evaluation of the validity of a proof. After re-writing the proof in list form, he was more willing to accept the proof as valid, but he still made the same error at the same place as before, writing,

"det(**A**) det(**A**) = (1)(-1), so det(**A**) = 1, [or] -1."

The proof by contradiction of Task 2 (Figure 25, page 80) was given in full, and S5's analysis of the proof concluded that it was a proof by contradiction because it ended with a contradiction. He was concerned about verifying the algebra steps in the middle, and said that he had not seen algebra arranged that way before, presumably referring to the fact that the more complicated side of the last equation was on the left instead of the right.

S5's list of the ingredients of a proof for Task 3 (Figure 29, page 86) was straightforward:

> Given a result, we have a statement.
>
> [Orally, he added, "And then we have to decide which method we are going to use to prove it. Either we're going to do it with using Direct proof, Contrapositive, Indirect, proof by Cases, by Lemma, any – sets, all of – any which way.]

Proof: Assume...

We wish to show that..., it follows that...

Work the proof.

Algebra, substitution

We have the something [Researcher infers " a contradiction"], or conclusion.

Unless a contradiction, we end with a [S5 drew a small square to signify the end of a proof.]

This puts into his own words what was observed about how he views the construction of a proof: the form of the proof is important to him, as in the notice of a statement of what to prove, having the word "Proof" underlined, having the word "Assume" as the first word of the proof, and having the box at the end of the proof. He showed by this list that he does understand the flow and process of making a proof, although he has not studied the process as an object.

The last interview task was to discuss three ways to prove that $6^n = 1 \pmod 5$ for positive integers n (Task 4, Figure 28, page 83). S5 requested assistance with the definition of "mod 5" in order to calculate the first few powers of 6 (mod 5). He then attempted to prove that $6^n = 1 \pmod 5$ for all positive integers n by an inductive argument using divisibility, but was unsuccessful. The proof using the binomial expansion required help from the interviewer to complete the calculation modulo 5, for he said he was unfamiliar with the binomial expansion. The proof by mathematical induction was a struggle: "This is not my strong order [*sic*] right now, mathematical induction." With guidance, he found the starting step, formed the induction assumption, and he completed the proof of the induction step. The induction step of

this particular proof may be completed by arithmetic modulo 5 or by using the divisibility definition of congruence. He did not know how to proceed, and at the researcher's suggestion, he used the fact $6^1 \equiv 1 \pmod 5$ from the beginning of Task 4 and the induction assumption $6^n \equiv 1 \pmod 5$ to conclude that $6^{n+1} \equiv 1 \bmod 5$. It could be argued that he was looking at the form, and not thinking in terms of previously proved results about arithmetic modulo 5.

In summary, S5 was cooperative and communicative, revealing substantial weaknesses in his use of algebra that hindered his progress. He could construct simple direct proofs, but encountered obstacles whenever technical definitions were involved. He did not express difficulties with the concepts of converse and contrapositive of a mathematical statement, but was not successful with proofs involving contradictions. Analyzing proofs with contradictions, he had difficulty identifying the starting assumptions and the forms of the proofs. He looked for keywords such as "contradiction" to aid him in deciding what proof strategy had been used. He could construct simple proofs by mathematical induction, but not proofs that required complete induction or the method of minimum counterexample.

His progress was quite small through the course of the semester. He continued to make the mistakes at the end of the semester that he had made at the beginning. He never did produce a valid proof by contradiction. On portions of two items of the final exam, he presented correct arguments leading to contradictions. His way of learning to construct proofs appeared to be to treat each new proof as a new start, with no antecedents and no connection with other proofs that he had seen or produced. This

very local view of the role of proof is also part of the way he views mathematics, namely, as a collection of facts that other people discover and tell him.

S5's Performance on the Final Exam. Student S5 wrote a direct proof of Final Exam Item 1 (Figure 29, page 86), which was to show that the function $f: \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 7x - 1$ was one-to-one. His first line was, "We wish to show that $f(a) = f(b)$, and finally $a = b$." But what he should have been saying was, "We wish to show that whenever $f(a) = f(b)$ then $a = b$." His failure to include quantifiers impeded both his understanding and his communication. What he actually proceeded to show was that if $7a$-$1 = 7b$-1, then $a = b$; and he included a statement of what he had proved. The second part of the problem was to show that this same function was onto. Again, he wrote a direct proof, but the lack of quantifiers obscured whether he understood what he wrote. His exam read, "We wish to show that $f(a) = b$, for some a, b in $\mathbf{Z}$ [*sic*, the integers]. Choose an a, then $7a$-$1 = b$." Then he solved for a, and concluded that f is onto. His algebra and conclusion are correct, but the initial assumption merely asserts that the function is defined on at least one point.

Item 2 was the proof that set intersection distributes over cartesian product of sets (Figure 30, page 87), and S5 had no difficulty producing a direct proof of the biconditional statement for set equality.

Item 3 (Figure 31, page 88) was to show that 1000 cannot be written as the sum of a three integers, an even number of which are odd. S5 constructed a valid proof by contradiction for the case where he assumed 1000 was the sum of three odd

numbers; but he omitted the case when 1000 is the sum of one odd integer and two even integers.

The problems that S5 had with arithmetic modulo n on the interviews re-surfaced on item 4 (Figure 32, page 89), where it was required to show that $x \sim y$ if and only if $x^2 = y^2$ mod 6 is an equivalence relation on the set of integers. All three parts (reflexive, symmetric, and transitive properties) began with a statement of what was to be shown; unfortunately, he then deduced the next step from this early statement of the conclusion. This may be an occurrence of the habit of looking up an answer to a problem, and writing it down before proceeding to the problem solution. This trait was not as obvious in the interviews, because many of the proofs there were often presented for discussion, and not generated by the student. The proofs that the relation was symmetric and transitive had this same feature; however, he recovered from this misstep immediately, and concluded with satisfactory proofs. The proof of reflexivity, on the other hand, was invalid:

> First, we wish to show that R is reflexive, or aRa. Therefore, $a^2 \equiv a^2$ (mod 6), and $0 \equiv 0$ (mod 6). Hence $6 \mid 0$, which is true and $\therefore$ R is reflexive.

The errors here are similar to the statements made in Interview 5, Item 4: (1) The inference "Therefore" is invalid, (2) $a^2 = a^2$ (mod 6) is true, but irrelevant, (3) $6 \mid 0$ is true, but irrelevant. He has essentially proved that if R is reflexive, then a true statement follows. This is a logical error that affected all of his work in this course in mathematical proof. As with most of the entire class, he could not determine the

equivalence classes. He wrote, "[1] = {1, 3, 6}, [2] = {2,4}, [5] = {5}." This might indicate that he was thinking that the relation R was defined on $\mathbf{Z}_6$, the integers mod 6, instead of the set $\mathbf{Z}$ of all integers. It is also possible that he memorized the text book's example of an equivalence relation on $A = \{1, 2, 3, 4, 5, 6)$ for which his answer is three of the six equivalence classes there (Chartrand, et al., 1999, 83).

Item 5 (Figure 33, page 89) required a proof that

If x is an integer, then x^3 *is even* $\Leftrightarrow$ *x is even* $\Leftrightarrow 5x^2$ *is even*

S5 constructed a correct proof of a different result. He proved these two results:

1. If x is an even integer, then $5x^2$ is even.
2. If x is an odd integer, then $5x^2$ is odd.

He may have been led to think of these because of the Lemmas that were suggested by the professor (Figure 34, page 90), but in fact he did not prove the required result.

The three proofs by mathematical induction were treated differently by S5. The proof that

$$2^n > n^2 + n \text{ for all positive integers } n \geq 5$$

(Item 6, Figure 35, page 90) was written in the appropriate form, but the induction assumption was written as a fact, not an assumption: "Observe that for n=5, $2^5 > 5^2 + 5$, we have that $2^k > k^2 + n$ [*sic*, $k^2 + k$], for some $k \geq 5$." He then proceeded to give a correct proof of the inductive step. The second proof, a result requiring a proof by complete induction (Item 7, Figure 36, page 91), he did not seriously attempt, simply writing one initial case and an apology that he had not studied complete induction proofs for this final exam. He did not do the proof by the method

of minimum counterexample (Item 8, Figure 37, page 91), offering the beginning of a proof by ordinary mathematical induction, but not doing the proof.

In Item 9, Part a (Figure 38, page 92), student S5 constructed a complete proof by contrapositive, lacking only a statement of the conclusion. He gave an appropriate counterexample to the mod 6 statement in Item 9, Part b, showing a confident understanding of the definition of congruence mod 6. In Item 9, Part c, he proved that the sum of two even integers is an even integer, which is neither the proposition nor the converse of the proposition under discussion. The false set equality proposed in Item 9, Part d was accepted by S5, who wrote a correct direct proof of one set inclusion, but made an error on the reverse inclusion proof. He deduced from $x \in A \cup B$ that $x \in A$ and that $x \notin B$. If he had drawn a Venn diagram, he might have avoided this mistake. He answered the question about well-ordered sets (Item 9, Part e) accurately and with an appropriate counterexample. This showed that he had paid attention to the subtlety in the definition of a well-ordered set.

Item 10 (Figure 39, page 94) required the recall of the proof strategies and the starting assumptions thereof. S5 only made one error, saying that an appropriate first step in proving $P \Rightarrow Q$ would be to "Assume $P \Rightarrow Q$." This is not the sort of mistake that one would make in the actual process of constructing proofs.

<u>Significance of S5's Performance.</u> The written assessments and the interviews gave consistent information about the ways that student S5 plans and constructs proofs. Two related difficulties that he has are that he is careless about details, and he

has weak concept images. When he omitted quantifiers in his definitions and proofs, he was not just committing errors, he was crippling his own opportunity to understand the mathematics he was doing. His regrettable habit of writing his conclusion first and then making deductions from it rendered his proofs incomplete at best, and invalid certainly. He was competent when constructing simple direct proofs and simple proofs by mathematical induction. He did not understand the structure of proofs by contradiction.

S5 could do the mathematics required in this transition course, but he lacked some self-monitoring skills to enable him to detect his own errors. His inattention to quantifiers and the structure of proofs by contradiction interfered with his ability to complete proofs in this transition course. S5 depends upon mnemonics to prompt his recollection of definitions and proof strategies. Certain difficulties were persistent from the pretest and early interviews through to the final exam. He did produce valid simple direct proofs on the final exam, which demonstrated some progress in his proof-constructing ability. However, in the proofs that required coordination of several concepts and quantifiers, he did not improve.

Student S6's Performance

S6 was a mathematics student who wanted to be successful at constructing proofs. She knew that there were some mathematical concepts that were difficult for her, but she was determined to learn from her mistakes. Although her pretest indicated that she was intuitive about assessing the validity of mathematical

statements, she was weak on providing reasons for her judgements. She did not hesitate to share what she was thinking, even when what she said was irrelevant to the task at hand. S6 maintained an informative and outgoing nature throughout the semester.

S6 expressed uncertainty about her statement of the converse of an implication (Interview 1, Task 1, Figure 11, page 65), even though it was correct. She wrote the truth tables (Task 2; Figure 12, page 66) with little trouble, and she did not need to write auxiliary columns for the negations $\sim P$ and for $\sim Q$. She chose the Pythagorean Theorem for discussion in Task 3 (Figure 13, page 67), and correctly stated its converse, inverse, and contrapositive.

In Interview 2, Task 1 (Figure 14, page 68), student S6 correctly defined even integer. She confidently stated the converse, inverse, and contrapositive of the given statement, and she correctly identified which were true and which were false.

Task 2 of Interview 2 (Figure 15, page 69) began to show some weaknesses. She identified the proof as a proof by contradiction primarily because of the keywords near the end of the proof, "This is a contradiction." She admitted that it was this keyword that induced her to say that it was a proof by contradiction. In other words, the uses of the word "contradiction" are confusing her. There is the strategy of "proof by contradiction", and there is the circumstance of a statement and its negation being simultaneously asserted. To follow up, the protocol asked for specification of the contradiction. She replied, " What is the contradiction in that proof? I don't know." She did understand that the proof was valid, and not invalidated by the contradiction.

Task 3 (Figure 16, page 70) required a proof that

If a and b are real numbers, then ab nonzero $\Rightarrow$ *a nonzero.*

The proof did not come automatically to her; but with some prompting which asked for the converse and contrapositive statements, she successfully completed the proof by contrapositive. She treated the matching of proof strategies to starting assumptions (Task 4, Figure 17, page 71) by first identifying the two items that were "not helpful", and she explained why one of them was not helpful. Then she correctly matched the three principal proof strategies with their corresponding starting assumptions.

Interview 3, Task 1 (Figure 18, page 72) invited her to construct a function from a four-point set to itself. Student S6 defined her first function to be the identity function, and the second to leave two points fixed and switch the other two points. When calculating the composite functions, she did it backwards, as some algebraists do. When she was asked for a definition of injective function (Task 2, Figure 18, page 72), she gave an appropriate verbal definition: The function $f: A \rightarrow A$ is one-to-one if and only if "all the elements map to different points." Her definition became an obstacle in the proof required in Task 3 (Task 3, Figure 19, page 73). She chose to attempt a proof by contradiction, but could not construct a proof. Her inability to do this was principally due to not knowing how to show that a function is one-to-one with regard to the definition that she had given. A secondary obstacle was her failure to relate the hypothesis to the definition of injective function, namely, understanding what it meant for $g \circ f$ to be injective. She suggested that a diagram might help her, and drew ovals for the sets A, B, and C, and drew arrows for the functions f, g, and $g \circ f$.

This enabled her to convince herself that the proposition was true, but she could not find a way to write a proof.

When S6 read Task 4 (Figure 20, page 74), she immediately said it was false, and provided two counterexamples. After restating the proposition as an inequality, she proceeded to prove it with a proof by contrapositive. When asked what proof strategy she had used, she was unable to distinguish between proof by contrapositive and proof by contradiction. At this point, S6 has some serious deficiencies about understanding proof strategies. To be optimistic, one might say she has room for growth in her understandings of how to construct proofs.

Interview 4 began with tasks that paralleled the opening tasks of Interview 3. The first task (Figure 21, page 75) asks for one function from a three-point set to a four-point set, and another from a four-point set to a three-point set. She constructed suitable functions, and found their composites as requested in Task 2 (Figure 22, page 76). As in the previous interview, when she was asked for the definition of a property of functions, she gave a verbal definition: A function is surjective if and only if "all range elements are hit by something from the domain." This is correct, but not ready to use in a proof unless one is proficient with sets and elements. In the proof for Task 3 (Figure 22, page 76), she discussed choosing "the right strategy", not understanding that there are alternative correct strategies for proofs, and any one might be successful. She began a proof by contradiction, then changed to direct proof after discussion and a diagram. Then she produced a "contradiction" which was not there. She wrote a paragraph proof by contrapositive. But the proof was not valid.

She arranged the cards with the ingredients of making proofs (Task 4, Figure 23, page 77) in the following way, observing that some of the items could go in more than one place:

1a. Algebra
1b. Number properties
1c. Math Knowledge
2. Prior theorems from the course
3a. Identify the hypothesis *P* and consequent *Q*
3b. Understand definitions of terms
4. Possibly restate in your own words what it is you are to prove
5. Choose a proof strategy
6. Choose the starting assumption
7. Mathematical Background Knowledge
    7a. Prior theorems from the course
    7b. Algebra
    7c. Number properties
    7d. Math Knowledge
8. Check your progress towards the goal of the proof

S6 reveals in this list how she thinks about the activity of constructing proofs, and it reflects what she has done in the series of interviews. She may list knowing definitions and prior results first because she does not have the confidence of these concepts.

When S6 dealt with functions given by formulas (Task 5, Figure 24, page 79), she solved for the inverse of a given function confidently. She was also able to calculate the composite of the original function with its inverse function and *vice versa*. She affirmed that the composite functions obtained were bijective, and concluded from Task 3 of Interviews 3 and 4 that the constituent functions were also bijective.

Interview 5, Task 1 (Figure 25, page 80), presented a direct proof of the linear algebra result that the determinant of an idempotent matrix is ± 1. Student S6 said the

result that was proved was, "The determinant of **A** is ±1", instead of stating a conditional statement. She verified the steps of the proof, and decided it was a direct proof since it did not have any stated contradiction. She was invited to rewrite the proof in list form, which she did easily. She said that both the paragraph form and the list form were understandable to her, but the list form seemed easier to read for this particular proof.

The proof by contradiction presented in Task 2 (Figure 26, page 81), was identified as a proof by contradiction because of the keyword "contradiction." S6 gave correct explanations for the step of the proof, and the starting assumption.

Task 3 (Figure 27, page 81) asked for the student to generate a list of ingredients for making proofs, similar to the task in Interview 4. S6 gave an abbreviated list that omitted many essential topics:

1. Apply the previous math knowledge to see how you are going to prove it.
2. You want to be sure that you are making progress towards the goal.
3. And then you put a little box when you are done. [the end-of-proof symbol]

Her list is consistent with her proof construction behavior, which emphasizes preliminary knowledge before decisions about a proof strategy. It was not clear why she omitted such decisions, or why she omitted the use of previous theorems.

The proof that $6^n = 1 \pmod 5$ (Task 4, Figure 28, page 83) involved discussion of two direct algebra proofs and a proof by mathematical induction. S6 relied on inductive reasoning from the four examples in Task 4, Part a, thereby producing a conjecture, not a calculation. She required prompting to simplify the binomial

expansion (mod 5). The prompting invited her to think of exchanging the number of pennies for nickels and noting the remainder in pennies. She did not use arithmetic modulo 5 in the early parts of this task. The student's proof by mathematical induction was complete and correct, following the form of such proofs. In the proof of the inductive step, for the first time in the interview, she used arithmetic modulo 5 without appealing to the divisibility of integers. Furthermore, she saw that the proofs did not require the fact that five is a prime number. In response to the invitation to generalize the result, she made an attempt, and obtained $6^n = 1$ (mod 13). She was using 13 because it is a different prime number. She did not see how to replace the six by anything related to the 13. The intent of the question was to suggest that $(1 + a)^n = 1$ (mod a) for any integers a and n.

In summary, S6 was proficient with algebra and constructing simple direct proofs, but was not secure in her knowledge of the three proof strategies and their uses. She had difficulty identifying the proof strategy of a given proof throughout the interview sequence. The only proof by contradiction that she tried to construct was not valid. S6 was in control of understanding the concepts of converse, inverse, and contrapositive of a mathematical statement, but had not learned how these were used in the choice of proof strategies.

S6 worked successfully with functions given by formulas which were familiar from earlier mathematics experiences, but was not comfortable trying to prove results about abstract functions. She was successful constructing direct proofs of results in number theory and set theory, but she continued to have trouble with proofs that were

not direct. She was not successful with constructing proofs concerning equivalence relations or functions. Her work showed that she learned from her mistakes, but that she did not have time in the semester to assimilate the material in the second half of the semester. This later material required synthesis and comparison of mathematical concepts, and she was not successful at constructing proofs by contradiction or the related method of minimum counterexample.

S6's Performance on the Final Exam. When S6 proved that the given affine function $f(x) = 7x - 1$ is one-to-one and onto (Final Exam, Item 1, Figure 29, page 86), she did not write quantifiers, but otherwise with correct direct proofs. In Item 2 (Figure 30, page 87), she had the correct structure of a proof of set equality, but misstated the definition of intersection of two sets by saying an element of the intersection was in one *or* the other of the two sets. For Item 3 (Figure 31, page 88), she gave a proof by contradiction, but omitted one possible case. As with her other proofs that involve contradictions, she began with a clear statement of purpose: "Assume, to the contrary, that 1000 can be written as the sum of three integers, an even number of which are even." In this, she is following the model proofs of the textbook and her professor.

Item 4 (Figure 32, page 89) required a proof that a relation defined in terms of an equality modulo 6 is an equivalence relation. Student S6 proved it clearly and completely. She used the divisibility definition of equality modulo 6, instead of using the previously-proved result that $a = b \pmod 5$ and $c = d \pmod 5$ implies

$ac = bd$ (mod 5). Her list of the equivalence classes was redundant in the sense that she listed the six equivalence classes in $\mathbf{Z}_6$, but not the four equivalence classes of the relation under discussion.

The proof in Item 5 (Figure 33, page 89) that

If x is an integer, then x^3 is even $\Leftrightarrow$ x is even $\Leftrightarrow 5x^2$ is even

was partitioned by cases depending on x being even or odd, with intermediate results proved by contrapositive, but four times the conclusion about even or odd integers are reversed. This anomaly even included the latter half of her proof contradicting the lemmas she had allegedly proved immediately before.

Her first proof by mathematical induction (Item 6, Figure 34, page 90) was in the appropriate form, and she included the quantifier in the induction assumption. There was a slight error in her statement of what she wished to show in the inductive step, and the algebra steps with inequalities are not complete. Her proof by complete induction (Item 7, Figure 35, page 90) only proceeded as far as a correct statement of the induction assumption. She then did not know what to do next, having written, "Observe that....?" Item 8 (Figure 36, page 91) involved the proof by the method of minimum counterexample, which she started in good fashion, but did not identify the minimum counterexample. Where she wrote, "Assume, to the contrary, there is a number where 6 does not divide $(n^3 + 5n)$. Let m be that number", she should have written, "Let m be the smallest such number." After a minor error with a quantifier, she proceeds with some algebra, making a conclusion which is contrary to what she has proved, and claiming she has found "a contradiction." She does not say what it

contradicts, nor how that completes the proof. This incomplete proof could be amended with three changes, but as it stands, it reveals a failure to understand the concepts of greatest lower bound of a set of integers, minimum counterexample, and proof by contradiction.

Item 9 (Figure 38, page 92) entailed five propositions to prove or disprove. S6 proved the first with a proof by contrapositive. The only error she made was in repeating the error from Item 5 of this Exam by proving that an integer expression is odd but claiming it has been shown to be even. This is either confusion about the definitions of even and odd integers, or more likely, a confusion about proofs which involve negations. Item 9, Part b was a false statement about congruence classes modulo 6, and she easily answered it with an appropriate counterexample. Item 9, Part c asked whether every even integer is the sum of two unequal odd integers. She presented 2 = 1 + 1 as a proposed counterexample. She is either thinking of positive integers only, or has confused the converse with the proposition. This happens with universal propositions when they are not stated in conditional if/then form. If the latter suggestion were the difficulty, then she may have treated the problem differently if it were stated, "If *n* is an even integer, then *n* is the sum of two unequal odd integers."

The proposed set equality in Item 9, Part d was answered with a counterexample consisting of finite sets. Except for a minor notational error, it was a well chosen example. The final part of this Item 9, Part e, was a question about well-ordered sets. She correctly saw that the proposition was false, but gave an inappropriate counterexample which did not illustrate what she claimed.

She presented perfectly correct answers to all eight of the matching items of 10 (Figure 39, page 94), including the starting assumptions for proofs by contrapositive and proofs by contradiction.

Significance of S6's Performance. Student S6 was a "C" student who showed that she was capable of learning mathematics that is new to her, but appeared to need clarifying opportunities to have her concept images challenged. She made errors identifying even and odd integers in her proofs involving contradictions, which, it is surmised, represent difficulties with understanding the contradictions, rather than difficulties with understanding even and odd integers. She was able to show that she understood the congruence classes modulo n, and could construct direct proofs using them via the definition of divisibility. Like most of the other students, she did not use the result $a = b \pmod 5$ and $c = d \pmod 5$ implies $ac = bd \pmod 5$. She produced several proofs in which she proved an integer quantity to be odd and declared that it was therefore even. These may have been instances of proofs by contrapositive, but the starting assumption was not there to clarify if this was what she intended. Whereas she showed numerous weakness at the beginning of the course, she demonstrated strengths on corresponding tasks on the final exam.

She did not show that she understood the logic of proofs by mathematical induction in any of its forms except the ordinary mathematical induction. She could reproduce the forms of the proofs, but did not show understanding of the inductive step and its purpose. The proof by minimum counterexample is a proof by

contradiction. She failed to make the appropriate starting assumption when attempting this proof, and was therefore unable to draw a contradiction.

Summary of Research Question 2: How Do Students Develop Skill in Planning and Reporting Proofs?

Measuring skill development always begins with measuring where the students are at the beginning. The pretest showed that many of the students did not yet understand the distinction between a mathematical statement and its converse, and the early interviews demonstrated that most of the interview students were not knowledgeable about proofs which were other than direct. Many did not realize the critical role that definitions played, the importance of quantifiers, nor the part played by previously proved results. The detailed discussion of the interview students in this chapter details how some, but not all, of them progressed in these areas.

It is to be expected that there will be variance in the ways that students perform in the passage of a semester course. The interviews together with the written assessments provided the more detailed information. The discussion in this chapter showed that S5, for example, changed very little in how he learned to construct proofs. He continued to make the same mistakes in the final exam that he had made throughout the semester, although he was beginning to write a form of proof by contradiction. At the other extreme was S2, who constructed flawless proofs and knew all the definitions from start to finish. A drastic conclusion would be that both

S5 and S2 showed no improvement during the semester! S1, S3, and S6 were similar in the kinds of misunderstandings they brought to the course, and similar in how they overcame them. They were all aware of the obstacles that appeared in their work, and were soliciting ways to overcome them. S4 was a diligent, but undisciplined, student who was satisfied with her trial and error way of learning. She performed best on written assessments, but was still tentative in her command of the material in the latter part of the course.

The interviews showed also that while most students accepted the idea that the three proof strategies were offered as feasible choices, a few students thought that there was only one right choice of proof strategy for each situation. The final exam provided evidence that some students would use direct proof by cases, whereas others would not widen their vision to permit that. Final Exam Item 9, Part a (Figure 38, page 92) invited the students to make such a choice. Half of them did a direct proof by cases, and half did a proof by contrapositive. All of them completed the proof.

The students in this class reproduced the writing style of the textbook and the instructor in useable and flexible fashion. They gave clear introductory statements of how their proofs were going to proceed, they chose appropriate variables to use, and they usually showed a clear idea of how the goal of the proof was promoting their choices of steps. The textbook and instructor wrote out quantifiers consistently, but this was one concept that was not acquired by all students, as was noted in the discussions of the interviews and the final exam in this chapter. One topic that was part of the course objectives, but that a majority of the students did not learn, was

proof by contradiction. Chapter Two mentioned the difficulties with proofs by contradiction that other researchers have noted. Those same difficulties were present in this class of students.

The proofs presented to the students in the textbook and in class were in either paragraph form or list form. Several times in the interviews the students were invited to rewrite a proof from one form to the other. They did not experience obstacles to doing this fluently. In proofs with several algebra steps, they preferred the list format. The course did not present diagrams or flowcharts to explain proofs.

Research Question 3: What are the Obstacles to Students Beginning Proofs?

The research cited in Chapter Two pointed to some of the difficulties students experience with starting proofs, and identified patterns in them that might be treatable by instruction. In this study, the evidence of a particular transition course was examined to develop a list of the obstacles to beginning the construction of proofs. This will serve to direct the focus of this analysis. The obstacles that surfaced in this study were (1) obstacles related to interpreting statements to be proved, (2) starting assumptions, and (3) the role of definitions.

Obstacles Related to Interpreting Statements to be Proved

There are several different kinds of obstacles to starting proofs that are related to elementary logic concepts. These elementary topics from logic are: interpreting the converse of a mathematical implication; conjunctions, disjunctions, and negations; and universal and existential quantifiers.

Obstacles Related to the Converse of a Statement. The performance of students on the pretest provided baseline data on their understanding of the concept of converse of a mathematical statement. As discussed on page 49, only two of the n=16 students showed a correct understanding of the converse of a statement at the beginning of the course. The instructor and the textbook treated this topic early, and reviewed it frequently. The interviews and written assessments showed that the students differed in their understandings of converse statements. In order to illustrate the variety among the students, here is a summary of their performance concerning the concept of converse in the interviews:

1. S1 did not know what the converse of an implication was in the first interview; but in subsequent interviews, quizzes, and the final exam, he showed a good understanding of converse.

2. S2 indicated from the very first interview and all the written assessments that she understood the concept of converse very well.

3. S3 revealed that he did not understand what the converse of an implication was in the first two interviews. Subsequently, however, in the interviews, quizzes, and final exam, he showed that he had appropriated the concept well.

4. S4 did not know what the converse was in the first interview, but by the second interview showed that she recognized a converse and the distinction between the converse and in the original implication. However, she made mistakes of interpreting the converse on the final exam.

5. S5 showed that he seemed to understand what the converse of an implication was in the early interviews, but subsequently did very poorly on tasks that required using or interpreting the converse. For example, even after completing the desired proof, he could not tell what strategy he had used in Interview 3, Task 4, (Figure 20, page 74). He required prompts to ask what were the converse, inverse, and contrapositive of the original implication.

6. S6 had an entirely different pattern than the others. She had a partial understanding of the concept of converse in the first interview, showed very good knowledge of converse in the second interview and subsequence interviews and quizzes, but misinterpreted the converse on one problem of the final exam.

This survey of the interview students shows that even a foundational subject such as the converse of an implication was unevenly understood by the students in the class.

Obstacles Related to the Use of Conjunctions, Disjunctions, and Negations. There were only a few instances where individual students made incorrect statements about negations. Student S1, for example, did not understand simple negation on the pretest (page 69), but later, in Interview 2 (Task 3, Figure 16, page 70) was able to state the negation of "*ab* is nonzero."

The most prominent use of conjunctions was in the compound hypothesis (P and $\sim Q$) of a proof by contradiction. Here students were involved in a tangled skein of cognitive obstacles that impeded quite a few of them from mastering this topic in the course. More will be said later about this difficulty, under the heading "Obstacles Related to Starting Assumptions."

Many propositions in mathematics have compound hypotheses. In this study the following examples have appeared:

(1) the local maximum theorem from calculus (Pretest, Items 1-2, Figures 5-6, page 56-57),
(2) the product of a nonzero rational number and an irrational number appeared in Interview 2 (Task 2, Figure 15, page 69), and
(3) the consequences of a composite function being injective or surjective appeared in Interview 3 (Task 3, Figure 19, page 73) and Interview 4 (Item 3, Figure 22, page 76)

Figure 40. Examples of Compound Hypotheses

However, it was not the compound nature of the hypotheses that was an obstacle to the students in these examples. The obstacles were understanding how to apply the

definition to the hypothesis, and knowing how to prove that a definition was satisfied at the conclusion of the proof.

A number of important concepts in this transition course are built from the atomic concepts of sets, functions, and numbers. Some prominent examples are: injective functions, surjective functions, equivalence relations, equivalence classes, and congruence mod n. Understanding the definition of equivalence relation involves a network of concepts, together with a coordinated schema to organize them mentally. The concepts are: viewing sets as whole units, absolute results (reflexivity for all points), conditional results (symmetry and transitivity when certain points are given), and the concept of the set of equivalence classes as a set whose elements are themselves sets. This transition course also illuminated the connections with partitions and canonical functions related to the set of equivalence classes.

Understanding the definition of injective function requires knowing what the definition says, what it means, and how it is used in proofs. The discussion of the interviews in this chapter included several instances where students could not prove results about injective functions because of faulty concept definitions. See especially the discussions of Interview 3, Task 3 (Figure 19, page 73).

Obstacles Related to the Use of Quantifiers. All of the students showed good symbol sense, although they frequently eschewed quantifiers, particularly the existential quantifiers ("there is"; "there exists"; "for some") and universal quantifiers ("for all", "for each", "for every"). When proving that given linear functions were

injective or surjective, they would include just enough correct use of symbols and algebra that a mathematician could extrapolate what they should have meant, but most of them omitted the quantifiers. For example, on the first problem of the final exam (Figure 29, page 86), ten of the 15 students who responded did not use quantifiers in their proofs that $f(x) = 7x - 1$ was injective, while the remaining five students did use them appropriately. On the same task, four of the five responding interview students did not use quantifiers. The second part of the same problem required a proof that f is surjective. Six of the 15 students did not use quantifiers; but nine of them did, although two of these students did not write correct statements using the quantifiers. On the same task, two of the responding interview students did not use quantifiers; but three of them did use them appropriately.

The students' work showed that they often did not write down the appropriate quantifiers, although their use of the variables that they introduced indicated that they may perhaps have said them to themselves. Some of the students were clearly just learning the subjects and verbs of the definitions, and not the quantifiers that restricted when the definitions applied. For proving that a function was one-to-one or onto, for example, two-thirds of the students provided abbreviated proofs which omitted all quantifiers. This was explained in the interviews when they used the same unstated abbreviations. For example, student S1 said, "Let's see – one-to-one is $f(a) = f(b)$, and onto is $f(a) = b$." (Interview 3, Task 2, Figure 18, page 72). This could be generously interpreted as shorthand for (1) "To show that a function f is one-to-one, the plan for the proof is 'Assume that for some numbers a, b, we have $f(a) = f(b)$. We

wish to show that $a = b$' "; and (2) "To show that a function f is onto, the starting assumption is 'Assume b is in the co-domain. We wish to locate an element a in the domain, for which $f(a) = b$.' "

There is another way that the example just given may be an obstacle. The definition of injective function f is required to be applied in the next interview task (Interview 3, Task 3, Figure 19, page 73) to the function $g \circ f$. Recall that in a similar situation on the pretest, student S1 wrote that there would be no connection between these functions. This is the error of not recognizing particularization, or instantiation, discussed in chapter Two: If the definition is given for any function, then it applies to the injective function $g \circ f$.

Although the students were careful to follow the text and instructor in most details of mathematical statements, they frequently avoided mention of quantifiers in common definitions, such as injective and surjective functions, equivalence relations, etc. This particular course did not emphasize the symbols for "there exists" or "for all", as some courses do.

<u>Obstacles Related to Starting Assumptions</u>

An obstacle to applying a starting assumption for two of the interview students was that they thought there was only one right choice for a proof strategy, and hence for a starting assumption. Since one-third of the interview population had this view, it

should be addressed. The textbook addressed this by examples of results (Chartrand, et al., 65) demonstrating that all three proof strategies were feasible (Figure 41).

If x is a nonzero real number and $x + (1/x) < 2$, then $x < 0$.

Figure 41. Example From the Textbook

Obstacles Related to Starting Assumptions of Direct Proofs. The most frequently assessed topic was the starting assumption for direct proofs, because most of the proofs in the course are direct proofs. There was only one final exam question that produced a number of incorrect responses (Item 9, Part c, Figure 38, page 92). On this item, there were eight correct and eight incorrect responses concerning starting assumptions of direct proofs. Other obstacles were encountered in this question, however, such as failure to consider negative numbers, failure to allow for unequal odd numbers, and similar errors that may be classified as oversights. There were 11 times in the written assessments and interviews concerning the starting assumptions of direct proofs that students were required to construct direct proofs. The individual students' performance varied (Figure 42), but only one (S7) was below 50%. The same data shows how the class performed over time during the semester (Figure 43). Assessments 6-11 are from the final exam, and show that there were two items on which everyone in the class was correct concerning the starting assumption.

These were the second part of Item 4, showing symmetry of a given relation, and Item 5, showing a number theoretic property.

The performance of the entire class through the passage of the semester cannot be expected to show monotonic improvement, because the course continuously introduced mathematical concepts that were new to the students. These new concepts carried with them additional opportunities for cognitive and semantic obstacles.

Obstacles Related to Starting Assumptions of Proofs by Contrapositive. There were not enough assessments spread throughout the semester to detect improvement in constructing proofs by contrapositive. There were four written assessments on the starting assumptions of proof by contrapositive, and five interview tasks. The four written assessment items were all on the final exam. Five students had correct responses to all four items, and three students were incorrect on at least two of the four items (Figure 44).

The performance of the entire class on the items attempted by proofs by contrapositive on the final exam was quite high (Figure 45). Final Exam Item 9, Part a required a proof of a result about odd integers: five students completed valid direct proofs, seven students completed valid proofs by contrapositive, three students completed proofs by contrapositive that were substantially correct, and one student's work was not available. The three exceptions were two students who made an incorrect deduction at the conclusion of their proofs (discussed on page 152); and one whose starting assumption and conclusion made it unclear whether he was attempting

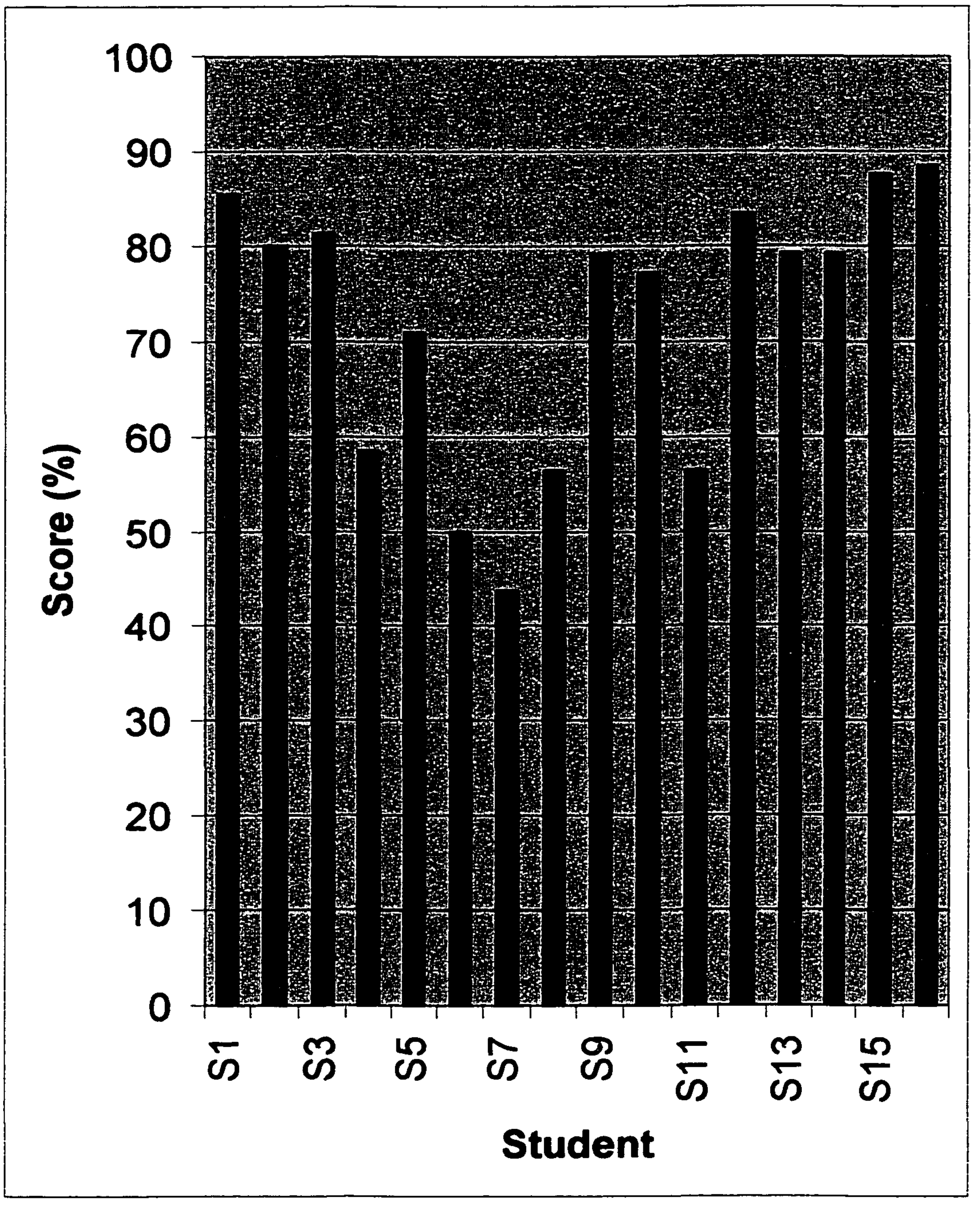

Figure 42. Starting Assumption of Direct Proofs by Student

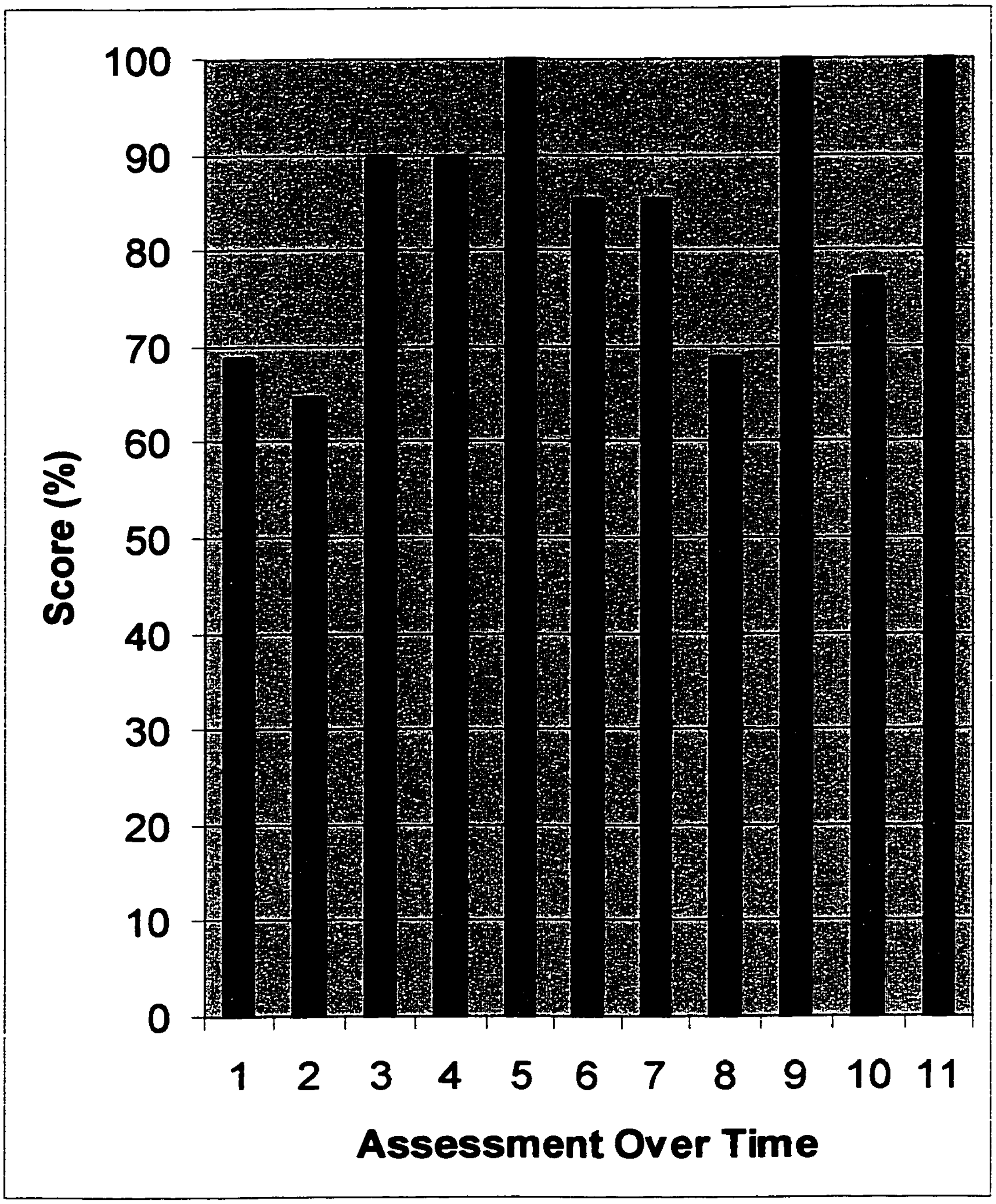

Figure 43. Starting Assumption of Direct Proofs by Item

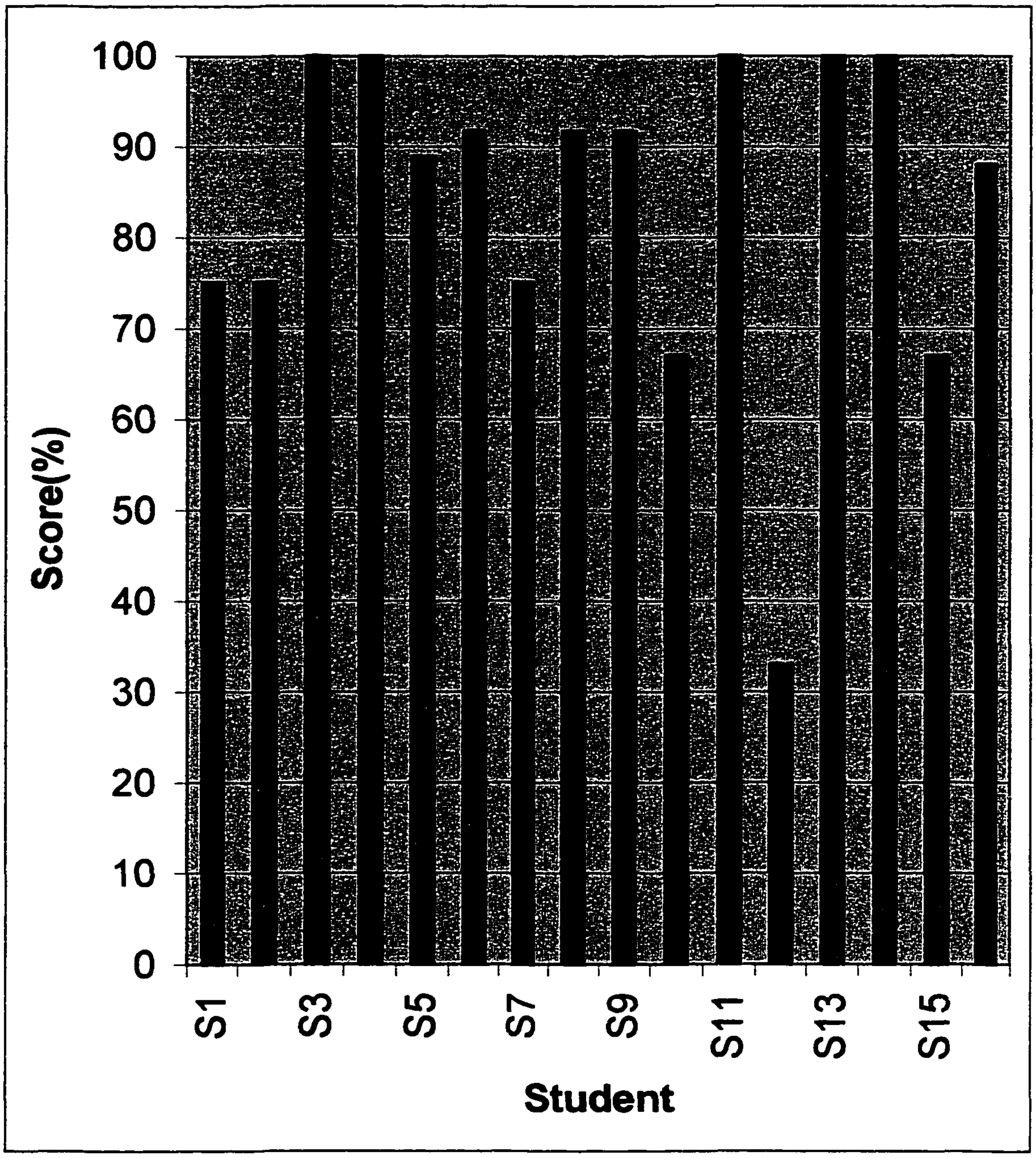

Figure 44. Starting Assumption of Proofs by Contrapositive by Student

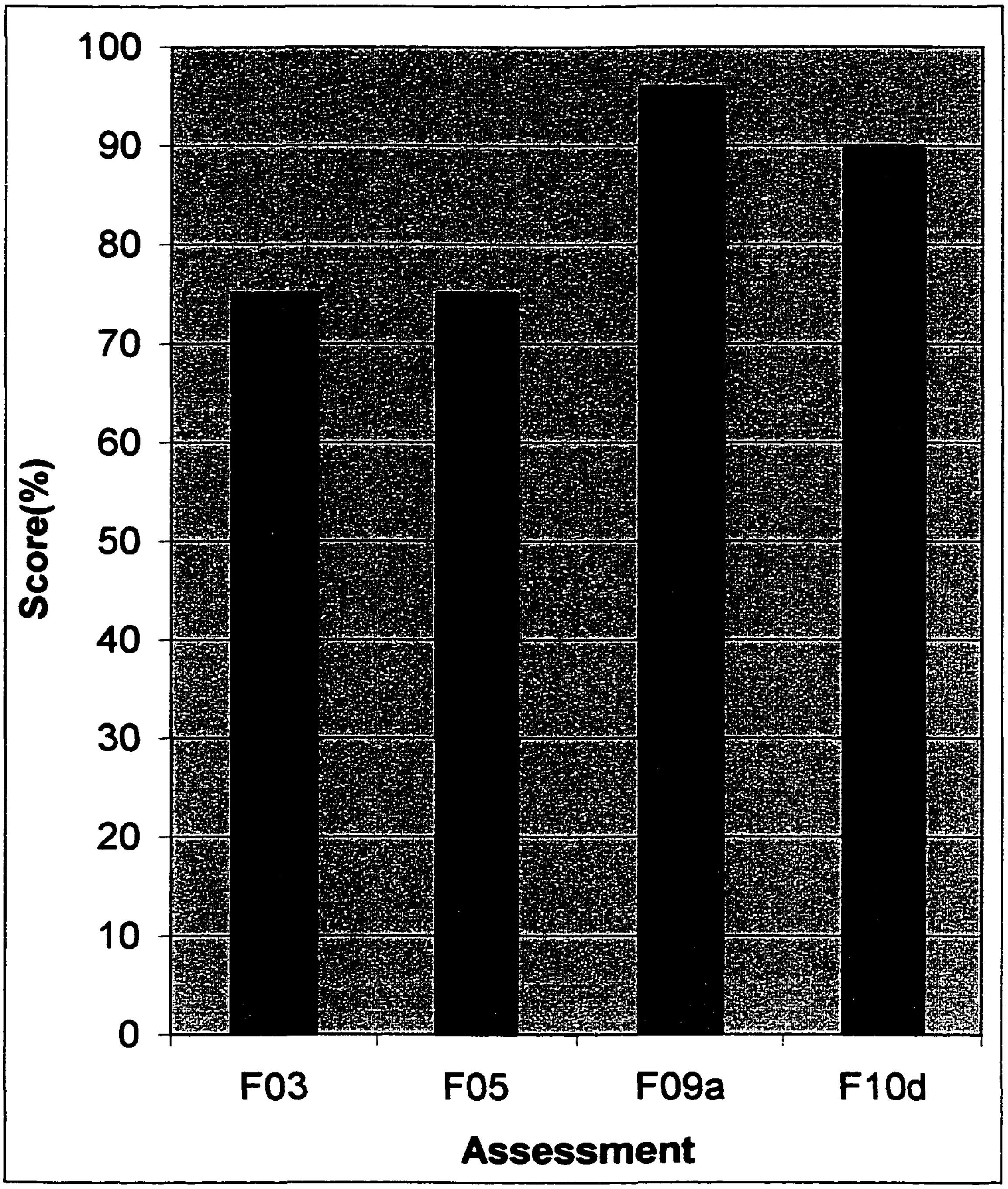

Figure 45. Starting Assumption of Proofs by Contrapositive by Item

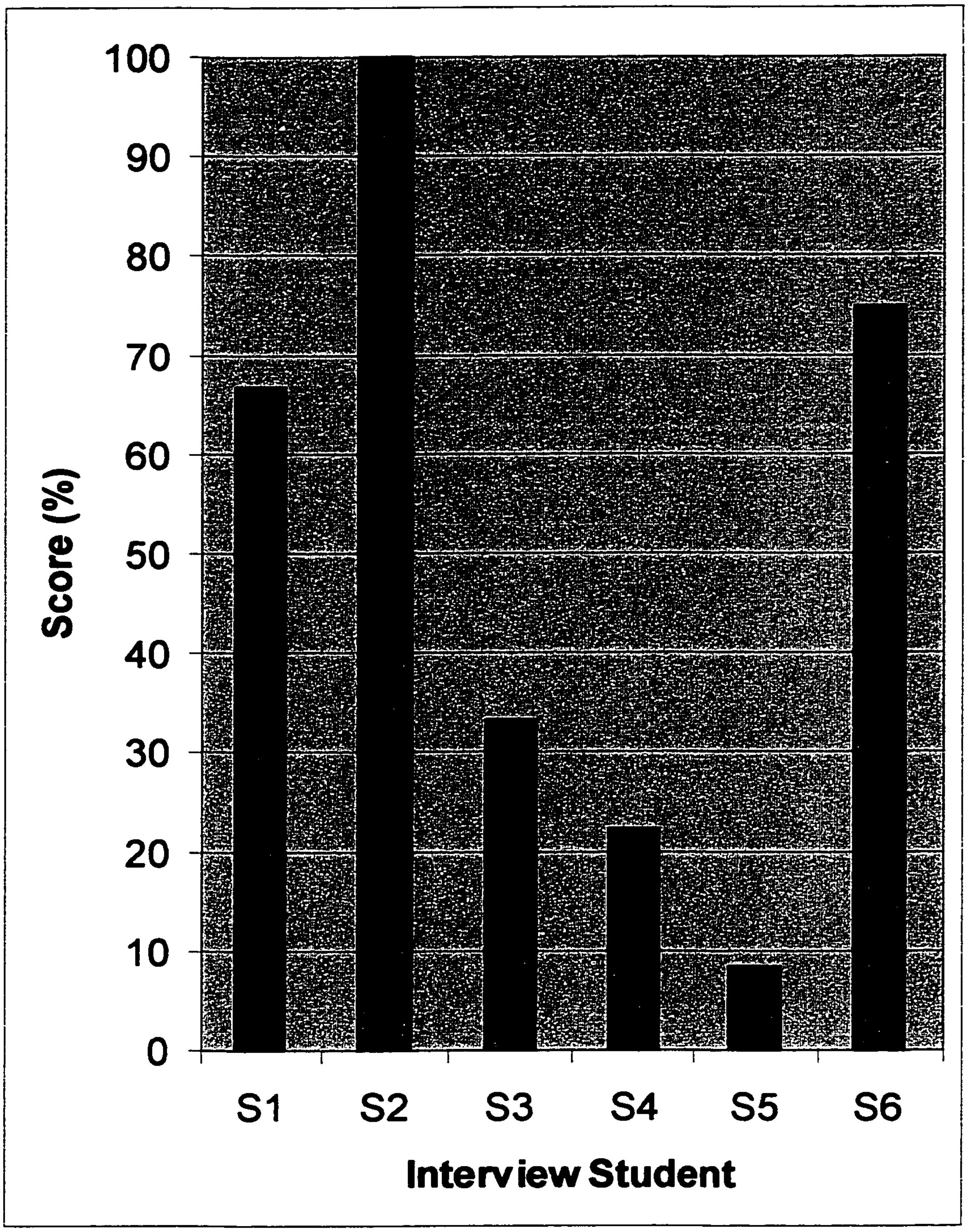

Figure 46. Starting Assumption of Proofs by Contradiction by Student

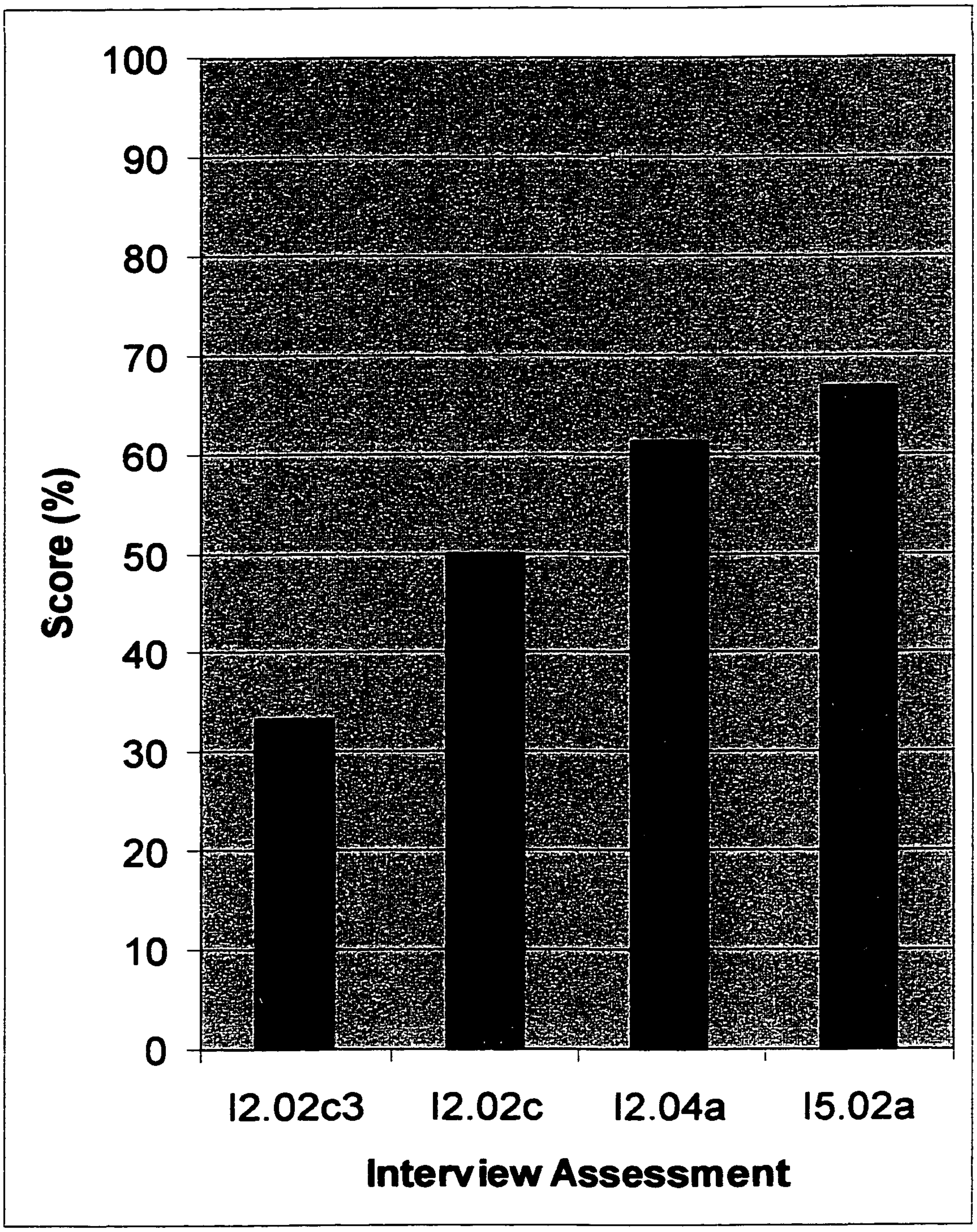

Figure 47. Starting Assumption of Proofs by Contradiction by Task

a proof by contrapositive or proof by contradiction. Item 10, part d simply required an identification of the starting assumption for proofs by contrapositive. Two of the students who had difficulty with the starting assumption for proof by contrapositive on other assessments avoided this opportunity to produce a proof by contrapositive by devising valid direct proofs.

Obstacles Related to Starting Assumptions of Proofs by Contradiction. Proofs by contradiction were only assessed once in the available written assessments, and four times in the interviews. The paucity of data is partly due to the small percentage of proofs by contradiction in the undergraduate curriculum. There are only about a dozen proofs by contradiction in standard calculus and linear algebra courses combined, and subsequently about another dozen in advanced calculus and abstract algebra. Additional complications are the form of the hypothesis, which is compound and involves a negation as well. In addition, the common language uses of the word "contradiction" predisposes the students to treat a proof as suspect if it contains the word "contradiction." One student, in fact, said that his first impression was that since a proof under discussion ended with a contradiction, the proof was invalid (page 96). The sole written item concerning the starting assumptions for a proof by contradiction was a recognition matching question on the final exam (Item 10g, Figure 39, page 94) on which eleven of the 16 students (69%) correctly matched the assumption to the proof strategy. Three of the six interview students did well on the interview questions about the starting assumptions of a proof by contradiction, but one of them missed the

final exam question. The other three interview students did not show any understanding of the starting assumptions or the aims of the proof strategy at mid-semester, and yet they did the matching question on the final exam correctly (Figure 46).

A principal theme of this research and of the transition course it studied is the progression in skill in recognizing and initiating the starting assumptions in direct proofs, proofs by contrapositive, and proofs by contradiction. When faced with a conditional statement, however, students have several other obstacles to overcome before choosing a starting assumption for a proof strategy. Mathematical notation, definitions of unfamiliar concepts, and use of quantifiers are prominent among these. These obstacles are discussed elsewhere in this section of this chapter. Mathematical concepts that are new to the students can also be obstacles to choosing starting assumptions. The students entering this particular class were unfamiliar with set operations, notation of abstract functions, and congruences. These concepts had not been stressed in the curriculum that these students had seen to this point.

The examples of students whose performance changed during the semester were interview student S3, who did not recognize any of the three opportunities to start direct proofs on the pretest but improved through the semester; and four others (not interview subjects) who were mixed in their success at starting direct proofs on the pretest, but improved afterwards. The interview students as a subset of n=6 students, showed increased performance on identifying completed proofs as proofs by contradiction (Figure 47).

The starting assumption for a proof by contradiction is a compound statement (P and $\sim Q$), which is different from those for direct proofs and for proofs by contrapositive. There was only one opportunity to assess starting proofs by contradiction on the written assessments (Final Exam, Item 10, Part g, Figure 39, page 94). That item involved only recognizing the starting assumption in terms of P and Q; there were no written assessments of the students' ability to generate such a starting assumption or to recognize the starting assumption within a proof. Eleven of the 16 students correctly matched the starting assumption with the proof strategy.

As shown by the responses to the eight parts of Final Exam Item 10, recognizing the starting assumption for direct proofs is most readily recognized by students (15 correct answers out of 16 students); for proofs by contrapositive less readily (13 out of 16); and for proofs by contradiction, least of all (11 out of 16). The difficulties students have with proofs by contradiction are complicated since the starting assumption involves negation, conjugation, and an expectation that a contradiction will be achieved.

Obstacles Related to Proofs by Mathematical Induction. Proofs by mathematical induction were not a primary target of this investigation, for they have been studied elsewhere (Dubinsky 1986, 1989; Ernest, 1984; Harel, 2000). The formation of the induction assumption was an object of interest for this study. The transition course studied ordinary mathematical induction, the method of minimum counterexample, and complete induction. Students were quite competent with

ordinary mathematical induction (Final Exam, Item 6, Figure 39, page 94), but had difficulty forming correct induction assumptions for the other forms (Final Exam, Items 7 and 8, Figures 36-37, page 91).

The specific obstacles observed in the final exam task for ordinary mathematical induction (Item 6, Figure 35, page 90) were: the use of quantifiers, the form of the induction assumption, and the inequalities to prove the induction step. One student (S8) left the problem blank; three students completed all but the correct inequalities (S1, S3, S9); the remaining 12 students constructed complete and valid proofs by ordinary mathematical induction.

Final Exam Item 7 (Figure 36, page 91) required a proof by complete induction. Eight students did only some preliminary calculations with small values of n; one misunderstood the statement of the problem; two made some progress but misunderstood the subscripts involved; and the remaining five students produced valid proofs.

Final Exam Item 8 (Figure 37, page 91) prescribed a proof by minimum counterexample. Three students said the form of the assumption was "Assume, to the contrary, that 6 does not divide $(n^3 + 5n)$ for *every* positive integer." [emphasis mine] In the language of the textbook, "an error has been made." De Morgan's rule is needed to find the correct negation of the statement, "$6 \mid (n^3 + 5n)$ for all positive integers n." Two students (S10, S11) attempted a proof by ordinary induction; one other (S3) attempted a proof by complete induction. Two students left this problem blank; eight did not state the correct assumption; three made some progress but did

not complete the proof of the inductive step; and three completed valid proofs. As noted, the difficulty of this form of induction is that it is a proof by contradiction, which has previously been identified as an obstacle.

Obstacles Related to the Role of Definitions

There is a well-known distinction between the concept definition, which is the literal definition of a concept; and the concept image, which is the mental view of the concept that an individual has. (Dreyfus, 1991) A further distinction, called concept usage, was introduced by Moore (1990), which drew attention to the overlooked part of explaining mathematics in providing information about how a definition will be used in later proofs. These ideas may be illustrated by the proof from Interview 3, Task 3 (Figure 19, page 73). The hypothesis is that $g \circ f$ is an injective function. This may mean any of the following: (1) Different values in the domain of $g \circ f$ have different images; or (2) If $a \neq b$ in the domain, then $g \circ f(a) \neq g \circ f(b)$, or (3) If there exist a,b in the domain with $g \circ f(a) = g \circ f(b)$, then $a = b$. The first two statements are, of course, the same thing; one without mathematical notation, and one with it. All of these are valid definitions of what it means for $g \circ f$ to be injective, but they are not equally useful for deducing something further. In this case, one must look to the goal of the proof, namely, to show that f is injective.

An additional obstacle which impeded the construction of proofs was the failure to know the definitions of technical concepts in which quantifiers were involved, such as injective function, surjective function, and congruence. Additional complications arose when a concept required the coordination of two or more concepts, for example, the starting assumption for proofs by contradiction, equivalence relations, equivalence classes, and the domain and range of a function.

The pretest showed that skill in using definitions within proofs was very weak, but the final exam showed instances of confident and correct usage of definitions of even and odd integers (Items 3, 5, and Item 9, Part a), unions of sets (Item 2), and properties of functions (Item 1).

In the pretest, an odd integer n was represented by $n = 2k + 1$ for some integer k. Several students called this an assumption instead of a definition. It is possible that their definition of odd integer would be a number like 1, 3, 5, 7, 9, ... This would not be a complete definition, for example, because it omits negative integers. Neither would such an inductive definition be suitable for proving general results about numbers.

By the time of the interviews, the students had encountered the use of the representation $n = 2k+1$ for odd integers n frequently in the transition course, and it was by then the immediate step that they used.

The issue is not *what* is the definition of odd integer? The issue is *how* is it used in constructing proofs? On the final exam, the students made correct usage of definitions of odd and even integers.

Table 3

Usage of Definitions of Odd and Even Integers

Final Exam Item:	3	5	9a	9c
No. Correct (n=16)	15	14	14	15

Two students, S1 and S6, did something unusual in Item 9, Part a (Figure 38, page 92). It appears that they proved $6x+7y$ is an odd integer, but concluded it was even. Here is S6's proof; S1 did essentially the same thing, but used cases, depending on whether x was odd or even.

> Result: Let $x,y \in \mathbf{Z}$. If $6x + 7y$ is even, then y is even.
> Proof: Assume y is odd. Let $y = 2a+1$ for some $a \in \mathbf{Z}$. Thus $6x+7y = 6x+7(2a+1) = 6x+14a+7 = 2(3x+7a+3)+1$. Since $3x+7a+3$ is an integer, $6x+7y$ is even [*sic!*].

It may mean that they left out a few phrases, namely, "... $6x+7y$ is odd. This contradicts the hypothesis of the problem. Therefore, y is even, and the proof by contrapositive is completed."

S1 said that the initial encounter with proofs with sets was very difficult (page 78). He said that the first few chapters had been prescriptive, so that the he knew how to start and pursue the proofs. One reason for his crisis was that the textbook's first proof with sets is not a set inclusion, but a set equality. To prove a set equality required a biconditional proof.

This particular transition course dealt most often with unions and intersections of only two or three sets at a time; there were only a few classroom examples of

infinite indexed families of sets. Although the concepts of union and intersection were obstacles as the students formed their concept images, by mid-semester they were constructing correct proofs of subset relationship, properties of unions and intersections, properties of cartesian products, etc. The only weakness in their proofs of elementary set theory was the final exam question about prove or disprove $(A \cup B) - B = A$ (Item 9, Part d, Figure 38, page 92). Only eight students constructed a correct counterexample to the assertion. Five students proved that the left-hand side is a subset of A. Six students made incorrect deductions about sets and their elements to conclude that the two sets were equal. Only one, student S15, drew a Venn diagram. Then S15 provided a correct counterexample.

Verifying that a given relation is an equivalence relation requires three direct proofs. The only obstacle to this verification that surfaced in this proof (Final Exam, Item 4, Figure 32, page 89) involved the five students of the 13 responding who erected the obstacle themselves because they failed to use quantifiers to keep their inferences correct.

There was only one problem on the final exam (Item 4, Part b, Figure 32, page 89) that required the students to find all the equivalence classes of an equivalence relation. Only one of the students was able to complete the task. Two left the problem blank; two wrote nonsensical sets; eight gave partial answers in the wrong set, namely $\mathbf{Z}_6$; and three students gave subsets of $\mathbf{Z}$, of which two students were, however, incorrect.

In this transition course, the notions of equivalence relation and its associated set of equivalence classes were new to the students. It was therefore important for them to see many examples and determine for themselves the equivalence classes. They were not often successful at this, for it requires the coordination of several cognitive associations.

Obstacles Related to Choosing Mathematical Notation and Representations. In this transition course, the abstract notions of injective and surjective were applied to functions that were not given by formulas, and this was an obstacle to several of the students. In a subsequent course in abstract algebra, the students will encounter a variety of representations of the elements of abstract groups by functions, geometric transformations, matrices, and permutations.

Summary of Research Question 3.

One student (S4) characterized her difficulty with starting a proof as "lack of knowledge", but there are many levels of knowledge required in constructing proofs. In addition to the basic knowledge of proof strategies, logical necessities, and quantifiers, making proofs requires the specific skills of definitions, algebra or other mathematical background, and the advanced skill of coordinating these with the creative acts of choosing notation, choosing representations for the objects, and bringing relevant theorems to bear.

The principal obstacles to starting proofs found in this study were the use of quantifiers, the starting assumption of proofs by contradiction, the methods of complete induction and minimum counterexample, the role of definitions within proofs, and choosing mathematical notation and representations.

Research Question 4: What are the Obstacles to Students Completing Proofs?

Three principal obstacles to completing proofs were identified in this study. The previous research question already discussed the use of definitions involved in the conclusion of the implication to be proved (pages 150ff.). Another obstacle was the awareness that previously proved results could be used within a proof in effective and efficient ways. The third obstacle was understanding how to treat a contradiction within a proof by contradiction.

In the interviews, when students were asked how far they were from completing a proof, they often had no sense of how close they were to resolving the distance from where they were to the goal of the proof. In the interviews for this study, students were repeatedly asked what the goal was towards which their incipient proof was aimed. After reviewing definitions, choosing representations, and knowing the goal, students would still be unable to continue their proof, and said they did not see any connection between the starting assumption and the conclusion they were to prove. The root of their difficulty turned out to be failure to know how to use the definition of the concept in the conclusion they were to prove. This was detailed in

the discussion of Interview 3, Item 3 (Figure 19, page 73) and Interview 4, Item 3 (Figure 22, page 76), earlier in this chapter.

A second obstacle was the realization that proofs can make efficient use of previously proved theorems. The importance of including previously proved results is not as pivotal in the transition course as in advanced courses, but it could still be seen here. The reason it is not quite as important here is that usually the necessary steps can be filled in with just the few steps that were used to prove the previous result. For example, the textbook proving that if the functions f and g are bijective, then so is $g \circ f$, uses the prior theorems that if f and g are injective (resp. , surjective), then $g \circ f$ is, too.

Another example of previously proved results that were ignored was modular arithmetic. The properties were proved in order to simplify later results, but the students did not appropriate or appreciate them as results that could be used later. Arithmetic with congruences (mod n) was new to these students. The properties of arithmetic mod n, such as

If a and b are integers and $a = b$ (mod n), then $a^2 = b^2$ (mod n)

were proved as exercises, but these results were not used afterwards in the students' proofs. For example, in Final Exam Item 4 (Figure 32, page 89), most of the students provided correct proofs that $x^2 = y^2$ (mod 6) defines an equivalence relation on the set of integers **Z**, but they wrote their proofs in terms of divisibility, not congruences. This is acceptable for correct proofs, but it is also an example of missed opportunities to use the previously proved results that $a = b$ (mod 5) and $c = d$ (mod 5) implies

$ac \equiv bd$ (mod 5). In their proofs of each of the three properties of an equivalence relation, they repeated the three lines that prove the previous result given above. This was confirmed by the Final Exam Item 9, Part b (Figure 38, page 92), in which they were asked about arithmetic mod 6, and only nine of the 16 students were successful, and for a very similar reason. Interview 5 (Task 4, Figure 28, page 83) also confirmed this by asking for proofs that $6^n \equiv 1$ (mod 5) in several different ways. None of the interview students thought in terms of congruences until prompted.

The third obstacle to completing proofs was understanding what to do when a contradiction had been achieved. Although templates for direct proofs and for ordinary mathematical induction were helpful aids to students, the form of proof by contradiction was clearly an obstacle to understanding. Students would often look for the key expressions, "Assume to the contrary" or "This is a contradiction." Some examples of this were quoted in the discussion of the interviews in this Chapter.

There was also some lingering confusion about the form of proof by contrapositive. Student S5 on Final Exam Item 9a, for example, began his proof of $P \Rightarrow Q$ by saying, "Assume, to the contrary, ~Q" and concluded his proof with "Therefore, ~P, which is a contradiction." He possibly intended a proof by contradiction but neglected to write the compound hypothesis "Assume P and assume, to the contrary, ~Q".

Research Question 4 asked, "What are the obstacles to students completing proofs?" A number of minor conceptual obstacles that interfere with the construction of proofs were identified in the discussion of Research Question 3 (pages 132-154).

The ones that remain obstacles to completing proofs are: knowing the definition of the object in the conclusion of the proof, knowing how and when to appeal to previously proved results, and understanding how to interpret a contradiction within a proof.

Research Question 5: How Does Instruction in a Course on Mathematical Proof Affect the Ability of Students to Understand What They are Trying to Prove?

The students made noticeable use of the same language as the textbook and instructor, leading one to the proposition that this influence was important. The students' behavior may have been simply learning those phrases and forms as the rote way to satisfy the teacher's expected proof forms on homework and tests, but they were correct forms, and they were helpful to some students to build schema for themselves. Examples of this were quoted earlier in this Chapter from the interviews and the quizzes and final exam. Although the following list was not written down for the students, some aspects of writing proofs that the textbook and instructor modeled were:

1. State the method to be attempted in the proof
2. State the hypotheses
3. If a proof by contradiction is chosen, then state the assumption with the words "Assume to the contrary"
4. If a contradiction is obtained, clearly state what it contradicts
5. End the proof with a statement of what has been shown

There were some procedural direct proofs that students assimilated successfully after class instruction and homework, namely, ordinary mathematical induction to prove a formula, verifying the definition of equivalence relations, and proving that specific nonsingular Möbius functions are one-to-one and onto. There was more variance in student performance with direct proofs of more challenging tasks, such as demonstrating inequalities and proving abstract results about sets and functions. Direct proofs involving cases were also taught, and the students used them successfully, for example on the final exam in Item 3, Item 5, and Item 9, Parts a and c, as discussed earlier in this chapter. One interview student (S1, page 78) expressed a need for more direction in how to choose the cases. When he employed the strategy of direct proof by cases, for example, in Final Exam Item 9, Part a (Figure 38, page 92), his proof was complete and valid, even though his choice of cases was unnecessary.

Although the direct proofs were essentially in a template form, the students did not appropriate the forms of proofs by contrapositive and proofs by contradiction as readily. In proofs by contrapositive, they often knew the form of the argument ($\sim Q \Rightarrow \sim P$). Many of them did not provide a statement of the conclusion of what had been proved, although this had been modeled for them in class and text. This omission was noted in the discussion of the final exams of the interview students S1, S2, S4, and S5.

The instruction concerning proofs by contradiction was threaded through the entire course, and yet it escaped the attention of many students. It was introduced

early, in the discussion of truth tables. The textbook and the instructor demonstrated that the statement ($P \Rightarrow Q$) was logically equivalent to the statement ((P and ~Q) $\Rightarrow$ (contradiction)). Subsequently, the proof strategy for proof by contradiction was explained, and it was used to obtain results concerning the number of prime integers, the solution of inequalities, and the form of mathematical induction called the method of minimum counterexample. The model proofs all had the same general form, although a strict template was not taught. Examples of successful and unsuccessful proofs by contradiction were cited in the discussion of Research Question 2 earlier in this chapter (pages 78-130).

Also in the earlier discussion of Research Question 2, the interviews revealed frequent instances of students who expressed insecurity of knowledge about the form of a proof or the steps within a proof, but showed more confidence as the semester progressed. The interviews asked repeatedly, "What is the goal of your proof?" and "What stands in the way of your completing this proof?" In many of these cases, the student had completed their proof, and the question served to invite them to look back at it to see whether it was valid. Students S1 and S4 especially benefited from this introduction to self-assessment of their own work.

Research Question 6: How Does Instruction in Proof Strategies Improve the Performance of Students in Solidifying Schema in Proof-planning and Proof-reporting?

The previous research question discussed the purposes of instruction in the transition course and the impact it has on students' ability to understand what they are proving. The present question is how that influence improved the understanding of the students regarding how they planned proofs and wrote them down. These understandings took different forms for the different students. S1 and S3 generally had difficulty with new concepts, but through repeated practice and sustained attention to the instruction performed well at the summary assessment in the final exam. S4 did not solidify her understandings until called to account at a quiz or exam. S5 never did assimilate some of the concepts about proofs or the new content in the course, even at the final exam. On the other hand, S2 expended effort to learn new concepts at their first inception, and actively pursued ways to validate the proofs that she wrote herself.

Interview tasks that invited the students to discuss their conceptions of the proof process (Interview 4, Task 4, Figure 23, page 77; and Interview 5, Task 3, Figure 27, page 81) revealed information on students' schema for constructing proofs. The students did not add any additional ingredients to the list of topics suggested by the researcher. If one separates the portions of these ingredients related to the proof techniques studied in the course from those that are more related to mathematical content, the following bipartite set of ingredients emerges:

Table 4

Ingredients in the Process of Constructing Proofs

a.	Choose a proof technique and starting assumption D. Direct Proof P. Proof by Contrapositive C. Proof by Contradiction	1.	Using Definitions
b.	Deductions	2.	Interpreting Statements, Converses, and Quantifiers
c.	Conclusion	3.	Using Prior Mathematical Knowledge
		4.	Applying Previously Established Results

Although there is an implied order in the use of the first list, there is no order intended in the second list. The process of constructing proofs involves a sequence of these seven ingredients. The sequence must begin with one of the techniques under "a", and it will almost always terminate with "c"; however, the remaining steps may each occur many times. The choice of proof technique (direct proof, proof by contrapositive, or proof by contradiction) is the same as the choice of starting assumption, so those have been listed together. Definition usages in this course included appropriate understandings of injective functions, surjective functions, congruences, and equivalence classes. Prior mathematical knowledge included results from arithmetic, algebra, geometry, trigonometry, calculus, linear algebra, and

elementary number theory. This list would, or course, be different in a study of students taking advanced courses or secondary school courses. The previously established results in the list of ingredients would also be different in another course than the transition course. Results proved in this course concerned divisibility, inequality, injective functions, surjective functions, and congruences.

Some possible ways to order these steps are listed here, but many more are feasible, in fact, infinitely many more, since there could be non-terminating loops with repeating steps: (1) D1b, (2) D1bc, (3) D3bc, (4) P4bc, (5) D1b1bc, (6) C1b3bc, (7) C1b4bc, (8) P11bbc, (9) D11b3bc, (10) D11b3b4bc, (11) C11b3b1bc.
For example, a direct proof that

$$g \circ f \text{ is injective} \Rightarrow f \text{ is injective}$$

might be D12b1b1c. Someone else might present a different valid proof, but it would be similar in the ingredients that are used. Proofs are not normally written in this fashion, but this coded representation of proofs permits visual comparison among different proofs by the same person as well as patterns in the proofs constructed among the students. Comparing the case study students' responses to the two tasks mentioned above allows a preliminary description of each one's schema for constructing proofs. Then by reconsidering their actual proofs in the interviews and written assessments demonstrates whether those schema are stable or whether they vary with the type of mathematical content that is involved.

Table 5

Actual Schema for Constructing Proofs

Task/Item	S1	S2	S3	S4	S5	S6
I4.4	a1432ac	a1a34bc	1a4a3bc	NA	3a1a43bc	341a43bc
I5.3	a3c	Dc	A3bc	NA	a3bc	ac
Divisors (I2.1)	D123bc	D1b3bc	D1bc	D13bc	D1b3bc	D13bc
Ordering (I2.3)	P1b23bc	C3bc	D123bc	D13bc	NA	P3bc
Injection (I3.3)	D1b3bc	D1b231bc	D1b1bc	C1SX	D11bc	D113bc
Ordering (I3.4)	P3b3bc	C23b3bc	P3b3bc	C3c	P3b3bc	P3bc
Surjection (I4.3)	D111SX	D1b23bc	D111bSX	NA	D1SX	D1131bc
Injection (F01a)	D13b1c	NA	D1b3bc	D13bc	D1b3bc	D1b13bc
Surjection (F01b)	D13b1c	NA	D1b3bc	D13bc	D1b3bc	D1b13bc
Sets (F02)	D12b1bc	NA	D1b3bc	D1bcX	D1bc	D1bcX
Numbers (F03)	C123bc	NA	C1b23bc	C1b23bc	C1b3bc	C123bc
Relations (F04)	D123bc	NA	D13bcX	D123bc	D1c3bcX	D1b23bc
Divisors (F05)	P123b3bc	P123bc	P123b1bc	P13bc	P13SX	P1b23bc
Ordering (F06)	D123bc	D1b23bc	D123bc	D13SX	D13c	D1b23bSX
Sequences (F07)	Blank X	D1b23bc	D123cX	D1SX	D1SX	D1SX
Divisors (F08)	Blank X	C1b23bc	C1SX	C13SX	CSX	C1b3bcX
Divisors (F09a)	P1b12bcX	P123bc	NA	D13bc	C13bc	P1b3bcX

Explanations:
"I" denotes Interview; "F" denotes Final Exam.
NA denotes not available.

"S" denotes that the student stopped and did not complete the proof; "c" would denote a concluded proof.
"X" denotes an invalid proof. The reasons were varied.

The first two rows were discussed earlier, concerning how the students perceive the process of constructing proofs. Notice first that the earlier task (Interview 4, Task 4) required students to arrange given ingredients, whereas the later task (Interview 5, Task 3) required the students to produce the ingredients in the proof-process on their own. The table shows, on the one hand, that their understandings in Interview 4, Task 4 are similar for S1, S2, and S3; and on the other hand, S5 and S6 have admitted to conceptual mathematical distractions before they even get started on the proof process proper. The responses to Interview 5, Task 3 are quite different, and demonstrate the variety of responses that can occur when a question is directive vs. non-directive. For Interview 5, Task 3, S3, S5, and S6 omitted definition usage and previously established results from their report at that time of their understanding of the proof-process on their own. These observations are a different perspective on the analysis of Research Question 2 earlier in Chapter Four (pages 56-131).

There were two proofs of injectivity. Interview 3, Task 3 was mentioned above:

$$\text{Prove: } g \circ f \text{ is injective} \Rightarrow f \text{ is injective}$$

Final Exam Item 1a, on the other hand, was to prove that the function f given by $f(x) = 7x - 1$ is injective. Comparing the students' proofs of these two results in this Table, the distinction is in the use of prior mathematical knowledge ("3"), but this is

for a mathematical reason – the final exam item involved a specific function, and algebra was required to complete the proof. The interview task involved an abstract function for which there was no defining formula. Comparing each students' approaches to these two proofs shows that each student who completed these two proofs did them the same essential way. The use of quantifiers would be signified in these proofs by "2" but it is absent in the students' work.

These examples of a few distinct schema are not all that are present in these students, but they illustrate the range of schema that they have. It would not be fair to the students, however, to label them as having just one kind of schema. An individual might express reasons for results in algebra as due to their mathematical authorities, such as textbooks, the instructor, or previous books or teachers. But they might at the same time be able to justify results in calculus by giving the derivations. Deciphering what forms of schema they have in the context of this course, however, is made difficult by the introduction of new content material. In this study, questions such as this were attended by asking their reasons for some mathematics about which they had previous knowledge (Pretest Item 1, Figure 5, page 56; and Interview 5, Task 1, Figure 25, page 80).

One of the planned ways to involve the students in thinking about their own proving was to compare more than one proof of the same result. The textbook and class instruction did this by contrasting proofs using the different proof strategies, as mentioned earlier (Figure 41, page 153). The interviews explored this with Interview 5, Task 4 (Figure 28, page 83). The depth of this question is that we must separate

the students' performance from the influence of the instruction. The pretest serves in this role as the situation before instruction; the final exam serves as the situation afterwards.

The Pretest showed that students had some start of ideas/schema on constructing direct proofs, but not proofs by contrapositive or proofs by contradiction. As the semester progressed, they improved, although some students' progress was erratic. The interview students were easier to track, since they were open to discussing their progress and responses to the instruction. Distinct improvement was evident in the performance of S1, S3, and S6. This improvement was certainly due to the students' seeing examples of properly written proofs, but also to their cognitive growth. The textbook modeled proofs and the professor modeled modes of reporting proofs.

The students used the wording, vocabulary, syntax and organization of the proofs as they were written by the textbook and the teacher. This was due to the consistent language in the book, and the teacher, being a co-author, used it consistently as well. The instructor wrote proofs on the blackboard with the same notation and sentences that the book used, but not just copying, so students could see what were the important ideas and how to communicate them when they wrote their own proofs.

According to Table 5, the students' initial perceptions concerning when to use definitions in proofs ("1") was to appeal to them once only, near the beginning of their proofs. This is generally how they did employ definitions, as shown by the coding of

their proofs. Students S1, S3, and S6 indicated in their descriptions of the process of constructing a proof that they anticipated using previously established results in the course early in their proofs; however, in their actual proofs coded in Table 5, they never used such results.

In actual usage, S1, S3, and S5 did not complete the proof that a function was surjective in the interview task (Interview 4, Task 3, Figure 22, page 76) due to unusable concept definitions of surjectivity. They did, however, complete the proofs that a given function was surjective on the final exam (Item 1, Part b, Figure 29, page 86), although none of them used quantifiers ("2"). Strict evaluators would judge their work incomplete due to the absence of quantifiers. These three students were also unable to complete the proofs by complete induction (Final Exam Item 7, Figure 36, page 91) and by the method of minimum counterexample (Final Exam Item 8, Figure 37, page 91). S1 left these two items blank on his exam.

S2 successfully completed all of the proof tasks in Table 5, and she followed the form of proof construction that she had described in Interview 4, Task 3. Her proofs also included the coding sequence "23" more frequently than any of the other students; in other words, she used quantifiers and prior mathematical knowledge in her proofs.

S4 seldom used quantifiers ("2") in her proofs, and her proofs were often abbreviated to short sequences "13bc" that indicate an appeal to a definitions, to prior mathematical knowledge, and deduction of the conclusion. The weak concept images she revealed in interviews 1-3 are consistent with this performance in proof

construction. Her proofs that were invalid, however, were not invalid due to difficulties with proof concepts, but rather due to these incomplete images of mathematical concepts.

S6 produced proofs that appear to have a deeper understanding of when to use prior mathematical knowledge, but he also eschewed quantifiers ("2"). His proofs that were invalid were essentially the same ones as produced by S4, and for the same reasons.

There is an apparent partition of these six students, then, into three classes. S1, S3, and S5 all demonstrated weak concept images of a specific mathematical feature (surjectivity) and also weak proof concepts concerning complete induction and the method of minimum counterexample. These topics could be planned targets of instruction to draw the attention of such students to their deficiencies. A second subset of the partition consists of S4 and S6, who did understand the concepts of the proof techniques of the course, but were hindered by their incomplete knowledge of definitions of mathematical concepts and use of quantifiers. These obstacles could also be addressed by instruction. The final subset of this partition consists of S2 alone, who had no apparent difficulties with proof concepts, quantifiers, definitions, prior mathematical knowledge, or usage of previously established results. These results are clearly seen when the invalid proofs of Table 5 are listed separately as Table 6.

Table 6

Nature of Errors in Invalid Proofs

Task/Item	S1	S2	S3	S4	S5	S6
Divisors (I2.1)						
Ordering(I2.3)						
Injection (3.3)				Def. Usage		
Ordering (I3.4)						
Surjection (I4.3)	Def. Usage		Def. Usage		Def. Usage	
Injection (F01a)						
Surjection (F01b)						
Sets (F02)				Def. Usage		Def. Usage
Numbers (F03)						
Relations (F04)			Def. Usage Arg.		Arg.	
Divisors (F05)						
Ordering (F06)						Algebra
Sequences (F07)	No Attempt		Arg.	Ind. Assump.	Ind. Assump.	Ind. Assump.
Divisors (F08)	No Attempt		Initial Assump.	Algebra	Initial Assump.	Algebra
Divisors (F09a)	Contra.					Contra.

Explanations:
The shaded cells represent valid proofs or NA from Table 5.
"Def. Usage" signifies an error in definition usage.
"Arg." indicates a logic error by arguing from the conclusion.
"No Attempt" means that the student left the item blank.
"Ind. Assump." signifies an error in forming the induction assumption.
"Initial Assump." signifies an error in forming the hypothesis about the minimum element in a proof by minimum counterexample.
"Contra." means an error in understanding a contradicition at the conclusion of a proof.

As to the affects of instruction on the changes in students' schema, this study did not evaluate the professor's grading of student work, nor how she communicated back to the individual students concerning it. The class observations and interviews were the only ways used to evaluate the input of instruction. Two observations are related to how students may think about proofs as a result of the instruction. One was a common prejudice for algebraic presentations. Geometric and structural approaches to proving were not generally a part of this course. The second observation is that there were very few times when results were first developed as conjectures. The results to be proved were most often given as statements concerning which the students may or may not have accepted as probable. The roles of proof as convincing and explaining are involved in understanding how this does have effects on how the students approach constructing proofs. The interview students did understand the value of examples in forming a conjecture, for example in Interview 5, Task 4 (Figure 28, page 83). They expressed varying degrees of confidence whether the statement

$$6^n = 1 \textit{ (mod 5) for all positive integers } n$$

could be proved, but they all thought that the evidence of five specific cases was enough to make it probable. This appears to be accepting inductive reasoning, but it may have been a way of saying that they did not immediately see a way to prove the general result.

The classification of student schema for understanding proofs which was found by Harel and Sowder (1998, 258) lists these terms for the schema:

I. External Conviction
    A. Ritual
    B. Authoritarian
II. Empirical
    A. Inductive
    B. Perceptual
III. Analytical
    A. Transformational
        1. Internalized
        2. Interiorized
        3. Restrictive
            a. Contextual
                i. Spatial
            b. Generic
            c. Constructive
    B. Axiomatic
        1. Intuitive Axiomatic
        2. Structural
            a. Structured Symbolic
        3. Axiomatizing

Figure 48. Proof Schema (Harel and Sowder, 1998)

The present research did not specifically assess the students thinking to classify their proof schema in the fine detail of this outline, but the first two levels do provide a framework for discussing the understandings of the interview students. There was very little evidence in this research of the External Conviction schema, and when it did occur it was in relation to recall of algebraic or geometric facts. These instances were cited in the discussion of the interviews earlier in this chapter.

The Empirical (Inductive) schema were noticed in a few specific tasks, for example, showing that $6^n = 1 \pmod 5$ for every positive integer n (Interview 5, Task 4, Figure 28, page 83). There were two students who presented Empirical

(Perceptual) proofs concerning injective and surjective functions (Interview 3, Task 3, Figure 19, page 73; and Interview 4, Task 3, Figure 22, page 76), namely, proofs that depended heavily on diagrams instead of algebraic notation. On the similar proofs of the final exam (Item 1, Figure 29, page 86), these same students presented entirely algebraic proofs in the Analytical (Transformational) schema.

The principle schema employed by all of the case study subjects in this study was Analytical (Transformational). "Transformational observations involve operations on objects and anticipation of the operations' results." (*ibid*, 258) There were not enough opportunities for these students to show understanding of Analytical (Axiomatizing) schema.

The question of how the students' schema for constructing proofs develops has another interpretation, too. It also refers to those external influences that impact the students in their work in the transition course during the passage of time in the semester. These influences are the models of valid proofs presented in the classroom, the textbook, and in any collateral reading the students may do; and it also includes the influences of how they study, with whom they study, and their participation in questioning about proof constructing and reporting. A deeper influence is the order of instructional topics, for in advanced courses it is common to use earlier concepts in the course to prove later results. Detecting this influence in a transition course is complicated by the continual introduction of mathematical material that is new to the students: sets, functions, number theory, equivalence relations, and the techniques of mathematical induction. Other implementations of transition courses ameliorate the

novelty of the mathematical material by review and reinforcement of the prerequisite calculus and linear algebra concepts.

In summary, the students in this transition course were alert to the proof techniques taught in this course, they would occasionally use inductive reasoning or accept symbolic proofs, and they would sporadically cite an authority for results. They were not in the category of those who accept inductive reasoning or external conviction schema as their primary modes of thinking, however. With one exception (S5), the entire class was attempting and succeeding to construct proofs in the context of the transition course and its choices of proof strategies.

Summary

The Research Questions were introduced in Chapter 1, contextualized in the review of the literature in Chapter 2, investigated by the methodology of Chapter 3, and analyzed in Chapter 4. This study grew from a perceived need to assess how students were learning to construct proofs (Research Questions 1 and 2, page 12) in a transition course (Chapter 1). Previous research suggests some real and potential obstacles (Research Questions 3 and 4) to such learning (Chapter 2). The study was designed to supplement the usual course written assessments with focused interviews that probed the students' schema (Research Questions 5 and 6) for planning proofs (Chapter 3). The analysis of these assessments revealed a partial pattern to the understandings of the students in a transition course (Chapter 4). The pretest and first

two interviews provided baseline information about the students' understandings of the three proof techniques. The objectives of this transition course centered on the three general proof techniques: direct proof, proof by contrapositive, and proof by contradiction. The course addressed the need for necessary logic background at the commencement, and demonstrated that the three proof techniques are logically equivalent. In this chapter, the research questions were addressed separately. The students' initial understandings about proofs were found to be appropriate, but mixed with some common misunderstandings. The interviews and written assessments followed the growth in the students' abilities to construct proofs.

Analyzing the obstacles to starting and completing proofs enriched the field of discussion for how students learn to construct proofs. The obstacles that surfaced in this study were obstacles related to (1) interpreting statements to be proved, (2) starting proofs by contradiction, (3) interpreting a contradiction within a proof, (4) understanding the role of definitions, (5) using previously proved results within proofs.

Further analysis investigated the ways in which instruction affected student learning of how to construct proofs. The textbook and instructor modeled the proof strategies and forms of reporting proofs, and this study noted the degree to which students used those models in their work. Finally, there was an effort to describe the effect of instruction on the ways the students think about proof construction. Students' schema are varied, but some aspects of the instruction had etched trails in their comments to show their impact.

The final chapter will relate the specific findings concerning these obstacles, derive conclusions of this study, admit some limitations of this study, and suggest some lines for further research.

CHAPTER FIVE

CONCLUSIONS

Introduction

This final chapter summarizes the findings of Chapter 4 and relates them to the previous research cited in Chapter 2. The Research Questions were introduced in Chapter 1, contextualized in the review of the literature in Chapter 2, investigated by the methodology of Chapter 3, and analyzed in Chapter 4. The findings relative to each research question are summarized first, followed by the conclusions which can be drawn. This chapter concludes by acknowledging the limitations of the study and offers suggestions for further research and for practice.

Primary Findings

Initial Conceptions of Proof

A pretest at the beginning of the course and the initial interviews provided measures of the students' understandings at the beginning of the course (Research Question 1). Pretest Item 2 (Figure 5, page 56) gave a familiar result from calculus, and asked the students to comment on how it applied to a particular situation. Nine

students incorrectly applied the false converse. Three students verified the hypothesis, but made no conclusion. Two students made comments that were not answering the question. Only one student made the correct observation that the result did not apply. This was an early indication that many students might have difficulty understanding what was to be proved.

Definition usage was assessed in two pretest items. Item 3 (Figure 7, page 58) used the definition of odd integers within a given proof; and Item 6 (Figure 10, page 61) required the use of the definition of an integer factor. These were correctly treated by only five students (Item 3) and six students (Item 6). This likewise portends that many students might have difficulty understanding what was to be proved.

Ten of the 16 students demonstrated by their work that they understood that negative results could be established by a counterexample (Pretest Item 5, Figure 9, page 61). This has bearing on their understanding of universal and existential quantifiers. This particular item, however, asked how they would disprove a specific result. It does not assess how they might decide to consider whether a counterexample exists.

Two items that only asked for recognition of a starting assumption for direct proofs (Item 3, Figure 7, page 58; and Item 4, Figure 8, page 60) were successfully done by 7 and 11 students, respectively. These were both problems with the context of even and odd integers, and understanding the definitions of these concepts may have interfered with some students' abilities to construct the proofs. Thirteen of the 16 students could start a direct proof (Pretest Item 6, Figure 10, page 61) with an

appropriate assumption, but only eight were able to complete the proof. The inference to be drawn is that they understood that a proof proceeds from the hypotheses towards the conclusion, but that for a variety of reasons, they were unable to obtain the conclusion.

The pretest, therefore, revealed the following conceptions of proof on the first day of the transition course. A direct proof proceeds by deductive steps from the hypothesis to the conclusion. A statement with a universal quantifier may be disproved by a single counterexample. Features that are potential difficulties include: understanding what is to be proved, understanding the converse of a statement, using a previously proved result, and using a definition.

Planning and Reporting Proofs

Since the initial conceptions of proof were weak in the specific areas mentioned, how do the students develop the skills of planning and reporting proofs? (Research Question 2) The sequence of five interviews elicited the students' comments while they confronted proofs in the three formats of direct proof, proof by contrapositive, and proof by contradiction. The students were able to recognize and to construct direct proofs; even with mathematical concepts that were new to them, such as divisibility, congruences, and elementary set theory. More of them had difficulties when abstract functions were involved in the concepts of injective

functions, surjective functions, and the composite of two functions. Progress in direct proofs involving these concepts was not assured. Of the six case study students, five of them could not construct such a direct proof (Interview 3, Task 3, Figure 19, page 73), and four of the five responding students could not construct the dual proof on the next interview (Interview 4, Task 3, Figure 22, page 76). In contrast, on a simpler related problem on the final exam (Items 1a and 1b, Figure 29, page 86), four out of five of these same students produced proofs that were nearly complete. The inference from this is that although individual students had some difficulties with abstract functions, or with the definitions of injective and surjective functions, or with composites of functions; the greater difficulty was with the coordination of the various concepts within a proof.

The method of proof by contradiction involves deducing a contradiction and interpreting it. There are three potential places where there could be difficulties in constructing a valid proof by these methods: the starting assumption(s), the derivation of a contradiction, and the interpretation of what the contradiction means for the proof. The first of these was discussed in the last chapter (pages 138-148). Different students encountered different difficulties in this study, but the most common hindrance was the last. The students knew from the examples of proofs by contradiction in class and textbook that achieving a contradiction in the course of a proof was not fatal, but was in fact near the end of the proof. Frequently, the statement of what had been proved was omitted by students in their proofs by contradiction, so that it was not clear whether they knew what the contradiction

signified. Since they were careful to include closing statements in their direct proofs, it may be inferred that the students in this study did not know how to make the conclusion of the proofs by contrapositive and proofs by contradiction.

The course also presented three forms of mathematical induction: ordinary mathematical induction, the principle of minimum counterexample, and complete induction. The principle of ordinary mathematical induction was successfully used by students when proving a formula. For example, the last task of the last interview (Interview 5, Task 4, Figure 28, page 83) asked for a proof by mathematical induction in Part e, and all five of the respondents provided valid proofs. Only one student required prompting from the researcher. On the other hand, 12 of the 16 students constructed valid proofs that an inequality was true for all positive integers (Final Exam Item 6, Figure 35, page 90).

The principle of minimum counterexample requires a proof by contradiction, and there were only three students who were able to construct such a proof on the final exam (Item 8, Figure 37, page 91). There were not enough assessments on this topic to measure the students' progress. Difficulties in developing skill with this method of proof include the several obstacles that are presented by proofs by contradiction: the compound hypothesis, the negation, the obtaining of a contradiction, and the interpretation of the contradiction.

The students had difficulty with complete induction in forming the induction hypothesis. Six of the 16 students were able to state the induction hypothesis correctly, but then only three were able to complete the proof (Final Exam Item 7,

Figure 36, page 91). Here again, there were not enough opportunities to measure changes in the students' skill with this method of proof.

The development of skill in planning proofs was perceptible in direct proofs, although this does not mean that students could always produce valid proofs. Later concepts in the course were challenging enough to impede their proof production. They developed the skill, as S5 said in Interview 4, by practice: "Not practicing enough. If you don't practice, you can't do it [construct proofs]." The forms of mathematical induction were introduced late in the semester, and from the point of view of research, there were not enough opportunities to assess student improvement with these proof methods.

Obstacles to Constructing Proofs

Undergraduate mathematics students encounter a variety of difficulties constructing proofs. The study exposed student difficulties with understanding the converse of a statement, using definitions properly when constructing proofs, and correctly stating the starting assumptions of proofs. The first two interviews revealed a fourth difficulty, namely, interpreting a contradiction obtained in the course of a proof.

The study identified five obstacles that hindered students in beginning their proofs (Research Question 3):

1. Interpreting statements to be proved
   a. Interpreting converses
   b. Interpreting conjunctions, disjunctions, and negations
   c. Interpreting universal and existential quantifiers
2. Understanding starting assumptions
3. Using definitions of terms in the hypotheses and conclusion
   a. Using previously proved results
   b. Choosing notation and representations

The research cited in Chapter Two noted the difficulties students have with interpreting statements to be proved, but none of those studies examined these particular obstacles. Selden and Selden (1987, 1995, 1997b) noted the same obstacles in their research on how students interpret more complicated statements, such as limits of functions or limits of Riemann sums. The usage of definitions has been the subject of two dissertations (Moore, 1990; Rin, 1982). Using previously proved results and choosing notation and representations are two obstacles observed in this study which have not been reported in the literature.

This study also identified four obstacles to the completion of students' proofs (Research Question 4):

1. Using definitions within proofs
2. Stating the induction hypothesis in proofs by complete induction
3. Using the method of minimum counterexample
4. Treating contradictions within proofs

There have been studies of ordinary mathematical induction (Dubinsky, 1986, 1989; Ernest, 1984; Harel, 2000), but not the forms identified here. The obstacle of treating contradictions within proofs affects the performance of students in their proofs by contradiction, and their proofs by the method of minimum counterexample. There has

not been a research study on how this difficulty of interpreting contradictions interferes with students completing proofs.

Understanding What Is to Be Proved

This study observed three special influences of instruction that appeared to effect the ability of students to understand what they are trying to prove (Research Question 5). First, the students appropriated the language and mathematical symbolism used in the textbook and by the professor. They wrote proofs on the written assessments and in the interviews that began with clear statements of the intended proof strategy, and of the hypotheses. This was most apparent in direct proofs and proofs by ordinary mathematical induction, but showed up clearly in the work of the eight students who wrote proofs by contrapositive on Final Exam Item 9a (Figure 38, page 92).

A second aspect of instruction was an appreciation for the forms in which proofs are written. The textbook and the instructor consciously drew attention to proofs written in paragraph form to prepare students to read proofs in advanced courses. Thorndike & O'Daffer (1993) summarize research which maintains that the learning of geometry proofs is not affected by whether the proof is presented in a list form or paragraph form. In this research, the interview students expressed a preference for a list form when the proof involved algebra steps. All of the interview students correctly rewrote a proof given in paragraph form into a list form (Interview

5, Task 1, Figure 25, page 80), and a given proof in list form into a paragraph form (Interview 3, Task 3, Figure 19, page 73; and Interview 4, Task 3, Figure 22, page 76) without hesitation or difficulty.

The third influence of instruction noted in this study was template proofs. The course did not teach specific forms for proofs, but the students used the forms that the instructor wrote in class. This appeared on the written assessments, as well as in the interviews when students constructed proofs. Direct proofs and proofs by ordinary mathematical induction were the principal models that were successfully followed. However, students did not emulate the models of proofs by complete induction and proofs by the method of minimum counterexample, particularly in making correct induction assumptions. There does not appear to have been any research of these influences. Objections that students will not think if they are given models to follow overlooks the pedagogical advantages for teaching and learning such models afford. With the difficulties identified here, templates may prove efficacious.

Effects of Instruction

A further purpose of this study was to examine how instruction in proof strategies improves the performance of students in solidifying schema in proof-planning and proof-reporting (Research Question 6). Two interview tasks in particular, asked students to give a holistic treatment of how they viewed the activity of producing a proof (Interview 4, Task 4, Figure 23, page 77; and Interview 5, Task

3, Figure 27, page 81). The three students who had been constructing valid proofs were unanimous in placing highest priority to the proof planning process, starting with the selection of a proof strategy and its incumbent starting assumption. The two students who had been having difficulty with technical definitions and using previously proved results, however, gave these skills pride of place and correspondingly lower emphasis on the planning and starting assumption features. These latter students also expressed that when they were asked for a suggested proof-strategy, they were being confronted by a multiple-choice question to which there was one "best" answer.

An individual student's schema for producing proofs is not immutable. It will continue to change throughout the course and throughout the student's mathematical career, as the student responds to advanced mathematical ways to look at this course's material, and as the student becomes aware of other mathematical connections among the concepts they have encountered in this course. One of the interview students, S5, appeared to look for linear organization of the concepts surrounding the process of producing proofs. He arranged the ingredients contributing to making proofs (Interview 4, Task 4, Figure 23, page 77) in terms of how he would organize a direct proof. In proofs by contradiction, he was unable to articulate what happened after a contradiction had been obtained. Although he knew from the examples in the textbook and in class that somehow the proof was at an end, he could not say why. He understood that the starting assumption for proofs by contradiction was more involved, but he did not know how to proceed with such a proof. On the final exam, he constructed one valid proof by contrapositive and two valid proofs by

contradiction, although one of the latter did not have the starting assumption correctly written.

Other students, such as S1, S3, and S6, showed indications of still modifying their schema as the course concluded. Each of them had shown areas of uncertainty or weakness during the semester that they remedied by the time of the final exam. This is part of the organic nature of schema, which continually change as people live and think. Students such as S2, however, had schema already formed at the beginning of the course. She understood the distinctions between hypothesis and conclusion, and a statement and its converse. She demonstrated that she could construct direct proofs and proofs by contrapositive correctly, and understood the significance of a contradiction obtained in a proof. She readily assimilated the concepts of proofs by contrapositive and proofs by contradiction, and used them effectively.

This present study builds on earlier research involving college students' abilities to construct proofs which examined student errors (Selden & Selden, 1987), attitudes (Moore, 1990, 1991, 1994), and expert vs. novice understandings (Hart, 1994). The empirical research of Harel and Sowder (1998; Sowder and Harel, 1999) provided a classification of students' schema for proof understanding, production, and appreciation. It was not the intent of this study to characterize the students according to this classification, but the work of Harel and Sowder informed the process of the interview protocols.

Conclusions

Responses to the pretest indicated that the students had appropriate conceptions of direct proofs, and the idea that a counterexample could disprove a universal statement. Their construction of proofs was, however, hindered by several factors. The obstacles to constructing proofs identified in this study were: understanding definitions, using definitions, and understanding an implication and its converse. These results confirm the work of Moore (1990, 1991, 1994), and Rin (1982). This replication of findings is confirmatory, not redundant. It indicates that these are genuine obstacles, not just anomalies in the research data. Martin and Harel (1989) found obstacles in understanding the converse of an implication and in applying implications properly. These also were evident in the pretest performance of the students in this study, but was not evident in the subsequent written assessments or interviews. The distinction may be that Martin and Harel were studying prospective elementary school teachers, whereas most of the students in this study were intending to major in mathematics, statistics, science, or computer science. The majors of the six interview students are listed in Table 3.1 (page 39).

The students in this study had difficulties with the following notions: converse, negations, conjunctions, universal quantifiers and existential quantifiers. Although the textbook and instructor modeled how to write correct quantifiers in mathematical sentences, many of the students either omitted them or misused them. These obstacles were not independent of each other, however. A proof by minimum counterexample

requires a proof by contradiction, whose hypothesis is compound and involves a negation. The particular problem on the final exam that prescribed proof by minimum counterexample also necessitated De Morgan's Laws in order to state correctly what had to be proved (Final Exam, Item 8, Figure 37, page 91). These results are consistent with prior research by Dubinsky and Lewin (1986), and by Selden and Selden (1987, 1990a, 1997b).

The difficulty which this study observed with proofs by contradiction has been noted by others (Friedlander and Herschkowitz, 1997; Roberti, 1987; Senk, 1985; Thompson, 1996), but these were reports of pedagogical practice, not research studies. The obstacles due to the form of the compound hypothesis were noted already. This study found that the conclusion of this form of proof is also an obstacle. When a contradiction is achieved in the proof, often students did not know how to complete the proof. In other words, they were unable to interpret what it means when deductions lead to a contradiction within a proof.

Limitations of This Study

This study is open to the criticism that the patterns it has identified may be just a set of particulars, due to the limited number of students. However, the triangulation of the entire class (n=16) with the interview subset (n=6) and the obstacles noted in the research literature give this study much more power than if it stood alone. The parameters of this study are openly given in this report, and may be replicated by

others. The qualitative nature of the investigation depended on the interactive interviews. Such interviews gave opportunities to probe more deeply for the students' explanations for their decisions, and therefore provide much more useful information than written assessments alone.

One could object that the chronological measure of the assessments planned in this study were not smooth enough to permit more detailed analysis of each student's progress. The original plan was to spread the interviews more evenly over the semester, and to sample the written assessments more frequently, but logistics and oversights interfered. It was anticipated that there might be students who would withdraw from the interview program; in fact, only one did. The interview protocols were kept flexible, but at the cost of persistent assessing of some individual concepts. The written assessments included questions submitted by the researcher at the courtesy of the instructor. However, difficulties obtaining copies of the student work arose at the middle of the semester. Some data was not made available to the researcher; these instances have been noted in Chapter 4.

It is true that there was not enough planned assessment of proofs by contradiction. That was considered early in the planning of this study. Proof by contradiction is rich enough in obstacles to merit a dedicated research study. The obstacles are several, and it would take such a study to separate their influences.

Similarly, this study did not intend to provide complete analysis of the difficulties arising in the various proofs by mathematical induction. The researcher and several members of the dissertation committee discussed these, and decided the

difficulties were too broad and pervasive for the kind of study being planned. Even though there has been research on the learning of mathematical induction (Dubinsky, 1986, 1989; Ernest, 1982; Harel, 2000), there are still enough questions about how to teach it in a way that fosters understanding, to invite another study.

Suggestions for Further Research

This study identified several important obstacles to students' successful construction of proofs; namely, using definitions, using previously proved results, using quantifiers, and interpreting contradictions within a proof. A subsequent research study could focus on just one of these, and plan a continuous assessment throughout a transition course to measure the students' cognitive growth.

As previously noted, mathematical induction and proofs by contradiction were not foci of this study. This was on purpose, since there have been other studies of mathematical induction, and there is room for more. Proof by contradiction is a candidate for a dedicated study. There are quite a few articles on how teachers perceive the obstacles involved in teaching and learning proofs by contradiction, but there has been no research study at the college level to test the effectiveness of various pedagogical approaches.

There has not been a longitudinal study of how students encounter these concepts of proof construction as they progress through their undergraduate career. The study of how students in various undergraduate mathematics courses view proofs

(Harel and Sowder, 1998; Sowder and Harel, 1999) is longitudinal in that it tracks students through several courses, but the research is concentrating on classifying the schema students have for accepting the validity of roofs, not how they construct proofs. A longitudinal study, of course, requires planning the assessments to repeatedly invite responses about the same limited set of concepts. The sample population would have to be large enough to allow for some attrition. The difficulties in designing such a study would also be related to the lack of a unified textbook series for the undergraduate mathematics curriculum.

A larger question is: What is the role of the transition course in the undergraduate mathematics curriculum? After defining its place, and setting it in the context of its prerequisite courses and subsequent courses, a research study could investigate whether it successfully fills that role. The usual justification for the transition course is that students in advanced calculus and abstract algebra do not have these tools for constructing proofs; and without the transition course, the advanced course must develop those tools. Does the transition course actually produce students who do better work at constructing proofs in advanced courses? Do subsequent courses in abstract algebra and advanced calculus require the same level of rigor? There has been no research on these important aspects of a transition course.

The problem of students needing better grounding in topics that they have not seen before, such as set theory and abstract functions, could be addressed by different forms of assessments such as class presentations or cooperative group work to involve them in debate over the meanings of definitions. For example, many students upon

first encountering set theory do not understand the set builder notation (Moore, 1990), and think it means any subset of the defined set, instead of meaning all elements that satisfy the conditions defining the subset.

A further research study could consider the effects of study habits on learning abstract notions in the transition course. Moore (1990, 144) made brief mention of this, but only in summary comments. How students study and whether they study what the textbook and instructor are trying to communicate should be related to how well they assimilate the concepts surrounding proof construction. There are objective measures such as class attendance and numbers of hours spent in study, but research could clarify how students view the productivity of their study time. A related variable in such a study could be modes of self-assessment employed by the students. There are numerous techniques for monitoring one's assimilation of new concepts, such as mnemonics, Polya's problem-solving strategies (Polya, 1957), and Schoenfeld's bird-on-your-shoulder monitor (Schoenfeld, 1995).

The advanced mathematical processes of generalization and abstraction are important in the undergraduate mathematics curriculum in all subsequent courses to the transition course. By having students look back at their proofs, students can learn ways to evaluate the validity of their own proofs and the proofs of others. A study, therefore, that concentrates on generalization and abstraction, is another research possibility.

Implications for Practice

When transition courses were initiated in the 1970s, the principal justification was that they would provide the grounding and practice in the construction of proofs that was lacking in students entering courses in advanced calculus and abstract algebra. The course that was sampled for this study is similar to courses implicitly described by textbooks commonly used for transition courses (Bond and Keane, 1999; Smith, Eggen, and St. Andre, 1997; Solow, 1990; Velleman, 1994). The textbook (Chartrand et al., 1999) and the instructor developed the course topics for this particular transition course on the theme of proof construction and understanding. The instructor used a typical lecture/discussion format for the class, gave frequent quizzes, and was available to the students for additional help. But the obstacles to students completely assimilating the concepts involved in constructing proofs are still there. Most of the students could construct simple direct proofs and proofs of formulas by mathematical induction. But definition usage, the treatment of contradictions, and the use of previously proved results within proofs persisted as obstacles. These topics would have to be targeted as essential in order to reduce the effect they have as obstacles to students' performance in constructing proofs.

The goal of the transition course is to teach the students what they need to know about constructing proofs to prepare them for advanced courses. What makes this goal ambitious is assuming that it can be accomplished by a single course. Conceptual learning of proofs for all of mathematics must be studied in the context of

all mathematics. The concept of a reasoning and proof strand throughout all undergraduate mathematics courses would be a much more effective treatment, in the opinion of this researcher and others (Moore, 1990, 1994; Leron, 1985; Selden and Selden, 1990a).

Summary

The early chapters of this study presented the position that proof-making is a complex problem-solving activity which requires the coordination of skill with variables and their meanings, definition usage, logic, prior theorems, and the crucial starting assumptions. The students in this one particular course in mathematical proof revealed in their writing and in interviews that there are a number of reasons for their successes and failures.

The study began with the premise that student responses in written work and interactive interviews could reveal their conceptions of proof, the difficulties in constructing proofs, and their growth in understanding the process of making and reporting proofs. It was expected that the difficulties that students encountered in constructing proofs could be discovered by expanding on the obstacles that research had already documented. In particular, starting proofs and definition usage within proofs have been the subject of previous research studies, and the difficulties with proofs by contradiction have been frequently mentioned in the literature. This study confirmed these obstacles, as well as the existence of others: notably, the role of

previously proved results within proofs, the interpretation of contradictions within proofs, and the role of mathematical notation and representations.

Furthermore, this study examined how the case study students transformed their thinking concerning the constructing and reporting proofs. Although complicated by the continual incursion of mathematical concepts that were new to the students, the students did have patterns of thinking which were particular to themselves.

Appendix A

Pretest

MATH 314 - Mathematical proof NAME: ____________________

1. Here is a known fact: If $a < 0$, then the quadratic function $y = f(x) = ax^2 + bx + c$ has a maximum y-value.

Given the quadratic function $g(x) = -4.9x^2 + 9.8x - 32.1$, what can you conclude from the known fact ?

2. Here is a known fact:

If the function $f(x)$ has a relative maximum at $x=a$, then $f'(a) = 0$.

Given that $g(x) = x^3 - 3x + 7$ has $g'(1) = 0$, what can you conclude from the known fact?

3. You have probably noticed that when you add two odd integers, the sum always seems to be an even integer. In mathematics, it is commonplace for observations of patterns like this to lead to conjectures and then to attempted proofs. The above statement leads to this conjecture:

If a and b are odd integers, then a + b (the sum) is an even integer.

One student's proof looked like this:

If a and b are odd integers, then a and b can be written $a = 2m + 1$ and $b = 2n + 1$, where m and n are other integers.
If $a = 2m + 1$ and $b = 2n + 1$, then $a + b = 2n + 1 + 2m + 1$.
If $a+b = 2m + 1 + 2n + 1$ then $a+b = 2m + 2n + 2$.
If $a+b = 2m + 2n + 2$ then $a+b = 2(m + n + 1)$.
If $a+b = 2(m + n + 1)$ then a+b is an even integer.

a. Look at the first statement. What does she assume?

What additional assumptions, facts, or algebraic properties did she use within her argument?

b. Does the student's argument prove the conjecture? Describe the features of the argument that support your position.

4. How would you begin a proof of the fact that the square of an even integer is an even integer?

5. Consider the expression $n^2 - n + 41$.

A student claimed that if n is an integer, then $n^2 - n + 41$ is a prime number.

How could you prove that this claim is not true?

6. Here is a definition: An integer m is a <u>factor</u> of the integer n if and only if there is a integer k such that $n = mk$.

Prove that if a is a factor of b, and if b is a factor of c, then a is a factor of $a + c$.

Appendix B

Quizzes

Quiz #1 – MATH 314 - Mathematical Proof
Dr. Ping Zhang
January 21, 2000

1. (3pts.) Let $A = \{x \mid x \in R \text{ and } x^2 - 2x + 1 = 0\}$ and $B = \{0, 1\}$.
(a) List the elements of A.

(b) Which of the following statements is true? Explain.
(i) $A = B$ (ii) $A \subset B$ (iii) $B \subset A$

(c) Determine the intersection and union of A and B.

2. (4 pts.) Give an example of
(a) a set S such that $S \subseteq \wp(S)$ and $|S| = 5$.

(b) two sets A and B such that $A \in B$ and $A \subseteq B$

3. (2 pts.) Let $A = \{1, \{1\}\}$. Determine $\wp(A)$.

Quiz #5 – MATH 314 - Mathematical Proof
Dr. Ping Zhang
March 24, 2000

1. (3pts.) Let A = {1, 2, 3, 4, 5}. Then the distinct equivalence classes resulting from an equivalence relation R on A is {1, 2}, {3,5}, {4}.
What is R ?

2. (7 pts.) Let R be the relation defined on **Z** by aRb if 2a + b = 0 (mod 3). Prove that R is an equivalence relation.

3. (4 pts.) In $\mathbf{Z}_9$, express the following sum and product as $[\gamma]$, where $0 \leq \gamma \leq 8$.

(a) [3] + [6]

(b) [-2] · [5]

(c) Let [a], [b] ε $\mathbf{Z}_9$. If [a] ·[b] = [0], does it follow that [a] = [0] or [b] = [0] ? Why?

4. (6 pts) Let the function f: **R** → **R** [be] defined by $f(x) = 3x - 1$.

(a) Show f is one to one.

(b) Show f is onto.

Appendix C

Final Exam

Final Exam
Mathematics 314
April 21, 2000
Dr. Ping Zhang

1. (5 pts) A function $f: \mathbf{R} \rightarrow \mathbf{R}$ is defined by $f(x) = 7x - 1$.
(a) Prove that f is one-to-one.

(b) Prove that f is onto.

2. (8 pts.) Let A, B, and C be sets. Prove that $(A \cap B) \times C = (A \times C) \cap (B \times C)$.

3. (8 pts.) Prove that 1000 cannot be written as the sum of three integers, an even number of which are even.

4. (10 pts.) A relation R is defined on $\mathbf{Z}$ by xRy if $x^2 = y^2 \pmod 6$
(a) Prove that R is an equivalence relation.

(b) Determine the distinct equivalence classes.

5. (8 pts) Prove the following:
<u>Result</u>: Let $x \varepsilon \mathbf{Z}$. Then x^3 is even if and only if $5x^2$ is even.
(The following two Lemmas were not printed on the Exam, but were written on the blackboard. Most students proved the Lemmas; others did not.)
Lemma 1: If x^3 is even then x is even.
Lemma 2: If $5x^2$ is even, then x is even.

6. (8 pts.) Prove that $2^n > n^2 + n$ for every integer $n \geq 5$

7. (10 pts.) A sequence $x_1, x_2, x_3, \ldots$ is defined recursively by $x_1 = 1$, $x_2 = 4$, and
$$x_n = -x_{n-1} + 2x_{n-2} + 6n - 7 \text{ for all } n \geq 3.$$
Use the strong form of induction to prove that $x_n = n^2$ for all positive integers n.

8. (10 pts.) Use the method of minimum counterexample to prove that $6 \mid (n^3 + 5n)$ for every positive integer n.
[Recall that $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$].

9. (25 pts.) Prove or disprove.

(a) Let $x, y \in \mathbf{Z}$. If $6x + 7y$ is even, then y is even.

(b) In $\mathbf{Z}_6$, if $[a] + [b] = 5$, then $a + b = 5$.

(c) Every even integer is the sum of two unequal odd integers.

(d) For every two sets A and B, $(A \cup B) - B = A$.

(e) If a set S of real numbers contains a least element, then S is well-ordered.

10. (8 pts.) A proof of $P \Rightarrow Q$ is to be given. If the first step of the proof is given below, then which of the following is true:

(1) a direct proof is being used.
(2) a proof by contrapositive is being used.
(3) a proof by contradiction is being used.
(4) an error has been made.

(a) Assume that Q is true _____.

(b) Assume that P is true _____.

(c) Assume that $P \Rightarrow Q$ is true _____.

(d) Assume that Q is false _____.

(e) Assume that P is false _____.

(f) Assume that $P \Rightarrow Q$ is false _____.

(g) Assume that P is true and Q is false _____.

(h) Assume that P is false and Q is true _____.

Appendix D

Interview Protocols

Interview #1 – INTERVIEWER'S COPY, WITH SUGGESTED PROMPTS
After Ch. 2 (Logic) and Ch. 3 (Direct Proof and Proof by Contrapositive)

1. In your own words, how would you describe the converse of an implication?
[The response will depend on their choice of representation, whether they use symbols or words, but the interviewer will ask a follow-up question to allow the student to clarify their response.]

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2. a. Fill in the Truth Table:

P	Q	$P \Rightarrow Q$ implication	$Q \Rightarrow P$ converse	$\sim P \Rightarrow \sim Q$ inverse	$\sim Q \Rightarrow \sim P$ contrapositive
T	T				
T	F				
F	T				
F	F				

2b. Which of the four items above (the implication, its converse, its inverse, its contrapositive) are logically equivalent?

2c. If you were starting a direct proof of $P \Rightarrow Q$, what would the starting assumption be?
If you were starting a proof by contrapositive of $P \Rightarrow Q$, what would the starting assumption be?

If you were starting a proof by contradiction of $P \Rightarrow Q$, what would the starting assumption be?

3a. You have seen implications proved in Algebra, Geometry, Trigonometry, Calculus, Linear Algebra, and Number Theory, such as the following. Choose one of them for us to discuss.

* If a triangle is a right triangle, then the sum of the squares of the legs equals the square of the longest side (Pythagorean Theorem).

* If $b^2 - 4ac > 0$ then the quadratic equation $a x^2 + b x + c = 0$ has two real roots (Quadratic Formula).

* If x is not 1, then $1 + x + x^2 + \ldots + x^{n-1} = (x^n - 1)/(x-1)$ (Geometric Series)

* If $f(x)$ is differentiable and $f(x)$ has a relative maximum at x=a then $f'(a) = 0$ [This one was on the pre-test]

* If $f(x)$ is twice differentiable and $f(x)$ has an inflection point at x=a then $f''(a) = 0$

* If $\lim f(x) = A$ and $\lim g(x) = B$ as $x \to a$, then $\lim (f(x) + g(x)) = A + B$

* If the series $a_1 + a_2 + \ldots$ converges, then $\lim a_n = 0$

* If the n x n matrix A has det(A) not zero, then A has an inverse matrix A^{-1}

* If the n x n matrix A has an inverse matrix A^{-1}, then det(A) is not zero.

* If B and C are inverses for the matrix A, then B = C (Uniqueness of Inverses)

3b. For the implication that you chose from the list above, state the converse, the inverse, and the contrapositive. Which of those four are true and which are false?

3c. How would you start a Proof by Contrapositive of the implication you chose above? Just talk about the beginnings of the proof, and why you would make that beginning.

Interview #2 - INTERVIEWER'S COPY, WITH SUGGESTED PROMPTS
After Ch. 4 (Proofs Involving Sets), Ch. 5 (Proof by Contradiction),
and Ch. 6 (Prove or Disprove)

1. a. How would you give a define an even integer?
Write it down.
[This is for reference at the end of discussing the following proof.]

Now consider this implication: **If the integer n is a multiple of 4 then n is even.**

b. Write or say: The converse, the inverse, and the contrapositive of the above statement.
Then for each of the four implications, tell which of them is true and which of them is false.
Explain your reasoning.

[If the student needs help, the interviewer will suggest,
"Would it help to write the implication as $P \Rightarrow Q$?"
"Circle the hypothesis P and the conclusion Q in the statement."]

c. If you were to prove the implication
If the integer n is a multiple of 4 then n is even
What would you write for a starting assumption?

d. What would you try to do next?

e. What would you ultimately want to show in such a proof?
[Prompt - how would you use your definition of even integers?]

---

2. a. Write down your definition of "rational number."

b. Here is a Result and a Proposed Proof. Is the proof correct? Why or why not?
Result: If x is a nonzero fraction and y is an irrational number, then xy is an irrational number.

Proposed Proof:
Assume x is a nonzero fraction and y is an irrational number and xy is a fraction. Then y = (xy)/x is a fraction. This is a contradiction. Therefore, xy is irrational.

[Note - Students may be more comfortable with writing x = a/b, y = c/d as fractions.]

c. What was the strategy for this proof - - - Direct Proof? Proof by Contrapositive? Proof by Contradiction?
[prompt: It may help to circle the P and the Q in the implication.]

d. Explain what the "contradiction" is in the proof. Does the contradiction mean that the proof is wrong?

e. Would the proof still be valid if the restriction that x is nonzero were removed? Discuss why or why not.

---

3. Result: **Let a and b be real numbers. If ab is nonzero, then a is nonzero.**

a. Discuss how you would begin a proof of this result.
[Allow time, then if necessary, use prompts:
 What starting assumption would you use?
 What proof strategy would you try first?
Note that this result may be proved directly, by contrapositive, or by contradiction.]

b. What would be your next step?

c. How would you know when your proof would be done?

d. Do you think now that the result is true?

e. State the converse of the Result. Is the converse true?

f. State the contrapositive of the Result. Is the contrapositive true?

---

4. Suppose you are trying to prove that $P \Rightarrow Q$ for some statements P and Q.
a. Match the following assumptions with the kind of proof it leads to

Assume P	direct proof
Assume Q	proof by contradiction
Assume ~P	proof by contrapositive
Assume ~Q	not helpful
Assume P and ~Q	

b. Explain the expression "not helpful" above. In what way is the assumption it refers to not helpful?

c. If you started a direct proof by assuming what you said above, what would be your ultimate goal that you wished to show? Why?

d. If you started a proof by contrapositive by assuming what you said above, what would be your ultimate goal that you wished to show? Why?

e. If you started a proof by contradiction by assuming what you said above, what would be your ultimate goal that you wished to show? Why?

Interview #3 – INTERVIEWER'S COPY, WITH SUGGESTED PROMPTS
After Ch. 6 (Prove or Disprove), Ch. 7 (Equivalence Relations), and Ch. 8 (Functions)

1. Let A = {r,s,t,u} be a set with four elements.

a. Give an example of a function $g : A \to A$ which is one-to-one, but not onto.
Give an example of a function $h : A \to A$ which is onto, but not one-to-one.
[Interviewer's comment: These are impossible tasks, intended to provoke discussion about the definitions of one-to-one and onto.]

	g			h	
r		r	r		r
s		s	s		s
t		t	t		t
u		u	u		u

[Prompts as needed:
is that a function?
Is that function one-to-one? Why?
Is that function onto? Why?]

b. Find the composition functions $g \circ h$ and $h \circ g$:

	$g \circ h$			$h \circ g$	
r		r	r		r
s		s	s		s
t		t	t		t
u		u	u		u

[Do you remember the definition of the composite function?]

---

2. Write down your definition of <u>one-to-one</u> function (same as the word <u>injective</u>.)
[Be sure they get this right, because they need it to do the following task.]
[Be sure they can tell the difference between the definition of a function and the definition of a one-to-one function.]
[Notice that this task is stated without mathematical notation, in order to allow the students to choose notation.]

---

3. Suppose f : A --> B and g : B --> C are functions.
Now suppose that you are required to prove the
Result: If $g \circ f : A \to C$ is one-to-one, then f is one-to-one.

a. What are your choices for a proof-strategy, and which one would you choose?
[Any of the three strategies may be claimed at first; they will be invited to look back on the proof they construct to see what strategy they did in fact use.]

b. Given the proof-strategy that you just chose, what would your starting assumption be?
[This is the main point of the semester course.]

c. Review the definition of one-to-one that you wrote above.
[They need to see that the definition of one-to-one function now needs to be applied to the function g∘f.]

d. How would you start writing the proof?
[If needed, comment on the need to look at what they are trying to prove.]

e. What would your next step be?
[Expected difficulty: Understanding the definition of one-to-one as applied to the function g∘f.]

f. What are you trying to get to as a goal?
[This is the reminder to keep their eyes on the goal.]

g. What is in the way of your completing the proof now?

- -

4. Prove or Disprove:

Result: If a and b are positive real numbers, then $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$

[The result is false, since the equality is only true if a=0 or b=0.]

a. How would you start on this?
[Expected response: Try some number values for a and b, and see if it is true. They will quickly find a counterexample. This proposed result is in fact false for all positive a and b.]

b. Do you think the equation is true for all a and b?

c. Decide whether to prove or disprove the Result.
[At this point, invite the student to state a revised proposition: If a and b are positive real numbers, then $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$.]
[If a prompt is needed, ask if there are any values of a and b for which it is true. They may suggest a = 0 or b = 0, and need to be reminded that positive numbers are not zero.]

d. Would you like to try a Direct Proof, a Proof by Contrapositive, or a Proof by Contradiction?
[Inequalities are usually proved with either proof by contrapositive or proof by contradiction. They will find that it is not clear where to start if they want to do a direct proof.]

e. What would your starting assumption be in such a proof?

f. What would your next step be?
[Prompt: If they do not know what to do next, lead them to the suggestion to square both sides of the equation.]

g. What would your goal be for the end of the proof?

h. What is in the way of your completing the proof now?

- -

Interview #4 – INTERVIEWER'S COPY, WITH SUGGESTED PROMPTS
After Ch. 8 (Functions)
[Items #1-3 build on Interview #3]

1. Let A = {a,b,c} be a set with three elements, and B = {r,s,t,u} be a set with four elements.

a. Give an example of a function $g : A \rightarrow B$ which is one-to-one, but not onto.
Give an example of a function $h : B \rightarrow A$ which is onto, but not one-to-one.

	g			h	
a		r	r		a
b		s	s		b
c		t	t		c
		u	u		

b. Find the composition functions $g \circ h$ and $h \circ g$:

$g \circ h$ $\qquad\qquad$ $h \circ g$

---

2. How would you define an <u>onto</u> function (same as the word <u>surjective</u>)?

---

3. Suppose $f : A \rightarrow B$ and $g : B \rightarrow C$ are functions.
Last time, you proved that if $g \circ f$ is one-to-oen, then f is one-to-one.
Now suppose that you wanted to prove this result:

Result: If $g \circ f : A \rightarrow C$ is one-to-one, then g is onto.

a. What are your choices for a proof-strategy, and which one would you choose?

b. Given the proof-strategy that you just chose, what would your starting assumption be?

c. Review the definition of onto that you wrote above.

d. How would you start writing the proof?

e. What would your next step be?

f. What are you trying to get to as a goal?

g. What is in the way of your completing the proof now?

h. Write the proof in a paragraph form, as the textbook has been doing.

- -

4. Here are cards with some of the ingredients of proofs that you have seen this semester:

[Prepared cards with the following items, one to a card. Students were asked to arrange the cards in any kind of linear, branching, or circular pattern that would show how they think of the process of proving.]

Identify the hypothesis P and consequent Q
Check Definitions
Choose Strategy (Direct Proof, Proof by Contrapositive, Proof by Contradiction)
Choose Starting Assumption
Mathematical Background Knowledge
Prior Theorems from the course
Check the goal of the proof
[Others?]

[Prompts - ask "why did you put them in this way?" "What is the relationship between these cards?"]
[After they are settled on an arrangement, make a copy of their diagram, and confirm it with them.]

5. Find the inverse function for $g(x) = 2/(x - 1)$
[Prompt: let $y = g(x)$, and solve for x]
[This is the same as some of the book's examples and exercises]
If the inverse function is $h(x)$, then show that
$g \circ h(x) = x$ for almost all x; and
$h \circ g(x) = x$ for almost all x.

- -

Interview #5 – INTERVIEWER'S COPY, WITH SUGGESTED PROMPTS
[This is the final Interview]

1. Someone wrote the following proof of a mathematical assertion.

Assume A is an nxn matrix and $A^2 = I$, where I is the nxn identity matrix. Then $\det(A^2) = \det(AA) = \det(A)\det(A) = \det(I) = 1$, so $(\det(A))^2 = 1$. Therefore, $\det(A) = \pm 1$.

a. What result is established by this proof?
[If the student has no response, suggest they look at the keywords, "Assume" and "Therefore."]
[If the student is unable to state the proposition as a conditional statement, suggest that they think of the language $P \Rightarrow Q$, and circle the portion of the proposition that represents P and the portion that represents Q. Then ask How is Q related to the assumptions?]

b. What proof strategy was used?
[There is no contradiction, so the expected response is "Direct Proof"]

c. What was the starting assumption?

d. Explain what the person was doing in the second sentence of the proof. What do you suppose was their reason for doing these steps?

e. Is the proof valid or not? Why?

---

2. Here is another proof of another mathematical assertion.

Assume that a and b are any odd integers, and that a^2+b^2 is a multiple of 4.
Then $a = 2k+1$ for some integer k,
$b = 2m+1$ for some integer m, and
$a^2+b^2 = 4n$ for some integer n.
It follows that

$$4n = a^2+b^2 = (2k+1)^2+(2m+1)^2 = 4k^2 + 4k + 1 + 4m^2 + 4m + 1$$
$$4n = 4(k^2 + k + m^2 + m) + 2.$$

This may be written

$4(n - k^2 - k - m^2 - m) = 2$, which says that 2 is a multiple of 4.

This is a contradiction. Therefore, a^2+b^2 is not a multiple of 4.

a. What result is established by this proof?
[The Assumption is a compound statement; have them state the assumptions separately]
[If they still have trouble with the hypotheses, ask for the conclusion; then as what relationship is there between the conclusion and the hypotheses.]
[If the student is unable to state the proposition as a conditional statement, suggest that they think of the language $P \Rightarrow Q$, and circle the portion of the proposition that represents P and the portion that represents Q. Then ask How is Q related to the assumptions?]

b. What was the proof strategy?
[If they need a prompt, ask what the three strategies are that they have studied.]

c. What was the starting assumption?
[This is asked again after the discussion of (a), to make sure they can state it.]

d. Explain what the person was doing in the sentence of the proof which starts with "It follows that...." What do you suppose was their reason for doing these steps?
[Prompt: Algebra.]

e. Is the proof valid or not? Why?

---

3. Describe the process of constructing a proof.
[This is an unstructured way of asking the same thing as the corresponding question in the previous interview with the slips of paper.]

---

4. Let's find out what we can about the expression $6^n \pmod 5$, where n is a positive integer.

a. First, calculate some examples:

$6^1 \pmod 5 = \underline{\quad 6\ } \pmod 5 = \quad \pmod 5$
$6^2 \pmod 5 = \underline{\quad 36\ } \pmod 5 = \quad \pmod 5$
$6^3 \pmod 5 = \underline{\ 216\ } \pmod 5 = \quad \pmod 5$
$6^4 \pmod 5 = \underline{1296\ } \pmod 5 = \quad \pmod 5$

b. Now, calculate $6^n \pmod 5 =$
Can you explain how you obtained your answer?
[Expected response: Because the four examples in part (a) always obtained the number 1 – this would be a response that shows inductive reasoning. Suggest they think of $6 = 1 \pmod 5$, and raise both sides to the nth power.]

c. Next, let's look at 6^n in a different way, writing it as $(1 + 5)^n$, and using the binomial expansion from Algebra:

$$(1 + 5)^n = 1 + n(5)^1 + (n(n-1)/2)(5)^2 + \ldots + (5)^n$$

Then calculate that expression (mod 5)
Can you explain how you obtained your answer?
[Expected response: What's the binomial expansion? –they should have seen it in algebra in high school, and in calculus when differentiating x^n.]
[If they cannot calculate the result, encourage them to think of arithmetic (mod 5).]

d. What have we established? Have we proved it?

e. So now you have two direct proofs from algebra which show that $6^n = 1 \pmod 5$. Now prove it by Mathematical Induction.
[Expected response: They should be able to construct a proof by mathematical induction without difficulty. They might use divisibility instead of congruence arithmetic, but that's Ok.]

f. Was your proof of the inductive step a Direct Proof, a Proof by Contrapositive, or a Proof by Contradiction?

g. Did any of the proofs that $6^n = 1 \pmod 5$ use the fact that 5 is a prime number? Can you generalize the result?
[The intended generalization is $(a + 1)^n = 1 \pmod a$, but they may not be looking for that kind of statement.]

h. Which of the above three proofs will work to prove your generalization? Explain. If you are not sure, write them out in terms of your variable.

Appendix E

Human Subjects Institutional Review Board Research Protocol Clearance

Human Subjects Institutional Review Board

Kalamazoo. Michigan 49008-5162
616 387-8293

WESTERN MICHIGAN UNIVERSITY

Date: 24 February 2000

To: Christian Hirsch, Principal Investigator
Pence Atwood, Student Investigator for dissertation

From: Sylvia Culp, Chair Sylvia Culp

Re: Extension and Changes to HSIRB Project Number 98-10-32

This letter will serve as confirmation that the extension and changes to your research project "Validating Proofs in Advanced Calculus" requested in your Project Approval Review Form received 7 January 2000 have been approved by the Human Subjects Institutional Review Board.

The conditions and the duration of this approval are specified in the Policies of Western Michigan University.

Please note that you may **only** conduct this research exactly in the form it was approved. You must seek specific board approval for any changes in this project. You must also seek reapproval if the project extends beyond the termination date noted below. In addition if there are any unanticipated adverse reactions or unanticipated events associated with the conduct of this research, you should immediately suspend the project and contact the Chair of the HSIRB for consultation.

The Board wishes you success in the pursuit of your research goals.

Approval Termination: 24 February 2001

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