6-2001

Cost Domination in Graphs

David John Erwin
Western Michigan University

COST DOMINATION IN GRAPHS

by

David John Erwin

A Dissertation
Submitted to the
Faculty of The Graduate College
in partial fulfillment of the
requirements for the
Degree of Doctor of Philosophy
Department of Mathematics and Statistics

Western Michigan University
Kalamazoo, Michigan
June 2001

UMI Number: 3017311

Bell & Howell Information and Learning Company
300 North Zeeb Road
P.O. Box 1346
Ann Arbor, MI 48106-1346

COST DOMINATION IN GRAPHS

David John Erwin, Ph.D.

Western Michigan University, 2001

Let G be a connected graph having order at least 2. A function $f : V(G) \to \{0, 1, \ldots, \text{diam } G\}$ for which $f(v) \leq e(v)$ for every vertex v of G is a cost function on G. A vertex v with $f(v) > 0$ is an f-dominating vertex, and the set $V_f^+ = \{v \in V(G) : f(v) > 0\}$ of f-dominating vertices is the f-dominating set. An f-dominating vertex v is said to f-dominate every vertex u with $\mathrm{d}(u, v) \leq f(v)$, while the vertices in $V(G) - V_f^+$, namely, those vertices of G that are not f-dominating, do not f-dominate any vertices of G. A cost dominating function on G is a cost function f in which every vertex is f-dominated by some vertex in the f-dominating set.

For a cost function f on a nontrivial connected graph G, let $\sigma(f) = \sum_{v \in V(G)} f(v)$. The cost domination number $\gamma_c(G)$ is the minimum value of $\sigma(f)$ over all cost dominating functions f on G and a cost dominating function f with $\sigma(f) = \gamma_c(G)$ is a minimum cost dominating function.

We establish several sharp upper and lower bounds on the cost domination number of a graph in terms of other well-known invariants. For example, $\gamma_c(G) \leq \min\{\gamma(G), \text{rad } G\}$, where $\gamma(G)$ is the domination number of G and rad G is the radius of G. It is shown that there exist infinitely many graphs G with $\gamma_c(G) = \gamma(G) < \text{rad } G$ and infinitely many graphs G with $\gamma_c(G) = \text{rad } G < \gamma(G)$. Those graphs G having $\gamma_c(G) \leq 3$ are determined.

A cost dominating function f is minimal if there is no cost dominating function g satisfying (i) $g(v) \leq f(v)$ for all $v \in V(G)$ and (ii) $g(u) < f(u)$ for some $u \in V(G)$. The structure of the f-dominating set for both minimal and minimum cost dominating functions is determined. The upper cost domination number, which is the maximum value of $\sigma(f)$ over all minimal cost dominating functions f on G, is also studied.

A cost function f is cost independent if there is no pair u, v of distinct vertices in V_f^+ such that u is f-dominated by v. It is proved that for every graph G, there is a cost function on G that is both minimum cost dominating and cost independent. The cost independence number, which is the maximum value of $\sigma(f)$ over all cost independent functions f, is investigated.

ACKNOWLEDGEMENTS

First and foremost I am indebted to my advisor, Professor Gary Chartrand, for his excellent supervision of both my dissertation and the other projects that we have collaborated on. His interest, enthusiasm, insight and red pen were all invaluable. I am also grateful to the members of my committee: Professors Clifton Ealy, Michael Raines, Ping Zhang and my outside reader, Professor Garry Johns. Although she had no direct involvement with my dissertation, Professor Henda Swart is principally responsible for my interest in Graph Theory and her advice was a major factor in my decision to come to Western Michigan University.

Thanks to the Department of Mathematics and Statistics and The Graduate College for their financial support and to Margo, Cheryl and Barb and Professors John Martino and Jay Treiman for making my life a little easier on a great many occasions.

To my wife, Andrea, and my parents, Geoff and Jan, for their unflagging love and support.

Lastly, thanks to Nancy and Bridget for taking time out from sampling Kalamazoo's culinary delights, John Daniels for enduring three years of close confinement, Rash for allowing me to spell his name this way, and the Drudge Monkey: *Muuuurr!*

David John Erwin

TABLE OF CONTENTS

Table of Contents - Continued

TABLE OF FIGURES

AN INTRODUCTION TO COST DOMINATION

1.1 The Basic Terminology

Suppose that a major real estate developer has decided to construct a number of shopping malls in a region of the country such that every resident of this region is within a reasonable driving distance of at least one of the malls. Because of the costs involved, the company would like to construct as few malls as possible. This situation can be modeled by a graph G whose vertices represent the various sections or neighborhoods of the region and in which two vertices are joined by an edge if the corresponding sections or neighborhoods are within reasonable driving distance of each other. The problem of choosing locations for the malls may then be rephrased as a domination problem: Find a minimum dominating set for G, that is, a set $S \subseteq V(G)$ of minimum cardinality such that every vertex not in S is adjacent in G to some vertex in S. The vertices in S are optimal locations for the malls, while the cardinality of S, the domination number $\gamma(G)$ of G, is the smallest possible number of malls.

Domination is an area of graph theory that has been studied extensively (a comprehensive overview of the subject can be found in [9]). A number of variations of domination involve modifying what 'reasonable distance' means. A subset S of $V(G)$ is a *distance-k dominating set* if every vertex not in S is within distance k of

some vertex of S. Distance domination is reviewed in [10]. Slater [12] considered the problem where not all of the vertices of G need be within the same distance of a dominating set. We now consider a domination problem in which a vertex in a dominating set can dominate more vertices in the graph if we are prepared to 'pay' more for this privilege. From the real estate developer's point of view, an analogous situation would be the option (perhaps through increased advertising or expanding the size of the development) to pay more so as to make some malls more attractive to consumers than others. What is meant by 'more attractive' is that a resident of the region would be prepared to drive further to visit a larger or more varied mall. In terms of our graphical model, this means that a mall built at the location represented by some vertex dominates (in general) a smaller number of vertices of the graph than a more expensive mall constructed at the same site. With this in mind, we make the following definitions.

Let G be a connected graph of diameter d. We write $\Delta(G)$ for the maximum degree of the vertices of G. For two integers a and b, we write $[a..b]$ for the integer interval $\{a, a+1, \ldots, b\}$ if $a \leq b$, or the empty set if $a > b$. The eccentricity $e(v)$ of a vertex v is the greatest distance between v and a vertex of G. Any function $f : V(G) \to [0..d]$ such that for every vertex v of G we have $f(v) \leq e(v)$ (the utility of this restriction will be discussed shortly) is referred to as a *cost function* on G. A vertex v for which $f(v) > 0$ is an *f-dominating vertex*, and the set $V_f^+(G) = \{v \in V(G) : f(v) > 0\}$ of f-dominating vertices is the *f-dominating set*. If there is no ambiguity, we write V_f^+ for $V_f^+(G)$. An f-dominating vertex v is said to *f-dominate* every vertex u with $\mathrm{d}(u, v) \leq f(v)$, while the vertices in $V(G) - V_f^+$, namely, those vertices of G that are not f-dominating, do not f-dominate any vertices of G. A *cost dominating function* on G is a cost function

f in which every vertex is f-dominated by some vertex in the f-dominating set.

From the manner in which we have defined cost domination, it is now evident why the restriction $f(v) \leq e(v)$ is placed on every vertex v; for if there were some vertex v for which $f(v) > e(v)$, then we would be paying more for v to f-dominate no additional vertices.

Notice that if $G = K_1$, then diam $G = 0$ and the single cost function defined on G is not cost dominating. Henceforth, we shall assume that every graph under consideration is both connected and nontrivial (that is, has order at least 2).

If f is a cost function on a connected graph G, let $\sigma(f) = \sum_{v \in V(G)} f(v)$. The *cost domination number* $\gamma_c(G)$ is the minimum value of $\sigma(f)$ over all cost dominating functions f on G. A cost dominating function f on G for which $\sigma(f) = \gamma_c(G)$ is called a *minimum cost dominating function* on G. Hence if f is a minimum cost dominating function on G, then $\gamma_c(G) \geq |V_f^+|$. If f is a cost function on G, then for $0 < i \leq d$ we let $V_f^i(G) = \{v \in V(G) : f(v) = i\}$, where $V_f^i = V_f^i(G)$ if there is no ambiguity. Hence, $\gamma_c(G) = \min_f \sum_{i=1}^{d} i|V_f^i|$, where the minimum is taken over all cost dominating functions f on G. If $\sigma(f) \leq k$ for some nonnegative integer k, then $|V_f^t| = 0$ for every positive integer $t > k$. We write $\mathbb{N}$ for the set of positive integers. For a cost function f, define $\vec{f} : V(G) \to \mathbb{N} \cup \{0\}$ by

$$\vec{f}(v) = |\{u \in V_f^+ : \mathrm{d}(u, v) \leq f(u)\}|,$$

that is, $\vec{f}(v)$ is the number of vertices that f-dominate v. Hence f is a cost dominating function if and only if $\vec{f}(v) \geq 1$ for all $v \in V(G)$. A cost dominating function f that has the property that $\vec{f}(v) = 1$ for every $v \in V(G)$ is *efficient*. A vertex u that is f-dominated by a vertex v is an *f-neighbor* of v. Thus, while

u is a neighbor of v if and only if v is a neighbor of u, it is possible that u is an f-neighbor of v while v is not an f-neighbor of u. For each $v \in V_f^+$, the set of f-neighbors of v is the *closed f-neighborhood* of v and is denoted $N_f[v]$. If $v \in V_f^1$, then $N_f[v] = N[v]$. If $S \subseteq V_f^+$, then $N_f[S] = \bigcup_{v \in S} N_f[v]$. Necessarily, then, a cost function f is a cost dominating function if and only if $N_f[V_f^+] = V(G)$.

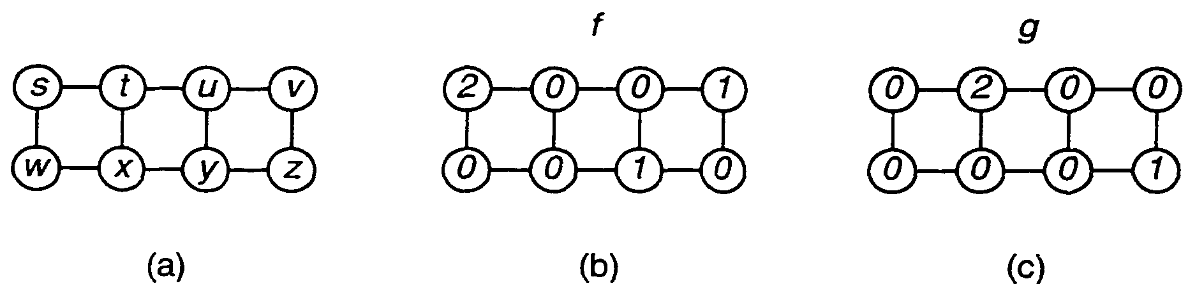


Figure 1: Cost Dominating Functions on $P_2 \times P_4$.

To illustrate these concepts, consider the graph $P_2 \times P_4$ of diameter 4 shown in Figure 1(a). Figures 1(b) and 1(c) show two cost dominating functions f and g, where the f-dominating set is $V_f^+ = \{s, v, y\}$, and the g-dominating set is $V_g^+ = \{t, z\}$. The vertex s f-dominates s, t, u, w, x and has closed f-neighborhood $N_f[s] = \{s, t, u, w, x\}$, but s does not g-dominate any vertices. Moreover, $V_f^1 = \{v, y\}$, $V_f^2 = \{s\}$ and $V_f^3 = \emptyset$; also, $\vec{f}(u) = 3$, $\vec{f}(x) = \vec{f}(z) = 2$ and every vertex $r \in \{s, t, v, w, y\}$ satisfies $\vec{f}(r) = 1$. Since $\vec{f}(u), \vec{g}(y) > 1$, neither f nor g is an efficient cost dominating function. The function f has $\sigma(f) = 4$ while $\sigma(g) = 3$. In fact, while we shall not prove this immediately, $\sigma(g) = \gamma_c(P_2 \times P_4)$ so that g is a minimum cost dominating function on $P_2 \times P_4$.

1.2 Some Fundamental Results

A vertex v in a connected graph G is a *central vertex* if $e(v) = \text{rad } G$, the radius of G. If v is a central vertex of a nontrivial connected graph G, then the function $f : V(G) \to [0..\text{rad } G]$ defined by

$$f(x) = \begin{cases} \text{rad } G & \text{if } x = v \\ 0 & \text{if } x \neq v \end{cases} \tag{1.1}$$

is a cost dominating function. Thus, $\gamma_c(G) \leq \text{rad } G$. Moreover, if S is a dominating set and $\chi_S : V(G) \to \{0, 1\}$ is the characteristic set function of S defined by

$$\chi_S(v) = \begin{cases} 1 & \text{if } v \in S \\ 0 & \text{if } v \notin S, \end{cases}$$

then χ_S is a cost dominating function. Hence, $\gamma_c(G) \leq \gamma(G)$. We summarize these two observations in the following result.

Proposition 1.2.1. *For every nontrivial connected graph G,*

$$\gamma_c(G) \leq min\{rad\ G, \gamma(G)\}.$$

While a cost function f on G may assign some values in the range $[\text{rad } G + 1..\text{diam } G]$ to the vertices of G, Proposition 1.2.1 shows that if f is a minimum cost dominating function, it does not do so. Consequently, when considering a minimum cost dominating function f, we shall henceforth assume that $f : V(G) \to [0..\text{rad } G]$.

Proposition 1.2.1 suggests the question: Does there exist a connected graph G with $\gamma_c(G) < \min\{\text{rad } G, \gamma(G)\}$? Not only does this question have an affirmative answer but in fact $\min\{\text{rad } G, \gamma(G)\} - \gamma_c(G)$ can be arbitrarily large, as we

now see. For $t \geq 2$, let $S(K_{1,t})$ be the subdivision graph of the star $K_{1,t}$. For a positive integer k, let H_k be the graph obtained by joining an endvertex of $S(K_{1,2+k})$ to an endvertex of P_{2k}. For example, the graph H_4 is shown in Figure 1.2.

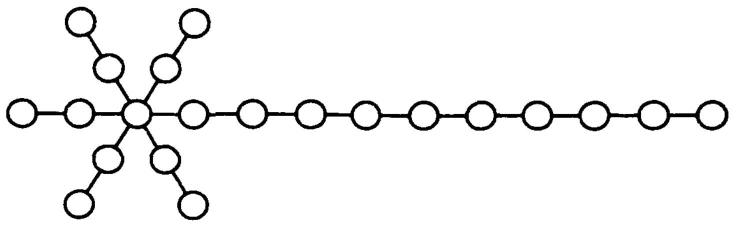


Figure 2: The Graph H_4.

Then $\gamma(H_k) \geq 2+k$ and rad $H_k = 2+k$. Let S be a minimum dominating set of P_{2k} and v the central vertex of $S(K_{1,2+k})$. Then the function $f : V(H_k) \to \{0,1,2\}$ defined by

$$f(x) = \begin{cases} 2 & \text{if } x = v \\ 1 & \text{if } x \in S \\ 0 & \text{if } x \in V(H_k) - (S \cup \{v\}) \end{cases}$$

is a cost dominating function on H_k and $\sigma(f) = 2 + \lceil 2k/3 \rceil$. If $k \geq 3$, then

$$\begin{aligned} \gamma_c(H_k) &\leq 2 + \lceil 2k/3 \rceil \\ &< 2 + k \\ &\leq \min\{\gamma(H_k), \text{rad } H_k\}. \end{aligned}$$

Consequently,

$$\min\{\gamma(H_k), \text{rad } H_k\} - \gamma_c(H_k) \geq \lfloor k/3 \rfloor.$$

We thus obtain the following result.

Proposition 1.2.2. *For every positive integer t, there exists a connected graph G for which*

$$min\{\gamma(G), rad\ G\} - \gamma_c(G) \geq t.$$

More can be deduced from the fact that the characteristic function associated with a minimum dominating set is a cost dominating function. It is well-known that if G is a connected graph of order $n \geq 2$, then $\gamma(G) \leq n/2$. Let f be a minimum cost dominating function on G. We consider two cases. Firstly, if every f-dominating vertex v satisfies $f(v) = 1$, then V_f^+ is a minimum dominating set and $|V_f^+| = \gamma(G)$. On the other hand, suppose there is at least one vertex $v \in V_f^+$ with $f(v) > 1$. Then no subset of $V(G)$ of cardinality $|V_f^+|$ is a dominating set, for otherwise $\gamma(G) < \gamma_c(G)$. Consequently, $|V_f^+| < \gamma(G)$ and hence, in either case, we have $|V_f^+| \leq \gamma(G)$. We thus obtain the following result.

Proposition 1.2.3. *If f is a minimum cost dominating function of a connected graph G of order $n \geq 2$, then*

$$|V_f^+| \leq \frac{n}{2}.$$

We now prove a useful lemma.

Lemma 1.2.4. *Let f be a minimum cost dominating function on a connected graph G. Then $V_f^+ = \{v\}$ if and only if*

1. *$f(v) = rad\ G$ and*

2. *v is a central vertex of G.*

Proof. First, assume that $V_f^+ = \{v\}$. Certainly, from Proposition 1.2.1, we have that $f(v) \leq \text{rad } G$. If $f(v) < \text{rad } G$, then there is a vertex w with $\text{d}(v, w) = e(v) \geq \text{rad } G > f(v)$. Hence f is not a cost dominating function on G. If v is not a central vertex of G, then once again f is not a cost dominating function on G. Suppose now that $f(v) = \text{rad } G$ and v is a central vertex of G. Then every vertex of G is f-dominated by v. Since f is a minimum cost dominating function it must therefore be the case that $V_f^+ = \{v\}$. □

If G is a graph and $S \subseteq V(G)$, then we denote by $\langle S \rangle$ the subgraph of G induced by S.

Theorem 1.2.5. *Let G be a connected graph and f a minimum cost dominating function on G. If $v \in V_f^+$, then*

$$f(v) = \gamma_c\left(\langle N_f[v] \rangle\right) = \textit{rad } \langle N_f[v] \rangle.$$

Proof. The restriction of f to $N_f[v]$ is necessarily a minimum cost dominating function on $\langle N_f[v] \rangle$. Consequently, $f(v) = \gamma_c(\langle N_f[v] \rangle)$. That $f(v) = \text{rad } \langle N_f[v] \rangle$ now follows immediately from Lemma 1.2.4. □

We claimed earlier that the graph $P_2 \times P_4$ shown in Figure 1 has cost domination number 3. We are now in a position to verify this. Assume, to the contrary, that there is a cost dominating function h on $P_2 \times P_4$ having $\sigma(h) = 2$. Since the radius of $P_2 \times P_4$ is 3, it follows from Lemma 1.2.4 that $|V_h^+| = 2$ and every h-dominating vertex a satisfies $h(a) = 1$. Consequently, V_h^+ is a dominating set of $P_2 \times P_4$ that consists of two vertices, contradicting the fact that $\gamma(P_2 \times P_4) = 3$. Hence $\gamma_c(P_2 \times P_4) = 3$.

1.3 Lower Bounds for the Cost Domination Number

It was shown in [9] that if G is a connected graph, then

$$\gamma(G) \geq \left\lceil \frac{\text{diam } G + 1}{3} \right\rceil . \tag{1.2}$$

In fact, an identical lower bound holds for the cost domination number.

Theorem 1.3.1. *If G is a nontrivial connected graph, then*

$$\gamma_c(G) \geq \left\lceil \frac{\textit{diam } G + 1}{3} \right\rceil .$$

Proof. Let u and v be two vertices in G such that $\mathrm{d}(u,v) = \text{diam } G$, and let $P : u = x_0, x_1, \ldots, x_{\text{diam } G} = v$ be a $u-v$ path of length diam G. Let f be a minimum cost dominating function on G. We now show that every f-dominating vertex x f-dominates at most $2f(x)+1$ vertices of P. Assume, to the contrary, that there exists a vertex $w \in V_f^+$ such that $N_f[w]$ contains at least $2f(w)+2$ vertices from P. Let s and t be the smallest and largest integers, respectively, for which $x_s, x_t \in N_f[w] \cap V(P)$. Then, by assumption, $t-s \geq 2f(w)+1$. The length of P is then at least $s+2f(w)+1+\text{diam } G-t$. Let P_s be a shortest x_s-w path in G, and let P_t be a shortest $w-x_t$ path in G. Since w f-dominates both x_s and x_t, both P_s and P_t have length at most $f(w)$. However, the $u-v$ path obtained by following P from u to x_s, then P_s from x_s to w, then P_t from w to x_t, and finally P from x_t to v has length at most $s+2f(w)+\text{diam } G-t$. This contradicts our choice of P, and consequently no such vertex w exists. Hence, every f-dominating vertex x f-dominates at most $2f(x)+1$ vertices of P. Moreover, since f is a cost

dominating function, every vertex of P is f-dominated. Consequently,

$$\begin{aligned}\text{diam } G+1 &\le \sum_{x\in V_f^+}[2f(x)+1]\\ &= 2\gamma_c(G)+|V_f^+|\\ &\le 3\gamma_c(G).\end{aligned}$$

□

As we now see, the bound of Theorem 1.3.1 is sharp.

Corollary 1.3.2. *For every integer $n \geq 2$,*

$$\gamma_c(P_n) = \gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil.$$

Proof. It is well-known that $\gamma(P_n) = \lceil \frac{n}{3} \rceil$. The result then follows from Proposition 1.2.1 and Theorem 1.3.1. □

It seems that a graph with a given cost domination number should not have an arbitrarily large radius. The next result establishes this. We say that v is an *eccentric vertex of* u if $d(u, v) = e(u)$.

Lemma 1.3.3. *Let G be a connected graph, f a minimum cost dominating function on G, and $M = max\{f(x) : x \in V_f^+\}$. Then*

$$rad\ G \leq 2\gamma_c(G) + |V_f^+| - M - 1.$$

Proof. Let $u \in V_f^+$ and let v be an eccentric vertex of u. Consider a $u - v$ geodesic P. The vertex u f-dominates $f(u)+1$ vertices of P, and, as in the proof of Theorem 1.3.1, every f-dominating vertex x apart from u f-dominates at most

$2f(x)+1$ vertices of P. Since every vertex of P is f-dominated, it follows that

$$e(u)+1 \leq f(u)+1+\sum_{x\in V_f^+-\{u\}}[2f(u)+1]$$
$$=2\gamma_c(G)-f(u)+|V_f^+|.$$

Since this is true for any vertex in the f-dominating set, the result follows. □

From Lemma 1.3.3 we immediately obtain the following result.

Corollary 1.3.4. *Let G be a connected graph, f a minimum cost dominating function on G, and $M = max\{f(x) : x \in V_f^+\}$. Then*

$$\gamma_c(G) \geq \left\lceil \frac{rad\ G + M + 1 - |V_f^+|}{2} \right\rceil.$$

1.4 The Cost Domination Number of Graphs Having Small Radius or Domination Number

It is well-known that for a nontrivial connected graph G, $\gamma(G) = 1$ if and only if rad $G = 1$. Graphs with cost domination number 1 admit an identical characterization.

Theorem 1.4.1. *Let G be a nontrivial connected graph. Then $\gamma_c(G) = 1$ if and only if rad $G = 1$.*

Proof. If rad $G = 1$, then $\gamma(G) = 1$ and the result follows directly from Proposition 1.2.1. For the converse, let G be a nontrivial connected graph for which $\gamma_c(G) = 1$. Then there is a cost dominating function f with $\sigma(f) = 1$. Hence $|V_f^+| = |V_f^1| = 1$. So from Lemma 1.2.4, there is a unique vertex $v \in V_f^1$ and

every vertex distinct from v is at distance 1 from v, that is, $e(v) = 1$ and so rad $G = 1$. $\square$

Theorem 1.4.1 could be restated as: $\gamma_c(G) = 1$ if and only if $\gamma(G) = 1$. There is a similar, though not identical, relationship between those graphs with cost domination number 2 and those with radius 2 or domination number 2.

Theorem 1.4.2. *Let G be a connected graph. Then $\gamma_c(G) = 2$ if and only if $min\{rad\ G, \gamma(G)\} = 2$.*

Proof. Certainly if $\gamma(G) = 2$ or rad $G = 2$, then Proposition 1.2.1 and Theorem 1.4.1 together imply that $\gamma_c(G) = 2$. Suppose then that $\gamma_c(G) = 2$ and rad $G \neq 2$. By Theorem 1.4.1, rad $G \geq 3$. Let f be a minimum cost dominating function on G. Since $\gamma_c(G) <$ rad G, no single vertex f-dominates $V(G)$. Hence $V_f^+ = \{u, v\}$ for some pair u, v of vertices of G and $f(u) = f(v) = 1$. Consequently, $\{u, v\}$ is a dominating set of G and, once again from Theorem 1.4.1, $\gamma(G) = 2$. $\square$

Neither of the two conditions (i) $\gamma(G) = 2$ and (ii) rad $G = 2$ implies the other. For example, $\gamma(P_6) = 2$ and rad $P_6 = 3$, while for every positive integer $n \geq 2$, rad $S(K_{1,n}) = 2$ and $\gamma(S(K_{1,n})) = n$. Moreover, there are graphs with cost domination number 2 and radius strictly greater than 2 as well as graphs with cost domination number 2 and domination number strictly greater than 2. For instance, let v be the central vertex of $S(K_{1,n})$, where $n \geq 2$. Then rad $S(K_{1,n}) = 2$ and the function $f : V(S(K_{1,n})) \to \{0, 1, 2\}$ defined by

$$f(u) = \begin{cases} 2 & u = v \\ 0 & u \neq v \end{cases}$$

is a cost dominating function with $\sigma(f) = 2$. Hence, from Theorem 1.4.1, $\gamma_c(S(K_{1,n})) = 2$ while $\gamma(S(K_{1,n})) = n$, so there exist graphs G such that $\gamma_c(G) = 2$ while $\gamma(G)$ is arbitrarily large. Similarly, a graph can have cost domination number 2 while having radius strictly greater than 2. For every positive integer $p \geq 2$, let M_p be the multigraph consisting of two vertices u and v joined by p parallel edges, and let G_p be the graph obtained by inserting two new vertices into every edge of M_p. For example, the graph G_3 is shown in Figure 3. For every p, the graph G_p has radius 3 and cost domination number 2.

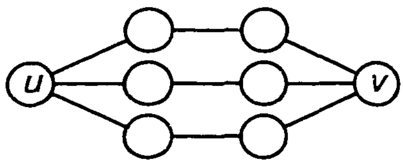


Figure 3: The Graph G_3.

We have now seen that there is an infinite class of graphs with cost domination number 2 and radius exceeding 2. The question now arises as to how large the radius of a graph having cost domination number 2 can be.

Proposition 1.4.3. *If G is a connected graph with $\gamma_c(G) = 2$, then*

1. *rad $G = 2$, or,*

2. *$\gamma(G) = 2$ and rad $G = 3$.*

Proof. Let G be a connected graph having cost domination number 2 and radius at least 3. If f is a minimum cost dominating function on G, then from Theorem 1.4.2 it follows that the f-dominating set is a minimum dominating set consisting

of two vertices, say u and v. Applying Lemma 1.3.3, we see that rad $G \leq 4$. Consequently, rad $G = 3$ or rad $G = 4$. In fact, rad $G = 3$. Assume, to the contrary, that rad $G = 4$. Thus $e(u) \geq 4$ and $e(v) \geq 4$. For $1 \leq i \leq e(u)$, let D_i be the set of vertices of G that are at distance i from u. The vertex u dominates itself and the vertices of D_1. As was demonstrated in the proof of Lemma 1.3.3, v dominates vertices in some set D_i for no more than three distinct, consecutive values of i. This implies that $e(u) = 4$ and, similarly, that $e(v) = 4$; so both u and v are central vertices. Moreover, since $\{u, v\}$ is a dominating set of G, we have that $N[v] = D_2 \cup D_3 \cup D_4$. Let w be some vertex in D_2. Then w is distance at most 2 from every vertex in $D_2 \cup D_3 \cup D_4$, distance 2 from u, and hence distance at most 3 from every vertex in D_1. Consequently, $e(w) < 4 =$ rad G, producing a contradiction. Therefore, no such graph G exists. From this argument and Theorem 1.4.2, the result follows. □

If G is a complete multipartite graph in which none of the partite sets is a singleton, then $\gamma(G) =$ rad $G = 2$. Hence, the following is a corollary of Theorem 1.4.2.

Corollary 1.4.4. *If $n_1, n_2, \ldots, n_t$ are integers satisfying $n_i \geq 2$ for all i with $1 \leq i \leq t$, then*

$$\gamma_c(K_{n_1, n_2, \ldots, n_t}) = 2.$$

Another consequence of Theorem 1.4.2 is the following.

Proposition 1.4.5. *If G is a connected graph and min$\{$rad $G, \gamma(G)\} = 3$, then $\gamma_c(G) = 3$.*

We summarize parts of Theorems 1.4.1 and 1.4.2 and Proposition 1.4.5 with the following result.

Proposition 1.4.6. *Let G be a connected graph. If $\min\{rad\ G, \gamma(G)\} = k$, where $1 \leq k \leq 3$, then $\gamma_c(G) = k$.*

It is natural to consider the question of whether Proposition 1.4.6 can be extended to include the case when $k = 4$. As we now see, this question has a negative answer. Consider the graph G of Figure 4 formed by joining an endvertex of $S(K_{1,4})$ to an endvertex of P_3.

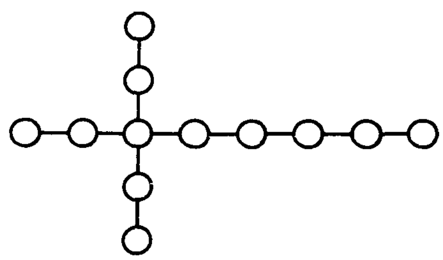


Figure 4: The Graph G.

The graph G has radius 4 and domination number at least 4. On the other hand, if u and v are the central vertices of $S(K_{1,4})$ and P_3, respectively, then the cost function $f : V(G) \to \{0, 2\}$ given by

$$f(x) = \begin{cases} 2 & \text{if } x = u \\ 1 & \text{if } x = v \\ 0 & \text{if } x \notin \{u, v\} \end{cases}$$

is a cost dominating function on G, so by Theorem 1.4.2 we have that $\gamma_c(G) = 3$ while the radius and domination number of G are both at least 4.

CLASSES AND PROPERTIES OF COST DOMINATING FUNCTIONS

2.1 Minimal Cost Dominating Functions

A dominating set S of a graph G is *minimal* if no proper subset of S dominates $V(G)$. Correspondingly, a cost dominating function f on a nontrivial connected graph G is said to be *minimal* if there is no cost dominating function f' satisfying (i) $f'(v) \leq f(v)$ for all $v \in V(G)$ and (ii) $f'(u) < f(u)$ for some $u \in V(G)$. For example, two different minimal cost dominating functions f and g on $S(K_{1,3})$ are shown in Figure 5.

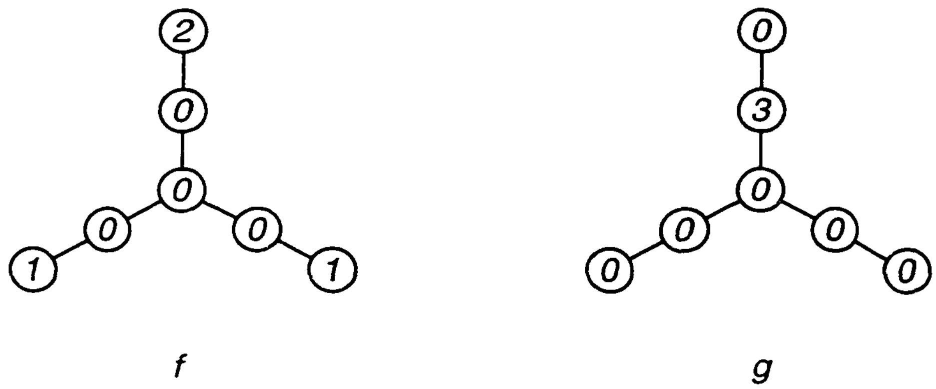


Figure 5: Two Minimal Cost Dominating Functions on $S(K_{1,3})$.

Since $\min\{\text{rad } S(K_{1,3}), \gamma(S(K_{1,3}))\} = 2$, by Theorem 1.4.2 we have that $\gamma_c(S(K_{1,3})) = 2$. Hence, since $\sigma(f) = 4$ and $\sigma(g) = 3$, both f and g are minimal cost dominating functions but not minimum cost dominating functions. Every minimum

cost dominating function is also minimal but, as the previous example shows, not every minimal cost dominating function is a minimum cost dominating function. As another example, the characteristic set function of any minimal dominating set is a minimal cost dominating function.

If G is a nontrivial connected graph and $v \in V(G)$, then the function $f : V(G) \to [0..\text{diam } G]$ given by

$$f(x) = \begin{cases} e(v) & \text{if } x = v \\ 0 & \text{if } x \neq v \end{cases}$$

is a minimal cost dominating function since $\mathrm{d}(u, v) \leq e(v)$ for every vertex u of G. Hence, we obtain the following result.

Theorem 2.1.1. *If f is a minimal cost dominating function on a connected graph G and $|V_f^+| \geq 2$, then for every vertex v, $f(v) < e(v)$.*

Proof. If there is a vertex v such that $f(v) = e(v)$, then v f-dominates every vertex of G. Since f is minimal, it follows that $V_f^+ = \{v\}$. □

Every vertex v in a minimal dominating set S has a private neighbor u, that is, a vertex u that is dominated only by v. Let f be a cost function on G and let $v \in V_f^+(G)$. A *private* f-neighbor of v is a vertex that is an f-neighbor of v but is not an f-neighbor of any vertex distinct from v; that is, a vertex u is a private f-neighbor of v if u is f-dominated only by v. Note that an f-neighbor u of v is a private f-neighbor of v if and only if $\vec{f}(u) = 1$. The following result shows that if f is a minimal cost dominating function, then every f-dominating vertex v has a private f-neighbor.

Theorem 2.1.2. *Let G be a connected graph and f a cost dominating function on G. Then f is minimal if and only if the following two conditions are satisfied.*

(a) *For every vertex v with $f(v) \geq 2$, there exists a private f-neighbor u of v that is distance $f(v)$ from v.*

(b) *If $f(v) = 1$, then v has a private f-neighbor $u \in N[v]$.*

Proof. We begin by assuming that f is minimal and proving that conditions (a) and (b) hold. We first verify (a). Let $v \in V_f^+ - V_f^1$, and let $S_v = \{u \in V(G) : \mathrm{d}(u, v) = f(v)\}$. Since f is minimal, it is immediate that $S_v \neq \emptyset$. If $\vec{f}(u) \geq 2$ for every $u \in S_v$, then the function $g : V(G) \to [0..\mathrm{rad}\ G]$ defined by

$$g(x) = \begin{cases} f(x) - 1 & \text{for } x = v \\ f(x) & \text{for } x \neq v \end{cases}$$

is a cost function in which (i) $\vec{g}(u) = \vec{f}(u) - 1 \geq 1$ for every $u \in S_v$ and (ii) $\vec{g}(u) = \vec{f}(u) \geq 1$ for every $u \in V(G) - S_v$. Hence, g is a cost dominating function such that $g(v) < f(v)$ and $g(x) \leq f(x)$ for all $x \in V(G)$, contradicting the minimality of f. We now verify (b). Let $v \in V_f^1$. If $\vec{f}(u) \geq 2$ for all $u \in N[v]$, then the function $h : V(G) \to [0..\mathrm{rad}\ G]$ defined by

$$h(x) = \begin{cases} 0 & \text{for } x = v \\ f(x) & \text{for } x \neq v \end{cases}$$

is a cost dominating function with $\sigma(h) < \sigma(f)$, contradicting the minimality of f. This concludes the first part of the proof.

To see that a cost dominating function satisfying conditions (a) and (b) must be minimal, assume, to the contrary, that f is a cost dominating function satisfying conditions (a) and (b) but that f is not minimal. Hence there is some

vertex $v \in V_f^+$ for which the cost function $g : V(G) \to [0..\text{diam } G]$ given by

$$g(x) = \begin{cases} f(v) - 1 & \text{if } x = v \\ f(x) & \text{if } x \neq v \end{cases}$$

is cost dominating. However, if $f(v) \geq 2$, by condition (a) there is some vertex y that is a private f-neighbor of v and, since $\mathrm{d}(v, y) = f(v)$, the vertex y is not g-dominated. Similarly, if $f(v) = 1$, then since g is a cost dominating function, every vertex in $N[v]$ is g-dominated. Consequently, every vertex in $N[v]$ is f-dominated by some vertex distinct from v. This contradicts condition (b). In either case a contradiction arises and so no such function f exists. □

In fact, a stronger result than Theorem 2.1.2 can be stated for minimum cost dominating functions.

Theorem 2.1.3. *Let G be a connected graph of order at least 2. Then there exists a minimum cost dominating function f on G such that for every vertex $v \in V_f^+$, there is a private f-neighbor u of v that is distance $f(v)$ from v.*

Proof. Among all minimum cost dominating functions on G, let f be one for which $|V_f^1|$ is a minimum. Let $v \in V_f^+$. If $f(v) \geq 2$, then the result follows directly from Theorem 2.1.2. Suppose then that $f(v) = 1$. From Theorem 2.1.2, v has a private f-neighbor. If some neighbor of v is f-dominated only by v, then once again the result follows. We claim that every vertex v with $f(v) = 1$ has a neighbor that is f-dominated only by v. To prove this, assume, to the contrary, that every neighbor of v is f-dominated by some vertex other than v. Let v' be some neighbor of v and w a vertex distinct from v that f-dominates v'. Since w does not f-dominate v, it follows that $f(w) = \mathrm{d}(w, v') = \mathrm{d}(w, v') - 1$. Furthermore, since $\sigma(f) = \gamma_c(G)$

and V_f^+ contains at least two vertices, it follows that $f(w) < \text{rad}\, G$. Consider the cost function $g : V(G) \to [0..\text{rad}\, G]$ defined by

$$g(x) = \begin{cases} f(w)+1 & \text{if } x = w \\ 0 & \text{if } x = v \\ f(x) & \text{if } x \notin \{v, w\}. \end{cases}$$

For the function g, it then follows that $V_g^+ = V_f^+ - \{v\}$. The vertex v is the only private f-neighbor of v, and v is g-dominated by w. Consequently, g is a cost dominating function on G. Moreover, we have that $\sigma(g) = \sigma(f) = \gamma_c(G)$. Thus, we have found a minimum cost dominating function g with $|V_g^1| < |V_f^1|$, contradicting our choice of f. Hence no such vertex v exists and the function f has the desired properties. □

2.2 The Upper Cost Domination Number

As we have said before, a dominating set S of a graph G is minimal if no proper subset of S is a dominating set of G. The *upper domination number* $\Gamma(G)$ of G is $\max\{|S| : S \text{ is a minimal dominating set of } G\}$. Since a minimum dominating set is also minimal it follows that $\gamma(G) \leq \Gamma(G)$ for every graph G. We define the *upper cost domination number* of a nontrivial connected graph G to be

$$\Gamma_c(G) = \max\{\sigma(f) : f \text{ is a minimal cost dominating function on } G\}.$$

If S is a minimal dominating set of a connected graph G, then the characteristic set function $\chi_S : V(G) \to \{0, 1\}$ of S is a minimal cost dominating function. From this observation and Proposition 1.2.1, we obtain the following result.

Proposition 2.2.1. *For every nontrivial connected graph G,*

$$\gamma_c(G) \leq \gamma(G) \leq \Gamma(G) \leq \Gamma_c(G).$$

A vertex v in a connected graph G is *peripheral* if $e(v) = \text{diam } G$. If v is a peripheral vertex of a nontrivial connected graph G, then the cost function $f : V(G) \to \{0, \text{diam } G\}$ given by

$$f(u) = \begin{cases} \text{diam } G & \text{if } u = v \\ 0 & \text{if } u \neq v \end{cases}$$

is a minimal cost dominating function on G. From this and Proposition 1.2.1, we obtain the following result.

Proposition 2.2.2. *For every nontrivial connected graph G,*

$$\gamma_c(G) \leq \textit{rad } G \leq \textit{diam } G \leq \Gamma_c(G).$$

Parts of Propositions 2.2.1 and 2.2.2 can be restated in a form similar to that of Proposition 1.2.1.

Proposition 2.2.3. *For every nontrivial connected graph G,*

$$\Gamma_c(G) \geq \textit{max}\{\textit{diam } G, \Gamma(G)\}.$$

Analogously with the discussion that followed Proposition 1.2.1, we now consider the question of whether there is a connected graph G for which $\Gamma_c(G) > \max\{\text{diam } G, \Gamma(G)\}$. To simplify the discussion that will follow, we mention that Cockayne, Favaron, Payan and Thomason [4] showed that if G is a bipartite graph, then $\Gamma(G) = \beta(G)$, where $\beta(G)$ is the maximum cardinality of an independent set of vertices of G. For a positive integer k, let H_k be the graph obtained by joining

an endvertex of $S(K_{1,2+k})$ to an endvertex of P_{2k}. This is the same class of graphs that we defined on page 6. Then $\Gamma(H_k) = \beta(H_k) = 2k+3$ and diam $H_k = 2k+4$. Let S be the set of vertices of $S(K_{1,2+k})$ that have degree 2 and v that endvertex of P_{2k} that is not joined in H_k to a vertex of $K_{1,2+k}$. Then the cost function $f : V(H_k) \to \{0, 1, 2k\}$ given by

$$f(x) = \begin{cases} 2k & \text{if } x = v \\ 1 & \text{if } x \in S \\ 0 & \text{if } x \in V(H_k) - (S \cup \{v\}) \end{cases}$$

is a minimal cost dominating function on H_k and $\sigma(f) = 3k+2$. Consequently,

$$\Gamma_c(H_k) - \max\{\Gamma(H_k), \text{diam } H_k\} \geq k-2.$$

We thus obtain the following result.

Proposition 2.2.4. *For every positive integer t, there exists a connected graph G for which*

$$\Gamma_c(G) - max\{\Gamma(G), diam\ G\} \geq t.$$

Even more can be deduced by studying the class $\{H_k\}$. We mentioned previously that for every positive integer k, $\gamma_c(H_k) \leq 2 + \lceil 2k/3 \rceil$. Consequently, $\Gamma_c(H_k) - \gamma_c(H_k) \geq 2k + \lfloor k/3 \rfloor$. From this we get the following proposition.

Proposition 2.2.5. *For every positive integer t, there exists a connected graph G for which*

$$\Gamma_c(G) - \gamma_c(G) \geq t.$$

It is perhaps worth noting that Proposition 2.2.5 can be independently derived from Proposition 2.2.1 and the well-known result [9] that for every positive integer t, there is a connected graph G for which $\Gamma(G) - \gamma(G) \geq t$.

2.3 Continuous Cost Dominating Functions

As an immediate consequence of Theorem 2.1.2, we obtain the following corollary.

Corollary 2.3.1. *If G is a connected graph and f is a minimal cost dominating function on G, then $|f(u) - f(v)| < d(u,v)$ for each pair u, v of distinct vertices in V_f^+.*

Proof. Assume to the contrary that there exists a minimal cost dominating function f on G and two distinct vertices u and v such that $\mathrm{d}(u,v) \leq |f(u) - f(v)|$. Assume, without loss of generality, that $f(v) \geq f(u)$. Then $f(u) \leq f(v) - \mathrm{d}(u,v)$. Let x be a vertex with $\mathrm{d}(x,u) \leq f(u)$. Then $\mathrm{d}(x,v) \leq \mathrm{d}(u,v) + \mathrm{d}(u,x) \leq f(v)$. Consequently, every vertex that is f-dominated by u is also f-dominated by v, contradicting Theorem 2.1.2. $\square$

If G is a connected graph, then a function $f : V(G) \to \mathbb{N} \cup \{0\}$ is *continuous* on $V(G)$ if for every pair u, v of adjacent vertices of G we have that $|f(u) - f(v)| \leq 1$. Two functions f and g on $S(K_{1,3})$ are shown in Figure 6.

The function f is not continuous since the vertex assigned the value 2 by f is adjacent to a vertex assigned the value 0. On the other hand, the function g is continuous on $S(K_{1,3})$.

An overview of continuity (in graphs) can be found in [2]. By means of the following lemma, we now give an equivalent condition for continuity.

Lemma 2.3.2. *Let G be a connected graph and $f : V(G) \to \mathbb{N} \cup \{0\}$. Then f is continuous on $V(G)$ if and only if $|f(u) - f(v)| \leq d(u,v)$ for every pair u, v of*

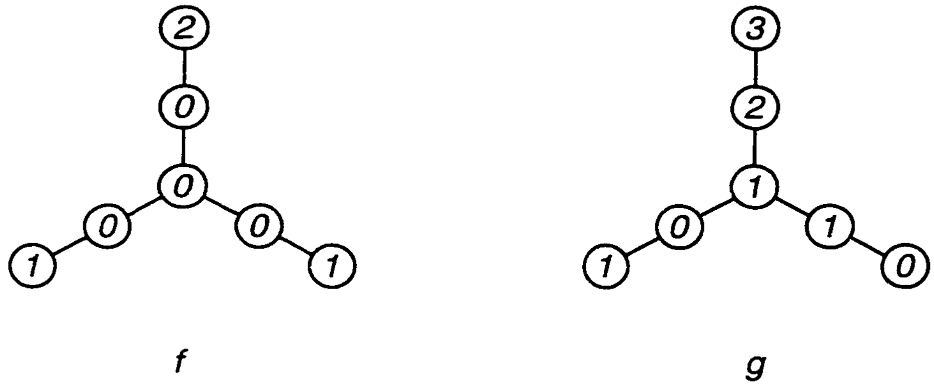


Figure 6: Two Functions on $S(K_{1,3})$.

vertices of G.

Proof. First, assume that f is continuous on $V(G)$. Let u and v be two vertices of G and assume without loss of generality that $f(u) \geq f(v)$. Let $P : v = v_0, v_1, \ldots, v_d = u$ be a $v - u$ geodesic, where $d = \mathrm{d}(u, v)$. Since f is continuous, we have that for every integer i with $1 \leq i \leq d$, $f(v_i) \leq f(v_{i-1}) + 1$. Consequently, $f(u) \leq f(v) + d$, or $|f(u) - f(v)| \leq d$. For the converse, it suffices to note that the condition $|f(u) - f(v)| \leq \mathrm{d}(u, v)$ applies to all pairs of adjacent vertices. □

In general, minimal cost dominating functions are not continuous. For example, the function f shown in Figure 6 is a minimal cost dominating function on $S(K_{1,3})$ that is not continuous. However, a minimal cost dominating function is continuous if restricted to the f-dominating set. Let G be a connected graph, f a minimal cost dominating function on G, and u and v two distinct vertices of V_f^+. From Corollary 2.3.1 we have that $|f(u) - f(v)| < \mathrm{d}(u, v)$ and so, by Lemma 2.3.2, the restriction of f to V_f^+ is continuous.

We now turn our attention to cost dominating functions f that we require to be continuous not only on the f-dominating set but on the entire vertex set.

Certainly every connected graph has a continuous minimal cost dominating function. If G is a nontrivial connected graph and S is a minimal dominating set of G, then the characteristic function $\chi_S : V(G) \to \{0,1\}$ of S is such a function. If G is a nontrivial connected graph, then the *continuous cost domination number* of G is

$$\gamma_{cc}(G) = \min\{\sigma(f) : f \text{ is a continuous cost dominating function on } G\}.$$

A continuous cost dominating function f with $\sigma(f) = \gamma_{cc}(G)$ is a *minimum continuous cost dominating function* on G. Also, the *upper continuous cost domination number* of G is

$$\Gamma_{cc}(G) = \max\{\sigma(f) : f \text{ is a continuous minimal cost dominating function on } G\}$$

Since every continuous cost dominating function is cost dominating and, as we have already remarked, the characteristic set function associated with any minimal dominating set is a continuous cost dominating function, from Proposition 2.2.2 we have that for every nontrivial connected graph G,

$$\gamma_c(G) \leq \gamma_{cc}(G) \leq \gamma(G) \leq \Gamma(G) \leq \Gamma_{cc}(G) \leq \Gamma_c(G). \tag{2.1}$$

Of course, $\gamma_c(G) = \gamma_{cc}(G)$ if and only if G has a continuous minimum cost dominating function. A stronger relationship exists between $\gamma_{cc}(G)$ and $\gamma(G)$ and between $\Gamma_{cc}(G)$ and $\Gamma(G)$.

Lemma 2.3.3. *If G is a connected graph and f is a continuous minimal cost dominating function on G, then*

$$max\{f(v) : v \in V(G)\} = 1.$$

Proof. Let $M = \max\{f(v) : v \in V(G)\}$. Certainly, $M \geq 1$. Assume, to the contrary, that $M \geq 2$. Let u be a vertex of G with $f(u) = M$. Since $M \geq 2$ and f is a minimal cost dominating function, u has at least one neighbor. Since f is continuous, every neighbor x of u satisfies $f(u) \geq f(x) \geq f(u) - 1 \geq 1$. Moreover, if z is any vertex of G distinct from u, then there is some neighbor x of u with $\mathrm{d}(x, z) = \mathrm{d}(u, z) - 1$. Consequently, every vertex that is f-dominated by u is f-dominated by at least one neighbor of u. Let $g : V(G) \to [0..\mathrm{rad}\, G]$ be the function defined by

$$g(y) = \begin{cases} f(u) - 1 & \text{if } y = u \\ f(y) & \text{if } y \neq u. \end{cases}$$

Then g is a continuous cost dominating function on G and $\sigma(g) < \sigma(f)$, contradicting the minimality of f. Consequently $M \leq 1$ and the result follows. □

Theorem 2.3.4. *For every nontrivial connected graph G,*

$$\gamma_{cc}(G) = \gamma(G) \qquad \textit{and} \qquad \Gamma_{cc}(G) = \Gamma(G).$$

Proof. If f is a continuous minimal cost dominating function on G, then by Lemma 2.3.3, V_f^+ is a minimal dominating set. Consequently,

$$\gamma(G) \leq \gamma_{cc}(G) \leq \Gamma_{cc}(G) \leq \Gamma(G)$$

and the result follows from equation 2.1. □

Corollary 2.3.5. *Let G be a connected graph. Then G has a continuous minimum cost dominating function if and only if $\gamma_c(G) = \gamma(G)$.*

Proof. If G has a continuous minimum cost dominating function, then from equation (2.1) we have that $\gamma_c(G) = \gamma_{cc}(G)$. From Theorem 2.3.4 it then follows

that $\gamma_c(G) = \gamma(G)$ and this concludes the first part of the proof. Suppose then that $\gamma_c(G) = \gamma(G)$. By equation (2.1), $\gamma_{cc}(G) = \gamma_c(G)$ so that G must have a continuous minimum cost dominating function. □

2.4 The Structure of f-Dominating Sets

If f is a minimal cost dominating function on a connected graph G, then we have already seen that every vertex in the f-dominating set has a private f-neighbor. If f is a minimum cost dominating function, then even more can be said.

Theorem 2.4.1. *Let G be a connected graph and f a minimum cost dominating function on G. Then for every pair u, v of distinct vertices with $f(u) \leq f(v)$,*

$$f(u) \leq \left\lceil \frac{d(u,v)}{2} \right\rceil .$$

Proof. Assume, to the contrary, that there is some pair u, v of distinct vertices with $f(u) \leq f(v)$ such that $f(u) > \lceil \mathrm{d}(u,v)/2 \rceil$. Let $d = \mathrm{d}(u,v)$ and let $P : u = v_0, v_1, \ldots, v_d = v$ be a $u - v$ geodesic. Consider the cost function $g : V(G) \to [0..\mathrm{rad}\, G]$ given by

$$g(w) = \begin{cases} f(v) + f(v_{\lceil d/2 \rceil}) + \lceil \frac{d}{2} \rceil & \text{if } w = v_{\lceil d/2 \rceil} \\ 0 & \text{if } w = u \text{ or } (w = v \text{ and } d > 1) \\ f(w) & \text{if } w \notin \{u, v, v_{\lceil d/2 \rceil}\}. \end{cases}$$

Since $g(v_{\lceil d/2 \rceil}) > f(v_{\lceil d/2 \rceil})$, every vertex that is f-dominated by $v_{\lceil d/2 \rceil}$ is g-dominated

by $v_{\lceil d/2 \rceil}$. Let x be a vertex of G that is f-dominated by u. Then

$$\begin{aligned} \mathrm{d}(x, v_{\lceil d/2 \rceil}) &\leq \mathrm{d}(x,u) + \mathrm{d}(u, v_{\lceil d/2 \rceil}) \\ &\leq f(u) + \lceil d/2 \rceil \\ &\leq g(v_{\lceil d/2 \rceil}). \end{aligned}$$

Consequently, x is g-dominated by $v_{\lceil d/2 \rceil}$. A similar argument shows that every vertex that is f-dominated by v is g-dominated by $v_{\lceil d/2 \rceil}$. Hence g is a cost dominating function on G. However,

$$\begin{aligned} \sigma(g) &= \sigma(f) - [f(u) + f(v)] + [f(v) + \lceil d/2 \rceil] \\ &= \sigma(f) + \lceil d/2 \rceil - f(u) \\ &< \sigma(f), \end{aligned}$$

which contradicts the fact that $\sigma(f) = \gamma_c(G)$. Thus no such pair u, v of vertices exists and the result follows. □

Note that Theorem 2.4.1 is trivially true whenever $f(u) = 0$. Hence, when applying Theorem 2.4.1, we need only consider pairs u, v of distinct vertices where both u and v are in the f-dominating set.

The converse of Theorem 2.4.1 is false, however. For some integer $k \geq 1$, let T be the set of vertices of degree 2 in the graph $S(K_{1,2+k})$. The function $f : V(S(K_{1,2+k})) \to \{0, 1\}$ defined by

$$f(v) = \begin{cases} 1 & \text{if } v \in T \\ 0 & \text{if } v \notin T \end{cases}$$

is a minimal cost dominating function but not a minimum cost dominating function. However, for every pair u, v of distinct vertices in T, we have that $\mathrm{d}(u, v) = 2$ while $f(u) = f(v) = 1 = \mathrm{d}(u, v)/2$.

Let G be a nontrivial connected graph, f a cost function on G and u and v two vertices of G. Then we define the function $\mathrm{d}_f : V(G) \times V(G) \to \mathbb{N} \cup \{0\}$ as

$$\mathrm{d}_f(u, v) = f(u) + \mathrm{d}(u, v).$$

The following theorem provides another necessary condition for a cost dominating function to be a minimum cost dominating function.

Theorem 2.4.2. *Let G be a connected graph and $v \in V(G)$. If f is a minimum cost dominating function on G, then for every nonempty set $S \subseteq V_f^+$,*

$$\sum_{u \in S} f(u) \leq max\{d_f(u, v) : u \in S\}.$$

Proof. Assume, to the contrary, that there is some nonempty set $S \subseteq V_f^+$ for which

$$\sum_{u \in S} f(u) > M,$$

where $M = \max\{\mathrm{d}_f(u, v) : u \in S\}$. Let $g : V(G) \to [0..\mathrm{rad}\, G]$ be the function given by

$$g(x) = \begin{cases} [1 - \chi_S(v)]f(v) + M & \text{if } x = v \\ 0 & \text{if } x \in S - \{v\} \\ f(x) & \text{if } x \notin S \cup \{v\} \end{cases}$$

in which $\chi_S : V(G) \to \{0, 1\}$ is the characteristic set function of S. If $v \in S$, then $g(v) = M \geq f(v)$. On the other hand, if $v \notin S$, then $g(v) = M + f(v) \geq f(v)$. Thus every vertex that is f-dominated by v is also g-dominated by v. Let $y \in$

$N_f[S]$ and let $y' \in S$ be a vertex that f-dominates y. Then

$$\begin{aligned} d(y,v) &\le d(y,y') + d(y',v) \\ &\le f(y') + d(y',v) \\ &\le M \le g(v) \end{aligned}$$

and so y is g-dominated by v. Consequently, g is a cost dominating function on G. Notice that

$$\begin{aligned} \sum_{u\in S} g(u) &= \chi_S(v)g(v) \\ &= \chi_S(v)\left([1-\chi_S(v)]f(v) + M\right) \\ &= \chi_S(v)M \end{aligned}$$

since $\chi_S^2 = \chi_S$. In addition,

$$\sum_{u\in V(G)-S} g(u) = \sum_{u\in V(G)-S} f(u) + [1-\chi_S(v)]M.$$

Hence,

$$\begin{aligned} \sigma(g) &= \sum_{u\in S} g(u) + \sum_{u\in V(G)-S} g(u) \\ &= \chi_S(v)M + \sum_{u\in V(G)-S} f(u) + [1-\chi_S(v)]M \\ &= \sum_{u\in V(G)-S} f(u) + M \\ &< \sum_{u\in V(G)-S} f(u) + \sum_{u\in S} f(u) \\ &= \sigma(f). \end{aligned}$$

This contradicts the fact that $\sigma(f) = \gamma_c(G)$ and hence no such set S exists. □

Immediately from Theorem 2.4.2 we obtain the following corollary.

Corollary 2.4.3. *Let G be a connected graph. If f is a minimum cost dominating function on G, then for every nonempty set $S \subseteq V_f^+$,*

$$\sum_{u \in S} f(u) \leq \min_{v \in S} \max_{u \in S} d_f(u,v).$$

Proof. Let S be a nonempty set of vertices of G and $v \in S$. From Theorem 2.4.2,

$$\sum_{u \in S} f(u) \leq \max\{\mathrm{d}_f(u,v) : u \in S\}. \tag{2.2}$$

Since the left hand side of (2.2) is independent of v and the right hand side remains true for every choice of $v \in S$, the result follows. □

For a nontrivial connected graph G, we shall denote by $\mathcal{F}_{\mathrm{mcd}}(G)$ the set of all minimum cost dominating functions on G, and let

$$\mathrm{d}_{\mathrm{mcd}}(G) = \min\left\{\max\{\mathrm{d}_f(u,v) : u \in V_f^+\} : v \in V_f^+ \text{ and } f \in \mathcal{F}_{\mathrm{mcd}}(G)\right\}.$$

We now have an upper bound on the cost domination number.

Corollary 2.4.4. *Let G be a nontrivial connected graph. Then*

$$\gamma_c(G) \leq d_{mcd}(G).$$

Proof. Let $f \in \mathcal{F}_{\mathrm{mcd}}(G)$ and $S = V_f^+$. Then from Corollary 2.4.3,

$$\gamma_c(G) = \sum_{x \in V_f^+} f(x) \leq \min\left\{\max\{\mathrm{d}_f(u,v) : u \in V_f^+\} : v \in V_f^+\right\}. \tag{2.3}$$

Since equation (2.3) holds for every minimum cost dominating function f on G, the result follows. □

By means of the next result, we now investigate the sharpness of the bound on $\gamma_c(G)$ given by Corollary 2.4.4.

Lemma 2.4.5. *If G is a nontrivial connected graph with $\gamma_c(G) = \mathrm{rad}\, G$, then $\gamma_c(G) = d_{mcd}(G)$.*

Proof. Let v be the central vertex of G and $f : V(G) \to \{0, \mathrm{rad}\, G\}$ the minimum cost dominating function defined by

$$f(x) = \begin{cases} \mathrm{rad}\, G & \text{if } x = v \\ 0 & \text{if } x \neq v. \end{cases}$$

Since $V_f^+ = \{v\}$, we have that

$$\begin{aligned} \mathrm{d}_{\mathrm{mcd}}(G) &\leq \min\left\{\max\{\mathrm{d}_f(u,v) : u \in V_f^+\} : v \in V_f^+\right\} \\ &= \mathrm{d}_f(v,v) \\ &= \mathrm{rad}\, G = \gamma_c(G). \end{aligned}$$

The result then follows from Corollary 2.4.4. □

For every positive integer t with $2 \leq t \leq 5$, let $G_t = P_t$; and for $t \geq 6$ let G_t be the graph obtained by joining every vertex of $\overline{K}_{t-5}$ to the central vertex of P_5. For example, the graph G_8 is shown in Figure 7.

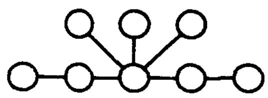


Figure 7: The Graph G_8.

Then

$$\mathrm{rad}\, G_t = \begin{cases} 1 & \text{if } t \in \{2,3\} \\ 2 & \text{if } t \geq 4. \end{cases}$$

For every integer $t \geq 2$, the graph G_t has order t and from Proposition 1.4.6 we have that $\gamma_c(G_t) = \text{rad } G_t$. Consequently, Lemma 2.4.5 provides the following result.

Proposition 2.4.6. *For every integer $n \geq 2$, there exists a connected graph G of order n having*

$$\gamma_c(G) = d_{mcd}(G).$$

THE COST DOMINATION NUMBER AND OTHER PARAMETERS

3.1 Graphs Having Equal Cost Domination and Domination Numbers

Corollary 1.3.2 states that $\gamma_c(P_n) = \gamma(P_n)$ for every positive integer n. We now show that cycles also have the property that their domination and cost domination numbers are equal. First, however, we establish a lemma.

Lemma 3.1.1. *Let G be a connected graph and f a minimum cost dominating function on G. If $\gamma(G) <$ rad G, then*

1. $\gamma_c(G) <$ *rad* G
2. $|V_f^+| \geq 2$
3. *max*$\{f(v) : v \in V_f^+\} \leq$ *rad* $G - 2$.

Proof. From Proposition 1.2.1, $\gamma_c(G) \leq \gamma(G)$. Thus $\gamma_c(G) <$ rad G. Suppose that $V_f^+ = \{v\}$. By Lemma 1.2.4, $f(v) =$ rad G and hence $f(v) > \gamma_c(G)$. From this, it follows that $|V_f^+| \geq 2$. Finally, suppose that there is a vertex u satisfying $f(u) =$ rad $G - 1$. Since $|V_f^+| \geq 2$, there is a vertex v in G distinct from u and satisfying $f(v) \geq 1$. However, $\sigma(f) \geq f(u) + f(v) \geq$ rad G which contradicts $\gamma_c(G) <$ rad G. Consequently, no such vertex u exists. □

We are now in a position to determine the cost domination numbers of cycles.

Theorem 3.1.2. *For every positive integer $n \geq 3$,*

$$\gamma_c(C_n) = \gamma(C_n) = \left\lceil \frac{n}{3} \right\rceil.$$

Furthermore, if f is a minimum cost dominating function on C_n, then

$$max\{f(v) : v \in V(C_n)\} \leq 3.$$

Proof. It is well-known that $\gamma(C_n) = \lceil n/3 \rceil$. From Theorem 1.4.1, $\gamma_c(C_3) = 1 = \gamma(C_3)$. If $n \in \{4, 5, 6\}$, then $\gamma(C_n) = 2$ and, by Theorem 1.4.2, $\gamma_c(C_n) = 2$. Moreover, $\gamma(C_7) = \text{rad } C_7 = 3$, so from Proposition 1.4.5 we have $\gamma_c(C_7) = 3$. Suppose then that $n \geq 8$. By Proposition 1.2.1, it suffices to prove that $\gamma(C_n) \leq \gamma_c(C_n)$. Let f be a minimum cost dominating function on C_n and $x \in V_f^+$. Since $n \geq 8$, we have

$$\gamma(C_n) = \left\lceil \frac{n}{3} \right\rceil < \left\lfloor \frac{n}{2} \right\rfloor = \text{rad } C_n.$$

From Lemma 3.1.1, $|V_f^+| \geq 2$ and so $N_f[x] \neq V(C_n)$. It follows that the subgraph $P(x)$ of C_n induced by $N_f[x]$ is a path of order $2f(x) + 1$. Since f is a cost dominating function, $N_f[V_f^+] = V(C_n)$. Let $S(x)$ be a minimum dominating set for $P(x)$. Then

$$|S(x)| = \left\lceil \frac{2f(x) + 1}{3} \right\rceil$$

and since $f(x) \geq 1$, $\left\lceil \frac{2f(x)+1}{3} \right\rceil \leq f(x)$. Let

$$S = \bigcup_{x \in V_f^+(C_n)} S(x).$$

Then S is a dominating set for C_n and

$$|S| \leq \sum_{x \in V_f^+(C_n)} f(x) = \gamma_c(C_n).$$

Consequently, $\gamma(C_n) \leq \gamma_c(C_n)$ and the first part of the result follows.

We now verify the second part of the result, namely that if f is a minimum cost dominating function on C_n, then $\max\{f(v) : v \in V(C_n)\} \leq 3$. Assume, to the contrary, that v is a vertex of C_n having $f(v) \geq 4$. Then from the fact that $\sigma(f) = \gamma_c(C_n)$, we have that $n \geq 8$. In addition, since $f(v) \geq 4$,

$$\left\lceil \frac{2f(v)+1}{3} \right\rceil < f(v)$$

so that in the first part of this proof, the dominating set $S(v)$ that we introduce has cardinality strictly less than $f(v)$, which contradicts our choice of f. Consequently, no such vertex v exists, and the second part of the result follows. □

Theorem 1.3.1 and equation (1.2) show that if G is a graph for which $\gamma_c(G) = \lceil(\text{diam } G + 1)/3\rceil$, then $\gamma_c(G) = \gamma(G)$. Theorem 3.1.2 demonstrates that the converse is not true, that is, there are graphs with equal domination and cost domination numbers, both strictly greater than the lower bound given by Theorem 1.3.1.

As was mentioned in the proof of Theorem 3.1.2, for every integer $n \geq 8$, we have

$$\gamma(C_n) = \left\lceil \frac{n}{3} \right\rceil < \left\lfloor \frac{n}{2} \right\rfloor = \text{rad } C_n.$$

Hence there is an infinite class of graphs G such that

$$\gamma_c(G) = \gamma(G) < \text{rad } G.$$

The following result provides a general (though somewhat trivial) condition for the domination and cost domination numbers of a graph to be equal.

Theorem 3.1.3. *Let G be a connected graph. Then $\gamma_c(G) = \gamma(G)$ if and only if there exists a minimum cost dominating function f on G such that V_f^+ is a dominating set.*

Proof. We suppose first that $\gamma_c(G) = \gamma(G)$. Let S be a minimum dominating set of G and f the characteristic set function associated with S. Since S is a minimum dominating set, it follows from the assumption that $\gamma_c(G) = \gamma(G)$ that f is a minimum cost dominating function. Furthermore, $S = V_f^+$. We now suppose that there exists a minimum cost dominating function f on G for which V_f^+ is a dominating set. Then $\gamma(G) \leq |V_f^+| \leq \gamma_c(G)$ and the result follows from Proposition 1.2.1. □

3.2 Graphs Having Equal Cost Domination Number and Radius

In this section, we establish the existence of two infinite classes of graphs G for which $\gamma_c(G) = \text{rad } G$. One of these classes consists of certain grid graphs, and the other we define now.

For a graph G and a positive integer k, we define $S_k(G)$ as that graph obtained from G by inserting k vertices into every edge of G. It follows that $S_1(G) = S(G)$ and $S_k(P_2) = P_{k+2}$. For positive integers k and t, we let

$$S_{k,t} = S_k(K_{1,t}).$$

The graph $S_{2,3} = S_2(K_{1,3})$ is shown in Figure 8.

The graph $S_{k,t}$ has been discussed previously for some values of k and t. Theorem 1.4.1 states that $\gamma_c(K_{1,t}) = \text{rad } K_{1,t} = 1$ for every positive integer t, and

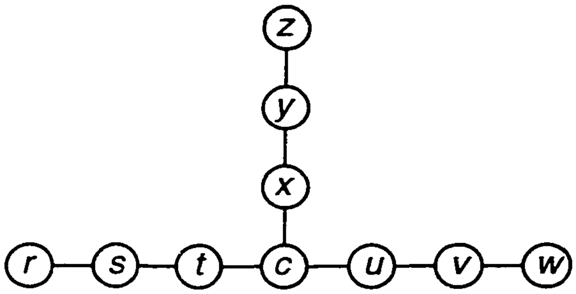


Figure 8: The Graph $S_{2,3}$.

from Proposition 1.4.6, $\gamma_c(S_{k,t}) = \text{rad } S_{k,t}$ for every pair k, t of integers with $t \geq 3$ and $k \in \{1, 2\}$. We shall prove that $\gamma_c(S_{k,t}) = \text{rad } S_{k,t}$ for every positive integer k and every integer $t \geq 5$. We now introduce the notation that we shall employ in the proof.

Every path in $S_{k,t}$ of length k that does not contain the central vertex of $S_{k,t}$ will be called an *arm* of $S_{k,t}$. Thus $S_{2,3}$ has three arms: the path r, s, t, the path u, v, w and the path x, y, z. The central vertex c is in none of the arms. In general, $S_{k,t}$ has t arms, each having $k + 1$ vertices. If v is a non-central vertex of $S_{k,t}$, then there is a unique arm of $S_{k,t}$ containing v that we denote $\text{arm}(v)$. For the graph $S_{2,3}$, $\text{arm}(r)$ is the path r, s, t and $\text{arm}(v)$ is the path u, v, w. Moreover, for every integer j with $0 \leq j \leq k$, that vertex of $\text{arm}(v)$ whose distance is j from the endvertex (in $S_{k,t}$) of $\text{arm}(v)$ is denoted $\text{arm}_j(v)$. Hence, for the graph $S_{2,3}$, $\text{arm}_0(r) = r$, $\text{arm}_1(r) = s$ and $\text{arm}_2(r) = t$. Note that $\text{arm}_0(r) = \text{arm}_0(s) = \text{arm}_0(t)$. In general, the endvertex (in $S_{k,t}$) of $\text{arm}(v)$ is $\text{arm}_0(v)$ and the vertex of $\text{arm}(v)$ that is adjacent to the central vertex of $S_{k,t}$ is $\text{arm}_k(v)$.

Before determining the cost domination number of $S_{k,t}$, we present a lemma that will prove useful in that calculation. Let f be a cost dominating function on

G and let $v \in V_f^+(G)$. A vertex x is said to be *f-essential* to v if there is a private f-neighbor u of v and a shortest $u - v$ path containing x. If f is minimal, then from Theorem 2.1.2 every f-dominating vertex has at least one private f-neighbor. Consequently, the vertex v is f-essential to v as is every private f-neighbor of v. For example, consider the graph G of Figure 9(a) and the minimal cost dominating function f on G shown in Figure 9(b).

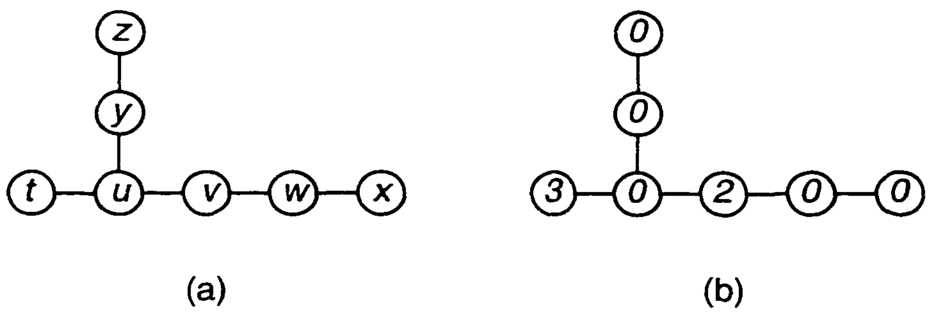


Figure 9: A Cost Dominating Function f.

Both of the f-dominating vertices t and v have only one private f-neighbor. The vertex t has the private f-neighbor z while the vertex v has the private f-neighbor x. However, the vertices t, u, y, z are all f-essential to t as they comprise the (unique) $t - z$ geodesic in G and, similarly, the vertices v, w, x are f-essential to v.

Lemma 3.2.1. *Let f be a cost dominating function on G, let x and x' be two vertices of G, and let v and v' be distinct members of $V_f^+(G)$. If x is f-essential to v and x' is f-essential to v', then $x \neq x'$.*

Proof. Let u be a private f-neighbor of v, let P be a shortest $v - u$ path, and let x be a vertex on P that is f-essential to v. Similarly, let u' be a private f-

neighbor of v', let P' be a shortest $v' - u'$ path, and let x' be a vertex on P' that is f-essential to v'. Suppose to the contrary that $x = x'$. Since u and u' are private f-neighbors of v and v', it is immediate that x is neither u nor u'. Since P is a shortest $v - u$ path and u' is a private f-neighbor of v',

$$\mathrm{d}(v,u) = \mathrm{d}(v,x) + \mathrm{d}(x,u) \leq f(v) < \mathrm{d}(v,u') \leq \mathrm{d}(v,x) + \mathrm{d}(x,u').$$

Similarly,

$$\mathrm{d}(v',u') = \mathrm{d}(v',x) + \mathrm{d}(x,u') \leq f(v') < \mathrm{d}(v',u) \leq \mathrm{d}(v',x) + \mathrm{d}(x,u).$$

Adding these two chains of inequalities, we obtain

$$\mathrm{d}(v,x) + \mathrm{d}(x,u) + \mathrm{d}(v',x) + \mathrm{d}(x,u') < \mathrm{d}(v,x) + \mathrm{d}(x,u') + \mathrm{d}(v',x) + \mathrm{d}(x,u),$$

an impossibility. □

Corollary 3.2.2. *Let f be a minimal cost dominating function on a connected graph G. For every vertex $v \in V_f^+$ and every private f-neighbor v' of v, no f-dominating vertex distinct from v lies on any $v - v'$ geodesic in G.*

Proof. Since f is minimal, every f-dominating vertex is f-essential to itself. The result now follows directly from Lemma 3.2.1. □

We are now ready to determine $\gamma_c(S_{k,t})$.

Theorem 3.2.3. *For every positive integer k and every integer $t \geq 5$,*

$$\gamma_c(S_{k,t}) = rad\ S_{k,t} = k + 1.$$

Proof. From Proposition 1.2.1, $\gamma_c(S_{k,t}) \leq k + 1$. We now show that $\gamma_c(S_{k,t}) \geq k + 1$. Assume, to the contrary, that $\gamma_c(S_{k,t}) < k + 1$. Then from Lemma 1.2.4,

every minimum cost dominating function on $S_{k,t}$ has at least two cost dominating vertices. Let c be the central vertex of $S_{k,t}$.

Among the minimum cost dominating functions on $S_{k,t}$, let f be one for which $\vec{f}(c)$ is a minimum. We claim that $\vec{f}(c) = 1$. Assume, to the contrary, that $\vec{f}(c) \geq 2$. Then there are distinct vertices $y_1, y_2 \in V_f^+$ such that $c \in N_f[y_1] \cap N_f[y_2]$. We note from Lemma 3.2.1 that c is f-essential to at most one of y_1 and y_2. We may assume that c is not f-essential to y_1. Since y_1 is f-essential to itself, $y_1 \neq c$. Hence y_1 is contained in an arm of $S_{k,t}$. Moreover, since c is not f-essential to y_1, every private f-neighbor of y_1 is contained in $V(\text{arm}(y_1)) \cap N_f[y_1]$. The graph $\langle V(\text{arm}(y_1)) \cap N_f[y_1] \rangle$ is a subpath of $\text{arm}(y_1)$. Let i be that integer with $0 \leq i \leq k$ such that $\text{arm}_i(y_1) = y_1$. We now consider two cases, according to whether $f(y_1) = 1$ or $f(y_1) \geq 2$.

Case 1: $f(y_1) = 1$. Since y_1 f-dominates c, it follows that $i = k \geq 1$. From Theorem 2.1.2, y_1 has at least one private f-neighbor in $N[y_1] - \{c\}$, and every private f-neighbor of y_1 is contained in $N[y_1] - \{c\}$. Thus $f(\text{arm}_{i-1}(y_1)) = 0$. The cost function $f' : V(S_{k,t}) \to [0..k+1]$ defined by

$$f'(x) = \begin{cases} f(x) & \text{if } x \notin \{y_1, \text{arm}_{i-1}(y_1)\} \\ 0 & \text{if } x = y_1 \\ 1 & \text{if } x = \text{arm}_{i-1}(y_1) \end{cases}$$

is a minimum cost dominating function on $S_{k,t}$. Since y_1 is not an f'-dominating vertex and $\text{arm}_{i-1}(y_1)$ does not f'-dominate c, we have $\vec{f'}(c) < \vec{f}(c)$. This contradicts our assumption about f and so $\vec{f}(c) = 1$.

Case 2: $f(y_1) \geq 2$. From Theorem 2.1.2, y_1 has a private f-neighbor at distance $f(y_1)$ from y_1. Since c is not f-essential to y_1, every private f-neighbor

of y_1 is a vertex of $\mathrm{arm}(y_1)$. However, c is f-dominated by y_1. Therefore, $\mathrm{arm}_{i-f(y_1)}(y_1)$ is the unique vertex of $S_{k,t}$ with this property; namely, $\mathrm{arm}_{i-f(y_1)}(y_1)$ is the only private f-neighbor of y_1 at distance $f(y_1)$. From this and Corollary 3.2.2, we have that $f(\mathrm{arm}_j(y_1)) = 0$ for every integer j with $i - f(y_1) \leq j \leq i-1$. Notice that

$$\mathrm{d}(y_1, c) = k + 1 - i \leq f(y_1)$$

and

$$i \leq k;$$

consequently,

$$i - f(y_1) \leq k - f(y_1) \leq i - 1.$$

Consider the cost function $f' : V(S_{k,t}) \to [0..k+1]$ given by

$$f'(x) = \begin{cases} f(y_1) & \text{if } x = \mathrm{arm}_{k-f(y_1)}(y_1) \\ 0 & \text{if } x = y_1 \\ f(x) & \text{if } x \notin \{y_1, \mathrm{arm}_{k-f(y_1)}(y_1)\}. \end{cases}$$

The distance from $\mathrm{arm}_{k-f(y_1)}(y_1)$ to c is $f(y_1) + 1$, so that c is not f'-dominated by either y_1 or $\mathrm{arm}_{k-f(y_1)}(y_1)$. However, $\mathrm{arm}_{k-f(y_1)}(y_1)$ f'-dominates every private f-neighbor of y_1. The minimum cost dominating function f' has $\vec{f'}(c) < \vec{f}(c)$, contradicting our assumption about f.

Thus $\vec{f}(c) = 1$, as claimed. Let v be that unique vertex of $S_{k,t}$ that f-dominates c. Then $S_{k,t} - N_f[v]$ is a disjoint union of paths, where each such path is a subpath of some arm of $S_{k,t}$. Let $\mathcal{P}$ be the collection of these paths and

$d = \mathrm{d}(v, c)$. One path in $\mathcal{P}$ has length $k - f(v) - d$ and every other path in $\mathcal{P}$ has length $k - f(v) + d$. Of course, if $v = c$ (and hence $d = 0$), then every path in P has the same length $k - f(v)$. Since f is a minimum cost dominating function on $S_{k,t}$,

$$\gamma_c(S_{k,t}) = f(v) + \gamma_c(P_{k-f(v)-d+1}) + (t-1)\gamma_c(P_{k-f(v)+d+1}). \tag{3.1}$$

From Corollary 1.3.2, $\gamma_c(P_n) = \gamma(P_n)$. Furthermore, since $\vec{f}(c) = 1$, every f-dominating vertex u distinct from v f-dominates vertices only in arm(u). We thus assume, without loss of generality, that every f-dominating vertex u distinct from v has $f(u) = 1$ and, further, that no f-dominating vertex u distinct from v is f-dominated by v. This second condition can always be met by choosing the members of V_f^+, apart from v, to be vertices in $\mathcal{P}$ that are not adjacent to c. Since $k \geq 1$, it follows from the preceding discussion that there is a minimum cost dominating function that satisfies this condition.

If $v = c$, let $\mathcal{A}$ be a collection of $t-1$ arms of $S_{k,t}$; while if $v \neq c$, then let $\mathcal{A}$ be the $t-1$ arms of $S_{k,t}$ that do not contain v. For each $A \in \mathcal{A}$, let u_A be that f-dominating vertex of A for which $\mathrm{d}(u_A, c)$ is a minimum. Every f-dominating vertex u in $S_{k,t} - N_f[v]$ has $f(u) = 1$. Let $\mathcal{P}'$ be the $t-1$ paths in $\mathcal{P}$ that are subpaths in $S_{k,t}$ of the $t-1$ arms in $\mathcal{A}$. From the manner in which we chose the arms in $\mathcal{A}$, the paths in $\mathcal{P}'$ are pairwise isomorphic. We may thus assume without loss of generality that there is some positive integer d' such that for every $A \in \mathcal{A}$,

$$d' = \mathrm{d}(u_A, c).$$

By assumption, there is no $A \in \mathcal{A}$ such that u_A is f-dominated by v. Also, in every $P \in \mathcal{P}$ there is a vertex y that is adjacent to a vertex in $N_f[v]$. Hence there must be some $A \in \mathcal{A}$ such that $y \in N[u_a]$. The vertex v f-dominates $f(v) - d$

vertices in each arm in $\mathcal{A}$. Consequently,

$$f(v) - d + 1 \leq d' \leq f(v) - d + 2. \tag{3.2}$$

Define the set $S = \{v\} \cup \left(\bigcup_{a \in \mathcal{A}} u_A\right)$. Then since $f(u_A) = 1$ for every $A \in \mathcal{A}$, we have

$$\sum_{x \in S} f(x) = f(v) + t - 1.$$

For $x \in S$,

$$\mathrm{d}_f(x, v) = \begin{cases} f(v) & \text{if } x = v \\ 1 + d + d' & \text{if } x \in S - \{v\}. \end{cases}$$

From equation (3.2), $f(v) - d - 1 \leq f(v) - d + 1 \leq d'$. Hence

$$\max_{x \in S} \mathrm{d}_f(x, v) = 1 + d + d'.$$

Let

$$m = \min_{y \in S} \max_{x \in S} \mathrm{d}_f(x, y).$$

Then

$$m \leq 1 + d + d'.$$

Since f is a minimum cost dominating function, it follows from Corollary 2.4.3 that

$$\sum_{x \in S} f(x) = f(v) + t - 1 \leq 1 + d + d'.$$

From equation (3.2), $f(v) \geq d' + d - 2$. Consequently,

$$d' + d + t - 3 \leq d' + d + 1.$$

This inequality holds if and only if $t \leq 4$. This a contradiction. Hence no such function f exists, every minimum cost dominating function on $S_{k,t}$ has exactly one cost dominating vertex, and $\gamma_c(S_{k,t}) = k + 1$. □

Corollary 3.2.4. *For every positive integer k and every integer $t \geq 5$, the graph $S_{k,t}$ has exactly one minimum cost dominating function.*

Proof. From the proof of Theorem 3.2.3, no minimum cost dominating function has more than one cost dominating vertex. The result follows immediately from Lemma 1.2.4. □

Conjecture 3.2.5. *For every positive integer k and $t \in \{3, 4\}$,*

$$\gamma_c(S_{k,t}) = rad\ S_{k,t} = k + 1.$$

We now consider the problem of determining the cost domination number of the grid graph $P_m \times P_n$. The domination number of $P_m \times P_n$ has been determined for small values of m and all values of n. Since $P_1 \times P_n = P_n$, $\gamma(P_1 \times P_n) = \gamma(P_n)$. Jacobson and Kinch [11] established $\gamma(P_m \times P_n)$ for $m \in \{2, 3, 4\}$, while Chang and Clark [1] proved formulas that Hare [8] had conjectured for $m = 5$ and $m = 6$. The domination numbers for $P_m \times P_n$, $1 \leq m \leq 6$, are listed below:

$$\gamma(P_1 \times P_n) = \left\lfloor \frac{n+2}{3} \right\rfloor, \ n \geq 1$$

$$\gamma(P_2 \times P_n) = \left\lfloor \frac{n+2}{2} \right\rfloor, \ n \geq 1$$

$$\gamma(P_3 \times P_n) = \left\lfloor \frac{3n+4}{4} \right\rfloor, \ n \geq 1$$

$$\gamma(P_4 \times P_n) = \begin{cases} n+1, & n \in \{1, 2, 3, 5, 6, 9\} \\ n, & \text{otherwise for } n \geq 1 \end{cases}$$

$$\gamma(P_5 \times P_n) = \begin{cases} \lfloor \frac{6n+6}{5} \rfloor, & n \in \{2,3,7\} \\ \lfloor \frac{6n+8}{5} \rfloor, & \text{otherwise for } n \geq 1 \end{cases}$$

$$\gamma(P_6 \times P_n) = \begin{cases} \lfloor \frac{10n+10}{7} \rfloor, & n \geq 6 \text{ and } n \equiv 1 \pmod 7 \\ \lfloor \frac{10n+12}{7} \rfloor, & \text{otherwise for } n \geq 4. \end{cases}$$

Since $P_1 \times P_n = P_n$ for every positive integer n, Corollary 1.3.2 gives the cost domination number of $P_1 \times P_n$. We shall thus be concerned with calculating the cost domination number of $P_2 \times P_n$ for $n \geq 2$ and, more generally, the cost domination number of $P_m \times P_n$ for $m, n \geq 2$. Throughout the discussion, we shall use the terminology that we now introduce.

For every pair m, n of positive integers, let $P_m : u_1, u_2, \ldots, u_m$ and $P_n : v_1, v_2, \ldots, v_n$. The vertex set of $P_m \times P_n$ is the set $\{(u_i, v_j) : 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$. For convenience, for every pair i, j of integers with $1 \leq i \leq m$ and $1 \leq j \leq n$, we shall let $w_{i,j} = (u_i, v_j)$. Furthermore, for every integer i with $1 \leq i \leq m$, the subgraph of $P_m \times P_n$ induced by the set $\{w_{i,j} : 1 \leq j \leq n\}$ is the *ith row* of $P_m \times P_n$, denoted $R_i(P_m \times P_n)$ or simply R_i if the context is clear. Likewise, for every integer j with $1 \leq j \leq n$, the subgraph of $P_m \times P_n$ induced by the set $\{w_{i,j} : 1 \leq i \leq m\}$ is the *jth column* of $P_m \times P_n$, denoted $C_j(P_m \times P_n)$ or simply C_j if there is no ambiguity. The row and column to which a vertex x belongs will be denoted $R(x)$ and $C(x)$, respectively. Hence if $x \in V(R_i) \cap V(C_j)$, then $R(x) = i$ and $C(x) = j$.

The radius of $P_m \times P_n$ will play an important part in what follows, so we

mention here that for every pair m, n of positive integers,

$$\text{rad } P_m \times P_n = \left\lceil \frac{m-1}{2} \right\rceil + \left\lceil \frac{n-1}{2} \right\rceil .$$

We now determine the cost domination number of $P_2 \times P_n$.

Theorem 3.2.6. *For every integer $n \geq 1$,*

$$\gamma_c(P_2 \times P_n) = \gamma(P_2 \times P_n) = \textit{rad } P_2 \times P_n = \left\lfloor \frac{n+2}{2} \right\rfloor .$$

Proof. Let f be a minimum cost dominating function on $P_2 \times P_n$ and $\{x_1, x_2, \ldots, x_t\}$ the f-dominating set, where $t = |V_f^+|$ and the vertices $x_1, x_2, \ldots, x_t$ are ordered such that, for every pair i, j of integers with $1 \leq i \leq j \leq t$, we have that $C(x_i) \leq C(x_j)$. We now define a cost function $g : V(P_2 \times P_n) \to [0..\text{rad } P_2 \times P_n]$ with g-dominating set $\{y_1, \ldots, y_t\}$. For each integer i with $1 \leq i \leq t$, we let $g(y_i) = f(x_i)$. We now specify to which vertices of $P_2 \times P_n$ the vertices $y_1, y_2, \ldots, y_t$ correspond, and we do this by stating, for each i with $1 \leq i \leq t$, which column and which row the vertex y_i belongs to. Firstly, if i is odd, then y_i will be in row 1; otherwise if i is even, then y_i will be in row 2. We let $C(y_1) = g(y_1)$. Hence y_1 g-dominates the two vertices in column 1, all those vertices z in row 1 with $C(z) \leq 2g(y_1)$, and all those vertices z in row 2 with $C(z) \leq 2g(y_1) - 1$.

We now choose y_2 such that $C(y_2) = C(y_1) + g(y_1) + g(y_2)$. Notice that by doing so, each vertex in $N_g[y_1] \cup N_g[y_2]$ is g-dominated exactly once. We continue this process, in general for $2 \leq i \leq t-1$, assigning vertex y_i to column $C(y_{i-1}) + g(y_{i-1}) + g(y_i)$ until we have placed all of the vertices y_i with $1 \leq i \leq t-1$, noting that (by virtue of the fact that f is a minimum cost dominating function of $P_2 \times P_n$ and $\sigma(g) = \sigma(f)$) each vertex y_i will be assigned to some well-defined vertex of $P_2 \times P_n$. The set $\{y_1, y_2, \ldots, y_{t-1}\}$ is not a g-dominating set of $P_2 \times P_n$.

We note that the largest integer i for which there is some vertex in column i that is g-dominated by some vertex y_i with $1 \le i \le t-1$ is $2\sum_{i=1}^{t-1} g(y_i)$, and that, due to the way in which we have chosen the vertices y_i, the largest integer i for which there is some vertex in column i that is f-dominated is at most $2\sum_{i=1}^{t-1} g(y_i)$. We now choose y_t to be in column $\min\{C(y_{t-1})+g(y_{t-1})+g(y_t), n\}$. Since $P_2 \times P_n$ is f-dominated, it must be g-dominated. Assuming without loss of generality that $R(y_t) = 1$, since y_t g-dominates the vertex w_{2n}, it follows that

$$2\gamma_c(P_2 \times P_n) - 1 = 2\sigma(g) - 1 \ge n.$$

The result now follows from the fact that

$$\left\lfloor \frac{n+2}{2} \right\rfloor = \left\lceil \frac{n+1}{2} \right\rceil.$$

□

Corollary 3.2.7. *For every positive integer r, there exists a graph G_r for which*

$$\gamma_c(G_r) = \gamma(G_r) = rad\ G_r = r.$$

Proof. Let $G_r = P_2 \times P_{2r-1}$. The result follows immediately from Theorem 3.2.6.

□

As an immediate consequence of Theorem 3.2.6, we can determine $\gamma_c(P_3 \times P_n)$ for every positive integer n.

Corollary 3.2.8. *For every integer $n \ge 1$,*

$$\gamma_c(P_3 \times P_n) = rad\ P_3 \times P_n = 1 + \left\lceil \frac{n-1}{2} \right\rceil.$$

Proof. Assume, to the contrary, that $\gamma_c(P_3 \times P_n) <$ rad $P_3 \times P_n$, and let f be a minimum cost dominating function on $P_3 \times P_n$. Since f cost dominates every

subgraph of $P_3 \times P_n$ isomorphic to $P_2 \times P_n$, it follows that there exists a cost dominating function g on $P_2 \times P_n$ with $\sigma(g) \leq \sigma(f)$. However,

$$\text{rad } P_3 \times P_n = 1 + \left\lceil \frac{n-1}{2} \right\rceil = \text{rad } P_2 \times P_n$$

so that, by Theorem 3.2.6,

$$\gamma_c(P_2 \times P_n) \leq \sigma(g) \leq \sigma(f) < \text{rad } P_2 \times P_n = \gamma_c(P_2 \times P_n).$$

This is a contradiction. Hence no such function f exists and the result follows. □

For $n \geq 1$, $\gamma_c(P_3 \times P_n) = 1 + \lceil (n-1)/2 \rceil$, while $\gamma(P_3 \times P_n) = \lfloor (3n+4)/4 \rfloor$. For $n \geq 3$,

$$1 + \left\lceil \frac{n-1}{2} \right\rceil < \left\lfloor \frac{3n+4}{4} \right\rfloor.$$

Hence, for $n \geq 3$ we have that rad $P_3 \times P_n = \gamma_c(P_3 \times P_n) < \gamma(P_3 \times P_n)$. Thus, there is an infinite class of graphs G for which

$$\gamma_c(G) = \text{rad } G < \gamma(G).$$

We conclude this discussion with a conjecture.

Conjecture 3.2.9. *For every pair m, n of integers with $\min\{m, n\} \geq 2$,*

$$\gamma_c(P_m \times P_n) = \textit{rad } P_m \times P_n = \left\lceil \frac{m-1}{2} \right\rceil + \left\lceil \frac{n-1}{2} \right\rceil.$$

For every positive integer k and every integer $t \geq 3$, the graph $S_{k,t}$ is separable. For every pair m, n of integers with $m, n \geq 2$, we have $\kappa(P_m \times P_n) = 2$, where $\kappa(P_m \times P_n)$ denotes the connectivity of $P_m \times P_n$. This gives us the following result.

Corollary 3.2.10. *For every positive integer r, there exist graphs G_r and H_r, both having radius r, such that $\kappa(G_r) = 1$, $\kappa(H_r) = 2$ and $\gamma_c(G_r) = \gamma_c(H_r) = r$.*

Proof. For $r \geq 2$, we let $G_r = S_{r-1,t}$ and $H_r = P_2 \times P_{2r-1}$ and the result follows from Theorems 3.2.3 and 3.2.6, respectively. Let $G_1 = K_{1,3}$ and $H_1 = K_3$. Since rad $K_{1,3}$ = rad $K_3 = 1$ and, from Theorem 1.4.1, $\gamma_c(K_1, 3) = \gamma_c(K_3) = 1$, this completes the proof. □

3.3 Relationships Between $\gamma_c(G)$ and Other Domination Parameters

Those functions that map the vertex set $V(G)$ of a graph G to the closed unit interval $[0, 1]$ are called *fractional set functions*. A *fractional dominating function* on G is a fractional set function $f : V(G) \to [0, 1]$ such that $\sum_{u \in N[v]} f(u) \geq 1$ for every $v \in V(G)$. The *fractional domination number* $\gamma_f(G)$ is the minimum value of $\sigma(f) = \sum_{v \in V(G)} f(v)$ over all fractional dominating functions f on G. Fractional domination can be viewed as a linear programming problem. Since $\gamma_f(G)$ is the value of a linear program with integer coefficients, $\gamma_f(G)$ is always a rational number. Moreover, for every graph G there exists a fractional dominating function f with $\sigma(f) = \gamma_f(G)$ such that every vertex of G is assigned a rational number in the interval $[0, 1]$ by f. An overview of fractional domination may be found in [5].

Not only is the characteristic set function associated with a dominating set a cost dominating function (as we remarked earlier), it is also a fractional dominating function. Consequently, we obtain the well-known result that for every graph G,

$$\gamma_f(G) \leq \gamma(G).$$

This inequality and the bound on $\gamma_c(G)$ stated in Proposition 1.2.1 gives rise to a natural question: Is $\gamma_f(G)$ either an upper or lower bound for $\gamma_c(G)$? In fact, as we now see, it is neither of these. It is well-known that for every integer $n \geq 3$, $\gamma_f(C_n) = n/3$ while Theorem 3.1.2 states that $\gamma_c(C_n) = \lceil n/3 \rceil$. Consequently, when $n \equiv 1, 2 \pmod 3$, the fractional domination number of C_n is strictly less than the cost domination number of C_n, and thus there is an infinite class of graphs G for which $\gamma_f(G) < \gamma_c(G)$.

We now show that there is an infinite class of graphs G for which $\gamma_c(G) < \gamma_f(G)$. First, however, we introduce the well-studied (a comprehensive discussion may be found in [9]) concept of efficient domination to aid in our analysis. For a graph G, an *efficient dominating set* of G is a dominating set of G such that every vertex of G is dominated exactly once. Not every graph has an efficient dominating set. However, if a graph G has an efficient dominating set S, then $|S| = \gamma(G)$. For every positive integer $t \geq 3$, let $G_t = S_{2,t}$ be the graph of radius 3 obtained by inserting two new vertices into every edge of $K_{1,t}$. For example, the graph G_4 is shown in Figure 10.

The graph G_t has an efficient dominating set W_t of cardinality $t + 1$, consisting of the central vertex of G_t together with the endvertices of G_t. Goddard and Henning [7] and, independently, Erwin [6] showed that if a graph G has an efficient dominating set, then $\gamma_f(G) = \gamma(G)$. Consequently, $\gamma_f(G_t) = t + 1 = \gamma(G_t)$. On the other hand, $\gamma_c(G_t) \leq 3$ so that $\gamma_c(G_t) < \gamma_f(G_t)$. In fact, since $\min\{\text{rad } G_t, \gamma(G_t)\} = 3$,

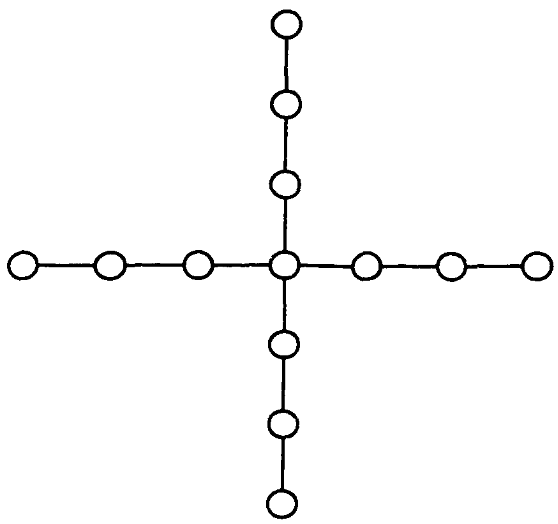


Figure 10: The Graph $G_4 = S_{2,4}$.

it follows from Proposition 1.4.5 that $\gamma_c(G_t) = 3$. Consequently, there is an infinite class of graphs G having $\gamma_c(G) < \gamma_f(G)$.

We now turn to another well-known variation of domination. As we mentioned previously, a subset S of $V(G)$ is a distance-k dominating set if every vertex not in S is within distance k of some vertex of S. The *distance-k domination number* $\gamma_k(G)$ is the minimum cardinality of a distance-k dominating set of G, and a distance-k dominating set S for which $|S| = \gamma_k(G)$ is a *minimum* distance-k dominating set. Let G be a connected graph and for some integer k with $1 \leq k \leq \text{rad } G$, let S be a minimum distance-k dominating set. Then the function $f : V(G) \to \{0, k\}$ defined by

$$f(v) = \begin{cases} 0 & \text{if } v \notin S \\ k & \text{if } v \in S \end{cases}$$

is a cost dominating function on G and $\sigma(f) = k\gamma_k(G)$. Consequently, we obtain the following upper bound on the cost domination number.

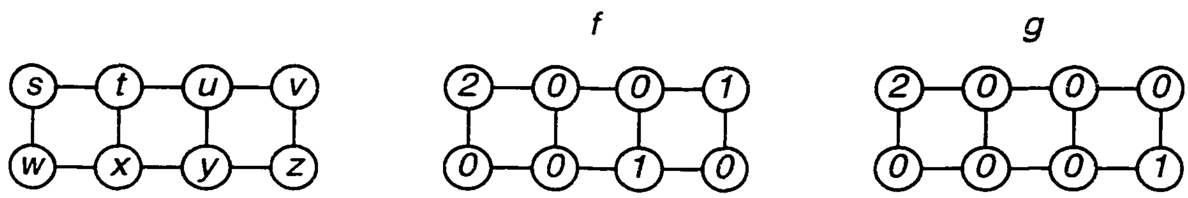
Proposition 3.3.1. *For every nontrivial connected graph G,*

$$\gamma_c(G) \leq min\{k\gamma_k(G) : 1 \leq k \leq rad\ G\}.$$

Of course, $\gamma_1(G) = \gamma(G)$. In addition, if rad $G = r$, then $\gamma_r(G) = 1$. Hence, the bound given by Proposition 3.3.1 is actually a generalization of the bound $\gamma_c(G) \leq \min\{\gamma(G), \text{rad } G\}$.

3.4 Efficient Cost Domination

We mentioned earlier that a dominating set S of a graph G is efficient if every vertex of G is dominated exactly once by the vertices of S. With this in mind, let f be a cost dominating function on a connected graph G. Then f is *efficient* if every vertex of G is f-dominated exactly once or, equivalently, if $\vec{f}(v) = 1$ for every vertex v of G. Consider the graph $P_2 \times P_4$ shown in Figure 11(a).



(a) (b) (c)

Figure 11: Cost Dominating Functions on $P_2 \times P_4$.

Two cost dominating functions f and g on $P_2 \times P_4$ are shown in Figures 11(b) and 11(c). Although the function f is not efficient as the vertex u is f-dominated

by more than one vertex, the function g is an efficient cost dominating function on $P_2 \times P_4$ because every vertex of $P_2 \times P_4$ is g-dominated exactly once.

As was remarked before, not every graph has an efficient dominating set. However, every graph has an efficient cost dominating function. For example, if v is a central vertex of G, then the function $f : V(G) \to [0..\text{rad } G]$ given by

$$f(u) = \begin{cases} \text{rad } G & \text{if } u = v \\ 0 & \text{if } u \neq v \end{cases}$$

is an efficient cost dominating function as every vertex of G is f-dominated exactly once. If S is an efficient dominating set of a graph G, then $|S| = \gamma(G)$; hence every efficient dominating set is also a minimum dominating set. On the other hand, not every efficient cost dominating function is a minimum cost dominating function. For every positive integer $t \geq 3$, let G_t be the graph of radius 3 obtained by inserting two new vertices into every edge of $K_{1,t}$. We noted that the graph G_t has an efficient dominating set W_t with $|W_t| = t + 1 = \gamma(G_t)$. The characteristic set function $h_t : V(G_t) \to \{0, 1\}$ of W_t is an efficient cost dominating function with $\sigma(h_t) = \gamma(G_t)$. However, for $t \geq 3$ we have that $\gamma_c(G_t) < \gamma(G_t)$; so h_t is an efficient cost dominating function that is not a minimum cost dominating function. While h_t is not an efficient minimum cost dominating function on G_t, the graph G_t does have such a function (the cost function given by (1.1)).

3.5 The Cost Domination Numbers of Subgraphs and Subdivisions of Graphs

In general, there is no relationship between the cost domination number of a graph G and the cost domination number of a subgraph of G, even if the subgraph is induced. Numerous examples of graphs G and H exist for which

$G \subseteq H$ and $\gamma_c(G) \leq \gamma_c(H)$. To see that this is not always the case, however, let G be a connected graph of order n and maximum degree at most $n-2$. By Theorem 1.4.1, $\gamma_c(G) \geq 2$. If we now form a new graph H from G by introducing a new vertex v and joining v to every vertex of G, then $\gamma_c(H) = 1$. Thus in this case, $G \subseteq H$ while $\gamma_c(G) > \gamma_c(H)$.

There is, however, a relationship between the cost domination number of a graph and the cost domination number of a subdivision of that graph.

Theorem 3.5.1. *If H is an elementary subdivision of a connected graph G, then $\gamma_c(G) \leq \gamma_c(H)$.*

Proof. If $\gamma_c(H) = \text{rad } H$ then the result follows trivially, so assume that $\gamma_c(H) < \text{rad } H$. Then $\gamma_c(H) \leq \text{rad } G$. Let h be a $\gamma_c(H)$-function and g its restriction to G. If $V_f^+(H) \subseteq V(G)$, then g is a cost dominating function on G. If $V_f^+(H)$ contains the vertex u introduced by the subdivision of G, then let v be one of the two neighbors in H of u. Define $g' : V(G) \to [0..\text{rad } G]$ as

$$g'(x) = \begin{cases} h(u) + h(v) & x = v, \\ h(x) & x \neq v. \end{cases}$$

Let $w \in V(G)$ and P be a shortest $u-w$ path in H. If P contains v, then $\text{d}(v, w) = \text{d}(u, w) - 1$; so v f-dominates w in G if u f-dominates w in H. If P does not contain v, then $\text{d}(v, w) \leq \text{d}(u, w)$ and once again v f-dominates w in G if u f-dominates w in H. Thus g' is a cost dominating function on G with $\sigma(g) = \gamma_c(H)$ and $\gamma_c(G) \leq \gamma_c(H)$. □

Every subdivision of a graph can be obtained by a succession of elementary subdivisions. Consequently, we have the following.

Corollary 3.5.2. *If H is a subdivision of G, then $\gamma_c(G) \leq \gamma_c(H)$.*

COST INDEPENDENCE IN GRAPHS

Let G be a connected graph. A set S of vertices in G is *independent* if no two vertices of S are adjacent or, equivalently, if no vertex in S dominates any other vertex in S. A cost function (not necessarily a cost dominating function) f on a nontrivial connected graph G such that no f-dominating vertex f-dominates any other f-dominating vertex is a *cost independent function* on G. Equivalently, a cost independent function is a cost function f that satisfies

$$\mathrm{d}(u, v) > \max\{f(u), f(v)\}$$

for every pair u, v of distinct vertices in V_f^+. For example, a cost independent function on $P_2 \times P_4$ is shown in Figure 12. If S is an independent set in G, then the characteristic set function of S is cost independent. In addition, if f is a cost independent function on a connected graph G, then V_f^+ is an independent set in G.

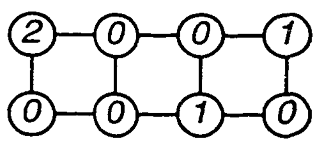


Figure 12: A Cost Independent Function on $P_2 \times P_4$.

Notice that if f is a cost function and u and v are two f-dominating vertices, then $f(u)$ and $f(v)$ are both positive and so $|f(u) - f(v)| < \max\{f(u), f(v)\}$. Thus, while Corollary 2.3.1 does not imply that every minimal cost dominating function is cost independent, the next result shows that for every cost dominating function f, there exists a cost function g that is both cost dominating and cost independent and for which $\sigma(g) \leq \sigma(f)$.

Theorem 4.1.1. *If f is a cost dominating function on a connected graph G, then there is a cost function g on G that is both cost dominating and cost independent and for which $\sigma(g) \leq \sigma(f)$. Furthermore, $V_g^+ \subseteq V_f^+$.*

Proof. Let f be a cost dominating function. If f is cost independent, then $g = f$ has the desired property. Hence we may assume that f is not cost independent. Consider the sequence $\{h_i\}$ of functions, where $h_0 = f$ and for $i \geq 1$, if h_{i-1} is not cost independent, then there exist distinct vertices u_i and v_i of $V_{h_{i-1}}^+$ for which $\mathrm{d}(u_i, v_i) \leq \max\{h_{i-1}(u_i), h_{i-1}(v_i)\}$. Assume, without loss of generality, that $\max\{h_{i-1}(u_i), h_{i-1}(v_i)\} = h_{i-1}(u_i)$. Hence the vertex v_i is h_{i-1}-dominated by u_i. Let

$$m_i = \min\{\mathrm{d}(u_i, v_i), h_{i-1}(v_i)\}$$

and define $h_i : V(G) \to [0..\mathrm{rad}\, G]$ by

$$h_i(x) = \begin{cases} h_{i-1}(u_i) + m_i & \text{if } x = u_i \\ 0 & \text{if } x = v_i \\ h_{i-1}(x) & \text{if } x \notin \{u_i, v_i\}. \end{cases}$$

From the definition of h_i,

$$\sigma(h_i) = \sigma(h_{i-1}) + m_i - h_{i-1}(v_i) \leq \sigma(h_{i-1}).$$

Let y be a vertex that is h_{i-1}-dominated by v_i. Hence, $\mathrm{d}(y, v_i) \leq h_{i-1}(v_i)$. We now consider two cases.

Case 1: If $m_i = \mathrm{d}(u_i, v_i)$, then since $h_{i-1}(v_i) \leq h_{i-1}(u_i)$,

$$\begin{aligned} \mathrm{d}(y, u_i) &\leq \mathrm{d}(y, v_i) + \mathrm{d}(v_i, u_i) \leq h_{i-1}(v_i) + \mathrm{d}(u_i, v_i) \\ &\leq h_{i-1}(u_i) + \mathrm{d}(u_i, v_i) = h_i(u_i). \end{aligned}$$

Case 2: If $m_i = h_{i-1}(v_i)$, then since $\mathrm{d}(u_i, v_i) \leq h_{i-1}(u_i)$ we have

$$\begin{aligned} \mathrm{d}(y, u_i) &\leq \mathrm{d}(y, v_i) + \mathrm{d}(v_i, u_i) \leq h_{i-1}(v_i) + \mathrm{d}(u_i, v_i) \\ &\leq h_{i-1}(v_i) + h_{i-1}(u_i) = h_i(u_i). \end{aligned}$$

Thus, in either case, y is h_i-dominated by u_i. Consequently, h_i is a cost dominating function on G and $V^+_{h_i} = V^+_{h_{i-1}} - \{v_i\} \subsetneq V^+_{h_{i-1}}$. Since G contains a finite number of vertices, there is some positive integer k such that h_k is both cost dominating and cost independent. By setting $g = h_k$, we obtain the desired result. □

There is no result analogous to Theorem 4.1.1 in the classical theory of domination. For a positive integer $t \geq 2$, let $S^*_{t,t}$ be the double star obtained from P_2 by adding t pendant edges at each vertex. The graph $S^*_{t,t}$ has exactly one minimum dominating set, consisting of the two central vertices, and this set is not independent. By way of contrast, Theorem 4.1.1 implies that for every connected graph G, we can always find a minimum cost dominating function f that is cost independent (and, hence, such that the members of V^+_f form an independent set of G).

The minimum cardinality of a set of vertices that is both dominating and independent in a graph G is the *independent domination number* $i(G)$. In the classical theory of domination, the domination number $\gamma(G)$ and the independent domination number $i(G)$ of a graph G are distinct invariants, that is, there are graphs G for which $i(G) \neq \gamma(G)$. Following the definition of the independent domination number, we define a *cost independent dominating function* to be a cost function that is both cost dominating and cost independent and the *cost independent domination number* $\gamma_{ci}(G)$ of a nontrivial connected graph G by

$$\gamma_{ci}(G) = \min\{\sigma(f) : f \text{ is a cost independent dominating function}\}.$$

An immediate consequence of Theorem 4.1.1 is the following.

Corollary 4.1.2. *For every nontrivial connected graph G,*

$$\gamma_{ci}(G) = \gamma_c(G).$$

Proof. Since every cost independent dominating function is cost dominating, $\gamma_c(G) \leq \gamma_{ci}(G)$. However, from Theorem 4.1.1, for every minimum cost dominating function f on G, there exists a cost independent dominating function g on G with $\sigma(g) \leq \sigma(f)$. Hence $\gamma_{ci}(G) \leq \gamma_c(G)$ and the result follows. □

Thus, while in general $\gamma(G) \neq i(G)$, the cost invariants γ_c and γ_{ci} are, in fact, synonymous.

Theorem 4.1.1 can be strengthened. Not only does every connected graph have a minimum cost dominating function that is cost independent, but in fact every connected graph has a minimum cost dominating function f that is cost independent and such that the f-dominating set does not include any endvertices.

Lemma 4.1.3. *Let f be a minimum cost dominating function on a connected graph G of order at least 3. If v is an endvertex of G and $v \in V_f^+$, then $f(v) = 1$.*

Proof. Suppose that $v \in V_f^+$ and $f(v) > 1$. Let u be the neighbor in G of v. By hypothesis, u is not an endvertex of G. Suppose that $u \in V_f^+$. However, since for every vertex $x \in V(G) - \{v\}$, $\mathrm{d}(v, x) = 1 + \mathrm{d}(u, x)$, the function $h : V(G) \to [0..\mathrm{rad}\, G]$ defined by

$$h(x) = \begin{cases} \max\{f(u), f(v)\} & \text{if } x = u \\ 0 & \text{if } x = v \\ f(x) & \text{if } x \notin \{u, v\} \end{cases}$$

is a cost dominating function on G and $\sigma(h) = \sigma(f) - \min\{f(u), f(v)\} < \sigma(f)$. This contradicts the assumption that f is a minimum cost dominating function. Consequently, $u \notin V_f^+$. Let $f' : V(G) \to [0..\mathrm{rad}\, G]$ be the function defined by

$$f'(x) = \begin{cases} 0 & \text{if } x = v \\ f(v) - 1 & \text{if } x = u \\ f(x) & \text{if } x \notin \{u, v\}. \end{cases}$$

Since $f(v) > 1$, it follows that $f'(u) \geq 1$ and u f'-dominates v. In addition, for every vertex w of G with $w \neq v$, $\mathrm{d}(w, u) = \mathrm{d}(w, v) - 1$; so f' is a cost dominating function on G. However, the function f' has $\sigma(f') = \sigma(f) - 1$, which contradicts the fact that $\sigma(f) = \gamma_c(G)$. Consequently, if $v \in V_f^+$, then $f(v) = 1$. □

Theorem 4.1.4. *Let G be a connected graph of order at least 3 containing an endvertex v, and let f be a minimum cost dominating function on G. Then there is a minimum cost dominating function g on G that is cost independent and such that $v \notin V_g^+$.*

Proof. Let f be a minimum cost dominating function on G. If $v \in V_f^+$, then, as established in Lemma 4.1.3, $f(v) = 1$. Let u be the neighbor in G of v. We now consider two cases.

Case 1: $\vec{f}(u) = 1$. Then v is the only vertex that f-dominates u. The function $g : V(G) \to [0..\text{rad } G]$ defined by

$$g(x) = \begin{cases} 0 & \text{if } x = v \\ 1 & \text{if } x = u \\ f(x) & \text{if } x \notin \{u, v\} \end{cases}$$

is a cost dominating function on G. Moreover, if w is any g-dominating vertex with $w \neq u$, then $\mathrm{d}(w, u) > g(w) \geq 1 = g(u)$ and so g is cost independent.

Case 2: $\vec{f}(u) > 1$. Let w be a vertex distinct from v that f-dominates u. Define a function $f' : V(G) \to [0..\text{rad } G]$ by

$$f'(x) = \begin{cases} 0 & \text{if } x = v \\ f(w) + 1 & \text{if } x = w \\ f(x) & \text{if } x \notin \{v, w\}. \end{cases}$$

Since u is f-dominated, v is f'-dominated. Hence, f' is a cost dominating function on G with $\sigma(f') = \sigma(f) = \gamma_c(G)$. We can now apply Theorem 4.1.1 to obtain a minimum cost dominating function g on G that is cost independent and for which $V_g^+ \subseteq V_{f'}^+$. Since $v \notin V_{f'}^+$, the result follows.

□

Corollary 4.1.5. *If G is a connected graph of order at least 3, then there exists a minimum cost dominating function f on G that is cost independent and such that no endvertex is in the f-dominating set.*

Let T be a nontrivial tree. If T has order 2, then every nontrivial subtree T' of T has $\gamma_c(T') = 1$. If, on the other hand, the order of T is at least 3, then by Corollary 4.1.5 there is a minimum cost dominating function f on T such that no endvertex of T is in V_f^+. Consequently, if T' is any subtree of T obtained by removing an endvertex of T, then f restricted to $V(T')$ is an f-dominating function on T' and $\gamma_c(T') \leq \gamma_c(T)$.

If T'' is any subtree of T, then T'' can be obtained from T by a sequence of successive deletions of endvertices. Hence, we obtain the following result.

Proposition 4.1.6. *If T is a tree and T' is a nontrivial subtree of T, then*

$$\gamma_c(T') \leq \gamma_c(T).$$

4.2 An Introduction to Maximal Cost Independent Functions

An independent set of vertices is *maximal* if it is not properly contained in another independent set of vertices. A cost independent function f on a nontrivial connected graph G is *maximal* if there is no cost independent function g satisfying (i) $g(v) \geq f(v)$ for all $v \in V(G)$ and (ii) $g(u) > f(u)$ for some $u \in V(G)$. For example, a maximal cost independent function f on P_4 is shown in Figure 13.

Before proceeding, we pause to revisit a convention that we introduced

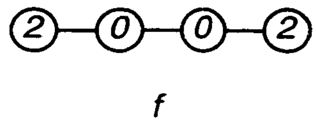


Figure 13: A Maximal Cost Independent Function.

earlier. Specifically, if f is a cost function on a nontrivial connected graph G and v is a vertex of G, then we required that $f(v) \leq e(v)$. This restriction is natural in the context of cost domination since we are interested in those cost dominating functions f for which $\sigma(f)$ is minimized (sometimes subject to certain conditions). When considering cost independence, however, we are looking for cost independent functions f for which $\sigma(f)$ is maximized (also, sometimes, under certain conditions). Why, then, should we still require that a cost independent function f satisfy $f(v) \leq e(v)$ for every vertex v?

To answer this question, we define a function $f : V(G) \to \mathbb{N} \cup \{0\}$ to be *nearly cost independent* if for every pair u, v of distinct vertices of G we have that $\mathrm{d}(u,v) > \max\{f(u), f(v)\}$. However, if G is any connected graph, $v \in V(G)$ and $k \in \mathbb{N}$, then the function $f : V(G) \to \{0, k\}$ defined by

$$f(x) = \begin{cases} k & \text{if } x = v \\ 0 & \text{if } x \neq v \end{cases}$$

is nearly cost independent. Hence,

$$\max\{\sigma(f) : f \text{ is nearly cost independent on } G\}$$

is, for every graph G, not well-defined. Since we're interested in 'large' cost independent functions, nearly cost independent functions are, from this perspective, uninteresting.

Once we accept that the set of values of a cost independent function should be bounded, a natural question arises: What bound should be used? The values assigned to the vertices of a graph by a cost function are meant to be distances in that graph. Since there is no pair u, v of vertices having $\mathrm{d}(u, v) > e(v)$, we could make the claim that it doesn't make sense to assign some vertex v a value greater than $e(v)$. Moreover, we wish to relate cost domination and cost independence to one another. This task will undoubtedly be easier if we adopt for cost independent functions the same restrictions that we have placed on cost dominating functions.

We now begin our discussion of maximal cost independent functions by establishing some conditions for maximality. If v is a vertex of G and $S \subseteq V(G)$, then

$$\mathrm{d}(v, S) = \min\{\mathrm{d}(v, x) : x \in S\}.$$

The following result characterizes most maximal cost independent functions.

Theorem 4.2.1. *Let G be a connected graph and f a cost independent function on G with $|V_f^+| \geq 2$. Then f is maximal if and only if the following two conditions are satisfied:*

(a) *f is cost dominating;*

(b) *for every $v \in V_f^+$,*

$$f(v) = d(v, V_f^+ - \{v\}) - 1.$$

Proof. First, assume that f is maximal. We show that (a) and (b) hold, beginning with (a). Assume, to the contrary, that f is not cost dominating. Then there is some vertex w that is not f-dominated, which implies that $f(w) = 0$. The function

$g : V(G) \to [0..\text{diam } G]$ defined by

$$g(x) = \begin{cases} 1 & \text{if } x = w \\ f(x) & \text{if } x \neq v \end{cases}$$

is cost independent as no neighbor w' of w has $f(w') > 0$. Moreover, $\sigma(g) > \sigma(f)$, contradicting the maximality of f. This verifies (a).

We now show that condition (b) holds, namely, for every $v \in V_f^+$, we have $\mathrm{d}(v, V_f^+ - \{v\}) = f(v) + 1$. Assume, to the contrary, that there is some vertex $v \in V_f^+$ for which $\mathrm{d}(v, V_f^+ - \{v\}) \neq f(v) + 1$. We now consider two cases.

Case 1: $\mathrm{d}(v, V_f^+ - \{v\}) < f(v) + 1$. Consequently, there is some vertex y distinct from v and having $f(y) > 0$ such that $\mathrm{d}(v, y) \leq f(v)$. This contradicts the assumption that f is cost independent.

Case 2: $\mathrm{d}(v, V_f^+ - \{v\}) > f(v) + 1$. Then every vertex $y \in V_f^+$ distinct from v satisfies $\mathrm{d}(v, y) \geq f(v) + 2$. Hence, the function $h : V(G) \to [0..\text{diam } G]$ given by

$$h(x) = \begin{cases} f(v) + 1 & \text{if } x = v \\ f(x) & \text{if } x \neq v \end{cases}$$

is cost independent and $\sigma(h) > \sigma(f)$. This contradicts the maximality of f.

This establishes (b). For the converse, let f be a cost independent function satisfying conditions (a) and (b) and assume, to the contrary, that f is not maximal. Hence, there is some vertex u such that the function $k : V(G) \to [0..\text{diam } G]$ defined by

$$k(x) = \begin{cases} f(u) + 1 & \text{if } x = u \\ f(x) & \text{if } x \neq u \end{cases}$$

is cost independent. Since f is cost dominating, every vertex of G is f-dominated, implying that $f(u) > 0$ and so $u \in V_f^+$. By assumption, there is some vertex z such that $\mathrm{d}(u, z) = f(u) + 1$. Hence z is k-dominated by u, contradicting the fact that k is cost independent. □

Immediately from Theorem 4.2.1 we obtain the following.

Corollary 4.2.2. *Let G be a connected graph and f a maximal cost independent function on G having $|V_f^+| \geq 2$. Then there are distinct vertices $u, v \in V_f^+$ with $f(u) = f(v)$.*

Proof. Let $u \in V_f^+$ be such that

$$\mathrm{d}(u, V_f^+ - \{u\}) = \min\{\mathrm{d}(x, V_f^+ - \{x\} : x \in V_f^+)\}.$$

Then there is some vertex $v \in V_f^+$ for which $\mathrm{d}(v, V_f^+ - v) = \mathrm{d}(u, V_f^+ - u)$. Hence by condition (b) of Theorem 4.2.1, we have that $f(u) = f(v)$. □

4.3 Maximal Cost Independent Functions and MCI-Admitting Sets

We now describe a consequence of Theorem 4.2.1. Let I be a nonempty subset of the vertices of G. If there exists a maximal cost independent function f on G with $V_f^+ = I$, then we say that I *admits* a maximal cost independent function, or that the set I is *MCI-admitting* (MCI for maximal cost independent). Necessarily, if I admits a cost independent function, then I is independent. However, not every independent set of vertices admits a maximal cost independent function. For example, consider the path $P_7 : t, u, v, w, x, y, z$ of Figure 14.

Figure 14: The Path P_7.

The set $\{w, z\}$ is independent but does not admit a maximal cost independent function, however. To see this, first observe that $\mathrm{d}(w, z) = 3$. If f is a cost independent function on P_7 with $V_f^+ = \{w, z\}$, then $f(w), f(z) \leq 2$. Consequently, the vertex t is not f-dominated and hence, by Theorem 4.2.1, f is not maximal cost independent.

Let G be a connected graph and I a set of vertices that admits a maximal cost independent function f. If V_f^+ consists of only one vertex, say w, then $f(w) = e(w)$. If, on the other hand, $|V_f^+| \geq 2$, then by Theorem 4.2.1 we have that for every $v \in I$,

$$f(v) = \mathrm{d}(v, I - \{v\}) - 1. \tag{4.1}$$

Notice that the quantities on the right hand side of equation (4.1) are completely determined by G and I. Denote by $\mathcal{S}_{\mathrm{MCI}}(G)$ the set of all MCI-admitting subsets of $V(G)$ and by $\mathcal{F}_{\mathrm{MCI}}(G)$ the set of all maximal cost independent functions on G. As a consequence of Theorem 4.2.1, we obtain the following

Corollary 4.3.1. *Let G be a nontrivial connected graph. Then there is a one-to-one correspondence between $\mathcal{S}_{MCI}(G)$ and $\mathcal{F}_{MCI}(G)$.*

Proof. We have already noted that for every MCI-admitting set of vertices there exists a unique maximal cost independent function. The fact that to every maximal cost independent function f there corresponds a unique MCI-admitting set follows from the fact that V_f^+ admits f. $\square$

Hence the study of the maximal independent functions defined on a connected graph G will be made somewhat easier if we can characterize $\mathcal{S}_{\mathrm{MCI}}(G)$. To this end, we state one such result.

Theorem 4.3.2. *Let G be a nontrivial connected graph and I a nonempty set of vertices of G. Then $I \in \mathcal{S}_{MCI}(G)$ if and only if the following two conditions are satisfied:*

(a) *I is independent;*

(b) *for every vertex $u \in V(G) - I$, there is a vertex $v \in I$ for which*

$$d(u, v) < d(v, I - \{v\}).$$

Proof. First we assume that $I \in \mathcal{S}_{\mathrm{MCI}}(G)$ and show that conditions (a) and (b) hold. We have already remarked that I must be independent. To see that condition (b) holds, let $u \in V(G) - I$. Since I is MCI-admitting, let f be a maximal cost independent function on G with $V_f^+ = I$. Since f is maximal cost independent, by Theorem 4.2.1, f is cost dominating. Hence there is some vertex v that f-dominates u. Moreover, since f is cost independent we have, once again from Theorem 4.2.1, that

$$\mathrm{d}(u, v) \leq f(v) < \mathrm{d}(v, I - \{v\}).$$

For the converse, assume that I is a nonempty set of vertices of G satisfying conditions (a) and (b). Define the function $g : V(G) \to [0..\mathrm{diam}\ G]$ by

$$g(x) = \begin{cases} \mathrm{d}(v, I - \{v\}) - 1 & \text{if } v \in I \\ 0 & \text{if } v \notin I. \end{cases}$$

To show that $I \in \mathcal{S}_{\mathrm{MCI}}(G)$, we shall show that g satisfies conditions (a) and (b) of Theorem 4.2.1. That condition (b) of Theorem 4.2.1 holds follows trivially from our definition of g in which, since I is independent, we have $g(w) \geq 1$ for every vertex $w \in I$. Let $u \in V(G) - I$. Then by assumption, there is some vertex $v \in I$ such that $\mathrm{d}(u, v) \leq \mathrm{d}(v, I - \{v\}) - 1 = g(v)$. Consequently, every vertex of G is g-dominated, g is cost dominating and condition (a) of Theorem 4.2.1 holds. Thus by Theorem 4.2.1 g is a maximal cost independent function with $V_g^+ = I$ and so $I \in \mathcal{S}_{\mathrm{MCI}}(G)$. □

4.4 Maximum Cost Independent Functions and the Cost Independence Number

If G is a connected graph, then the maximum cardinality of an independent set of vertices of G is the *independence number* $\beta(G)$. The *cost independence number* $\beta_c(G)$ is the maximum value of $\sigma(f)$ over all cost independent functions f on G, and a cost independent function f with $\sigma(f) = \beta_c(G)$ is a *maximum cost independent function.* Obviously, every maximum cost independent function is maximal. However, not every maximal cost independent function is a maximum cost independent function. For example, if v is the central vertex of P_3, then the function $f : V(P_3) \to \{0, 1\}$ given by

$$f(x) = \begin{cases} 1 & \text{if } x = v \\ 0 & \text{if } x \neq v \end{cases}$$

is maximal cost independent but not maximum cost independent since (as is easily seen) $\beta_c(P_3) = 2$.

Proposition 1.2.1 states that $\gamma_c(G) \leq \gamma(G)$. Moreover, the characteristic set function $\chi_S : V(G) \to \{0, 1\}$ associated with an independent set S of vertices

of G is cost independent. Hence, $\beta(G) \leq \beta_c(G)$. It is well-known [3] that $\gamma(G) \leq \beta(G)$. We thus obtain the following chain of inequalities.

Proposition 4.4.1. *If G is a nontrivial connected graph, then*

$$\gamma_c(G) \leq \gamma(G) \leq \beta(G) \leq \beta_c(G).$$

Every maximal independent set is a minimal dominating set. However, while Theorem 4.2.1 shows that every maximal cost independent function is cost dominating, a maximal cost independent function need not be minimal cost dominating. For example, the maximal cost independent function f shown in Figure 13 is not a minimal cost dominating function since the cost function $g : V(P_4) \to \{0, 1\}$ defined by

$$g(v) = \begin{cases} 1 & \text{if } f(v) = 2 \\ 0 & \text{if } f(v) = 0 \end{cases}$$

is a cost dominating function.

We now present a lower bound on the cost independence number of a graph.

Theorem 4.4.2. *If G is a connected graph of diameter $d \geq 1$, then*

$$\beta_c(G) \geq 2(d-1).$$

Proof. Let u and v be antipodal vertices in G, that is, $\mathrm{d}(u, v) = d$. Then the function $f : V(G) \to \{0, d-1\}$ defined by

$$f(x) = \begin{cases} d-1 & \text{if } x \in \{u, v\} \\ 0 & \text{if } x \notin \{u, v\} \end{cases}$$

is cost independent. □

Hence, if G is a graph of diameter $d \geq 3$, then no maximum cost independent function f has $|V_f^+| = 1$. For if $v \in V(G)$, then $f(v) \leq d$, while from Theorem 4.4.2 we have that $\beta_c(G) \geq 2d - 2$ which, for $d > 2$, is strictly greater than d.

While Theorem 4.4.2 is a consequence of a simple observation, the next result demonstrates that the bound given by Theorem 4.4.2 is, in fact, sharp for some classes of graphs.

Theorem 4.4.3. *For every integer $n \geq 3$,*

$$\beta_c(P_n) = 2n - 4.$$

Proof. It is easily seen that $\beta_c(P_3) = 2$. We assume, then, that $n \geq 4$. From Theorem 4.4.2, it is sufficient to prove that $\beta_c(P_n) \leq 2n-4$. Let $P_n : v_1, v_2, \dots, v_n$ and let f be a maximum cost independent function on P_n. Since diam $P_n \geq 3$, $|V_f^+| > 1$.

We claim that $f(v_1) > 0$. Assume, to the contrary, that $f(v_1) = 0$ and let i be the smallest integer such that $f(v_i) > 0$. Then the function $g : V(P_n) \to [0..n-1]$ defined by

$$g(x) = \begin{cases} f(v_i) + i - 1 & \text{if } x = v_1 \\ 0 & \text{if } x = v_i \\ f(x) & \text{if } x \notin \{v_1, v_i\} \end{cases}$$

is cost independent, since if j is the smallest positive integer with $j > i$ such that $f(v_j) > 0$, then

$$\mathrm{d}(v_1, v_j) = \mathrm{d}(v_1, v_i) + \mathrm{d}(v_i, v_j) > i - 1 + f(v_i) = g(v_1).$$

Further, $\sigma(g) > \sigma(f)$, which contradicts $\sigma(f) = \beta_c(P_n)$. Consequently $f(v_1) > 0$ and, similarly, $f(v_n) > 0$.

Lastly, it must be the case that for every integer i with $2 \leq i \leq n-1$, $f(v_i) = 0$. Assume, to the contrary, that there is some smallest integer j with $1 < j < n$ such that $f(v_j) > 0$. Since f is a maximum cost independent function, from Theorem 4.2.1 we have that $f(v_1) = j - 2$. Consequently, the function $h : V(P_n) \to [0..n-1]$ given by

$$h(x) = \begin{cases} f(v_1) + 1 + f(v_j) & \text{if } x = v_1 \\ 0 & \text{if } x = v_j \\ f(x) & \text{if } x \notin \{v_1, v_j\} \end{cases}$$

is cost independent because $N_h[v_1] = N_f[v_1] \cup N_f[v_j]$. However, $\sigma(h) = \sigma(f) + 1$, which contradicts $\sigma(f) = \beta_c(P_n)$. It follows that no such integer j exists. Hence, $V_f^+ = \{v_1, v_n\}$ and consequently $\beta_c(P_n) \leq 2n - 4$. $\square$

BIBLIOGRAPHY

[1] T.Y. Chang and W.E. Clark, *The domination numbers of the* $5 \times n$ *and* $6 \times n$ *grid graphs*, J. Graph Theory **17** (1993), 81–107.

[2] G. Chartrand, L. Eroh, M. Schultz, and P. Zhang, *An introduction to analytic graph theory*, Util. Math., to appear.

[3] G. Chartrand and L. Lesniak, *Graphs and Digraphs*, 3 ed., Chapman and Hall, New York, 1996.

[4] E.J. Cockayne, O. Favaron, C. Payan, and A.G. Thomason, *Contributions to the theory of domination, independence and irredundance in graphs*, Discrete Math. **33** (1981), 249–258.

[5] G.S. Domke, G.H. Fricke, R.R. Laskar, and A. Majumdar, *Fractional domination and related parameters*, in Domination in Graphs: Advanced Topics (T.W. Haynes, S.T. Hedetniemi, and P.J. Slater, eds.), Marcel Dekker, New York, 1998, pp. 61–89.

[6] D.J. Erwin, *Parameters Related to Fractional Domination in Graphs*, Master's thesis, University of Natal, Durban, 1995.

[7] W. Goddard and M.A. Henning, *Real and integer domination in graphs*, Discrete Math. **199** (1999), 61–75.

[8] E.O. Hare, *Algorithms for grids and grid-like graphs*, Ph.D. dissertation, Clemson University, 1989.

[9] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[10] M.A. Henning, *Distance domination in graphs*, in Domination in Graphs: Advanced Topics (T.W. Haynes, S.T. Hedetniemi, and P.J. Slater, eds.), Marcel Dekker, New York, 1998, pp. 321–349.

[11] M.S. Jacobson and L.F. Kinch, *On the domination number of products of graphs: I*, Ars Combin. **18** (1984), 33–44.

[12] P.J. Slater, *R-Domination in graphs*, J. Assoc. Comput. Mach. **23** (1976), 446–450.