Surface Models of Finite Geometries

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SURFACE MODELS OF FINITE GEOMETRIES

by

Ramón Manuel Figueroa-Centeno

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Faculty of The Graduate College
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requirements for the
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The standard non-Euclidean geometries, hyperbolic geometry and elliptical
gometry, both arise by negating the parallel postulate of Euclid. Both these
gometries share with Euclidean geometries an infinitude of points and lines. But also
possible are many finite geometries. Among these are the class of projective
gometries $PG(m,q)$ of projective dimension $m$ ($m \geq 2$) and the $k$-configurations.
These mathematical objects, although primarily geometric in nature, provide related
structures of combinatorial interest: block designs. These have applications in
scheduling problems and the design of experiments for statistical analysis. Recently,
A. T. White has added a topological flavor to the study of the geometries $PG(m,q)$.
In particular, when $m = 2$ he found models, as imbeddings of Cayley graphs on
surfaces and pseudosurfaces, using voltage graphs (obtained from the classical field
construction of $PG(m,q)$). In this dissertation we have extended his work for the case
$m + 1$ a prime number in a similar manner. We have also used this approach (and ad
hoc methods at times) to successfully provide concise voltage graph constructions for
$3$-configurations (a $k$-configuration is a finite geometry such that exactly $k$ points lie
on any line, exactly $m$ lines intersect at any given point, where $m$ is a constant, and
any two points determine at most one line). The class of $3$-configurations is
interesting since it contains some classical finite geometries, e.g., the Fano plane and
the Desargues and Pappus configurations.
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Ramón Manuel Figueroa-Centeno
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CHAPTER I

INTRODUCTION

1.1 Block Designs, Geometries, and Graph Imbeddings

A basic problem of pharmacological research is that of the study of drug interactions, in which, ideally one would like to test every possible combination of drugs. However, this presents a practical impossibility. Consider ten drugs which we wish to test two at a time, three at a time, and so on up to all ten at a time. In this case we would need to perform 1013 experiments. Now, assume we want to do this for 100 drugs. In this case we would require

\[2^{100} - 100 - 1 = 1,267,650,600,228,229,401,496,703,205,275\]

experiments!

So what can we do? The answer lies in the theory of combinatorial block designs. Since, obviously, we cannot perform all experiments, let us perform a representative subset of them. So consider the following definition ([23] p.189):

**Definition 1.1** A \((v,b,r,k,\lambda)\)-balanced incomplete block design (BIBD) is a set of \(v\) objects and a collection of \(b\) subsets of the object set, each subset being called a block, satisfying: (a) each object appears in exactly \(r\) blocks; (b) each block contains exactly \(k\) \((k < v)\) objects; and (c) each pair of distinct objects appears together in exactly \(\lambda\) blocks.

So if we let each block be an experiment and each point be a drug then finding BIBD's will give us good ways of testing drugs.

Now, the following well-known result follows immediately from elementary
counting arguments (see for example [23] p. 190):

**Theorem 1.1** If a \((v, b, r, k, \lambda)\)-BIBD exists then \(vr = bk\) and \(\lambda(v - 1) = rt(k - 1)\).

We will also be interested in a more general kind of design: partially balanced incomplete block designs, whose definition in graph theoretical terms follows:

**Definition 1.2** Let \(G\) be a graph. We say \(G\) is strongly regular if: (1) it is regular, but neither complete nor empty; and (2) for each pair of adjacent vertices \(u\) and \(v\), there are the same number \(p_{22}^1\) of vertices adjacent to both \(u\) and \(v\); and (3) for each pair of non-adjacent vertices \(w\) and \(x\), there are the same number \(p_{22}^2\) of vertices adjacent to both \(w\) and \(x\). Strongly regular graphs underlie \((v, b, r, k; \lambda_1, \lambda_2)\) partially balanced incomplete block designs (PBIBD's); instead of one \(\lambda\) value as for BIBD's, now there are two: each pair of non-adjacent vertices (objects) appears together in exactly \(\lambda_1\) blocks, and each pair of adjacent vertices appears together in exactly \(\lambda_2\) blocks.

For example, the graph \(K_{m,n} \equiv n \overline{K}_m\ (m, n \geq 2)\) is strongly regular, with \(p_{22}^1 = (n - 1)m\) and \(p_{22}^2 = (n - 2)m\).

Let us digress for a moment to mention one of the great discoveries of nineteenth century Mathematics: Non-Euclidean Geometries. These geometries hinge upon the basic realization that other geometries, besides the one described in Euclid's *Elements*, are equally valid and logically consistent. This insight led to Einstein's theory of relativity.

So freed from their euclidean constraints, mathematicians began to explore other axiom systems that led to different geometries. Then the following question arose naturally: can we stipulate a set of axioms out of which a finite geometry arises (i.e., a logically consistent system in which all lines have a finite number of points
and all points lie on a finite number of lines). Moreover, can we construct such a
gamey with the added property that its points and lines are respectively the objects
and the blocks of a BIBD? This indeed can be done! So we can get block designs
with "geometric flavor".

A whole infinite class of such finite geometries can be constructed in theory
using abstract algebra (the actual constructions can be done using a computer algebra
system, such as Maple V). They are each denoted $PG(2,q)$, where $q$ is a prime
power, and are the best known of the so called finite projective planes. With them in
hand we can then obtain finite projective spaces $PG(m,q)$ (see Section 1.3), where $m$
($m \geq 2$) is an integer and $q$ is a prime power, which are also block designs.

Depicting, by means of topological graph theory, some of these (see Chapter II) and
other finite geometries (3-configurations; see Chapter 3) is going to be the aim of this
dissertation.

What follows is a description the smallest finite projective plane $PG(2,2)$.
also known as the Fano Plane. Its axioms are:

1. There is at least one line.
2. Every line contains exactly three points.
3. Not all points lie on one line.
4. Two distinct points belong to exactly one common line.
5. Two distinct lines contain exactly one common point.

We can construct a model for the Fano plane algebraically as follows. Let $Z$,
be the set of points of the plane and let the lines be $a = \{0,1,3\}$, $b = \{1,2,4\}$,
$c = \{2,3,5\}$, $d = \{3,4,6\}$, $e = \{4,5,0\}$, $f = \{5,6,1\}$, and $g = \{6,0,2\}$. Notice that all can
be generated cyclically from any particular one using $Z$; e.g.

$$g = \{6,0,2\} = \{0 + 6, 1 + 6, 3 + 6\} = a + 6.$$ 

We have thus a $(7,7,3,3,1)$-BIBD. The classic way of depicting the Fano plane is
shown in Figure 1.

Figure 1. The Classic Depiction of the Fano Plane.

This depiction, although geometric, possesses some deficiencies:

1. The line \{1,2,4\} is differently shaped, yet that line is not distinguishable from the others \textit{via} the axioms.

2. Each other line has two “end” points and one “middle” point, yet there is no axiom for “betweenness” here.

3. There are three “crossings” in the figure, that have no significance in the geometry.

4. One cannot discern that \( r = 3 \), by looking at small neighborhoods of points 1, 2, 4, and 0 in the figure.

5. It does not generalize into depictions of the other projective planes \( PG(2,q) \).

To overcome these deficiencies, we first need the notion of graph imbedding (for a formal and extensive development see Gross and Tucker [12] or White [23]). Consider the problem of drawing a graph \( G \) on the plane without extraneous edge crossings (equivalent to drawing it on a sphere since it is the one point compactification of the plane), it is well known that this is not always possible, i.e. not all graphs are \textit{planar}. So the natural approach is to add \( k \) “handles” to the sphere
until we can draw $G$ on it without any extraneous crossings (such a drawing is called an imbedding of $G$). The minimum such $k$ is the genus of $G$, denoted $\gamma(G)$. Now a sphere with $k$ handles is what is called a compact connected orientable 2-manifold, or orientable surface for short, denoted $S_k$ ($k \geq 0$) (e.g. $S_0$ is the sphere and $S_1$ is the torus). So the problem that preoccupies topological graph theorists is: given a graph $G$ what is $\gamma(G)$? This interest extends naturally to looking at imbeddings of $G$ that have each region a 2-cell (homeomorphic to a disk), as are all genus imbeddings of connected graphs. Over the years many techniques have been developed to lay siege to the genus problem (which is NP-complete) for graphs; we will use a very powerful one: the theory of voltage graph imbeddings, as developed by Gross and Alpert (see [2], [9], [10] and [11]). We explain the rudiments of this theory in Section 1.2. Now, we observe that the fundamental parameter when dealing with an imbedding with $r$ regions of a graph of order $p$ and size $q$ (not only necessarily into orientable surfaces) is the euler characteristic $\chi = p - q + r$; its value is $2 - 2k$ for $S_k$ (see [23]).

Associate a graph with the Fano plane which Coxeter (see [7]) calls the Menger graph of the geometry (see Chapter III for a more general description). The points of the geometry become the vertices of the graph, and two vertices are adjacent if the points they represent are collinear (belong to a common line). So we get $K_7$, the complete graph of order seven. Then, we imbed $K_7$ on a surface $S_k$ so that certain regions depict the lines of the geometry. The remaining regions are hyperregions of the imbedding of the geometry. The graph imbedding will have bichromatic dual, reflecting a $K_7$-decomposition of the graph (edges partitioned into 3-cycles). That is, each edge will bound one region depicting a line of the geometry and one hyperregion. Any imbedding of $K_7$ on the torus models the Fano plane in this fashion; see Figure 2 (a). The unshaded regions model the lines, while the shaded regions are what remains in the torus after the geometry is modeled. As there is no
imbedding of the Fano plane in $S_4$, (we cannot do better than having all hyperregions triangular, for $\lambda = 1$), we say that this geometry has genus one. Note that the deficiencies (1) through (4) of the model of Figure 1 have all been remedied by the model of Figure 2 (a). In 1992 White (see [21]) noticed that deficiency (5) disappeared also since he was able to extend this construction to depict $PG(2,q)$; i.e. he found efficient imbeddings of $PG(2,q)$ into orientable surfaces and pseudosurfaces (for a definition of pseudosurfaces see Section 1.2), and thus solved a long standing problem. To do this he used the index 1 voltage graph imbeddings of Cayley graphs for the groups $\mathbb{Z}_{q^i \cdot q^{i-1}}$, where $q$ is a prime power, to depict $PG(2,q)$ (for definitions of Cayley graph and voltage graph see Section 1.2 too).

Subject to the condition that $\mathbb{Z}_{q^i \cdot q^{i-1}}$ acts regularly not only on the images of the points and the lines of $PG(2,q)$, but also on each orbit of the region set for the imbedding, the imbeddings constructed have maximum efficiency, as they attain the upper bound for the euler characteristic of an ambient space. We will extend his constructions further (see Chapter II) to $PG(m,q)$ when $m+1$ is a prime.

Now White (see [21]) also defined the genus $\gamma(\mathfrak{G})$ of a geometry $\mathfrak{G}$, with point set $\Psi$ and line set $\mathcal{L}$. First define the bipartite incidence graph $G(\mathfrak{G})$ of $\mathfrak{G}$, by taking $\Psi \cup \mathcal{L}$ as the vertex set, and all pairs $\{p,l\}$, where $p \in \Psi$, $l \in \mathcal{L}$ and $p \in l$, as edges. (Coxeter calls this the Levi graph in [7], where he constructs regular maps for it.) Then we define $\gamma(\mathfrak{G}) = \gamma(G(\mathfrak{G}))$. We observe that this parameter is well defined when the geometries considered are complete (any two models are isomorphic, so that every statement in the axiom system of the geometry can be proven or disproven), as is the case for all $PG(2,q)$. Then a natural modification of the imbedding of $G(\mathfrak{G})$ produces a model of $\mathfrak{G}$ on $S_4$, and conversely. For $\lambda \leq 1$, as for all $PG(2,q)$ (but not $PG(m,q)$ with $m > 2$), the girth of $G(\mathfrak{G})$ is at least six so that a
hexagonal imbedding is necessarily minimal. For the $K_7$ imbedding of Figure 2 (a) modeling the Fano plane $\hat{\mathcal{F}}_7$, $(\hat{\mathcal{F}}_7, \hat{\mathcal{E}})$ can be obtained by inserting a new (line) vertex in each unshaded region, joining it by an edge to each (point) vertex in the boundary of that region, and then deleting all the $K_7$ edges (see Figure 2 (b)). The result is a hexagonal imbedding of the Heawood graph in the torus.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{The Fano Plane as $K_7$ on the Torus.}
\end{figure}

1.2 Topological Graph Theory and Voltage Graphs

In this section we present a few definitions and the bare essentials of voltage graph theory (see Gross and Tucker [12] or White [23]). First let us define a pseudosurface:

**Definition 1.3** Let $A$ denote a set of $\sum_{i=1}^{t} n_i m_i \geq 0$ distinct points of $S^2$ (the sphere with $k$ handles), with $1 < m_1 < m_2 < \ldots < m_t$. Partition $A$ into $n_i$ sets of $m_i$ points each, $i = 1, 2, \ldots, t$. For each set of the partition, identify all the points of that set. The resulting topological space is called a pseudosurface. Each point resulting from an identification of $m_i$ points of $S^2$ is called a singular point. If a graph $G$ is imbedded
in a pseudosurface we assume that every singular point is occupied by a vertex of $G$; such a vertex is called a singular vertex of $G$. A generalized pseudosurface results when finitely many identifications, of finitely many points each, are made on a topological space of finitely many components, each of which is a pseudosurface, with a connected topological space resulting.

It is clear that every surface is a generalized pseudosurface. An example of a disjoint union of generalized pseudosurfaces is shown in Figure 3.

![Figure 3. A Union of Disjoint Generalized Pseudosurfaces.](image)

Next comes the definition of a Cayley graph:

**Definition 1.4** Given a group $\Gamma$ and a generating set $\Delta$ the *Cayley graph* $G_\Delta(\Gamma)$ has vertex set $\Gamma$ and edge set $\{\{g, g\delta\} \mid g \in \Gamma, \delta \in \Delta \cup \Delta^{-1}\}$.

Now, we are ready to define voltage graphs:

**Definition 1.5** Let $K$ be a pseudograph, with $K^* = \{(u, v) \mid (u, v) \in E(K)\}$. Let $\Gamma$ be a group and let $\phi: K^* \to \Gamma$ satisfy $\phi(e^{-1}) = (\phi(e))^{-1}$, for all $e \in K^*$. Then the triple
\( (K, \Gamma, \phi) \) is called a \textit{voltage graph}. Let \((K, \Gamma, \phi)\) be 2-cell imbedded in an orientable surface \(S\), with \(R\) a \(k\)-gonal region for this imbedding having boundary described by the closed walk \(c_1, c_2, \ldots, c_m\) in \(K\). Let \(|R|_\phi\) be the order of \(\prod_{i=1}^m \phi(c_i)\) in \(\Gamma\). Define a \textit{covering graph} \(K \times_{\phi} \Gamma\) for \((K, \Gamma, \phi)\) with vertex set \(V(K) \times \Gamma\) and edges \(\{(u, g), (v, g\phi(e))\}\) for each edge \(e = [u, v]\) of \(K\) and each \(g \in \Gamma\). Then we have

\textbf{Theorem 1.2} Let \((K, \Gamma, \phi)\) be 2-cell imbedded in the orientable surface \(S\). Then there exists a 2-cell imbedding of \(K \times_{\phi} \Gamma\) into an orientable surface \(\tilde{S}\) and a (possibly branched) covering projection \(\rho : \tilde{S} \to S\) such that (a) \(\rho^{-1}(K) = K \times_{\phi} \Gamma\); (b) if \(b\) is a branch point of multiplicity \(m\), then \(b\) is in the interior of a region \(R\) such that \(|R|_\phi = m\); and (c) if \(R\) is a \(k\)-gonal region for \(K\) in \(S\), then \(\rho^{-1}(R)\) has \(\frac{|R|}{|R|_\phi}\) components, each a \(k|R|_\phi\)-gonal region for \(K \times_{\phi} \Gamma\) in \(\tilde{S}\).

If \(|R|_\phi = 1\), we say that \(R\) satisfies the \textit{Kirchhoff Voltage Law} (KVL). If the KVL holds for every \(R\) determined in \(K\) in \(S\), then we say that the imbedding satisfies the KVL. Let \(p\), \(q\) and \(r\) denote the number of vertices, edges and regions for \(K\) in \(S\), and similarly \(\tilde{p}\), \(\tilde{q}\) and \(\tilde{r}\) for \(\tilde{K} = K \times_{\phi} \Gamma\) in \(\tilde{S}\). We always have \(\tilde{p} = p|\Gamma|\) and \(\tilde{q} = q|\Gamma|\). If the KVL holds for the imbedding then (by the previous theorem) \(\tilde{r} = r|\Gamma|\) too. In this situation it follows immediately that the euler characteristics \(\tilde{X}\) and \(X\) of \(\tilde{S}\) and \(S\) respectively are related by: \(\tilde{X} = X|\Gamma|\). Now, we say that the imbedding (either above or below) is of \textit{index} \(i\) \((i \in \mathbb{N})\) if \(|V(K)| = i\). Finally, notice that if \((K, \Gamma, \phi)\) is an index 1 voltage graph then \(K \times_{\phi} \Gamma = G_{\delta}(\Gamma)\).

1.3 The Finite Projective Spaces \(PG(m, q)\)

In this section we present the classical construction of the finite projective spaces \(PG(m, q)\) (with \(q\) a prime power and \(m \geq 2\)) (see [3]; and for the axiomatic
Let \( V \) be an \((m + 1)\)-dimensional vector space over a field \( K \) with origin 0. Then let \( W = V - \{0\} \) and \( \mathcal{B} \) be a basis for \( V \).

Consider \( \bar{x} \) and \( \bar{y} \) in \( W \). Then we say that \( \bar{x} \) and \( \bar{y} \) are equivalent if and only if there exists \( \lambda \neq 0 \) in \( K \) such that \( \bar{y} = \lambda \bar{x} \), we denote this by \( \bar{x} \sim \bar{y} \). Then this gives an equivalence relation, and we let \( [\bar{x}] = \{ \bar{y} \mid \bar{x} - \bar{y} \} \) be the equivalence class of \( \bar{x} \).

Then the finite projective space \( PG(m,K) \) is the set \( \{ [\bar{x}] \mid \bar{x} \in W \} \); and each equivalence class \( [\bar{x}] \) is called a point of \( PG(m,K) \). Now, a hyperplane \( [\bar{\alpha}] \).

for \( \bar{\alpha} \in W \) (line when \( m = 2 \) and plane when \( m = 3 \)), is defined as follows:

\[
[\bar{\alpha}] = \{ \bar{x} \in W \mid \bar{\alpha} \cdot \bar{x} = 0 \},
\]

i.e. it is a set of points whose representing vectors and the origin form a subspace of dimension \( m - 1 \) of \( V \). Notice then that if \( \bar{x}, \bar{y} \in W \) such that \( [\bar{x}] = [\bar{y}] \) then \( \bar{x} \in [\bar{\alpha}] \) if and only if \( \bar{y} \in [\bar{\alpha}] \) for any \( \bar{\alpha} \in W \). Thus we say \( [\bar{x}] \) lies on \( [\bar{\alpha}] \) if \( \bar{x} \in [\bar{\alpha}] \).

As an example notice that if \( m = 2 \) and \( K = \mathbb{R} \) then \( PG(2,\mathbb{R}) \) is the classical real projective plane.

Now, let \( PG(m,q) \) denote \( PG(m,GF(q)) \) where \( GF(q) \) is the finite field of order \( q \) (hence \( q \) is a prime power). Then let \( v = \left( q^{m+1} - 1 \right) / (q - 1) \), \( k = \left( q^m - 1 \right) / (q - 1) \) and \( \lambda = \left( q^{m-1} - 1 \right) / (q - 1) \). Then it can be shown (see [3]) that \( PG(m,q) \) has \( v \) points and hyperplanes, each hyperplane has \( k \) points; each point lies on exactly \( k \) hyperplanes and given any two points they lie together in exactly \( \lambda \) hyperplanes. So we have the following theorem,

**Theorem 1.3** The points and hyperplanes of \( PG(m,q) \) form a symmetric \((v,k,\lambda)\)-design, where the objects and blocks are the points and hyperplanes of \( PG(m,q) \) respectively.

Actually more is true:
Theorem 1.4 (Singer's Theorem; see [3] or [19]) \( PG(m,q) \) (as a design) is cyclic, and the points in any hyperplane determine a \((v,k,\lambda)\)-difference set, i.e. the group \( \mathbb{Z}_v \) acts regularly (one orbit of length \( v \)) on both the points and hyperplanes of \( PG(m,q) \) by addition as a group of collineations.

Now, since we have specified \( K = GF(q) \) we can be more detailed on how the above construction works for \( PG(m,q) \). We wish to construct an \((m+1)\)-dimensional vector space over \( GF(q) \); but, recall that \( GF(q^{m+1}) \) is an \((m+1)\)-dimensional vector space over \( GF(q) \), so let us take it as our vector space. Now, \( GF(q^{m+1}) \) is constructed by taking a monic irreducible polynomial \( f \) of degree \( m+1 \) over \( GF(q) \), and letting \( GF(q^{m+1}) = GF(q)(\omega) \), with \( f(\omega) = 0 \). So \( W = GF(q)(\omega) - \{0\} \), and \( \mathcal{W} = \{\omega^i\}_{i=0}^{m} \) is a basis for \( GF(q^{m+1}) \). Also, we can now be more precise as to what we mean by \( \bar{\alpha} \cdot \bar{x} \); let \( \bar{\alpha} \cdot \bar{x} = \sum_{i=0}^{m} \alpha_i x_i \) if \( \bar{\alpha} = \sum_{i=0}^{m} \alpha_i \omega^i \) and \( \bar{x} = \sum_{i=0}^{m} x_i \omega^i \). Thus we have that
\[
[\omega^i] = \{\omega^j \mid j \equiv i \pmod{v}\},
\]
and the set of points of \( PG(m,q) \) is
\[
\mathscr{P} = \{[\omega^i] \mid i = 0, \ldots, v-1\}.
\]

Now, by an abuse in notation, take \( i \) to represent the point \([\omega^i]\), for \( i = 0, \ldots, v-1 \), so we take \( \mathscr{P} = \mathbb{Z}_v \). To explain why this abuse of notation makes sense we refer to Marshall Hall’s (see [13]) proof of Singer’s Theorem. There one sees that the group generated by the function \( \theta: W \rightarrow W \), such that \( \theta([\omega^i]) = [\omega^{i+1}] \) (composition being the operation of course), acts regularly on the points and hyperplanes of \( PG(m,q) \); but \( \langle \theta \rangle \cong \mathbb{Z}_v \) (letting \( \theta^i \) correspond to \( i \)). Thus the abuse in notation is explained as it allows us to perform all our operations modulus \( v \).

Let us now look at the set \( \mathcal{L} \) of hyperplanes of \( PG(m,q) \). To do this pick an arbitrary element \( \bar{\alpha} \) of \( W \), say \( \bar{\alpha} = \omega^2 \), and define \( S_0 \) as follows,
\[ S_n = \{ \bar{x} \in W | \bar{x} \cdot \bar{x} = 0 \} = \{ \bar{x} = \sum_{i=0}^{n} x_i \omega^i \in W | x_i = 0 \} \]

but \( W = \langle \omega \rangle \). So,

\[ S_n = \{ \omega^i : 0 \leq i \leq v-1 \} ( \text{coefficient of } \omega^i) = 0 \text{ in } \omega^i (\bmod f(\omega)) \} \]

By our abuse of notation this becomes,

\[ S_n = \{ i \in \mathbb{Z}_v | (\text{coefficient of } \omega^i) = 0 \text{ in } \omega^i (\bmod f(\omega)) \} \]

Thus \( S_n = \{0, 1, \ldots \} \) is one hyperplane of \( PG(m, q) \) and by Singer's Theorem a cyclic difference set modulus \( v \) (each element of \( \mathbb{Z}_v \) occurs exactly \( \lambda \) times, as a difference of distinct elements of modulus \( v \)). All other hyperplanes of \( PG(m, q) \) are the translates of \( S_0 \) modulus \( v \), i.e. the set of hyperplanes of \( PG(m, q) \) is,

\[ \mathcal{U} = \{ S_0 + i (\bmod v) \mid i \in \mathbb{Z}_v \} \]

Each of these hyperplanes is a cyclic difference set too. Notice that \( S_0 \) is a hyperplane such that 0 and 1 lie in it (the only such one when \( m = 2 \)), so we say that \( S_0 \) is in **standard position**. Let \( S_j = S_0 + j (\bmod v) \). So

\[ \mathcal{U} = \{ S_j \mid j \in \mathbb{Z}_v \} \]

Let \( S \in \mathcal{U} \); then we say that \( t \in \mathbb{Z}_v - \{0\} \) is a **multiplier** of \( S \) if \( tS \in \mathcal{U} \). So in such a case multiplication by \( t \) serves as an automorphism of the design.

Now, \( q = p^n \) for some prime \( p \) and natural \( n \). Then the multiplier group of \( S_0 \) is \( \langle p \rangle \) (see Baumert for example). Also, notice that \( (v, k) = 1 \) (as we shall show in Chapter II) so there exists \( k^{-1} \in \mathbb{Z}_v \). So it can be shown that if \( j \equiv k^{-1} \left( - \sum_{i \in S_0} s \right) (\bmod v) \)

then \( tS_j = S_j \) for every multiplier of \( S_0 \) (see [3]). We then set \( S = S_j \); so it follows that \( qS = S \), since \( q \in \langle p \rangle \).

Let us present an example: \( PG(4, 2) \). First we find a monic irreducible polynomial of degree 5 over \( GF(2) = \mathbb{Z}_2 \), say \( f(x) = x^5 + x^2 + 1 \). Let then \( \omega \) be a root of \( f \), so
Compute $\omega^4 = \omega^2 + 1$. Then construct the field $GF(32)$ by adjoining $\omega$ to $\mathbb{Z}/2 \mathbb{Z}$, i.e.

$GF(32) = \mathbb{Z}/2 \mathbb{Z}(\omega)$. We present the field elements in Figure 4 (notice there that $GF(32) - \{0\} = \langle \omega \rangle$). Now, let

$$S_0 = \{\omega^0, \omega^1, \omega^2, \omega^3, \omega^4, \omega^5, \omega^6, \omega^7, \omega^8, \omega^9, \omega^{10}, \omega^{11}, \omega^{12}, \omega^{13}, \omega^{14}, \omega^{15}, \omega^{16}, \omega^{17}, \omega^{18}, \omega^{19}, \omega^{20}, \omega^{21}, \omega^{22}, \omega^{23}, \omega^{24}, \omega^{25}, \omega^{26}, \omega^{27}, \omega^{28}, \omega^{29}, \omega^{30}\}.$$

(i.e. the set of all elements of $GF(32)$ that do not have a $\omega^2$ term in their expansion).

which with our abuse of notation (i.e. $\omega' \leftrightarrow i$) becomes

$$S_0 = \{0, 1, 3, 4, 6, 9, 10, 16, 17, 18, 21, 25, 27, 29, 30\}.$$

Therefore, for $PG(4, 2)$, the points are $Y = \mathbb{Z}_{11}$ and the hyperplanes are $\mathcal{L} = \{j + S_0 | j \in \mathbb{Z}_{11}\}$.

\[\begin{array}{c|c|c}
0 & \omega^{10} = \omega^4 + 1 & \omega^{20} = \omega^3 + \omega^2 \\
\omega^0 = 1 & \omega^{11} = \omega^2 + \omega + 1 & \omega^{21} = \omega^4 + \omega^3 \\
\omega^1 = \omega & \omega^{12} = \omega^3 + \omega^2 + \omega & \omega^{22} = \omega^4 + \omega^2 + 1 \\
\omega^2 = \omega^2 & \omega^{13} = \omega^4 + \omega^3 + \omega^2 & \omega^{23} = \omega^3 + \omega^2 + \omega + 1 \\
\omega^3 = \omega^3 & \omega^{14} = \omega^4 + \omega^3 + \omega + 1 & \omega^{24} = \omega^4 + \omega^3 + \omega^2 + \omega \\
\omega^4 = \omega^4 & \omega^{15} = \omega^4 + \omega + \omega^2 + \omega + 1 & \omega^{25} = \omega^4 + \omega^3 + 1 \\
\omega^5 = \omega^5 + 1 & \omega^{16} = \omega^4 + \omega^3 + \omega + 1 & \omega^{26} = \omega^4 + \omega^2 + \omega + 1 \\
\omega^6 = \omega^3 + \omega & \omega^{17} = \omega^4 + \omega + 1 & \omega^{27} = \omega^3 + \omega + 1 \\
\omega^7 = \omega^4 + \omega^2 & \omega^{18} = \omega + 1 & \omega^{28} = \omega^4 + \omega^2 + \omega \\
\omega^8 = \omega^3 + \omega^2 + 1 & \omega^{19} = \omega^2 + \omega \\
\omega^9 = \omega^5 + \omega^3 + \omega & \omega^{20} = \omega^4 + \omega \\
\end{array}\]

Figure 4. The Field $GF(32)$.

Now, $15 \cdot 29 = 435 \equiv 1 \pmod{31}$, so $15^{-1} = 29$ in $\mathbb{Z}_{11}$. We then have that

\[
 j \equiv 15^{-1} \left( -\sum_{s \in S_0} s \right) \pmod{31} = 29,
\]

so $S = S_{29} = 29 + S_0$. 

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\[ S = \{29, 30, 1, 2, 4, 7, 8, 14, 15, 16, 19, 23, 25, 27, 28\} \]

is fixed by multiplication by \( q = 2 \). Furthermore, multiplication by \( q = 2 \) breaks \( S \) into the following three orbits (operations carried \( \mod 31 \) of course): \{1, 2, 4, 8, 16\}, \{7, 14, 28, 19\} and \{15, 30, 29, 27, 23\}. This is a fact that we will find useful in Chapter II.
CHAPTER II

IMBEDDINGS OF $PG(m,q)$ WITH $m+1$ A PRIME

2.1 Technical Results

In this chapter we extend the work of White (see [21]) for the finite projective planes $PG(2,q)$ to the finite projective spaces $PG(m,q)$ with $m+1$ a prime. We also improve on his notation by utilizing modular arithmetic. But first let us state and prove some preliminary results, about the multiplicative order of certain elements of $\mathbb{Z}_q$.

Lemma 2.1 For $q = p^n$, $\text{ord}(p) = (m+1)n$ in $\mathbb{Z}_q$.

Proof First notice that

$$v = (q^{n-1} - 1)/(q - 1) = q^n + q^{n-1} + \cdots + 1;$$

then

$$q^{n+1} - 1 = (q - 1)v \equiv 0 \pmod{v},$$

and then

$$p^{(m+1)n} = q^{n+1} \equiv 1 \pmod{v}.$$ 

Thus, if $\sigma = \text{ord}(p)$ then $\sigma$ divides $n(m+1)$.

Now, $p^\sigma \equiv 1 \pmod{v}$ so $v$ divides $p^\sigma - 1$. Thus there exists $s \in \mathbb{N}$ such that

$$p^\sigma - 1 = sv = s(p^m + p^{m-1}n + \cdots + 1);$$

then we have that

$$p^\sigma - 1 \geq p^m + p^{m-1}n + \cdots + 1,$$

so
Thus \( \sigma > mn \geq n \), which implies that \((n, \sigma) < 1\) and \(\sigma + n > mn + n = (m + 1)n\). This leads us to

\[
\frac{\sigma + n}{\sigma} > \frac{(m + 1)n}{\sigma} \in \mathbb{R}.
\]

since \(\sigma\) divides \((m + 1)n\). Then we have that

\[
2 > 1 + \frac{n}{\sigma} > \frac{(m + 1)n}{\sigma}.
\]

Therefore,

\[
\frac{(m + 1)n}{\sigma} = 1.
\]

Q.E.D.

The following corollary follows immediately from the previous lemma.

**Corollary 2.1** \(\text{ord}(q) = m + 1\) in \(\mathbb{F}_q\).

Now, let \(S\) be the hyperplane of \(PG(m, q)\) fixed by multiplication by \(q\), that we found above, and \(x \in S\). Let \(\Gamma = \langle q \rangle \cong \mathbb{F}_q\), and consider the action of \(\Gamma\) on \(S\). So,

\[
m + 1 = |\Gamma| = |\Gamma x||\Gamma_x|.
\]

where \(\Gamma x\) is the orbit containing \(x\) and \(\Gamma_x \leq \Gamma\) is the stabilizer of \(x\). Also, \(\Gamma x \subseteq S\).

Thus \(|\Gamma x| = 1\) or \(m + 1\), since \((m + 1)\) is a prime.

Recall the following well known result (see [15]).

**Theorem** Let \((a, b) = 1\) and \(p\) be an odd prime. Then

\[
(a + b, \frac{a^n + b^n}{a + b}) = 1\text{ or } p.
\]

So if we let \(a = q\) and \(b = -1\) we get that \((a, b) = (q, -1) = 1\), since \(q\) is a prime.
power. Also, $m + 1$ is an odd prime (since $m \geq 2$) and

$$v = \frac{q^{m+1} - 1}{q - 1}.$$ 

So we get the following lemma.

**Lemma 2.2** \((v, q - 1) = 1\) or \(m + 1\).

**Lemma 2.3** If \((v, q - 1) = 1\) and \(x \neq 0\) then \(|\Gamma x| = m + 1\).

**Proof** If \(|\Gamma_i| = 1\) then \(|\Gamma x| = m + 1\) follows from \(m + 1 = |\Gamma x| |\Gamma_i|\). Now, if \(|\Gamma_i| = m + 1\) then \(\Gamma_i = \Gamma\) and \(q \in \Gamma_i\), so that \(qx = x\) in \(\mathbb{Z}_v\); that is \((q - 1)x = 0\). But, since \((v, q - 1) = 1\) there exists \((q - 1)^{-1} \in \mathbb{Z}_v\) and thus \(x = 0\).

Q.E.D.

We will find useful the contrapositive of the previous lemma.

**Lemma 2.4** If \(|\Gamma x| = 1\) then \(x = 0\) or \((v, q - 1) = m + 1\).

**Proof** We have that \(|\Gamma x| = 1\) so \(|\Gamma_i| = m + 1\). This implies that \(\Gamma_i = \Gamma\) so that \(q \in \Gamma_i\). Then \(q \equiv qx (\mod v)\) and thus \((q - 1)x \equiv 0 (\mod v)\). Recall now that \(ax \equiv ay (\mod m)\) if and only if \(x \equiv y (\mod \frac{m}{\gcd(m, \alpha)})\). Therefore, \(x \equiv 0 (\mod \frac{v}{\gcd(v, m + 1)})\).

Q.E.D.

**Lemma 2.5** If \(|\Gamma x| = 1\) and \((v, q - 1) = m + 1\) then

$$x \equiv t \frac{x}{m+1} (\mod v), \text{ for some } t \in \mathbb{Z}_{m+1}.$$

**Proof** We have that \(|\Gamma x| = 1\) so \(|\Gamma_i| = m + 1\). This implies that \(\Gamma_i = \Gamma\) so that \(q \in \Gamma_i\). Then \(q \equiv qx (\mod v)\) and thus \((q - 1)x \equiv 0 (\mod v)\). Recall now that \(ax \equiv ay (\mod m)\) if and only if \(x \equiv y (\mod \frac{m}{\gcd(m, \alpha)})\). Therefore, \(x \equiv 0 (\mod \frac{v}{\gcd(v, m + 1)})\).

Q.E.D.

**Lemma 2.6** If \(q \equiv 0, 1 (\mod m + 1)\) then \(m + 1\) divides \(k\).

**Proof** Recall, \(k = \frac{q^m - 1}{q - 1}\). Now, \(q \equiv 0 (\mod m + 1)\) and \(m + 1\) is a prime so, by Fermat's little theorem, \(q^m \equiv 1 (\mod m + 1)\); that is \(q^m - 1 \equiv 0 (\mod m + 1)\). Also,
$q - 1 \not\equiv 0 \pmod{m + 1}$. Therefore, $k \equiv 0 \pmod{m + 1}$, since $m + 1$ is prime.

**Q.E.D.**

Lemma 2.7 If $\Gamma v = m + 1$ for some $v \in S$ then

$$\sum \{ v \mid v \in \Gamma x \} \equiv 0 \pmod{v}.$$

**Proof** Let $x \in S$ such that $|\Gamma x| = m + 1$. Then

$$\Gamma x = \{ x, qx, q^2x, \ldots, q^m x \}.$$

Thus

$$\sum \{ v \mid v \in \Gamma x \} = x(q^m + \cdots + q^2 + q + 1) = xv.$$

**Q.E.D.**

Theorem 2.1 $q \not\equiv 1 \pmod{m + 1}$ if and only if $(v, q - 1) = 1$.

**Proof** By Lemma 2.2 $(v, q - 1) = 1$ or $m + 1$. So if we assume that $q \not\equiv 1 \pmod{m + 1}$ we have that $(v, q - 1) = 1$. Now, for the converse assume, to the contrary, that $(v, q - 1) = 1$ and $q \equiv 1 \pmod{m + 1}$. Notice that $(v, q - 1) = 1$ if and only if there exist $r, s \in \mathbb{Z}$ such that $rv + s(q - 1) = 1$. This in turn is true if and only if there exists $r \in \mathbb{Z}$ such that

$$\frac{rv - 1}{q - 1} = r \frac{v}{q - 1} - \frac{1}{q - 1} \in \mathbb{Z}.$$

Now, $v = q^m + q^{m-1} + \cdots + q + 1$ can be thought of a polynomial in $q$, so we can use polynomial long division to compute $v/(q - 1)$. In this way we get

$$\frac{v}{q - 1} = \sum_{i=0}^{m-1} (m-i)q^i + \frac{m+1}{q - 1}.$$

Let $\theta = \sum_{i=0}^{m-1} (m-i)q^i \in \mathbb{Z}$.

Then
\[
\frac{r^V - 1}{q - 1} = r \left( \frac{\theta + m + 1}{q - 1} \right) - \frac{1}{q - 1} = r\theta + \frac{r(m + 1) - 1}{q - 1} \in \mathbb{Z}.
\]

So \((v, q - 1) = 1\) if and only if there exists \(r \in \mathbb{Z}\) such that
\[
\frac{r(m + 1) - 1}{q - 1} \in \mathbb{Z}.
\]

Now, since \(q \equiv 1 \pmod{m + 1}\) there exists \(t \in \mathbb{Z}\) such that \(q - 1 = t(m + 1)\).

So \((v, q - 1) = 1\) if and only if there exists \(r \in \mathbb{Z}\) such that
\[
\frac{r(m + 1) - 1}{t(m + 1)} \in \mathbb{Z},
\]
which is true if and only there exists \(u \in \mathbb{Z}\) such that
\[
ut(m + 1) = r(m + 1) - 1,
\]
and hence \((r - ut)(m + 1) = 1\), which is a contradiction since \(m + 1 > 1\).

Q.E.D.

Now we are ready, in light of the above results, to divide our study of how to imbed \(PG(m,q)\) (with \(m + 1\) a prime) into three cases,

\textbf{Case 0} \quad q \equiv 0 \pmod{m + 1},

\textbf{Case 1} \quad q \equiv 1 \pmod{m + 1}, and

\textbf{Case 2} \quad q \not\equiv 0, 1 \pmod{m + 1}.

The nicest case is the last (as we shall see soon) and hence we start with it.

In these three cases let \(s\) denote \(\left\lfloor \frac{x}{m+1} \right\rfloor\).

2.2 Case 2

In this case \(q \not\equiv 0, 1 \pmod{m + 1}\) so \((v, q - 1) = 1\) by Theorem 2.1; and if \(x \neq 0 \in S\) and \((v, q - 1) = 1\) then \(|\Gamma x| = m + 1\), by Lemma 2.3. Also, by Lemma 2.6.
$m + 1$ divides $k$ so $0 \not\in \mathcal{S}$. Therefore, we can decompose $\mathcal{S}$ into $s$ orbits of length $m + 1$.

$$\mathcal{S} = \bigcup_{i} \Gamma_{\mathcal{V}}.$$ 

for some $\{v_i, x_1, \ldots, x_i\} \in \mathcal{S}$. Notice then that $\Gamma_{\mathcal{V}} = \{v_i, qx_i, q^2x_i, \ldots, q^m x_i\}$ for $i = 0, 1, \ldots, s - 1$. This suggests the following ordering of the elements of $\mathcal{S}$.

$$\mathcal{T} = \{x_n, x_1, \ldots, x_{i-1}, qx_n, qx_1, \ldots, qx_{i-1}, q^2x_n, q^2x_1, \ldots, q^2x_{i-1}, \ldots, q^mx_n, q^mx_1, \ldots, q^mx_{i-1}\};$$

i.e., $\mathcal{T} = (a_i)^{i-1}_{a_0}$ such that $a_i = q^i x_n$ with $i = sa + b$ and $0 \leq b < s$.

Now, set $\delta_i = a_i - a_{i-1}$, $0 \leq i \leq k - 1$, taking subscripts modulus $k$, and set

$$\Delta = \{\delta_i\}_{i=0}^{k-1}.$$ 

The following lemma follows directly from the definition of the $\delta_i$'s.

**Lemma 2.8** \[ \sum_{i=0}^{k-1} \delta_i = 0. \]

**Lemma 2.9** \[ \sum_{i=0}^{m} \delta_{q^i} \equiv 0 \text{ (mod } v) \text{ with } 0 \leq i \leq s. \]

**Proof** Here we have that

$$\sum_{i=0}^{m} \delta_{q^i} = \sum_{i=0}^{m} (a_{q^i} - a_{q^{i-1}}) = \sum_{i=0}^{m} (q^i x_i - q^i x_{i-1}) = \sum \{y | y \in \Gamma_{x_i}\} - \sum \{y | y \in \Gamma_{x_{i-1}}\}. $$

So the result follows from Lemma 2.7.

Q.E.D.

We are now ready to construct our imbedded voltage graph. To do this we take a $k$-gon labeled clockwise with $(\delta_0, \delta_1, \ldots, \delta_{k-1})$ with $(m + 1)$-gons $\mathcal{R}_i$ ($0 \leq i \leq s - 1$) labeled counterclockwise with $(\delta_0, \delta_1, \ldots, \delta_{m-1})$ attached to the $k$-gon along the directed edge labeled $\delta_i$. So orientability follows immediately (all edges appear exactly twice and with opposite orientations); and by Lemmas 2.8 and
By Theorem 1.2 we have an imbedding of $G_A(Z^*)$ above with $v$ $k$-gons and all other regions $(m + 1)$-gons. Also, by this theorem we see that if $R$ is the single $k$-gon below then $\rho_i(R) = \{S_i\}_{i=1}^{N}$ is the set of hyperplanes of $PG(m,q)$. In this context $S_i$ denotes the hyperplane obtained by adding $i$ to every element of $S$. We obtain an imbedding of $G(PG(m,q))$ by inserting a vertex in the middle of every $S_i$ above, joining it with all the vertices of that $k$-gon and then deleting all the edges in $G_A(Z^*)$. All the regions of this imbedding are $2(m+1)$-gons. Now, notice that even though $G_A(Z_v)$ might be a multigraph (since $\lambda \geq 1$) we have that $G(PG(m,q))$ is a graph, since there are no repeated vertices in any $k$-gon in the imbedding of $G_A(Z_v)$.

To investigate when our voltage graph is imbedded into a surface we look at the permutation $\pi$ of $\Lambda' = \Lambda \cup \Lambda'$ induced by the clockwise ordering of directed edges leaving the vertices. The voltage graph imbedding is on a surface if and only if $\pi$ acts transitively on $\Lambda'$. So consider the vertex $x$ at the tail of the edge labeled $\delta_i$ for a fixed $i \in Z_4$; the clockwise rotation starting at $x$ begins: $(-\delta_m, \delta_i, -\delta_{i-1}, \delta_{i+1}, \ldots)$. as can be seen in Figure 5. Then notice that the next vertex in the rotation on the $k$-gon (the one at the head of the edge labeled $-\delta_{i-1}$ and the tail of the one labeled $\delta_{i+1}$), is separated from $x$ by a path of length $(s + (i-1)) - i = s - 1$ lying entirely on the $k$-gon. So if we continue in this fashion we will visit vertices that are at a clockwise distance of $s - 1$ from the preceding one in the rotation, until we come back to $x$. Now, notice that if $(k,s-1) = 1$ then we will visit all the vertices before returning back to $x$ and hence will have only one vertex in our voltage graph and thus a surface imbedding above. Therefore, let us analyze when is $(k,s-1) = 1$ in the following lemma.

**Lemma 2.10** In Case 2 we have,
(a) \( (k, s - 1) = 1 \) or \( m + 1 \).

(b) \( (k, s - 1) = m + 1 \) if and only if \( s \equiv 1 \pmod{m + 1} \), and

(c) \( (k, s - 1) = 1 \) if and only if \( s \not\equiv 1 \pmod{m + 1} \).

**Proof**  
(a) First, recall that in this case \( k = s(m + 1) \). Let \( \alpha = (k, s - 1) \). Then \( \alpha \) divides \( s(m + 1) \) and hence it divides \( m + 1 \) since \( (s, s - 1) = 1 \); but \( m + 1 \) is a prime so \( \alpha = 1 \) or \( m + 1 \).

(b) Assume \( \alpha = m + 1 \) then \( s - 1 = (m + 1)\beta \) for some \( \beta \in \mathbb{Z} \). Thus \( s \equiv 1 \pmod{m + 1} \). Conversely, if \( s \equiv 1 \pmod{m + 1} \) then \( m + 1 \) divides \( s - 1 \); but, \( k = s(m + 1) \) so \( (k, s - 1) > 1 \), which implies \( (k, s - 1) = m + 1 \) by part (a).

(c) Follows immediately from (a) and (b).

Q.E.D.

![Clockwise Rotation at Vertex x](image)

**Figure 5.** Clockwise Rotation at Vertex \( x \).
The above lemma implies that only in the case when \( s \equiv 1 \pmod{m+1} \) do we get a pseudosurface imbedding above from our voltage graph below. Now, since we prefer surfaces to pseudosurfaces, we modify our voltage graph to obtain a surface imbedding when \( s \equiv 1 \pmod{m+1} \). To do this we study more carefully the action of the rotation \( \pi \) on \( \Delta^* \), which in this case has \( (k, s - 1) = m + 1 \) orbits of length
\[
2s = 2 \frac{k}{m+1}
\]
(the 2 comes from taking both positive and negative \( \delta \)'s). Let \( O_i \) denote the orbit containing \( \delta_i \) \((0 \leq i \leq m)\). Then \( O_i \) is the cycle \( \{d_n\}_{n=0}^{m-1} \) with
\[
d_n = \begin{cases} 
\delta_{i,i+2} & \text{if } j \text{ is even} \\
-\delta_{i,i+2} & \text{if } j \text{ is odd}
\end{cases}
\]
From this description it follows that each of these orbits contains exactly one edge of the \((m + 1)\)-gon \( R_0 \) (these are any negative \( \delta \)'s with \( h \equiv 0 \pmod{s} \)); namely, \( O_i \) \((0 \leq i \leq m)\) contains \( -\delta_{i,i+1} \). So
\[
O_i = \langle \delta_{i,i+2}, -\delta_{i,i+1}, \ldots, -\delta_{i,1}, -\delta_{i,0} \rangle \text{ for } 0 \leq i \leq m.
\]
Then we replace in our voltage graph \( R_0 \) by \( \hat{R}_0 \), the \((m + 1)\)-gon whose boundary is labeled counterclockwise by \( \langle \delta_{m}, \ldots, \delta_{2}, \delta_{1}, \delta_{0} \rangle \). i.e. we reverse the order of the edges in the boundary of \( R_0 \) without reversing their orientation. This concatenates orbit \( O_i \) to orbit \( O_{i-2} \) for all \( i \)'s taken modulus \( m + 1 \), as seen in Figure 6; but, the number of \( O_i \)'s is \( m + 1 \) which is odd and this implies that all orbits merge into one (even though we are skipping \( O_{i-1} \) between \( O_i \) and \( O_{i-2} \)). Our new unique orbit \( O \) can be then expressed as
\[
O = O_m O_{m-2} \cdots O_2 O_0 O_{m-1} \cdots O_1.
\]
Now, notice that when replacing \( R_0 \) by \( \hat{R}_0 \) we did not change the orientation of any edge so the imbedding is still orientable, and \( \mathbb{Z}_k \) is abelian so the KVL still holds in \( \hat{R}_0 \) as it did in \( R_0 \).
Finally, let us compute the genus of the surface in which $G_{3}(\mathbb{Z})$ is imbedded above. To do this we need the euler characteristic of our voltage graph below:

\[\chi = 1 - s(m + 1) + (1 + s) = 2 - ms.\]

So the euler characteristic above is $\tilde{\chi} = (2 - ms)\nu$ and hence the genus is

\[\gamma = 1 + \frac{tms-21\nu}{2}.\]

Let us put the above into use by constructing a voltage graph for $PG(4,2)$ ($\nu = 31$, $k = 15$ and $\lambda = 7$). Thus $q \equiv 2 (\text{mod } 5)$, which implies that we are indeed in Case 2; and $s \equiv 3 (\text{mod } 5)$, which means our construction yields a surface without modifying region $R_0$. The field construction of $PG(4,2)$ yields a hyperplane $S$ (see Chapter 1) which is fixed by multiplication by $q = 2$. We order $S$ according to the orbits of the action of $q$ on it and get
\( S = (1, 7, 15, 2, 14, 30, 4, 28, 29, 8, 25, 27, 16, 19, 23). \)

Now, we label the vertices of a 15-gon clockwise with the elements of \( S \) (see Figure 7).

![Figure 7. A Voltage Graph for PG(4,2).](image)

Then we label its edges with the clockwise differences of the vertex labels modulus 31. We then follow our general construction above and attach \( s = 3 \) 5-gons labeled counterclockwise as indicated in the figure and delete the vertex labels. Notice now that every edge occurs exactly twice and with opposite directions. So we get an orientable surface, as shown above. Also, we get the KVL (modulus 31) in all 4 regions. Furthermore, here we were able to pick an ordering of the elements of \( S \) in
which all successive differences are different (in spite of the fact that \( \lambda > 1 \)), and hence this voltage graph will lift to voltage graph, not a multigraph! So this voltage graph lifts to an imbedding in \( S_{156} \) (a sphere with 156 handles) with all regions 15-gons and 5-gons. The vertices in the boundary of each 15-gon form a hyperplane of \( PG(4,2) \) and vice-versa.

2.3 Case 0

In this case \( q \equiv 0 \pmod{m+1} \) so \( (\nu, q - 1) = 1 \) by Theorem 2.1. Now

\[
k = q^{n-1} + q^{n-2} + \cdots + q + 1,
\]

so \( k = qr + 1 \) for some \( r \in \mathbb{Z} \); then \( k - rq = 1 \) and hence \( (k, q) = 1 \). Thus \( m + 1 \) does not divide \( k \) since \( m + 1 \) divides \( q \). Hence \( 0 \in S \), for we know by Lemma 2.3 that if \( (\nu, q - 1) = 1 \) and \( x \neq 0 \) then \( |\Gamma x| = m + 1 \) for all \( x \in S \). Therefore

\[
S = \{0\} \cup \bigcup_{i=0}^{\nu-1} \Gamma x_i.
\]

for some \( \{x_0, x_1, \ldots, x_{\nu-1}\} \subset S \). This suggests the following ordering of the elements of \( S \),

\[
\bar{S} = (x_0, x_1, \ldots, x_{\nu-1}, q x_0, q x_1, \ldots, q x_{\nu-1}, q^2 x_0, q^2 x_1, \ldots, q^2 x_{\nu-1}, \ldots q^\nu x_0, q^\nu x_1, \ldots, q^\nu x_{\nu-1}, 0);
\]

i.e., \( \bar{S} = (\bar{a}_i)_{i=0}^{\nu-1} \) such that \( a_i = q^a x_n \) for \( i = sa + b \) (\( 0 \leq b < \nu; \ 0 \leq a \leq m \)) and \( a_{\nu-1} = 0 \). Now, set \( \delta_i = a_i - a_{i-1}, \ 0 \leq i \leq k - 1 \), taking subscripts modulus \( k \), and set \( \Delta = \{\delta_i\}_{i=0}^{k-1} \).

Then the construction of the voltage graph follows exactly as in Case 0. We get orientability immediately as above. Now, notice that

\[
R_0 = (x_0 - 0, q x_0 - x_{\nu-1}, q^2 x_0 - q x_{\nu-1}, \ldots, 0 - q^\nu x_{\nu-1}),
\]

and

\[
R_i = (x_i - x_{i-1}, q x_i - q x_{i-1}, \ldots, q^\nu x_i - q^\nu x_{i-1}) \text{ for } i \neq 0.
\]

So the KVL holds by Lemma 2.7 for the \( R_i \)'s and by Lemma 2.8 for the \( k \)-gon.
The following lemma implies that in this case our voltage graph below has to
lif into a pseudosurface and no reordering of the generators on the region boundaries
can remedy this.

**Lemma 2.11** In Case 0 the euler characteristic \( \tilde{\chi} \) of the embedding above is odd.

**Proof** First, the euler characteristic below, \( \chi = 1 - k + \frac{k}{2} + \frac{1}{m + 1} = \frac{-m - 1}{m + 1} \), is odd
since \( m \) is even. Now, the parity of \( q^{1} \) is constant for any \( i \in \mathbb{Z} \), so \( q^{1} + q^{1} + q^{1} \) is
even since \( m \) is even. Hence \( v = q^{1} + q^{1} + \cdots + q^{1} + 1 \) is odd too. Therefore \( \tilde{\chi} = v \chi \) is
odd.

Q.E.D.

2.4 Case 1

In this case \( q = 1 \pmod{m + 1} \), so \( (v, q - 1) = m + 1 \) by Theorem 2.1 and Lemma
2.2, and hence \( m + 1 \) divides both \( q - 1 \) and \( v \). Now, \( v = qk + 1 \), so \( (v, k) = 1 \).
Therefore, \( m + 1 \) does not divide \( k \) since it does divide \( v \).

Here if we consider the action of \( \Gamma \) on \( S \) there are fewer than \( m + 1 \) orbits of
length 1, since we have shown, in Lemma 2.5, that if \( |\Gamma x| = 1 \) and \( (v, q - 1) = m + 1 \)
then \( x \in \{\alpha \in \mathbb{Z}_{m + 1} \mid \alpha = 0, \ldots, m\} \), and we know that \( m + 1 \) does not divide \( k \), the length of
\( S \). This means that if \( t \) and \( s \) are the number of orbits of length 1 and \( m + 1 \)
respectively then \( k = s(m + 1) + t \) with \( 0 < t < m + 1 \). Therefore

\[
S = \left( \bigcup_{i} \Gamma x_{i} \right) \cup \left( \bigcup_{j} \{x_{j}\} \right).
\]

for some \( \{x_{0}, x_{1}, \ldots, x_{s}, y_{0}, y_{1}, \ldots, y_{1}\} \subset S \). This suggests the following ordering of the
elements of \( S \).
\[ S = \{ x_0, x_1, \ldots, x_{i-1}, q x_0, q x_1, \ldots, q x_{i-1}, q^2 x_0, q^2 x_1, \ldots, q^2 x_{i-1}, \ldots, q^n x_0, q^n x_1, \ldots, q^n x_{i-1}, \} ; \]

i.e., \( S = \{ a_i \}_{i=0}^{k-1} \) such that \( a_i = q^i x_0 \) for \( i = sa + b \) (\( 0 \leq b < s ; 0 \leq a \leq m \)) and

\( a_i = y_{i-k-1} \) for \( i \geq k - t \).

Now, set \( \delta_i = a_i - a_{i-1} \), \( 0 \leq i \leq k - 1 \), taking subscripts modulus \( k \), and set \( \Delta = \{ \delta_i \}_{i=0}^{k-1} \).

Then the construction of the voltage graph follows as in the previous two cases, but the expression for \( R_0 \) is more complicated.

\[ R_0 = (\delta_0, \delta_1, \delta_2, \ldots, \delta_{m-1}, \delta_{m+1}, \delta_{m+2}, \delta_{m+3}, \ldots, \delta_{m+k+1}) \]

We get orientability immediately as before. Now, notice that

\[ R_0 = (x_0 - y_1) q x_0 - x_{i-1}, q^2 x_0 - q x_{i-1}, \ldots, q^m x_0 - q^m x_{i-1}, y_0 - q^n x_{i-1}, \]

\[ y_1 - y_0, y_2 - y_1, \ldots, y_{i-1} - y_{i-2} \]

and

\[ R_i = (x_r - x_{i-1}, q x_i - q x_{i-1}, \ldots, q^m x_i - q^m x_{i-1}) \] for \( i \neq 0 \).

So the KVL follows as well, from Lemmas 2.7 and 2.8.

**Lemma 2.12** In Case I the euler characteristic \( \chi \) of the imbedding above has the same parity as \( t \).

**Proof** First \( \chi = v \chi = v(l - k + (1+s)) = v(2 - k + s) \); but, as we observed in the proof of Lemma 2.11, \( v \) is odd. So the parity of \( \chi \) is the parity of \( s - k \). Now \( k = s(m+1) + t \), so \( s - k = -t - sm \), which has the parity of \( t \) since \( m \) is even.

Q.E.D.

Therefore when \( t \) is odd the imbedding has to be on a pseudosurface. Now, when \( t \) is even we have found it difficult to analyze when it is that a surface imbedding is attained, since we do not know the exact value of \( t \) in this case.
Therefore, we leave this question open to further study.

With this done we conclude our study of the imbeddings of $PG(m,q)$ with $m+1$ a prime.

2.5 Remarks on $PG(2,q)$

Finally, let us remark that whereas White (see [21]) was able to compute in some cases the genus of the geometries $PG(2,q)$ we are not able to do this from our constructions; since, among other things, he uses that for $m = 2$ we get $\lambda = 1$ and his extraneous regions (our $R'$s) are triangles or nearly so. Also, he was able to guarantee that in his work corresponding to our Case 1 he obtains a surface. As remarked above we were not able to do so; however, as a tradeoff our $R_0$'s there is easier to describe than his. His work corresponding to our Cases 0 and 2 follows as a direct corollary from our constructions.
CHAPTER III

3-CONFIGURATIONS OF SMALL ORDER

3.1 Preliminary Definitions and Results

In this chapter we introduce a general technique for describing a family of finite geometries, the 3-configurations, by means of graph imbeddings as we did in the previous chapter for $PG(m,q)$. After doing so we present examples (most of which are constructed with the aid of voltage graphs) of these imbeddings.

First we will need a few definitions.

**Definition 3.1** A $k$-configuration ($k \geq 3$) is a finite geometry such that exactly $k$ points lie on every line, exactly $m$ lines intersect at each given point, where $m$ is a constant, and every two points determine at most one line. We call $m$ the replication number of the $k$-configuration.

Here we will study the case for which $k = 3$. We then associate to each 3-configuration $C$ a graph which Coxeter (see [7]) calls the Menger graph. The points of $C$ become the vertices of its Menger graph, and two vertices are adjacent if the points they represent are collinear (belong to a common line). So the set of lines in $C$ is in bijective correspondence with the 3-cycles of a $K_3$-decomposition of its Menger graph. It is now natural to call the order of $C$ the number of points in $C$. Also, if $m$ is the degree of $C$ then its Menger graph is $2m$-regular.

Notice now that any disjoint union of Menger graphs is a Menger graph, and conversely the connected components of a Menger graph are Menger graphs. So we need only search for connected Menger graphs, and then combine these to form
others. We say then that a \( k \)-configuration is \textit{connected} when its Menger graph is connected.

So as to proceed as in the previous chapter we would like to have that given a 3-configuration there is a bicolored (say into shaded and unshaded) 2-cell imbedding of its Menger graph into a surface, so that there is a bijective correspondence between unshaded regions (which are triangles) and the lines of \( C \). This is however \textbf{not} always possible, since for example a Menger graph need not be connected, so a single surface would not always suffice! So some compromise has to be made, in the guise of relaxing our requirements. Fortunately for us, it suffices to require that the imbeddings are into disjoint unions of orientable surfaces, as we shall see in the following theorem. For the converse of our result we will be able to substitute generalized pseudosurfaces for surfaces. This strengthens the result since all surfaces are generalized pseudosurfaces.

\textbf{Theorem 3.1} Every 3-configuration, of replication number \( m \), can be modeled by an orientable bichromatic dual imbedding of its Menger graph (say into shaded and unshaded regions) into a disjoint union of orientable surfaces, where the unshaded regions are all triangles depicting the lines of the 3-configuration (we call the shaded regions the \textit{hyperregions} of the imbedding of the 3-configuration). Conversely, every such imbedding onto a disjoint union of generalized pseudosurfaces (of a \( 2m \)-regular graph) is a model for a 3-configuration of replication number \( m \), having that graph as its Menger graph.

\textbf{Proof} First notice that we need only show that the result is true for connected graphs on connected pseudosurfaces; and conversely for connected 3-configurations. It is easy to see that an imbedding on a generalized pseudosurface as described in the
statement of the theorem yields a 3-configuration, since: (a) every unshaded region is a triangle, and hence gives a 3 point line; (b) the graph is $2m$-regular and the imbedding has bichromatic dual so $m$ unshaded regions meet at each vertex; and (c) given any two vertices they are joined by at most one edge (since it is a graph) and this edge bounds exactly one unshaded region, since the imbedding has bichromatic dual.

Conversely, each connected 3-configuration yields an imbedding on an orientable surface as follows (see Figure 2 for an example). Let $L$ be the (necessarily connected) Levi graph of the 3-configuration (see Chapter 1) and 2-cell imbed it onto an orientable surface (this is always possible since the genus parameter is well defined for any connected graph). For every vertex $v$ of $L$ that represents a line, do the following: join in pairs the three adjacent vertices to $v$ (which represent points) and then delete $v$ (and all three incident edges to it of course), and thus form an unshaded triangle. Shade all other regions of the imbedding. We need then to show that the resulting graph $G$ is the Menger graph of the 3-configuration and that the imbedding has bichromatic dual. We indeed have the Menger graph since the vertices correspond to the points and any two are adjacent if and only if they are collinear. Now, assume to the contrary that in the imbedding two (not necessarily distinct) similarly shaded regions are adjacent to one another; i.e., there exists $uv \in E(G)$ such that either it is in the boundary of two unshaded regions or two shaded regions. If both regions are shaded then the edge $uv$ joins two non-collinear points, a contradiction. If both are unshaded and distinct then it means that two points lie on at least two different lines. Finally, notice that (for $G \not\cong K_3$) an unshaded region cannot bound itself (the graph $G \cong K_3$ can trivially be imbedded on the sphere).

Q.E.D.
Our goal is to present some examples of 3-configurations of order 12 or less obtained and presented as imbeddings of their Menger graphs onto a disjoint union of orientable surfaces. To do this we divide our examples into two classes: (a) those we obtain from voltage graphs (all but one are of index 1); and (b) those that we obtain from *ad hoc* methods (e.g. via a computer search).

We need then to acquire some basic notation and general results that will aid us in our search for 3-configurations.

Let us list six useful parameters for a given $k$-configuration $C$ and its associated imbedded Menger graph $G$:

1. $k$, the number of points that lie on any given line;
2. $m$, the number of lines that intersect on any given point;
3. $l$, the number of lines;
4. $p$, the order of $C$ and hence the order of $G$;
5. $q$, the size of $G$; and
6. $r$, the number of regions of the imbedding of $G$.

We have then that $k = 3$ and $G$ is a $2m$-regular graph with $3l = mp$ (which comes from the fact that a $k$-configuration is a block design). Thus $2q = p(2m)$, and $q = mp$ is divisible by 3. Now, $q \leq \frac{p(p-1)}{2}$ and we let $m \neq 0$ (to avoid trivialities). So $1 \leq m \leq \left\lfloor \frac{p-1}{2} \right\rfloor$ and hence $3 \leq p \leq 12$.

Table 1 shows the different possible values of $p$ and $m$.

When $m = 1$ we need a 2-regular graph that can be decomposed into triangles and this can only be $nK_1$ ($n \geq 1$). Now, $nK_1$ can be easily be imbedded into a disjoint union of $n$ spheres. Thus we need not worry anymore about these! So without loss of generality let $m > 1$. 

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Table 1
Possible Values of $p$ and $m$

<table>
<thead>
<tr>
<th>$p$</th>
<th>$m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>{1}</td>
</tr>
<tr>
<td>4</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>5</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>6</td>
<td>{1,2}</td>
</tr>
<tr>
<td>7</td>
<td>{3}</td>
</tr>
<tr>
<td>8</td>
<td>{3}</td>
</tr>
<tr>
<td>9</td>
<td>{1,2,3,4}</td>
</tr>
<tr>
<td>10</td>
<td>{3}</td>
</tr>
<tr>
<td>11</td>
<td>{3}</td>
</tr>
<tr>
<td>12</td>
<td>{1,2,3,4,5}</td>
</tr>
</tbody>
</table>

Now, involutions cannot be used in constructing our index 1 voltage graphs. To explain this first notice that when drawing a voltage graph, involutions are depicted by half-edges as shown in Figure 8 (a). Replace, for argument sake, a half-edge on the boundary of an unshaded region $R$ by a loop (undirected since it denotes an involution) that bounds a shaded 1-gon (See Figure 8 (b)). This 1-gon lifts to $\frac{p}{2}$ shaded 2-gons above surrounded on both sides by regions in $\rho^{-1}(R)$ (see Figure 8 (c)) with $\rho$ being the covering projection. This 2-gon then collapses into an edge which has unshaded regions on both sides (see Figure 8 (d)). Therefore the imbedding above cannot have bichromatic dual.

The above implies that given a fixed $m$ and a given group $\Gamma$ we need at least $2m$ elements of order greater than 2 in $\Gamma$ to be able to obtain an index 1 voltage graph imbedding that yields a 3-configuration.
Figure 8. Involutions in Index 1 Voltage Graphs and Their Lifts.

Notice now that, given a voltage graph imbedding below, we get triangles above only from either triangles whose boundaries satisfy the KVL or 1-gons whose boundaries are loops labeled with an element of order 3 below.

Now the situation for 4-regular \((m = 2)\) index 1 voltage graphs that yield 3-configurations is easily described. There are essentially only 5 possible imbeddings of 4-regular index 1 voltage graphs (without specifying a particular finite group), and of these only one can give rise to 3-configurations, since it is the only one that does not use involutions and has 1-gons. We show it in Figure 9 and baptize it the 2-leafed clover (it is an imbedding of what Gross [9] calls a bouquet of two circles). Notice that the directions of the arrows are irrelevant in the sense that they can be inverted by using inverses. Therefore if a group only has two elements of order 3 (which must then be inverses of one another, and can then only be used one at a time) then no 4-regular 3-configuration arising from index 1 voltage graphs is possible.

Notice that when several examples of index 1 voltage graphs that yield
3-configurations exist for a given group and a fixed $m$ we will limit ourselves to looking for the ones whose imbeddings lift to the surface of least genus possible; in this sense we say that such voltage graphs are the best we can find. To find these we wrote a lengthy computer program that found all possibilities given a particular group. Unfortunately this program is inefficient and has an output that is extremely difficult to read, since it generates very many useless results that require tedious elimination (e.g. to find a single small voltage graph one may have to read 20 or more pages of cryptic output). Because of these shortcomings we have decided not to include this program among our appendices.

![Figure 9. The 2-Leafed Clover.](image)

We comment here that we have chosen to multiply permutations from right to left (since we view multiplying them as composing functions; $(f \circ g)(x) = f(g(x)))$.

3.2 3-Configurations of Order $p = 6$

Here we need only consider the case when $m = 2$. The only groups of order 6 are $\mathbb{Z}_6$ and $S_3$ (in this context the symmetric group on 3 elements). Each of these contains exactly 2 elements of order 3; so, as we have explained above, no index 1 voltage graph is going to yield a 3-configuration. We use in this case then an index 2 voltage graph for $\mathbb{Z}_6$ (see Figure 10), which lifts to a triangular imbedding of the octahedron graph $K_{3(2)}$ on the sphere (see Figure 11). This gives a
(6,4,2,3;0,1)-PBIBD, based upon the strongly regular graph $K_{4,2}$.

Figure 10. An Index 2 Voltage Graph for $Z_6$.

Figure 11. The Octahedron Graph $K_{3,2}$ on the Sphere.
3.3 3-Configurations of Order $p = 7$

Here we have only $m = 3$; and $Z_7$ (the only group of order 7) gives us the first instance of a very interesting situation. Take the voltage graph imbedding in Figure 12 on the torus and let $(\alpha, \beta, \gamma) = (1, 2, 3)$; so we get a voltage graph in which the KVL holds in both triangles, since $1 + 2 - 3 = 0$. So we get two 3-configurations above with $m = 3$ and $l = 7$, one from the unshaded regions and one from the shaded ones. Each is a $(7,7,3,3,1)$-BIBD, a Steiner Triple System. But, $1 + 2 - 3 = 0$ is true in $Z_n$ for all $n \geq 7$, so the KVL is satisfied in both regions. We get two 3-configurations (on the torus, since $p - q + r = n - 3n + 2n = 0$ always) with $m = 3$ and $l = n$, for all $n \geq 7$, from this voltage graph imbedding. Indeed this example has been remarked upon before by White (see White [23]); moreover, when $n = 7$ it is the same voltage graph imbedding that he uses for his construction for the Fano plane $PG(2,2)$, that we generalized in the previous chapter.

Figure 12. A Voltage Graph on the Torus.

In view of the above we have the following definitions.
**Definitions**  If there exists an $N \in \mathbb{N}$ such that a region in a voltage graph imbedding for $\mathbb{Z}_p$ satisfies the KVL for all $n \geq N$ then we say that the region satisfies the KVL in $\mathbb{Z}_p$. If all the regions of a voltage graph imbedding for $\mathbb{Z}_p$ satisfy the KVL in $\mathbb{Z}_p$ we say that this voltage graph imbedding is in $\mathbb{Z}_p$.

So the above 6-regular voltage graph imbedding for $\mathbb{Z}_p$ and its mirror image are in $\mathbb{Z}_p$.

### 3.4 3-Configurations of Order $p = 8$

Here too $m = 3$ is the only possibility. Now, let us remark that our exhaustive computer search found no 3-configurations that come from imbeddings of index 1 voltage graphs for $\mathbb{Z}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $D_4$. So we are left with $\mathbb{Z}_8$ and $Q$ (the quaternions). For $\mathbb{Z}_8$ we cannot do better than the voltage graph in $\mathbb{Z}_8$ that we found when looking at $\mathbb{Z}_7$. This imbeds the strongly regular graph $K_{4,2}$, on $S_3$, yielding two $(8,8,3,3;0,1)$-PBIBD's.

Recall that the quaternions $Q$ is the group whose elements are $\{\pm 1, \pm i, \pm j, \pm k\}$ and $1, -1$ and $i$ behave as in $\mathbb{C}$, with $i^2 = j^2 = k^2 = -1$, $ij = k$, $jk = i$, $ki = j$, $ik = -j$, $kj = -i$ and $ji = -k$. Notice then that $i$, $j$ and $k$ have all order 4 in $Q$. So our 6-regular voltage graph imbeddings are obtained from the 3-leaved clover in Figure 13 with $\alpha = i$ and $(\beta, \gamma) \in \{(-k, -j), (-j,k), (j,-k),(k,j)\}$ (so the exterior triangle satisfies the KVL) or is a mirror image of one obtained this way. We get then a 3-configuration with $m = 3$ and $l = 8$ from the unshaded regions of an imbedding of $K_{4,2}$, on $S_3$.

For example consider the case when $(\alpha, \beta, \gamma) = (i, -k, -j)$, then $i(-k)(-j)=1$ and we get the 3-configuration in Figure 14, a $(8,8,3,3;0,1)$-PBIBD. Notice that all entries in the table can be obtained from any particular one by translations by the group elements, e.g. $-j\{1,i,j\} = \{-j,k,1\}$. So it can be said that we have a
non-abelian difference set.

\[ \alpha \]
\[ \gamma \]
\[ \beta \]

Figure 13. The 3-Leafed Clover.

<table>
<thead>
<tr>
<th>{1, i, j}</th>
<th>{-1, -i, -j}</th>
<th>{i, -1, k}</th>
<th>{-i, 1, -k}</th>
</tr>
</thead>
<tbody>
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<td>{-j, k, 1}</td>
<td>{k, j, -i}</td>
<td>{-k, -j, i}</td>
</tr>
</tbody>
</table>

Figure 14. A 3-Configuration Obtained From the Quaternions.

3.5 3-Configurations of Order \( p = 9 \)

There are only two groups of order 9: \( Z_9 \) and \( Z_3 \times Z_3 \).

Let us start with \( Z_9 \), which contains only 2 elements of order 3 and no involutions. So no 4-regular voltage graph based on it yields 3-configurations.

For the 6-regular ones we have the one we got in \( Z_9 \) when looking at \( Z_3 \), and four more. For the first two let \((\alpha, \beta, \gamma) = (1, 3, 4)\) on the voltage graph imbedding in Figure 12, so it and its mirror image are in \( Z_9 \) since \( 1 + 3 - 4 = 0 \). For the last two let \((\alpha, \beta, \gamma) = (2, 3, -4)\) in Figure 12 too, so we get the KVL for it and its mirror image, but it is not in \( Z_9 \), since \( 2 + 3 - (-4) = 9 \equiv 0 \pmod{9} \). All of these give \((9, 9, 3, 3; 0, 1)\) designs, but they are not partially balanced, as the Menger graph, \( \overline{C_9} \), is not strongly

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regular.

For the 8-regular case we have found no index 1 voltage graph imbedding obtained from \( \mathbb{Z}_8 \), after our exhaustive computer search.

In the group \( \mathbb{Z}_4 \times \mathbb{Z}_2 \), all non-identity elements are of order 3. So we get many index 1 voltage graphs whose imbeddings yield configurations here!

The 4-regular ones arise from the 2-leafed clover by letting
\[
\alpha, \beta \in \{ \pm(0,1), \pm(1,0), \pm(1,1), \pm(1,2) \},
\]
such that \( \alpha \neq \pm \beta \). We get above a 3-configuration with \( m = 2 \) from an imbedding of the graph \( C_1 \times C_1 \) on \( S_1 \). As this graph is strongly regular these are \((9,6,2,3;0,1)\)-PBIBD's.

Notice that it is very easy to have the KVL hold on the boundary of any triangle whose boundary is, say, labeled clockwise \( \alpha, \beta, \) and \( -\gamma \), by just letting \( \alpha \) and \( \beta \) be as above and \( \gamma = \alpha + \beta \). There are then essentially two ways of getting a 6-regular index 1 voltage graph imbeddings that lift to \( K_{3,3} \) on the torus. The first one comes from taking the voltage graph in Figure 12 in such way that the KVL holds (if it holds for one triangle it will hold for both since we are in an abelian group). These are interesting to us since the configurations they yield are instances of the well known Geometry of Pappus (Pappus of Alexandria, c. 300-350 a.d.) which comes from the following Theorem of Pappus, as part of plane Euclidean geometry:

**Theorem of Pappus.** If \( A, B, \) and \( C \) are three distinct points on line \( L \) and \( A' \), \( B' \), and \( C' \) are three different distinct points on line \( L' \neq L \), then the points \( \overline{AB} \cap \overline{A'B}, \overline{AC} \cap \overline{A'C}, \) and \( \overline{BC} \cap \overline{B'C} \) are collinear.

One such situation is illustrated in Figure 15. Immediately we get a block design of nine objects (points) and nine blocks (lines), given in Figure 16. Now, its
Menger graph $K_{v,v}$ is **strongly regular**, so Table 2 gives a $(9, 9, 3, 3; 0, 1)$-PBIBD. Our model on the torus is shown on Figure 17 (notice that we write $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ as $ab$ for legibility).

![Diagram](image)

**Figure 15.** The Pappus Geometry.

<table>
<thead>
<tr>
<th>$A$</th>
<th>$B$</th>
<th>$C$</th>
<th>$A'$</th>
<th>$D$</th>
<th>$B'$</th>
<th>$C'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$'</td>
<td>$B'$</td>
<td>$C'$</td>
<td>$A$'</td>
<td>$E$</td>
<td>$C$</td>
<td>$A$'</td>
</tr>
<tr>
<td>$D$</td>
<td>$E$</td>
<td>$F$</td>
<td>$D$</td>
<td>$E$</td>
<td>$F$</td>
<td>$D$</td>
</tr>
</tbody>
</table>

**Figure 16.** The Pappus Geometry.

![Diagram](image)

**Figure 17.** The Pappus Geometry as Modeled on the Torus.
The second way of getting an imbedding of a 6-regular index 1 voltage graph for $\mathbb{Z}_3 \times \mathbb{Z}_3$, which lifts to an imbedding of $K_{6,1}$, on the torus that yields a 3-configuration is obtained adding a leaf to any one of the 2-leafed clovers in the 4-regular case, in such a way that the KVL holds for the exterior triangle in the resulting 3-leafed clover (with shaded exterior region and unshaded leaves). An example can be seen in Figure 18 (again we write $(a,b) = ab$ for legibility), that comes from the equation $(1,0) + (0,1) - (1,1) = (0,0)$.

![Figure 18. An Imbedding of $K_{6,1}$ on the Torus.](image_url)

Notice also that the shaded regions in the two 6-regular cases for $\mathbb{Z}_3 \times \mathbb{Z}_3$ form 3-configurations too.

The imbeddings of the 8-regular index 1 voltage graphs that come from $\mathbb{Z}_3 \times \mathbb{Z}_3$ are obtained by taking any of the 3-leafed clovers that we found in the 6-regular case above and placing a loop inside any of its three loops labeled with one of the two unused non-identity elements (when we label a loop we effectively use up
the label and its inverse). The resulting voltage graph imbedding (see Figure 19) lifts to $K_n$ in $S_4$ and gives us one 3-configuration with $m = 4$ and $l = 12$, which contains the 3-configuration with $m = 3$ that we got before from the exterior region of the 3-leaved clover that we modified. The configuration is a $(9.12.4.3.1)$-BIBD, another Steiner Triple System. This is also the affine geometry $AG(2,3)$.

Figure 19. A 3-Leafed Clover with one Double Leaf.

Other (genus-wise less efficient) 8-regular voltage graph imbeddings are possible; for example place a loop in either of the triangles of the 6-regular voltage graphs on the torus that we described above for the Pappus Geometry and label it with either one of the non-identity elements that we have not used. This gives us an imbedding of $K_n$ on $S_7$ that yields one 3-configuration.

3.6 3-Configurations of Order $p = 10$

Here $m = 3$ is the only possibility, so we need at least $2 \cdot 3 = 6$ elements of order greater than 2 to have an index 1 voltage graph whose imbedding yields a 3-configuration. Now, $D_5$ (the dihedral group of order 10) has only 4 elements of
order greater than 2. So we are left with \( Z_m \) whose situation is completely described by using the two (up to mirror images) voltage graphs in \( Z_m \) that we have from \( Z_m \) and \( Z_m \), which lift to the torus. We remark that the two resulting Cayley graphs, \( G_{12,11}(Z_m) \) and \( G_{14,1}(Z_m) \), are isomorphic and are not strongly regular, so we do not get a PBIBD here.

However, there is a famous 3-configuration of order 10 that cannot come from a voltage graph: the Geometry of Desargues. It arises from the following Theorem of Desargues (1593 - 1662), as part of Euclidean plane geometry.

**Theorem of Desargues.** If two triangles are perspective from a point, then they are perspective from a line.

In Figure 20, triangles \( BCD \) and \( EFG \) are perspective from point \( A \). Let \( H = \overline{BC} \cap \overline{EF} \), \( I = \overline{CD} \cap \overline{FG} \), and \( J = \overline{BD} \cap \overline{EG} \). Then the claim of Desargues' Theorem is that \( H, I, \) and \( J \) are collinear.

![Figure 20. The Geometry of Desargues.](image)

Now we get a block design of ten objects (points) and ten blocks (lines), given in Figure 21. The Menger graph is \( \overline{\Pi} \), the complement of the Petersen graph \( \Pi \); the
latter is shown in Figure 22. The graph $\overline{\Pi}$ is strongly regular, with $p^1_{ij} = 3$ and $p^2_{ij} = 4$.

<table>
<thead>
<tr>
<th>$ABE$</th>
<th>$CBH$</th>
<th>$FEH$</th>
<th>$H1J$</th>
<th>$EGJ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ACF$</td>
<td>$CDI$</td>
<td>$FGI$</td>
<td>$BDJ$</td>
<td>$ADG$</td>
</tr>
</tbody>
</table>

Figure 21. The Geometry of Desargues.

Thus Figure 21 gives a $(10, 10, 3, 3; 0, 1)$-PBIBD.

Figure 22. The Petersen Graph.

The two models above for the geometry of Desargues have the same deficiencies as did the geometry of Pappus. So now we seek a suitable imbedding of $\overline{\Pi}$. The voltage graph construction, that served so efficiently to model the Fano and Pappus geometries via Figure 12, does not apply now. One reason is that, as is well-known (see, for example, Problem 4-10 of [23]), the Petersen graph $\Pi$ is not a Cayley graph; thus $\overline{\Pi}$ is not a Cayley graph either. Moreover, although a triangular imbedding of $\overline{\Pi}$ in $S_5$ is compatible with the euler identity ($p - q + r = 2 - 2 \cdot 1$, where $p = 10$, $q = 30$, and $r = r_1 = 20$), it is straightforward to show that no such imbedding
exists: the neighbors of vertex $A$, for example, form no wheel graph $W_5$ with line
triangles alternating with hyper triangles. (The Petersen graph, a frequent
counterexample in graph theory, strikes again!) Thus there seems to be insufficient
symmetry to allow a useful voltage graph/covering space construction.

Instead, we found the imbedding of $\Pi$ in $S_2$ of Figure 23 by ad hoc methods.
So we can say that the geometry of Desargues has genus two, since we cannot model
it in a surface of lesser genus.

Figure 23. The Desargues Geometry Modeled on $S_2$. 

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Note that the hyperregions include five triangles and three pentagons; the latter share point A. We remark that a polygon of \( n \geq 3 \) sides bounding a region in a surface model of a geometry is just what it is in the Euclidean plane: a topological disk, bounded by \( n \) line segments, where a line segment is a connected portion of a line having finite length.) This facilitates the discovery of a 3-fold rotational symmetry (about A). Now if \( M \) denotes the \textit{map} consisting of the imbedding of the Menger on \( S_2 \) then this 3-fold rotational symmetry generates the automorphism group of \( M \), i.e. the graph automorphisms which preserve oriented region boundaries (which also here preserve region colors so the lines of the geometry are preserved as well). Note that some regions in Figure 23 (labeled I, II and III) appear three times to better illustrate this symmetry.

\[ \text{3.7 3-Configurations of Order } p = 11 \]

Here again \( m = 3 \) is the only possibility; and \( \mathbb{Z}_{11} \) is the only group of order 11; and, as usual for cyclic groups, \( \mathbb{Z}_{11} \) inherits all the voltage graph imbeddings in \( \mathbb{Z}_p \) that we have so far. We add two more voltage graphs to \( \mathbb{Z}_{11} \) (up to mirror images). Let \( (\alpha, \beta, \gamma) = (1, 4, 5) \) or \( (2, 3, 5) \) in the voltage graph in Figure 12, so we get the KVL in both triangles, since \( 1 + 4 - 5 = 2 + 3 - 5 = 0 \). There is also one (up to mirror images) that is not in \( \mathbb{Z}_{11} \): let \( (\alpha, \beta, \gamma) = (2, 4, -5) \) in Figure 12, so we get the KVL in both regions since \( 2 + 4 + 5 \equiv 0 \pmod{11} \). All of these lift to imbeddings on the torus, each yielding two 3-configurations with \( m = 3 \) and \( l = 11 \). As for \( \mathbb{Z}_{10} \), we remark that all Cayley graphs are isomorphic and not strongly regular.

\[ \text{3.8 3-Configurations of Order } p = 12 \]

There are 5 groups of order 12, which is divisible by 3. So we have many
cases to consider.

We start with the group $\mathbb{Z}_{12}$, which has only two elements of order 3, i.e. 4 and -4. Thus any triangles above come from KVL triangles below or a loop labeled 4 (clockwise or counterclockwise). This also implies that no imbedding of a 4-regular index 1 voltage graph here can yield a 3-configuration.

All the 6-regular voltage graph imbeddings that yield three configurations and come from a surface of lowest possible genus for $\mathbb{Z}_{12}$ can be described by means of the voltage graph drawn on the torus in Figure 12. There we make sure that both triangles satisfy the KVL; so above we get two 3-configurations (with $m = 3$), one from the shaded and one from the unshaded regions. Here we make as before the distinction between when the KVL is obtained from adding the labels along the region boundaries and getting 0 or getting a non-zero multiple of 12. All of the former are already described for $\mathbb{Z}_n$ (see the previous section) and the latter can only happen (up to mirror images) if $(\alpha, \beta, \gamma) = (-4, -5, 3)$ (here $-5 - 4 - 3 \equiv 0 \pmod{12}$). Notice that here we see that not all of the Cayley graphs obtained above are isomorphic; there are three graphs up to isomorphism: $G_{[1,4,5]}(\mathbb{Z}_{12})$, $G_{[1,3,4]}(\mathbb{Z}_{12}) \equiv G_{[3,4,5]}(\mathbb{Z}_{12})$ and $G_{[1,2,3]}(\mathbb{Z}_{12}) \equiv G_{[2,3,5]}(\mathbb{Z}_{12})$ (none of which is strongly regular).

There are 8 index 1 voltage graph imbeddings that are 8-regular for $\mathbb{Z}_{12}$ and lift to a graph on $S_6$ (which is the best we can do) whose unshaded regions yield a 3-configuration. They are obtained from letting in Figure 19 $\alpha = 2$, $\beta = 4$ and $(\gamma, \delta) \in \{(3, -5), (-5, 3), (1, -3), (-3, 1)\}$ or are mirror images of ones obtained this way. Thus we get KVL for the outside triangle and the loop labeled with 4. Notice that these graphs cannot yield 3-configurations for $\mathbb{Z}_n$, since 4 is of order 3 only in $\mathbb{Z}_6$ and $\mathbb{Z}_{12}$.

The group $\mathbb{Z}_6 \times \mathbb{Z}_2$ has only two elements of order 3, i.e. $\pm(2, 0)$; and three involutions $(0, 1)$, $(3, 0)$ and $(3, 1)$. So all triangles above can only come from a loop
labeled with (2,0) (clockwise or counterclockwise) or from a KVL triangle below. Furthermore, the only way of getting the KVL on a triangle, without using an involution, is by labeling its boundary (1,0), (1,1) and -(2,1), clockwise or counterclockwise. Also, no imbedding of a 4-regular index 1 voltage graph will yield a 3-configuration.

So the only two imbeddings of 6-regular index 1 voltage graphs that yield 3-configurations for \( \mathbb{Z}_n \times \mathbb{Z}_2 \) (that are best genus wise: the torus) is either the one obtained from the voltage graph in Figure 12 by letting \( \alpha = (1,0), \beta = (1,1) \) and \( \gamma = \alpha + \beta = (2,1), \) so that \( \alpha + \beta - \gamma = (0,0), \) or its mirror image. The resulting Cayley graph above is not isomorphic to any of the ones found for \( \mathbb{Z}_{12} \), but, it is not strongly regular either.

The following imbeddings of 8-regular index 1 voltage graphs for \( \mathbb{Z}_n \times \mathbb{Z}_2 \) lift to graph imbeddings on \( S_6 \). Take the voltage graph in Figure 19 with \( \beta = \pm(2,0) \) and the exterior region labeled (clockwise or counterclockwise) with (1,0), (1,1) and -(2,1), in such a way that \( \alpha + \beta \) is an involution. This can be done in 12 different ways, e.g. let

\[
(\alpha, \beta, \gamma, \delta) = ((1,1),(2,0),(1,0),-(2,1)).
\]

Let us now see what happens for \( A_4 \). This group is generated by the four 3-cycles \( a = (1 \ 2 \ 3), \ b = (1 \ 2 \ 4), \ c = (1 \ 3 \ 4) \) and \( d = (2 \ 3 \ 4) \) (notice that their inverses are their squares). There are three involutions in this group, which can be written as \( ab, \ ad \) and \( ba \). Recall that we are multiplying permutations from right to left, e.g. \( ad^{-1}c = (1 \ 2 \ 3)(2 \ 4 \ 3)(1 \ 4 \ 2) = e. \)

The 4-regular index 1 voltage graphs for \( A_4 \) whose imbeddings yield 3-configurations and lift to the sphere are described using the 2-leaved clover, by letting \( \alpha \) and \( \beta \) be of order 3 such that \( \alpha \beta \) is of order 2. Thus the voltage graph
imbeddings come from taking \((\alpha, \beta) \in \{(a, b), (a, c^{-1}), (a, d), (b, c), (b, d^{-1}), (c, d)\}\) or are mirror images of ones obtained that way. All Cayley graphs obtained above are isomorphic and not strongly regular.

The 6-regular index 1 voltage graphs for \(A_4\), whose imbeddings give us lowest genus (the torus) are the ones obtained from the 3-leafed clover, by letting \(\alpha, \beta\) and \(\gamma\) be any elements of order 3 in \(A_4\) such that \(\alpha \beta \gamma = e\). Thus the voltage graph imbeddings either come from taking

\[(\alpha, \beta, \gamma) \in \{(a, d^{-1}, b^{-1}), (a, c, d^{-1}), (a, b^{-1}, c), (b, c^{-1}, d)\}\]

or are mirror images of ones obtained that way. So the exterior region satisfies the KVL and the loops have boundary of order 3. Hence all regions lift to triangles. This means that above the shaded regions form a 3-configuration too. Notice that the resulting Cayley graph above is not isomorphic to the ones obtained for either \(Z_{12}\) or \(\mathbb{Z}_6 \times \mathbb{Z}_2\); and it is not strongly regular.

The situation for the imbeddings of 8-regular index 1 voltage graphs for \(A_4\) can be completely described by looking at the generating set \(\Delta = \{a, b, c, d\}\), i.e. the only generating set of \(A_4\) (up to inverses) that has four elements and no involutions. To do this consider the voltage graph depicted on the sphere in Figure 19, with \(\text{ord}(\alpha \beta) = 2\) and \(\text{ord}(\alpha \gamma \delta) = 1\). Thus the region bounded by \(\alpha \beta\) lifts to 6 squares and the one labeled \(\alpha \gamma \delta\) lifts to 12 triangles. Each loop lifts to 4 triangles. So above we get \(r = 30\) and hence we get an imbedding of \(K_{3(4)}\) on \(S_4\), whose triangles obtained from the unshaded regions form a 3-configuration, which is a \((12, 16, 4, 3; 0, 1)\)-PBIBD. Our computer search shows that we cannot do better, so \(S_4\) is the surface of lowest possible genus for \(\Delta = \{a, b, c, d\}\). Now, these voltage graphs are obtained by letting \((\alpha, \beta, \gamma, \delta)\) be chosen in one of the 12 different ways given in Figure 24, or by taking the mirror image of one obtained that way. Furthermore, no other generating set of \(A_4\)
(up to taking inverses) yields a configuration for which \( m = 4 \), since it is the only one without involutions.

\[
\begin{array}{|c|c|c|c|}
\hline
(a,c^{-1},d^{-1},b^{-1}) & (a,b,c,d^{-1}) & (b,c,d,a^{-1}) & (b,a,c^{-1},d) \\
\hline
(c,a^{-1},b^{-1},d^{-1}) & (c,b,d^{-1},a) & (d,c,a^{-1},b) & (d,a,b,c^{-1}) \\
\hline
(c,d,a,b^{-1}) & (a,d,b^{-1},c) & (d,b^{-1},c^{-1},a^{-1}) & (b,d^{-1},a^{-1},c^{-1}) \\
\hline
\end{array}
\]

Figure 24. Possible Values of \( (\alpha,\beta,\gamma,\delta) \) when \( \Delta = \{a,b,d,c\} \) in \( A_4 \).

Let us consider an example: let \( (\alpha,\beta,\gamma,\delta) = (a,c^{-1},d^{-1},b^{-1}) \), then we have the 3-configuration shown in Figure 25 (the first row comes from the unshaded loop).

\[
\begin{array}{|c|c|c|c|}
\hline
\{e,c,c^{-1}\} & \{a,d,ba\} & \{b,ab,d^{-1}\} & \{a^{-1},ad,b^{-1}\} \\
\hline
\{a^{-1},a,ab\} & \{ba,b,b^{-1}\} & \{b,c,ad\} & \{ab,d,c\} \\
\hline
\{e,a^{-1},d^{-1}\} & \{d,b^{-1},e\} & \{ad,c^{-1},a\} & \{c^{-1},d^{-1},ba\} \\
\hline
\{b^{-1},ab,c^{-1}\} & \{c,ba,a^{-1}\} & \{d^{-1},ad,d\} & \{a,e,b\} \\
\hline
\end{array}
\]

Figure 25. The 3-Configuration Obtained From \( (\alpha,\beta,\gamma,\delta) = (a,c^{-1},d^{-1},b^{-1}) \) in \( A_4 \).

The group \( T = \langle s,t \mid s^6 = e, s^3 = t^2, sts = t \rangle \) has as its elements the set \( \{e,s,s^{-1},s^2,(s^2)^{-1},t,t^{-1},t^2,sts,(st)^{-1},s^3t,(s^2t)^{-1}\} \); of these only two are of order 3 (i.e. \( s^2 \) and its inverse) and only one is an involution (i.e. \( t^2 \)). So any triangles above come from either a triangle below that satisfies the KVL or a loop labeled \( s^2 \) (clockwise or counterclockwise). Furthermore, there cannot be a 4-regular index 1 voltage graph that yields a 3-configuration.
The best (genus-wise above: $S_2$) 6-regular graph imbedding that yields a 3-configuration is given by the 3 leafed clover with one loop labeled with $s^7$ (clockwise or counterclockwise) in such a way that the exterior region (which is unshaded) satisfies the KVL. Thus the voltage graphs come from either taking $(\alpha, \beta, \gamma)$ in
\[
\left\{ (s^2, (s^2t)^{-1}, (st)^{-1}), (s^2, s^2t, st), (s^2, t, (s^2t)^{-1}), (s^2, t^{-1}, (s^2t)^{-1}), (s^2, (st)^{-1}, t^{-1}), (s^2, st, t) \right\}
\]
or the mirror image of one obtained that way. This gives us a 3-configuration for which $m = 3$.

Now, for 8-regular index 1 voltage graphs for $T$ we have two distinct families of imbeddings. First, the voltage graph in Figure 19 labeled in such a way that $(\alpha, \beta) = (s^2, s^2)$ or $(s^{-1}, (s^3)^{-1})$, and $\gamma$ and $\delta$ are of degree 4 (the elements of degree 4 in $T$ are $t$, $st$ or $s^2t$, and their inverses) such that $\alpha\gamma\delta = e$, so the KVL is satisfied. This lifts to an imbedding on $S_2$ and thus we get a 3-configuration ($m = 4$) from the exterior region and the loop labeled with $s^7$ (clockwise or counterclockwise). The voltage graphs are obtained from taking $(\alpha, \beta) = (s^2, s)$ and $(\gamma, \delta)$ in the set
\[
\left\{ (t^{-1}, t^{-1}), (s^2t, t), (t^{-1}, st), (t, (st)^{-1}), (st, (s^2t)^{-1}), ((st)^{-1}, s^2t) \right\},
\]
or are the mirror images of one obtained this way. The second family of 8-regular index 1 voltage graph imbeddings that yields 3-configurations is described by means of Figure 26 which lifts to $S_2$ too. Here label the edges such that the KVL holds in the triangle and the square with $a$ being $s^2$ or its inverse. So the voltage graph imbeddings are obtained by letting $\alpha = s^2$ and $(\beta, \gamma, \delta)$ be in the set
\[
\left\{ (s^2t)^{-1}, s^{-1}, t^{-1}), (s^2t, s^{-1}, t), (s, (s^2t)^{-1}, t^{-1}), ((s^2t)^{-1}, t^{-1}, s),
\right. \\
\left. (s^2t, t, s), (t, (st)^{-1}, s), (s, t, (st)^{-1}), (s, t^{-1}, (st)^{-1}), (s, t^{-1}, st), (t^{-1}, s^{-1}, st), (t^{-1}, st, s) \right\},
\]
or taking a mirror image of one of these.
Figure 26. An 8-Regular Voltage Graph for $T$.

The group $D_6$ (the group of symmetries of the regular hexagon) does not yield 3-configurations from imbeddings of index 1 voltage graphs. If we label the vertices of such a hexagon clockwise 1 through 6 we get that two generators are $\rho = (1\ 2\ 3\ 4\ 5\ 6)$ and $\phi = (2\ 6)(3\ 5)$. It can be easily seen then that $D_6$ has only two elements of order 3 ($\rho^3$ and its inverse), only two of order 6 ($\rho$ and its inverse) and all others are involutions. So for $D_6$ we have that there are no index 1 voltage graphs whose imbeddings yield a 3-configuration (for $m = 2$ since we only have two elements of order 3, and for $m > 2$ since we have too many involutions).

We remark that our computer search found no index 1 voltage graph construction for $m = 5$. We leave it for further study to determine if there are constructions of greater index that work, or failing this to find a suitable ad hoc imbedding of $K_{6(2)}$.

Finally, notice that here we can form a disconnected 3-configuration for $m = 2$ by taking two disjoint spherical imbeddings of the octahedron graph that we found for $p = 6$. 

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To finish we present the following table that summarizes the best (genus-wise) connected 3-configurations that we have found for all $p$ and $m$ ($m > 1$) that we have considered above (notice that the genus is the lowest genus we obtained and need not coincide with the genus of the graph utilized).

Table 2
Summary of Results

<table>
<thead>
<tr>
<th>$p$</th>
<th>$m$</th>
<th>graph(s)</th>
<th>group(s)</th>
<th>genus</th>
<th>block design(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>2</td>
<td>$K_{4(2)}$</td>
<td>$Z_6$</td>
<td>0</td>
<td>(6,4,2;3;0,1)-PBD</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>$K_7$</td>
<td>$Z_7$</td>
<td>1</td>
<td>(7,7,3,3;0,1)-PBIBD</td>
</tr>
<tr>
<td>8</td>
<td>3</td>
<td>$K_{4(2)}$</td>
<td>$Z_8$ and $Q$</td>
<td>1 and 2</td>
<td>(8,8,3;3;0,1)-PBIBD</td>
</tr>
<tr>
<td>9</td>
<td>2</td>
<td>$C_3 \times C_3$</td>
<td>$Z_3 \times Z_3$</td>
<td>1</td>
<td>(9,6,2;3;0,1)-PBIBD</td>
</tr>
<tr>
<td>9</td>
<td>3</td>
<td>$\overline{C}<em>9$ and $K</em>{3(3)}$</td>
<td>$Z_9$ and $Z_3 \times Z_3$</td>
<td>1</td>
<td>(9,9,3;3;0,1)-PBIBD*</td>
</tr>
<tr>
<td>9</td>
<td>4</td>
<td>$K_9$</td>
<td>$Z_3 \times Z_3$</td>
<td>4</td>
<td>(9,12,4;3,1)-PBIBD</td>
</tr>
<tr>
<td>10</td>
<td>3</td>
<td>$G_{11,2,31}(Z_{10})$</td>
<td>$Z_{10}$</td>
<td>1</td>
<td>the Menger graph is not strongly regular</td>
</tr>
<tr>
<td>11</td>
<td>3</td>
<td>$G_{11,2,31}(Z_{11})$</td>
<td>$Z_{11}$</td>
<td>1</td>
<td>the Menger graph is not strongly regular</td>
</tr>
<tr>
<td>12</td>
<td>2</td>
<td>$G_{4,21}(A_4)$</td>
<td>$A_4$</td>
<td>0</td>
<td>the Menger graph is not strongly regular</td>
</tr>
<tr>
<td>12</td>
<td>3</td>
<td>5 Graphs</td>
<td>$Z_{12}$, $Z_6 \times Z_2$ and $A_4$</td>
<td>1</td>
<td>the Menger graphs are not strongly regular</td>
</tr>
<tr>
<td>12</td>
<td>4</td>
<td>$K_{3(4)}$</td>
<td>$A_4$</td>
<td>4</td>
<td>(12,16,4;3;0,1)-PBIBD</td>
</tr>
</tbody>
</table>

*For $K_{3(3)}$ only, since $\overline{C}_9$ is not strongly regular.
Appendix A

Maple V Code to Find $PG(m,p)$ With $p$ a Prime
The following Maple V code will find $S$ and $S_n$ (as defined in Section 1.3) for $PG(m, p)$ with $p$ a prime. We remark that our code for $PG(m, q)$ for any prime power $q$ is considerably longer and convoluted (which is why we do not present it here).

First, we define $m$, $p$, $\nu$ and $k$.

```maple
> m := 4;
p := 2;
nu := ((p^(m+1) - 1)/(p-1));
k := ((p^m-1)/(p-1));
```

Then we find an irreducible polynomial of degree $m+1$ over $\mathbb{Z}_p$, and let one of its roots be $\omega$.

```maple
> f := Randprime((m+1) * 1, x) mod p;
h := f:
alias(omega = RootOf(h)):
```

Then we compute the elements of $GF(p^{m+1}) \equiv \mathbb{Z}_p(\omega)$ and with them we obtain $S_0$.

```maple
> S[0] := {0}:
f1 := 1:
for i from 1 to nu-1 do
omega^i;
f1 := sort(collect(evala(f1*omega) mod p, omega), omega);
print(omega^i, f1);
if coeff(f1, omega^2) = 0 then S[0] := S[0]
if f1 = 1 then print(warning) fi:
od:
S[0];
```

Finally, we compute $S$, a hyperplane of $PG(m, p)$ such that $pS = S$.

```maple
> j := -sum(S[0][j'], j'=1..k)*k^(-1) mod nu;
S[j] := (seq(S[0][i]+j mod nu, i = 1..k));
```
Appendix B

All Cayley Graphs of Order 12 and Degree of Regularity 8 or 10
A byproduct of the computer search that we performed when obtaining some of the results in Chapter 3 is a list of all the non-isomorphic Cayley graphs of order 12 and degree of regularity 4, 6, 8 or 10. Here we present the graphs of order 8 and the only graph of order 10 as examples from that list. We label them with the groups that they can be obtained. We have made no effort to identify them as well-known graphs (when possible); or draw them with the least number of crossings; or to list the generators that give rise to them; or order in any meaningful way (besides the order in which we found them).

1. $D_n$

2. $D_n$, $C_{12}$, $T$ or $C_n \times C_2$

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3. $D_6$ or $C_6 \times C_2$

4. $D_6$ or $C_{12}$

5. $D_6$, $T$, $C_6 \times C_2$ or $C_{12}$
6. \( A_4 \)

7. \( D_6, T, C_6 \times C_2, C_{12} \) or \( A_4 \)

8. \( D_6, T, C_6 \times C_2, C_{12} \) or \( A_4 \)
BIBLIOGRAPHY


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