6-1998

Fostering Reflective Thinking in First-Semester Calculus Students

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by

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A Dissertation
Submitted to the
Faculty of The Graduate College
in partial fulfillment of the
requirements for the
Degree of Doctor of Philosophy
Department of Mathematics and Statistics

Western Michigan University
Kalamazoo, Michigan
June 1998

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Western Michigan University, 1998

This study focuses on the fostering of reflective thinking in students in a reform calculus course through completion of homework assignments incorporating reflective tasks, and the effect of these assignments on student understandings of calculus and conceptions of mathematics. The study, conducted in Fall 1997, involved two sections of first-semester calculus at a large midwestern university and used quantitative (n = 25, 18) and qualitative (n = 7) techniques.

Homework assignments incorporating reflective tasks included asking students to compare and contrast textbook ideas; to write about how obstacles were overcome as they attempted exercises; to develop concept maps organizing and relating course material; and to explain, in writing, strategies regarding specified tasks. Analysis of covariance with pretest achievement scores as covariate was used to analyze student performance on four examinations by section. Student responses at the beginning and end of the semester to an inventory of mathematical conceptions were analyzed by section using a two-sample t-test. Audiotaped "think aloud" problem sessions were conducted with selected treatment section students and analyzed by category of thought using time-line graphs, which provided detail on reflective thinking used during problem solving unavailable from in-class examinations.

No significant differences in adjusted means were determined on the four examinations. Inspection of regression lines of examination scores and intersection

points revealed an interaction between treatment and precalculus achievement. Students scoring at the 12th percentile had better achievement on Exam 1 than control students. Students had better achievement than the control students at the 28th percentile for Exam 2, at the 32nd percentile for Exam 3, and at the 44th percentile for the Final examination. As the semester progressed, an increasing number of students appeared to benefit from the treatment.

The "think aloud" problem sessions supported this benefit of treatment. By the end of the semester, students exhibited categories of reflective thinking, such as Direction of Thinking, which were virtually absent at the beginning of the semester and exhibited more repetition and variety in their categories of thought.

UMI Number: 9835498

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ACKNOWLEDGMENTS

I wish to express my sincere appreciation to my dissertation advisor, Christian Hirsch, for his time, support, and guidance throughout my studies at Western Michigan University. My work is dedicated to my family and friends who stayed with me during my long graduate student career. Specifically,

to Fran, Charlotte, and Don — who did not live to see the completion,

to Barbara — who showed me mathematics through the eyes of one whose interests and talents lie elsewhere,

to Connie, Brenda, Susan and Rod — who have never wavered in their support and belief in me,

to Plummie, Toot, and Elizabeth — who are my role models,

to Melanie and Rebecca — who provided much needed and appreciated encouragement through the dark days,

to Jimmy, Becky, Susan, Brenda and Jim, and Audra — who were unexpected and very pleasant surprises along the way,

and, finally, to Doug — whose role in this degree was unique and significant.

To all of you, my heartfelt thanks!

Pamela Crawford

TABLE OF CONTENTS

Table of Contents—continued

CHAPTER

CHAPTER

LIST OF TABLES

LIST OF FIGURES

CHAPTER I

INTRODUCTION

"Education in any discipline helps students learn to think, but education also must help students take responsibility for their thoughts" (Everybody Counts, p. 3). Undergraduate calculus reform efforts mirror this belief by an increased concentration on students' conceptual understanding and development of mathematical thinking developed through extensive numerical, graphical, algebraic, and modeling interpretations (Assessing Calculus Reform Efforts, 1995).

> [Mathematics] instruction should be aimed at conceptual understanding rather than at mere mechanical skills, and at developing in students the ability to apply the subject matter they have studied with flexibility and resourcefulness. (p. 9)

Students in reform calculus courses are now expected to learn to use and to demonstrate the thinking processes of reflecting, explaining, and summarizing course material (Student Assessment in Calculus, 1997). These processes are one characteristic of students' development of advanced mathematical thinking, particularly when students use them when attempting to solve non-trivial problems (Dreyfus, 1991). Instructional strategies that enable students to develop such thinking processes are essential, therefore, so that students gain the ability to use their mathematical knowledge in new contexts when endeavoring to solve problems that require more than rote calculations (Calculus: The Dynamics of Change, 1996).

> [The] focus is on concepts, helping students take them apart, understand where they came from, see how their elements are inter-related, and ultimately to see how they might be used in a new context to build insights that are, at least for that individual, new and significant. (p. 1)

Students taking responsibility for their thoughts involves self-regulation, which is a student's ability to monitor the course of her/his learning and to adjust or revise strategies during this learning (Borkowski, 1992 and Lawson & Wollman, 1975). Students' increased awareness of, and sensitivity to, their own thought processes improves their learning (Schunk, 1982 and Weaver, 1987). Increased awareness includes the ability to monitor and control to some degree one's own thought processes, particularly how well one can organize and execute courses of action that may be required in problems containing novel and unpredictable elements. Other terms used to refer to the examination of one's knowledge and thoughts are "reflective intelligence" (Skemp, 1979, p. 175) and "reflective abstraction" (Dubinsky, 1991, p. 95 and Piaget, 1976, p. 45).

A student's sudden anxious feeling that she/he is not understanding some material and wants to understand the material is a metacognitive experience, according to Flavell (1976), who developed the now generally accepted description of metacognition:

> "Metacognition" refers to one's knowledge concerning one's own cognitive processes and products or anything related to them, e.g., the learning-relevant properties of information or data....Metacognition refers, among other things, to the active monitoring and consequent regulation and orchestration of these processes in relation to the cognitive objects or data on which they bear, usually in the service of some concrete goal or objective. (p. 232)

Note the two separate but related aspects of metacognition: (1) knowledge about cognitive processes and products, and (2) monitoring and control of cognitive actions.

Garofalo and Lester (1985) believe that metacognitive beliefs, decisions, and actions are important, albeit frequently overlooked, contributors to the success or failure of a wide variety of cognitive activities. In particular, students' successful

cognitive performance on tasks depends not only on their having sufficient knowledge but also on their having an awareness and control of that knowledge.

Rationale

A central focus of mathematics instruction is to help students learn to think mathematically, which includes not only mastering various facts and procedures but also understanding connections among them (Schoenfeld, 1988). Mathematics requires large bodies of organized information for understanding yet, as Novak (1996) notes, as students progress through our education programs they often move from predominantly high levels of meaningful learning to predominantly rote-mode learning. Students view relationships between concepts as arbitrary and not constructed by the students themselves. Students' metacognitive knowledge improves their strategy use by providing them with knowledge about when, where, and why they should use different strategies; with understanding about various tasks; and with information on their own capabilities and inadequacies (Schneider & Pressley, 1989).

> Self-interrogation is an important metacognitive technique. By asking questions of themselves, students can monitor themselves, predict and hypothesize, assess feelings of understanding or lack of understanding in order to choose and employ a self-correction strategy, and integrate new information with existing information. (Ganz & Ganz, 1990, p. 181)

Students who can direct their own thinking can connect new information with existing information, can purposely select their thinking strategies, and can relate time spent and degree of certainty to purpose (Dirkes, 1985).

Ganz and Ganz (1990) believe that even after learning new information, students' self-interrogation should not stop. Students should continue to reflect on this knowledge by asking themselves such questions as "Can I make some

generalizations, and are they fitting? Can I draw some conclusions, and are they plausible? Is this similar to anything I already know?" (p. 182). Ganz and Ganz characterize mature learners as those who treat studying as "a purposeful, attention-directing, self-questioning act" (p. 182) while less mature learners possess naive theories about what it takes for them to learn. These learners are often hindered by inferior, inefficient strategies which yield limited successes but which the learners consistently apply in a variety of situations. "[They] act as though the test of truth is that a proposition makes intuitive sense, sounds right, rings true. They see no need to criticize or revise accounts that do make sense — the intuitive feel of fit suffices" (Perkins, Allen & Hafner, 1983, p. 186).

As early as 1976, Flavell (1976) believed that the nature and development of metacognition and of cognitive monitoring and regulation was emerging as an interesting and promising area of research. Yet, there have been few studies of the effect of metacognition on mathematical performance. (For exceptions, see Schoenfeld, 1985, 1987, 1992 and Silver, Branca & Adams, 1980) This research will add to the existing body of knowledge about the reflective component of metacognition.

Definitions

Metacognition

The act of thinking about one's own thinking is metacognition (Ganz & Ganz, 1990). Although it is not always easy to distinguish what is metacognitive from what is cognitive, generally cognition is involved in doing, while metacognition is involved in choosing and planning what to do and monitoring what is being done (Garofalo & Lester, 1985). Expanding on ideas of Manning (1984), specific

examples of metacognitive strategies in first-semester reform calculus include students realizing they are not understanding instructions on how to find an antiderivative, students noticing they do not have all the derivatives necessary to complete a related rate problem, students realizing that they have more difficulty understanding the symbolic approach to the derivative concept than the graphical approach, and students realizing they need to slow down when computing a Riemann sum or they may make an error.

Reflective Thinking

That part of metacognition which focuses on an awareness of one's own knowledge about a particular concept or method is known as reflective thinking. Such an awareness includes, for example, the ability to discuss the meaning of a concept (method), to compare and/or contrast a concept (method) with other concepts (methods), to analyze obstacles and perhaps strategies for their removal in the learning of a concept (method), and to connect a concept (method) with other concepts (methods).

Reflective Tasks

Reflective thinking may be fostered through the use of reflective tasks, that is, tasks requiring a student to ponder and think over various actions or concepts used either as she/he completes exercises or by the author(s) in the development of new material in the textbook. Examples of such reflective tasks in homework assignments include asking the student to compare and contrast ideas in the textbook, to write about obstacles encountered as she/he attempted exercises and how such obstacles were overcome, to develop concept maps (see below) organizing and

relating their understanding of course material, or to explain in writing procedures regarding specified tasks.

Concept Maps

Concept maps are two-dimensional graphic representations of concepts, propositions, and their relationships. (For an example, see p. 129 in Appendix B.) These organizers represent super ordinate versus subordinate relationships, and interrelationships between concepts. In constructing concept maps, students are required to externalize their thinking by mapping out their conceptual structure of a topic (Park & Travers, 1996). Thus, concept maps are straightforward representations of the concepts and propositions a student holds (Smith, 1987).

Questions

The focus of this research is on the fostering of reflective thinking in first-semester reform calculus students by their completion of regular homework assignments which include reflective tasks and the effect of these assignments on student understandings and conceptions of mathematics. The questions addressed by this research are:

1. Will completion of homework assignments involving reflective tasks contribute to better student understanding of first-semester calculus than completion of homework assignments not involving such tasks?

2. Is there a relationship between the nature of reflective activity used during problem solving and student level of performance in first-semester calculus?

3. Will the amount of reflective activity used during problem solving increase as a result of student completion of homework assignments involving reflective tasks?

4. Will the nature of reflective activity used during problem solving change as a result of student completion of homework assignments involving reflective tasks?

5. Will student beliefs about mathematics change as a result of the completion of homework assignments involving reflective tasks?

Theoretical Framework

Constructivism

Constructivism has emerged as a major research paradigm in mathematics education (Schoenfeld, 1992). Constructivists view understanding as the building of mental frameworks from already existing pieces, that is, previously built frameworks become the content in subsequent constructions (Ernest, 1996 and Weaver, 1987). Note the emphasis in Flavell's (1976) statement concerning

> what many of us think to be perhaps _the_ central emphasis in learning and development, namely, how and under what conditions the individual assembles, coordinates, or integrates his already existing knowledge and skills into new functional organizations. (p. 231)

An individual's knowledge is transformed as new ways are created to observe and organize experiences. New mental representations are constructed through different but related mental processes such as encoding, combining, or comparing concepts (Davidson, Deuser & Sternberg, 1994). An important educational goal should be to help students and instructors understand this constantly evolving constructed nature of knowledge (Novak, 1996).

An individual's misconceptions can occur, however, in incorporating existing knowledge into new mental structures. These misconceptions can be localized and exist in an otherwise satisfactory mental structure (Mansfield & Happs, 1991). Students' misconceptions and misunderstood facts are the tools they bring to problem situations (Harris, Wallace & Rudduck, 1995 and Schoenfeld, 1992). These misconceptions and misunderstood facts are not isolated pieces of framework (Novak, 1996).

> In the majority of cases, the student, in reviewing the path (i.e., reflecting on the operations carried out), will either discover a hitch or give the teacher a clue to a conceptual connection that does not fit into the procedure that is to be learned. The first is an invaluable element of learning: It provides students with an opportunity to realize that they themselves can see what works and what does not. The second provides the teacher with an insight into the student's present way of operating and thus with a clearer idea of where a change might be attempted. (von Glaserfeld, 1996, p. 313)

As Schoenfeld states (1985, p. 368), "One cannot simply assume that students have not 'gotten it' one may have to discover what they have 'gotten' and 'debug' it."

Mathematics instruction, particularly at the college level, however, has traditionally focused on the content aspect of knowledge (Schoenfeld, 1992).

> From this perspective, "learning mathematics" is defined as mastering, in some coherent order, the set of facts and procedures that comprise the body of mathematics. The route to learning consists of delineating the desired subject-matter content as clearly as possible, carving it into bite-sized pieces, and providing explicit instruction and practice on each of those pieces so that students master them. From the content perspective, the whole of a student's mathematical understanding is precisely the sum of these parts. (p. 342)

According to Schoenfeld (1992), this carving of content into bite-sized pieces has at least two consequences. First, students come to expect that answers and methods to problems will be provided to them, that is, students believe they are not expected to figure out methods themselves. Second, students come to believe that there should be a ready method for the solution of any problem, and that the method should produce

an answer to the problem in a few minutes. Students often develop beliefs that they cannot attempt problems for which they have no discernible method and that they should curtail their efforts at solving problems if they cannot achieve success after a few minutes. Students gain, at best, a fragmented sense of mathematics and understand few (if any) of the connections that tie together the procedures they study (Schoenfeld, 1988).

Students are often unaware of their own thinking processes and unable to describe the strategies they use during problem solving. Students seldom evaluate the quality of their own thinking skills (Ganz & Ganz, 1990). Students retreat from intellectual challenges when they are asked to think harder and they have no available means for thinking harder (Harris, Wallace & Rudduck, 1995 and Scardamalia & Bereiter, 1983). Even when students realize they are approaching a problem incorrectly, they may not be able to break from their fixation with old irrelevant problem-solving procedures and develop new strategies (Davidson, Deuser & Sternberg, 1994).

Unfortunately for most students, knowledge of appropriate facts and procedures alone cannot guarantee a student success in mathematics (Odafe, 1994). Also, traditional examination systems have promoted within society the view that learning consists of "cramming," which inhibits the development of metacognition (Baird & White, 1996). According to Dirkes (1985), "metacognition is most useful as a buffer for uncertainty and a catalyst for response to opportunities where students do not have a learned response" (p. 97). Further, Dirkes believes that much of the knowledge that students learn in the future will not be as important as the facility with which they can generate ideas and find ways to interpret and link them.

Metacognition

Lester (1994) believes that good problem solvers are distinguishable from poor problem solvers. One distinguishing aspect is that not only do good problems solvers know more than poor problem solvers but what they know, they know differently. Good problem solvers make strong connections when they incorporate new knowledge into their existing mental frameworks. Another distinguishing aspect is that good problem solvers are more aware than poor problem solvers of their strengths and weaknesses as problem solvers. Good problem solvers are better also at monitoring and regulating their problem-solving efforts. They effectively use metacognition during problem solving — they know not only what and when to monitor, but also how to monitor.

One of the core components of metacognition is self-regulation, that is, the ability to monitor and assess progress 'on line,' and to adjust or revise the strategy in response to these assessments (Borkowski, 1992, Lawson & Wollman, 1975 and Schoenfeld, 1992). Self-regulated learners plan, organize, self-instruct and self-evaluate at various stages during the learning process (Zimmerman, 1986). Students can monitor their own planning behavior by asking themselves such reflective questions as "Is this a reasonable approach?" and "Am I getting too far away from the original question?". Unless students recognize the importance of this reflective monitoring, they may "get their solution process rolling like a freight train and then run out of track" (Shaughnessy, 1985, p. 403), which may explain why students often feel that they are "not in control" when they work mathematics problems. Metacognition can help a student recognize that there exists a problem, figure out exactly what is the problem, and understand how to accomplish a solution. Many of the most important steps in problem solving are not solution steps, but ones

that direct and guide the problem solving — the metacognitive processes are as important in problem solving as the cognitive ones (Davidson, Deuser & Sternberg, 1994).

To study effectively, students must be able to distinguish what they have understood from what they still need to clarify (Manning, 1984). They must be sensitive to the level of their comprehension to know what and when to reread, when to ask questions, and what additional information is needed. When students develop a metacognitive awareness of their own capacities, they gain the ability to predict what will and will not cause them difficulty (Scardamalia & Bereiter, 1983). They become aware of their own comprehension failure (Schoenfeld, 1985).

Instruction in mathematics should include activities that bring more of the cognitive processes into the open where instructors and students can examine them and try to understand them (Scardamalia & Bereiter, 1983). As discussed previously, students' success depends not only on having adequate mathematical knowledge, but also on having sufficient awareness and control of that knowledge (Garofalo & Lester, 1985).

> In other words, although it may be useful to understand "This is how I do it and this is how an expert does it," a sizable increment in self-knowledge occurs if the terms can be shifted to "This is how I usually do it and this is how it feels to do it like an expert. (Scardamalia & Bereiter, 1983, p. 74)

Observable behavior of cognitive strategies, however, gives only limited and often misleading insight into the underlying mental operations. The rapid and fleeting nature of mental events ensures that the data is elusive. Also, when people are engaged in mental activity, their attention is normally taken up with their immediate task rather than being directed toward the process itself (Scardamalia & Bereiter, 1983). Additionally, experts may have integrated their metacognitive strategies so neatly into their problem-solving routines that such behaviors become

difficult to observe, or perhaps their metacognitive strategies are needed only at certain points in the development of their expertise (Silver, Branca & Adams, 1980). To Corno (1986), as learners gain experience, they develop an awareness of strategy use and utility. Their metacognitive skills become more efficient and take on the character of automatic processing.

Even with Flavell's description of metacognition, it is not easy to distinguish between cognition and metacognition (Brown, 1987). Weinert (1987) defines metacognitions as second-order cognitions — "thoughts about thoughts, knowledge about knowledge, or reflections about actions" (p. 8). Flavell (1987) states "the purpose is no longer to reach the goal (cognitive strategy), but rather to feel absolutely confident that it has been reached (metacognitive strategy)" (p. 23), while, as noted previously, Garofalo and Lester (1985) believe one way of viewing the relationship between cognition and metacognition "is that cognition is involved in doing, whereas metacognition is involved in choosing and planning what to do and monitoring what is being done" (p. 164).

> Learning how to learn is metacognition. It involves enabling the mind to search out pattern, to analyze and solve problems, to summarize results, to check conclusions and to establish associations. It involves learning strategies for receiving, processing, storing, and recalling information. It requires skills in selfquestioning as well as awareness of failure. It is not just study skills. (White & Denny, 1983, p. 2)

The purpose of any metacognitive strategy is to generate information that will help the user be knowledgeable about and be in control of whatever she/he is doing. According to Flavell (1979) and Manning (1984), metacognitive strategies are higher order processes that include the knowledge or awareness that specific cognitive strategies will be of use in solving a problem. Students are able to learn material faster, understand it better, and retain the information longer using metacognitive strategies (Mayo, 1993). In addition, with the use of these strategies,

students become active participants in their learning as they organize information and build schemata, integrating their new knowledge with prior information and experiences. Knowledge is not just what a student knows — it is also how, when, and whether the student uses what she/he knows (Schoenfeld, 1992).

For Baird and White (1996), metacognitive strategies are employed by a person engaged in a process of purposeful inquiry, and comprise reflection (to determine purpose) and action (to generate information). Better learning will follow improvement in the implementation of appropriate strategies. Also, a student's academic benefits of enhanced metacognition are matched by affective benefits such as greater satisfaction, fulfillment, and sense of purpose, control, and self-worth. Learning with understanding is fostered when the learners engage in informed, deliberate activity, to the extent that they exert sufficient control over their personal learning approach, progress, and outcomes. Metacognitive strategies are initiated by asking oneself reflective questions such as "What am I doing?" and "Why am I doing this?"; selecting procedures to answer these questions; evaluating the results of the applied procedures; and then deciding what to do next.

Ganz and Ganz (1990) believe that students need to develop the ability to discern if they should reread, continue, slow their speed, find assistance, or choose a different strategy. Students assessing their feelings of understanding is also important for their understanding and learning. Students should ask themselves, "Did this make sense?", "Can I explain this using my own words?", "Can I make a judgment now?". To assess their lack of understanding, students should ask "Is this part harder than previous section?", "Do I understand the author's point?", "What is the meaning of this word as it is used here?", "Will I make sense of this later?". Students who develop metacognitive skills are much more likely to be able to make changes in their study habits and learning strategies when they encounter unfamiliar

tasks or challenges than students who do not develop metacognitive skills. Students who are developing metacognitive skills will be able to describe their thinking processes, mental frameworks, and future strategies when coping with a problem.

> When you are reading along and suddenly find yourself reading more slowly, the slowdown in processing may function as a cue that the material is getting difficult, or that something is puzzling, etc. Similarly, individuals may become aware that they have just read a sentence for the second time, and that awareness may serve as a metacognitive signal that the material is difficult, or that attention has wandered, etc. (Flavell, 1987, p. 28)

Homework Tasks

In mathematics, homework tasks serve at least three purposes: (1) as means of instruction, (2) as instruments of practice, and (3) as tests for the acquisition of mathematical skills (Schoenfeld, 1992, p. 337). Numerous authors (Austin, 1979, 1980; Betsinger, Cross, & DeFiore, 1994; Dangel & Hopkins, 1979; Elawar & Corno, 1985; Flavell, 1976; Goldin, 1992; Hembree, 1992; Hirsch, Kapoor, & Laing, 1983; Lester, 1994; Lucas, 1974; Miller, Duffy, & Zane, 1993; Schoen & Kreye, 1974; Schoenfeld, 1992; Suydam, 1985; and Wiebe, 1982) have studied the effect of independent problem-solving tasks on students. Problem solving is viewed as a complex process involving cognitive operations such as visualization, association, abstraction, comprehension, manipulation, reasoning, analysis, synthesis, and generalization, "each of which needs to be managed and all of which need to be coordinated" (Garofalo & Lester, 1985, p. 169). Schoenfeld (1981) believes college students lack the ability during problem solving to make managerial decisions such as

> selecting perspectives and frameworks for a problem; deciding at branch points which direction a solution should take; deciding whether, in light of new information, a path already taken should be abandoned; deciding what (if anything) should be salvaged from

> attempts that are abandoned or paths that are not taken; monitoring tactical implementation against a template of expectations for signs intervention might be appropriate, and much, much more. (p. 20)

Many instructors, however, pay little attention to the structure of the tasks which might develop these abilities in students (Hirsch, Kapoor, & Laing, 1983).

Scardamalia and Bereiter (1983) believe that a necessary condition for homework tasks to be of benefit to students is the presence of a large amount of redundancy in operations and in content. Students, however, often streamline the redundant parts into procedures that are insensitive to novel elements of problems, and therefore yield incorrect answers. If a procedure is not grounded in an understanding of the underlying principles, the procedure is error-prone and easily forgotten (Schoenfeld, 1992). Students often cannot use a poorly understood procedure, or they transform the procedure into one that will incorrectly work within their existing strategies (Scardamalia & Bereiter, 1983).

The types of questions that instructors ask their students greatly influence the type of questions that students internalize and ask themselves (Williamson, 1996). To Scardamalia and Bereiter (1983), as students try to incorporate their existing knowledge into new mental structures, the use of homework tasks can initiate interactions between students and subject matter that will allow their self-regulation to gradually restore equilibrium. Two ways of achieving this are to turn problems into discrimination or comparison tasks, and to have students provide procedural support for other students. Lawson and Wollman (1975) believe two factors should be present in homework tasks: (1) problems must be chosen so that students can partially, but not completely, understand them in terms of their existing knowledge; and (2) students must be allowed sufficient time to grapple with the new situations. Typical homework problems, however, seldom require a student to examine her/his own thinking, to make comparisons, or to raise questions. Unless

a student can discern basic ideas, her/his study time becomes an exercise in passive rereading or in rote memorization devoid of understanding (Ganz & Ganz, 1990).

Success in mathematics now requires that students learn how to learn — they must learn to organize information, to be patient, and to pay attention to detail (White & Denny, 1983). No longer does doing mathematics mean following the rules laid down by the instructor, does knowing mathematics mean remembering and applying the correct rule, and does verifying mathematical truth mean asking the instructor (Schoenfeld, 1992). Thought-provoking homework tasks provide students an opportunity for self-regulation. Students not only develop deeper understandings of the concepts involved, but they progress from relatively concrete and limited modes of thinking to more abstract and generalizable modes of thinking (Lawson & Wollman, 1977). Typical problem-solving questions, however, are usually not thought provoking, being instead more procedural than conceptual (Beidleman, Jones & Wells, 1995).

Writing Assignments

No other task forces students to clarify their personal understanding nor reveals misconceptions in their understanding as effectively as the requirement that they explain their understanding in writing (Calculus: The Dynamics of Change, 1996).

> If we are to take seriously our goal of having [students] understand concepts, we need a window on their minds. They have to tell us what and how they are thinking. We can listen to them talk, and we can ask them questions. That's a start, but our own thoughts are inevitably part of that process. The cleanest window we have is student writing. (Smith, 1996, p. 32)

Waller (1994) indicates that small reflective writing assignments can help students develop their metacognition yet, in instructional settings and in class, students are

seldom asked by their instructor what they know about themselves as problem solvers and as learners. Not only is reflection desirable as an aid to writing, but writing is desirable as an aid to reflection (Scardamalia, Bereiter & Steinbach, 1984).

Writing assignments in mathematics classes afford students the opportunity to organize their thoughts and to internalize and evaluate concepts (Nahrgang & Petersen, 1986). If a student is not able to effectively explain a concept or an idea in writing then the student does not have a firm understanding of that concept or idea (Beidleman, Jones & Wells, 1995). To Flavell (1987), writing affords practice and experience in metacognition by allowing one to critically inspect one's own thoughts. "Metacognition, like everything else, undoubtedly improves with practice" (p. 26). One way to become better at metacognition, therefore, is to practice it.

Reflective Tasks

Reflective tasks can involve students turning problems into discrimination or comparison tasks, providing procedural support for other students, and describing obstacles they encounter as they attempt to solve problems. These types of reflective tasks serve to put students in touch with the cognitive strategies they presently use (Scardamalia & Bereiter, 1983). A fourth type of reflective task involves drawing concept maps (see p. 129 in Appendix B for an example). Concept maps are a useful tool in identifying students' misconceptions or gaps in students' comprehension, in evaluating learning, and in aiding students to understand the constructed nature of knowledge (Mansfield & Happs, 1991 and Novak, 1996). Nevertheless, there have been few studies on the use of concept maps in mathematics (Williams, 1995). When students draw concept maps, they depict relational aspects of their

mathematical understandings that other instruments are unable to reveal (Hasemann and Mansfield, 1995). Students can visualize how they are connecting mathematical concepts and can then adjust these connections, if necessary.

Students' use of propositions linking the concepts is one of the most important components in the construction of the concept map as these propositions indicate explicitly the degree of differentiation students are making between concepts (Bartels, 1995 and Malone & Dekkers, 1984). Drawing a concept map necessitates the student being able to think in multiple directions at the same time, which is not easy for the student unless she/he has a deep understanding of the concepts being mapped (Merrill, 1987). Concept maps provide highly specific information to instructors in identifying certain places where instruction has failed to teach important concepts or connections (Novak, 1996). The incapacity of low ability students to remember the structural aspects of mathematics may be evidence of a more general failure to notice structure. If a student believes there is an underlying structure to mathematics, that student is more likely to believe the problems she/he solves today are apt to be useful tomorrow (Silver, Branca & Adams, 1980).

CHAPTER II

REVIEW OF RELATED RESEARCH

Research studies at the college and at the K-12 levels on metacognition, homework tasks, writing assignments, and reflective tasks were reviewed for their contributions to reflective thinking. Along with mathematics, research on learning in science disciplines such as physics, chemistry, and computer science, as well as in non-science disciplines such as english, reading, and social sciences, were examined. Williams (1995) states that few studies exist of the effect of metacognitive knowledge on students' performances, especially in mathematics, while Garofalo and Lester (1985) note the absence of studies concerning the nature and development of monitoring, assessing, and strategy-selecting behaviors. Very few studies were found concerning the use of reflective tasks, particularly in mathematics, at either the college, high school, or lower levels.

Metacognition

While most mathematics instructors would agree that developing student understanding of concepts is an important goal, enabling students to become good mathematical thinkers may depend as much upon students' acquisition of the habits of interpretation and sense-making as of their acquisition of any particular sets of skills, strategies, or knowledge (Scardamalia & Bereiter, 1983, and Schoenfeld, 1992). As Weinert (1987) asks students, "Do you learn more effectively when you know how to learn?" (p. 1). Metacognitive knowledge can improve with practice and will lead students to select, evaluate, revise, and abandon cognitive strategies used

during independent problem solving (Flavell, 1979 & 1987). As students develop metacognition, they gain unique problem-solving strategies that become aptitudes for learning (Dirkes, 1985).

Romainville (1993 and 1994) investigated the relationship between metacognition and academic performance of 35 Belgium first-year economics students by using data from structured interviews to examine the nature of the students' descriptions, judgments, and justifications of their cognitive strategies. The author developed metacognitive profiles that associated different performance levels with students' metacognitive knowledge characteristics, their learning conception, and their attribution modes. Romainville found that high-achieving students' metacognitive knowledge was more structured and hierarchically organized than that of low-achieving students. High-achieving students divided their strategies into rules, their rules into subrules, etc. but they did not describe their strategy in any more detail than did low-achieving students. The conditions governing high-achieving students' use of rules did not differ from those of low-achieving students, nor did high-achieving students judge their strategies more precisely than low-achieving students. However, high-achieving students did evoke metacognitive knowledge of cognitive process and cognitive results more often, and they did justify their strategies more often using intricate connected sequences of reasons than did low-achieving students.

Betsinger, Cross, and DeFiore (1994) compared the problem solving abilities and metacognitive functions of 19 mathematics and computer science majors to those of 19 english majors at a private midwestern university. Students described their thought processes as they were audiotaped and timed completing the Tower of Hanoi problem. They then responded to a series of follow-up questions and completed a paper-and-pencil survey regarding their use of five metacognitive

functions: (1) working backward to solve the problem, (2) setting subgoals to reduce the difference between the initial and the goal state, (3) breaking the problem into a set of differences and trying to eliminate each difference in search of a goal state, (4) using the structure of another problem to guide the solution of the present problem, and (5) attempting solutions in an unsystematic (trial-and-error) manner. Statistical analyses were performed on variables such as the student's success or lack of success in solving the problem, the number of steps away from the problem's perfect solution, the amount of time in seconds to complete the problem, and use of each of the five above-mentioned metacognitive functions. No statistically significant differences in problem solving were found on any of these variables between the mathematics and computer science majors and the english majors.

Bookman (1993) examined the differences between metacognitive behaviors displayed by six experts (first and second year graduate students in mathematics) and nine novices (college freshmen). The students were asked to "think aloud" while solving four mathematics problems: (1) a routine problem that used skills that reappeared in a later problem, (2) a problem with more than one obvious path to a solution, (3) a nonroutine problem that involved the use of the skills used in the routine problem but in a new and unfamiliar form, and (4) a problem with insufficient information. Of the two aspects of metacognition, beliefs about cognition were found to play a more important role than control of cognition. Bookman concluded that even the scarcity of appropriate managerial behavior can lead students to failure. The researcher also noted that verbal protocols, that is, transcripts of audiotaped or videotaped sessions, "are the most common way to study nonroutine problem solving" (p. 289).

Bookman's conclusion regarding the relationship between students' scarcity

of appropriate managerial behavior and their success in mathematics supports a similar conclusion by Schoenfeld (1992). In Schoenfeld's study, college and high school students were videotaped as they worked on unfamiliar mathematics problems. Approximately 60 % of the students' solution attempts were of the "read, make a decision quickly, and pursue that direction come hell or high water" type (p. 356). Schoenfeld analyzed the students' solution attempts by means of a time-line graph in which the activities of Read, Analyze, Explore, Plan, Implement, and Verify were timed and plotted. Notation was made by Schoenfeld of any specific remarks students made regarding these activities. Their solution attempts were compared to those of a mathematician trying to solve a difficult two-part problem. The mathematician spent more than half the allotted time trying to make sense of the problem, and spent a significant amount of time analyzing and exploring solution attempts. Schoenfeld concluded that students (novices) are unaware of or fail to use the executive skills used by a mathematician (an expert) but Schoenfeld believes that students can learn these skills through explicit instruction that focuses on the metacognitive aspects of mathematical thinking. Students need to ask themselves questions such as: "What good will this approach do [me]?", "Is this approach (or representation) justified?", "Am I making enough progress to continue this approach?", "How does this fit into the solution?", "What (exactly) [am I] doing?", and "What will [I] do with the outcome when [I] obtain it?" (p. 367 and p. 374). Schoenfeld's results support those of Eisenberg and Dreyfus (1985) who also studied students solving mathematics problems and found that

> students' backgrounds, courses taken, and grades could all be ignored, because all students rushed towards an answer, used known procedures uncritically, seldom questioned whether alternative solutions were available, and did not generalize unless asked to do so. Each problem was a separate entity, with little perception of similarity in process or problem-structure. (Becker & Pence, 1994, pp. 7-8)

Zimmerman and Pons (1986) investigated the learning strategies used by high school students in various contexts such as in classroom situations, at home, when completing writing assignments outside of class, when completing mathematics assignments outside of class, when preparing for and taking tests, and when poorly motivated. Eighty students were directed to rate their consistency in using each of their learning strategies. From these interviews, fourteen categories of self-regulated learning strategies were identified by the authors, which included self-evaluation; organizing and transforming; subgoal setting and planning; seeking information; keeping records and self-monitoring; environmental structuring; self-consequences; rehearsing and memorizing; seeking peer, teacher, or adult assistance; and reviewing notes, tests, or textbooks. As did Romainville above, these authors found that students' use of metacognitive strategies was highly related to the students' achievement level. Success in school was determined to be highly dependent on student self-regulation, especially in unstructured settings where studying often occurred.

Silver, Branca, and Adams (1980) studied the problem solving processes of 29 fifth-grade and 36 sixth-grade students as part of a longitudinal study. "The metacognitive relationship between problem posing, planning, and problem solving is an important one" (p. 215). Instruction was designed with a stronger emphasis on process than product. The instructional component of the project was written to integrate themes drawn from the work of Polya and of Krutetskii with themes drawn from developmental cognitive psychology such as metacognition. Specific attention was given to certain heuristic strategies such as drawing a diagram to aid in understanding a problem, making a chart or table to aid in detecting patterns, searching for patterns in problem data, identifying irrelevant (and, as a result, relevant) information in the problem statement, judging the sufficiency of problem

data, proposing a simpler problem that may be easier to solve, and thinking of previously solved related problems that may suggest useful processes. Heavy emphasis was placed on the creation of a student-generated list of useful problem solving processes that was discussed and modified throughout the school year. Silver's, Branca's, and Adams' work supports that of Gurova (1969) who reported success in observing the link between problem solving ability and awareness of one's mental processes and success in instructing students to be more aware of their problem solving processes.

Another longitudinal study on the role of metacognitive knowledge in mathematics-strategy use was that of Carr, Alexander, and Folds-Bennett (1994). They studied 39 second-grade students at selected points during the school year regarding the students' use of retrieval, internal, and external strategies while solving mathematics problems. In addition, they measured the students' metacognitive knowledge about mathematics. Their results were consistent with previous results that metacognition, motivation, and strategy use work together to promote learning.

Williamson (1996) examined many studies that have been performed regarding teachers' questioning in the classroom, particularly in the area of reading. Williamson noted the importance of the use of self-questioning as an aid to metacognition.

> Self-questioning then is a metacognitive process of reading which enables students to become independent in their understanding of text, because they are actively engaged through goal-directed, organized thinking. (p. 31)

Williamson's findings suggest that when teachers engage their students in metacognitive processing through questioning, the students will become more productive learners capable of assuming greater responsibility for their own

learning. Williamson described the K-W-L (Know-Want to Know-Learned) Plus strategy developed by Carr and Ogle (1987) which focused on the student as learner. There are three principal components in K-W-L Plus: (1) having students recall what is already known about a topic, (2) having students determine what they want to learn, and (3) having students identify what they have learned. Williamson's findings support those of Carr and Ogle (1987), who found that many students

> fail to realize that good reading means asking questions and thinking about ideas while reading. When they begin to read, they do not perceive that they should learn, rather than simply "look at text." They are unaware of basic techniques, such as identifying key ideas and summarizing. (p. 626)

In summary, research into metacognition suggests that instructors can assist students in improving their performance, particularly in mathematics, by developing students' knowledge about and use of metacognitive skills. Explicit instruction that concentrates on the metacognitive aspects of mathematical thinking seems to enable students to develop metacognitive skills, which evolve into successful problem-solving strategies and managerial techniques.

Homework Tasks

Among the many themes of the calculus reform movement, as delineated by Park and Travers (1996), are those of:

> involving students in doing mathematics instead of lecturing at them; stressing conceptual understanding, rather than only computation; developing meaningful problem-solving abilities, not just "plug and chug"; exploring patterns and relationships, instead of just memorizing formulas; becoming engaged in open-ended, discovery-type problems, rather than doing routine, closed form exercises; and approaching mathematics as a live exploratory subject, not merely a description of past work. (p. 156)

Homework tasks are one means of facilitating student achievement of these goals (Schoenfeld, 1992).

Published research on homework in school mathematics through 1977 was reviewed by Austin (1979), with a concentration on studies from 1960 to 1977. Austin found no studies for grades 1, 2, 11, or 12. While attitudes toward homework wavered considerably, Austin states that comparison results tend to show that homework can significantly improve academic achievement in mathematics; homework has a cumulative effect on students' performance, particularly in subsequent grades; routine drill homework does not seem to be of much value, though homework does generally seem to improve computational skill; there is little information on the length of homework assignments; and not every homework problem needs to be graded although comments on homework papers can improve student achievement. Suydam (1985) concurs with Austin's conclusions and notes further that students' completion of homework assignments may be useful since many studies seem to indicate that mathematics achievement is higher when students are given homework than when they are not, and no studies have shown that homework has a negative effect on students' achievements in mathematics.

Schoen and Kreye (1974) investigated the effects of five forms of written feedback to homework related to specificity and personalization in an elementary mathematics concepts course. The study involved 147 prospective elementary school teachers enrolled in two large sections of the course at a large state university. The forms of written feedback were:

1. Stating specifically why a student's answer was incorrect and then giving the correct response combined with the use of the student's first name in some of the feedback.

2. Stating simply that the student had given an incorrect answer and then giving the correct answer combined with the use of the student's first name in some of the feedback.

3. Stating specifically why a student's answer was incorrect and then giving the correct response but never using the student's first name.

4. Stating simply that the student had given an incorrect answer and then giving the correct answer but never using the student's first name.

5. Just marking each incorrect response with an "X".

The homework assignments were required to be handed in but were not graded. The investigators found that the feedback variations did not appear to cause any significant differences either in students' attitudes towards mathematics nor their achievement scores. They also found that there was a significant difference, however, in retention scores that favored the feedback specific to the student's error.

Wiebe's (1982) research supports the assertion that homework and tests can improve students' achievement in college mathematics courses. Weibe's study involved six classes of a mathematics content course designed for elementary education majors at a large state university. Two treatments were studied: (1) Teacher-imposed study and attendance, and (2) Student self-determined study and attendance. In the first treatment the final grades were derived from examination scores, homework that had been collected and graded, unannounced quizzes given at least once a week, and attendance. In the second treatment, final grades were based only on the examinations, although homework had been assigned but not collected and the unannounced quizzes from the treatment above had been distributed in class as study guides whose answers had been discussed in class. No statistically significant difference in student achievement was found between the two treatments, nor was any statistically significant difference found in student attitude toward mathematics.

Although there is a broad body of research on homework in pre-college mathematics, there appears to be minimal research reported on the effectiveness of

homework at the collegiate level. For best retention by students, the research literature supports the practice of instructors responding to homework errors with comments specific to the errors.

Writing Assignments

Student writing has received increased emphasis in mathematics instruction, particularly in the calculus reform movement (Student Assessment in Calculus, 1997). "Yet the precise contributions of writing to mathematical understanding, and ways to enhance those contributions, are not well understood" (p. 31). Although learning is variable and complex (Weinert, 1987) and thus difficult to study, Smith (Calculus: The Dynamics of Change, 1996) states that writing is the cleanest window instructors have on their students' minds.

Nahrgang and Petersen (1986) described their use of writing assignments in college mathematics courses at a large state university. They used journals, which they found allowed students to proceed at their own rates and to use their own experiences to develop an understanding of mathematical concepts. The authors' research indicates that journals provided instructors with a diagnostic tool since students' writings revealed areas of confusion and misunderstanding of mathematical concepts. To facilitate this diagnosing of students' misunderstandings, Nahrgang and Petersen encouraged their students to answer questions with words and sentences rather than with equations. The authors' findings demonstrate that, when writing, students had to relate information from the lecture to what they already knew, and then had to organize and synthesize that information so that concepts became their own.

Beidleman, Jones, and Wells (1995) reported on their evaluation of a variety of writing assignments in a "traditional" first-semester college calculus

course at a large state university. The authors noted that the quality of the students' writing assignments improved as the semester progressed. Also, the students stated on their mid-term and end-of-term evaluations that the writing assignments helped them gain a conceptual understanding of calculus rather than just memorizing formulas. The authors also observed that students needed time to ponder an assignment before completing it for submission.

Watson (1980) recounted the use of journal writing in a junior high school mathematics class. At times the author provided the students with lead sentences such as "This is how to ...," "The problem I am having with ...," and "My feelings about ..." (p. 519). Other times the students were instructed to write a paragraph on a topic of their choice. In these instances, almost all the students wrote a page or more about mathematics. Watson's results suggest that the students "seem to have looked inside themselves and to have seen what they could do to help solve their mathematical problems. Many of their grades improved" (pp. 518-519).

Although the exact benefit of writing to mathematical understanding and the most effective means to promote that benefit are not yet fully understood, writing does provide insight into students' minds that is not available by many other methods of inquiry. Students disclose areas of confusion and misunderstanding as they attempt to combine and consolidate information. Hence, writing can be a useful diagnostic tool of students' thinking.

Reflective Tasks

Students' development of mathematical strategies and retrieval from memory of number facts are usually thought of as automatic, non-reflective skills (Carr, Alexander & Folds-Bennett, 1994) yet students must possess knowledge of "how to do" as well as "why" when learning mathematics (Kaur & Sharon, 1994). The

"why" knowledge enhances students' adaptability and flexibility to complete new mathematical tasks (Kaur & Sharon, 1994). Often students are able to apply rules without being able to consider whether it is legitimate to do so or not.

Waller (1994) conducted a college-level study on the promotion of metacognition in engineering courses through the use of reflective tasks. Weekly reflective writing assignments were part of the point system for the course grade. The content of the writing was not graded, only the completion of all the assignments.

> The most important difference that I have seen between students who are merely excellent at pattern matching — that is, at solving problems by matching them to analogous problems solved in class, in the book, in old exams, etc. — and those who can apply concepts with originality to new situations and unfamiliar problems is metacognition. (p. 736)

In particular, Waller instructed her students to ask themselves questions such as "What are the three most important characteristics of the material being studied?", "What skills do I need in this instance?", and "When [I] solve these types of problems, which parts of the problem solving process do [I] do well? Not so well?" (pp. 736 and 737). Waller's results suggest that successful students are aware of themselves as problem solvers, know their strengths and weaknesses, are able to articulate (and often provide for themselves) the conditions under which they learn best. Student feedback to Waller's reflective tasks indicate that students appreciate their instructor taking an interest in the learning skills as well as whether they were learning the content of the course. "The majority of the students begin to realize that they need to become reflective, life-long learners" (p. 737).

Many studies of reflection that dealt specifically with concept mappings were found during the literature review. Novak (1996) originated concept mapping in 1972 in connection with a research program that needed an instrument to represent knowledge structures of students in a first- through twelfth-grade longitudinal

study. Students were interviewed periodically to monitor changes in their conceptual understanding.

> It is not uncommon for children to be high performers in routine classroom activities and yet derive little or no meaning from the instruction. They can perform very well on typical classroom tests or assignments, but gain little or no understanding of basic concepts and principles. Concept maps can serve as a diagnostic tool to identify students who are suffering from a pattern of rote-mode learning. (p. 35)

Novak's studies indicate that if students are provided with ten to twenty concepts on a given topic to map, they must evaluate the concepts to determine which are the most important concepts, which are the subordinate concepts, and how the concepts can be linked appropriately to describe the concept interrelationships. This necessitates successive efforts by the students at synthesis and evaluation, as well as knowledge of the concepts. When students are then asked to add several more concepts to their concept maps, the challenge of recall, synthesism, and evaluation is strengthened further.

Hasemann and Mansfield (1995) reported on the use of concept maps in two studies. The first study involved 25 fourth- and 26 sixth-grade students in Germany, who were interviewed twice during the school year regarding their classification of and relationships between concepts involved in word problems dealing with fractions. Four characteristics for evaluating concept maps from a holistic point of view were discussed by the authors, whereby they rated maps according to (1) the degree with which concepts and figures from the context were grouped together; (2) the degree with which concepts and figures from the mathematical content of the problem were grouped together; (3) the number of relationships, regardless of whether these relationships were meaningful or not to an expert; (4) whether there were references to something to do.

Hasemann and Mansfield (1995) also used concept maps in a long-term study

to investigate eighth-grade Australian students' understanding of geometry. The maps were used to monitor changes in students' conceptual frameworks and provided information about students' conceptual understanding that would not have been obtained using alternate methods. Hasemann's and Mansfield's results indicate that concept maps are an effective instrument to determine "whether a student prefers to construct mental models of a problem using mathematical terms and structures or whether his or her thinking is guided mainly by the situations in which the mathematics occurs" (p. 69). The researchers noted that concept maps are idiosyncratic since the maps depend on the student's prior learning experiences and reflections on those experiences. As does Moreira's (1979) studies, Hasemann's and Mansfield's work demonstrates there is no single correct or ideal concept map.

Park and Travers (1996) studied differences in the conceptual understanding of second-semester students enrolled in "traditional" and in "reform" (Calculus & Mathematica, [C&M]) calculus. The researchers used achievement tests, attitude surveys, and concept mappings in their studies. In analyzing the achievement tests and attitude surveys, analysis of covariance was used as the students may have had inherent differences in achievement and attitude that might have led to a bias in the researchers' analysis. For the achievement and attitude data, pre-tests were used as the covariates — on prior knowledge of mathematics for the achievement data and on prior attitudes towards mathematics for the attitude data. Analysis of the concept maps revealed that the C&M calculus students' scores were generally higher than those of the "traditional" calculus students' scores. C&M calculus students tended to display more concepts in their concept maps and to list relevant interrelationships between concepts while the "traditional" calculus students tended to draw relatively simplistic unrelated views of calculus. Park and Travers found a strong positive correlation between the use of cross links and achievement test scores, regardless of

type of calculus in which students were enrolled, suggesting that students who did well in linking concepts in different areas of calculus generally did well on achievement tests. Their analysis of attitude scores found that the C&M students were more positive towards mathematics as a process, towards computers, and towards cooperative learning than "traditional" students.

Park's and Travers' conclusions regarding cross links and achievement test scores support those of Carpenter (1985), who studied expert and novice problem solvers. Carpenter found that expert problem solvers tend to organize their knowledge in large related chunks on the basis of fundamental mathematical properties while novices tend to store their knowledge in more isolated bits or sort it on the basis of superficial characteristics that have no mathematical significance.

Another college-level concept mapping study is that of Bartels (1995), who studied concept maps drawn by 19 students regarding five topics in an elementary mathematics methods course. Concept maps were used as a research tool and as an instructional tool. Bartels concludes that four themes affected the connections students made: (1) compatibility of new knowledge with their prior knowledge, (2) explicitness and frequency of the connection between new and prior knowledge, (3) consistency in students' terminology and presentation, and (4) recency of the connection between new and prior knowledge. Students' understanding of what it means to make mathematical connections increased during the study to include a greater variety of connections. Bartels' findings indicate that the simplest method for using concept maps is to furnish students with a list of concepts and an incomplete map. A more difficult method for using concept maps is to give students a list of concepts but no map. Finally, the most challenging method of concept mapping is to have the students themselves determine the key concepts and construct a map using their concepts.

Bartels' results support those of Williams (1995), whose results found that concept mappings are beneficial in determining the organization and structure of a student's knowledge base as well as the fluency and efficiency with which the student employs that knowledge. Within studies using concept mapping, Williams notes, those in mathematics are conspicuous by their absence.

Merrill (1987) used concept maps to study the understanding of division by preservice elementary teachers. Merrill's research shows that concept maps have an advantage over more traditional outlining because of their added dimension of allowing the student to show the relationships that can exist between concepts of approximately equal generality. The student must think in multiple directions at the same time, a task not easily accomplished unless the student has a deep understanding of the concepts being mapped. In Merrill's study, students using concept maps scored higher on novel problem solving tests, which measured higher-level cognitive processes, than did students taught to organize information using more traditional outlining methods. To Merrill, the major concern in evaluating concept maps should be the connections that students draw and the linking words they place between concepts since these facets of the map indicate whether the student has an understanding of the concepts depicted in the map.

On the basis of research reported to date, reflective tasks can assist students in acquiring "why" knowledge about mathematical concepts and in learning about their strengths and weaknesses as problem solvers. This knowledge improves students' adaptability and flexibility of thinking when they are faced with unfamiliar tasks. Concept maps have been shown to be an effective means of detailing conceptual understanding as they require students to evaluate interrelationships among concepts. And finally, student ability to recognize cross links in mathematics has been positively correlated to achievement.

Summary

Several issues regarding reflective thinking in mathematics were revealed in the review of the research literature. Additional research is needed on the effect of routinely assigned reflective tasks, particularly those requiring students to make a written response. Also needed is research into the effect of the timing of reflective tasks in homework assignments, the effect of various types of reflective tasks and, more generally, how the use of such tasks relates to the process of developing metacognition in college-level students.

CHAPTER III

RESEARCH DESIGN AND METHODOLOGY

Introduction

This study focuses on (a) the fostering of reflective thinking in first-semester reform calculus students through their completion of homework assignments which included reflective tasks, and (b) the effect of these assignments on student understandings and conceptions of mathematics. This study has several components: quantitative analyses of student performance on four examinations and changes in conceptions of mathematics between the beginning and end of the semester, quantitative and descriptive analyses of student concept mappings, and a qualitative analysis of student use of reflective thinking in problem solving.

Design

Pilot Study

In the spring of 1997, two pilot studies were conducted to determine students' responses to the format of the reflective tasks. One pilot study was performed in a Winter semester Calculus I class at a large midwestern university while the other study was performed in a Spring quarter Calculus II class at a small midwestern liberal arts college. The textbook used in Calculus I was Calculus from Graphical, Numerical, and Symbolic Points of View by Ostebee and Zorn (1997), while the textbook used in Calculus II was Calculus by Hughes-Hallett, Gleason, et al. (1994).

In addition, two students in the liberal arts college pilot study participated in "think aloud" problem sessions several weeks into their quarter. The students were given three tasks modeled on problems in their textbook and were asked to "think aloud", that is, to describe the inner conversation they held with themselves as they worked the problems. The problem sessions were audiotaped, transcribed, and then analyzed by the investigator using a time-line graph similar to that developed by Schoenfeld (1992). While Schoenfeld categorized participants' "thinking aloud" into Read, Analyze, Explore, Plan, Implement, and Verify, the investigator in the pilot study categorized student "thinking aloud" into Read, Direction of Thinking, Compare/Contrast, Analysis of Obstacles, Connection with Other Concepts, and Explanation of Method. These categories were chosen to mirror the type of reflective tasks assigned the students. As a result of the analysis of the problem sessions, the investigator included in the time-line graphs a further category of Explanation of Calculations.

The reflective tasks assigned as homework in the pilot study revealed that students had varying levels of understanding of calculus concepts. Several reflective tasks were reworded to decrease the likelihood of misinterpretation by students. Some students in both pilots did not take the reflective tasks seriously (and so either did not complete the tasks or did not complete the tasks with an honest effort). The investigator determined that the tasks in the research study needed to count as part of the homework grade for the course so that the students would deem the tasks important to complete.

Several additional difficulties were encountered in the liberal arts college pilot study. First, some students worked together on an assignment, submitting identical papers for that assignment. Secondly, two of the four volunteers neglected to appear for their audiotaped problem sessions, alerting the investigator that

students needed to be reminded to appear for their problem sessions and needed to value the problem sessions. As a result, the investigator decided to compensate the volunteers for their participation, with the final problem session having the largest compensation so students would be enticed to complete all three sessions.

In summary, the pilot studies informed the present study in the following ways: (a) the addition of the category Explanation of Calculations to the time-line graph analysis, (b) the rewording several tasks to avoid misinterpretation by students, (c) instructors counting the reflective task assignments as part of student homework grade for the course, and (d) the compensation of volunteers for the "think aloud" problem sessions.

Description of Subjects

The subjects for this research reported here were students who self-selected into two sections of first-semester calculus at a large midwestern university during the Fall 1997 semester. Each section was taught by a calculus-experienced instructor, one of whom was the investigator. The investigator's section was the treatment section, the other section was the control section. Each section consisted originally of approximately forty students. Thirty-eight students in the treatment section and twenty-five students in the control section chose to participate in the research by signing consent forms. Of those who chose to participate, twenty-five students in the treatment section and eighteen students in the control section actually completed the course, defined as the completion of the four course examinations (Exam 1, Exam 2, Exam 3, and the Final examination).

The treatment and the control sections met at 8 am for 50-minute periods four days each week. (No class was held on Wednesdays.) Each section used Calculus

from Graphical, Numerical, and Symbolic Points of View by Ostebee and Zorn (1997) as their textbook, and followed the same traditional syllabus in covering the material. The instructors presented the in-class material in a lecture/discussion format from notes previously written by the instructor of the control section, and assigned the same daily homework problems from the textbook. Whereas the two sections had different instructors, every effort was made to otherwise ensure that the major difference between the two sections was the nature of the assigned homeword. Reflective tasks were incorporated in the homework assigned to the treatment section.

Students submitted solutions to assigned homework problems weekly, which were graded as part of their course grade. Students were encouraged to work with partners on these weekly assignments and to submit one set of solutions per partners. If they did, the partners received the same grade on their solution set. The instructors jointly wrote the four course examinations, administered them in class at the same times, and collaborated on the grading of the examinations. The four examinations are reproduced in Appendix C. In each section, students' course grades were determined using the weighting: Graded homework — 40 %; Exam 1, Exam 2, and Exam 3 — 10 % each; and the Final examination — 30 %.

Treatment Section

Reflective tasks were incorporated into homework assigned to students in the treatment section as part of the weekly homework assignments while others reflective tasks were stand-alone assignments. In total, twenty-three reflective tasks were assigned as homework — seventeen as part of weekly homework assignments and six as stand-alone assignments. If reflective tasks were to foster

reflective thinking, students needed time to think over, or ponder, their responses to reflective tasks. Therefore, students were given one week to respond to reflective tasks included in weekly homework assignments, and could respond in conjunction with their homework partner. Students were given at least five days to complete stand-alone reflective tasks. Stand-alone reflective tasks differed from weekly reflective tasks in several aspects. Compared with reflective tasks assigned as part of the weekly homework assignments, stand-alone assignments were often longer to state and worth more homework points; required students to submit individual responses; and involved concepts from several sections in the textbook. See Table 1 for samples of weekly reflective tasks by selected textbook chapters and Figure 1 for an example of a stand-alone reflective task. Refer to Appendix B for a complete listing of weekly and stand-alone reflective tasks.

The investigator attempted to keep the weekly homework assignment workload for the treatment section close to that of the control section by the selective replacement, when necessary, of a few of the homework problems in the assignment given in the control section with reflective tasks in the assignment given in the treatment section. These weekly assignments were lessened in number of problems assigned from the textbook so as not to introduce a time-devoted-to-task variable into the study, which would confound the results. Students in the treatment section were informally monitored through conversations with the investigator as to the amount of time spent completing homework assignments, including the amount of time spent on stand-alone reflective tasks and on reflective tasks incorporated into weekly homework assignments.

Of the twenty-five students in the treatment section who actually completed the course, nine of these students failed to submit at least one reflective task assignment. Of the twenty-three reflective tasks assigned as homework, three

Table 1

Sample Reflective Tasks From Selected Chapters

Chapter	Task
Chapter 1 Functions in Calculus	Draw an example of a graph that is not the graph of a function. Explain your reasoning.
Chapter 4 Applications of the Derivative	What are the similarities between exponential growth problems and their solutions and exponential decay problems and their solutions? What are the differences?
Chapter 5 The Integral	Look at your work on problem 10, § 5.3. What topics were present in this problem that were covered in this section? In this chapter?

Math 122 -- Fall 1997
Writing Assignment # 2 -- 15 points

Date Due: Monday, October 6th at the beginning of the class period.

Assignment:

> Look back on your test paper for Exam 1. What do you now see as the major misconception you had concerning specific test material? When did you realize it was a misconception? When was the misconception created, how was the misconception created, and what caused you to correct the misconception?

Figure 1. Example of a Stand-Alone Reflective Task.

students failed to submit only one assignment while six students failed to submit between two and four of them. All six students who failed to submit more than one reflective homework assignment also failed the course. The nine students who failed to submit at least one reflective task assignment were still included in the study as the investigator felt that students choosing not to submit assignments was a standard occurrence in college classes and so reflected typical classroom circumstances.

Assessment of Outcomes

Since the purposes of this study were to examine the fostering of reflective thinking in first-semester reform calculus students through their completion of homework assignments which included reflective tasks and the effect of these assignments on student understandings and conceptions of mathematics, both quantitative and qualitative information was collected in this study. The quantitative information was intended to provide measurements at various points throughout the semester of differences in student understanding of calculus concepts between those who had completed homework assignments which included reflective tasks and those who had not. The qualitative information was intended to provide details on the nature of any such differences that was not available from analysis of student performance on examinations.

Quantitative Measures

Achievement Pretest

During the second class meeting, the Mathematical Association of America [MAA] Calculator-based Calculus Readiness Test was administered as an achievement pretest to students in the treatment and control sections. Any student in either

section not present during the second class meeting completed the pretest as soon as possible outside of class. The pretest consisted of twenty multiple-choice questions and was designed by the MAA's Committee on Testing to measure a student's potential success in calculus. As recommended in the test administration guide, students were given thirty minutes to complete the pretest. The results of the pretest were not reported to the students.

Students had been told during the first class meeting that the pretest would be administered during the second class meeting and that they should not study for the pretest as the results would not be part of their grade but would provide the instructor with information as to individuals' preparedness for calculus. The reliability coefficient of the pretest was 66 % (Garrett & Woodworth, 1966, p. 341). The investigator had no reason to doubt that students in both sections took the pretest seriously and answered the questions to the best of their abilities.

In-class and Final Examinations

Four examinations, including a comprehensive Final examination, were administered in-class during the semester. Each of the first three examinations were 50-minute examinations. Exam 1 occurred during the fourth week of classes and covered material on Chapters 1 and 2 from the textbook on precalculus topics and on an intuitive introduction to the derivative. Exam 2 occurred during the eighth week of classes, just prior to the last official day to drop the course without receiving a grade, and covered material from Chapters 2 and 3 on the concept of the derivative and on formulae for derivatives of elementary functions. Exam 3 occurred during the thirteenth week of classes and covered material from Chapters 3 and 4 on rules for, and applications of, derivatives. The Final examination was

administered during the final examination period as scheduled by the university, occurred approximately three weeks after Exam 3, and was a comprehensive two-hour examination of material in Chapters 1 through 5.

Following each examination, the two instructors met to develop a common scoring rubric. The instructors established guidelines of how to distribute the points on each problem, which included specifying points for the correct start-up of the problem and for partial credit by error type. Both instructors graded the examinations from their own section, but, for each of the examinations, the control section instructor also graded copies of test papers from the treatment section. The control section instructor's scores on each question of each test were compared to those of the treatment section instructor to establish inter-rater reliability. The instructors agreed on 61 of 78 questions mutually graded for an inter-rater reliability of 78 %.

Concept Mappings

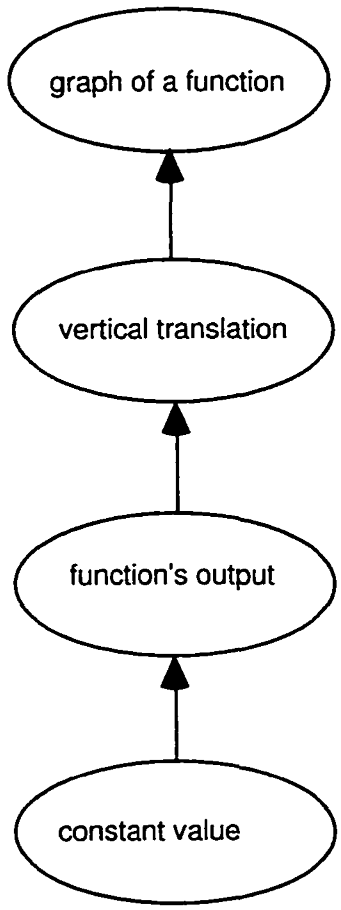
Students in the treatment section were assigned two concept mapping reflective tasks during the semester. See Figure 2 for the first assignment. See p. 131 in Appendix B for the second assignment. The first task was due the second week of the semester, the second task was due the fifth week of the semester. The control section instructor and the investigator also completed the concept mapping tasks separately, in order to form a combined final concept map for each task. The two instructors' concept maps were virtually identical. Each of the students' concept maps then was scored as to its correspondence with the instructors' combined concept map.

Concept Mapping Assignment #1 -- 10 points

Due: Friday, September 12th at the beginning of class

Tasks:

1) Connect the concepts in the ovals below to make a suitable sentence when read in the direction of the arrows.



2) Add the following two sets of concepts (and any necessary arrows and words) to the above concept map to make two other suitable sentences. If it is not possible to add a set of concepts, explain your thinking. You may reorganize the map and/or concepts as you need.

<u>Set 1</u>: horizontal translation
function's input

<u>Set 2</u>: vertical stretching and/or reflection

Figure 2. First Concept Mapping Assignment.

The instructors' and the students' concept maps were scored using the rubric described by Park and Travers (1996), which is based on five criteria: meaningful propositions, valid hierarchies, significant cross links, additional meaningful concepts, and misconceptions. Maps earned two points for each meaningful proposition, five points for each valid hierarchy level or significant cross link, and three points for each additional meaningful concept included in the map. One, three, or five points were deducted from the score for each misconception, depending on the extent of the misconception (see Figure 13 on p. 75). The students' final score on each assignment was then computed based on the ratio of their score to the instructors' score, with a maximum of 1.0.

The student-to-instructor ratios on the two concept mapping assignments were examined as to their relationship with the students' pretest achievement scores. If the reflective tasks were effective in fostering students' use of reflective thinking and increasing student understandings of first-semester calculus, the ratios on the second concept mapping task should show an increase in the student-to-instructor ratio when compared to the ratios of the first concept mapping task. The investigator examined the ratios for emerging patterns in support of completion of reflective tasks resulted in increased student understandings of calculus.

Five of the twenty-five students participating in the research declined to submit the first concept mapping assignment, receiving zeros for that assignment. Three additional students declined to submit the second concept mapping assignment, receiving zeros for that assignment. All eight of these students were dropped from the concept mapping portion of the research.

Conceptions of Mathematics Inventory

During the second class meeting, students in both sections also completed a survey concerning their conceptions about mathematics. As with the achievement pretest, any student in either section absent from the second class meeting completed the survey as soon as possible outside of class. The items in the survey were selected from the Conceptions of Mathematics Inventory [CMI] developed by Grouws, Howald, and Colangelo and described in Grouws, Howald, and Colangelo (1996). The first eight items concerned students' conceptions of the Structure of Mathematical Knowledge, the next eight of Doing Mathematics, and the final eight of Learning Mathematics. A sample item from each section of this inventory may be found in Figure 3.

Structure of Mathematical Knowledge

There is little in common between the different mathematical topics you have studied, like functions and graphs.

Doing Mathematics

Knowing why an answer is correct in mathematics is as important as getting a correct answer.

Learning Mathematics

Memorizing formulas and steps is not that helpful for learning how to solve mathematics problems.

Figure 3. Sample Items from the Conceptions of Mathematics Inventory.

Students had approximately 15 minutes to respond to the 24 items. Responses were based on a six-part Likert scale that ranged from 1 = Strongly Disagree to 6 = Strongly Agree. Twelve of the items were negatively worded. The scoring of the attitude survey is given in Table 2. For each student, a maximum total score on the CMI of 144 points was possible.

Table 2

Scoring of the Conceptions of Mathematics Inventory

Response	Positive Item	Negative Item
Strongly Agree	6	1
Agree	5	2
Slightly Agree	4	3
Slightly Disagree	3	4
Disagree	2	5
Strongly Disagree	1	6

At the end of the semester, students completed the same survey again in class. Three of the twenty-five students in the treatment section and two of the eighteen students in the control section failed to complete the end-of-semester survey. The scores of these five students were not included in the CMI portion of the research.

Qualitative Measures

Individual interviews with nine students from the treatment section were conducted by the investigator to provide further detail on students' use of reflective thinking during problem solving that was not available from the results of the in-class examinations. Each student was interviewed on three occasions, with

interviews occurring as soon as possible following each of the three in-class examinations. The interviews were audio taped "think aloud" problem sessions (Schoenfeld, 1992) and took place outside of class. Students were paid for their participation, receiving $ 7 for the first two interviews and $ 15 for the final interview.

Selection of Students

Three sets of students from the treatment section were individually asked to voluntarily participate in the "think aloud" problem sessions. Each set consisted of three students and was based on their individual scores on the achievement pretest and Exam 1. The first set was chosen from among those students who performed well (upper 30 %) on the two tests, the second set chosen from among those who performed average (middle 40 %), and the final set chosen from among those who performed poorly (lower 30 %). The investigator made it clear to the selected students that their participation or nonparticipation in the interviews was disjoint from the investigator's treatment of them during class. None of the students declined to participate in the interviews though two of the students failed to complete the course, dropping the class after the second examination, which dropped them at that time from the interviews. Two of the nine selected students were female.

The Interview Sessions

Each interview session was a "think aloud" problem session in which students were instructed to "continually think aloud" as they worked on tasks. The tasks (see Appendix D) were based on material covered on the examination that had occurred in class just prior to the interview. The investigator discussed with the students at the

first interview that the purpose of the interviews was to determine the nature of student thinking while problem solving in calculus and was not to judge the students on their problem-solving skills. They were informed that they might be challenged by some of the tasks so that their thinking could be studied but that the investigator was focusing on the thinking they expressed as they worked the tasks.

The first interview session dealt with functions and their slopes, the second session with formulae for finding derivatives of elementary functions and interpretations of those derivatives, and the third session with applications of derivatives. To encourage their verbalization, students were asked to first read the tasks aloud. Following any extended periods of silence (defined as lasting more than two seconds), the investigator prompted students to express their thinking.

Transcripts of the interview sessions were analyzed for student use of reflective thinking. The interviews were evaluated using a time-line graph technique as to their mirroring of the various types of reflective tasks that were assigned during the semester. It was hypothesized that if the homework tasks were effective in fostering students' reflective thinking, each successive interview with a particular student should show an increase in the use of the categories of Direction of Thinking, Compare/Contrast, Analysis of Obstacles, Connection with Other Concepts (Methods), and Explanation of Method.

Null Hypotheses

Examination Scores

The four sets of examination scores (Exam 1, Exam 2, Exam 3, and the Final examination) were analyzed on the basis of the pretest achievement score. The four null hypotheses tested were:

Hypothesis I. There is no statistically significant difference in the adjusted mean Exam 1 scores between the treatment and control sections.

Hypothesis II. There is no statistically significant difference in the adjusted mean Exam 2 scores between the treatment and control sections.

Hypothesis III. There is no statistically significant difference in the adjusted mean Exam 3 scores between the treatment and control sections.

Hypothesis IV. There is no statistically significant difference in the adjusted mean Final examination scores between the treatment and control sections.

To test these hypotheses, the scatterplots of the pretest achievement scores versus each set of examination scores were first examined for evidence of linear relationships. In addition, linearity was investigated by attempting to fit quadratic curves to the sets of data. Statistical rejection of a quadratic relationship was interpreted as strengthening the possibility of a linear relationship between each set of examination scores and the pretest achievement scores.

Next, the two sections' regression lines for the pretest achievement scores versus each set of examination scores were examined for equal slopes through regression with the use of a Treatment Indicator (0 = control section and 1= treatment section) and a two-tailed t-test. If the possibility of equal slopes was rejected for an examination ($p < .05$), the regression lines for the two sections for that examination were not parallel. The regression lines for that examination were examined, however, to determine the nature of their intersection.

If the possibility of equal slopes was not rejected ($p \geq .05$) for an examination, then conditions for the use of analysis of covariance (ANCOVA) had been met. ANCOVA was then used on that set of examination scores with the pretest achievement scores as covariate to determine possible treatment effects. If the null

hypothesis regarding adjusted means (that is, the regression lines having equal intercepts) was rejected for an examination ($p < .05$), then the difference in the sections' adjusted means was used as evidence that students' completion of homework assignments which included reflective tasks had significantly affected their scores on that examination.

Conceptions of Mathematics Inventory

The CMI beginning-of-semester and end-of-semester scores were analyzed using a two-tailed paired t-test. The null hypothesis tested was:

Hypothesis. There is no statistically significant difference in the mean change from the beginning-of-semester to the end-of-semester CMI scores of students between the treatment and control sections.

The scores were analyzed in total and individually by item. A rejection of the null hypothesis either in total or individually by item was interpreted by the investigator as evidence that the completion of homework assignments involving reflective tasks significantly contributed to a change in student beliefs about mathematics.

CHAPTER IV

QUANTITATIVE RESULTS

Introduction

This chapter reports the results of quantitative analyses performed on four sets of examination scores (Exam 1, Exam 2, Exam 3, and the Final examination) versus pretest achievement scores and on student responses to the Conceptions of Mathematics Inventory [CMI] survey taken at the beginning and end of the semester. Analysis of the two concept mapping assignments is also included in this chapter.

Possible treatment effects are investigated for each of four examinations with pretest achievement scores as covariate. The null hypothesis for each set of examination scores is studied in several steps. Scatterplots of examination scores versus pretest achievement scores are examined for linear patterns and are tested for the existence of a quadratic (i.e., nonlinear) relationship. If linearity exists, then homogeneity of regression line slopes with pretest achievement scores as covariate is tested. If equal regression slopes exist, analysis of covariance (ANCOVA) is used to examine treatment effects. If the regression line slopes are not homogeneous, however, the regression lines for that examination are examined for the nature of the intersection.

The student-to-instructor ratios for the two concept mapping assignments are investigated for emerging patterns that should appear if completion of homework incorporating reflective tasks resulted in students' increased understanding of calculus, particularly increased understanding of connections among calculus concepts and procedures. The null hypothesis regarding change in the means of

student responses to the CMI from the beginning to the end of the semester is analyzed by performing a two-tailed t-test on end-of-semester minus beginning-of-semester scaled scores by section.

Characteristics of the Two Sections

This study used two intact section of Calculus I. To examine the initial comparibility of students in the treatment and control sections, data regarding prior mathematics achievement, gender, age, year in college, and number of previous college mathematics courses were collected and analyzed using two-sample two-tailed t-tests. The data on the pretest achievement scores are given in Table 3; the other data are given in Table 4.

Table 3

Pretest Achievement Scores by Section

Section	Pretest Achievement Scores	
	Mean	SD
Treatment (N = 25)	12.60	3.46
Control (N = 18)	11.67	2.83

At an $\alpha = .05$ level of significance, there was no significant difference between the two sections with regard to pretest achievement scores ($p = .34$), gender ($p = .26$), age ($p = .13$), year in college ($p = .19$), nor number of previous college mathematics courses ($p = .86$). Hence, the students in the two sections appeared to be similar in regards to these aspects.

As reported in Chapter III, of approximately forty students enrolled in each section at the beginning of the semester, thirty-eight students in the treatment

Table 4

Characteristics of the Sections

Section	Age		Year in College*		Previous College Math Courses	
	Mean	SD	Mean	SD	Mean	SD
Treatment	21.48	8.35	1.52	0.872	0.88	1.39
Female = 9 (36%) Male = 16 (64%)						
Control	19.50	1.98	2.00	1.08	1.50	1.54
Female = 6 (33%) Male = 12 (67%)						

(*) 1 = Freshman, 2 = Sophomore, 3 = Junior, 4 = Senior

section and twenty-five students in the control section chose to participate in the research. Of those, twenty-five students in the treatment section and eighteen students in the control section actually completed the course, defined as the completion of the four course examinations (Exam 1, Exam 2, Exam 3, and the Final examination). The university at which the study was conducted had switched from a traditional calculus textbook to the Ostebee and Zorn textbook beginning with the Fall 1996 semester. Information regarding distribution of student grades and number of students in 8 am sections of first-semester calculus are reported in Table 5.

It is noted that, in the treatment section, 24/40 = 60 % and, in the control section, 17/37 = 62 % of the students failed to successfully complete the course (i.e., received a grade lower than a C). This compared to 21/77 = 27 % and 15/35 = 43 % for other 8 am sections of first-semester reform calculus.

Table 5

Percent Distribution of Student Grades in 8 am First-Semester Calculus Sections

Term	A	BA	B	CB	C	DC	D	Failed or Withdrew
Fall 1996								
(N = 77)	13 %	11 %	13 %	14 %	22 %	3 %	5 %	19 %
Winter 1997								
(N = 35)	25 %	9 %	14 %	9 %	0 %	0 %	11 %	32 %
Fall 1997								
Treatment (N = 40)	7 %	5 %	8 %	15 %	5 %	5 %	7 %	48 %
Control (N = 37)	5 %	5 %	8 %	8 %	11 %	3 %	6 %	54 %

Achievement Pretest

During the second class meeting both sections were administered the Mathematical Association of America Calculus Readiness Test to measure students' preparation for calculus. The pretest achievement scores were used as the covariate (uncontrolled variable) in the ANCOVA performed on examination scores. Before using ANCOVA, however, two conditions were checked for each examination to determine whether ANCOVA was appropriate: (1) the existence of a linear relationship between dependent variables (the examination scores) and the covariate, and (2) the homogeneity of the slopes of the pair of regression lines for the treatment and control sections.

Exam 1

Existence of Linear Relationship

To initially investigate the linearity between Exam 1 scores and pretest achievement scores, scatterplots of these scores were examined (see Figure 4). The regression lines for Exam 1 with scatterplots by section are shown in Appendix A, Figures 22 and 23.

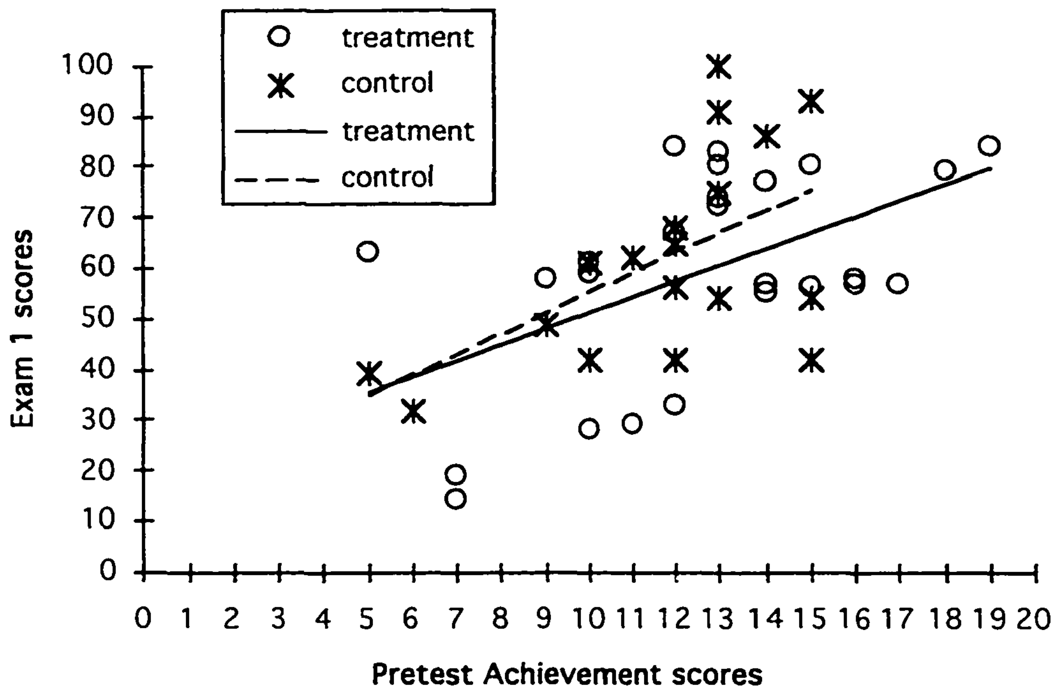


Figure 4. Scatterplot of Exam 1 and Pretest Achievement Scores With Regression Lines for Treatment and Control Sections.

While the large influence of the observations with pretest achievement scores of 5 and 6 was noted in the scatterplots of the control section, these observations did provide information about the lower end of the control section that would not have been available otherwise. No observations from either section were omitted. The

scatterplots for both treatment and control sections seemed to indicate the existence of a linear relationship between Exam 1 scores for each section and corresponding pretest achievement scores.

As an additional test of the linearity between Exam 1 scores and pretest achievement scores, a fit of Exam 1 versus pretest achievement scores using a quadratic model was investigated. The model used in the regression test for the existence of a quadratic relationship was

$$\text{Exam score} = \beta_0 + \beta_1 * \text{Pretest} + \beta_2 * (\text{Pretest})^2.$$

The coefficient of the quadratic term (the square of the pretest achievement scores) in the model was analyzed using a two-tailed t-test by testing the hypotheses

$$H_0\text{: } \beta_2 = 0 \text{ versus } H_A\text{: } \beta_2 \neq 0,$$

that is, by testing

$$H_0\text{: Linear relationship versus } H_A\text{: Quadratic relationship.}$$

The p-value was .832 for the treatment section and .661 for the control section. Using an $\alpha = .05$ level of significance, therefore, there was evidence to not reject the null hypothesis H_0: $\beta_2 = 0$ for Exam 1 scores for both sections since each of the p-values was larger than .05. This indicated that there did not exist support for a quadratic model for either treatment or control section for Exam 1 versus corresponding pretest achievement scores. Thus, based on examination of scatterplots and regression lines, and the above test results regarding quadratic models, there most likely was a linear relationship between Exam 1 and corresponding pretest achievement scores for each section.

Test for Homogeneous Slopes

The homogeneity of regression slopes was checked by creating a section treatment variable where Treatment Indicator = 0 for the control section and Treatment Indicator = 1 for the treatment section. The model used for analysis of Exam 1 versus pretest achievement scores was

$$\text{Exam score} = \beta_0 + \beta_1 \cdot \text{Pretest} + \beta_2 \cdot \text{Treatment} + \beta_3 \cdot (\text{Pretest} \cdot \text{Treatment}).$$

For the treatment section, therefore, the model amounted to

$$\text{Exam score} = (\beta_0 + \beta_2) + (\beta_1 + \beta_3) \cdot \text{Pretest},$$

while for the control section the model became

$$\text{Exam score} = \beta_0 + \beta_1 \cdot \text{Pretest}.$$

To investigate the homogeneity of regression slopes, the coefficient of the pretest achievement scores in the model was analyzed using a two-tailed t-test by testing the hypotheses

$$H_0\colon\ \beta_3 = 0 \quad \text{versus} \quad H_A\colon\ \beta_3 \neq 0$$

that is, by testing

$$H_0\colon\ \text{Control slope} = \text{Treatment slope} = \beta_1$$

$$\text{versus}$$

$$H_A\colon\ \beta_1 = \text{Control slope} \neq \text{Treatment slope} = \beta_1 + \beta_3.$$

Using an $\alpha = .05$ level of significance, there was evidence ($p = .628$) to not reject the null hypothesis H_0: $\beta_3 = 0$ for Exam 1 scores, which indicated that the

slopes of the regression lines for treatment and control sections for Exam 1 might indeed be homogeneous, that is, the regression lines for Exam 1 might be parallel. When this result was combined with the test for linearity, ANCOVA with pretest achievement scores as covariate was a legitimate analysis method for Exam 1 scores because conditions (1) and (2) listed above for use of ANCOVA were satisfied.

Results of ANCOVA Test

As stated in Chapter III, the null hypothesis addressed in the ANCOVA analysis of Exam 1 versus corresponding pretest achievement scores was:

Hypothesis I: There is no statistically significant difference in the adjusted mean Exam 1 scores between the treatment and control sections.

The ANCOVA results for Exam 1 are given in Table 6; the means, standard deviations, and adjusted means for Exam 1 are given in Table 7. The adjusted mean is the mean calculated from the ANCOVA which considers the pre-existing differences reflected in the pretest achievement scores. A boxplot of the Exam 1 scores is provided in Figure 5.

The calculated p-value of 0.312 (Table 6) for Exam 1 scores by Treatment source was greater than the $\alpha = .05$ level of significance; therefore, there was no evidence of significant difference between the two sections. Examination of the graphs of the regression lines for Exam 1 scores indicated that the regression line for the control section rose more sharply than that for the treatment section. The equations for the regression lines and their intersection point are given in Table 8. The relative position of these regression lines suggests that students in the treatment section did not do better on Exam 1 as a result of treatment when compared to students in the control section.

Table 6

Summary ANCOVA Results for Exam 1 Scores

Source	Df	Adj SS	MS	F	p
Covariate	1	4999.4	4999.4	16.52	0.000
Treatment*	1	317.5	317.5	1.05	0.312
Error	40	12106.0	302.6		
Total	42	17163.8			

(*) 0 = control section, 1 = treatment section

Table 7

Means, Standard Deviations and Adjusted Means for Exam 1

Section	Exam 1 Mean	Exam 1 SD	Exam 1 Adjusted Mean
Treatment	59.36	20.52	58.02
Control	61.72	20.29	63.58

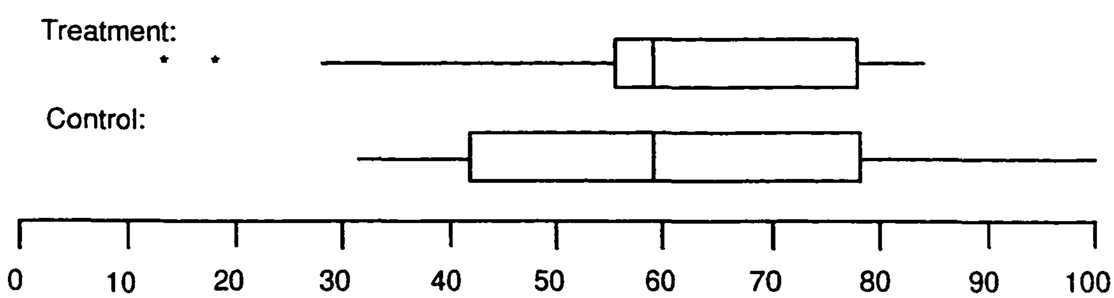


Figure 5. Boxplot of Treatment and Control Section Exam 1 Scores.

Table 8

Regression Line Equations and Their Intersection Point for Exam 1

Source	Regression Lines	Intersection Point
Exam 1 scores	$\hat{Y}_T$ = 3.148 • Pretest + 19.70	(5.724, 37.719)
	$\hat{Y}_C$ = 4.039 • Pretest + 14.60	

Exam 2

Existence of Linear Relationship

As with Exam 1 scores discussed above, to initially investigate the linearity between Exam 2 scores and pretest achievement scores, scatterplots of these scores were examined (see Figure 6). The regression lines for Exam 2 with scatterplots by section are shown in Appendix A, Figures 24 and 25. As with Exam 1, the scatterplots for both sections seemed to indicate the existence of a linear relationship between Exam 2 scores of each section and corresponding pretest achievement scores.

A fit of Exam 2 versus pretest achievement scores using a quadratic model was also investigated. The model used in the regression test for the existence of a quadratic relationship was the same as described above for Exam 1. The p-value of the treatment section was .849 and that of the control section was .950. Using an $\alpha = .05$ level of significance, therefore, there was no evidence to reject the null hypothesis H_0: $\beta_2 = 0$ for Exam 2 scores for both sections since each of the p-values was larger than .05. This indicated that there did not exist evidence to reject a linear model for either treatment or control section for Exam 2 versus corresponding pretest achievement scores. As with Exam 1, based on examination of

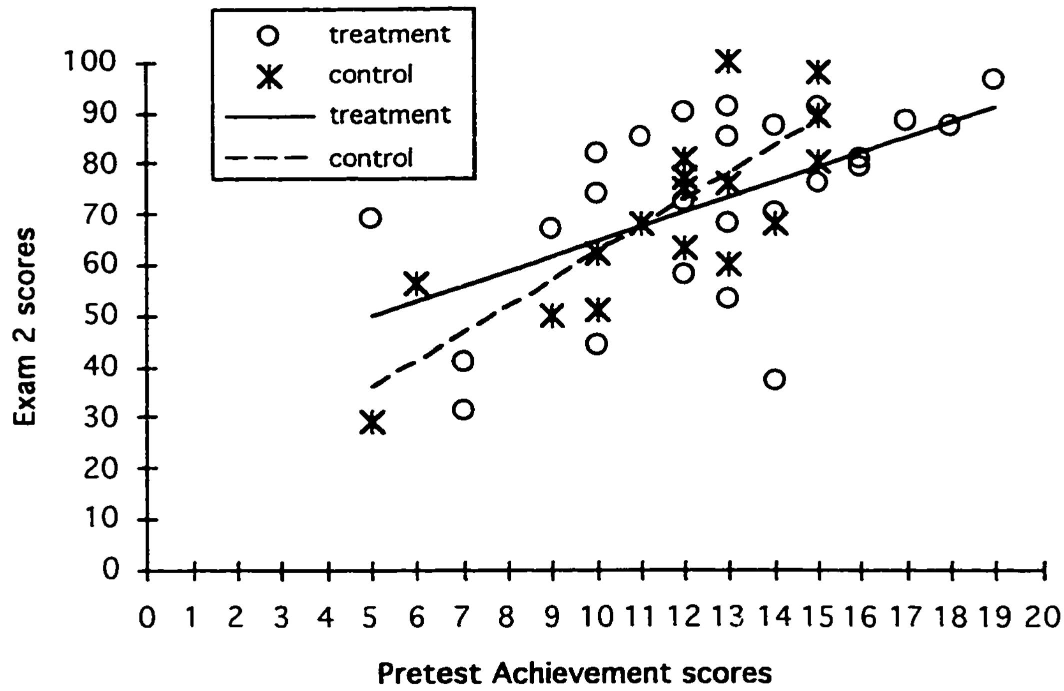


Figure 6. Scatterplot of Exam 2 and Pretest Achievement Scores With Regression Lines for Treatment and Control Sections.

scatterplots and regression lines, and the above test results regarding quadratic models, there most likely was a linear relationship between Exam 2 and corresponding pretest achievement scores for each section.

Test for Homogeneous Slopes

The homogeneity of regression slopes was checked through the use of a Treatment Indicator variable and model as discussed above for Exam 1. Using an $\alpha = .05$ level of significance, there was no evidence ($p = .126$) to reject the null hypothesis H_0: $\beta_3 = 0$ for Exam 2 scores, which indicated that the slopes of the regression lines for treatment and control sections for Exam 2 might indeed be homogeneous, that is, the regression lines for Exam 2 might be parallel. When this

result was combined with the test for linearity, ANCOVA with pretest achievement scores as covariate was a legitimate analysis method for Exam 2 scores because conditions (1) and (2) listed above for use of ANCOVA were satisfied.

Results of ANCOVA Test

The null hypothesis addressed in the ANCOVA analysis of Exam 2 versus corresponding pretest achievement scores was:

Hypothesis II: There is no statistically significant difference in the adjusted mean Exam 2 scores between the treatment and control sections.

The ANCOVA results for Exam 2 are given in Table 9; the means, standard deviations, and adjusted means for Exam 2 are given in Table 10. A boxplot of Exam 2 scores is given in Figure 7.

The calculated p-value of 0.564 (Table 9) for Exam 2 scores by Treatment source was greater than the $\alpha = .05$ level of significance; therefore, there was no evidence of significant difference between sections. Examination of the graph of the regression lines for Exam 2 scores indicated that, as with Exam 1, the control section regression line rose more sharply than that for the treatment section. The equations for the regression lines and their intersection point are given in Table 11. Students scoring lower than 11 on the pretest seemed to profit more from the calculus course which included reflective tasks in their assignments. This accounted for 7/25 = 28 % of students in the treatment section.

Table 9

Summary of ANCOVA Results for Exam 2 Scores

Source	Df	Adj SS	MS	F	p
Covariate	1	5794.7	5794.7	27.31	0.000
Treatment*	1	71.8	71.8	0.34	0.564
Error	40	8488.8	212.2		
Total	42	14290.2			

(*) 0 = control section, 1 = treatment section

Table 10

Means, Standard Deviations and Adjusted Means for Exam 2

Section	Exam 2 Mean	Exam 2 SD	Exam 2 Adjusted Mean
Treatment	72.08	18.46	70.64
Control	71.28	18.95	73.29

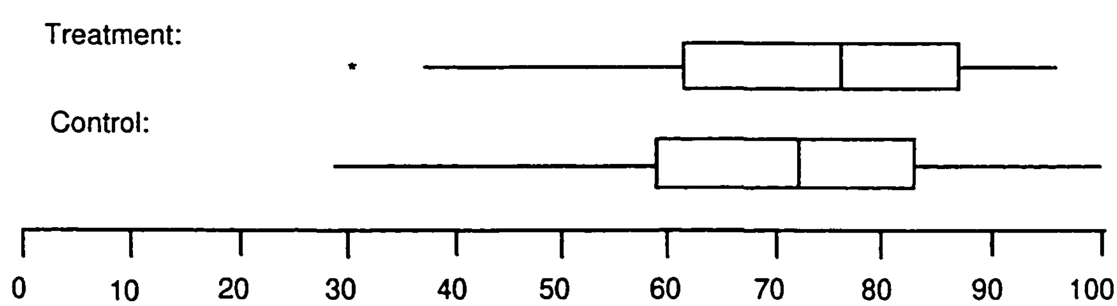


Figure 7. Boxplot of Treatment and Control Section Exam 2 Scores.

Table 11

Regression Line Equations and Their Intersection Point for Exam 2

Source	Regression Lines	Intersection Point
Exam 2 scores	$\hat{Y}_T = 2.951 \cdot \text{Pretest} + 34.90$ $\hat{Y}_C = 5.277 \cdot \text{Pretest} + 9.71$	(10.830, 66.859)

Exam 3

Existence of Linear Relationship

As with the examination scores discussed above, the linearity between Exam 3 scores and pretest achievement scores was investigated by examining scatterplots of these scores (see Figure 8). The regression lines for Exam 3 with scatterplots by section are shown in Appendix A, Figures 26 and 27. Once again, the scatterplots for both treatment and control sections seemed to indicate the existence of a linear relationship between Exam 3 scores corresponding pretest achievement scores.

A fit of Exam 3 versus pretest achievement scores using a quadratic model was investigated, also. The same model was used in the regression test for the existence of a quadratic relationship as that described above. The p-value was .962 for the treatment section, .755 for the control section. Using an $\alpha = .05$ level of significance, therefore, there was no evidence to reject the null hypothesis H_0: $\beta_2 = 0$ for Exam 3 scores for both sections since each of the p-values was larger than .05. This indicated that for Exam 3 versus corresponding pretest achievement scores, as with Exam 1 and Exam 2, support for a quadratic model did not exist for either treatment or control section. As with prior examination scores, the above test

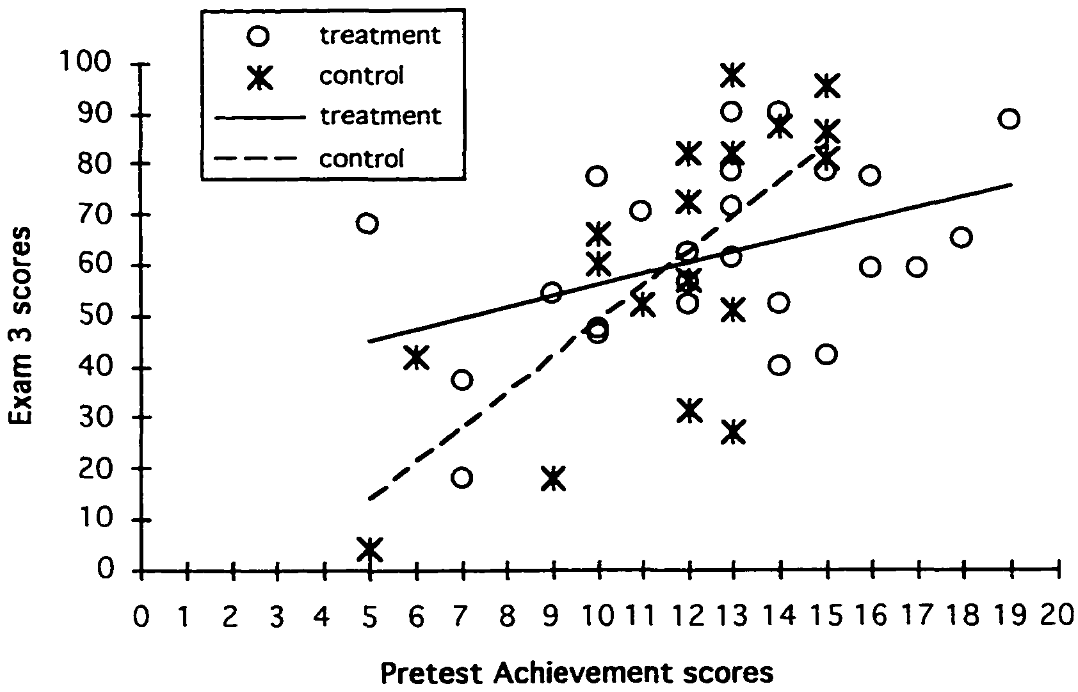


Figure 8. Scatterplot of Exam 3 and Pretest Achievement Scores With Regression Lines for Treatment and Control Sections.

results regarding quadratic models combined with examination of the scatterplots and regression lines suggested there most likely was a linear relationship between Exam 3 and corresponding pretest achievement scores for each section.

Test for Homogeneous Slopes

The homogeneity of regression slopes was checked again through the use of a Treatment Indicator variable and model discussed above. The null hypothesis H_0: $\beta_3 = 0$ for the Exam 3 scores was rejected at the $\alpha = .05$ level of significance ($p = .015$). The slopes of the regression lines for the treatment and control sections for Exam 3 were not homogeneous, that is, the regression lines for Exam 3 were not parallel. Therefore, ANCOVA with pretest achievement scores as covariate was not a

valid analysis method for Exam 3 scores because condition (2) listed above for use of ANCOVA was not satisfied. The regression lines for this examination were inspected, however, for the nature of their intersection. The means and standard deviations for Exam 3 are given in Table 12. Since ANCOVA could not be performed on Exam 3 scores, there were no adjusted means. A boxplot of Exam 3 scores is provided in Figure 9.

Table 12

Means and Standard Deviations for Exam 3

Section	Exam 3 Mean	SD
Treatment	61.48	17.84
Control	60.56	27.54

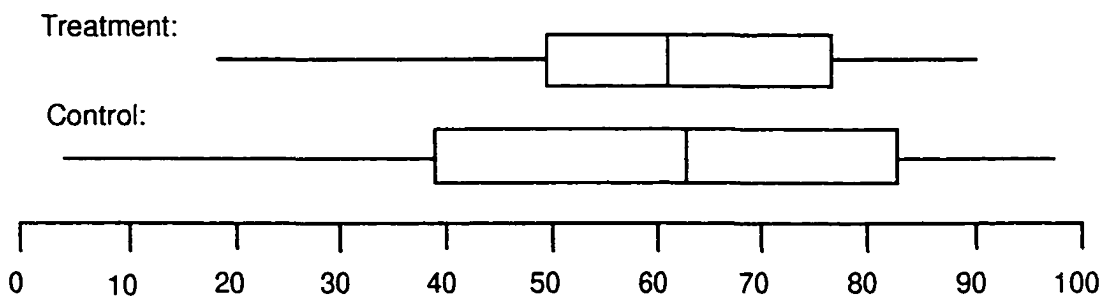


Figure 9. Boxplot of Treatment and Control Section Exam 3 Scores.

Examination of the graph of the regression lines for Exam 3 scores indicated that, as with the regression lines for Exam 2, the control section regression line rose more sharply than that for the treatment section. Students scoring lower than

12 (see Table 13) on the achievement pretest appearred to benefit more from the treatment. In the treatment section, 8/25 = 32 % of the students scored less than 12 on the pretest.

Table 13

Regression Line Equations and Their Intersection Point for Exam 3

Source	Regression Lines	Intersection Point
Exam 3 scores	$\hat{Y}_T$ = 2.166 • Pretest + 34.19	(11.436, 58.961)
	$\hat{Y}_C$ = 6.929 • Pretest - 20.28	

Final Examination

Existence of Linear Relationship

As with the examination scores discussed above, to initially investigate the linearity between Final examination scores and pretest achievement scores, scatterplots of these scores were examined (see Figure 10). The regression lines for the Final examination with scatterplots by section are shown in Appendix A, Figures 28 and 29. As with prior examination scores, the scatterplots for both treatment and control sections seemed to indicate the existence of a linear relationship between the Final examination and corresponding pretest achievement scores of each section.

As with the other examination scores, a fit of Final examination versus pretest achievement scores using a quadratic model was investigated using the model discussed for Exam 1. The p-value was .687 for treatment section and .986 for the control section. There was evidence to not reject the null hypothesis H_0: $\beta_2 = 0$

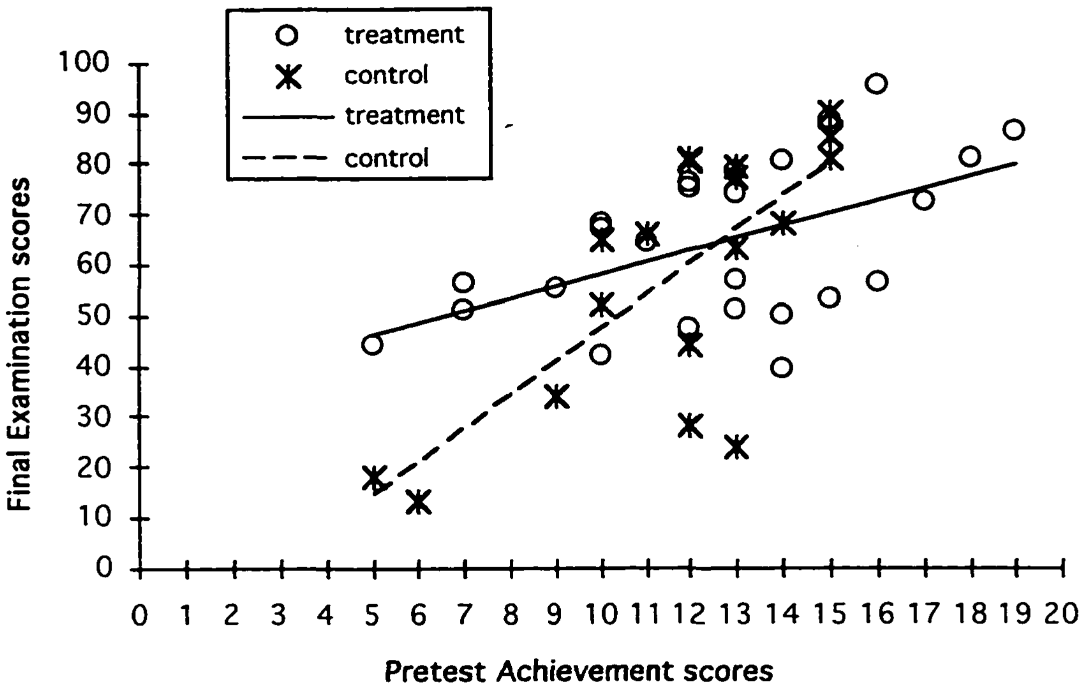


Figure 10. Scatterplot of Final Examination and Pretest Achievement Scores With Regression Lines for Treatment and Control Sections.

for Final examination scores at the α = .05 level of significance for both sections since each of the p-values was larger than .05. As a result, support for a quadratic model for either treatment or control section for Final examination versus corresponding pretest achievement scores did not exist. As with prior examination scores, there most likely was a linear relationship between Final examination and corresponding pretest achievement scores for each section, based on the scatterplots, regression lines, and the above test results regarding quadratic models.

Test for Homogeneous Slopes

Using a Treatment Indicator variable and the model discussed above, the homogeneity of regression slopes was tested. At an α = .05 level of significance,

there was enough evidence (p = .013) to reject the null hypothesis H_0: $\beta_3 = 0$ for the Final examination scores, indicating that the regression lines for the Final examination versus pretest achievement scores were not parallel. As with Exam 3 scores, ANCOVA with pretest achievement scores as covariate was not a valid analysis method for Final examination scores because condition (2) listed above for use of ANCOVA was not satisfied. The regression lines for this examination were inspected, however, for the nature of their intersection. The means and standard deviations for the Final examination scores are given in Table 14. Since ANCOVA could not be performed on the Final examination scores, there were no adjusted means. A boxplot of the Final examination scores is given in Figure 11. The means for all four examinations, including the comprehensive final examination, are graphed by section in Figure 12.

Table 14

Means and Standard Deviations for the Final Examination

Section	Final Examination Mean	SD
Treatment	64.20	15.71
Control	58.22	25.23

Examination of the graph of the regression lines for the Final examination scores indicated that, as with the previous examinations, the regression line rose more sharply for the control section than for the treatment section. For the Final examination, students scoring lower than 13 (see Table 15) on the achievement

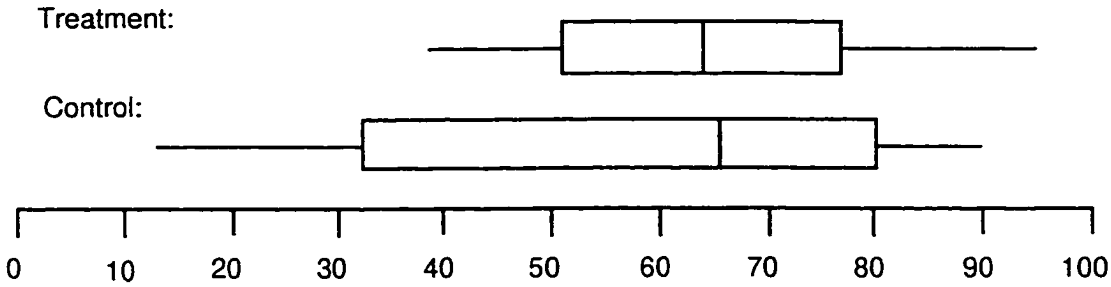


Figure 11. Boxplot of Treatment and Control Section Final Examination Scores.

Table 15

Regression Line Equations and Their Intersection Point for the Final Examination

Source	Regression Lines	Intersection Point
Final Examination Scores	$\hat{Y}_T$ = 2.375 * Pretest + 34.28 $\hat{Y}_C$ = 6.547 * Pretest - 18.15	(12.567, 64.127)

pretest seemed to profit more from homework assignments involving reflective tasks, which accounted for 11/25 = 44 % of the students in the treatment section.

Concept Maps

Only the treatment section completed the two stand-alone homework assignments that involved concept maps. The first assignment was submitted by students in the second week of the semester, the second assignment in the fifth week of the semester, just prior to Exam 1. Prior to completing any of the concept maps, students were provided examples of concept maps using every day concepts not from mathematics (see p. 129 in Appendix B). For their first concept mapping

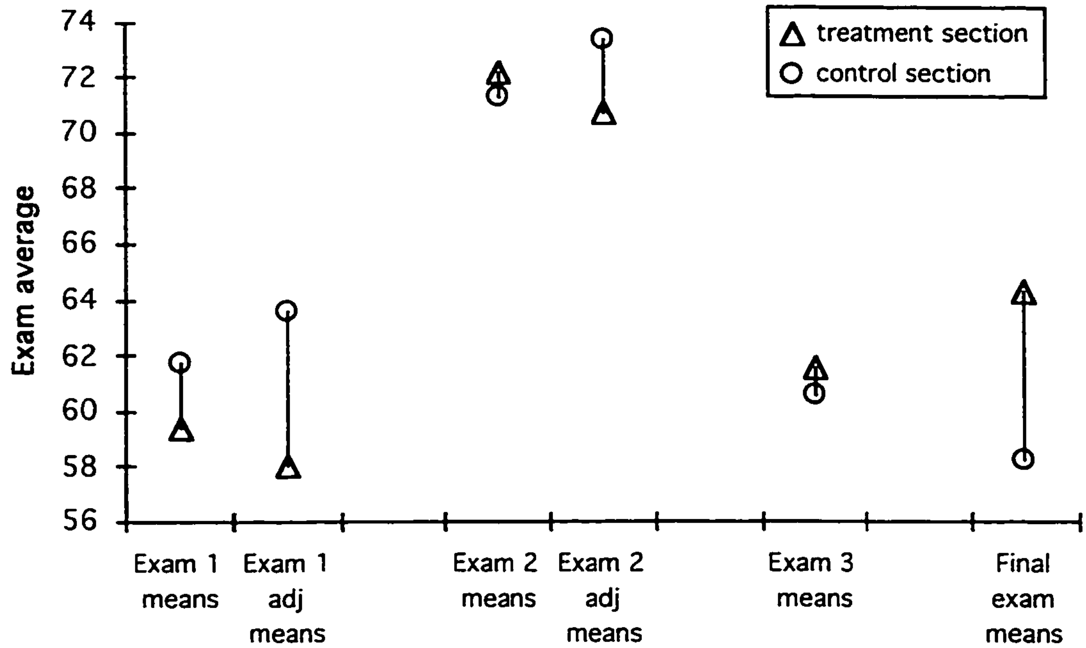


Figure 12. Means and Adjusted Means by Examination for Treatment and Control Sections.

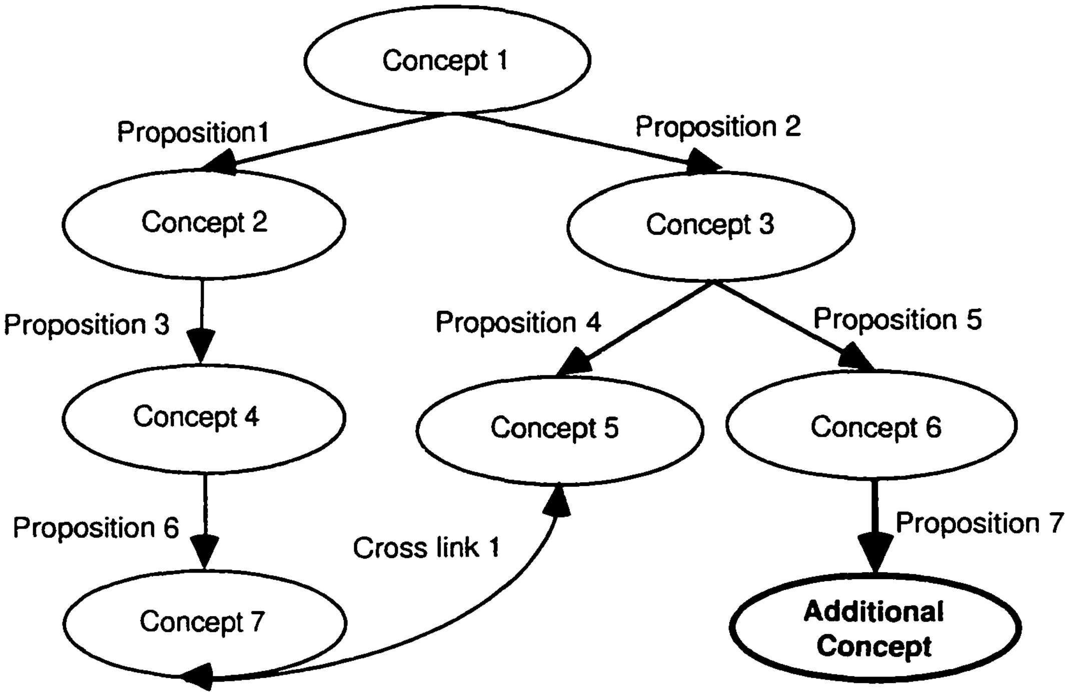
assignment (see p. 129 in Appendix B), students were required to connect concepts dealing with horizontal translations of graphs. As a further part of this assignment, students were asked to place additional given concepts dealing with vertical translations and with reflections on their maps, if possible. For their second assignment (see p. 131 in Appendix B), students received a list of concepts regarding velocity, rate of change, position, and derivative, and were asked to create their own concept maps. Seventeen students in the treatment section completed both concept mapping assignments. See Figures 13-16 for concept maps from the two assignments drawn by two students and discussed below.

<u>Analysis of Concept Maps</u>

The students' concept maps were graded according to five criteria: meaningful propositions, valid hierarchies, significant cross links, additional meaningful concepts, and misconceptions. As described in Chapter III, a student earned two points for each meaningful proposition, five points for each valid hierarchy level or significant cross link, and three points for each additional meaningful concept included in the map (see Figure 13). One, three, or five points were deducted from a student's score for each misconception, depending on the extent of the misconception.

A combined concept map was also drawn by the two instructors and scored in the same manner as those of the students'. The students' final score on each assignment was then computed based on the ratio of their score to the instructors' score, with a maximum of 1.0. A listing of student ratios for the two concept mapping assignments along with their pretest achievement scores is given in Table 16.

Several observations were noted by the investigator concerning the concept mapping ratios and pretest achievement scores. Of the six students scoring at least 14 on the achievement pretest, four achieved a 1.0 on the first concept mapping assignment and five achieved a 1.0 on the second assignment. Of these six students, only the student scoring highest grade in the section, a 19, on the achievement pretest failed to achieve a 1.0 on either assignment. Of the eleven students scoring at most 13 on the achievement pretest, only three scored a 1.0 on either assignment, with two of those students scoring a 1.0 on both assignments. Four students (numbers 1, 10, 11, and 17) expressed less organization and relations among



Meaningful propositions (2 points each)	14 points
Valid hierarchy levels (5 points each)	15 points
Significant cross links (5 points each)	5 points
Additional meaningful concepts (3 points each)	3 points
Misconceptions (1, 3, or 5 points each)	0 points
Total score	37 points

Figure 13. Example of a Concept Map and Its Scoring.

concepts when asked to construct their own concept maps relating provided concepts, as evidenced by a decrease in their student-to-instructor ratios between the first and second assignments.

Student Comments

At the end of the semester, students were asked whether they thought the concept mapping assignments were worthwhile and should be assigned by the instructor in future first-semester calculus courses. (Copies of student responses

Table 16

Student-to-Instructor Concept Mapping Ratios Ordered by Pretest Achievement Scores

Student	Concept Mapping Ratio Map #1	Map #2	Pretest Score
1	1.00	.79	5
2	.61	.92	7
3	.82	.92	7
4	1.00	1.00	9
5	.73	.77	10
6	1.00	1.00	10
7	.61	.95	12
8	.82	.90	12
9	.61	.97	13
10	.94	.85	13
11	.97	.90	13
12	.52	1.00	14
13	1.00	1.00	14
14	1.00	1.00	15
15	1.00	1.00	16
16	1.00	1.00	17
17	.97	.95	19
Mean	.85	.89	12.11
Standard Deviation	.17	.19	3.64

are in the possession of the investigator.) Students' comments regarding the concept mapping assignments (with their pretest achievement scores) were:

> Student 1 (pretest score = 7): "This was a good assignment because it got all the information in one place for me to look at. It was a very good and important one. It definitely helped me to make the connections between each concept."

> Student 2 (pretest score = 10): "Yes! [The concept mapping assignments] really helped my understanding of the relationships between concepts."

> Student 3 (pretest score = 10): "These concept maps were quite creative and I feel that they were very worthwhile. The concept maps opened me up to some relationships that I never would have recognized unless I did them."

> Student 4 (pretest score = 14): "I liked the concept maps. They forced me to go back into the sections and look in detail to understand the concepts and not just the steps to do the problems."

> Student 5 (pretest score = 15): "Very worthwhile in helping to delineate how concepts were related to one another and that there wasn't just one right way."

> Student 6 (pretest score = 16): "Very good! This helped connect ideas, and was very useful and beneficial."

These comments provide evidence that students' completion of concept mapping assignments enabled them to detect and express connections between topics in first-semester calculus. Only two of the students did not feel positively toward the concept mapping assignments. Their comments were:

> Student 7 (pretest score = 17): "Not worthwhile — this was supposed to tie ideas together, but it did not sink in."

> Student 8 (pretest score = 13): "These assignments could be good for some people so that they can see the relationships, but this on [sic] didn't really help me."

Examples of Students' Concept Maps

Examples of concept maps drawn by two students are provided in Figures 14-17. The first two concept maps were drawn by Edward, who had a pretest achievement score of 10, the second two by Audra, who had a pretest achievement score of 19. For Concept Mapping Assignment #1 (see p. 129 in Appendix B), students were provided the ovals, concepts, and arrows drawn in plain text. Each student's additional concepts are shown in bold ovals. For Concept Mapping Assignment #2 (see p. 131 in Appendix B), the students were provided with two ovals, one enclosing the phrase "original function" at the top of the map and the other oval enclosing the phrase "derivative" at the bottom. Students then were asked to incorporate several listed concepts in their map.

Edward's Concept Map #1 (Figure 14) contained several misconceptions, such as a "function's input changes the vertical translation of a function." Concept Map #2 (Figure 15) drawn by this same student showed more connections but was still simplistic. It lacked two of the provided concepts, and involved no additional concepts, while that drawn by Audra (Figure 17) involved all of the provided concepts as well as three additional concepts (shown in bold) and several cross links.

Analysis of the Conceptions of Mathematics Inventory

Parts of the CMI were administered to students in both the treatment and control sections at the beginning and at the end of the semester. The CMI consisted of twenty-four items and had a six-part Likert scale for responses: 1 = Strongly Disagree, 2 = Disagree, 3 = Slightly Disagree, 4 = Slightly Agree, 5 = Agree, and 6 = Strongly Agree. A paired two-tailed t-test was used to analyze the differences between the end-of-semester and beginning-of-semester scores, both in total (see

Tasks:

1) Connect the concepts in the (plain text) ovals below to make a suitable sentence when read in the direction of the arrows.

2) Add the following two sets of concepts (and any necessary arrows and words) to the concept map to make two other suitable sentences.

Set 1: horizontal translation
function's input

Set 2: vertical stretching and/or reflection

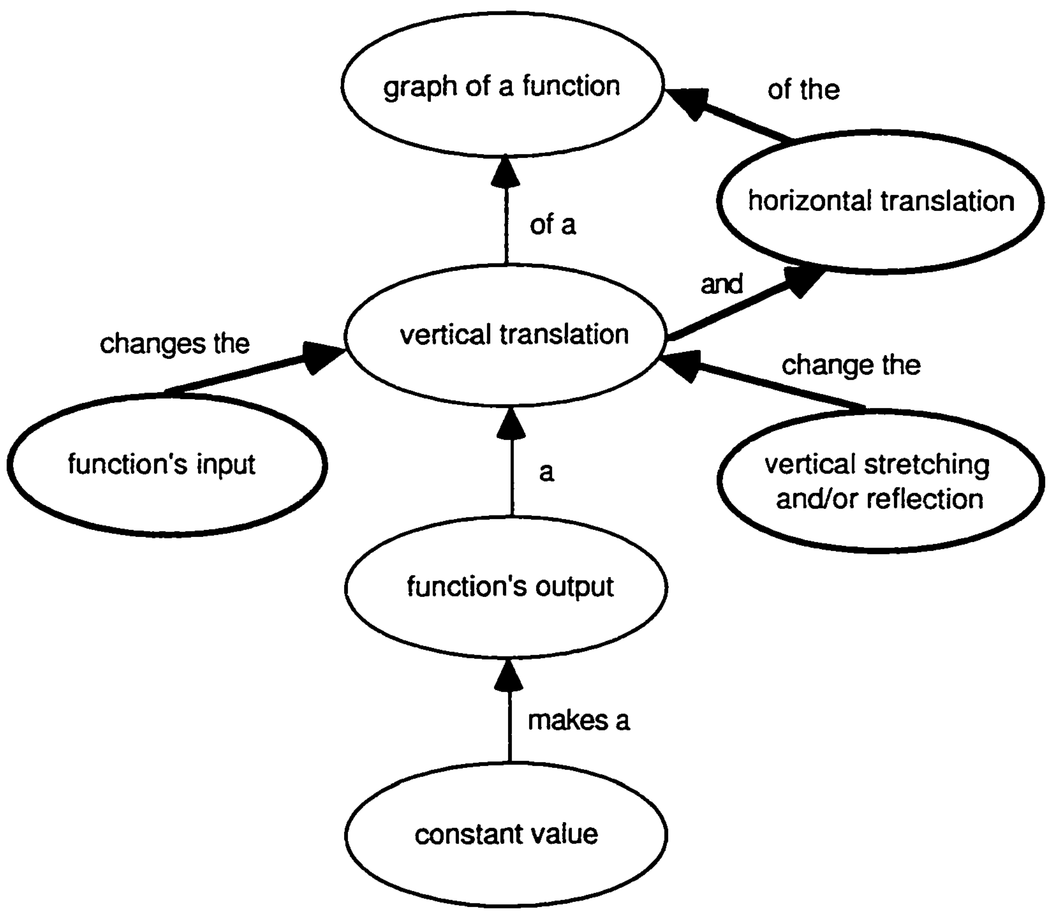


Figure 14. Edward's Concept Map #1.

Task:

Connect the following concepts in a concept map (enclosing the concepts in ovals and using arrows worded with propositions) to make one or more suitable sentences when read in the direction of your arrows. It may help to think how these terms are related to each other. You may add additional concepts, if you desire.

Amount function	Position
Slope function	Velocity
Rate function	Tangent line
Local linearity	Instantaneous rate of change

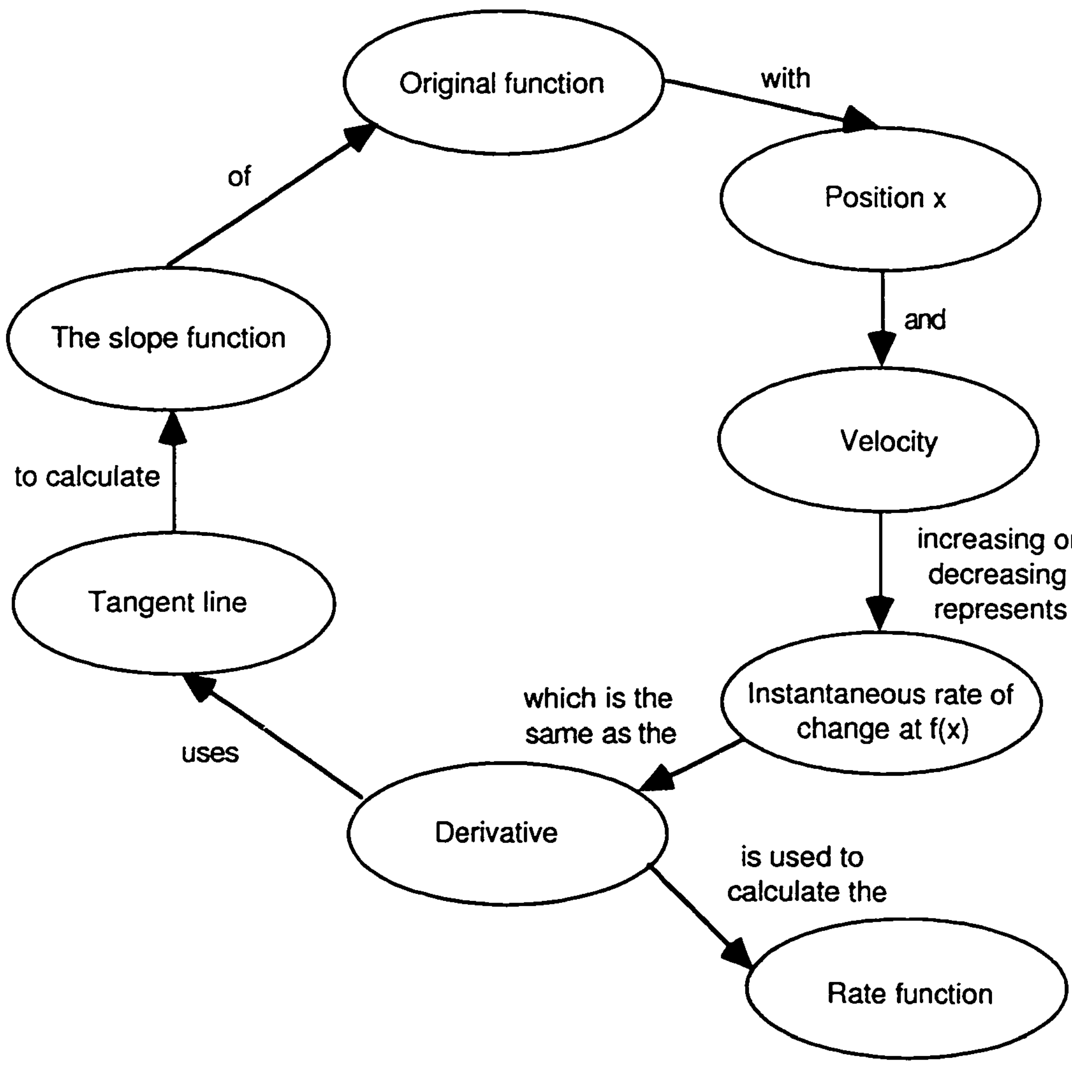


Figure 15. Edward's Concept Map #2.

Tasks:

1) Connect the concepts in the (plain text) ovals below to make a suitable sentence when read in the direction of the arrows.

2) Add the following two sets of concepts (and any necessary arrows and words) to the concept map to make two other suitable sentences.

Set 1: horizontal translation
function's input

Set 2: vertical stretching and/or reflection

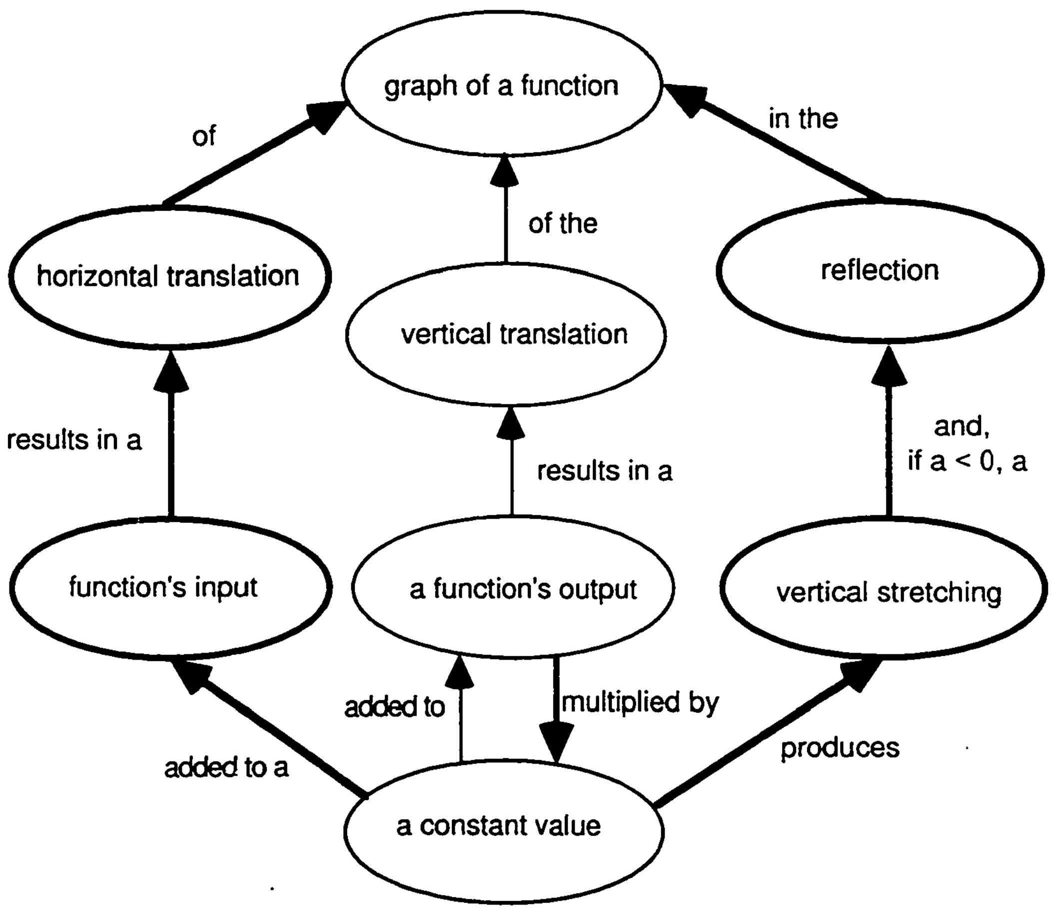


Figure 16. Audra's Concept Map #1.

Task:

Connect the following concepts in a concept map (enclosing the concepts in ovals and using arrows worded with propositions) to make one or more suitable sentences when read in the direction of your arrows. It may help to think how these terms are related to each other. You may add additional concepts, if you desire.

Amount function	Position
Slope function	Velocity
Rate function	Tangent line
Local linearity	Instantaneous rate of change

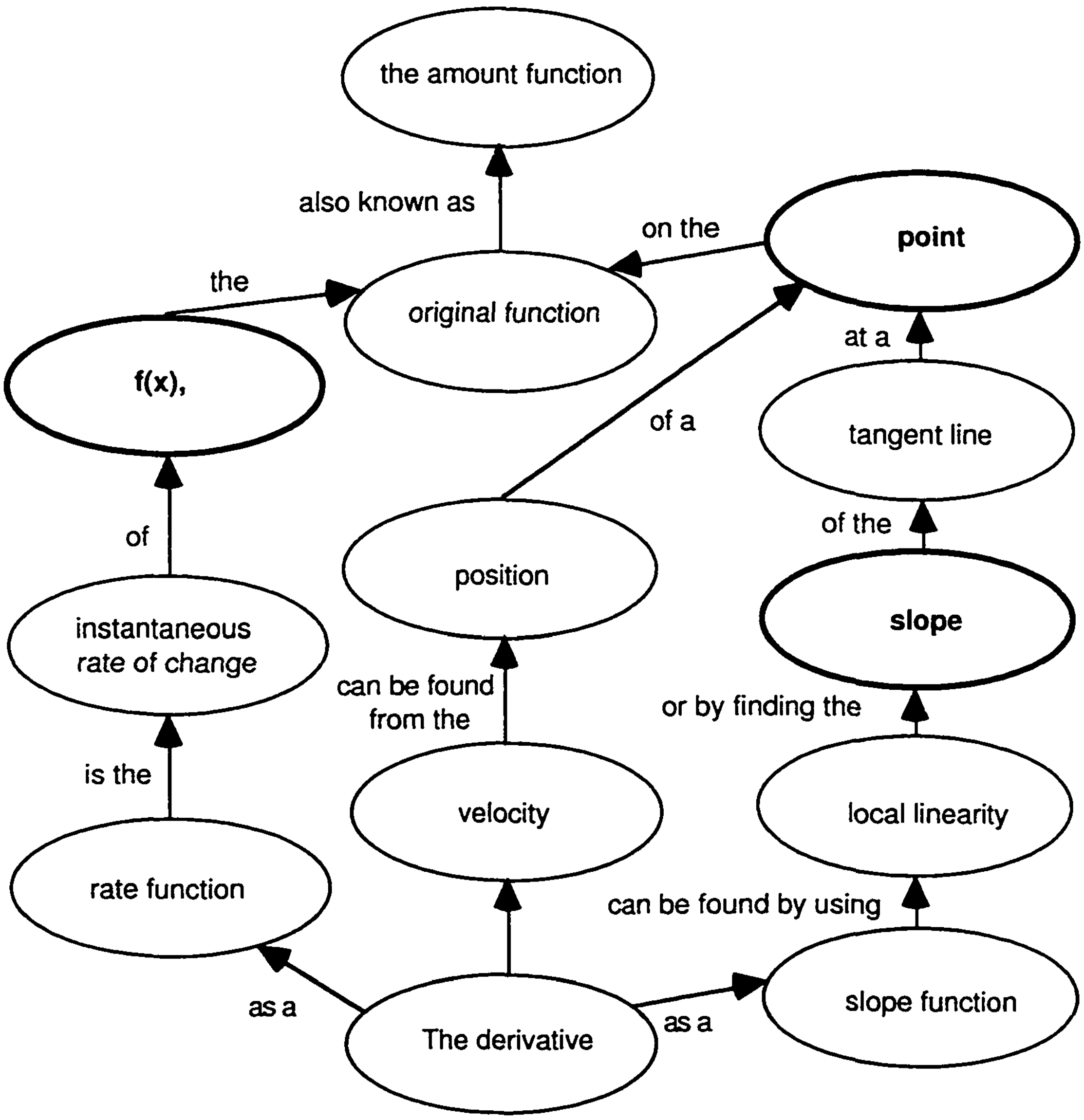


Figure 17. Audra's Concept Map #2.

Table 17) and for each item. Three students in the treatment section and two in the control section failed to complete the CMI at the end of the semester, resulting in a treatment section of size N = 22 and a control section of size N = 16 for the CMI analysis.

As discussed in Chapter III, eight CMI items concerned student conceptions of the Structure of Mathematical Knowledge. Students were asked to respond to statements regarding their thoughts on the existence of connections between topics in mathematics such as functions, graphs, equations, points, and lines. Another eight of the items concerned student conceptions of Doing Mathematics, in which students were asked to respond to statements regarding their thoughts on mathematical formulae, and on solving and understanding mathematical problems. Eight more of the items concerned student conceptions of Learning Mathematics, in which students were asked to respond to statements regarding their thoughts on memorizing and learning mathematics, and asking questions in mathematics classes. Twelve of the items, four in each group listed above, were negatively worded and were scored by reversing the scale, as presented in Chapter III, Table 2. Under this scaling, the highest score possible was a score of 144 .

Beginning-of-semester, end-of-semester, and end-of-semester minus beginning-of-semester means and standard deviations of total scaled scores are given in Table 21 as well as the results of the two-tailed t-test on end-of-semester minus beginning-of-semester total scaled scores by section. In particular, the following null hypothesis was tested:

Hypothesis: There is no statistically significant difference in the mean change from the beginning-of-semester to the end-of-semester CMI scores of students between the treatment and control sections.

At the $\alpha = .05$ level of significance, there was no statistically significant difference ($p = 0.88$) between the two sections with regards to the students' end-of-semester minus beginning-of-semester conceptions about mathematics. Examination of the p-values for the scores by individual item and by section (Structure of Mathematical Knowledge, Doing Mathematics, and Learning Mathematics) also did not indicate any statistically significant difference between the two sections.

Table 17

Means and Standard Deviations for Scaled Responses to the CMI

Source	Treatment Section (N = 22) Mean	SD	Control Section (N = 16) Mean	SD	p-value from t-test
Beginning of Semester					
Structure of Math	28.27	3.06	27.38	1.67	.25
Doing Mathematics	29.95	2.94	30.56	3.44	.57
Learning Mathematics	28.00	3.10	29.19	3.71	.31
Total*	86.23	7.43	87.12	5.78	.68
End of Semester					
Structure of Math	40.14	3.34	39.81	4.32	.80
Doing Mathematics	38.50	4.62	38.63	3.48	.92
Learning Mathematics	33.77	3.41	35.06	3.75	.29
Total*	112.41	9.33	113.50	8.79	.72
End Minus Beginning					
Structure of Math	11.86	2.95	12.44	5.05	.69
Doing Mathematics	8.55	5.69	8.06	5.31	.79
Learning Mathematics	5.77	5.09	5.87	5.25	.95
Total	26.2	11.5	26.4	11.4	.96

(*) Maximum = 144

CHAPTER V

QUALITATIVE RESULTS

Introduction

This chapter is a summary of the "think aloud" problem sessions conducted with seven students from the treatment section. Although nine students were originally invited to participate in the problem sessions, two of these nine students failed to complete the course and were dropped from the problem sessions when they dropped the course. Each of the seven students who completed the course participated in three individual problem sessions, each of which consisted of three tasks, as described in Chapter III. The students were audiotaped by the investigator during each session, and the audiotapes were analyzed.

Selection of students was based on their individual scores on the achievement pretest and Exam 1. Scores on both tests were used in choosing students so as to offset students who may have scored low on the Calculus Readiness pretest, yet scored high on Exam 1, or vice versa. The first set was chosen from among those students who performed in the upper 30 % on the two tests, the second set from among those who performed in the middle 40 %, and the final set from among those who performed in the lower 30 %. The investigator made it clear to the selected students that their participation or nonparticipation in the interviews was disjoint from the investigator's treatment of them during class. Students were compensated for their participation so that they would treat the problem sessions seriously and would continue their participation through all three sessions. None of the students asked declined to participate in the interviews. Of the two students who failed to complete

the course, one was from the middle 40 % of the section while the other was from the lower 30 % of the section, based on achievement pretest and Exam 1 scores.

The first problem sessions occurred within one week following Exam 1, and consisted of tasks covering material on functions and their slopes. The second problem sessions were held within one week after Exam 2, and involved tasks dealing with formulae for finding derivatives of elementary functions and interpretations of those derivatives. The third problem sessions occurred within one week after Exam 3, and incorporated tasks involving applications of derivatives.

Two professional transcriptionists were unable to transcribe two students' third session audiotapes. One student spoke much too softly to be audiotaped even though the investigator frequently reminded the student to speak louder. The other student mumbled so badly the student's words are not discernible. As a result, the investigator was unable to analyze the third sessions for these two students, but did include their other two sessions in the analysis below. Both students were in the top 40 % of the section based on achievement pretest and Exam 1 scores.

Classification of Student Responses

Students were instructed by the investigator during each "think aloud" problem session to "continually think aloud" as they worked on the tasks, which were based on material covered on the examination that had occurred in class just prior to the each problem session. The first task was intended as a warm-up task to the material just covered on the examination, with the second task of slightly more difficulty, and the third task still more difficult. The investigator discussed with students at the first session that the purpose of the sessions was to determine the nature of student thinking while problem solving in calculus and was not intended to be a judgment of the students' problem-solving skills. They were informed that they

might be challenged by some of the tasks so that their thinking could be studied but that the investigator was focusing on the thinking they expressed as they worked the tasks. To encourage their verbalization, students were asked to first read the tasks aloud. Following any extended periods of silence (defined as more than two seconds), the investigator prompted students to express their thinking.

As discussed in Chapter III, problem sessions with students were conducted to provide details on the nature of student thinking that was not available from analysis of student performance on examinations. Specifically, the investigator was interested in student abilities to direct thinking and use the types of reflective thinking modeled in reflective tasks assigned as homework. Student use of reflective thinking would be indicated by their comparing and/or contrasting concepts (or methods), analyzing obstacles encountered, connecting concepts (or methods) with other concepts (or methods), or explaining concepts (or methods) used as they completed the "think aloud" tasks. The investigator hypothesized if homework tasks incorporating reflective tasks were effective in fostering student reflective thinking, each successive problem session with a particular student should show an increase in the use of categories from among Direction of Thinking, Compare/Contrast, Analysis of Obstacles, Connection with Other Concepts, and Explanation of Method. Two additional categories were included in the classification, namely, Read (reading the problem) and Explanation of Calculations.

An example of one of the tasks from the first problem session, along with the investigator's classification of selected student comments, is provided in Figure 18. (All problem session transcripts are in the possession of the investigator.) Student responses within each task were categorized and then timed. Each response's categorization is depicted in the time-line graphs by its relative amount of time

"Think Aloud" Session #1, Task 2

Suppose f(0) = 2 and |f'(x)| < 1 for all x in [-5, 5].

a) Find upper and lower bounds on the value of f(1); that is, find numbers U and L so that L < f(1) < U.

b) Find upper and lower bounds on the value of f(-3); that is, find new numbers U and L so that L < f(-3) < U.

Selected Student Responses

1) Explanation of Calculations: "I guess the upper and lower bounds would have to be between 1 and 3 on the interval [-5, 5]. If [the slope] doesn't go greater than one, doesn't go less than one, it has to be that. So, L would have to be ..., L would have to be 1 and U would have to be 3."

2) Direction of Thinking: "I'm just trying to think of a way to set this up."

3) Compare/Contrast: "Now we're going back down. We were at f(0) before. Now we've gotta go back to -3 and we're still dealing with the same slope value. The absolute value has to be less than 1. So, basically, it should be the same thing."

4) Analysis of Obstacles: "Oh, wait. Gotta think about this for a second ... f(0) is 2. We have to think about f(-3) ..."

5) Connections with Other Concepts: "The slope is between negative 1 and positive 1 because the absolute value of f'(x) cannot be greater than 1, so it has to be between negative 1 and positive 1."

6) Explanation of Method: "Drawing a graph here. At x equals 0, f(0) is 2. The absolute value of the slope of f(x) is less than 1 for all x in the interval [-5, 5]. So, now I have to extend my graph to -5 and 5."

Figure 18. Sample Task and Selected Student Responses.

within that task (see Appendix D). The total amount of time a student spent working on a task is reported at the top of each graph. Composites of students' time-line graphs are provided in Appendix D, Figures 87-93.

No student was able to correctly solve all the tasks. The investigator remained silent if a student incorrectly completed a task. If a student appeared unable to complete a task and fell silent, the investigator would prompt the student until the investigator felt that the student could not complete the task. Prompts to students by the investigator were not included in the timing of the tasks, but their occurrences in the problem sessions are depicted in the last rows of the time-line graphs. Incorrect or incomplete tasks, or both, are marked on the time-line graphs in Appendix D by double-bars at the right-ends of the graphs.

Analysis of Student Responses

Student responses are individually analyzed below, followed by a discussion of commonalities and differences among the subjects.

Audra's Responses

Audra was chosen from the upper 30 % of the section based on pretest achievement and Exam 1 scores. Her time-line graphs are given in Appendix D, Figures 30-38. Her composite of her time-line graphs is given in Appendix D, Figure 87. She correctly solved all but two tasks, and appeared during the sessions to struggle only with the third tasks in the first and third problem sessions. She correctly solved Task 3 in the first session but incorrectly solved Task 3 in the third session. Audra responded with a variety of categories when struggling with tasks. She would respond with multiple uses of the same categories within a task though

often these categories were Explanation of Calculations and Explanation of Method. Her varied and repeated use of categories supports Lester's (1994) comments discussed in Chapter I that good problem solvers are better at monitoring and regulating their problem-solving efforts than poor problem-solvers. Good problem-solvers effectively use metacognition during problem solving since they appear to know not only what and when to monitor, but also how to monitor.

During the first problem session, Audra had responses categorized in each of the seven categories, with Direction of Thinking utilized just once, in Task 2, and Analysis of Obstacles occurring only in Task 3. In the second session, none of Audra's responses were categorized as Compare/Contrast and only one, in Task 3, as Analysis of Obstacles, although tasks promoting this type of reflective thinking had often been assigned as homework and discussed in the section. In the third session, she had no responses categorized as Compare/Contrast. Her wide variety of responses when struggling with a task are apparent in Task 3, which she was unable to complete correctly.

Barbara's Responses

Barbara was also chosen from the upper 30 % of the section. See Appendix D, Figures 39-44 for her time-line graphs for her problem sessions that could be transcribed. See Appendix D, Figure 88 for her composite of those time-line graphs. She spent a significant portion of each task reading the task. From her problem sessions that could be transcribed, she appeared to struggle only with the tasks in the first session. Task 1 in this session was the only task in which she had a response in the Analysis of Obstacles category. Similar to the case of Audra discussed above, she responded only twice in the Compare/Contrast category, with both responses

occurring in the same task. Although Barbara used every category at least once, with the exception of Direction of Thinking, she seldom repeated categories nor displayed a wide variety of responses within a task.

Her responses to Task 3 in the second session are interesting in that, with the exception of reading the problem, all of her responses fell in the Explanation of Calculations category. She was unable to correctly solve this task, yet the absence of prompt marks reveals that she continually expressed her thoughts as she worked the task. Her responses provide evidence to support Shaughnessy's (1985) statement cited in Chapter I that unless students recognize the importance of reflective monitoring, they may "get their solution process rolling like a freight train and then run out of track" (p. 403).

<u>Carl's Responses</u>

Carl was the third student chosen from the upper 30 % of the section. His time-line graphs for his problem sessions that could be transcribed are given in Appendix D, Figures 45-50. His composite of his time-line graphs for those sessions is given in Appendix D, Figure 89. Like the case of Barbara, Carl also spent a significant amount of time reading each task. His responses were seldom categorized as either Explanation of Calculations or Explanation of Method. Most often following his initial reading of a task, Carl's responses would be categorized as Direction of Thinking, Analysis of Obstacles, or Connect with Other Concepts. He appeared from the transcripts to struggle with Task 2 and Task 3 in the first session, incorrectly solving the first but correctly solving the second. He was unable to complete Task 2 and Task 3 in the second session. As with the other two students discussed previously, he did respond in all categories, usually displaying a

variety of categories within a single task, with the exception of Task 3 of the second session. In particular, the variation of response categories is noted in Task 3 of the first session, a task with which he struggled but solved correctly.

David's Responses

David was one of the two students from the middle 40 % of the section who completed the course. His time-line graphs are given in Appendix D, Figures 51-59, and his composite of his time-line graphs is given in Appendix D, Figure 90. Although David very seldom responded in the Compare/Contrast category, he usually varied his type of responses and frequently displayed a category more than once within a task. From the transcripts, he struggled with the second task in the first session, which he solved incorrectly, and the third tasks in the second and third sessions, both of which he solved correctly.

David would often follow his initial reading of a task with an Explanation of Method response. His responses in Task 1 in the first session are similar to Barbara's responses to Task 3 of her second session discussed above in that following his initial reading of the task, all his responses were categorized as Explanation of Calculations. Unlike the case of Barbara, though, David correctly solved his task, and from the transcripts, fully understood the task and a method for its solution. His responses to this task provide support for the hypothesis by Silver, Branca, and Adams (1980) that "metacognitive processes [for a high-ability learner] may become generally automatic and hence inaccessible during verbalization either during or after problem solving" (p. 218).

Edward's Responses

Edward was the other student chosen from the middle 40 % of the section who completed the course. See Appendix D, Figures 60-68 for his time-line graphs, and Appendix D, Figure 91 for his composite of his time-line graphs. Following his initial reading of the first tasks in each problem session, his responses were almost exclusively in the Explanation of Calculations category. Also, he rarely supplied Connection With Other Concepts category responses and seldom used the Compare/Contrast category. Edward did show an increase in the use of some categories of responses, such as Direction of Thinking, as the semester progressed.

Edward failed to respond in the Connection With Other Concepts category in four tasks, and in three of those four tasks his solutions were incorrect or incomplete. As with other students previously discussed, Edward seldom used the Compare/Contrast category. The Direction of Thinking response category was virtually absent in Edward's responses in the first problem session and in the first two tasks of the second session, but then was present in the Task 3 of the second session as the only response category represented, other than the Read category. Edward was unable to complete this task, however. Following its appearance in the second session, the Direction of Thinking response category was present in all of his responses in the third session. Edward's increased use of the Direction of Thinking response category, particularly as the tasks became more involved and required more knowledge of first-semester calculus concepts, supports Dirkes' (1985) comments in Chapter I that students who can direct their own thinking will be able to connect new information with existing information and can purposely select thinking strategies to solve problems.

Fran's Responses

Fran was one of the two students chosen from the lower 30 % of the section who completed the course. See Appendix D, Figures 69-77 for his time-line graphs. His composite of his time-line graphs is given in Appendix D, Figure 92. From the problem sessions transcripts, he struggled with the last two tasks in each of the first two sessions and with Task 3 in the third session. He was able to successfully complete the last two tasks in the first session, but not those in the second session, nor Task 3 in the third session. Fran only responded once in the Direction of Thinking category in the first session but responded at least once in this category in all tasks in the other sessions, including three times in Task 1 in the second session and five times in Task 3 in the third session. He, too, seldom used the Compare/Contrast category.

It is interesting to compare his responses in Task 3 in the first session with those of Task 3 in the third session. He had difficulties with both tasks and often fell silent, requiring prompts from the investigator. In Task 3 of the first session, when prompted to continue talking aloud, Fran would very often return to the same category of thinking he had expressed just prior to the prompt. By Task 3 of the third session, however, prompts by the investigator usually were followed by a change in response categories. These categories also alternated much more with each other in the third session than in the first session.

George's Responses

George was the other student chosen from the lower 30 % of the section who completed the course. His time-line graphs are found in Figures 78-86, and his composite of his time-line graphs is found in Figure 93 in Appendix D. As indicated

on these graphs, George failed to successfully complete any of the tasks in any of the problem sessions. In the first session, none of his responses were categorized as Direction of Thinking, and most of his responses were categorized as Explanation of Calculations or Explanation of Method. In the second session, he twice responded in the Direction of Thinking category, and much less often in the explanation categories. In the third session, however, the Direction of Thinking category was again absent in all his responses. He had great difficulty with the last two tasks in the third session, and returned to an extensive use of Explanation of Method category responses. His difficulties with the tasks in the second and third sessions are apparent from the relatively short time he worked on several of the tasks before giving up completely on their solution.

Commonalities and Differences Among the Student Responses

As discussed in several of the analyses above, there were commonalities and differences among the students. Visual displays of these similarities and differences are presented in Figures 19-21. In Figure 19, student responses are graphed by percent of total responses within each category for each session for each of the three precalculus achievement levels of students (upper 30 %, middle 40 %, and lower 30 % of the section) who participated in the problem sessions. Responses are graphed by session in Figure 20. Student responses were also analyzed for patterns in the response trails. For each group of students, the investigator began with the Read category and determined the most frequent category to follow the Read category. The most frequent category to follow that category was determined next, and the process continued until the trail cycled back to previously depicted categories. The response trails for each group are depicted by session in Figure 21.

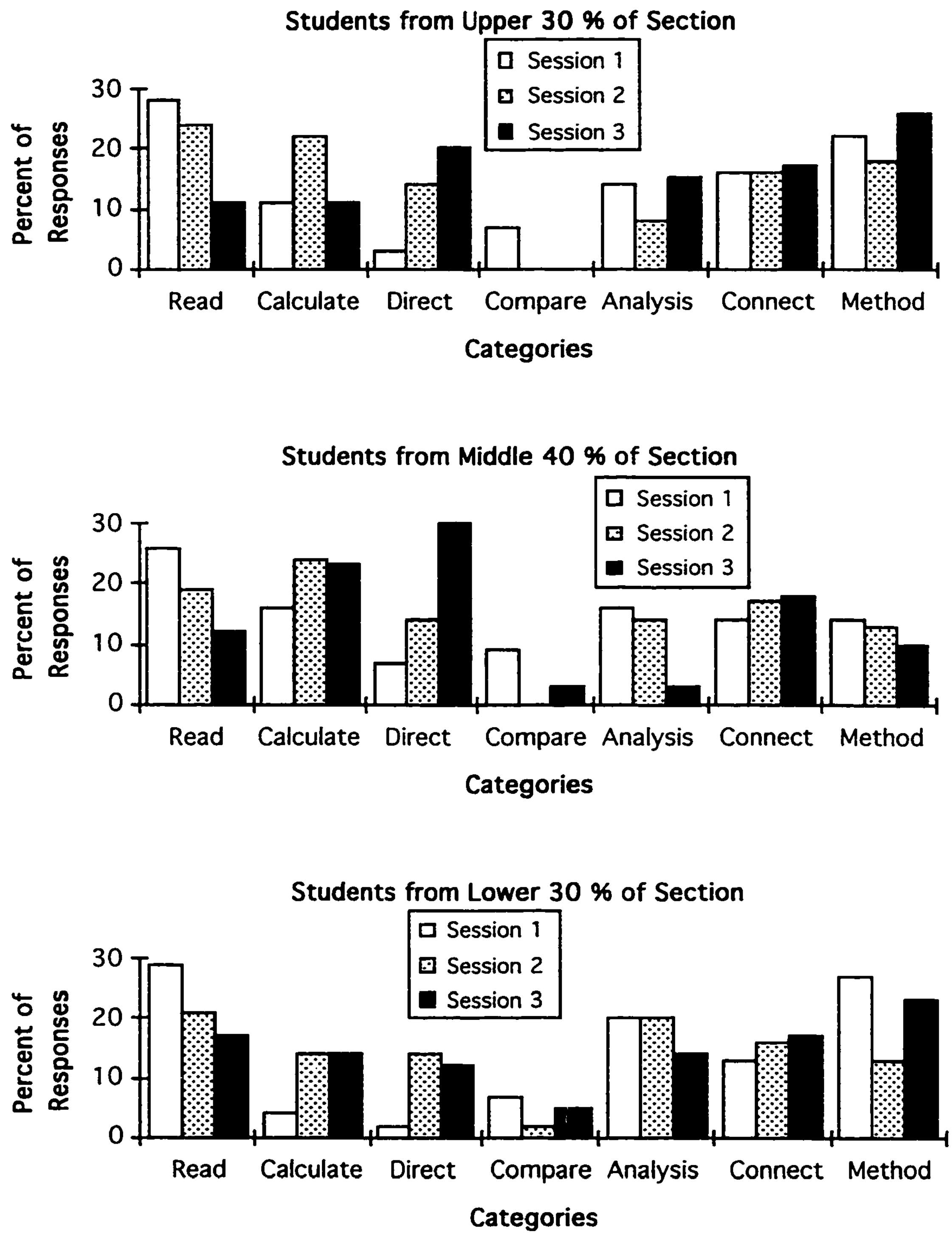


Figure 19. Comparison Graphs of Percent of Responses in Each Category by Precalculus Achievement.

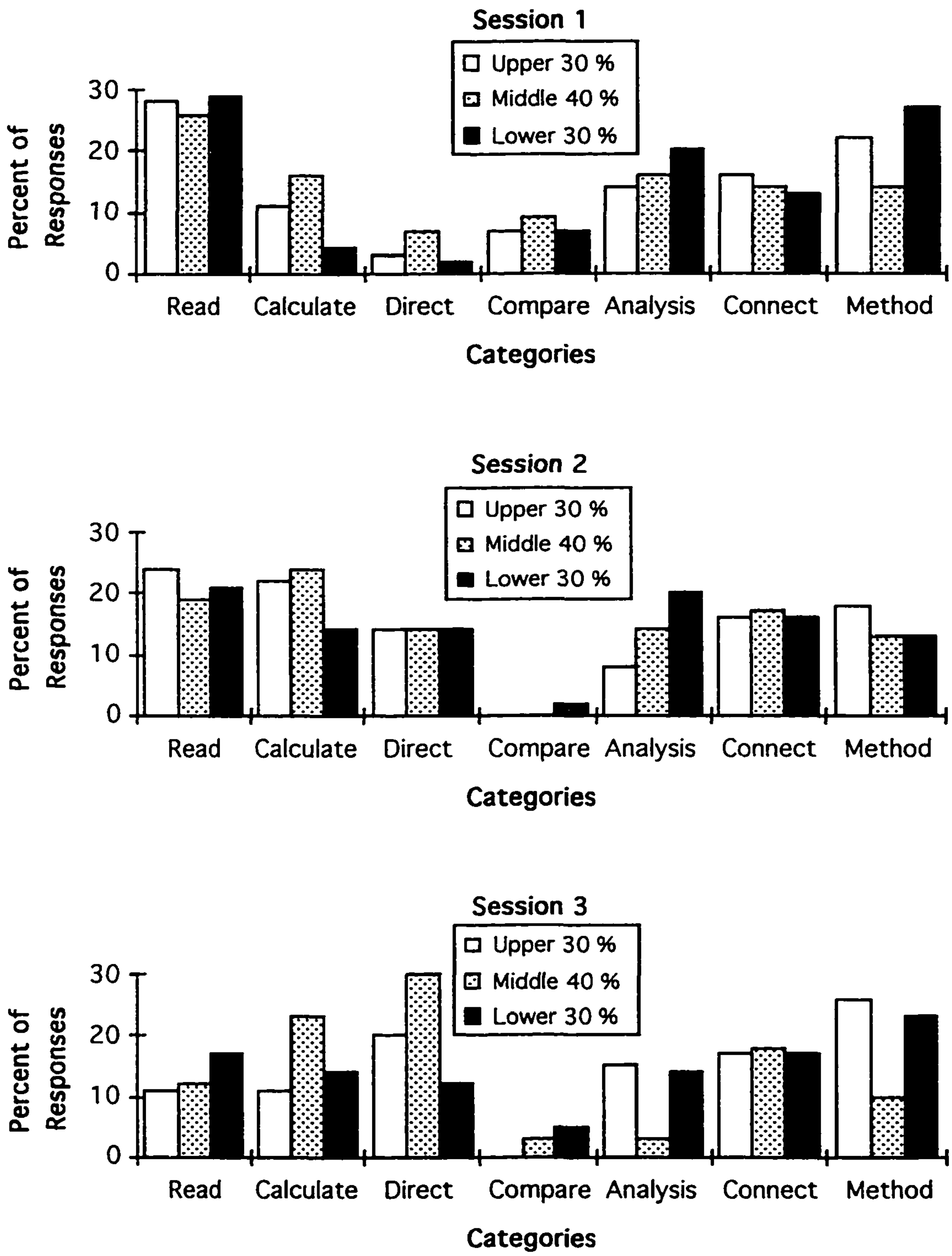


Figure 20. Comparison Graphs of Percent of Responses in Each Category by Session.

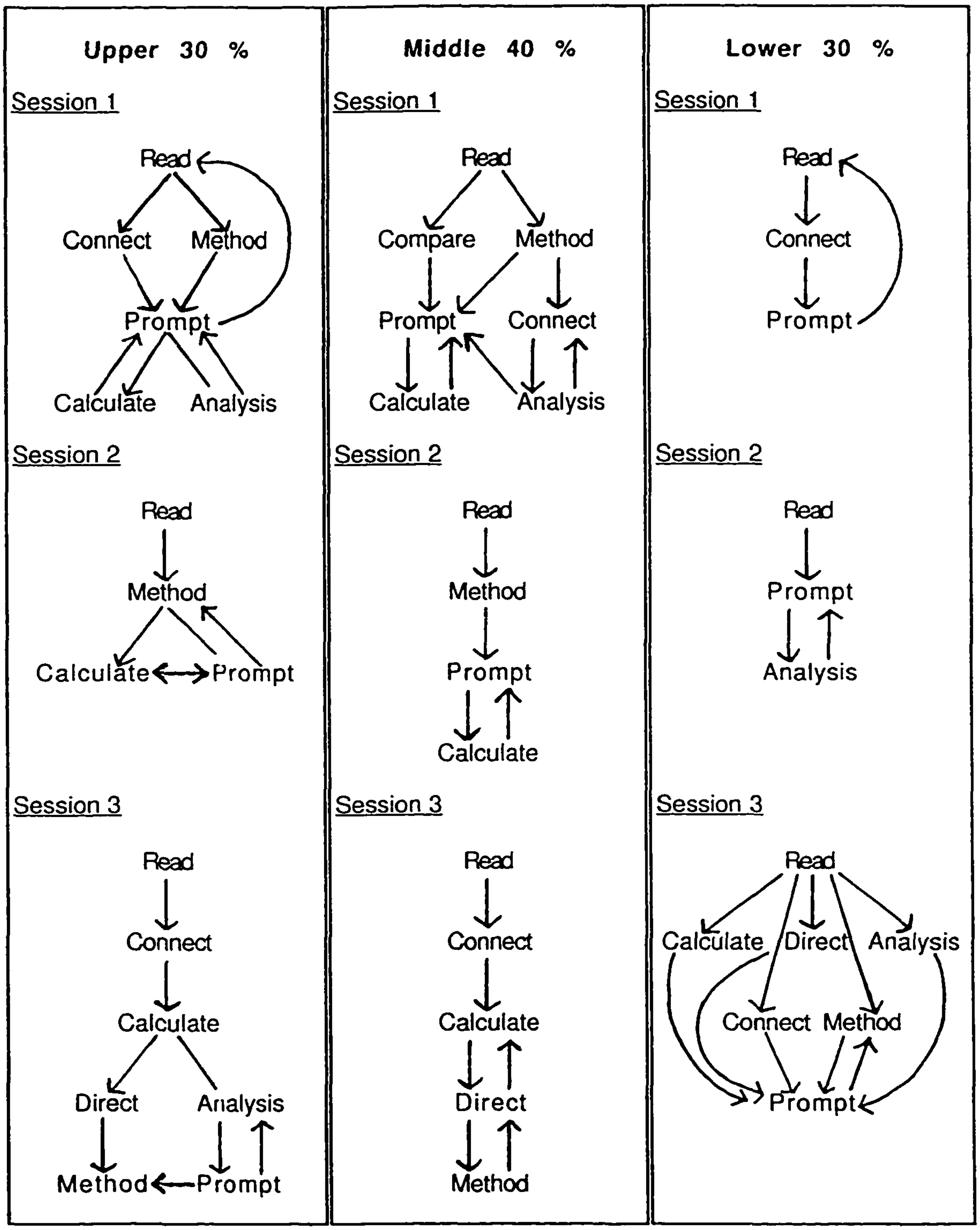


Figure 21. Comparison Diagrams of Progression Among Categories by Session by Precalculus Achievement.

Commonalities

Few of the students utilized the category of Compare/Contrast, although tasks requiring this type of reflective thinking had often been assigned as homework and discussed in class. In the "think aloud" tasks, none of the tasks specifically asked students to compare or contrast concepts or methods (see Appendix D) as had several reflective tasks incorporated into homework assignments (see Appendix B). In the first sessions, all of the students compared or contrasted subtasks, e.g., comparing the method they wanted to use in part b) of Task 2 with the method they had used in part a). Only one student compared subtasks in the second session (in Task 2). In the third sessions, three students used the Compare/Contrast category, comparing and contrasting their work in either Task 2 or Task 3 with their work in previous tasks assigned as homework or with material discussed in class. None of the students compared or contrasted "think aloud" tasks with any of the reflective tasks that had been incorporated into their homework assignments.

In the first problem sessions, the most frequent category of response for any group was the Read category. This may be due to each task in this session consisting of subtasks, while one of the tasks in the second session and two of the tasks in the third session did not have explicit subtasks labeled a), b), etc.

Many of the students, particularly in the first problem sessions, offered responses categorized as explanatory, whether Explanation of Calculations or Explanation of Method. This is evidenced in Figure 19 and Figure 20, where for each group of students in the first session, the Explanation of Calculation and Explanation of Method bars are among the tallest following the bars for the Read category. The Read and Explanation of Calculation categories are not associated with reflective

thinking. As discussed above in the case of Barbara, students may slip into the use of explanatory categories without realizing that they are not correctly solving a task. Or, as in the case of David, students may understand a task so well that their use of reflective thinking is not evident when expressing their thinking.

With the exception of George, who gave up after a short time on three of the tasks in the second and third problem sessions, other students began to display variety and alternation of response categories when struggling with solutions to tasks by the third problem session. The variety of responses in the third session is depicted in Figure 20 by the decrease in the heights of bars for the Read category and the increase in the heights of bars for the Direction of Thinking category.

The increase in variety is also evident in examination of the response trails for the three groups presented in Figure 21. The trails for the students in the upper 30 % and middle 40 % precalculus achievement levels of the section are remarkably similar in use of and connections between categories. The increased use of the Direction of Thinking category is evident in its inclusion for the first time in the third session trails for both groups. The increase in use of categories by the third problem session by students in the lower portion of the section with regard to the pretest achievement and Exam 1 scores is apparent by the increased complexity and interconnectedness of their response trail as compared to their trails for the first two sessions.

Differences

Several differences were also detected among students. Students who performed well in the course, such as Audra and David, responded using very few categories yet were often able to present correct solutions to tasks. Other students,

such as Barbara and George, would at times utilize only a few response categories before unsuccessfully concluding their work on a task. Also, while some students displayed a variety of response categories in the first problem sessions, some of the students chosen from the lower portion of the class with regard to the pretest achievement and Exam 1 scores did not display the variety of response categories until after the first problem session. Many of these students, in particular, began to display responses in the Direction of Thinking category as the semester progressed.

Examination of Figure 20 for the third problem session also displays differences among student groups. The students in the middle 40 % of the section with regard to the pretest achievement and Exam 1 scores had more responses in the Explanation of Calculations and Direction of Thinking categories than students in the other two groups, as demonstrated by the tall bars in the third session for this group in these categories in Figure 20. In contrast, the bars in this same session for this group in the categories of Analysis of Obstacles and Explanation of Method are smaller than those in these two categories for the other two groups.

As mentioned above, the response trails for students in the lower 30 % precalculus achievement level of the section are less involved than those for students in the other two parts of the section. Students in the lower 30 % of the section had response trails that involved fewer categories and had more straightforward connections when compared to those for students in the upper 30 % and middle 40 % of the section.

CHAPTER VI

SUMMARY AND CONCLUSIONS

Introduction

The purpose of this study was to investigate the fostering of reflective thinking in first-semester reform calculus students through their completion of homework assignments which included reflective tasks, and the effect of these assignments on student understandings and conceptions of mathematics. Chapter I described current perspectives on, and defined the terminology of, metacognition and reflective thinking, stated questions to be addressed by the study, and discussed the theoretical framework concerning constructivism, metacognition, homework tasks, writing assignments, and reflective tasks. Chapter II discussed related research on metacognition, homework tasks, writing assignments, and reflective tasks. Chapter III described the research design and methodology, and stated the null hypotheses to be tested by the study. Chapter IV presented the quantitative analyses performed on scores from an achievement pretest, four course examinations, and an inventory of student conceptions of mathematics, while Chapter V described the qualitative analysis performed on student responses during "think aloud" problem sessions. This chapter will connect the questions posed in Chapter I to the analyses detailed in Chapters IV and V.

Summary

The subjects involved in this study were students who self-selected into two sections of first-semester reform calculus at a large midwestern university during

the Fall 1997 semester. The investigator taught the treatment section, and a colleague taught the control section. Of those students who originally chose to participate in the study, twenty-five students in the treatment section and eighteen students in the control section actually completed the course, defined as the completion of the four course examinations, namely, Exam 1, Exam 2, Exam 3, and the Final examination.

The treatment and the control sections met at the same time for 50 minutes four days each week, used Calculus from Graphical, Numerical, and Symbolic Points of View by Ostebee and Zorn (1997) as their textbook, and followed the same traditional syllabus in covering the material. The instructors presented the in-class material in a lecture/discussion format from notes previously written by the instructor of the control section, and assigned the same daily homework problems from the textbook. Both instructors had experience teaching calculus. Although the instructors of the two sections differed, every effort was made to otherwise ensure that the major difference between the two sections was the assignment of homework which incorporated reflective tasks for the treatment section but not for the control section.

Students submitted solutions to selected homework problems once a week, and these problems were graded as part of their course grade. Students were encouraged to work with partners on these weekly assignments. If they did, the partners submitted one set of solutions and received the same grade for the assignment. The instructors jointly wrote the four course examinations, administered them in class at the same times, and collaborated on the scoring of the examinations. Graded homework accounted for 40 %, each of the three 50-minute in-class examinations accounted for 10 %, and the two-hour comprehensive Final examination accounted for 30 % of a student's course grade.

Some of the reflective tasks incorporated into homework assigned to students in the treatment section were part of the weekly homework while others were stand-alone assignments. The investigator attempted to keep the workload from the weekly homework assigned in the treatment section equivalent to that assigned the control section by the selective replacement of some of the homework problems from the textbook assigned the control section.

Since the purpose of this study was to examine the fostering of reflective thinking in first-semester reform calculus students through their completion of homework assignments which included reflective tasks and the effect of these assignments on student understandings and conceptions of mathematics, both quantitative and qualitative information was gathered. At the beginning of the semester, all students completed the MAA Calculus Readiness Test as an achievement pretest. The four course examinations served as posttests spaced throughout the semester. Students also completed selected portions of the Conceptions of Mathematics Inventory [CMI] developed by Grouws, Howald, and Colangelo (1996) at the beginning and end of the semester. Two of the reflective homework tasks assigned in the treatment section were concept mapping assignments, which were examined by the investigator for information regarding student ability to make connections and describe relations among calculus concepts. Qualitative information was gathered through "think aloud" problem sessions with seven students from the treatment section and was intended to provide details on the effect of homework assignments incorporating reflective tasks on student understandings of calculus that were not available from analyses of examination scores. Originally three students were selected from the upper 30 %, three from the middle 40 %, and three from the lower 30 % of the section based on their pretest achievement and Exam 1 scores. Of these nine students, one student from the middle 40 % and one from the lower 30 %

of the section dropped the course and were omitted from the study.

This study was conducted in an attempt to answer the five research questions posed in Chapter I. A restatement of each question, with any related null hypotheses, and pertinent results from the quantitative and qualitative studies follow.

Question 1

Question 1 is stated as: Will completion of homework assignments involving reflective tasks contribute to better student understanding of first-semester calculus than completion of homework assignments not involving such tasks?

This question was addressed by both quantitative and qualitative analyses. Quantitative analysis involved the following null hypotheses by examination.

Hypothesis I

Hypothesis I is stated as: There is no statistically significant difference in the adjusted mean Exam 1 scores between the treatment and control sections. This hypothesis was not rejected at the $\alpha = .05$ level of significance ($p = .312$), indicating that the students in the treatment section did not perform significantly better than the control students on Exam 1. Analysis of the "think aloud" problem sessions conducted with selected students shortly after Exam 1 provided some details on student thinking during problem solving, and indicated that students who scored in the lower portion of the section did not respond with the variety of reflective categories of thought generally displayed by students participating in the problem sessions who scored in the upper portion.

Qualitative analysis involved examination of the two stand-alone concept mapping assignments completed just prior to Exam 1. The ratios of scores on the two

concept maps to the joint concept maps created by the two instructors were examined as to their relationship with student pretest achievement scores. As discussed in the sample maps examined in Chapter IV, maps from students scoring in the lower part of the treatment section with regard to pretest achievement scores contained several misconceptions, often lacked the inclusion of provided concepts and involved no additional concepts, contained no crosslinks, and, in general, were more simplistic than maps constructed by students scoring in the upper part of the section with regard to pretest achievement scores.

Hypothesis II

Hypothesis II is stated as: There is no statistically significant difference in the adjusted mean Exam 2 scores between the treatment and control sections. This hypothesis was also not rejected at the $\alpha = .05$ level of significance ($p = .564$). The analysis of the "think aloud" problem sessions conducted shortly after Exam 2 displayed many of the same results found in the first problem session. Students who scored in the lower portion of the section still used fewer reflective categories of thought than did students who scored in the upper portion, although some students in the lower portion began to change their types of responses. This relative simplicity of reflective thinking is apparent upon examination of the trails of most frequent responses beginning with the Read category for students in the upper portion versus students in the lower portion of the section. As evident in their trails and time-line graphs, students in the lower portion of the section began increasing the number of their responses which indicated they were directing their thinking.

Hypothesis III

Hypothesis III is stated as: There is no statistically significant difference in the adjusted mean Exam 3 scores between the treatment and control sections. This hypothesis could not be tested using analysis of covariance with pretest achievement scores as covariate since the slopes of the regression lines for the two sections were not homogeneous. Examination of regression lines as to the slopes and intersection point revealed that the regression line for the control section rose more sharply than that for the treatment section. Students scoring lower than 12 (maximum = 20) on the achievement pretest appeared to profit from homework assignments which incorporated reflective tasks. This was 8/25 = 32 % of students in the treatment section.

In the "think aloud" problem sessions conducted shortly after Exam 3, other than one student who gave up after a short time on the last task in the second problem session and the last two tasks in the third problem session, students began to display variety in, and alternation of, response categories when struggling with solutions to tasks. Also, students in the third problem session continued to use categories of reflective thinking, such as that indicating direction of thinking, that had been virtually absent in the first problem sessions but had begun to appear by the second problem sessions.

Hypothesis IV

Hypothesis IV is stated as: There is no statistically significant difference in the adjusted mean Final examination scores between the treatment and control sections. This hypothesis also could not be tested using analysis of covariance with pretest achievement scores as covariate since the slopes of the regression lines for

the treatment and control sections were not homogeneous. Examination of regression lines showed that the regression line for the control section, similar to that for the Exam 3 scores, rose more sharply than that for the treatment section. For the Final examination, students scoring lower than 13 on the achievement pretest appeared to profit more from homework assignments which incorporated reflective tasks. This was 11/25 = 44 % of students in the treatment section. No "think aloud" problem sessions were conducted with students following the Final examination, which took place approximately three weeks after Exam 3.

Question 2

Question 2 is stated as: Is there a relationship between the nature of reflective activity used during problem solving and student level of performance in first-semester calculus?

Qualitative analysis indicated that there was a relationship between the nature of student reflective activity used during problem solving and level of performance. As discussed in Chapter V, two of the three students chosen for the "think aloud" problem sessions from the upper 30 % precalculus achievement level of the section displayed multiple uses of the same response category within individual tasks and variety in response categories, particularly when struggling with tasks. For one of these students, however, the categories displayed were explanation categories — explanations of calculations and explanations of method. These facets about responses were also displayed by one of the students from the middle 40 % of the section. The third student from the upper 30 % of the section seldom repeated categories or revealed variety of type of responses within a task. The four students earned course grades of A, BA, or B. As discussed in Chapter V,

students may easily slip into the use of explanatory categories without realizing they are incorrectly solving a task, demonstrating their neglect of reflective thinking to monitor their progress while completing the task. Other students may understand a task so well that their use of reflective thinking is not evident when expressing their thinking.

The other three students participating in the problem sessions, one from the middle 40 % and two from the lower 30 % of the section, rarely provided the variety and multiple occurrences of response categories discussed above. The student from the middle 40 % of the section almost always followed his initial reading of a task with an explanation of calculations. He seldom reported the use of connections with other concepts or the comparison or contrast among concepts. One of the two students from the lower 30 % of the section did not begin to vary categories of responses following periods of silence until the third problem session. The other student from the lower 30 % of the section was unable to successfully complete any of the tasks, and quickly gave up on completing the last two tasks in the third session. While he did begin to exhibit direction of thinking by the second session, by the third session, he returned to the frequent use of explanations of methods which he had displayed in the first problem session. These three students earned course grades of C, D, or E.

Question 3

Question 3 is stated as: Will the amount of reflective activity used during problem solving increase as a result of student completion of homework assignments involving reflective tasks?

This question was addressed by qualitative analysis, also. None of the students participating in the "think aloud" problem sessions displayed frequent or systematic use of the Compare/Contrast response category, although this type of reflective task had often been assigned as homework and discussed in class. This may be because the Compare/Contrast tasks assigned as part of the homework specifically asked students to make comparisons and contrasts. None of the tasks in the "think aloud" sessions directed students to this type of thinking although understandings of concepts and methods necessary to successfully complete part a of a task were often necessary to successfully complete part b of that same task. Students often missed that connections existed among subtasks within a task. Also, to utilize comparison and contrast types of thinking, students needed to find targets — tasks to which they could compare and contrast their present tasks. Students may not have sufficient repertoires of previous examples or tasks which they thoroughly understand and can call upon in these situations. Without adequate knowledge about previous examples or tasks, students would have difficulties discerning any similarities and differences among previous examples or tasks and their assigned tasks.

Of the categories used in the analysis of the "think aloud" problem sessions, the categories corresponding to reading the problem, explaining calculations, and prompting by the investigator were not based on reflective thinking. While students from the upper 30 % of the section often provided responses categorized as explanations of method, which involves reflective thinking, students from the lower 30 % of the section frequently displayed responses categorized as explanations of calculations, which does not involve reflective thinking. Students from the middle 40 % gave responses in both categories. This varied use by students of the two explanation categories did not appear to significantly change as the semester progressed.

Question 4

Question 4 is stated as: Will the nature of reflective activity used during problem solving change as a result of student completion of homework assignments involving reflective tasks?

Results of qualitative analysis indicated that the nature of students' reflective activity used during problem solving did change for some students, but that the change may not have been permanent. The categories of reflective thinking displayed by students from the upper 30 % of the section did not appear to significantly change as the semester progressed, though the investigator was hampered here by the absent transcripts for two students for the third problem session. For the student from the middle 40 % of the section who earned a BA for the course, there also did not appear to be any noticeable change in the nature of his reflective activity. For the other student from the middle 40 % of the section, however, his reliance on responses explaining his calculations continued throughout the three problem sessions but he did begin to provide more responses indicating his attempts to direct his thinking by the third problem session. This increase in responses categorized as students' direction of their thinking was evident in the students from the lower 30 % of the section. By the second problem session, both students had begun to increase their use of the direction of thinking category. This increase continued for the student from the lower 30 % who ultimately earned a C for the course, but declined for the student from the 30 % who earned a D for the course. This was the student who was unable to correctly solve any of the tasks.

Question 5

Question 5 is stated as: Will student conceptions about mathematics change as a result of the completion of homework assignments involving reflective tasks?

This question was addressed by quantitative analysis. Specifically, this question was addressed by the analysis involving the null hypotheses:

Hypothesis: There is no statistically significant difference in the mean change from the beginning-of-semester to the end-of-semester CMI scores between the treatment and control sections.

This hypothesis was not rejected using an $\alpha = .05$ level of significance ($p = 0.88$) for differences in total scaled scores. Analyses of the end-of-semester minus beginning-of-semester CMI scores by individual item using an $\alpha = .05$ level of significance also did not indicate any significant differences between the treatment and the control sections.

Conclusions

Student Achievement

Although no statistically significant differences in adjusted mean scores were found between the treatment and control sections for any of the examinations, other results suggested that students needed time to develop reflective thinking as a tool, that is, it was not until after numerous homework assignments that included reflective tasks that students, particularly those in the lower portion of the section with regard to prior mathematics achievement, were able to call upon these skills themselves. When intersection points for the regression lines for Exam 1 were

studied, the investigator noted that students in the treatment section scoring lower than 6 on the achievement pretest (3/25 = 12 %) appeared to benefit more from homework assignments that incorporated reflective tasks. Students in the treatment section scoring lower than 11 on the achievement pretest (7/25 = 28 %) appeared to benefit more on Exam 2 from homework assignments that incorporated reflective tasks. As mentioned above in the discussion of results from Exam 3 and the Final examination, students scoring lower than 12 (8/25 = 32 %) and lower than 13 (11/25 = 44 %), respectively, appeared to benefit more from homework assignments that involved reflective tasks. In these two cases, the treatment intervention with precalculus achievement was significant. The steady increase in the percentage of students appearing to benefit from homework assignments involving reflective tasks suggests that a longer time frame for exposure to such tasks may result in significant differences in student achievement when compared with students not assigned any reflective homework tasks.

Examination of the adjusted means for Exam 1 and Exam 2 and the means for Exam 3 and the Final examination also supports the above conclusion. For Exam 1, there is a difference of 5.56 points in the adjusted mean examination scores, with the difference favoring the control section. For Exam 2, this difference in adjusted means is 2.65 points and still favors the control section. For Exam 3, however, the difference in mean scores is .92 and favors the treatment section, while for the Final examination, the difference in mean scores is 5.98 and again favors the treatment section. These increasing mean score differences that favor the treatment section indicate that perhaps with a longer exposure to homework assignments involving reflective tasks significant differences in adjusted mean scores would occur.

Results from the "think aloud" problem sessions appear to support the conclusion that students need a longer exposure to homework assignments involving

reflective tasks if significant differences are to happen. Students scoring in the upper 50 % of the treatment section on the Calculus Readiness achievement pretest (median = 12) provided responses indicating they already used a variety of reflective thinking, and frequently alternated the type of reflective thinking used during problem solving. Students scoring low on the Calculus Readiness achievement pretest began by the third problem session to provide responses indicating their use of some hitherto unused categories of reflective thinking, such as direction of their thinking.

Student Conceptions of Mathematics

There was no significant change in student CMI scores between beginning and end of semester. Nothing in the study indicated that longer exposure to homework assignments incorporating reflective tasks would cause significant differences to appear. In student evaluations of the concept mapping assignments mentioned in Chapter IV, students noted that these assignments had helped them to see some of the connections between concepts. Yet in completing the CMI at the end of the semester, particularly in their responses to the items concerning Structure of Mathematical Knowledge, students gave no indication that seeing connections between calculus concepts had changed any of their conceptions about mathematics.

Pedagogical Implications

This study has added to the existing body of knowledge concerning the reflective component of metacognition, particularly in regards to students achievement and conceptions in first-semester calculus. In addition, several pedagogical implications may be derived from the findings. Reflective writing tasks

included as part of weekly graded homework assignments and as stand-alone tasks throughout a semester foster reflective thinking. Students need time, however, to develop this type of thinking, particularly students in the lower portion of a class with regard to prior mathematics achievement

The use of stand-alone reflective tasks such as concept mappings can also contribute to better student understandings of connections among first-semester calculus concepts. Assignments that specifically require students to organize their thinking about course material are appreciated by students, who recognize the benefit they derive from completing such assignments.

Finally, in-class discussions of specific reflective thinking techniques may assist students in their understandings of first-semester calculus concepts and methods. These techniques include comparison and contrast, connection with previous examples and tasks, analysis of obstacles and means to overcome them, and explanations of concepts or methods.

Suggestions for Further Research

The results of this study indicate the following possible directions for future research:

1. Due to the high number of students in both the treatment and control sections failing to complete, much less successfully complete, this first-semester calculus course, a similar study involving a larger number of subjects who complete the course would be of interest. Also, since populations in Calculus II and Calculus III courses tend to be more stable, a similar study of subjects in these courses might be revealing.

2. The above discussion on student achievement suggests that a closer examination of the possible cumulative effect of completion of homework assignments

involving reflective tasks on student understandings and conceptions of mathematics is needed. A study of student completion of such types of homework assignments across more than one semester seems to be warranted. This could be accomplished by studying the same body of students in a first-semester and then a second-semester calculus class, for example.

3. A closer examination of the effect of different types of reflective tasks incorporated into homework assignments on student understandings and conceptions of mathematics is needed. In particular, from their comments as reported in Chapter IV, students appeared to appreciate the organization and relation of concepts required for the completion of the concept mapping assignments. Yet only one student, as observed by the investigator, continued to use concept maps to organize course material when no mapping assignments were made later in the semester. Future research concentrating on a specific type of reflective task would be of benefit.

4. Finally, studies of the effects in courses other than calculus of homework assignments which include reflective tasks on student understandings of mathematics are indicated.

Appendix A

Scatterplots of Examination Scores Versus Pretest Achievement Scores by Section

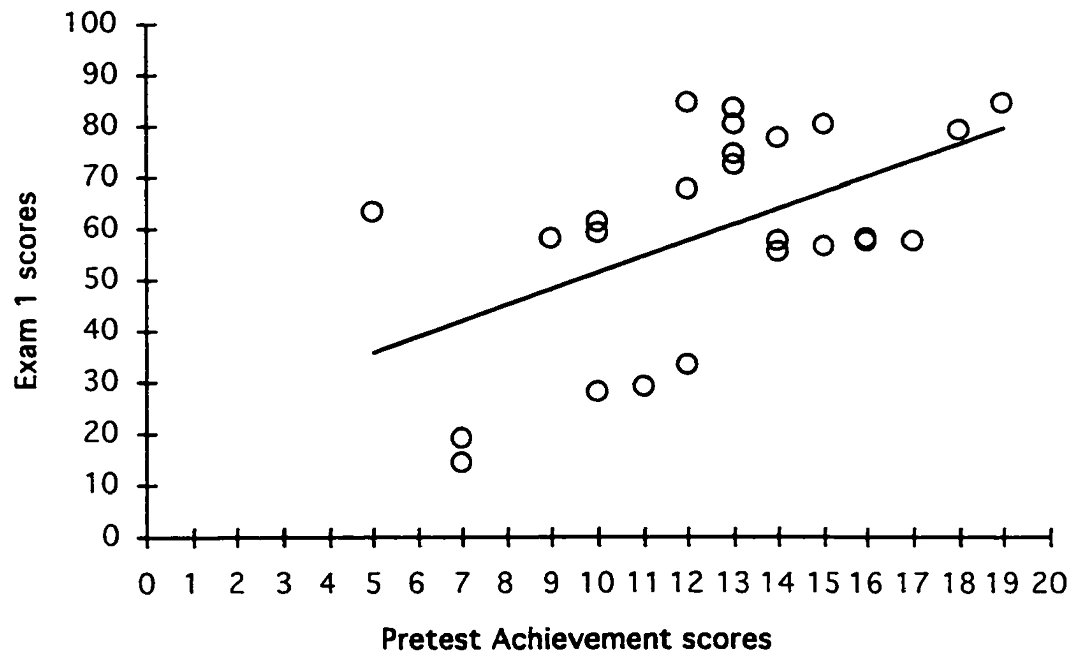


Figure 22. Scatterplot and Regression Line for Treatment Section Exam 1.

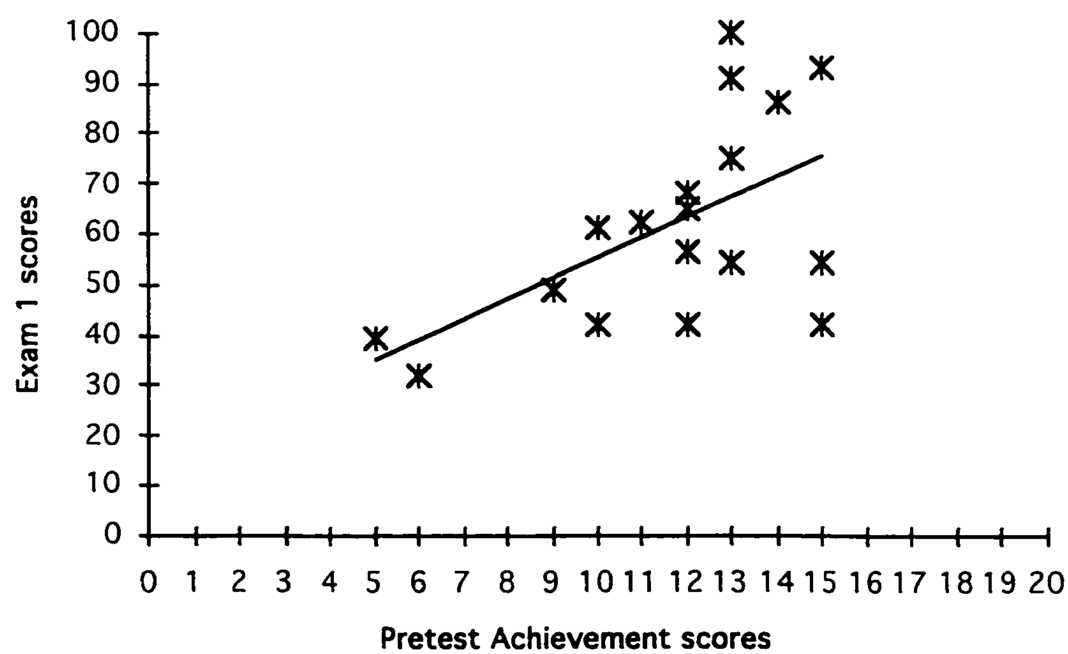


Figure 23. Scatterplot and Regression Line for Control Section Exam 1.

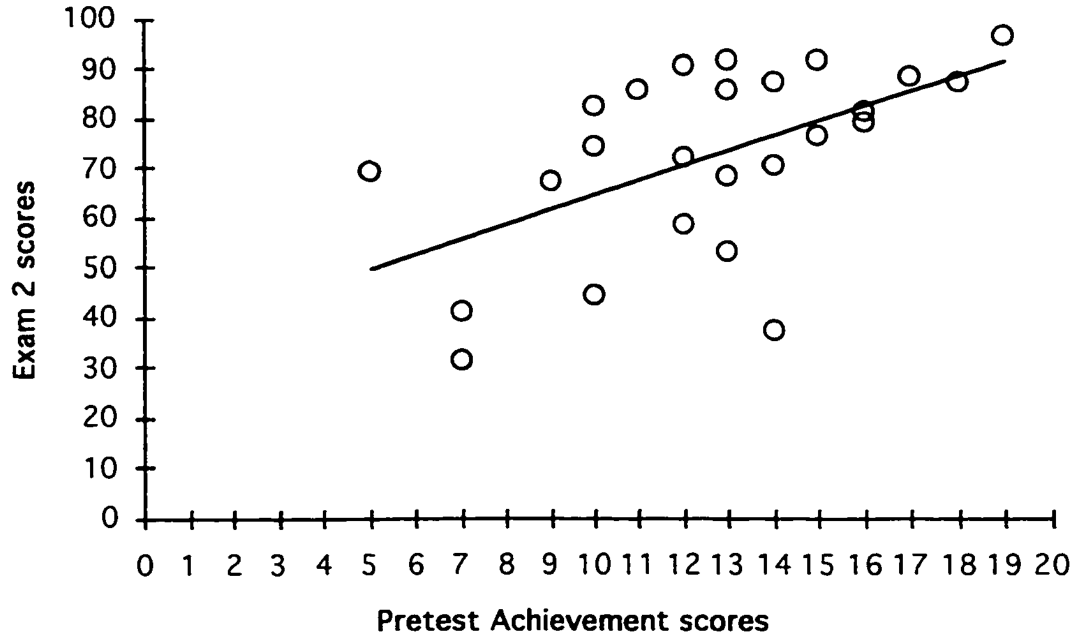


Figure 24. Scatterplot and Regression Line for Treatment Section Exam 2.

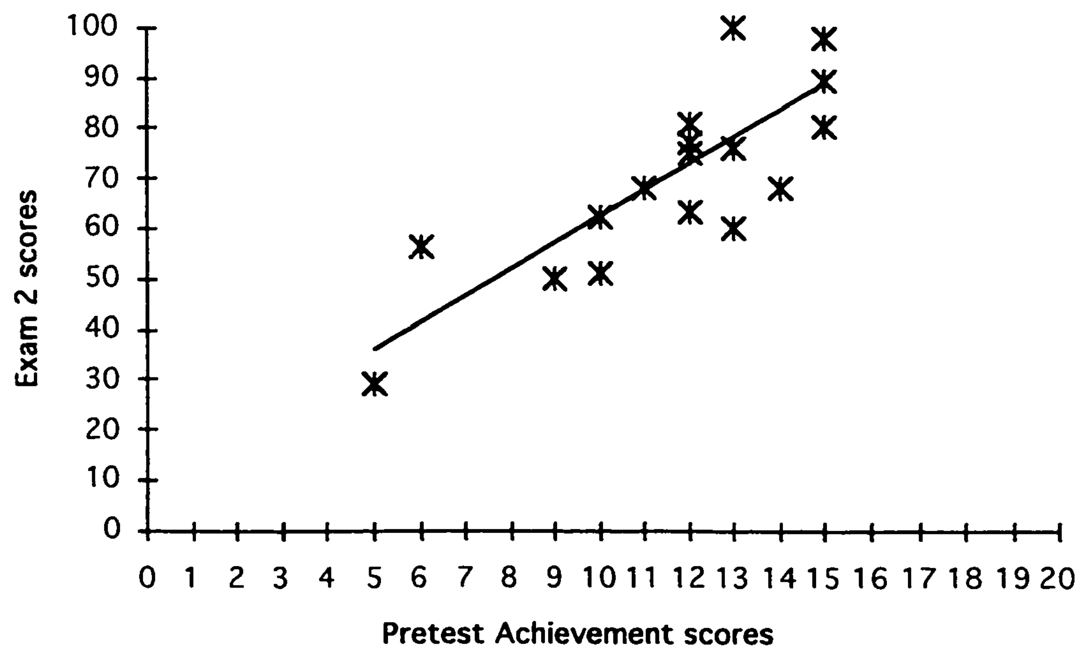


Figure 25. Scatterplot and Regression Line for Control Section Exam 2.

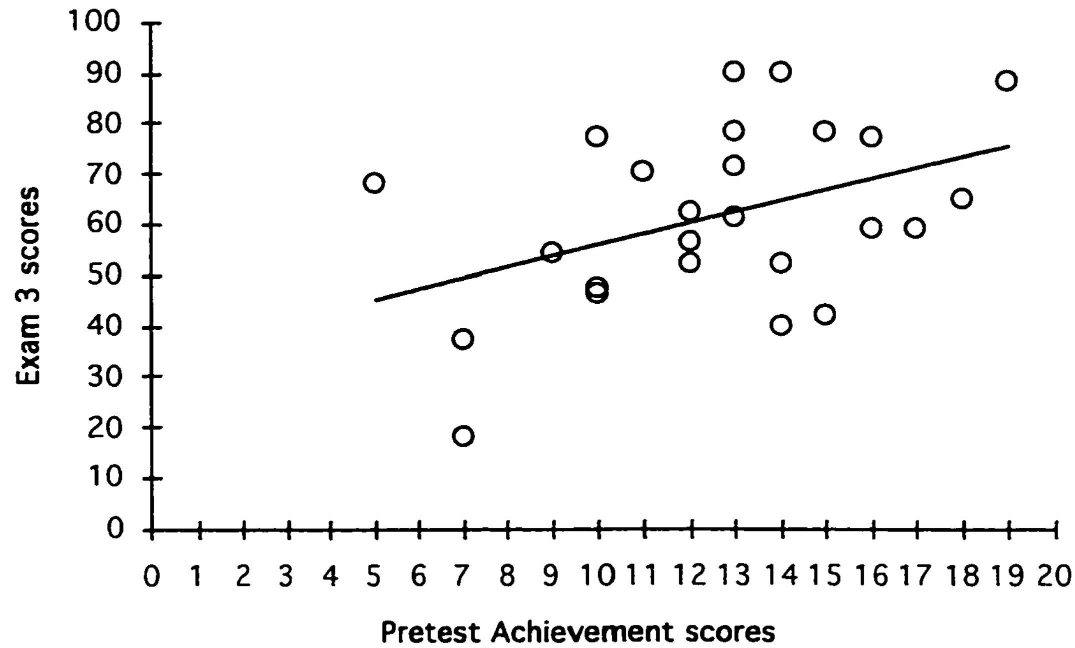


Figure 26. Scatterplot and Regression Line for Treatment Section Exam 3.

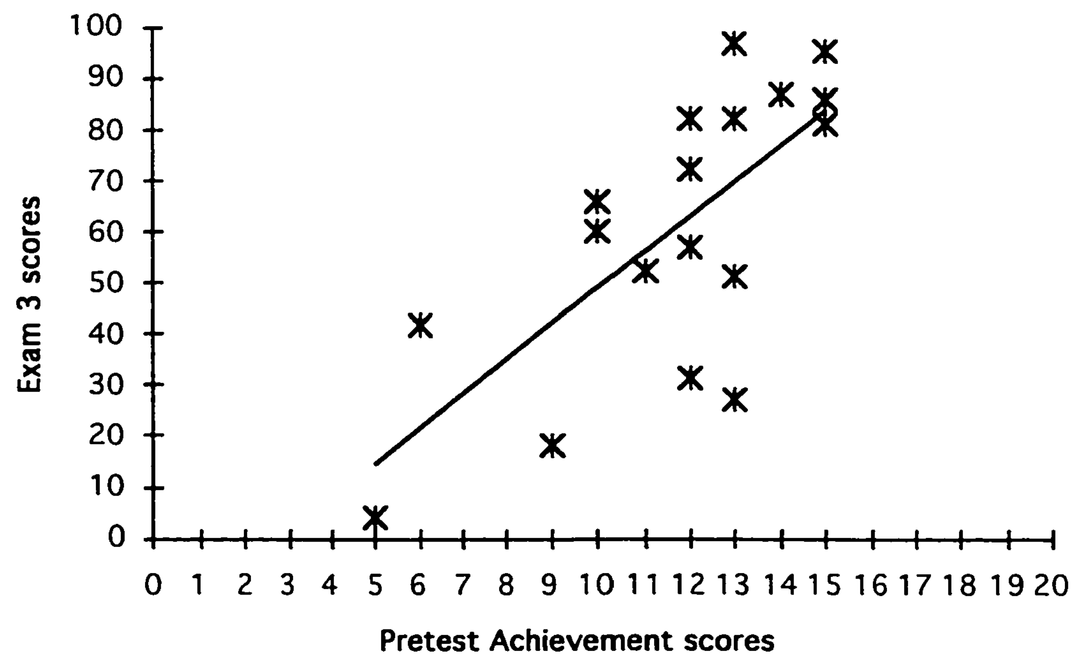


Figure 27. Scatterplot and Regression Line for Control Section Exam 3.

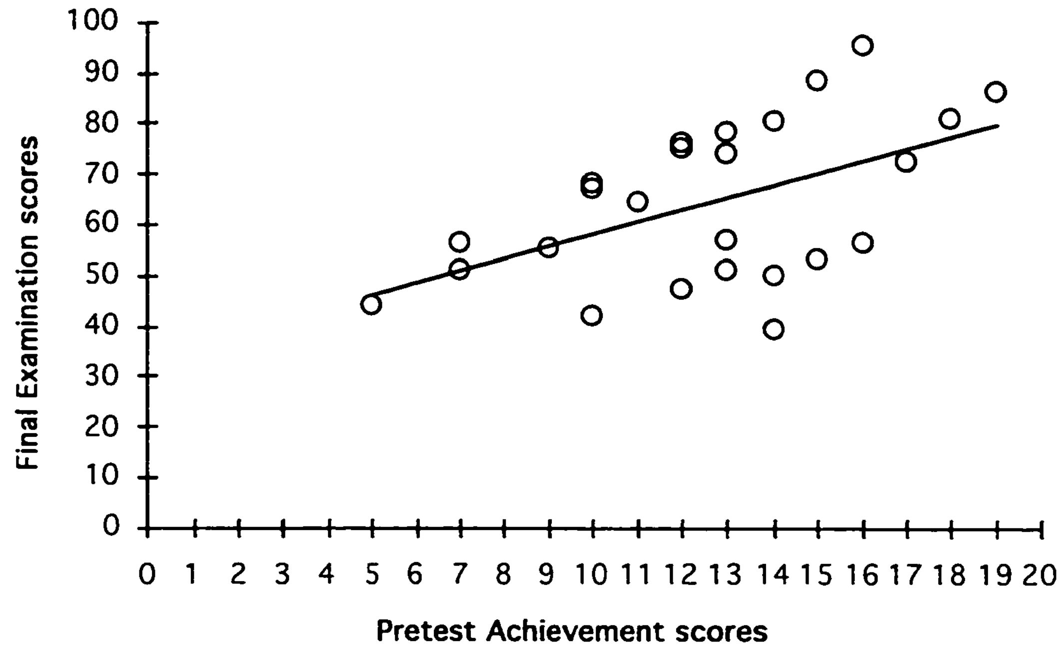


Figure 28. Scatterplot and Regression Line for Treatment Section Final Examination.

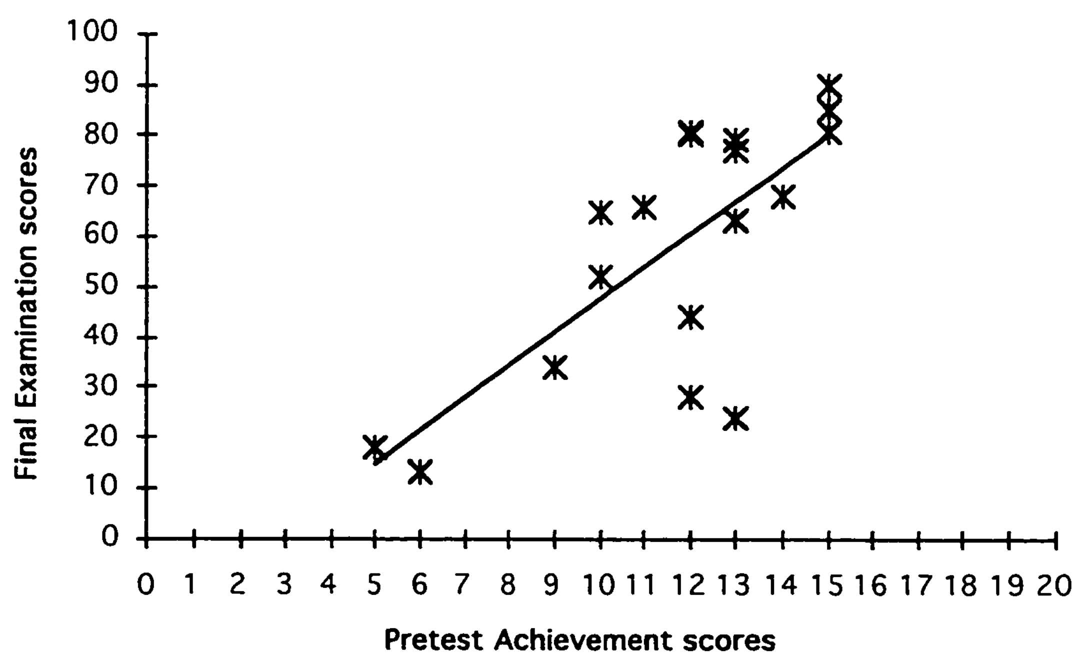


Figure 29. Scatterplot and Regression Line for Control Section Final Examination.

Appendix B

Reflective Tasks

Turn-In Homework Assignment #1

Problems from the textbook:

§1.1	Functions, Calculus Style	# 15, 22
§1.2	Graphs	# 13, 24, 53
§1.3	Machine Graphics	# 5
§1.4	What *Is* a Function?	# 11, 28, 36

Answer the following:

1) When do you think it might be useful to approximate one function by another function?

2) Draw an example of a graph that is not the graph of a function. Explain your reasoning.

Turn-In Homework Assignment #2

Problems from the textbook:

§1.5	A Field Guide to Elementary Functions	# 6, 12, 30
§1.6	New Functions from Old	# 22, 26, 30

Answer the following:

1) How might you identify a rational function from its graph? An exponential function? A logarithmic function? A trigonometric function?

2) How are $(f * g)(x) = f(x) * g(x)$ and $(f \circ g)(x) = f(g(x))$ similar? How do $(f * g)(x)$ and $(f \circ g)(x)$ differ?

Turn-In Homework Assignment #3

Problems from the textbook:

§2.1	Amount Functions and Rate Functions: The Idea of the Derivative	# 28
§2.2	Estimating Derivatives: A Closer Look	# 10, 30
§2.3	The Geometries of Derivatives	# 26, 10

Turn-In Homework Assignment #4

Problems from the textbook:

§2.4	The Geometry of Higher-Order Derivatives	# 4, 14
§2.5	Average and Instantaneous Rates: Defining the Derivative	# 6, 10

Turn-In Homework Assignment #5

Problems from the textbook:

§2.5	Average and Instantaneous Rates: Defining the Derivative	# 6, 12
§2.6	Limits and Continuity	# 20, 30, 34
§2.7	Limits Involving Infinity; New Limits from Old	# 4 (a, c), 10, 20

Turn-In Homework Assignment #6

Problems from the textbook:

§3.1	Derivatives of Power Functions and Polynomials	# 4, 24, 80
§3.2	Using Derivative and Antiderivative Formulas	# 20, 26
§3.3	Derivatives and Exponential and Logarithm Functions	# 50, 72

Turn-In Homework Assignment #7

Problems from the textbook:

§3.4	Derivatives of Trigonometric Functions	# 14, 42
§3.5	New Derivatives from Old: The Product and Quotient Rules	# 42, 48

Turn-In Homework Assignment #8

Problems from the textbook:

§3.6	New Derivatives from Old: The Chain Rule	# 14, 24, 32, 44, 48, 52
§3.7	Implicit Differentiation	# 2, 12
§3.8	Inverse Trigonometric Functions and Their Derivatives	# 18, 32

Turn-In Homework Assignment #9

Problems from the textbook:

§4.1	Differential Equations and Their Solutions	# 10, 20
§4.2	More Differential Equations: Modeling Growth	# 6, 10
§4.3	Linear and Quadratic Approximation; Taylor Polynomials	# 14, 18
§4.4	Newton's Method: Finding Roots	# 2, 10

Answer the following:

1) What are the similarities between exponential growth problems and their solutions and exponential decay problems and their solutions? What are the differences?

2) What does it mean for a function to approximate another function? How do you know when you have a "good" approximation? How are the graphs of the original function and a "good" approximation similar? How are they different?

Turn-In Homework Assignment #10

Problems from the textbook:

§4.6	Optimization	# 18
§4.8	Related Rates	# 2, 14

Answer the following:

1) What is the role of the first derivative of a function when solving an optimization problem? The role of the second derivative?

2) Why are the problems in this section called "related" rate problems? What exactly is the objective in solving these problems?

3) What concepts from previous chapters are used in solving:

 i) optimization problems? How are they used?

 i i) related rate problems? How are they used?

Turn-In Homework Assignment #11

Problems from the textbook:

§4.9	Parametric Equations, Parametric Curves	# 4, 6(a)
§4.10	Why Continuity Matters	# 6
§4.11	Why Differentiability Matters; The Mean Value Theorem	# 4
§5.1	Areas and Integrals	# 4(a)

Answer the following:

1) Suppose the coordinates of a point P are given by the parametric equations $x = f(t)$ and $y = g(t)$ for $a \le t \le b$. <u>Without looking at the graph</u>, describe how you can determine the direction of P.

2) Suppose f is a function defined on [a, b] such that f(a) = A and f(b) = B. If f assumes every value in [A, B], must f be continuous on [a, b]? Why or why not? How is the Intermediate Value Theorem involved in this question?

Turn-In Homework Assignment #12

Problems from the textbook:

§5.1	Areas and Integrals	# 4(a), 6(c, e, g), 12
§5.2	The Area Function	# 4
§5.3	The Fundamental Theorem of Calculus	# 10

Answer the following:

1) Look at your work on problem 10, section 5.3. What topics were present in this problem that were covered in this:

    i) section?

    ii) chapter?

    iii) course?

2) How is the relationship between f and A_f similar to that between f' and f? How is it different?

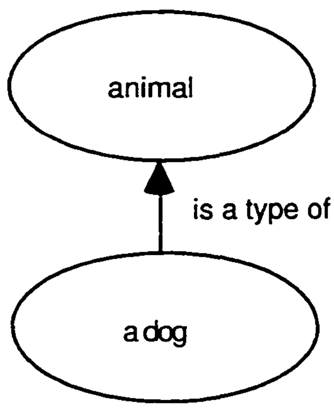
3) Why are Theorems 3 and 4 (pgs. 366 and 368) called The <u>Fundamental</u> Theorems of Calculus?

Concept Mapping Task #1

§ 1.2 -- Graphs

Definition -- A concept map is a diagram in which important concepts are placed within ovals and are linked by arrows to other concepts by means of phrases. The concepts and phrases combine (using the direction of the arrows) to form suitable sentences.

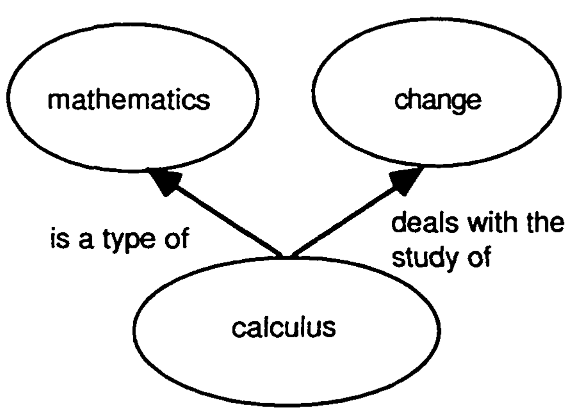
Example One:



yields the sentence:

"a dog is a type of animal"

Example Two:



yields two sentences:

"calculus is a type of mathematics"

and

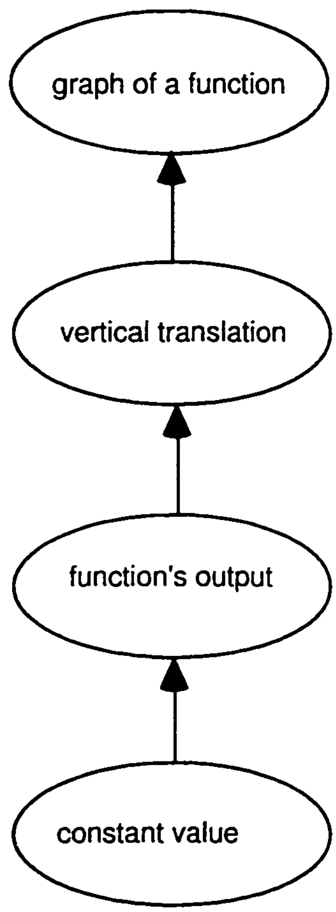
"calculus deals with the study of change"

Concept Mapping Assignment #1 -- 10 points

Due: Friday, September 12th at the beginning of class

Tasks:

1) Connect the concepts in the ovals below to make a suitable sentence when read in the direction of the arrows.



2) Add the following two sets of concepts (and any necessary arrows and words) to the above concept map to make two other suitable sentences. If it is not possible to add a set of concepts, explain your thinking. You may reorganize the map and/or concepts as you need.

<u>Set 1</u>: horizontal translation
function's input

<u>Set 2</u>: vertical stretching and/or reflection

Concept Mapping Assignment # 2 -- 20 points

Due: Monday, September 29th at the beginning of class

Task: Connect the following concepts in a concept map (enclosing the concepts in ovals and using arrows worded with propositions) to make one or more suitable sentences when read in the direction of your arrows. (Your arrows may either point from the top of the page to the bottom of the page, or vice versa.) It may help to begin by thinking how these terms are related to each other. You may add additional concepts, if you desire.

Amount function	Position
Slope function	Velocity
Rate function	Tangent line
Local linearity	Instantaneous rate of change

Original function

Derivative

Math 122 -- Fall 1997
Writing Assignment # 2 -- 15 points

Date Due: Monday, October 6th at the beginning of the class period.

Assignment:

a) Look back on your test paper for Exam 1. What do you now see as the major misconception you had concerning specific test material? When did you realize it was a misconception? When was the misconception created, how was the misconception created, and what caused you to correct the misconception?

b) Choose one of the test problems from Exam 1 that involved your misconception. Write a one-to-two paragraph solution for the problem as if you were explaining the solution to the problem to a classmate. Identify which problem you are explaining. Be sure to include graphical sketches, if appropriate, and to watch your terminology. Explain to your classmate your original solution and your corrected solution. Please do not just give me back my answer key -- I'm interested in your understanding and your explanations!

Please type as much of this assignment as possible. You may draw any sketches or symbols that you can not type. Use standard paper with standard fonts and margins.

Making Sense of Derivatives
Writing task for Math 122
(30 points)

Due: Monday, October 20th at the beginning of class

In the past two weeks, we have studied the derivative of a function from a variety of perspectives. In particular, we have used graphical, numerical, and symbolic approaches. Imagine you know a classmate who has missed class for the past two weeks, and is anxious to catch up on the material. He needs a summary of the ideas we have covered dealing with derivatives that includes a discussion of the relationships among the ideas.

Write this classmate a letter filling him in on what he has missed. Specifically, your paper should include a brief explanation of the derivative from each of the three perspectives, with an example of each. You will want to choose a nonlinear function *f* and a point *a* at which to evaluate *f'(a)*, and illustrate the three perspectives individually. Be sure to make the relationships among the three approaches clear to your classmate.

Include any graphs, tables, or pictures that illustrate the points you are making. Your paper must be word-processed and will probably be about two pages in length. The assignment points will be distributed as follows:

6 points	Choosing a nonlinear function *f* and selecting the point *a*
8 points	Summary of graphical approach, including relationships
8 points	Summary of numerical approach, including relationships
8 points	Summary of symbolic approach, including relationships
30 points	

Working with Derivatives

I. In the space provided, take the derivative of each of the following functions.
(2 points apiece.)

1. $y = \cos(\tan 3x^2)$

2. $y = (\cos x)(\tan x)$

3. $g(x) = 3/x^2$

4. $y = 3e^x - 4 (\sin x)(\ln x)$

5. $G(x) = 5^{\cos x}$

6. $y = x^2/\sin(4x)$

7. $F(x) = (1 - 4x^3)^2$

8. $H(x) = \sin\ (5x^4)$

9. $y = \cos^3(t)(\sin\ t)$

10. $y = e^{-t^2}$

11. $f(x) = \ln \sqrt{5x^3 - 7x + \pi^3}$

12. $g(x) = \ln (5x^3 (\sin x))$

II. Considering the product rule, quotient rule, and chain rule, divide the 12 functions you just differentiated into categories depending on what rule you used in finding the derivative. You come up with your own category headings. A function may be (and often will be!) in more than one category. (6 points)

Quiz # 4
Chain Rule for Derivatives
(20 points)

Due: Thursday, October 30th at the beginning of class

You are to work the two chain rule problems in the left-hand column. Corresponding to each step you take in the left-hand column, you are to write in the right-hand column your thoughts, justifications, and/or questions. I am interested in whether you understand what you are writing in the left-hand column, not just whether you get the correct answer. When you have finished each problem, the right-hand column should read like a conversation you have with yourself as you are working the problem. This is one way you can write me about what you are fuzzy about, where you need help, what you do and do not understand, etc.

Please be sure I can read your writing!

1) Find the derivative of
$h(x) = (\ln x + 6x^3 + 9)^5$

2) Using the fact that $\sin^{10}(7x^2) = [\sin(7x^2)]^{10}$, find the derivative of

$$k(x) = \sin^{10}(7x^2)$$

Appendix C

Course Examinations

Math 122 Exam # 1

Fall 1997

Show all your work for maximum credit.

1) Suppose the minimum value of f occurs at $x = 4$, and the maximum value of f occurs at $x = 3$. Furthermore, the domain of f is the interval [0, 6] and the range of f is the interval [-5, 2].

   a) Where does the minimum value of $g(x) = -3\,f(x - 1) + 2$ occur? The maximum value?

   b) Determine the domain and the range of $g(x)$.

2) Suppose f' is positive on the interval [-3, 6]. Indicate whether the following statements **must** be true, **might** be true, or **cannot** be true. Justify each answer with a sentence or a sketch.

   a) $-f(x)$ is a decreasing function on [-3, 6].

   b) $f(1) > 0$

3) Give a numerical approximation of how close $f(x) = e^x$ is to $g(x) = \frac{x^2}{2} + x + 1$ over the interval [-2, 2]. (In other words, what is the worst possible error over that interval?)

4)

x	1	2	3	4	5	6	7
$f(x)$	3	-7	19	4	178	2	1

a) Write a table for f^{-1}, where f is given as above.

b) The domain of f is the integers from 1 to 7. State the domain of f^{-1}.

5) If the price of wheat is increasing by 5 % a year, in how many years will the price of wheat double?

6) Complete the following table to show values for functions f, g, and h given the following conditions:

a) f is symmetric about the y-axis

b) g is an odd function

c) $h(x) = g(f(x))$

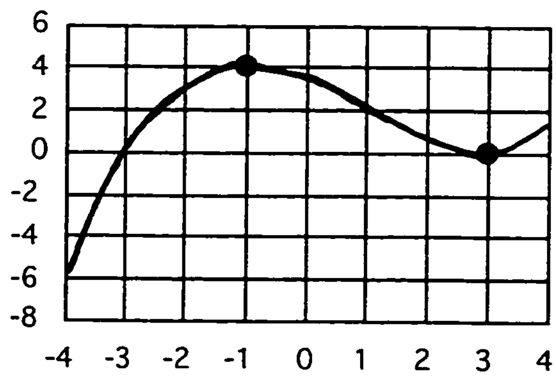
x	f(x)	g(x)	h(x)
-3	0	0	
-2	2	2	
-1	2	2	
0	0	0	
1			
2			
3			

7) Suppose that $f'(3) = -2$. Assume that f is an even function. Explain why $f'(-3) = 2$. (Give a graphical argument.)

8) Let $G(v)$ be the number of miles per gallon that a vehicle gets as a function of its speed, v, in miles per hour. Interpret the statement $G'(35) = 0.4$ in plain English using terms such as gas mileage and speed.

9) Suppose that $f(-1) = -2$ and that f' is the function shown below. Answer the following questions about f. (Note that the graph of f is not shown.)

Graph of f'



a) Could $f(3) = -6$? Justify your answer.

b) Could $f(3) = 2$? Justify your answer.

c) Where in the interval $[-4, 4]$ does f achieve its smallest value?

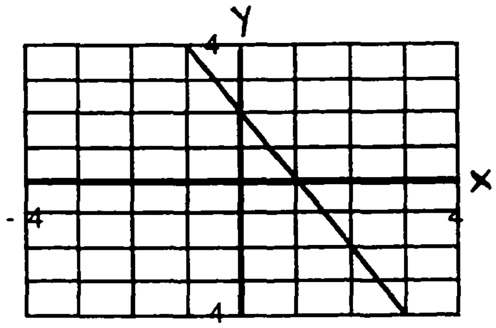
Math 122 Exam # 2

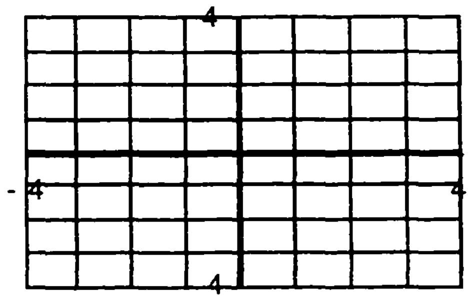
Fall 1997

Show all your work for maximum credit.

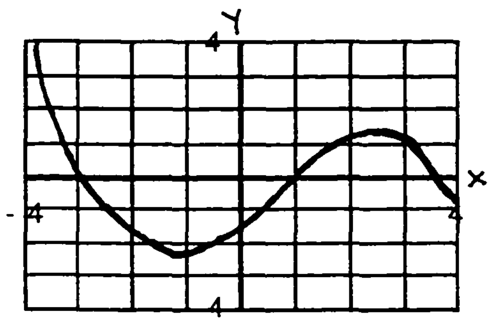
1) Sketch a graph of the derivative function of each of the following functions:

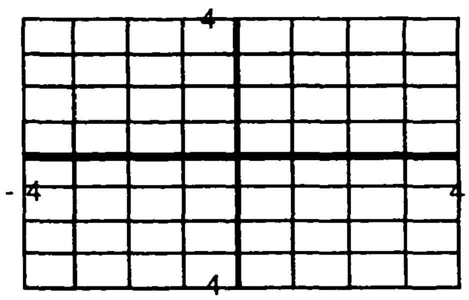
a)





b)





2) Use the limit definition of the derivative to find the derivative of $f(x) = 3x^2 + x$.

3) The graph of the derivative of a function g is shown below. The graph of g' has a local maximum at x = 3.8 and a local minimum at x = 7.4. Use the graph of g' to answer the following questions about g. <u>There is no need to justify your work</u>. [NOTE: The graph of g is not shown.]

a) On which intervals is g increasing?

Graph of g'

b) Where does g have stationary points?

c) Where does g have a local minimum?

d) On which intervals is g concave down?

e) Where does g have a point of inflection?

f) Where does g achieve its maximum value on [0, 7]?

g) Where does g achieve its minimum value on [6, 10]?

4) Suppose that the line $y = 2x - 1$ is tangent to the graph of f at $x = 4$ and that $f''(x) > 0$ on [0, 6].

a) Find f(4).

b) Find f'(4).

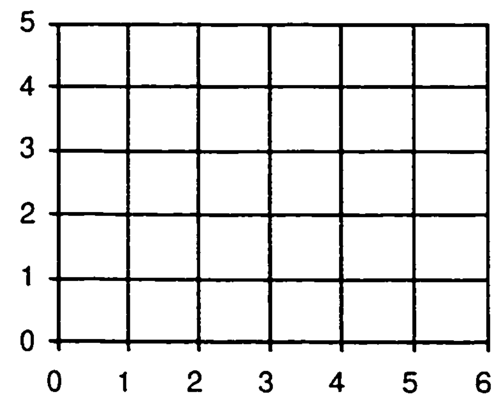
c) Find a lower bound on the value of f(3).

5) Let $f(x) = \begin{cases} x & \text{if } x \text{ is not an integer} \\ 0 & \text{if } x \text{ is an integer} \end{cases}$

a) Draw a graph of f over the interval $[1,\ 4]$.

b) Evaluate $\lim\limits_{x \to 2} f(x)$

c) Evaluate $\lim\limits_{x \to \frac{5}{2}} f(x)$

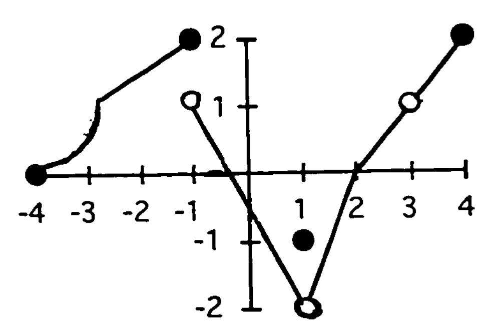


d) For which values of a in the interval $(1,\ 4)$ does $\lim\limits_{x \to a} f(x)$ exist?

6) Let f be the function whose graph is shown below. Evaluate each of the limits that follow or explain why the limit does not exist.

a) $\lim\limits_{x \to -1^-} f(x)$

Graph of f



b) $\lim\limits_{x \to -1^+} f(x)$

c) $\lim\limits_{x \to -1} f(x)$

d) $\lim\limits_{x \to 1^-} f(x)$

e) $\lim\limits_{x \to 1} f(x)$

f) $\lim\limits_{x \to 0} f(x)$

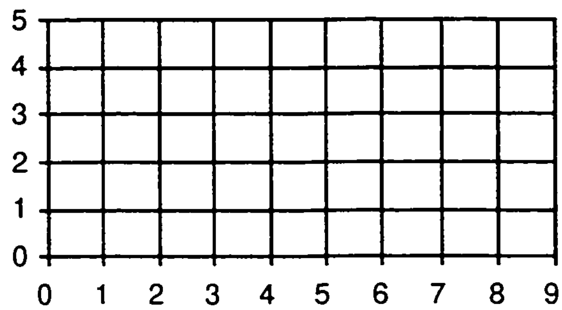
7) In order for a function f to be continuous at a point $x = a$, all three of the following must be satisfied:

i) $f(a)$ must be defined (i.e., a must be in the domain of f)

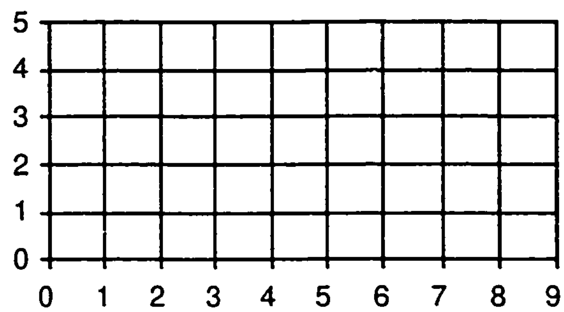
ii) $\lim_{x \to a} f(x)$ must exist

iii) $\lim_{x \to a} f(x) = f(a)$

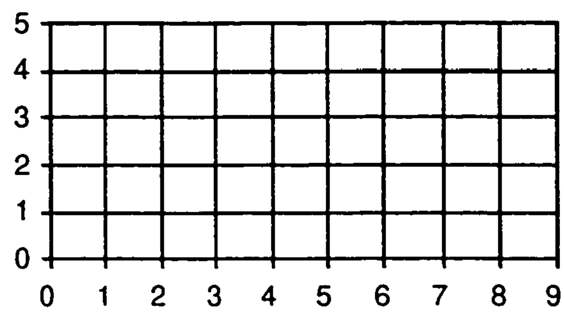
a) Draw the graph of a function f for which condition i) fails for $a = 3$.



b) Draw the graph of another function f for which condition i) holds but condition ii) fails for $a = 3$.



c) Draw the graph of another function f for which conditions i) and ii) hold but condition iii) fails for $a = 3$.



8) Complete the following table:

Function f(x)	f'(x), the derivative of f	F(x), an antiderivative of f
3^x		
sin x		
$4x^5 - 3x^4 + 6x - 9$		
$6e^x$		

9) Some numbers are smaller than their squares (2 is smaller than 4), some are larger than their squares (1/3 is larger than 1/9), and some are equal to their squares (0 and 1). Any number x with $0 \le x \le 1$ will be greater than or equal to its square. Among all such numbers, find the ones that differ from their squares the most, that is, for which the function $s(x) = x - x^2$ is a maximum.

Math 122 Exam # 3

Fall 1997

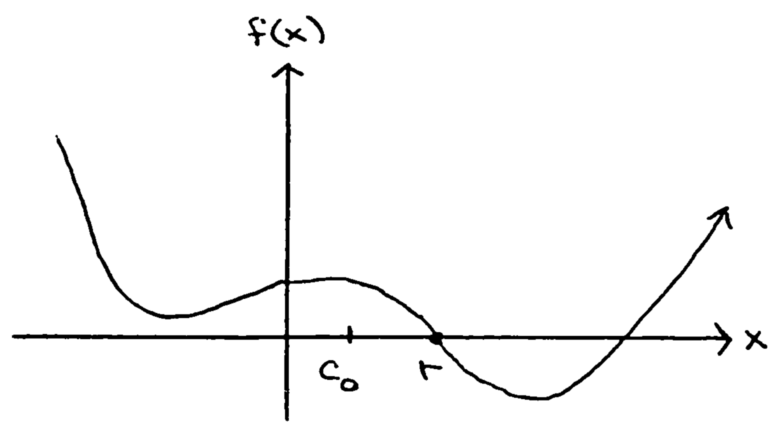
Show all your work for maximum credit.

1) Is the function $y(x) = \sqrt{4x + 3}$ a solution to the differential equation $y \cdot y' = 2$? Explain.

2) Tomato soup poured into a ceramic mug has a temperature of 200° F. Assume the room temperature is a constant 68° F, and that after 3 minutes, the soup has cooled to 160° F. Find a formula for the temperature of the soup at time t.

3). Find the second degree Taylor polynomial that approximates $f(x) = \ln x$ at $x = 2$.

4). To find the value of the root r, Newton's Method is used, with a starting value at $x = c_0$.



a) Show the next two approximations c_1 and c_2 on the graph.

b) Explain what will happen as we continue the process.

5) Find the dimensions of the rectangle of largest area with base on the x-axis and upper vertices on the graph of $y = 5 - x^2$. (Sketch it first!)

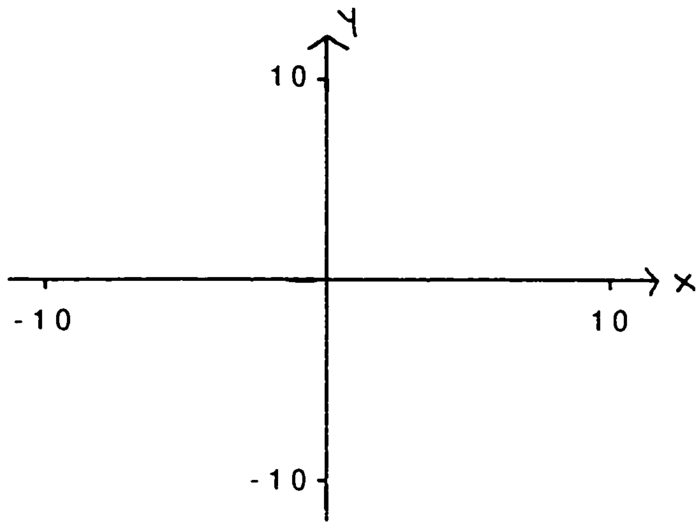
6) A girl starts at a point A and runs east at a rate of 10 feet per second. One second later, another girl starts at A and runs north at a rate of 8 feet per second. At what rate is the distance between them changing two seconds after the second girl starts?

7) Find a number c in the interval (0, 5) that satisfies the Mean Value Theorem for $f(x) = x^2 + 3x - 1$.

8a) Sketch the parametric curve given by

$x = 4(1 - \sin t)$

$y = 4(1 - \cos t)$ for $0 \leq t \leq 2\pi$.



b) Mark the direction of travel and label the point corresponding to $t = \pi$.

c) What is the slope of the curve at $t = \pi/4$?

9) Determine whether the following statement **must** be true, **might** be true, or **cannot** be true. Mention the name of the appropriate theorem and justify your conclusion. (You may also find diagrams useful.)

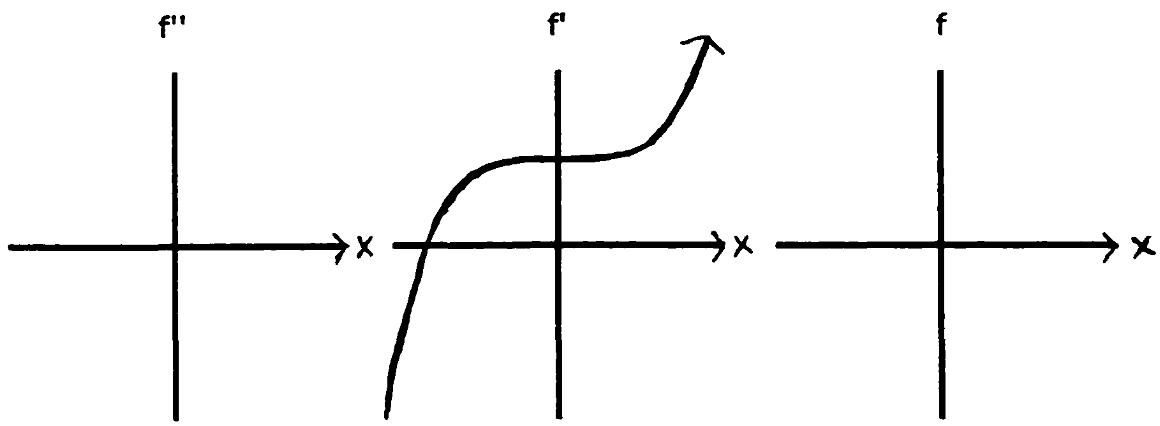
If f is continuous and has no roots in [2, 7], then $f(2) \cdot f(7) < 0$.

Math 122 Final Exam

Fall 1997

Show all your work for maximum credit.

1) Given the graph of f' below, sketch curves representing f" and f, respectively. Assume f goes through the origin. No scale is given, so simply sketch an estimate.



2) Let f(t) represent the number of books that are sold at a new bookstore t days after it opens. Explain the equations f(10) = 260 and f'(10) = -20 in terms of sales.

3) Find $\int_{-3}^{3} x(\cos x)dx$. (Hint: Think of the geometrical interpretation of the definite integral.)

4) Let $f(x) = 2x^3 - 12x^2 + 18x - 3$. Find the stationary points, inflection points, and the maximum and minimum values of f over the interval [0, 5].

5) Find the third-order Taylor polynomial for $f(x) = \sin x + \cos x$ at the point $x_0 = \pi$.

6) Find the average value of the function $g(x) = 2^x$ over the interval [2, 7].

7) What is the equation of the tangent to the function $y = \ln [3x (\cos x)]$ at $x = 1$? (Rounding during the calculation process is acceptable.)

8) Let f be a function that is continuous and differentiable on the interval [2, 7]. Furthermore, let $f'(2) = 5$ and $f'(7) = -1$. Which of the MVT, IVT, or EVT guarantees a stationary point in (2, 7), and why?

9) Given $x^2y + y^2x = 2$, find $\frac{dy}{dx}$. What is the slope of the curve at the point (1, 1)?

10) Find the area bounded by the curves $y = x^2 - 4$ and $y = 2 - x$.

11) Approximate $\int_0^2 \sin(x^2)dx$ using the right sum rule with 4 subdivisions.

12) Use the difference quotient with $h = .01$ to approximate $f'(2)$ for $f(x) = \frac{\ln x}{x^2}$.
Then find the exact value.

Appendix D

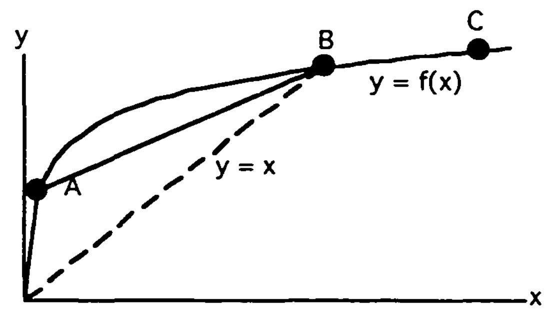
"Think Aloud" Problem Session Tasks and Time-Line Graphs

"Think Aloud" Session #1

Task 1:

For the graph y = f(x) shown below, arrange the following numbers in ascending (i.e. smallest to largest) order:

- The slope of the curve at A.
- The slope of the curve at B.
- The slope of the curve at C.
- The slope of the line AB.
- The number 0.
- The number 1.



Total Time = 2 minutes 24 seconds

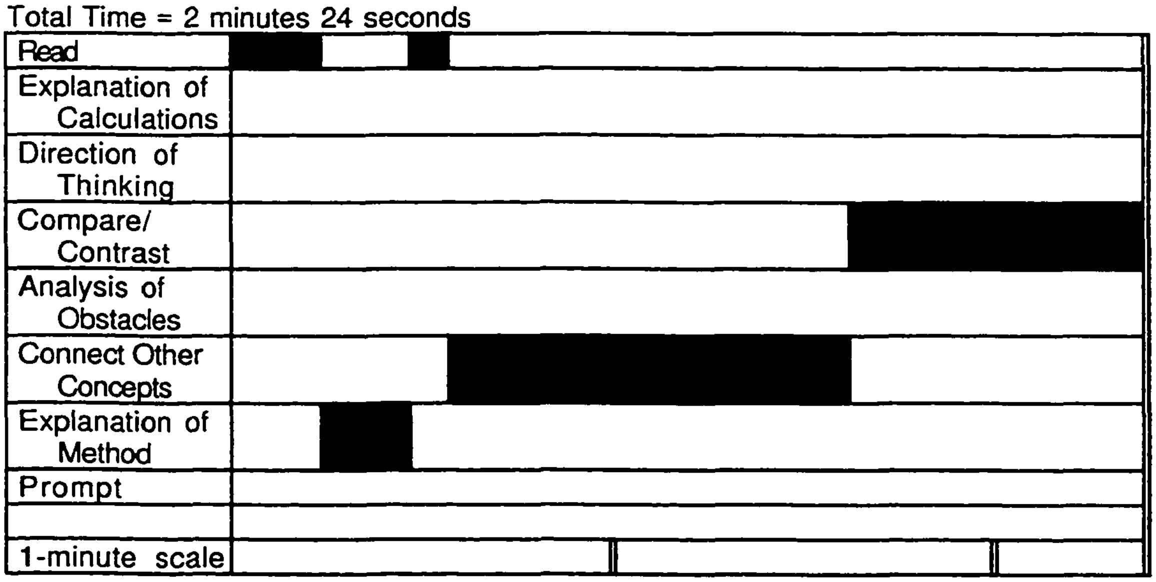


Figure 30. Audra's Time-Line Graph for Problem Session #1, Task 1.

"Think Aloud" Session #1

Task 2:

Suppose $f(0) = 2$ and $|f'(x)| < 1$ for all x in $[-5, 5]$.

a) Find upper and lower bounds on the value of $f(1)$, that is, find numbers U and L so that $L < f(1) < U$.

b) Find upper and lower bounds on the value of $f(-3)$, that is, find new numbers U and L so that $L < f(-3) < U$.

Total Time = 5 minutes 6 seconds

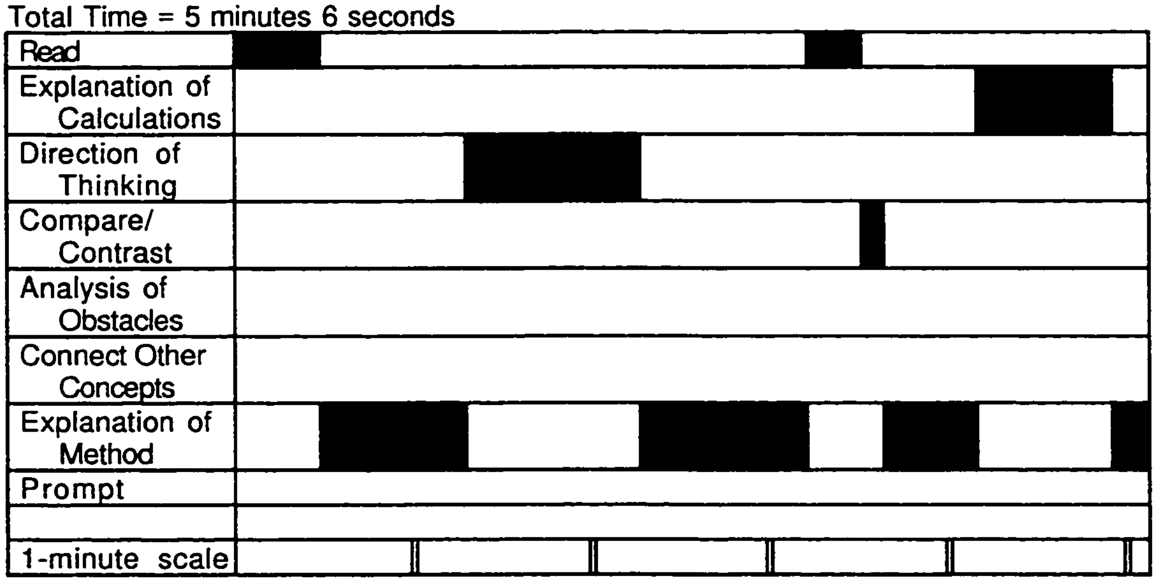


Figure 31. Audra's Time-Line Graph for Problem Session #1, Task 2.

"Think Aloud" Session #1

Task 3:

A company's revenue from car sales, C (measured in thousands of dollars), is a function of advertising expenditure, a, also measured in thousands of dollars. Suppose $C = f(a)$.

a) What does the company hope is true about the sign of f'? Explain.

b) What does the statement $f'(100) = 2$ mean in practical terms? How about $f'(100) = 0.5$?

c) Suppose the company plans to spend about $ 100,000 on advertising. If $f'(100) = 2$, should the company spend slightly more or slightly less than $ 100,000 on advertising? What if $f'(100) = 0.5$? Explain your answers.

Total Time = 7 minutes 59 seconds

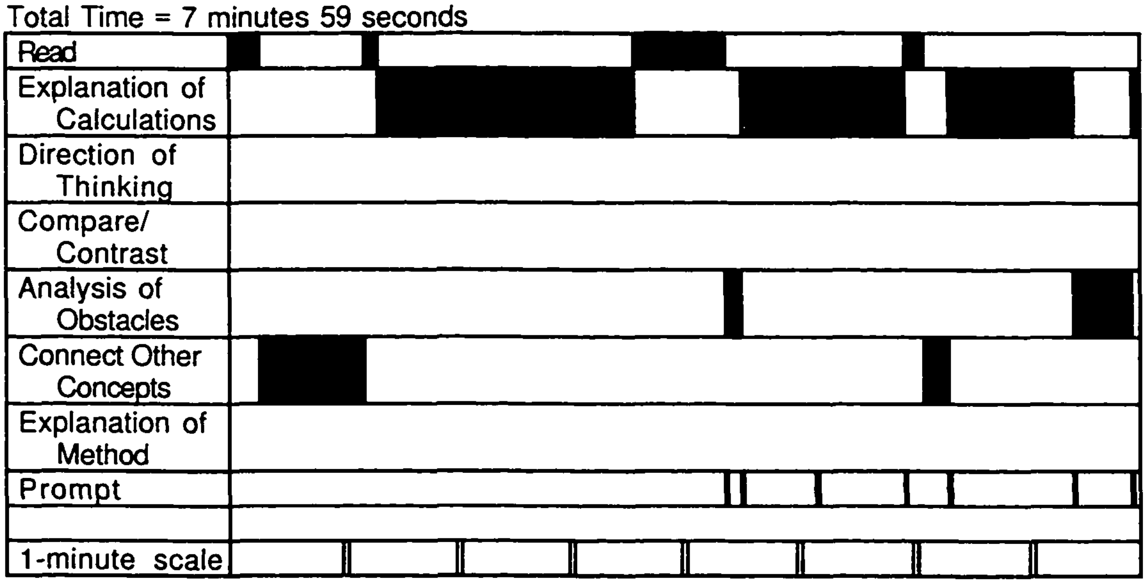


Figure 32. Audra's Time-Line Graph for Problem Session #1, Task 3

"Think Aloud" Session #2

Task 1:

If $f(x) = 13 - 8x + \sqrt{2}x^2$ and $f'(r) = 4$, find r.

Total Time = 2 minutes 21 seconds

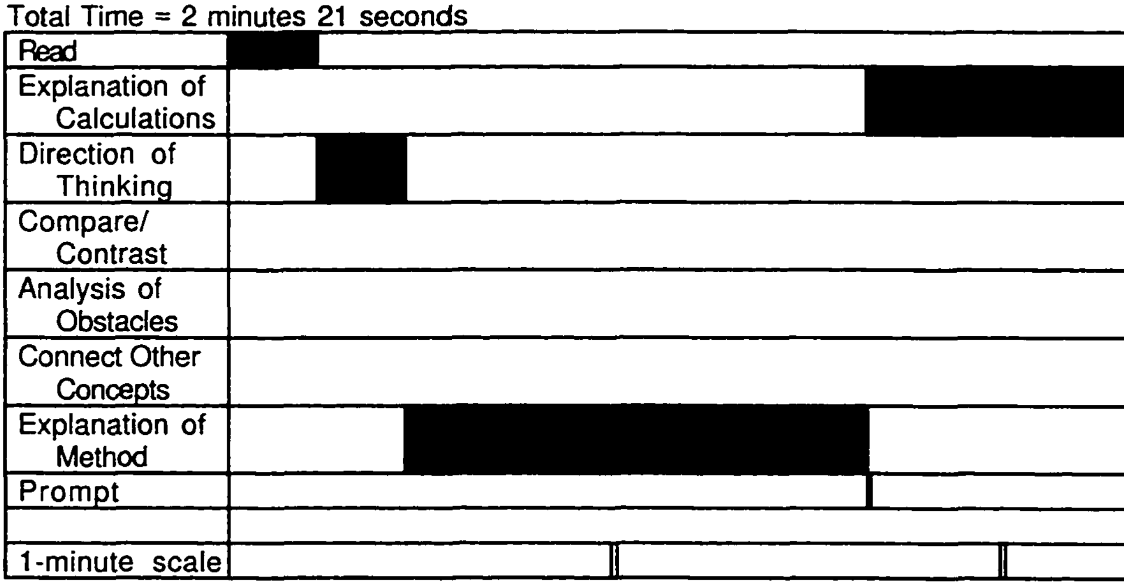


Figure 33. Audra's Time-Line Graph for Problem Session #2, Task 1.

"Think Aloud" Session #2

Task 2:

Suppose the slope of the tangent line at any point P on the graph of an equation f equals the square of the x-coordinate of the point P. Find the equation for f if the graph contains:

a) the origin

b) the point (3,6)

Total Time = 3 minutes 29 seconds

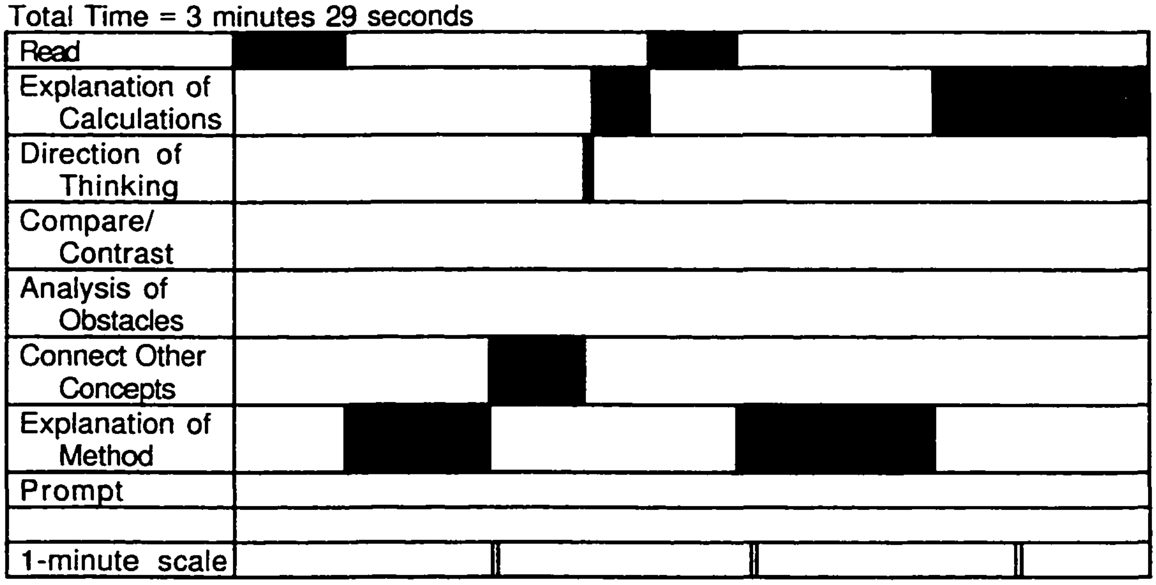


Figure 34. Audra's Time-Line Graph for Problem Session #2, Task 2.

"Think Aloud" Session #2

Task 3:

The *Global 2000 Report* gave the world's population, P, as 4.1 billion in 1975 and growing at 2% annually.

a) Give a formula for P in terms of time, t, measured in years since 1975.

b) Find each of the following and discuss what each represents in practical terms:

i) $P'(t)$

ii) $P'(0)$

iii) $P'(15)$

Total Time = 7 minutes 19 seconds

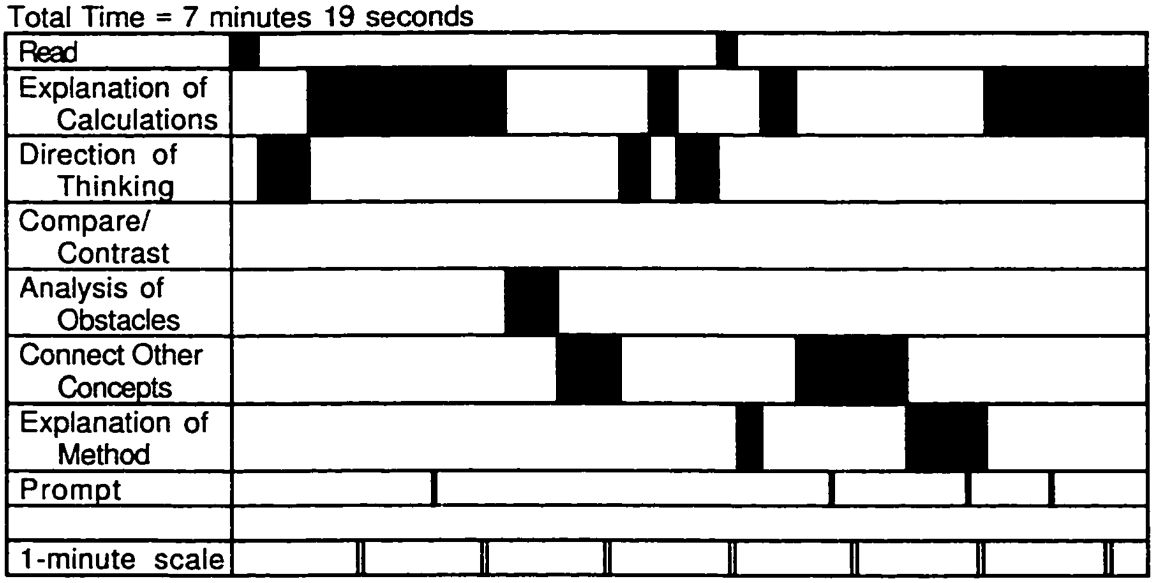


Figure 35. Audra's Time-Line Graph for Problem Session #2, Task 3.

"Think Aloud" Session #3

Task 1:

Plot the curve defined by the equation $y^2 = x^3$ and find the equations of the two lines tangent to the curve at $x = 1$.

Total Time = 5 minutes 16 seconds

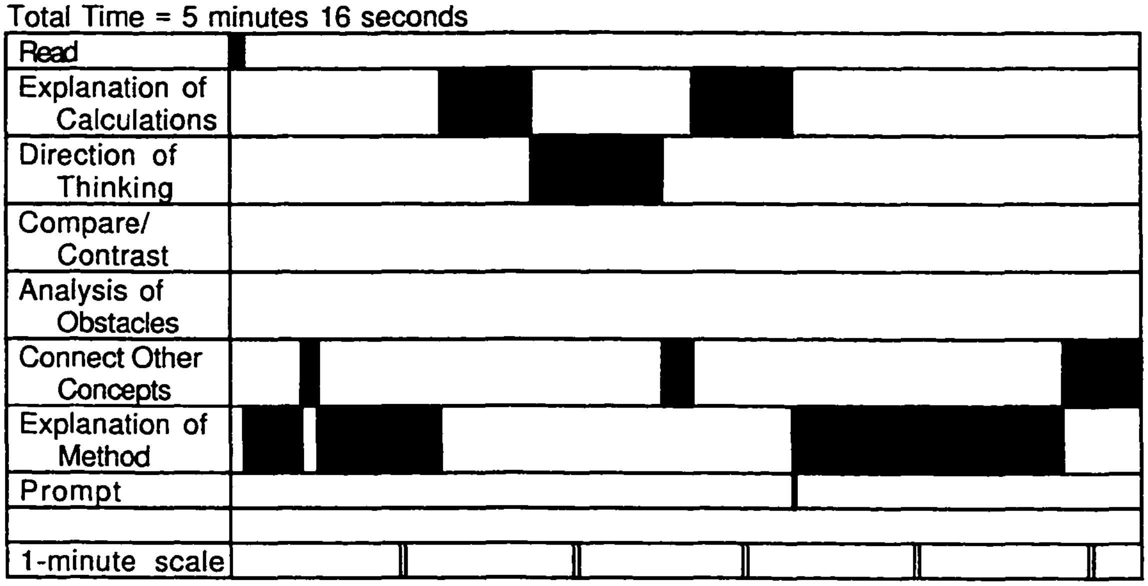


Figure 36. Audra's Time-Line Graph for Problem Session #3, Task 1.

"Think Aloud" Session #3

Task 2:

A mold grows at a rate proportional to the amount present. Initially, its weight is 2 grams; after two days, it weighs 5 grams. How much does it weigh after eight days?

Total Time = 4 minutes 12 seconds

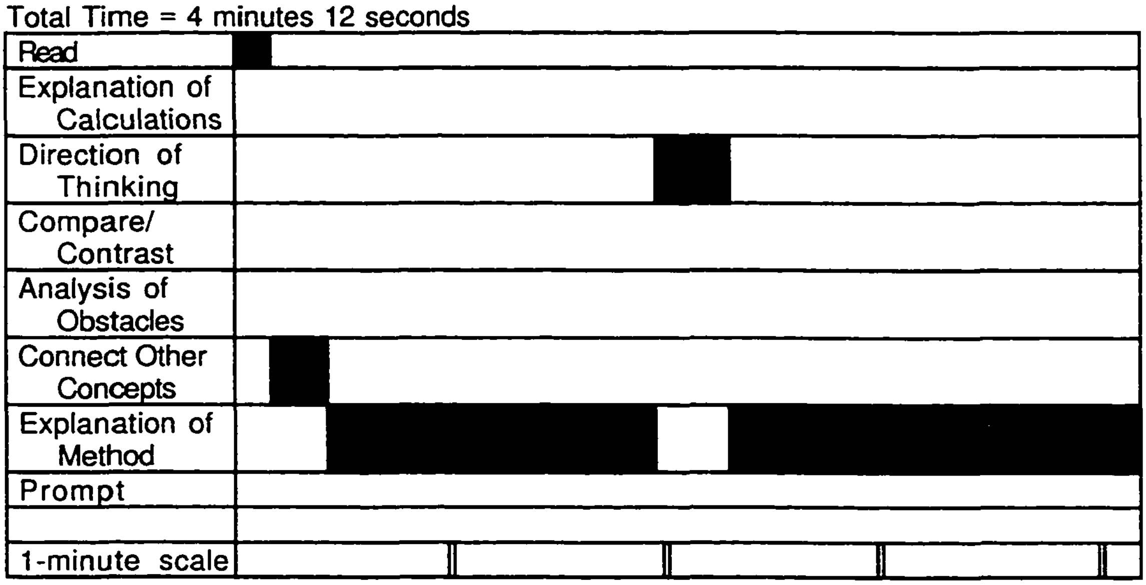


Figure 37. Audra's Time-Line Graph for Problem Session #3, Task 2.

"Think Aloud" Session #3

Task 3:

A rectangle has its base on the x-axis, a vertex on the y-axis, and a vertex on the curve $y = e^{-x^2}$.

a) What choice of vertices gives the largest area?

b) Show that one of the vertices found in part (a) is at an inflection of the curve.

Total Time = 16 minutes 30 seconds

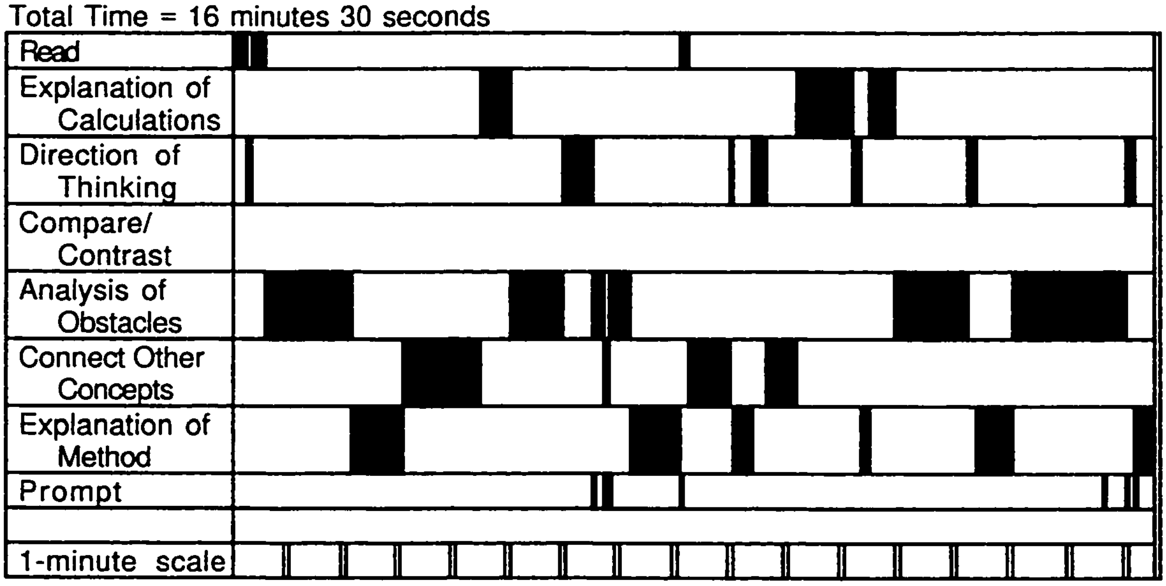


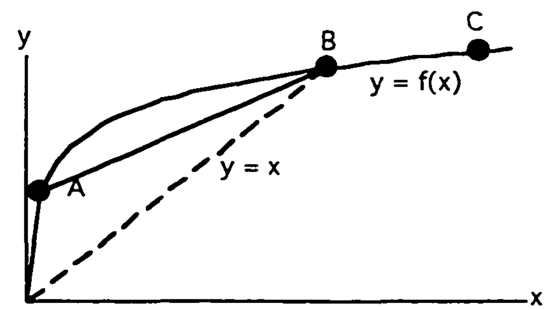
Figure 38. Audra's Time-Line Graph for Problem Session #3, Task 3.

"Think Aloud" Session #1

Task 1:

For the graph y = f(x) shown below, arrange the following numbers in ascending (i.e. smallest to largest) order:

- The slope of the curve at A.
- The slope of the curve at B.
- The slope of the curve at C.
- The slope of the line AB.
- The number 0.
- The number 1.



Total Time = 3 minutes 58 seconds

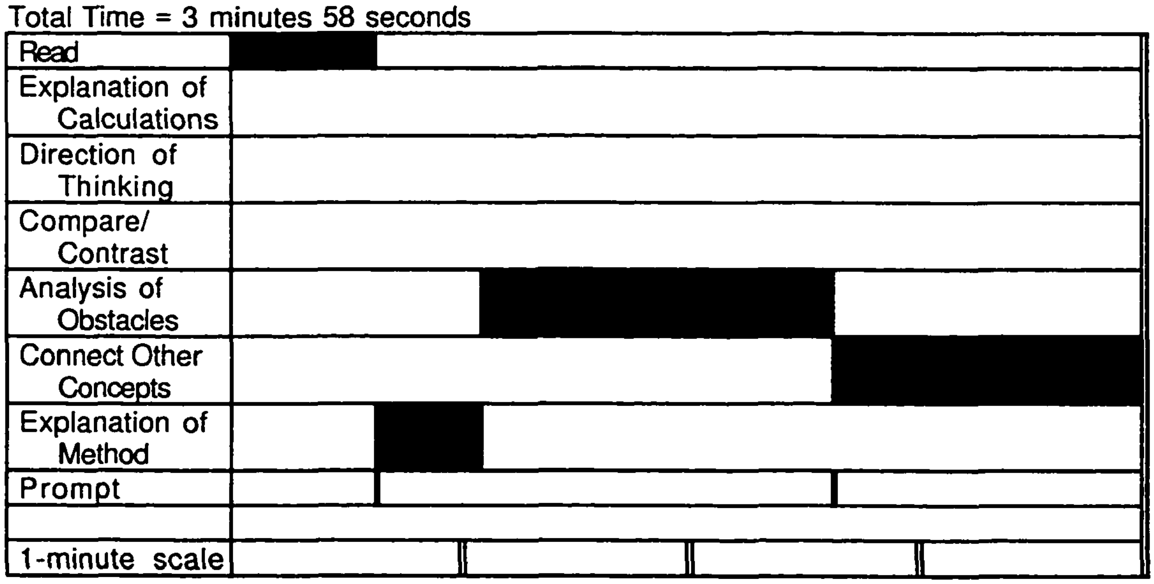


Figure 39. Barbara's Time-Line Graph for Problem Session #1, Task 1.

"Think Aloud" Session #1

Task 2:

Suppose f(0) = 2 and |f'(x)| < 1 for all x in [-5, 5].

a) Find upper and lower bounds on the value of f(1), that is, find numbers U and L so that L < f(1) < U.

b) Find upper and lower bounds on the value of f(-3), that is, find new numbers U and L so that L < f(-3) < U.

Total Time = 8 minutes 59 seconds

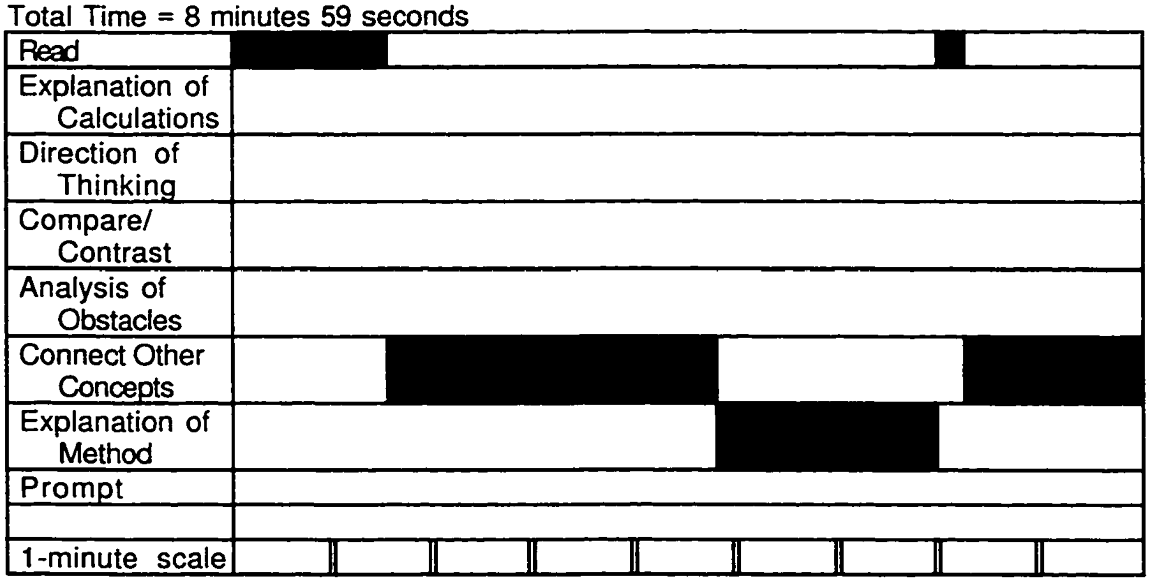


Figure 40. Barbara's Time-Line Graph for Problem Session #1, Task 2.

"Think Aloud" Session #1

Task 3:

A company's revenue from car sales, C (measured in thousands of dollars), is a function of advertising expenditure, a, also measured in thousands of dollars. Suppose $C = f(a)$.

a) What does the company hope is true about the sign of f'? Explain.

b) What does the statement $f'(100) = 2$ mean in practical terms? How about $f'(100) = 0.5$?

c) Suppose the company plans to spend about $ 100,000 on advertising. If $f'(100) = 2$, should the company spend slightly more or slightly less than $ 100,000 on advertising? What if $f'(100) = 0.5$? Explain your answers.

Total Time = 14 minutes 57 seconds

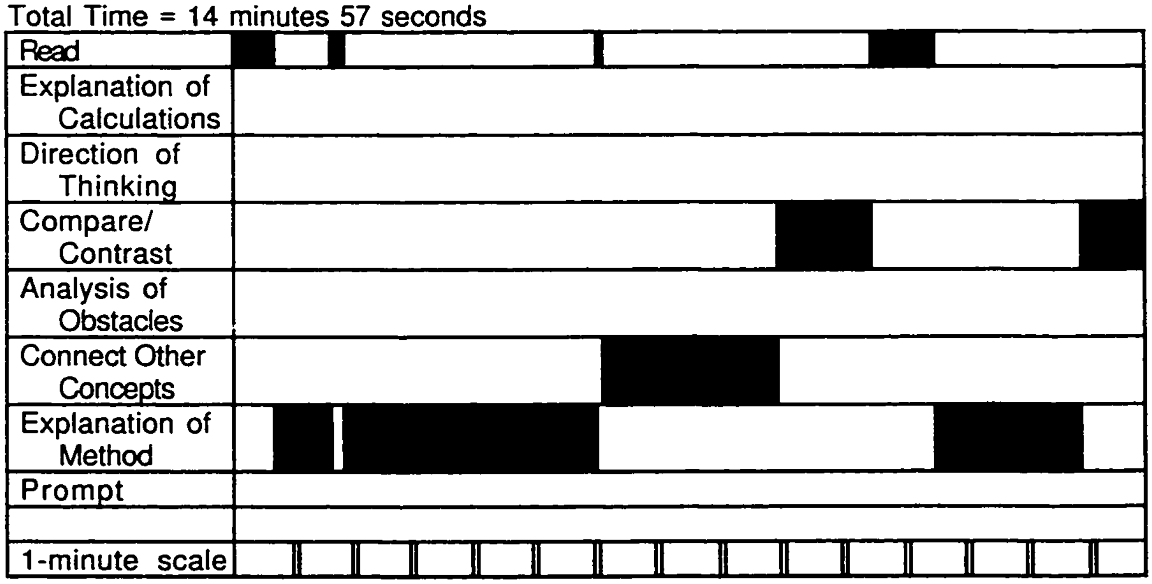


Figure 41. Barbara's Time-Line Graph for Problem Session #1, Task 3.

"Think Aloud" Session #2

Task 1:

If $f(x) = 13 - 8x + \sqrt{2}x^2$ and $f'(r) = 4$, find r.

Total Time = 3 minutes 26 seconds

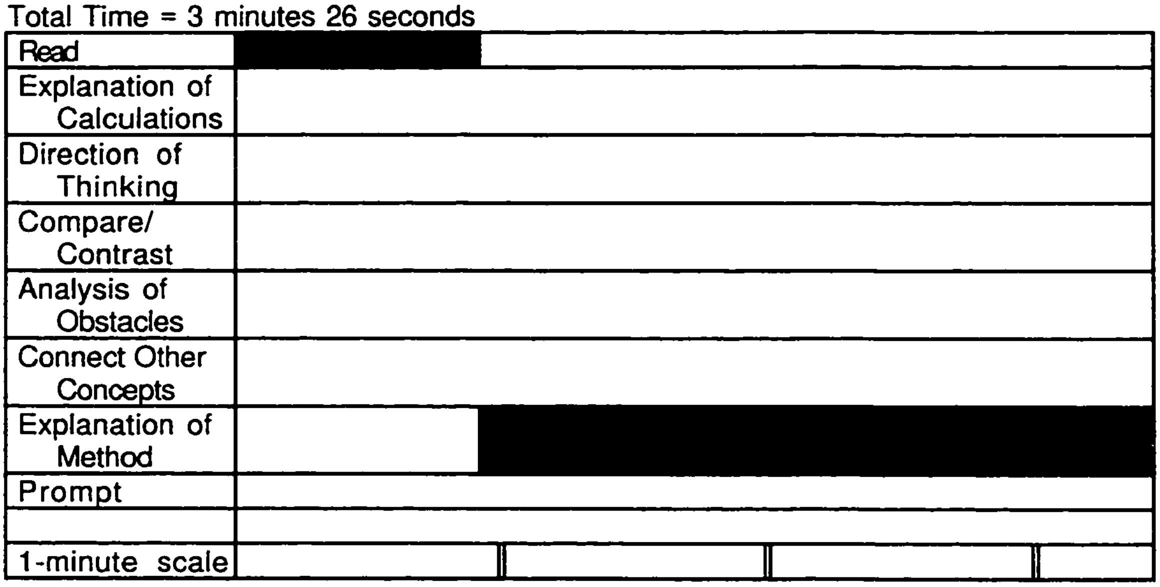


Figure 42. Barbara's Time-Line Graph for Problem Session #2, Task 1.

"Think Aloud" Session #2

Task 2:

Suppose the slope of the tangent line at any point P on the graph of an equation f equals the square of the x-coordinate of the point P. Find the equation for f if the graph contains:

a) the origin

b) the point (3,6)

Total Time = 9 minutes 8 seconds

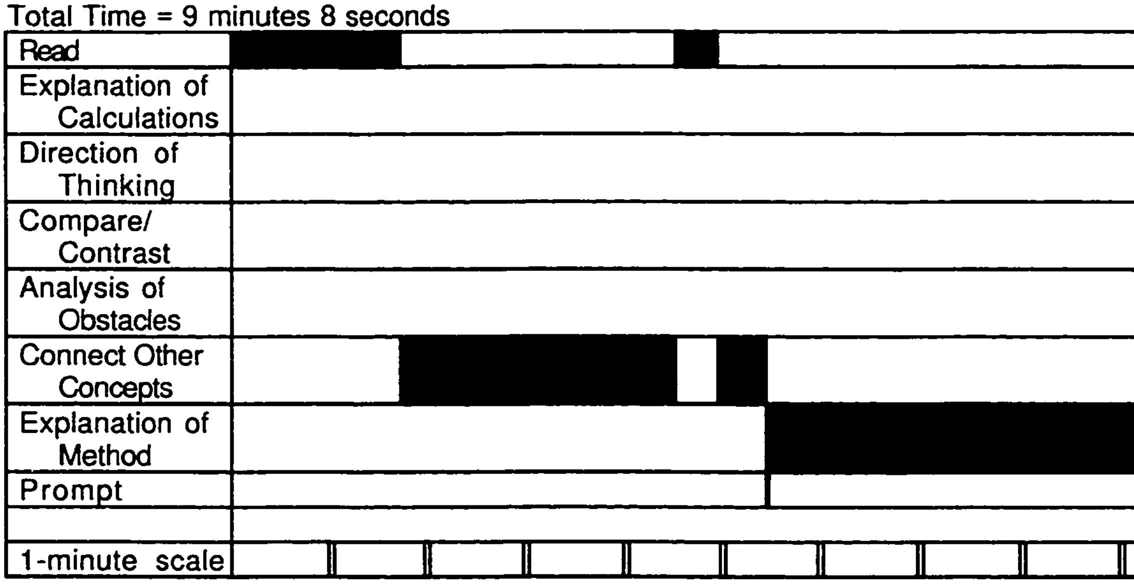


Figure 43. Barbara's Time-Line Graph for Problem Session #2, Task 2.

"Think Aloud" Session #2

Task 3:

The *Global 2000 Report* gave the world's population, P, as 4.1 billion in 1975 and growing at 2% annually.

a) Give a formula for P in terms of time, t, measured in years since 1975.

b) Find each of the following and discuss what each represents in practical terms:

i) $P'(t)$

ii) $P'(0)$

iii) $P'(15)$

Total Time = 7 minutes 22 seconds

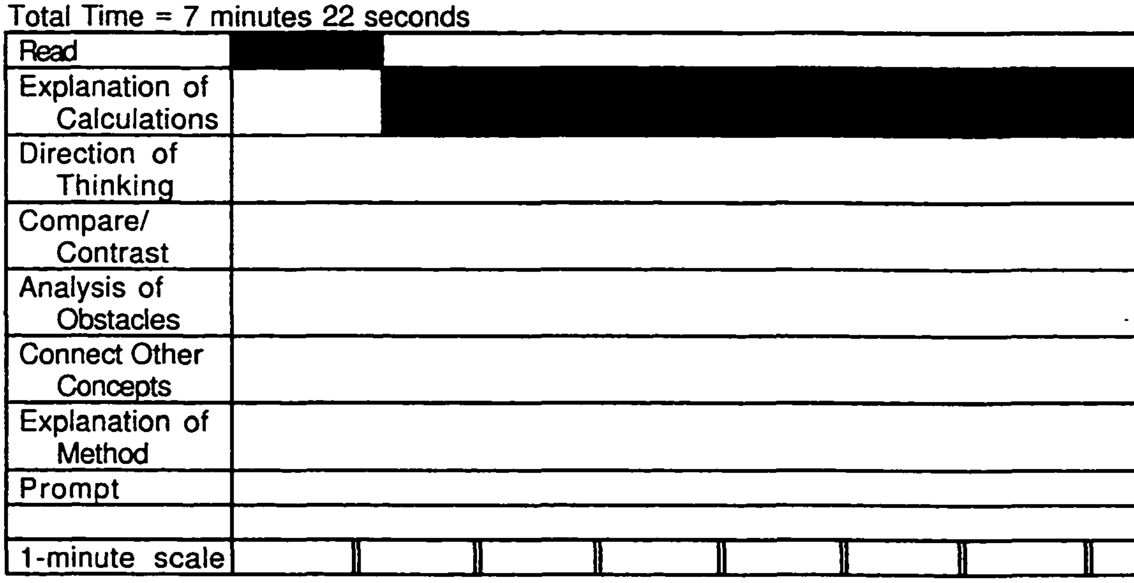


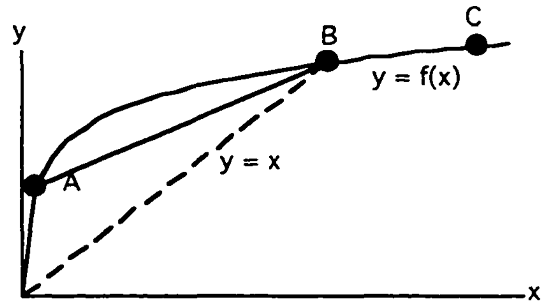
Figure 44. Barbara's Time-Line Graph for Problem Session #2, Task 3.

"Think Aloud" Session #1

Task 1:

For the graph $y = f(x)$ shown below, arrange the following numbers in ascending (i.e. smallest to largest) order:

- The slope of the curve at A.
- The slope of the curve at B.
- The slope of the curve at C.
- The slope of the line AB.
- The number 0.
- The number 1.



Total Time = 2 minutes 59 seconds

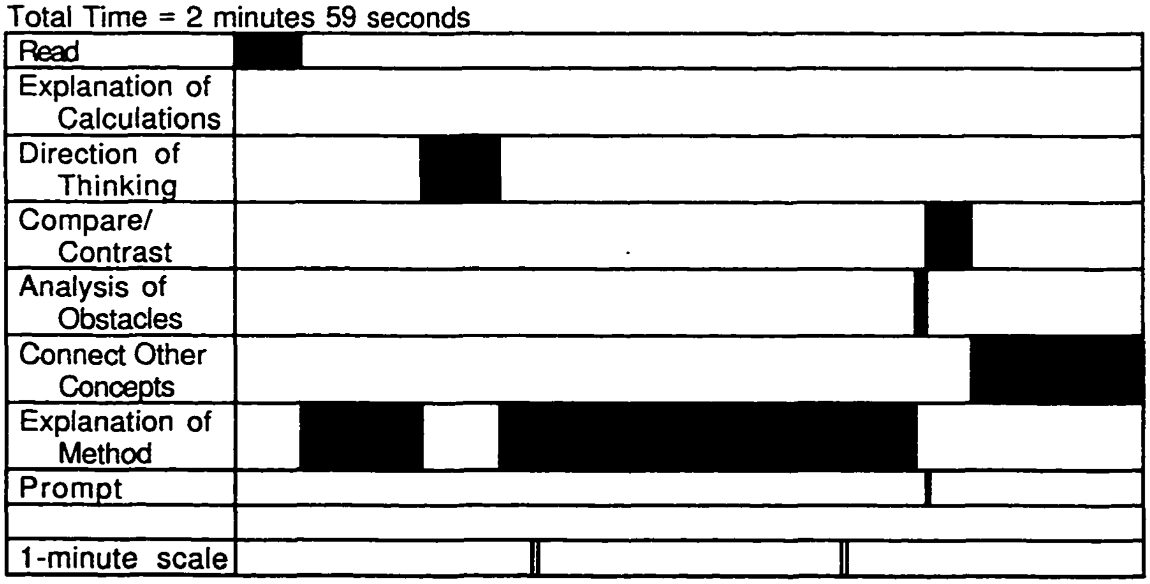


Figure 45. Carl's Time-Line Graph for Problem Session #1, Task 1.

"Think Aloud" Session #1

Task 2:

Suppose f(0) = 2 and $|f'(x)| < 1$ for all x in [-5, 5].

a) Find upper and lower bounds on the value of f(1), that is, find numbers U and L so that $L < f(1) < U$.

b) Find upper and lower bounds on the value of f(-3), that is, find new numbers U and L so that $L < f(-3) < U$.

Total Time = 10 minutes 28 seconds

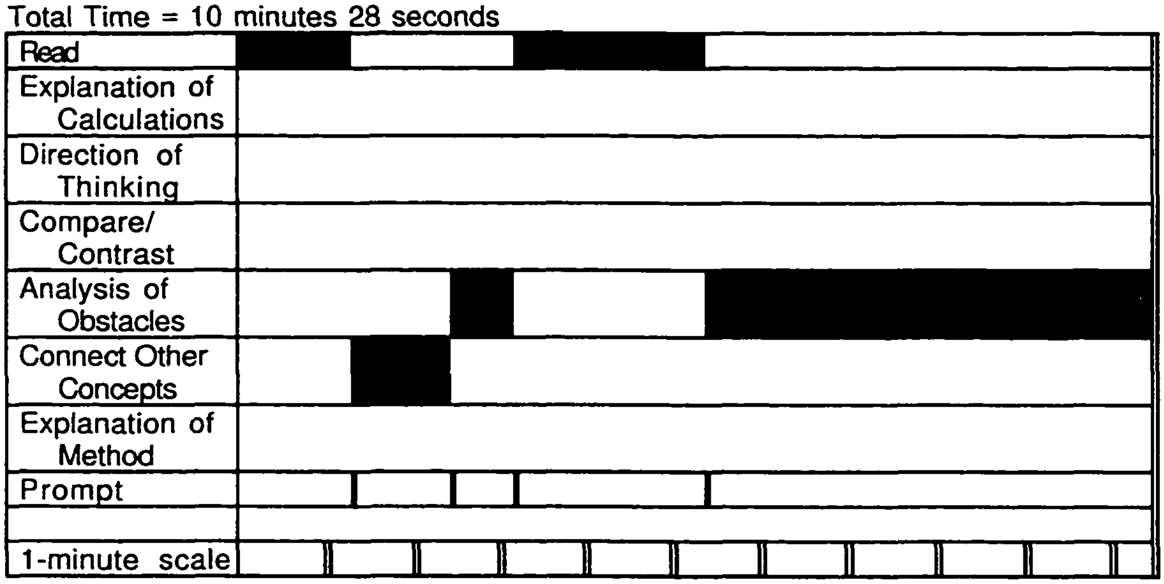


Figure 46. Carl's Time-Line Graph for Problem Session #1, Task 2.

"Think Aloud" Session #1

Task 3:

A company's revenue from car sales, C (measured in thousands of dollars), is a function of advertising expenditure, a, also measured in thousands of dollars. Suppose $C = f(a)$.

a) What does the company hope is true about the sign of f'? Explain.

b) What does the statement $f'(100) = 2$ mean in practical terms? How about $f'(100) = 0.5$?

c) Suppose the company plans to spend about \$ 100,000 on advertising. If $f'(100) = 2$, should the company spend slightly more or slightly less than \$ 100,000 on advertising? What if $f'(100) = 0.5$? Explain your answers.

Total Time = 8 minutes 48 seconds

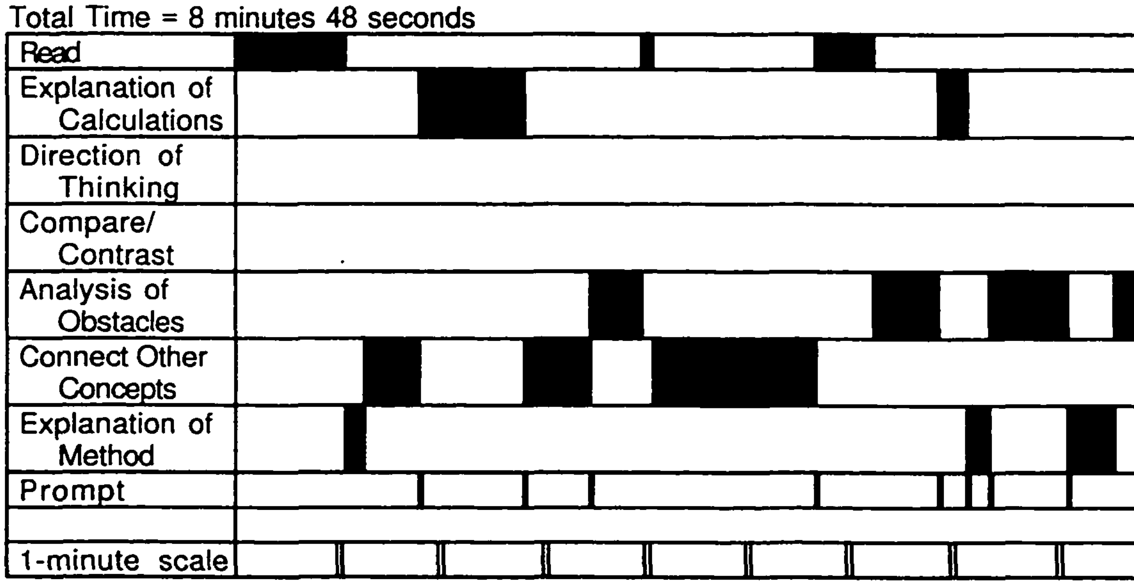


Figure 47. Carl's Time-Line Graph for Problem Session #1, Task 3.

"Think Aloud" Session #2

Task 1:

If $f(x) = 13 - 8x + \sqrt{2}x^2$ and $f'(r) = 4$, find r.

Total Time = 3 minutes 12 seconds

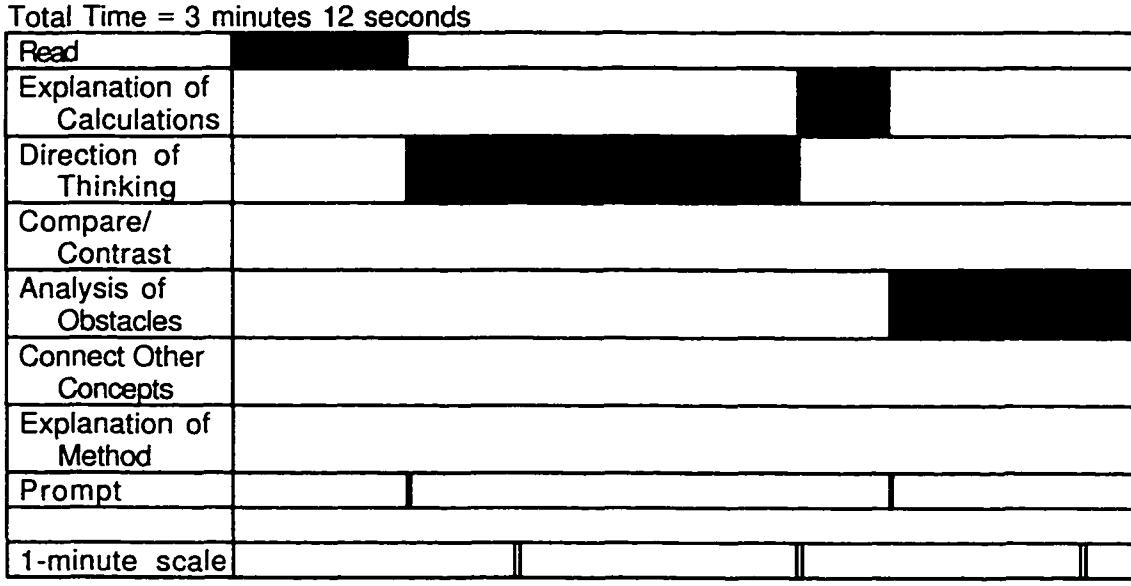


Figure 48. Carl's Time-Line Graph for Problem Session #2, Task 1.

"Think Aloud" Session #2

Task 2:

Suppose the slope of the tangent line at any point P on the graph of an equation f equals the square of the x-coordinate of the point P. Find the equation for f if the graph contains:

a) the origin

b) the point (3,6)

Total Time = 4 minutes 1 second

Figure 49. Carl's Time-Line Graph for Problem Session #2, Task 2.

"Think Aloud" Session #2

Task 3:

The *Global 2000 Report* gave the world's population, P, as 4.1 billion in 1975 and growing at 2% annually.

a) Give a formula for P in terms of time, t, measured in years since 1975.

b) Find each of the following and discuss what each represents in practical terms:

i) $P'(t)$

ii) $P'(0)$

iii) $P'(15)$

Total Time = 1 minute 10 seconds

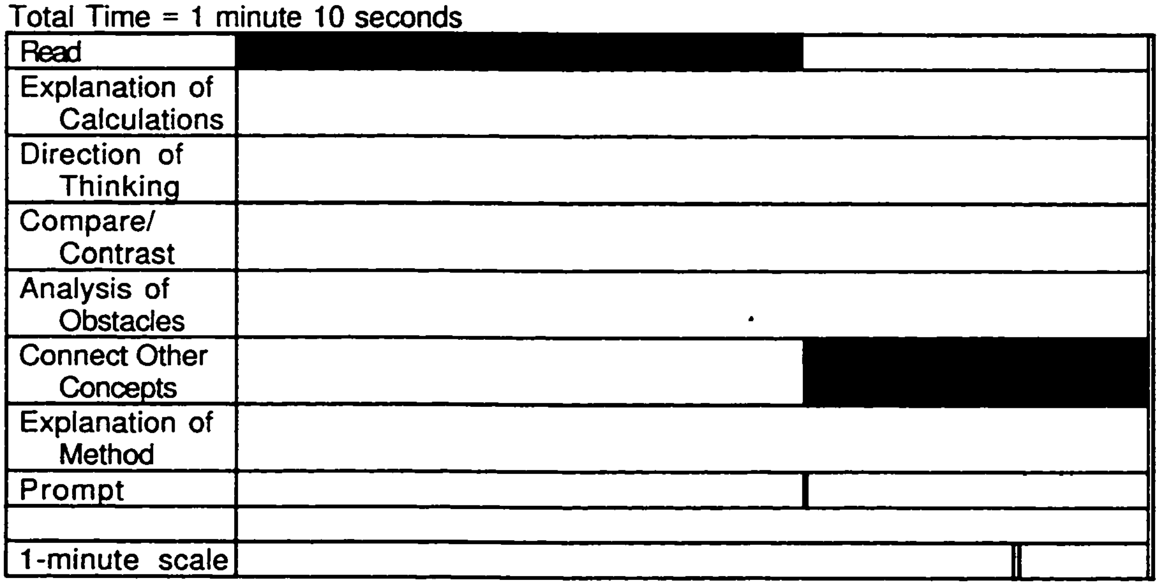


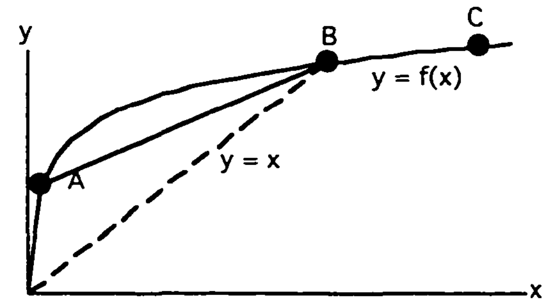
Figure 50. Carl's Time-Line Graph for Problem Session #2, Task 3.

"Think Aloud" Session #1

Task 1:

For the graph y = f(x) shown below, arrange the following numbers in ascending (i.e. smallest to largest) order:

- The slope of the curve at A.
- The slope of the curve at B.
- The slope of the curve at C.
- The slope of the line AB.
- The number 0.
- The number 1.



Total Time = 1 minute 50 seconds

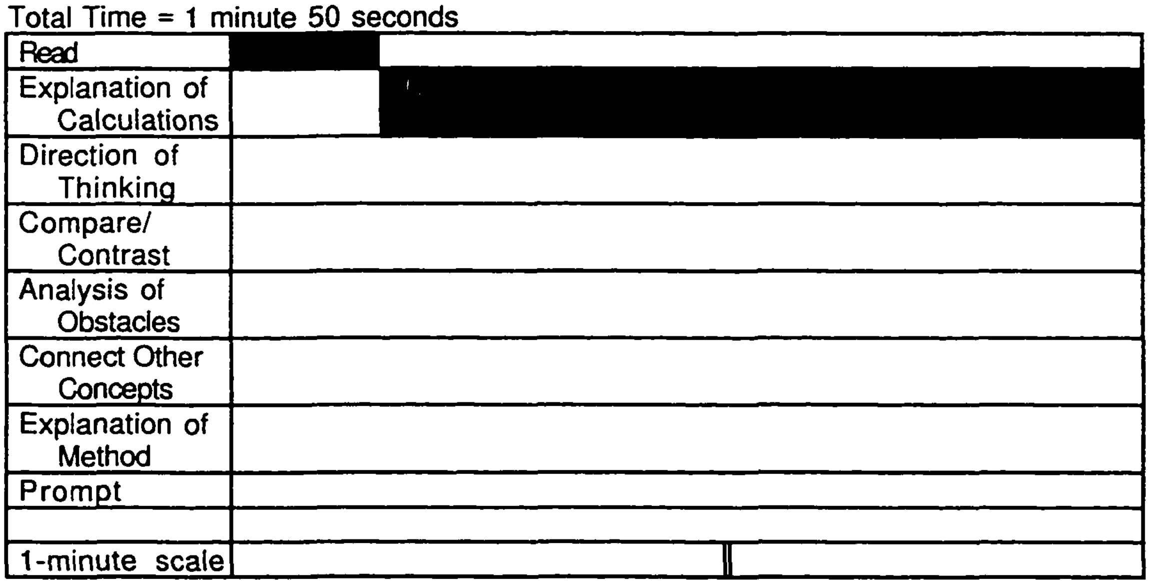


Figure 51. David's Time-Line Graph for Problem Session #1, Task 1.

"Think Aloud" Session #1

Task 2:

Suppose $f(0) = 2$ and $|f'(x)| < 1$ for all x in $[-5, 5]$.

a) Find upper and lower bounds on the value of $f(1)$, that is, find numbers U and L so that $L < f(1) < U$.

b) Find upper and lower bounds on the value of $f(-3)$, that is, find new numbers U and L so that $L < f(-3) < U$.

Total Time = 7 minutes 9 seconds

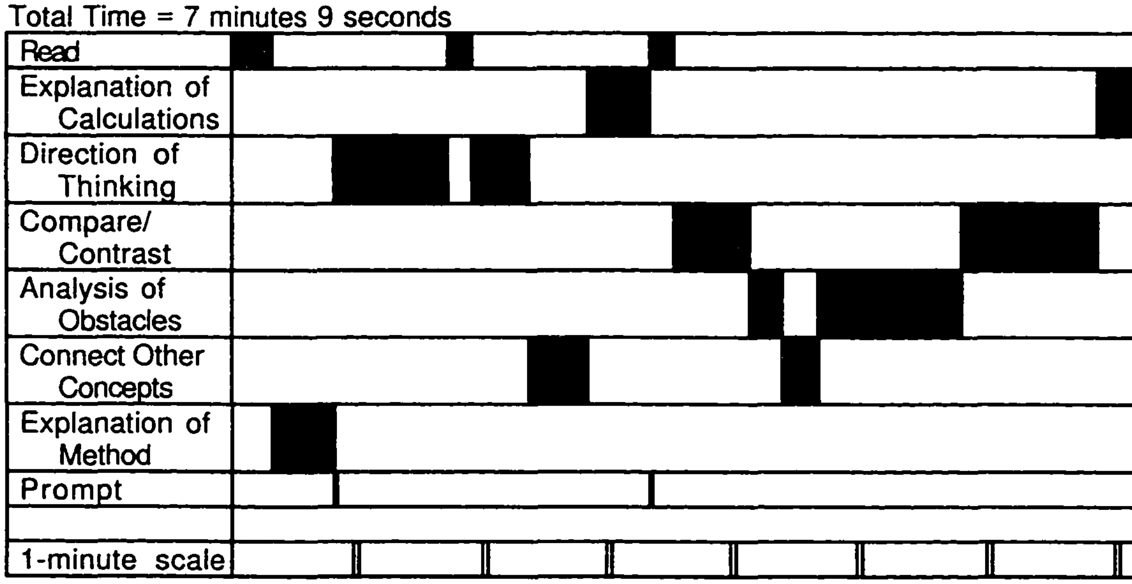


Figure 52. David's Time-Line Graph for Problem Session #1, Task 2.

"Think Aloud" Session #1

Task 3:

A company's revenue from car sales, C (measured in thousands of dollars), is a function of advertising expenditure, a, also measured in thousands of dollars. Suppose $C = f(a)$.

a) What does the company hope is true about the sign of f'? Explain.

b) What does the statement $f'(100) = 2$ mean in practical terms? How about $f'(100) = 0.5$?

c) Suppose the company plans to spend about \$ 100,000 on advertising. If $f'(100) = 2$, should the company spend slightly more or slightly less than \$ 100,000 on advertising? What if $f'(100) = 0.5$? Explain your answers.

Total Time = 12 minutes 56 seconds

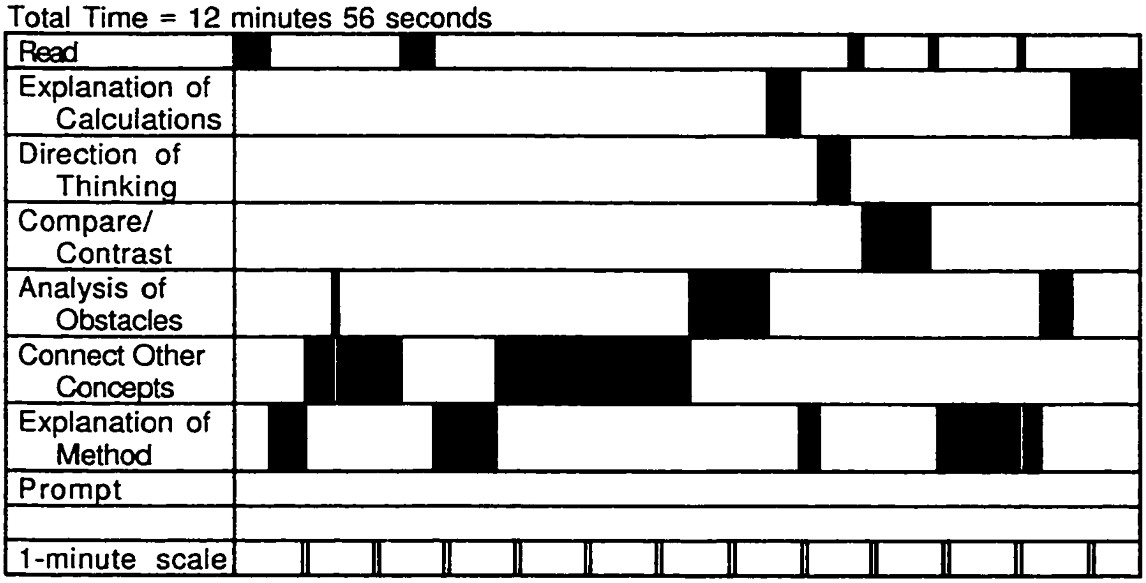


Figure 53. David's Time-Line Graph for Problem Session #1, Task 3.

"Think Aloud" Session #2

Task 1:

If $f(x) = 13 - 8x + \sqrt{2}x^2$ and $f'(r) = 4$, find r.

Total Time = 3 minutes 3 seconds

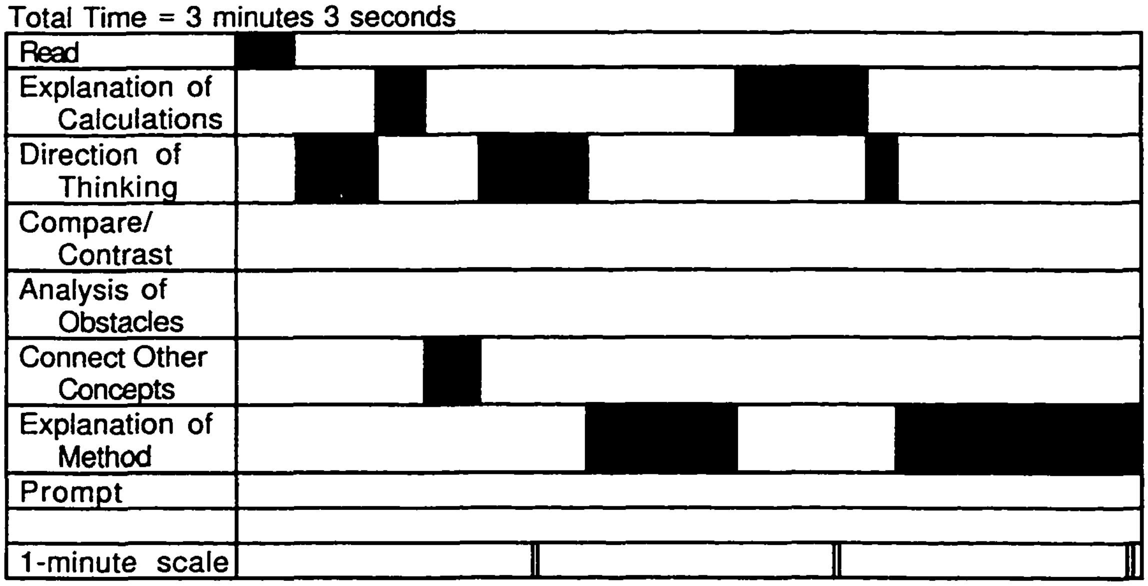


Figure 54. David's Time-Line Graph for Problem Session #2, Task 1.

"Think Aloud" Session #2

Task 2:

Suppose the slope of the tangent line at any point P on the graph of an equation f equals the square of the x-coordinate of the point P. Find the equation for f if the graph contains:

a) the origin

b) the point (3,6)

Total Time = 9 minutes 48 seconds

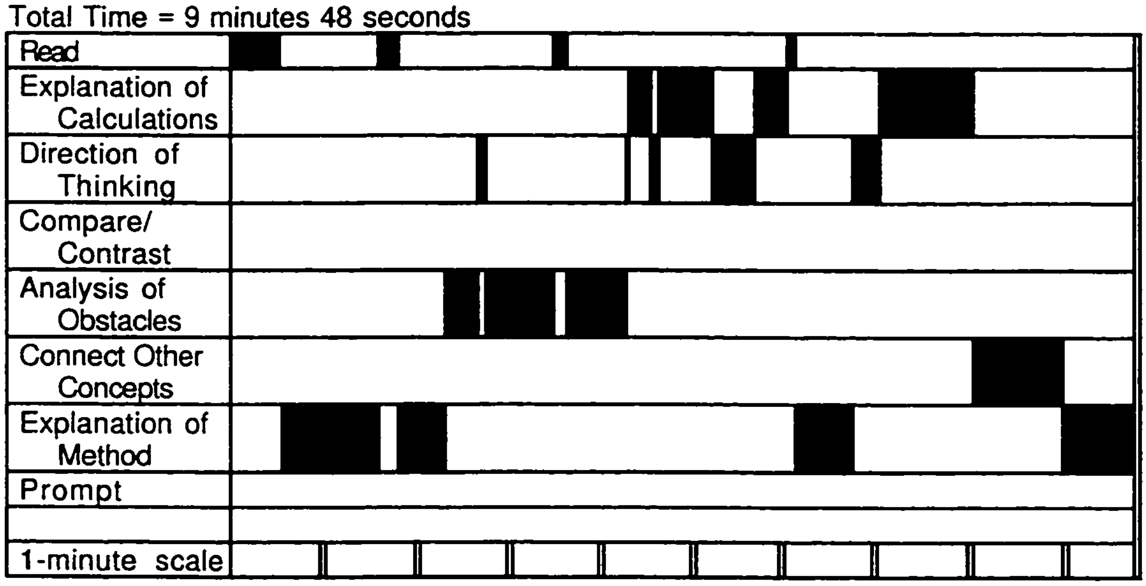


Figure 55. David's Time-Line Graph for Problem Session #2, Task 2.

"Think Aloud" Session #2

Task 3:

The *Global 2000 Report* gave the world's population, P, as 4.1 billion in 1975 and growing at 2% annually.

a) Give a formula for P in terms of time, t, measured in years since 1975.

b) Find each of the following and discuss what each represents in practical terms:

i) $P'(t)$

ii) $P'(0)$

iii) $P'(15)$

Total Time = 9 minutes 1 second

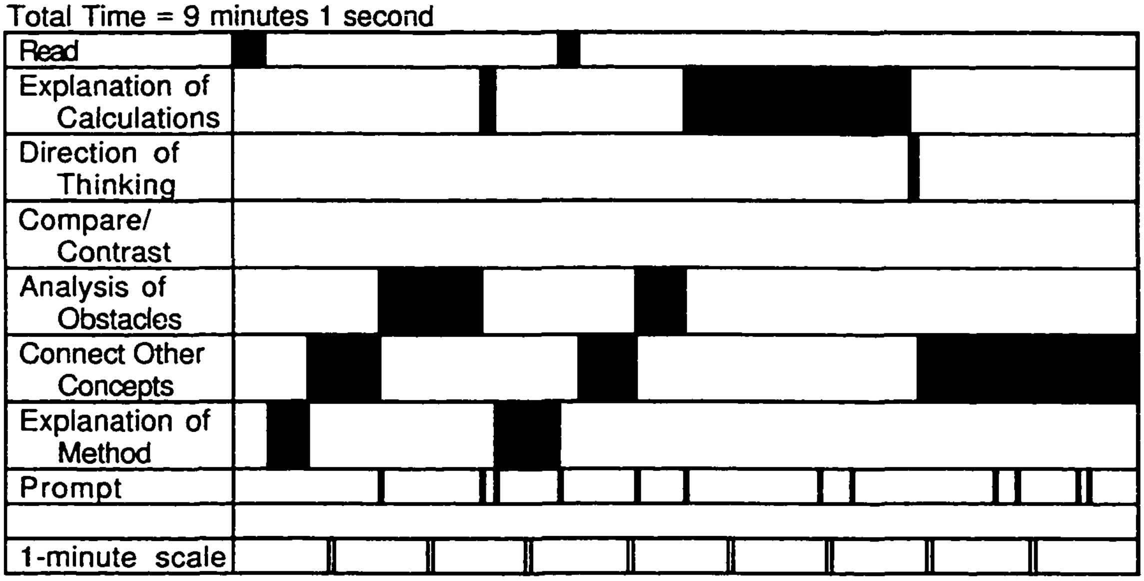


Figure 56. David's Time-Line Graph for Problem Session #2, Task 3.

"Think Aloud" Session #3

Task 1:

Plot the curve defined by the equation $y^2 = x^3$ and find the equations of the two lines tangent to the curve at $x = 1$.

Total Time = 5 minutes 22 seconds

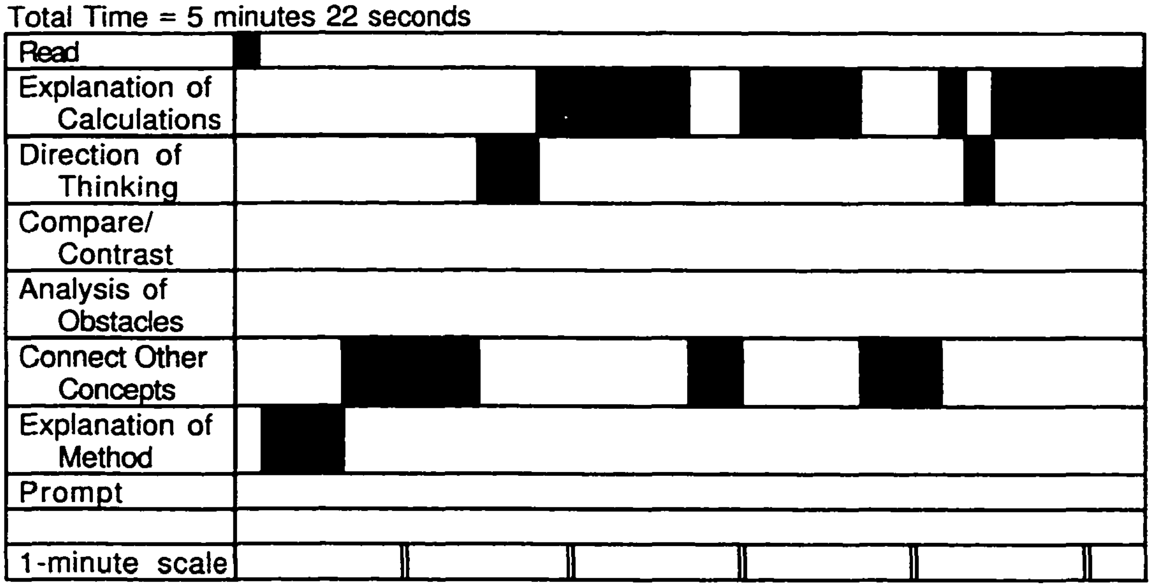


Figure 57. David's Time-Line Graph for Problem Session #3, Task 1.

"Think Aloud" Session #3

Task 2:

A mold grows at a rate proportional to the amount present. Initially, its weight is 2 grams; after two days, it weighs 5 grams. How much does it weigh after eight days?

Total Time = 4 minutes 17 seconds

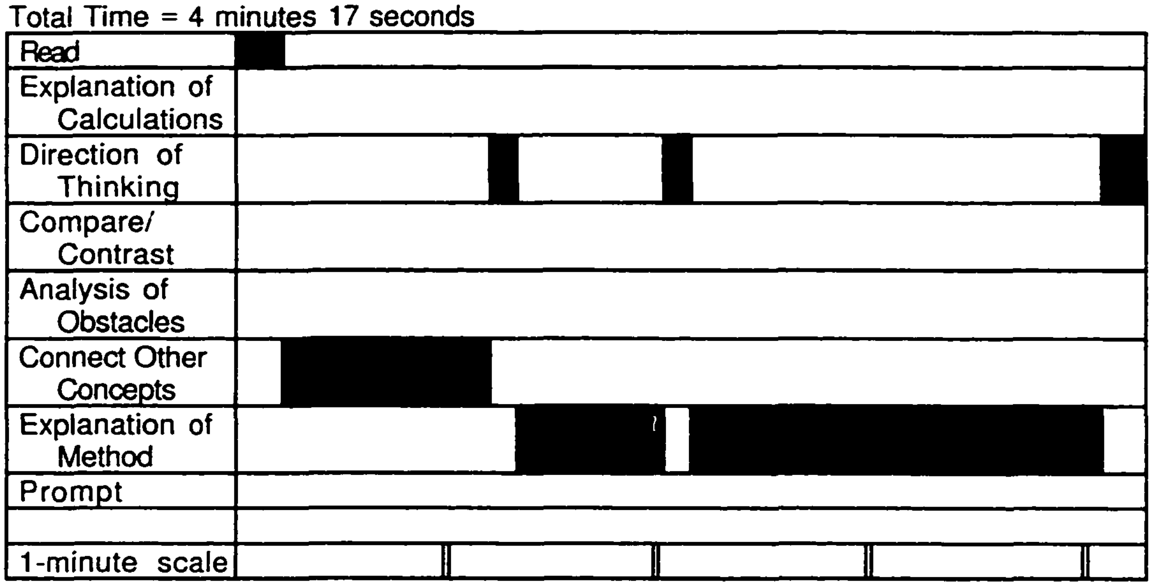


Figure 58. David's Time-Line Graph for Problem Session #3, Task 2.

"Think Aloud" Session #3

Task 3:

A rectangle has its base on the x-axis, a vertex on the y-axis, and a vertex on the curve $y = e^{-x^2}$.

a) What choice of vertices gives the largest area?

b) Show that one of the vertices found in part (a) is at an inflection of the curve.

Total Time = 22 minutes 15 seconds

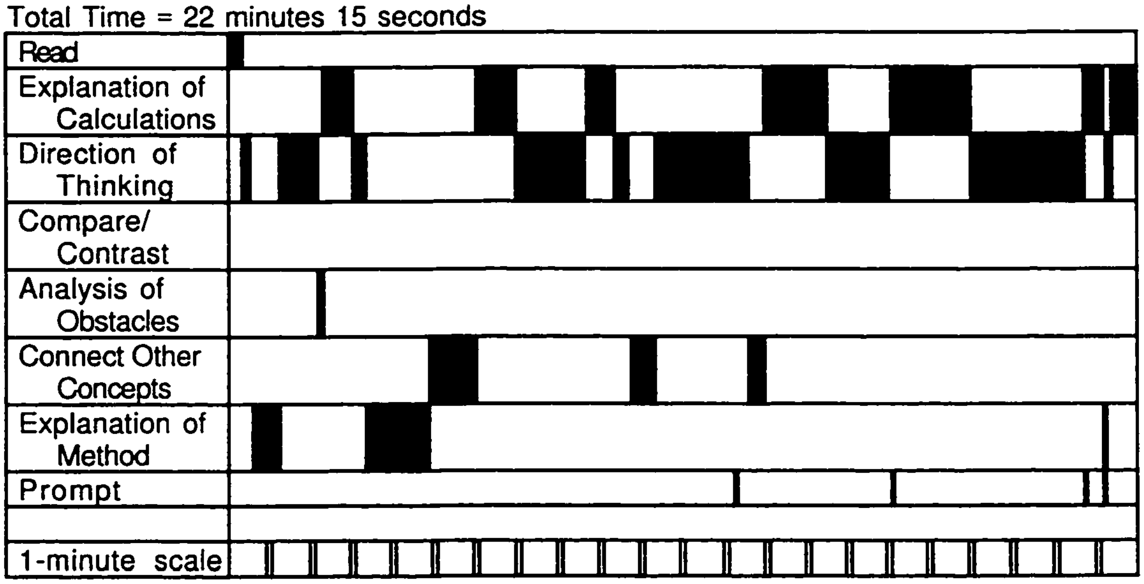


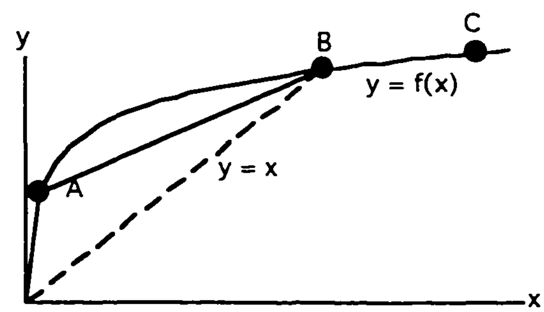
Figure 59. David's Time-Line Graph for Problem Session #3, Task 3.

"Think Aloud" Session #1

Task 1:

For the graph y = f(x) shown below, arrange the following numbers in ascending (i.e. smallest to largest) order:

- The slope of the curve at A.
- The slope of the curve at B.
- The slope of the curve at C.
- The slope of the line AB.
- The number 0.
- The number 1.



Total Time = 2 minutes 33 seconds

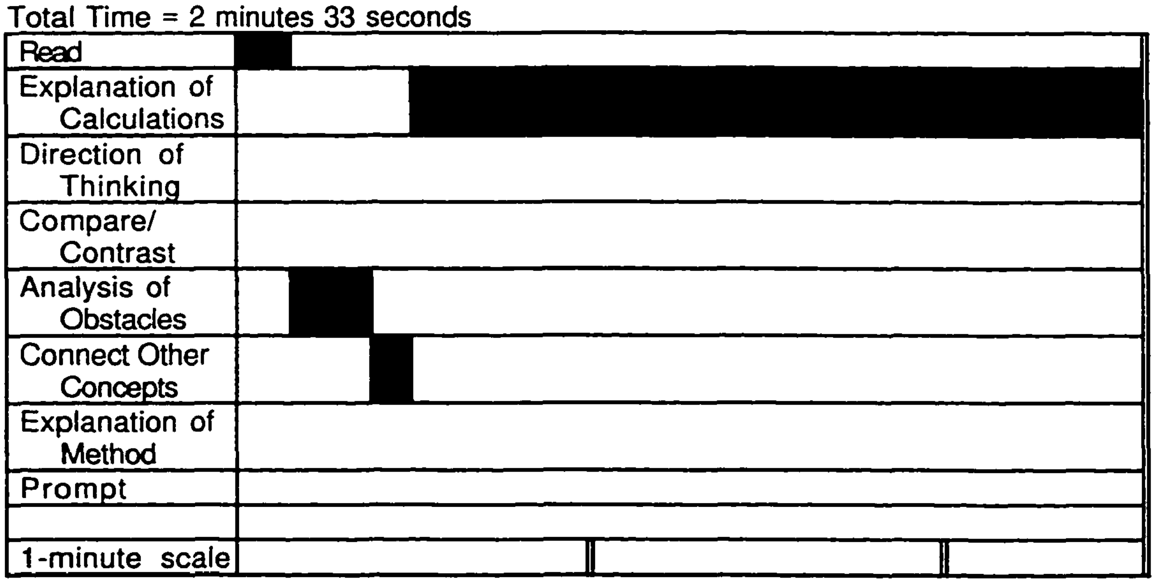


Figure 60. Edward's Time-Line Graph for Problem Session #1, Task 1.

"Think Aloud" Session #1

Task 2:

Suppose $f(0) = 2$ and $|f'(x)| < 1$ for all x in $[-5, 5]$.

a) Find upper and lower bounds on the value of $f(1)$, that is, find numbers U and L so that $L < f(1) < U$.

b) Find upper and lower bounds on the value of $f(-3)$, that is, find new numbers U and L so that $L < f(-3) < U$.

Total Time = 4 minutes 11 seconds

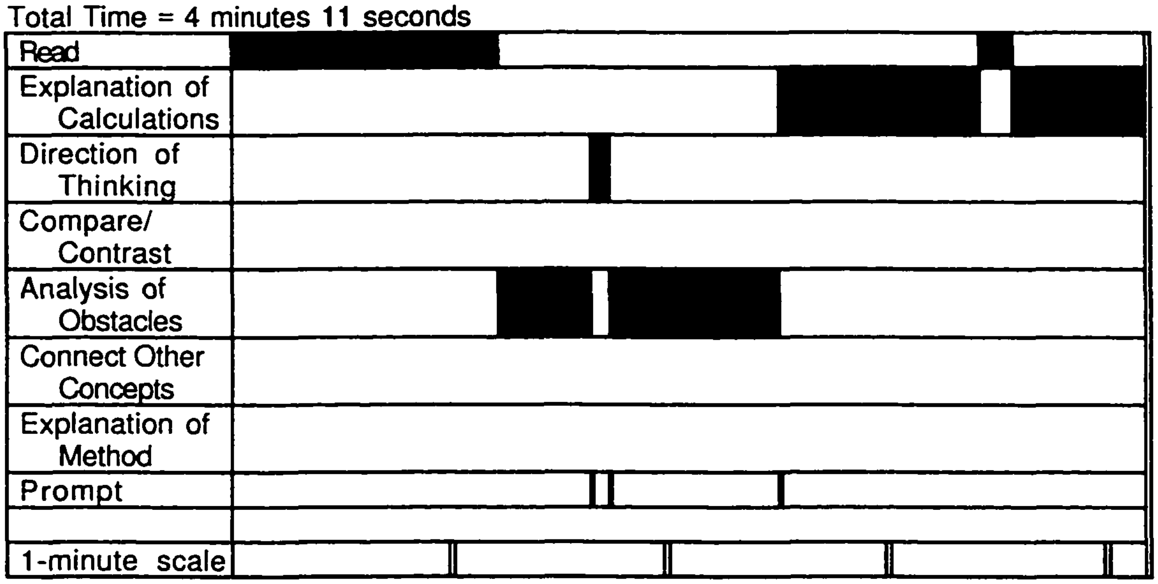


Figure 61. Edward's Time-Line Graph for Problem Session #1, Task 2.

"Think Aloud" Session #1

Task 3:

A company's revenue from car sales, C (measured in thousands of dollars), is a function of advertising expenditure, a, also measured in thousands of dollars. Suppose $C = f(a)$.

a) What does the company hope is true about the sign of f'? Explain.

b) What does the statement $f'(100) = 2$ mean in practical terms? How about $f'(100) = 0.5$?

c) Suppose the company plans to spend about $ 100,000 on advertising. If $f'(100) = 2$, should the company spend slightly more or slightly less than $ 100,000 on advertising? What if $f'(100) = 0.5$? Explain your answers.

Total Time = 4 minutes 17 seconds

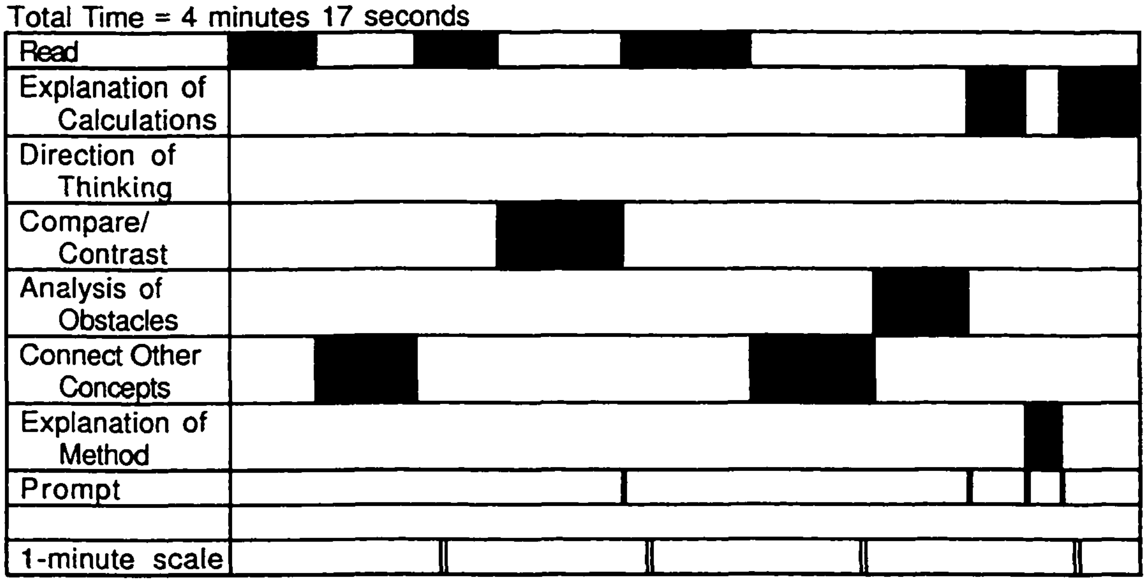


Figure 62. Edward's Time-Line Graph for Problem Session #1, Task 3.

"Think Aloud" Session #2

Task 1:

If $f(x) = 13 - 8x + \sqrt{2}x^2$ and $f'(r) = 4$, find r.

Total Time = 1 minute 48 seconds

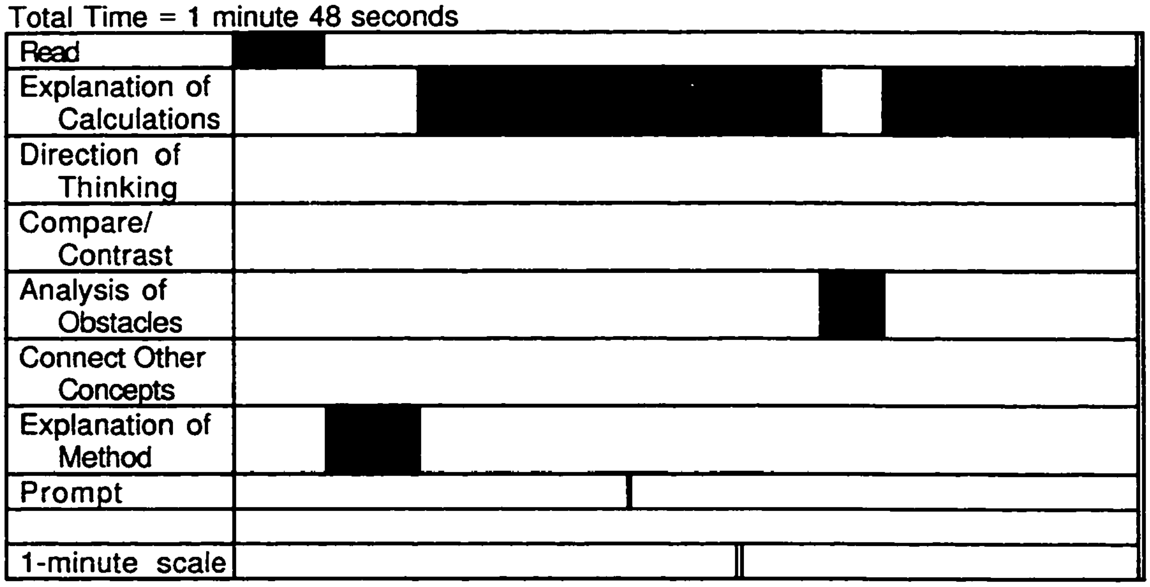


Figure 63. Edward's Time-Line Graph for Problem Session #2, Task 1.

"Think Aloud" Session•#2

Task 2:

Suppose the slope of the tangent line at any point P on the graph of an equation f equals the square of the x-coordinate of the point P. Find the equation for f if the graph contains:

a) the origin

b) the point (3,6)

Total Time = 4 minutes 46 seconds

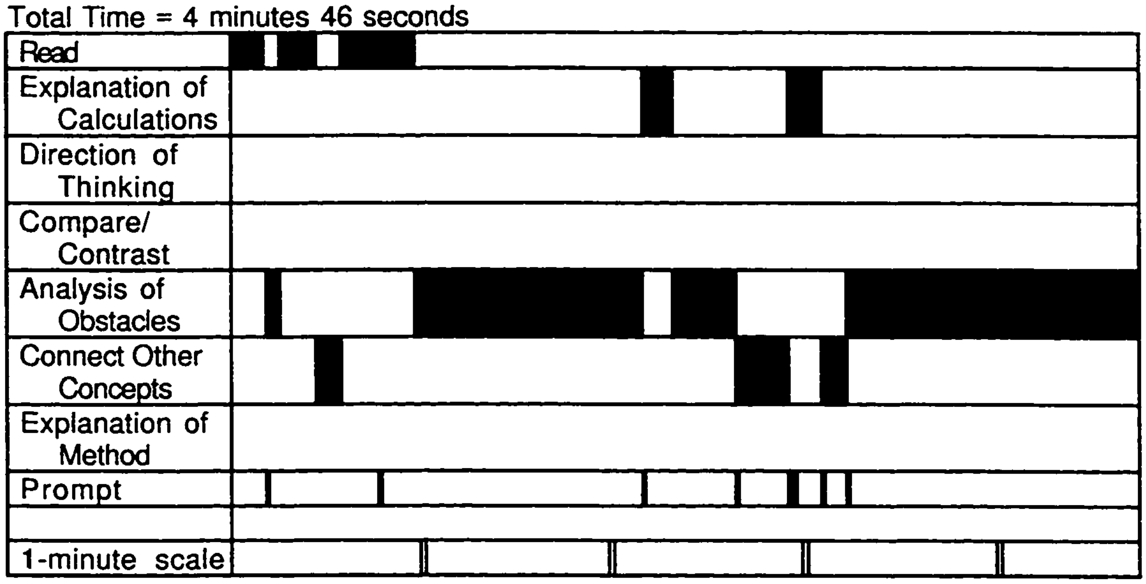


Figure 64. Edward's Time-Line Graph for Problem Session #2, Task 2.

"Think Aloud" Session #2

Task 3:

The *Global 2000 Report* gave the world's population, P, as 4.1 billion in 1975 and growing at 2% annually.

a) Give a formula for P in terms of time, t, measured in years since 1975.

b) Find each of the following and discuss what each represents in practical terms:

i) $P'(t)$

ii) $P'(0)$

iii) $P'(15)$

Total Time = 1 minute 43 seconds

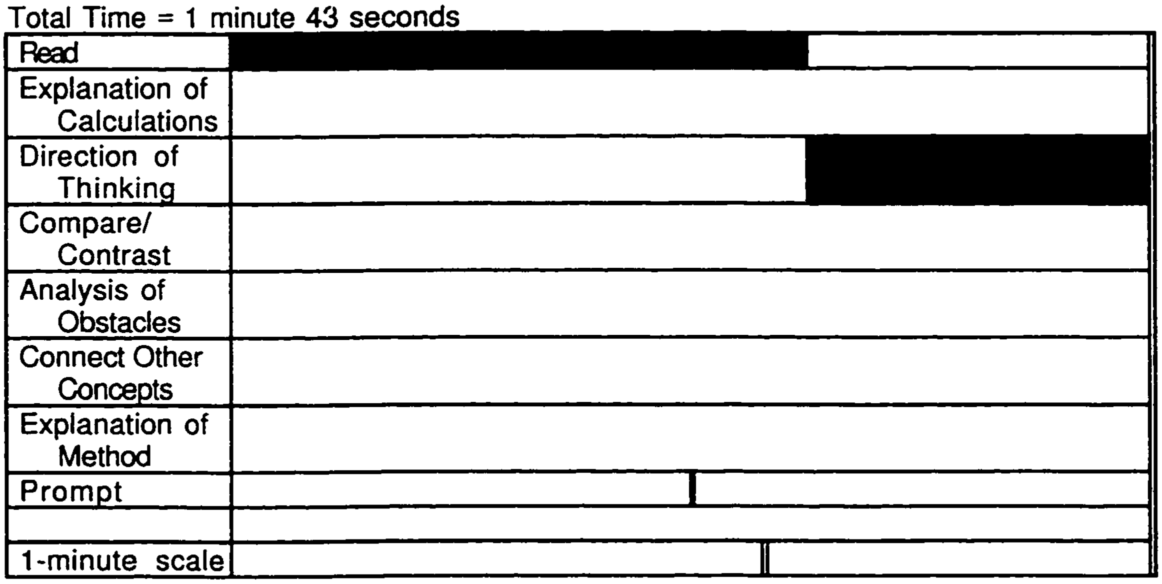


Figure 65. Edward's Time-Line Graph for Problem Session #2, Task 3.

"Think Aloud" Session #3

Task 1:

Plot the curve defined by the equation $y^2 = x^3$ and find the equations of the two lines tangent to the curve at $x = 1$.

Total Time = 3 minutes 52 seconds

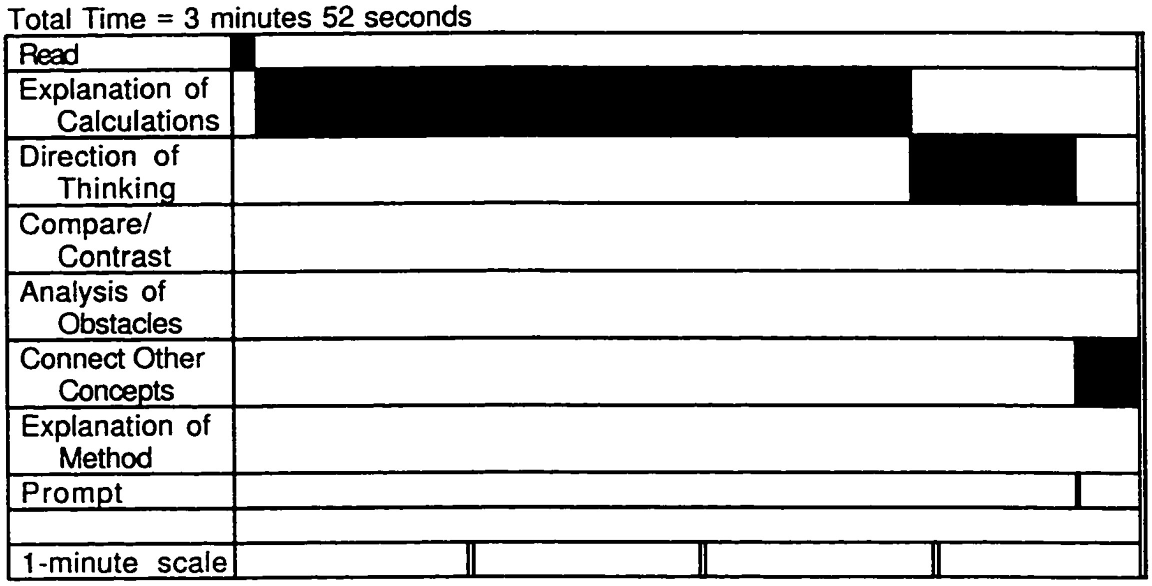


Figure 66. Edward's Time-Line Graph for Problem Session #3, Task 1.

"Think Aloud" Session #3

Task 2:

A mold grows at a rate proportional to the amount present. Initially, its weight is 2 grams; after two days, it weighs 5 grams. How much does it weigh after eight days?

Total Time = 4 minutes 19 seconds

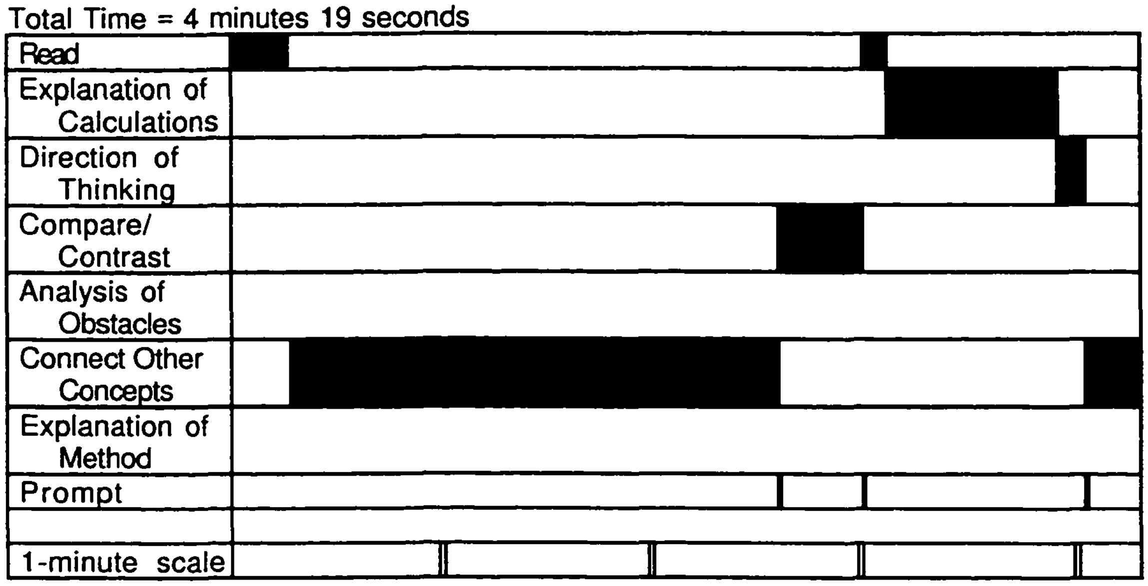


Figure 67. Edward's Time-Line Graph for Problem Session #3, Task 2.

"Think Aloud" Session #3

Task 3:

A rectangle has its base on the x-axis, a vertex on the y-axis, and a vertex on the curve $y = e^{-x^2}$.

a) What choice of vertices gives the largest area?

b) Show that one of the vertices found in part (a) is at an inflection of the curve.

Total Time = 3 minutes 17 seconds

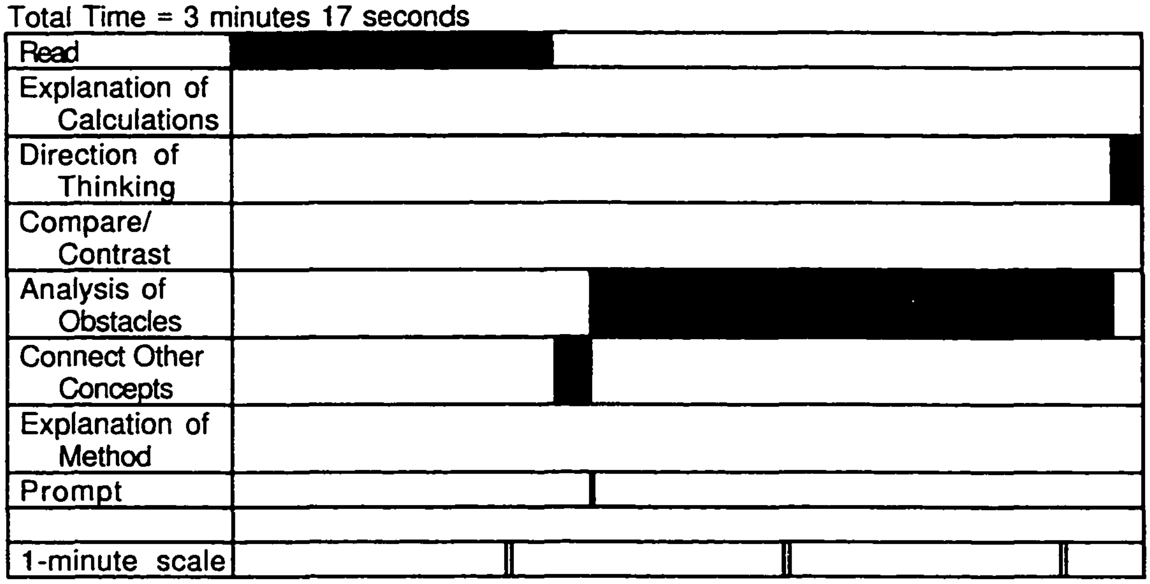


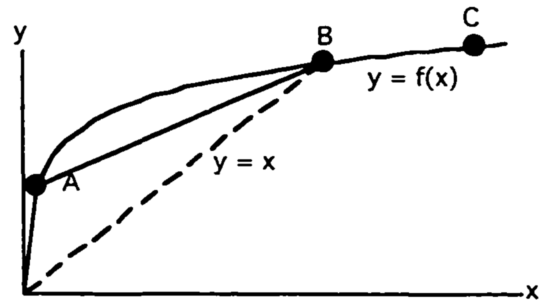
Figure 68. Edward's Time-Line Graph for Problem Session #3, Task 3.

"Think Aloud" Session #1

Task 1:

For the graph y = f(x) shown below, arrange the following numbers in ascending (i.e. smallest to largest) order:

- The slope of the curve at A.
- The slope of the curve at B.
- The slope of the curve at C.
- The slope of the line AB.
- The number 0.
- The number 1.



Total Time = 1 minute 15 seconds

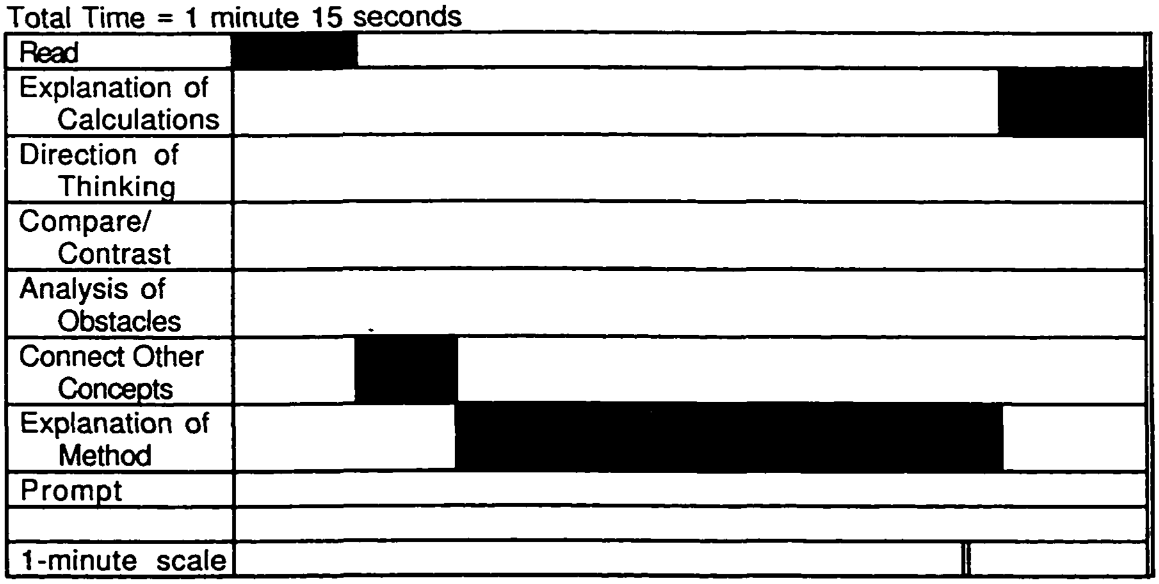


Figure 69. Fran's Time-Line Graph for Problem Session #1, Task 1.

"Think Aloud" Session #1

Task 2:

Suppose $f(0) = 2$ and $|f'(x)| < 1$ for all x in $[-5, 5]$.

a) Find upper and lower bounds on the value of $f(1)$, that is, find numbers U and L so that $L < f(1) < U$.

b) Find upper and lower bounds on the value of $f(-3)$, that is, find new numbers U and L so that $L < f(-3) < U$.

Total Time = 5 minutes 6 seconds

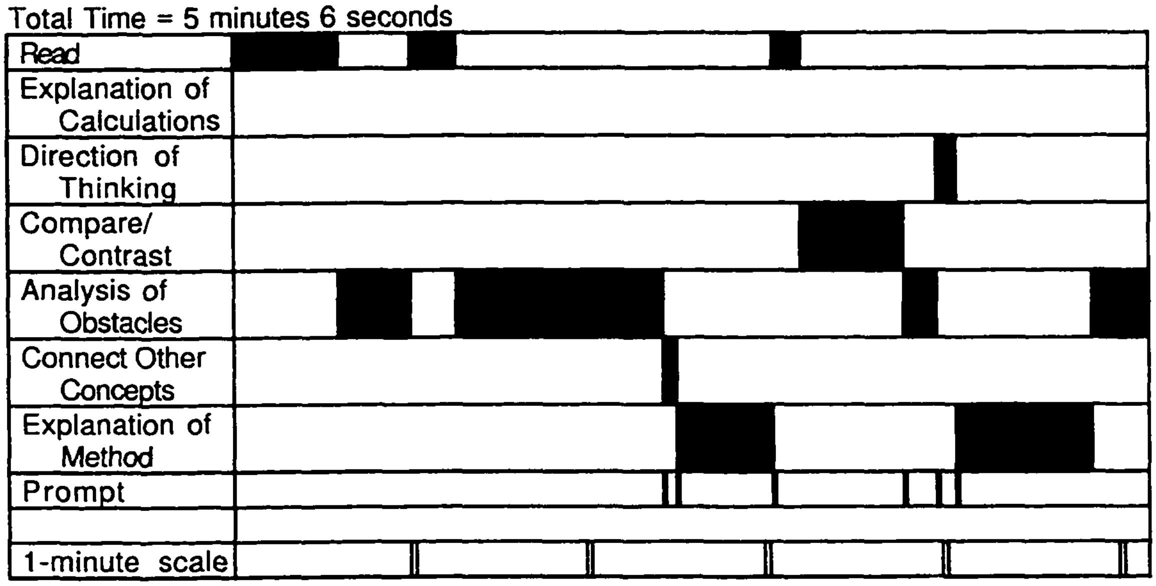


Figure 70. Fran's Time-Line Graph for Problem Session #1, Task 2.

"Think Aloud" Session #1

Task 3:

A company's revenue from car sales, C (measured in thousands of dollars), is a function of advertising expenditure, a, also measured in thousands of dollars. Suppose $C = f(a)$.

a) What does the company hope is true about the sign of f'? Explain.

b) What does the statement $f'(100) = 2$ mean in practical terms? How about $f'(100) = 0.5$?

c) Suppose the company plans to spend about \$ 100,000 on advertising. If $f'(100) = 2$, should the company spend slightly more or slightly less than \$ 100,000 on advertising? What if $f'(100) = 0.5$? Explain your answers.

Total Time = 5 minutes 52 seconds

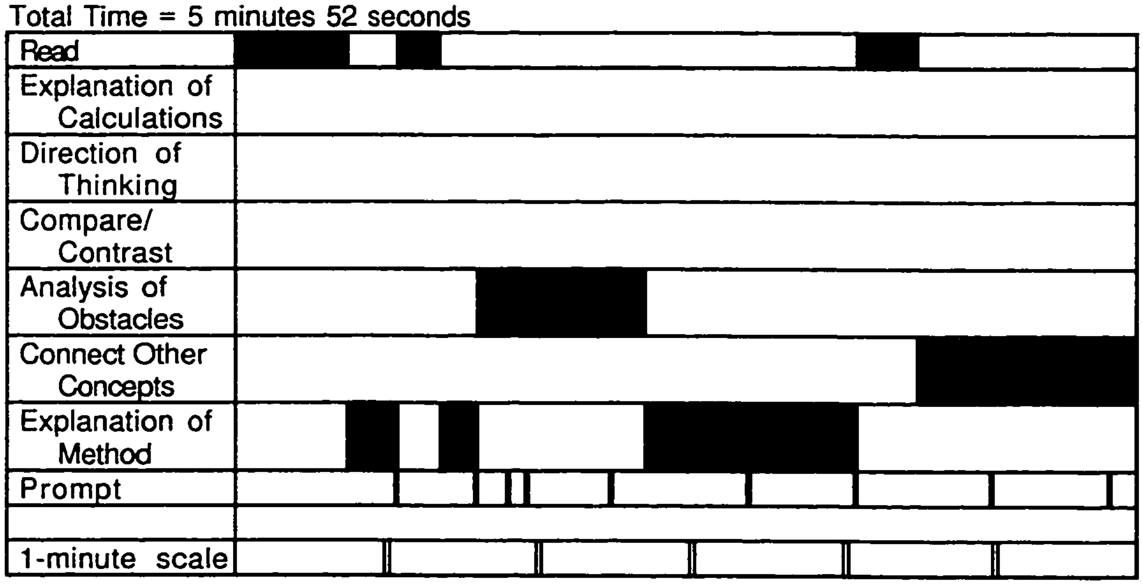


Figure 71. Fran's Time-Line Graph for Problem Session #1, Task 3.

"Think Aloud" Session #2

Task 1:

If $f(x) = 13 - 8x + \sqrt{2}x^2$ and $f'(r) = 4$, find r.

Total Time = 6 minutes 24 seconds

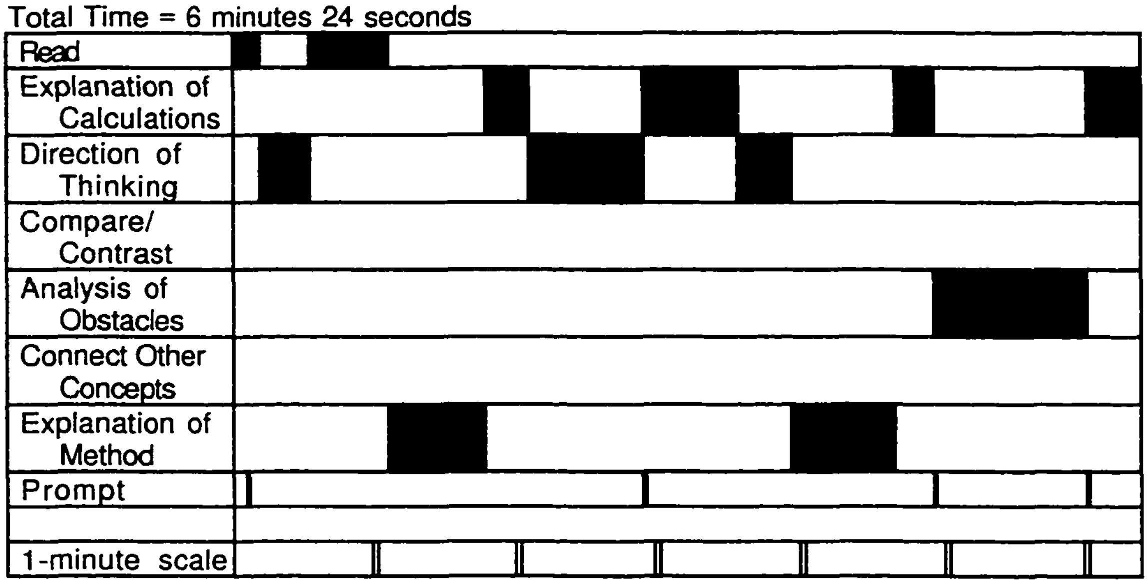


Figure 72. Fran's Time-Line Graph for Problem Session #2, Task 1.

"Think Aloud" Session #2

Task 2:

Suppose the slope of the tangent line at any point P on the graph of an equation f equals the square of the x-coordinate of the point P. Find the equation for f if the graph contains:

a) the origin

b) the point (3,6)

Total Time = 5 minutes 51 seconds

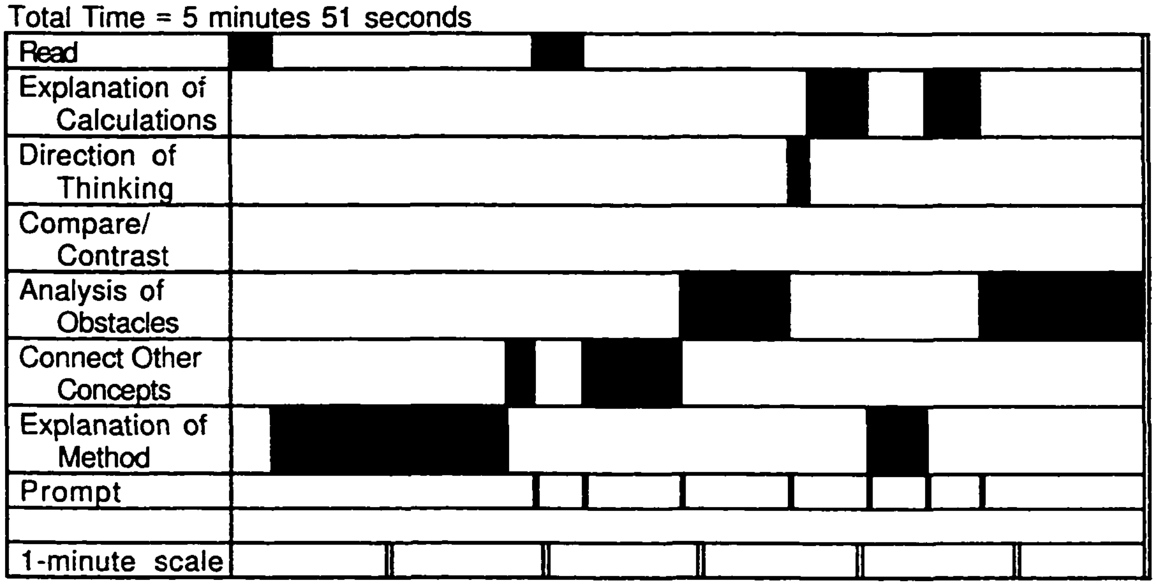


Figure 73. Fran's Time-Line Graph for Problem Session #2, Task 2.

"Think Aloud" Session #2

Task 3:

The *Global 2000 Report* gave the world's population, P, as 4.1 billion in 1975 and growing at 2% annually.

a) Give a formula for P in terms of time, t, measured in years since 1975.

b) Find each of the following and discuss what each represents in practical terms:

i) P'(t)

ii) P'(0)

iii) P'(15)

Total Time = 4 minutes 22 seconds

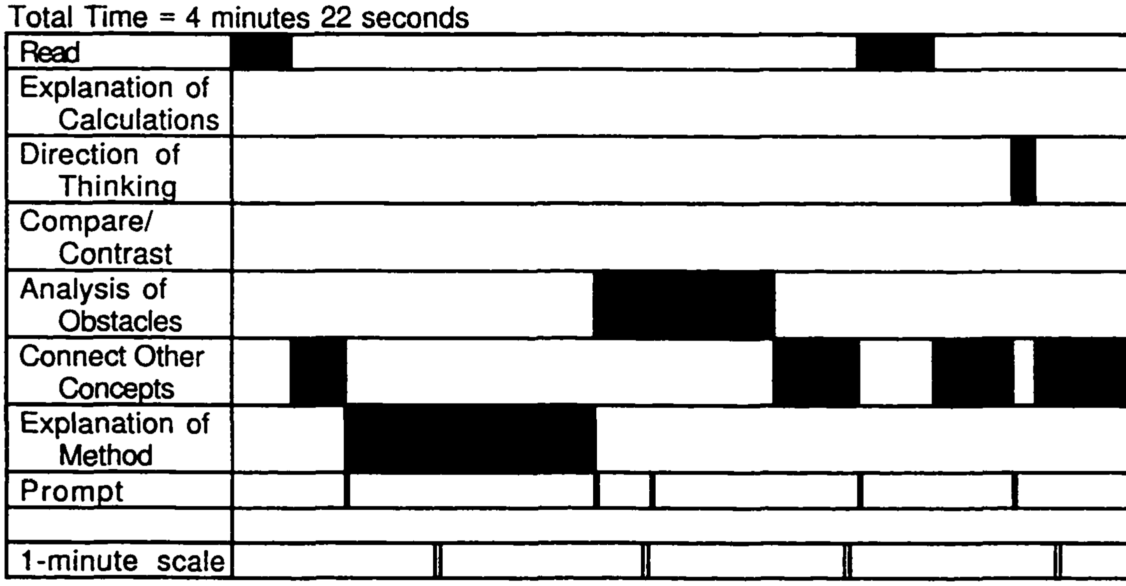


Figure 74. Fran's Time-Line Graph for Problem Session #2, Task 3.

"Think Aloud" Session #3

Task 1:

Plot the curve defined by the equation $y^2 = x^3$ and find the equations of the two lines tangent to the curve at $x = 1$.

Total Time = 7 minutes 7 seconds

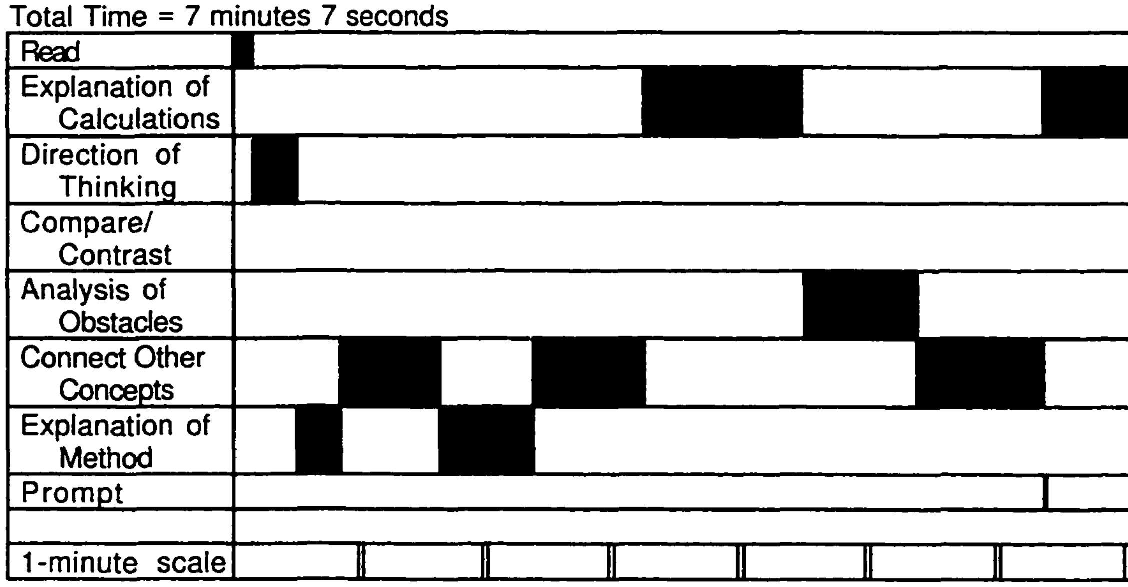


Figure 75. Fran's Time-Line Graph for Problem Session #3, Task 1.

"Think Aloud" Session #3

Task 2:

A mold grows at a rate proportional to the amount present. Initially, its weight is 2 grams; after two days, it weighs 5 grams. How much does it weigh after eight days?

Total Time = 2 minutes 51 seconds

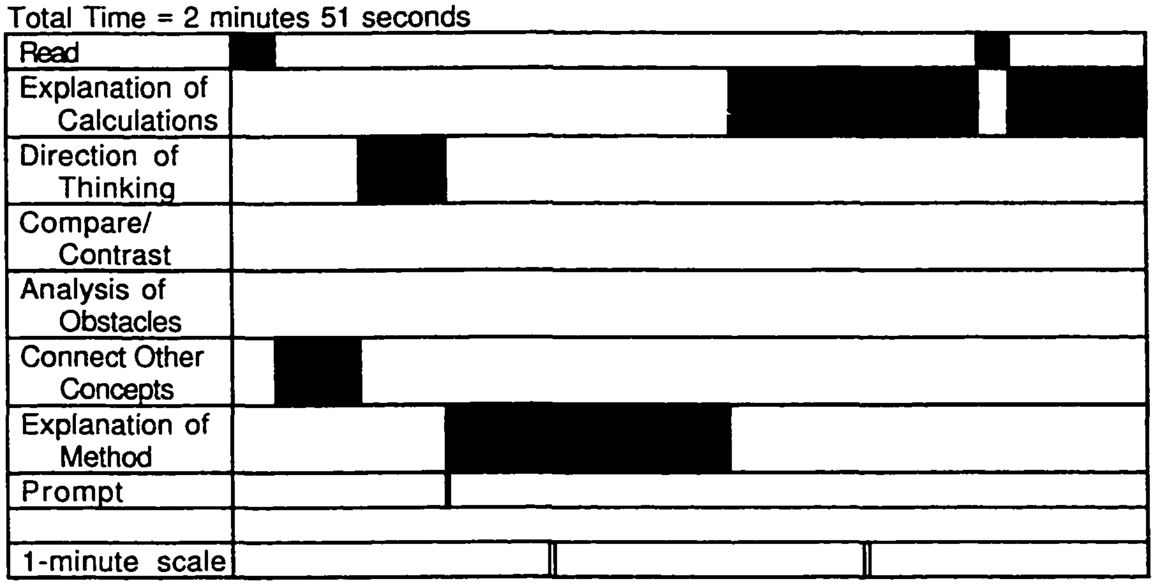


Figure 76. Fran's Time-Line Graph for Problem Session #3, Task 2.

"Think Aloud" Session #3

Task 3:

A rectangle has its base on the x-axis, a vertex on the y-axis, and a vertex on the curve $y = e^{-x^2}$.

a) What choice of vertices gives the largest area?

b) Show that one of the vertices found in part (a) is at an inflection of the curve.

Total Time = 18 minutes 45 seconds

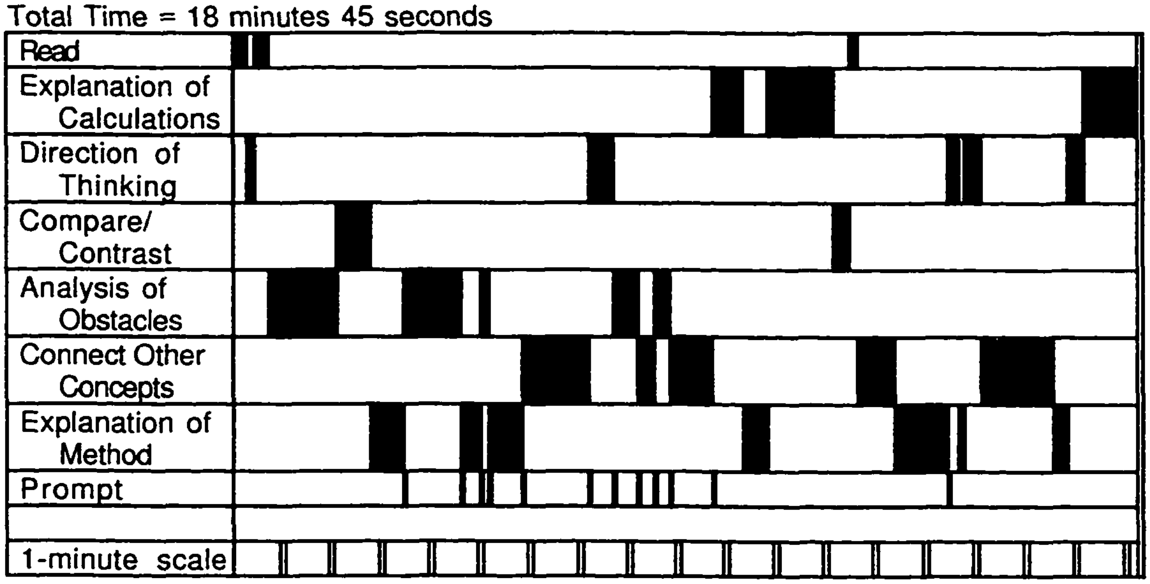


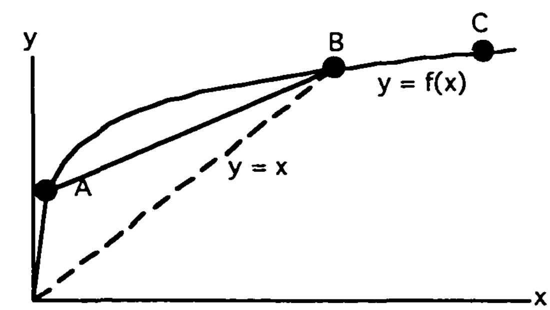
Figure 77. Fran's Time-Line Graph for Problem Session #3, Task 3.

"Think Aloud" Session #1

Task 1:

For the graph y = f(x) shown below, arrange the following numbers in ascending (i.e. smallest to largest) order:

- The slope of the curve at A.
- The slope of the curve at B.
- The slope of the curve at C.
- The slope of the line AB.
- The number 0.
- The number 1.



Total Time = 4 minutes 9 seconds

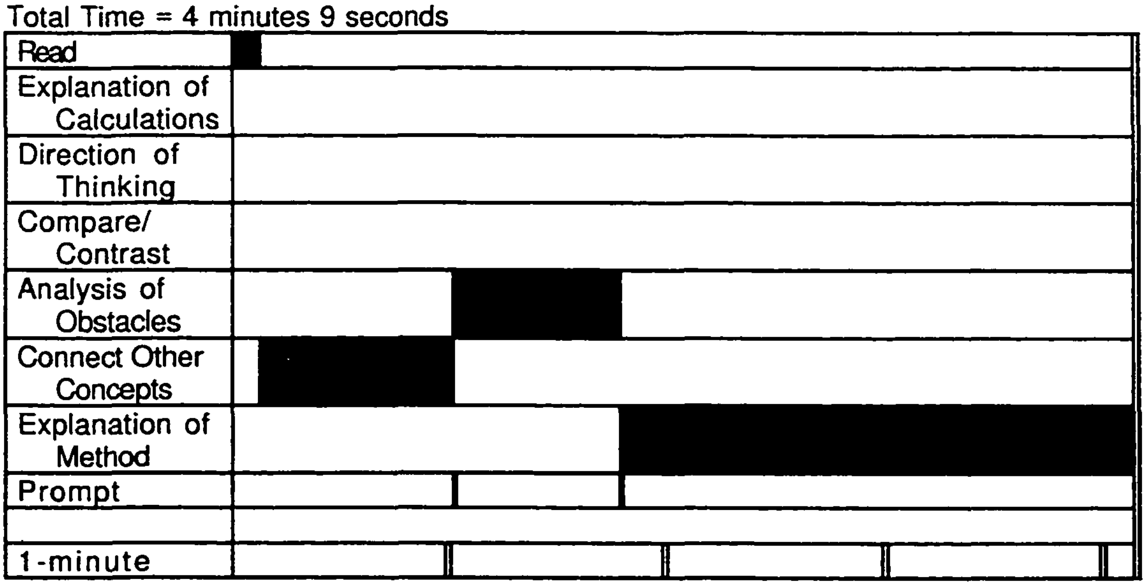


Figure 78. George's Time-Line Graph for Problem Session #1, Task 1.

"Think Aloud" Session #1

Task 2:

Suppose $f(0) = 2$ and $|f'(x)| < 1$ for all x in $[-5, 5]$.

a) Find upper and lower bounds on the value of $f(1)$, that is, find numbers U and L so that $L < f(1) < U$.

b) Find upper and lower bounds on the value of $f(-3)$, that is, find new numbers U and L so that $L < f(-3) < U$.

Total Time = 4 minutes 44 seconds

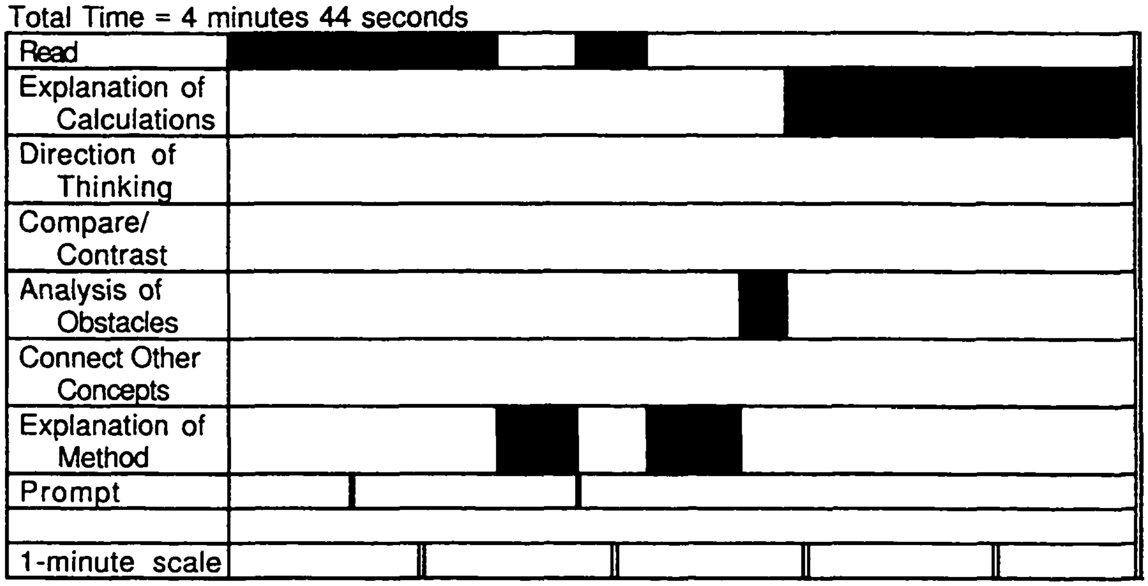


Figure 79. George's Time-Line Graph for Problem Session #1, Task 2.

"Think Aloud" Session #1

Task 3:

A company's revenue from car sales, C (measured in thousands of dollars), is a function of advertising expenditure, a, also measured in thousands of dollars. Suppose $C = f(a)$.

a) What does the company hope is true about the sign of f'? Explain.

b) What does the statement $f'(100) = 2$ mean in practical terms? How about $f'(100) = 0.5$?

c) Suppose the company plans to spend about \$ 100,000 on advertising. If $f'(100) = 2$, should the company spend slightly more or slightly less than \$ 100,000 on advertising? What if $f'(100) = 0.5$? Explain your answers.

Total Time = 7 minutes 17 seconds

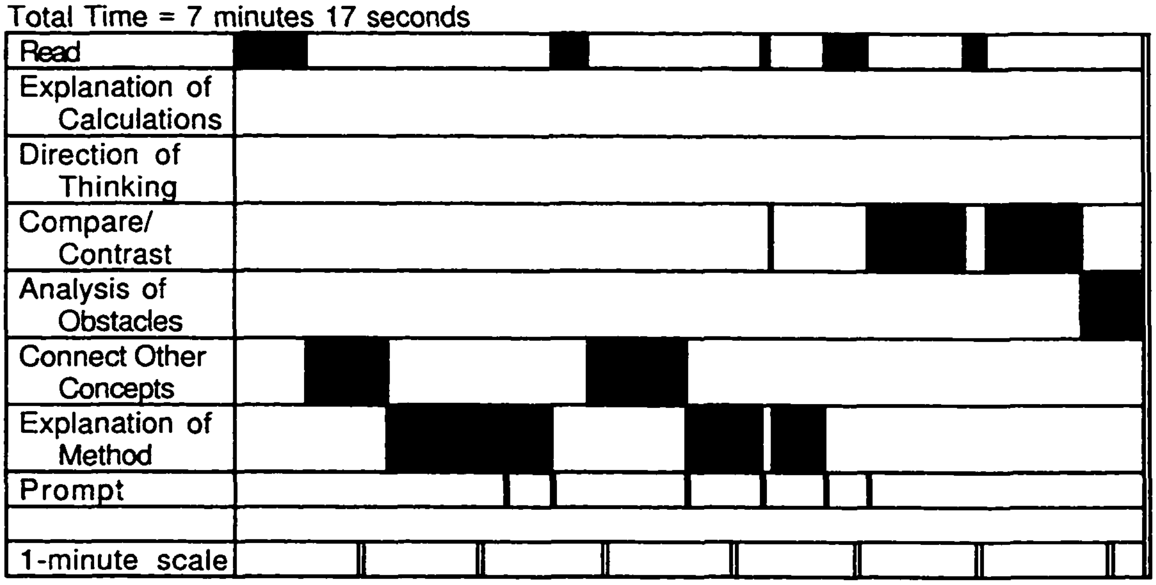


Figure 80. George's Time-Line Graph for Problem Session #1, Task 3.

"Think Aloud" Session #2

Task 1:

If $f(x) = 13 - 8x + \sqrt{2}x^2$ and $f'(r) = 4$, find r.

Total Time = 4 minutes 35 seconds

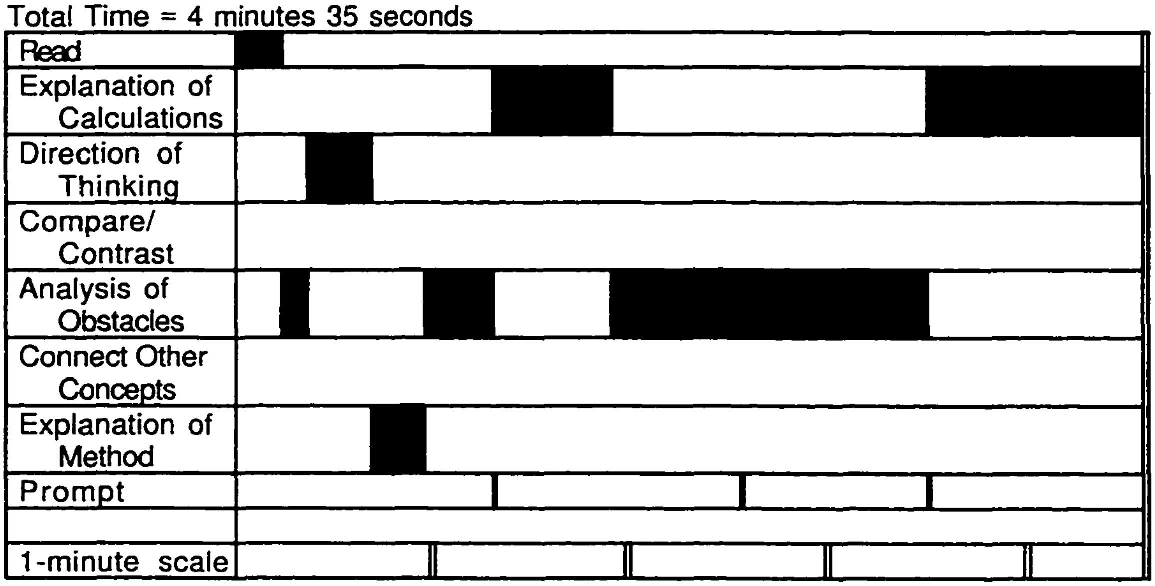


Figure 81. George's Time-Line Graph for Problem Session #2, Task 1.

"Think Aloud" Session #2

Task 2:

Suppose the slope of the tangent line at any point P on the graph of an equation f equals the square of the x-coordinate of the point P. Find the equation for f if the graph contains:

a) the origin

b) the point (3,6)

Total Time = 6 minutes 5 seconds

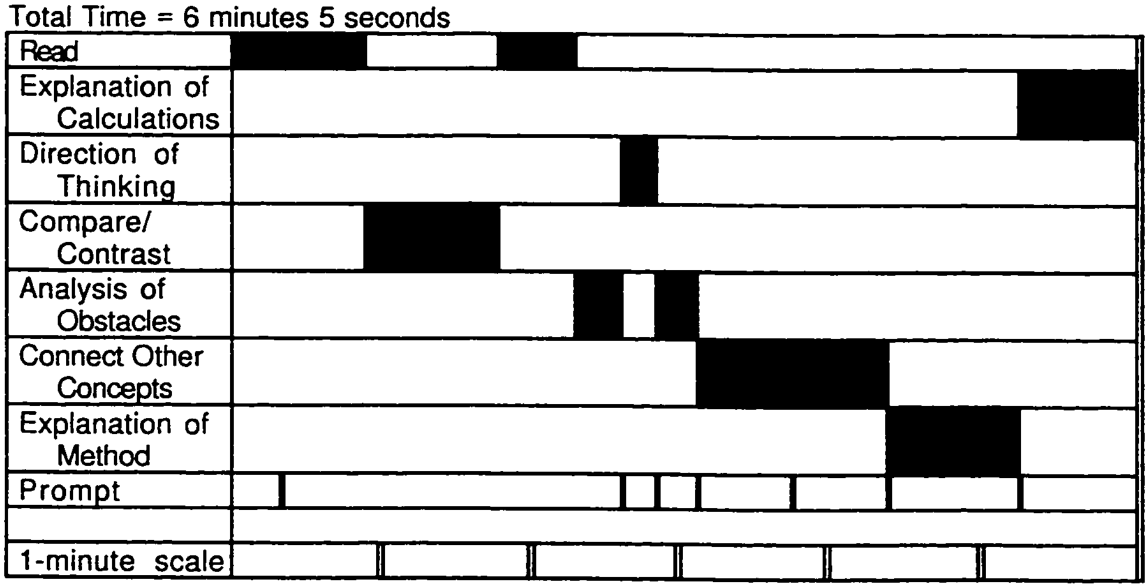


Figure 82. George's Time-Line Graph for Problem Session #2, Task 2.

"Think Aloud" Session #2

Task 3:

The *Global 2000 Report* gave the world's population, P, as 4.1 billion in 1975 and growing at 2% annually.

a) Give a formula for P in terms of time, t, measured in years since 1975.

b) Find each of the following and discuss what each represents in practical terms:

i) $P'(t)$

ii) $P'(0)$

iii) $P'(15)$

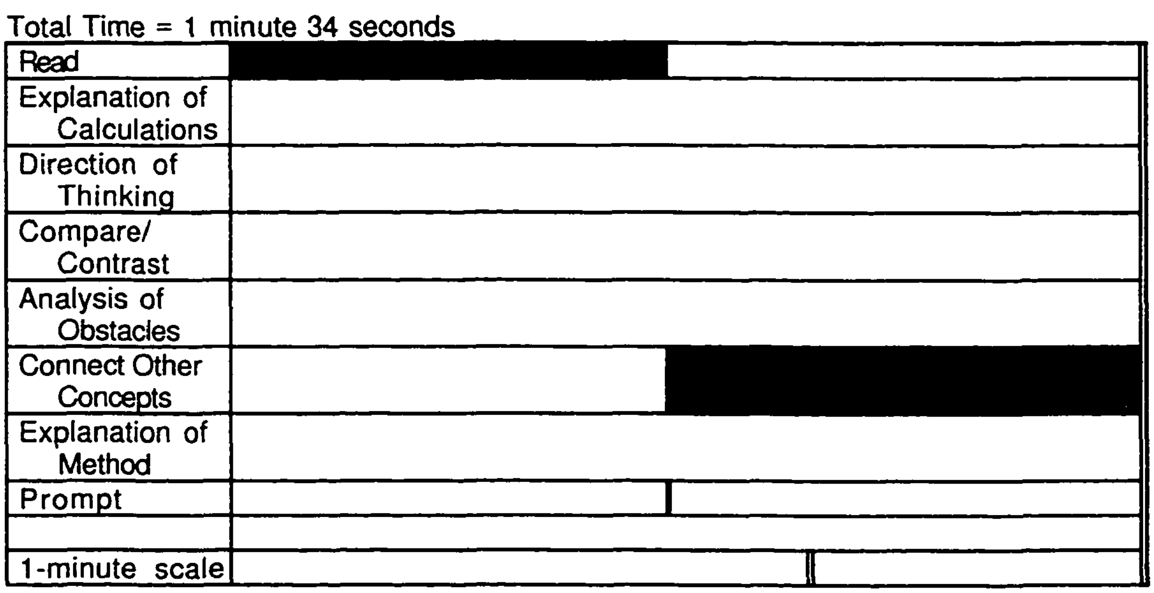


Figure 83. George's Time-Line Graph for Problem Session #2, Task 3.

"Think Aloud" Session #3

Task 1:

Plot the curve defined by the equation $y^2 = x^3$ and find the equations of the two lines tangent to the curve at $x = 1$.

Total Time = 10 minutes 6 seconds

Figure 84. George's Time-Line Graph for Problem Session #3, Task 1.

"Think Aloud" Session #3

Task 2:

A mold grows at a rate proportional to the amount present. Initially, its weight is 2 grams; after two days, it weighs 5 grams. How much does it weigh after eight days?

Total Time = 1 minute 10 seconds

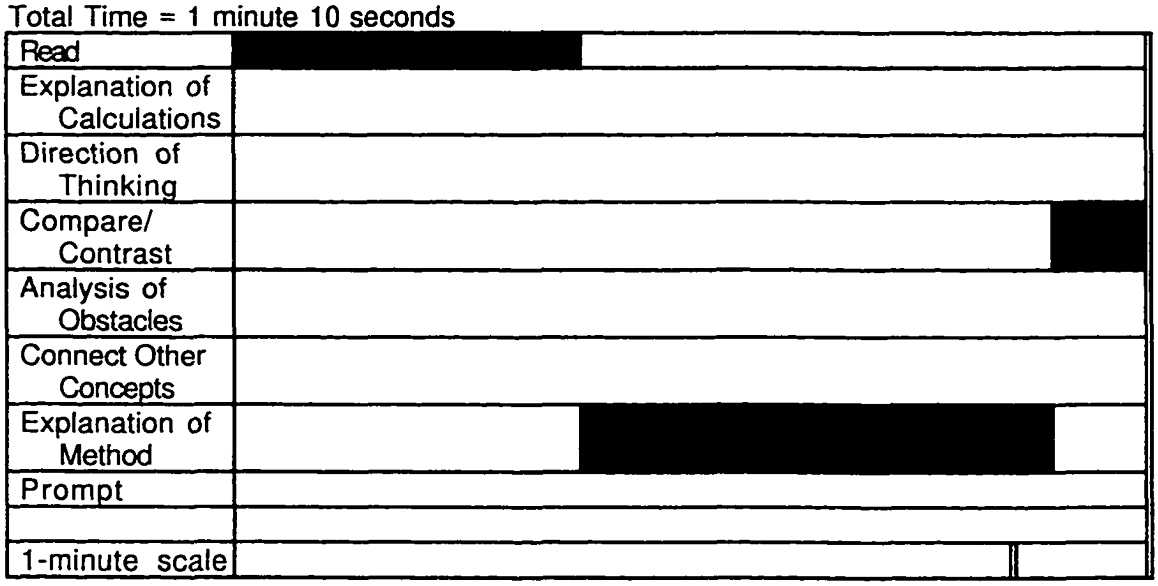


Figure 85. George's Time-Line Graph for Problem Session #3, Task 2.

"Think Aloud" Session #3

Task 3:

A rectangle has its base on the x-axis, a vertex on the y-axis, and a vertex on the curve $y = e^{-x^2}$.

a) What choice of vertices gives the largest area?

b) Show that one of the vertices found in part (a) is at an inflection of the curve.

Total Time = 2 minutes 12 seconds

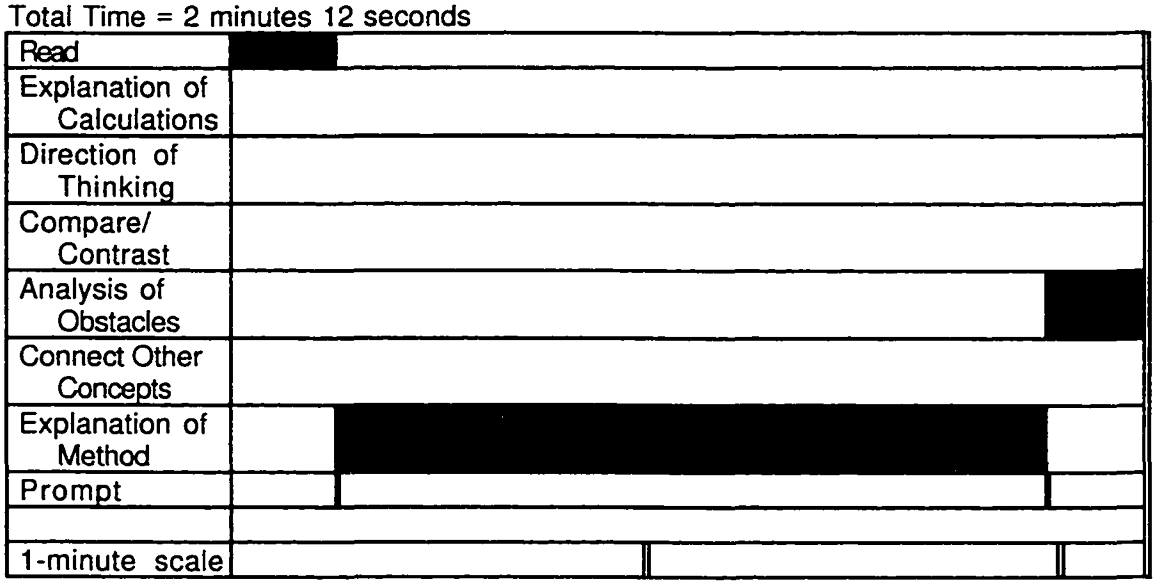


Figure 86. George's Time-Line Graph forProblem Session #3, Task 3.

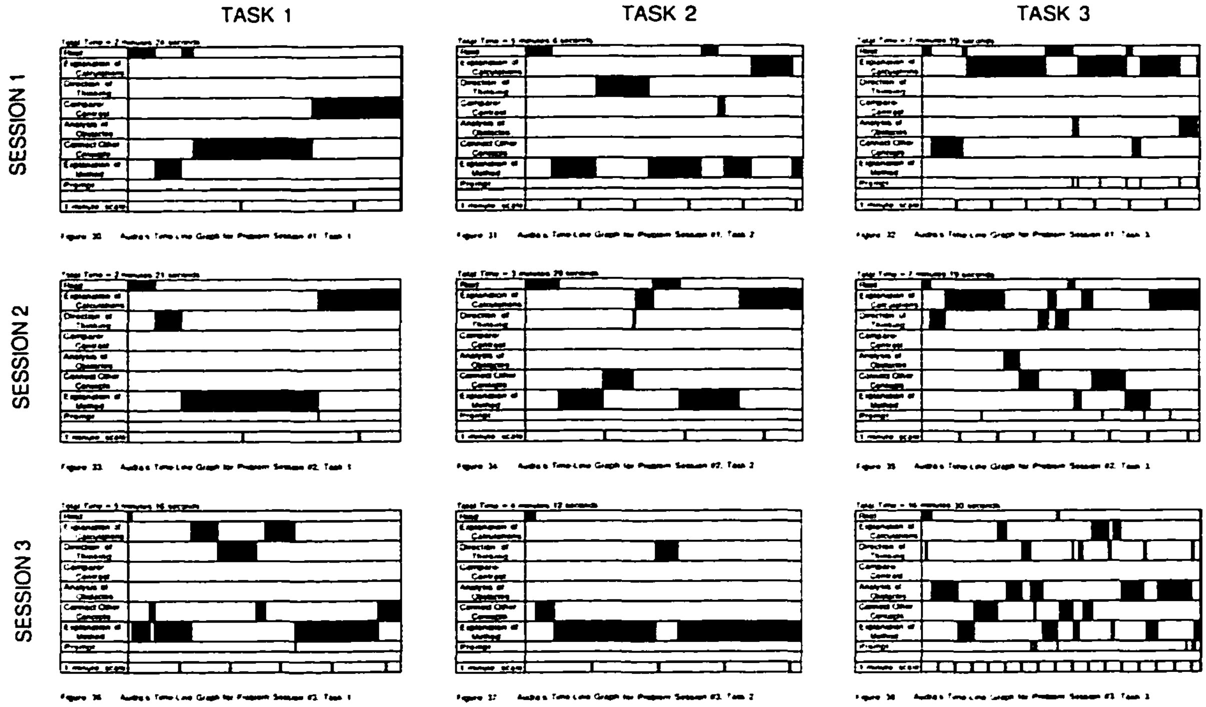


Figure 87. Composite of Audra's Time-Line Graphs.

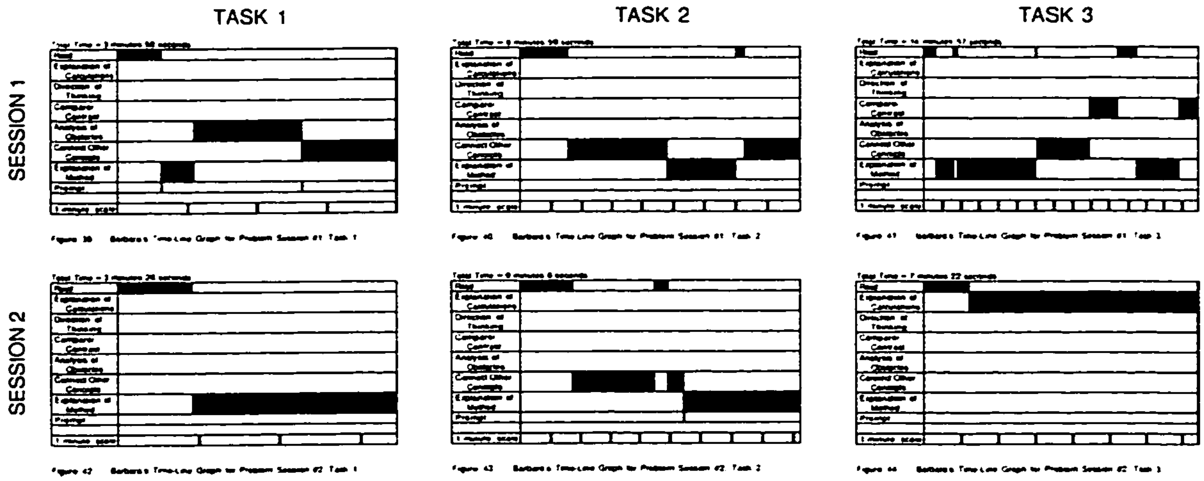


Figure 88. Composite of Barbara's Time-Line Graphs.

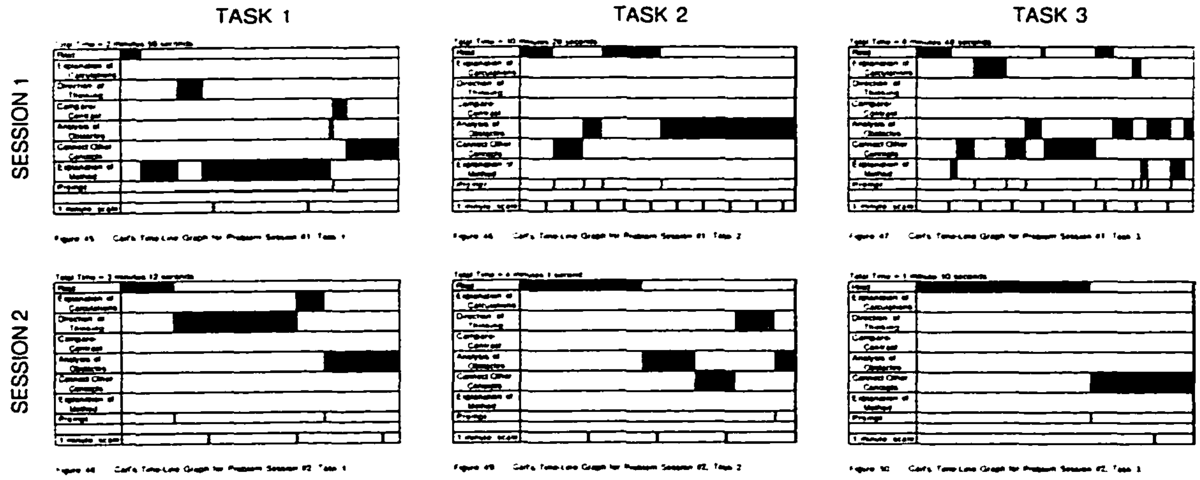


Figure 89. Composite of Carl's Time-Line Graphs.

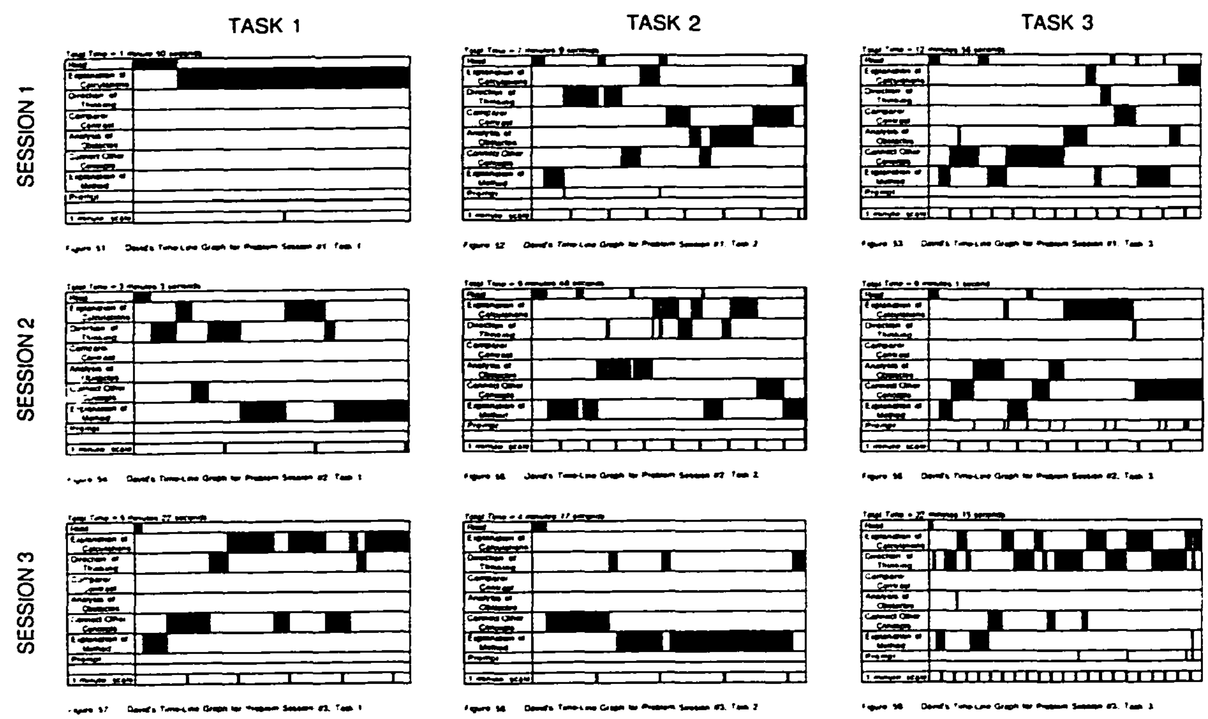


Figure 90. Composite of David's Time-Line Graphs.

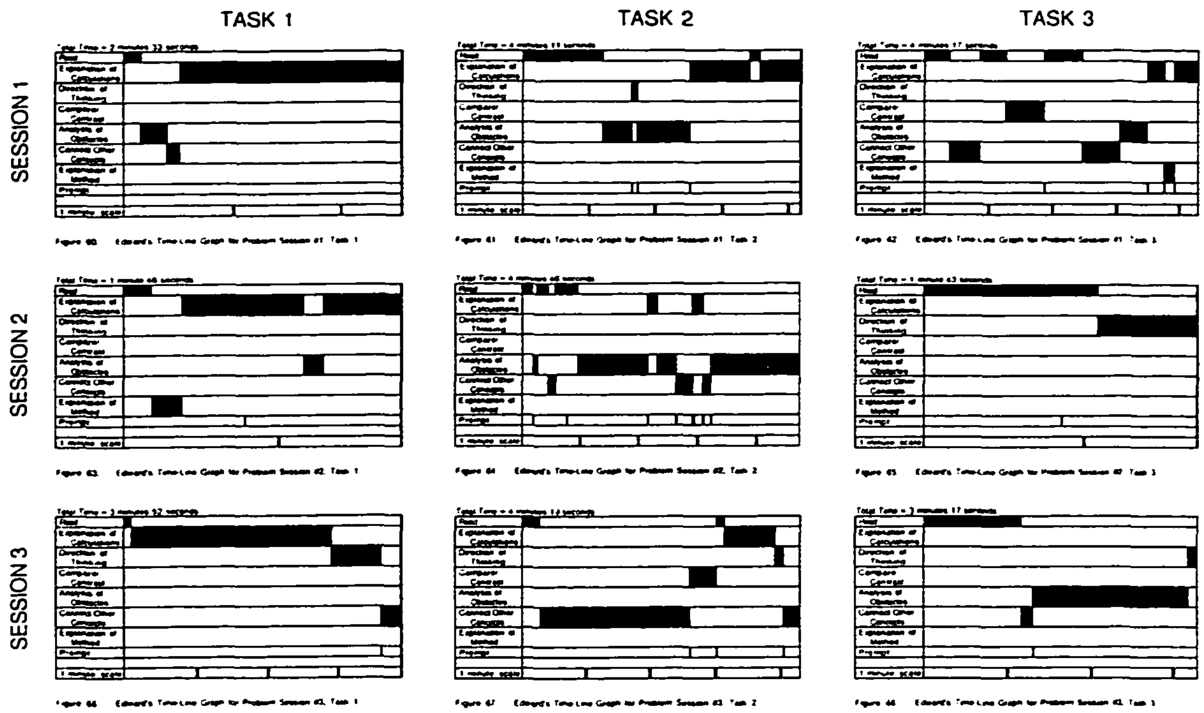


Figure 91. Composite of Edward's Time-Line Graphs.

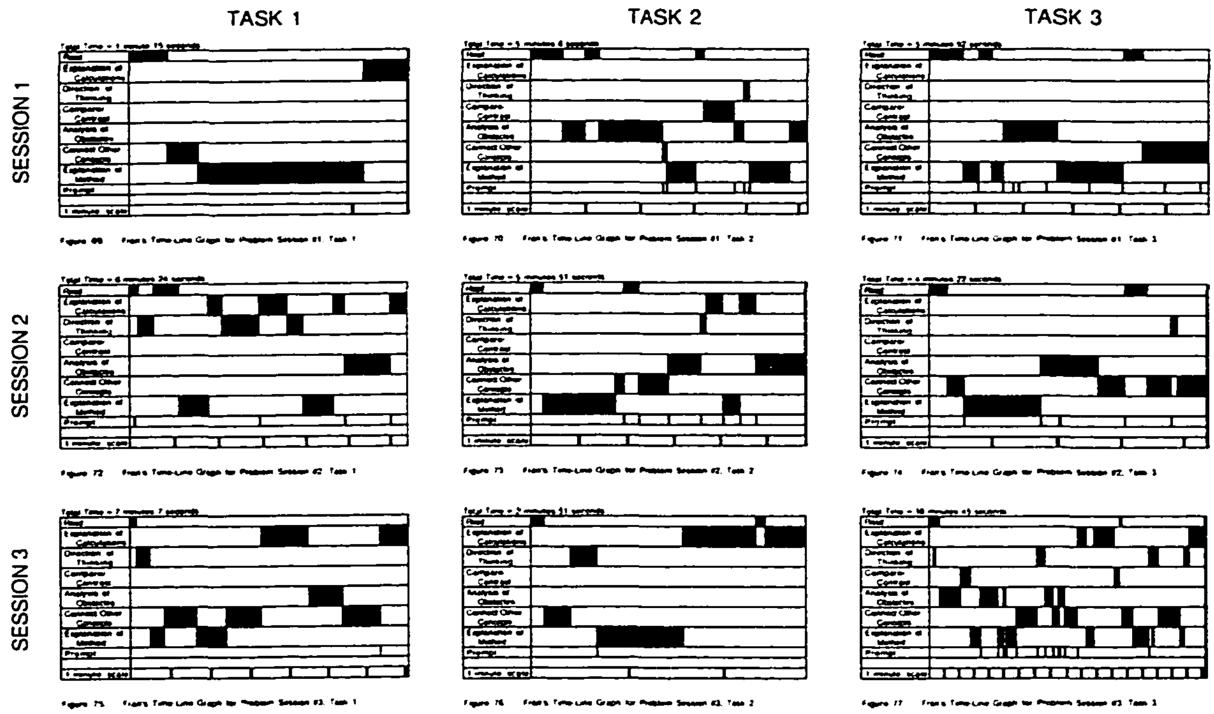


Figure 92. Composite of Fran's Time-Line Graphs.

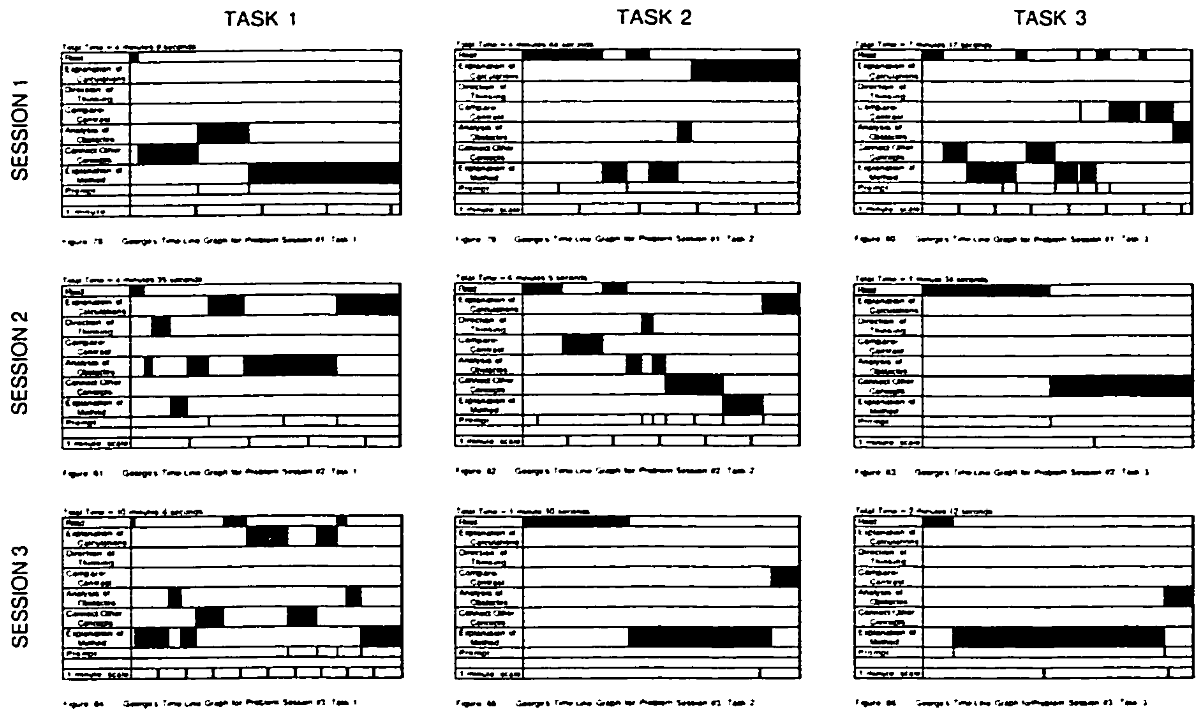


Figure 93. Composite of George's Time-Line Graphs.

Appendix E

Human Subjects Institutional Review Board Research Protocol Clearance

Human Subjects Institutional Review Board

Kalamazoo Michigan 49008-3899

WESTERN MICHIGAN UNIVERSITY

Date: 3 September 1997

To: Christian Hirsch, Principal Investigator
Pamela Crawford, Student Investigator

From: Richard Wright, Chair

Re: HSIRB Project Number 97-08-11

This letter will serve as confirmation that your research project entitled "Fostering Reflective Thinking in First-Semester Calculus Students" has been **approved** under the **expedited** category of review by the Human Subjects Institutional Review Board. The conditions and duration of this approval are specified in the Policies of Western Michigan University. You may now begin to implement the research as described in the application.

Please note that you may **only** conduct this research exactly in the form it was approved. You must seek specific board approval for any changes in this project. You must also seek reapproval if the project extends beyond the termination date noted below. In addition if there are any unanticipated adverse reactions or unanticipated events associated with the conduct of this research, you should immediately suspend the project and contact the Chair of the HSIRB for consultation.

The Board wishes you success in the pursuit of your research goals.

Approval Termination: 3 September 1998

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