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Domination in Digraphs

Lisa Hansen
Western Michigan University

DOMINATION IN DIGRAPHS

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Lisa Hansen

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Lisa Hansen, Ph.D.

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In graph theory, domination in graphs has been studied extensively. In contrast, there has been relatively little research involving domination in digraphs. In a digraph D, a vertex v openly (or 1-step) out-dominates every vertex to which v is adjacent and openly in-dominates every vertex from which v is adjacent. More generally, a vertex v k-step out-dominates every vertex w such that $d(v, w) = k$ and k-step in-dominates every vertex u such that $d(u, v) = k$. A set S of vertices of D is a k-step out-dominating set if every vertex of D is k-step out-dominated by some vertex of S, and the minimum cardinality of a k-step out-dominating set is the k-step out-domination number $\rho_k^+(D)$. Similar definitions are made for a k-step in-dominating set and the k-step in-domination number $\rho_k^-(D)$. A set S of vertices of D is a k-step twin dominating set of D if every vertex of D is k-step in-dominated by some vertex of S and is k-step out-dominated by some vertex of S. The k-step twin domination number $\rho_k^*(D)$ of D is defined in the expected manner.

We determine the values of these parameters for various digraphs. We also study the minimum and maximum of the k-step twin domination numbers, where the minimum and maximum is taken over all possible orientations of a graph. Furthermore, bounds on these parameters are determined and some Nordhaus-Gaddum-type results are presented.

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UMI
300 North Zeeb Road
Ann Arbor, MI 48103

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CHAPTER I

INTRODUCTION

1.1 Domination in Graphs

Historically, the study of domination in graphs originated with a problem posed by de Jaenisch [8] in 1862. The problem is that of finding the minimum number of queens that can be placed on a chessboard so that each square of the chessboard is attacked or "dominated" by at least one of the queens. Other applications of domination in graph theory are in communication and network theory as well as coding theory. As a result, the study of domination in graphs has received much attention in recent years. In fact, Haynes [11] has organized an extensive bibliography consisting of articles and books concerning domination. There is even a book entirely devoted to domination, written by Haynes, Hedetniemi, and Slater [12]. The theory of domination in graphs was formally introduced by Berge [2] in 1958 and by Ore [22] in 1962. In this introductory chapter we describe some of the key terms and notation in the theory of domination in graphs and digraphs.

For a graph G and a vertex v of G, the (*open*) *neighborhood of* v is $N(v) = \{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood of* v is $N[v] = \{v\} \cup N(v)$. Each element of $N(v)$ is called a *neighbor* of v. For a set S of vertices, the *open neighborhood of* S is $N(S) = \bigcup_{v\in S} N(v)$ and the *closed neighborhood of* S is $N[S] = \bigcup_{v\in S} N[v] = S \cup N(S)$. A vertex v in a graph G is said to *dominate* itself and all of its neighbors, that is, v dominates the vertices in $N[v]$. A set S of vertices of G is a *dominating set* of G if every vertex of G is dominated by some vertex of S or, equivalently, $N[S] = V(G)$. A dominating set of

minimum cardinality is called a *minimum dominating set.* The *domination number* $\gamma(G)$ of G is the cardinality of a minimum dominating set. A *minimal dominating set* of G is a dominating set S of G such that no proper subset of S is a dominating set of G. A minimum dominating set is thus a minimal dominating set; indeed, it is one of minimum cardinality. The *upper domination number* $\Gamma(G)$ is the maximum cardinality among all minimal dominating sets of G. Of course, $\Gamma(G) \geq \gamma(G)$ for every graph G. Indeed, it is possible for $\Gamma(G) > \gamma(G)$. The graph G of Figure 1 illustrates this fact. Specifically, $\{y\}$ is a minimum dominating set of G and thus $\gamma(G) = 1$. Also, $S = \{v, w\}$ is a minimal dominating set of G that is not a minimum dominating set of G. In fact, the set S is a minimal dominating set of maximum cardinality, so $\Gamma(G) = 2$.

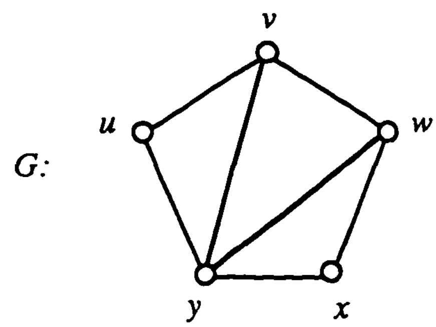

Figure 1. A Graph G Such That $\Gamma(G) > \gamma(G)$.

The following result due to Ore [22] characterizes minimal dominating sets.

Theorem 1.A A dominating set S of a graph G is a minimal dominating set of G if and only if every vertex v of S satisfies at least one of the following two properties:

(i) there exists a vertex w in $V(G) - S$ such that $N(w) \cap S = \{v\}$,

(ii) v is adjacent to no vertex of S.

Another classical result regarding minimal dominating sets due to Ore [22] is the following.

Theorem 1.B If G is a graph with no isolated vertices and S is a minimal dominating set of G, then $V(G) - S$ is a dominating set of G.

Several bounds exist for the domination number of a graph. The following upper bound is due to Ore [22].

Theorem 1.C If G is a graph of order n without isolated vertices, then $\gamma(G) \le n/2$.

Other bounds exist for the domination number of a graph in terms of its order and maximum degree. The lower bound is due to Walikar, Acharya, and Sampathkumar [25], and the upper bound is given by Berge [3].

Theorem 1.D If G is a graph of order n with maximum degree $\Delta(G)$, then

$$\left\lceil \frac{n}{1 + \Delta(G)} \right\rceil \le \gamma(G) \le n - \Delta(G).$$

Yet another set of bounds exist in terms of the order and size of the graph. The lower bound is given by Berge [3] and the upper bound by Vizing [24].

Theorem 1.E If G is a graph of order n and size m, then

$$n - m \le \gamma(G) \le n + 1 - \sqrt{1 + 2m}\ .$$

Furthermore, $\gamma(G) = n - m$ if and only if each component of G is a star.

A more general type of domination is *k-domination*. In a graph G, a vertex v *k-dominates* any vertex w such that $d(v, w) \le k$. Notice, then, that

1-domination is equivalent to ordinary domination. A *k-dominating set* of G is a set S of vertices of G such that every vertex of G is k-dominated by some vertex of S. The minimum cardinality of the k-dominating sets of G is called the *k-domination number* $\gamma_k(G)$ of G. For the graph G of Figure 2, $\gamma_2(G) = 2$.

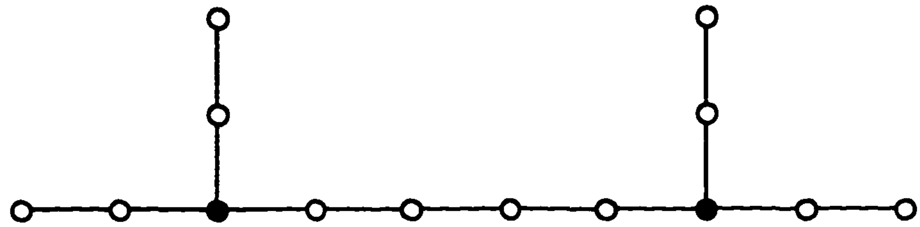

Figure 2. A Graph G Such That $\gamma_2(G) = 2$.

The following upper bound for the k-domination number of a graph is due to Henning, Oellermann and Swart [15].

Theorem 1.F If G is a connected graph of order $n \geq k + 1 \geq 2$, then

$$\gamma_k(G) \leq \frac{n}{k+1}.$$

A more restrictive type of domination is called *open domination.* A vertex v *openly dominates* each vertex of its open neighborhood $N(v)$. A set S of vertices of G is an *open dominating set* of G if every vertex of G is adjacent to some vertex of S, or equivalently, if $N(S) = V(G)$. The *open domination number* $\rho_1(G)$ is then the minimum cardinality among all open dominating sets of G. For the graph G of Figure 1, the set $\{u, y\}$ is a minimum open dominating set and thus $\rho_1(G) = 2$.

Yet another type of domination is *step domination.* For a positive integer k, a vertex v is said to *k-step dominate* each vertex at distance exactly k from v. We can analogously define *k-step dominating sets* and the *k-step domination number*

$\rho_k(G)$ of G. Observe that 1-step domination is equivalent to open domination. For the graph G of Figure 3, $\rho_2(G) = 5$.

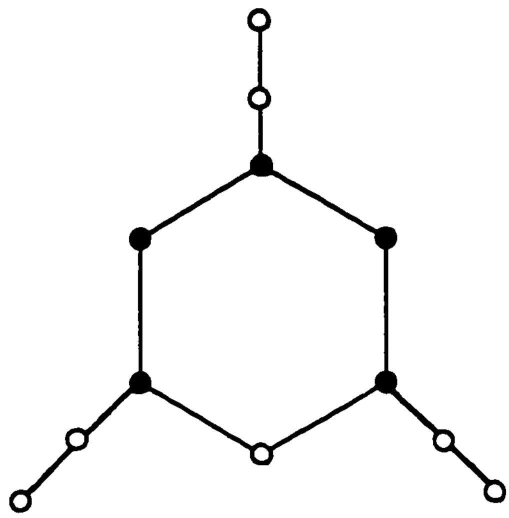

Figure 3. A Graph G Such That $\rho_2(G) = 5$.

The parameter $\rho_k(G)$ may not exist for certain graphs G and certain integers k. The following result of Hayes, Schultz, and Yates [10], however, determines exactly when $\rho_k(G)$ does exist.

Theorem 1.G Let G be a connected graph. The k-step domination number of G is well-defined if and only if $\operatorname{rad} G \geq k$.

Also, Chartrand, Henning, and Schultz [6] showed that, in general, there is no relationship among the numbers $\rho_i(G)$, $i = 1, 2, \ldots$, for an arbitrary graph G.

Theorem 1.H For positive integers i, j, and k with $i < j$, there exist graphs G and H such that $\rho_i(G) - \rho_j(G) \geq k$ and $\rho_j(H) - \rho_i(H) \geq k$.

Let G be a graph of order n and size m with $V(G) = \{v_1, v_2, \ldots, v_n\}$. For an integer $k \geq 1$, the *k-neighborhood* of v is the set $N_k(v) = \{u \in V(G) \mid d(v, u) =$

$k\}$. A sequence $s\colon \ell_1, \ell_2, ..., \ell_n$ of positive integers is called a *universal dominating sequence* for G if every vertex of G is ℓ_i-step dominated by v_i for some i, $1 \le i \le n$, or equivalently, if $\bigcup_{i=1}^{n} N_{\ell_i}(v_i) = V(G)$. Observe that the constant sequence $1, 1, ..., 1$ of length n is a universal dominating sequence of every nontrivial connected graph. Also if $\ell_i = 1$ for each i, then $\sum_{i=1}^{n} |N_{\ell_i}(v_i)| = 2m$ by the First Theorem of Graph Theory. The number $\sum_{i=1}^{n} |N_{\ell_i}(v_i)|$ counts the total number of times the vertices of a graph are step dominated and this number is called the *universal value* of the universal dominating sequence. The graph G of Figure 4 has universal dominating sequence $1, 2, 2, 2, 2, 4$ and universal value 10 as can be seen from Table 1.

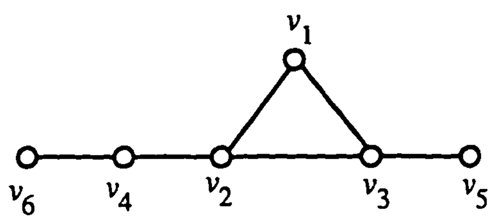

Figure 4. A Graph With Universal Dominating Sequence $1, 2, 2, 2, 2, 4$.

A graph G is a *constant universal graph* if every universal dominating sequence of G has the same universal value. Hayes, Schultz, and Yates [10] characterized constant universal graphs.

Theorem 1.I A connected graph G is a constant universal graph if and only if for all $v \in V(G)$ and for all integers j and k such that $1 \le j, k \le e(v)$,

$$|N_j(v)| = |N_k(v)|.$$

Table 1

Verification That the Sequence 1, 2, 2, 2, 2, 4 is a Universal Dominating Sequence of the Graph of Figure 4

vertex	is ℓ_i-step dominated	by v_i
v_1	2	v_4
v_2	2	v_5
v_3	1	v_1
v_4	2	v_3
v_5	4	v_6
v_6	2	v_2

A concept similar to universal domination is *planetary domination.* For a graph G of order n, a sequence of positive integers s: $\ell_1, \ell_2, \ldots, \ell_k$ $(k \leq n)$ is a *planetary domination sequence* for G if G contains distinct vertices $v_1, v_2, \ldots, v_k$ such that every vertex of G is ℓ_i-step dominated by v_i for some i, $1 \leq i \leq k$. For the graph G of Figure 4, the sequence 3, 1, 1 is a planetary dominating sequence of G as can be seen from Table 2.

A planetary dominating sequence s is *minimal* if no proper subsequence of s is a planetary dominating sequence of G. A *minimum planetary dominating sequence* for a graph G is a planetary dominating sequence of minimum length and the length of such a sequence is the *planetary domination number* $\mathrm{pl}(G)$ of G. For every nontrivial connected graph G and every integer i with $1 \leq i \leq \mathrm{rad}\, G$, it is straightforward to show that $\mathrm{pl}(G) \leq \rho_i(G)$. In fact, Chartrand, Henning, and Schultz [6] showed that this inequality can be strict for each i.

Theorem 1.J There exist graphs G of arbitrarily large radius such that $\mathrm{pl}(G) < \rho_i(G)$ for every integer i $(1 \le i \le \mathrm{rad}\, G)$.

Table 2

Verification That the Sequence 3, 1, 1 is a Planetary Dominating Sequence of the Graph of Figure 4

vertex	is ℓ_i-step dominated	by v_i
v_1	1	v_2
v_2	1	v_3
v_3	1	v_2
v_4	1	v_2
v_5	1	v_3
v_6	3	v_1

In 1956, Nordhaus and Gaddum [21] presented several bounds on the sum and product of the chromatic numbers of a graph and its complement. Nordhaus-Gaddum-type results exist for domination parameters as well. The following result is due to Jaegar and Payan [16].

Theorem 1.K If G is a graph of order $n \ge 2$, then

(a) $3 \le \gamma(G) + \gamma(\overline{G}) \le n + 1$, and

(b) $2 \le \gamma(G)\, \gamma(\overline{G}) \le n$.

The upper bound in (a) can be improved if neither G nor $\overline{G}$ contains isolated vertices. The result is due to Joseph and Arumagam [17].

Theorem 1.L If G is a graph of order $n \geq 2$ such that neither G nor $\overline{G}$ has isolated vertices, then

$$\gamma(G) + \gamma(\overline{G}) \leq \frac{n+4}{2}.$$

1.2 Domination in Digraphs

The concepts of domination easily generalize to digraphs. Digraphs are, in a sense, a more natural setting for domination since domination is, intuitively, a nonsymmetric idea. However, domination in digraphs has not been studied nearly as extensively as domination in graphs. In fact, Lee [19] wrote the first Ph.D. dissertation devoted to the study of domination in digraphs in 1994. A survey of topics studied in domination of digraphs can be found in Ghoshal, Laskar, and Pillone [9].

Let D be a digraph and let v be a vertex of D. The *(open) out-neighborhood* of v is $N^+(v) = \{u \in V(D) \mid (v, u) \in E(D)\}$ and the *(open) in-neighborhood* of v is $N^-(v) = \{u \in V(D) \mid (u, v) \in E(D)\}$. For a set S of vertices, the *out-neighborhood* of S is $N^+(S) = \bigcup_{v \in S} N^+(v)$ and the *in-neighborhood* of S is $N^-(S) = \bigcup_{v \in S} N^-(v)$. The *minimum indegree* and *minimum outdegree* of a digraph D are defined, respectively, as $\delta^-(D) = \min_{v \in V(D)}\{\,|N^-(v)|\,\}$ and $\delta^+(D) = \min_{v \in V(D)}\{\,|N^+(v)|\,\}$. The *maximum indegree* $\Delta^-(D)$ and *maximum outdegree* $\Delta^+(D)$ of D are defined in the expected manner.

A vertex v in a digraph D *dominates* itself and all vertices of $N^+(v)$, while v *openly dominates* only the vertices belonging to $N^+(v)$. The *domination number* $\gamma(D)$ is the minimum cardinality of a dominating set of D, and the *open domination number* $\rho_1(D)$ is the minimum cardinality of an open dominating set of D. For the digraph D of Figure 5, it is straightforward to show that $\{u, w\}$ is a minimum

dominating set of D and $\{u, w, x\}$ is a minimum open dominating set of D. Therefore, $\gamma(D) = 2$ and $\rho_1(D) = 3$.

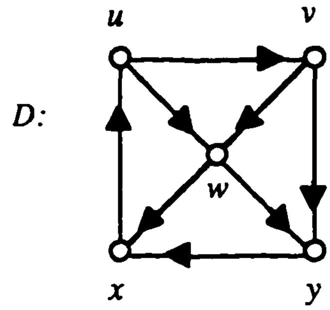

Figure 5. A Digraph D Such That $\gamma(D) = 2$ and $\rho_1(D) = 3$.

Two well-known digraphs are the *directed n-cycle* $\vec{C}_n$ of order n and the *directed path* $\vec{P}_n$ of order n. The digraphs $\vec{C}_6$ and $\vec{P}_5$ are shown in Figure 6. It is straightforward to show that $\gamma(\vec{C}_n) = \lceil n/2 \rceil$, $\gamma(\vec{P}_n) = \lceil n/2 \rceil$, $\rho_1(\vec{C}_n) = n$, and $\rho_1(\vec{P}_n)$ is not defined.

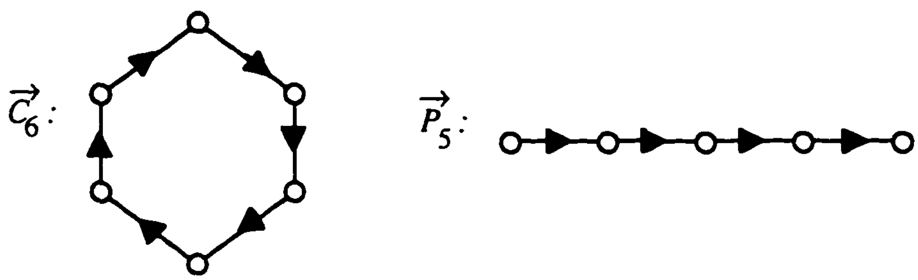

Figure 6. The Directed Cycle $\vec{C}_6$ and the Directed Path $\vec{P}_5$.

Some bounds have been found for the domination number of a digraph. Lee [19] determined an upper bound in terms of the minimum indegree of a digraph D.

Theorem 1.M If D is a digraph of order n and minimum indegree $\delta^-(D) \geq 1$, then

$$1 \leq \gamma(D) \leq \frac{\delta^-(D) + 1}{2\delta^-(D) + 1}\, n.$$

Ghoshal, Laskar, and Pillone [9] determined lower and upper bounds in terms of the maximum outdegree of a digraph.

Theorem 1.N If D is a digraph of order n, then

$$\frac{n}{1 + \Delta^+(D)} \leq \gamma(D) \leq n - \Delta^+(D).$$

Lee [19] determined an upper bound for the domination number of a tournament.

Theorem 1.O If T is a tournament of order n, then $1 \leq \gamma(T) \leq \lfloor \log_2(n + 1) \rfloor$.

Recall, as in Theorem 1.C, that Ore [22] showed $\gamma(G) \leq n/2$ for any graph G of order n without isolated vertices. Ghoshal, Laskar and Pillone [9] showed a similar result for strongly connected digraphs.

Theorem 1.P If D is a strongly connected digraph of order n, then $\gamma(D) \leq \lceil n/2 \rceil$.

An *out-dominating set* of a digraph D is a set S of vertices of D such that every vertex of $D - S$ is adjacent from some vertex of S. The minimum cardinality of an out-dominating set of D is the *out-domination number* $\gamma^+(D)$ of D. Out-domination, then, is the same as ordinary domination in digraphs and, in fact, $\gamma^+(D) = \gamma(D)$. However, an *in-dominating set* of D is a set S of vertices of D such that every vertex of $D - S$ is adjacent to some vertex of S. The *in-domination number*

$\gamma^-(D)$ of D is the minimum cardinality of an in-dominating set of D. For the digraph D of Figure 7, the set $\{u\}$ is a minimum out-dominating set of D while the set $\{v, w, x, y\}$ is a minimum in-dominating set of D. Thus $\gamma^+(D) = 1$ and $\gamma^-(D) = 4$.

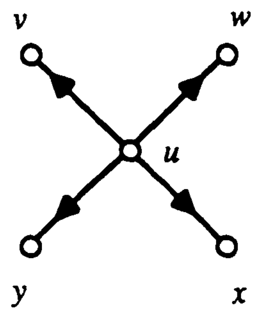

Figure 7. A Digraph D Such That $\gamma^+(D) = 1$ and $\gamma^-(D) = 4$.

Chartrand, Harary and Yue [5] presented a sharp bound for the sum of these two parameters.

Theorem 1.Q If D is a digraph of order n, every component of which is nontrivial, then $\gamma^+(D) + \gamma^-(D) \leq 4n/3$.

A set S of vertices of D is a *twin dominating set* of D if every vertex of D is out-dominated by some vertex of S and in-dominated by some vertex of S. The minimum cardinality of a twin dominating set is the *twin domination number* $\gamma^*(D)$ of D. It is shown in [4] that $\gamma^*(D) \leq 2n/3$ for every nontrivial strong digraph D of order n.

1.3 Orientable Domination of Graphs

One way in which the study of domination in graphs can be related to the study of domination in digraphs is through orientations of graphs. The digraphs D_1 and D_2 of Figure 8 are orientations of the same underlying graph $K_{1,5}$. However, it is straightforward to show that $\gamma(D_1) = 1$ and $\gamma(D_2) = 5$.

In the sense of minimizing dominating sets, the digraph D_1 is the optimal way to orient the edges of $K_{1,5}$ and the digraph D_2 is the worst. This concept has led to the study of *orientable domination*. For a graph G, Chartrand, VanderJagt and Yue [7] defined the *lower orientable domination number* $\text{dom}(G)$ of a graph G to be the minimum domination number among all orientations of G. The *upper orientable domination number* $\text{DOM}(G)$ is the maximum domination number among all orientations of G. For the graph $K_{1,5}$, the digraphs of Figure 8 are orientations of $K_{1,5}$ showing that $\text{dom}(K_{1,5}) = 1$ and $\text{DOM}(K_{1,5}) = 5$. The following results regarding orientable domination numbers are due to Chartrand, VanderJagt and Yue [7].

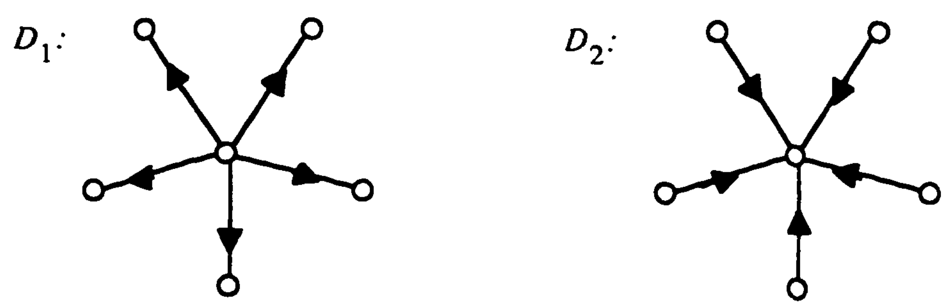

Figure 8. Orientations of $K_{1,5}$ Having Different Domination Numbers.

Theorem 1.R For any graph G, $\text{dom}(G) = \gamma(G)$.

Theorem 1.S For every graph G and integer c with $\mathrm{dom}(G) \leq c \leq \mathrm{DOM}(G)$, there exists an orientation D of G such that $\gamma(D) = c$.

These parameters were also calculated for certain classes of graphs such as cycles, paths and stars.

This work was extended to *orientable k-step domination* by Holley, Lai, and Yue [13, 14]. For instance, the following result characterizes those connected graphs for which the orientable 1-step domination numbers exist.

Theorem 1.T Let G be a connected graph. Then $\mathrm{dom}_1(G)$ and $\mathrm{DOM}_1(G)$ exist if and only if G is not a tree.

For k-step domination, the following sufficient condition was found by Holley, Lai, and Yue [13, 14].

Theorem 1.U If G is a 2-edge-connected graph for which $\mathrm{rad}\, G \geq k$, then $\mathrm{dom}_k(G)$ and $\mathrm{DOM}_k(G)$ are defined.

The upper orientable domination numbers of complete graphs were studied in [7] and [13]. No known formula exists for $\mathrm{DOM}(K_n)$ or $\mathrm{DOM}_1(K_n)$. The following two results were reported in [14].

Theorem 1.V If $n \geq 3$ is an integer, then $\mathrm{DOM}_2(K_n) = 3\lfloor n/3 \rfloor$.

Theorem 1.W If k and n are integers with $3 \leq k \leq n$, then $\mathrm{DOM}_k(K_n)$ is not defined.

Chartrand, Dankelmann, Schultz, and Swart also introduced *orientable twin domination* and presented the following result in [4].

Theorem 1.X For each graph G and integer c with $\text{dom}^*(G) \le c \le \text{DOM}^*(G)$, there exists an orientation D of G such that $\gamma^*(D)$.

CHAPTER II

OPEN TWIN DOMINATION

2.1 Open Twin Domination in Digraphs

In a digraph D, a vertex v *openly out-dominates* the vertices of $N^+(v)$. Open out-domination, then, is synonymous with open domination. An *open out-dominating set* of D is a set S of vertices such that every vertex of D is adjacent from some vertex of S, that is, $N^+(S) = V(D)$. The minimum cardinality among the open out-dominating sets of D is called the *open out-domination number* of D and is denoted by $\rho_1^+(D)$.

A vertex v *openly in-dominates* the vertices belonging to $N^-(v)$. An *open in-dominating set* of D is a set S of vertices such that every vertex of D is adjacent to some vertex of S, or equivalently, $N^-(S) = V(D)$. The minimum cardinality among the open in-dominating sets of D is called the *open in-domination number* and is denoted by $\rho_1^-(D)$.

For example, consider the digraph D in Figure 9. We show that $\rho_1^+(D) = 4$. Observe that $\mathrm{id}\, v_i = 1$ for $i = 4, 5, 6, 7$. This implies that there exists a unique vertex that out-dominates each of v_4, v_5, v_6, and v_7, namely, v_3, v_6, v_7, and v_4, respectively. Thus v_3, v_6, v_7, and v_4 must belong to every open out-dominating set. Since $S = \{v_3, v_4, v_6, v_7\}$ is an open out-dominating set, it is a minimum such set. Therefore $\rho_1^+(D) = |S| = 4$.

However, for the digraph D of Figure 9, $\rho_1^-(D)$ is not defined since the vertex v_1 has outdegree 0. That is, v_1 is not adjacent to any vertex and cannot be in-dominated by any vertex of D.

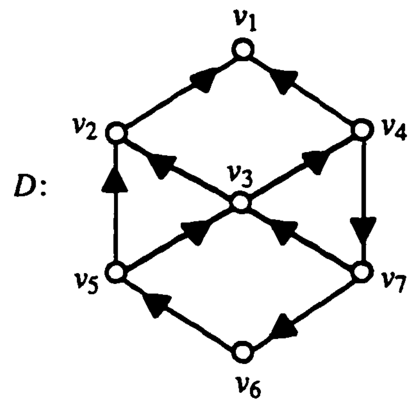

Figure 9. A Digraph D Such That $\rho_1^+(D) = 4$.

In fact, this brings us to our first observation. For a digraph D, a necessary and sufficient condition for $\rho_1^+(D)$ to be defined is that each vertex has positive indegree. Similarly, $\rho_1^-(D)$ is defined if and only if every vertex has positive outdegree. We begin this section with a number of observations.

Proposition 2.1 Let D be a digraph with $\delta^-(D) \geq 1$, and let D' be the converse of D. Then $\rho_1^+(D) = \rho_1^-(D')$.

Proof A vertex u out-dominates a vertex v in D if and only if u in-dominates v in D'. Consequently, an open out-dominating set of D is an open in-dominating set of D' and vice versa. Therefore $\rho_1^-(D') \leq \rho_1^+(D)$ and $\rho_1^+(D) \leq \rho_1^-(D')$, implying that $\rho_1^+(D) = \rho_1^-(D')$. ❑

Proposition 2.2 If a digraph D is a directed tree, then neither $\rho_1^+(D)$ nor $\rho_1^-(D)$ is defined.

Proof Suppose that D is a digraph with n vertices and m arcs such that $\rho_1^+(D)$ is defined. Then every vertex of D has positive indegree. Thus

$$m = \sum_{v \in V(D)} \text{id}\, v \geq \sum_{v \in V(D)} 1 = n,$$

implying that D is not a directed tree.

A similar argument shows that if D is a digraph such that $\rho_1^-(D)$ is defined, then D is not a directed tree. ❑

For a digraph D and a set W of vertices of D, we say that a set S of vertices of D *openly twin dominates* W if every vertex belonging to W is both out-dominated by some vertex of S and in-dominated by some vertex of S. An *open twin dominating set* of a digraph D is a set S of vertices of D such that S openly twin dominates $V(D)$. The *open twin domination number* $\rho_1^*(D)$ of D is defined as the minimum cardinality among the open twin dominating sets of D. It is clear that $\rho_1^*(D)$ is defined if and only if every vertex of D has both positive indegree and positive outdegree.

For example, let D be the digraph in Figure 10. Observe that the vertex u is uniquely out-dominated by v and is uniquely in-dominated by y. Thus v and y must belong to every open twin dominating set. Also the vertex w is uniquely in-dominated by x, and x is uniquely out-dominated by w. So w and x must also belong to every open twin dominating set. Since $S = \{v, w, x, y\}$ is an open twin dominating set of D, it follows that $\rho_1^*(D) = |S| = 4$.

In the case that D is a directed cycle, $\rho_1^*(D)$ is easily determined.

Proposition 2.3 For every integer $n \geq 3$, $\rho_1^*(\vec{C}_n) = n$.

Proof Since each vertex of $\vec{C}_n$ is uniquely out-dominated by the vertex that precedes it and is uniquely in-dominated by the vertex that follows it, every vertex must belong to every open twin dominating set, implying that $\rho_1^*(\vec{C}_n) = n$. ❑

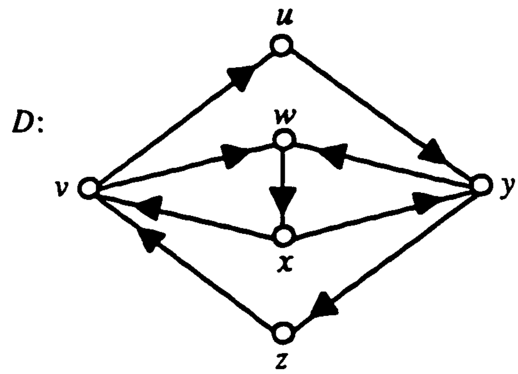

Figure 10. A Digraph D Such That $\rho_1^*(D) = 4$.

Next we present a simple lower bound for the open twin domination number of a digraph.

Proposition 2.4 If D is a digraph with $\delta^-(D) \geq 1$ and $\delta^+(D) \geq 1$, then $\rho_1^*(D) \geq 2$.

Proof Let D be a digraph and let S be a minimum open twin dominating set of D. Since a vertex v in S cannot out-dominate or in-dominate itself, there exists a vertex u $(\neq v)$ belonging to S and so $|S| \geq 2$. ❑

There exists an infinite class of digraphs that attain this lower bound.

Proposition 2.5 For every integer $n \geq 2$, there exists a digraph D_n of order n such that $\rho_1^*(D_n) = 2$.

Proof We construct D_n by letting $V(D_n) = \{u, v\} \cup \{w_1, w_2, ..., w_{n-2}\}$ and $E(D_n) = \{(u, v), (v, u)\} \cup \{(u, w_i) \mid 1 \leq i \leq n-2\} \cup \{(w_i, v) \mid 1 \leq i \leq n-2\}$. The digraph D_6 is shown in Figure 11. Since u is adjacent to every vertex of D_n

and v is adjacent from every vertex of D_n, it follows that $\{u, v\}$ is a minimum open twin dominating set of D_n and thus $\rho_1^*(D_n) = 2$. ❑

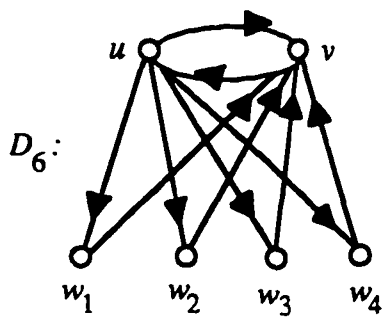

Figure 11. An Example of the Digraph Constructed in the Proof of Proposition 2.5.

In fact, the digraph D_n of Proposition 2.5 is the unique digraph of order n and minimum size such that $\rho_1^*(D_n) = 2$.

Lower and upper bounds for $\rho_1^*(D)$ in terms of $\rho_1^+(D)$ and $\rho_1^-(D)$ are presented next.

Proposition 2.6 If D is a digraph with $\delta^-(D) \geq 1$ and $\delta^+(D) \geq 1$, then

$$\max\{\rho_1^+(D), \rho_1^-(D)\} \leq \rho_1^*(D) \leq \rho_1^+(D) + \rho_1^-(D).$$

Proof The lower bound follows from the fact that every open twin dominating set of D is both an open out-dominating set of D and an open in-dominating set of D. To prove the upper bound, let S^+ be a minimum open out-dominating set of D and S^- be a minimum open in-dominating set of D. Then $S = S^+ \cup S^-$ is an open twin dominating set of D. Thus $\rho_1^*(D) \leq |S| \leq |S^+| + |S^-| = \rho_1^+(D) + \rho_1^-(D)$. ❑

The following result is now immediate.

Corollary 2.7 If D is a digraph of order n such that $\delta^-(D) \geq 1$ and $\delta^+(D) \geq 1$,

then

$$\rho_1^+(D) + \rho_1^-(D) \leq 2\rho_1^*(D) \leq 2n. \tag{1}$$

The bounds in (1) are sharp since for $n \geq 3$, $\rho_1^+(\vec{C}_n) = n$ and $\rho_1^-(\vec{C}_n) = n$. Thus $\rho_1^+(\vec{C}_n) + \rho_1^-(\vec{C}_n) = 2n$.

For a digraph D with $\delta^-(D) \geq 1$ and $\delta^+(D) \geq 1$, let $a = \rho_1^-(D)$, $b = \rho_1^+(D)$, and $c = \rho_1^*(D)$. It then follows from our previous results that $a \geq 2, b \geq 2, c \geq \max\{a, b\}$, and $c \leq a + b$. The next result shows that all such possible triples a, b, c of integers are attainable.

Theorem 2.8 Let a, b, and c be integers such that $a \geq 2, b \geq 2, c \geq \max\{a, b\}$ and $c \leq a + b$. Then there exists a digraph D such that $\rho_1^-(D) = a$, $\rho_1^+(D) = b$, and $\rho_1^*(D) = c$.

Proof Without loss of generality, we assume that $b \geq a$. We consider three cases.

Case 1 $c = b$. Define a digraph D by letting $V(D) = \{u_1, u_2, ..., u_b, u_{b+1}\}$ and letting $u_1, u_2, ..., u_b, u_{b+1}, u_1$ be a $(b + 1)$-cycle of D. Also let $(u_i, u_1) \in E(D)$ for $i = a, a + 1, ..., b$. Then $S_1^- = \{u_1, u_2, ..., u_a\}$ is a minimum open in-dominating set and $S = \{u_1, u_2, ..., u_b\}$ is both a minimum open out-dominating set and a minimum open twin dominating set of D.

Case 2 $c = a + b$. Define a digraph D such that $V(D) = \{u_1, u_2, ..., u_a, v_1, v_2, ..., v_b\}$. Let $u_1, u_2, ..., u_a, u_1$ be a directed a-cycle of D and let $v_1, v_2, ..., v_b, v_1$ be a directed b-cycle of D. Let (v_i, u_i) be an arc of D for each $i = 1, 2, ..., a$ and let (v_j, u_a) be an arc of D for $j > a$. Then $S_1^- = \{u_1, u_2, ..., u_a\}$ is a minimum open in-dominating set, $S_1^+ = \{v_1, v_2, ..., v_b\}$ is a minimum open out-dominating set, and $S_1^* = V(D)$ is a minimum open twin dominating set of D.

Case 3 $b < c < a + b$. Let D be a digraph of order $a + b$ with vertices labeled 1, 2, ..., $a + b$. Define $E(D) = \{(j, (a + b - c + j) \bmod a) \mid j = 1, 2, ..., c\} \cup \{(j, j \bmod c) \mid j = c + 1, c + 2, ..., a + b\} \cup \{(c - b + j, a + j) \mid j = 1, 2, ..., b\}$. It can easily be verified that since $b < c < a + b$, the corresponding relation defined by E(D) is irreflexive. For this digraph D the set $S_1^- = \{1, 2, ..., a\}$ is a minimum open in-dominating set, $S_1^+ = \{c - b + 1, c - b + 2, ..., c\}$ is a minimum open out-dominating set, and $S_1^* = \{1, 2, ..., c\}$ is a minimum open twin dominating set of D. This construction is shown for the case $a = 5, b = 8$, and $c = 12$ in Figure 12. ❑

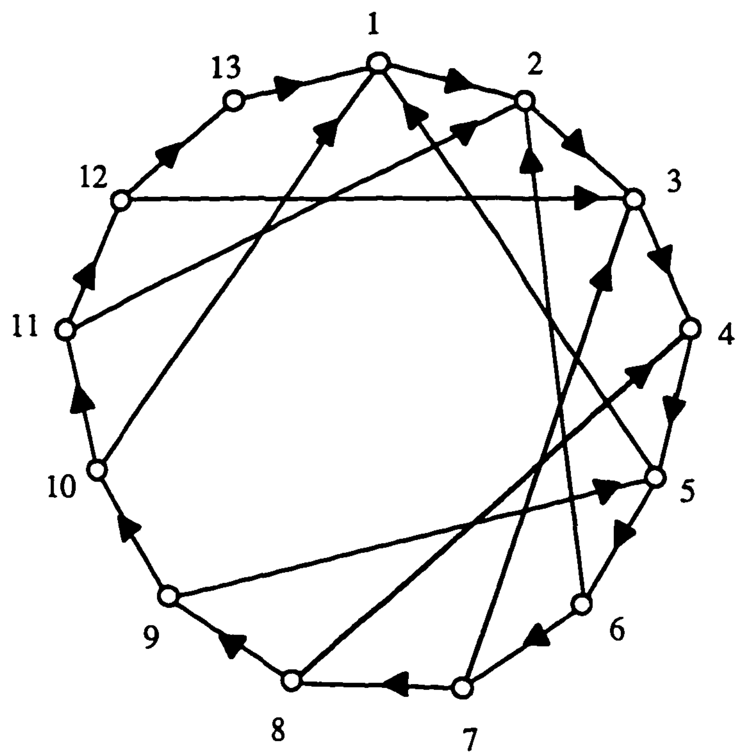

Figure 12. An Example of the Digraph Constructed in the Proof of Theorem 2.8.

Let S be a minimum open twin dominating set of a digraph D. What structure does the subdigraph $\langle S \rangle$ have? The answer to this question can be found in the next theorem.

Theorem 2.9 If D is a digraph with $\delta^-(D) \geq 1$ and $\delta^+(D) \geq 1$, then there exists a digraph D' such that D is an induced subdigraph of D' and $V(D)$ is a minimum open twin dominating set of D'.

Proof Let $V(D) = \{v_1, v_2, ..., v_n\}$ and let $k = \lceil n/2 \rceil$. Construct a digraph D' from D by adding vertices $u_1, u_2, ..., u_k$ and arcs (u_i, v_{2i}) and (v_{2i-1}, u_i) for $i = 1, 2, ..., k$, where the subscripts are expressed modulo n. Then for each $i = 1, 2, ..., k$, the vertex u_i is uniquely in-dominated by v_{2i} and is uniquely out-dominated by v_{2i-1}. Thus any open twin dominating set of D' must contain $V(D)$. And since $\delta^-(D) \geq 1$ and $\delta^+(D) \geq 1$, in fact, $V(D)$ is a minimum open twin dominating set of D'. ❑

As an immediate consequence, we have the following.

Corollary 2.10 For every digraph D with $\delta^-(D) \geq 1$ and $\delta^+(D) \geq 1$, there exists a digraph D' with minimum open twin dominating set S such that $\langle S \rangle = D$.

2.2 Orientable Open Twin Domination of Graphs

Much of our attention will be focused on asymmetric digraphs and, in general, on orientations of a given graph G. Recall that, for a digraph D, the open twin domination number $\rho_1^*(D)$ is defined if and only if every vertex has both positive indegree and positive outdegree. Thus, in this case, the underlying graph must have minimum degree at least 2. For a graph G with minimum degree at least 2, an orientation D of G is called a *valid orientation* if every vertex of D has both

positive indegree and positive outdegree. In order to show that every graph with minimum degree at least 2 has a valid orientation, we will use the following theorem, due to Robbins [23].

Theorem 2.A Every 2-edge-connected graph has a strong orientation.

Proposition 2.11 Every graph with minimum degree at least 2 has a valid orientation.

Proof Let G be a graph with $\delta(G) \geq 2$. Since G is not a tree, G contains at least one cyclic block. Let $B_1, B_2, ..., B_k$, $k \geq 1$, be the cyclic blocks of G. We proceed by induction on k. If $k = 1$, then G is 2-edge-connected. By Theorem 2.A, the graph G has a strong orientation, and, clearly, a strong orientation is a valid orientation.

Now we assume that every graph with minimum degree at least 2 containing at most $k \geq 1$ cyclic blocks has a valid orientation. Let G be a graph with minimum degree at least 2 such that G has $k + 1$ cyclic blocks. Since $\delta(G) \geq 2$, we know G contains a cyclic end-block B. Form an orientation of G by first assigning a strong orientation to B. Since B is an end-block, there is a shortest path P (possibly of length 0) between a vertex v of B and a vertex u of one of the other k cyclic blocks, say B'. Direct the edges of P from B to B'. Then the subgraph $H = G - ((V(P) \cup V(B)) - \{u\})$ is a graph with k cyclic blocks. By the inductive hypothesis, we can assign a strong orientation to H, producing a valid orientation of G. ❑

In the proof of Proposition 2.11, we used the fact that a strong orientation is a valid orientation. However, not every valid orientation is a strong orientation, as can be seen in the digraph of Figure 13.

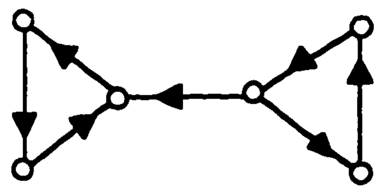

Figure 13. A Valid Orientation That is Not a Strong Orientation.

We next present a lower bound for the open twin domination number of a valid orientation of a graph.

Proposition 2.12 Let G be a graph with $\delta(G) \geq 2$. If D is a valid orientation of G, then $\rho_1^*(D) \geq 3$.

Proof Let S be a minimum open twin dominating set of D and let u be a vertex belonging to S. Then u is out-dominated by some vertex v of S and is in-dominated by some vertex w of S. Since D is asymmetric, $w \neq v$. Thus $|S| \geq 3$ and so $\rho_1^*(D) \geq 3$. ❑

Graphs can have more than one valid orientation. The graph G of Figure 14 has, up to isomorphism, nine valid orientations, three of which are D_1, D_2, and D_3.

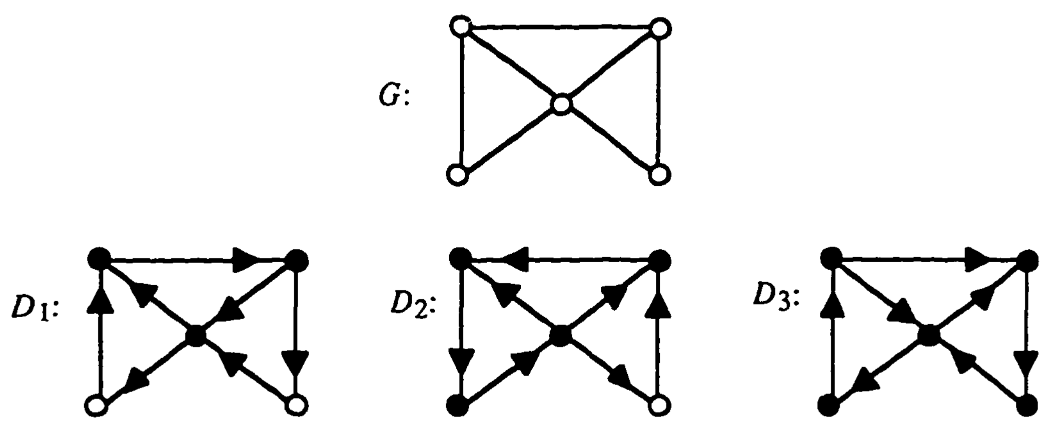

Figure 14. A Graph G and Three of Its Valid Orientations.

It is straightforward to show that $\rho_1^*(D_1) = 3$, $\rho_1^*(D_2) = 4$, and $\rho_1^*(D_3) = 5$. This leads us to other concepts.

Let G be a graph with $\delta(G) \geq 2$. Define the *lower orientable open twin domination number* of G and the *upper orientable open twin domination number* of G, respectively, as

$$\mathrm{dom}_1^*(G) = \min\{\rho_1^*(D)\}$$

and

$$\mathrm{DOM}_1^*(G) = \max\{\rho_1^*(D)\},$$

where the minimum and maximum, respectively, is taken over all valid orientations D of G.

2.3 The Lower Orientable Open Twin Domination Number

We begin this section by relating $\mathrm{dom}_1^*(G)$ to the parameter $\mathrm{dom}_1(G)$, which was first introduced in [13].

Proposition 2.13 If G is a graph with $\delta(G) \geq 2$, then

$$\mathrm{dom}_1(G) \leq \mathrm{dom}_1^*(G).$$

Proof Let $\mathcal{D}^+ = \{D \mid D \text{ is an orientation of } G,\ \delta^-(D) \geq 1\}$ and let

$$\mathcal{D}^* = \{D \mid D \text{ is an orientation of } G,\ \delta^-(D) \geq 1 \text{ and } \delta^+(D) \geq 1\}.$$

Then $\mathrm{dom}_1(G) = \min_{D \in \mathcal{D}^+}\{\rho_1^+(D)\}$ and $\mathrm{dom}_1^*(G) = \min_{D \in \mathcal{D}^*}\{\rho_1^*(D)\}$. Clearly $\mathcal{D}^* \subseteq \mathcal{D}^+$. Also, by Proposition 2.6, for each digraph $D \in \mathcal{D}^*$, it follows that $\rho_1^+(D) \leq \rho_1^*(D)$. Therefore, $\mathrm{dom}_1(G) \leq \mathrm{dom}_1^*(G)$. ❑

Since there is only one valid orientation of a cycle, namely a directed cycle, both the lower and upper orientable open twin domination numbers are easy to calculate for cycles. In particular, it follows from Proposition 2.3 that $\mathrm{dom}_1^*(C_n) = \mathrm{DOM}_1^*(C_n) = n$. We next calculate the lower orientable twin domination number for complete graphs.

Proposition 2.14 For $n \geq 3$, $\mathrm{dom}_1^*(K_n) = 3$.

Proof By Proposition 2.12, we know $\mathrm{dom}_1^*(K_n) \geq 3$. We obtain an orientation of K_n by forming a directed 3-cycle among three vertices u, v, and w. Let X be the set of remaining $n-3$ vertices. For each vertex $x \in X$, direct the edge from u to x, direct the edge from x to w, and orient the edge vx arbitrarily. (See Figure 15.) The resulting orientation is a valid orientation and has open twin dominating set $\{u, v, w\}$. Therefore $\mathrm{dom}_1^*(K_n) = 3$. ❑

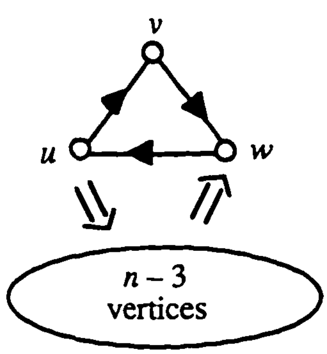

Figure 15. An Illustration of the Construction in the Proof of Proposition 2.14.

For a digraph D, a lower bound for $\rho_1^*(D)$ can be found in terms of its order and maximum degree.

Theorem 2.15 Let G be a graph of order n with minimum degree at least 2. If D is a valid orientation of G, then

$$\rho_1^*(D) \geq \left\lceil \frac{2n}{\Delta(G)} \right\rceil.$$

Proof Let S be a minimum open twin dominating set of D. Then $\sum_{v \in S} \text{od } v \geq n$ and $\sum_{v \in S} \text{id } v \geq n$. So

$$\begin{aligned} 2n \leq \sum_{v \in S} \text{od } v + \sum_{v \in S} \text{id } v &= \sum_{v \in S} (\text{od } v + \text{id } v) \\ &= \sum_{v \in S} \deg_G v \\ &\leq \sum_{v \in S} \Delta(G) \\ &= |S| \; \Delta(G). \end{aligned}$$

So $\rho_1^*(D) = |S| \geq \frac{2n}{\Delta(G)}$. Thus $\rho_1^*(D) \geq \left\lceil \frac{2n}{\Delta(G)} \right\rceil$. ❑

From Theorem 2.15, we immediately obtain the following.

Corollary 2.16 For each graph G of order n with minimum degree at least 2, $\text{dom}_1^*(G) \geq \left\lceil \frac{2n}{\Delta(G)} \right\rceil$.

It is interesting to note that the bound in Corollary 2.16 is attained for complete graphs and for cycles. For complete bipartite graphs, we obtain the following bound from Corollary 2.16.

Corollary 2.17 For $m \geq n \geq 2$,

$$\text{dom}_1^*(K_{m,n}) \geq \left\lceil \frac{2(m+n)}{m} \right\rceil.$$

The next theorem shows that the bound in Corollary 2.17 is not, in general, attained for complete bipartite graphs.

Theorem 2.18 For $m \geq n \geq 2$,

$$\operatorname{dom}_1^*(K_{m,n}) = 4.$$

Proof Let D be a valid orientation of $K_{m,n}$. Let $U = \{u_1, u_2, \ldots, u_m\}$ and $V = \{v_1, v_2, \ldots, v_n\}$ be the partite sets of D, and let S be a minimum open twin dominating set of D. Since the vertices belonging to V can only be out-dominated by the vertices of U, it follows that at least one vertex of U, say u_i, belongs to S. Now since $\operatorname{id} u_i \geq 1$ and $\operatorname{od} u_i \geq 1$, we know that u_i out-dominates a proper subset of the vertices of V. Thus there exists another vertex $u_j \in S$, for some $j \neq i$. A symmetric argument shows that S contains at least two vertices belonging to V. Hence $|S| \geq 4$, implying that $\rho_1^*(D) \geq 4$ for every valid orientation D of $K_{m,n}$. Therefore $\operatorname{dom}_1^*(K_{m,n}) \geq 4$.

Finally we exhibit an orientation D of $K_{m,n}$ in Figure 16 such that $\rho_1^*(D) = 4$.

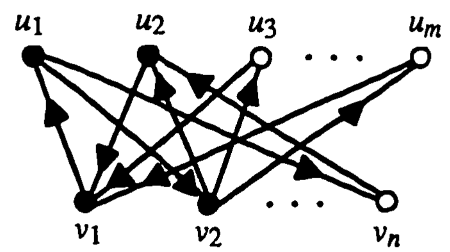

Figure 16. An Orientation D of $K_{m,n}$ Such That $\rho_1^*(D) = 4$.

In particular, $E(D)$ contains the subset

$$\{(u_1, v_j), (v_j, u_2) \mid 2 \leq j \leq n\} \cup \{(u_i, v_1), (v_2, u_i) \mid 2 \leq i \leq m\} \cup \{(v_1, u_1)\}.$$

All other edges are oriented arbitrarily. Then $\{u_1, u_2, v_1, v_2\}$ is a minimum open twin dominating set. Thus $\rho_1^*(D) = 4$ and $\operatorname{dom}_1^*(K_{m,n}) = 4$. ❑

The lower orientable twin domination number can also be determined for complete t-partite graphs, $t \geq 3$.

Theorem 2.19 For $t \geq 3$, $\mathrm{dom}_1^*(K_{n_1,n_2,\ldots,n_t}) = 3$.

Proof By Proposition 2.12, we know $\mathrm{dom}_1^*(K_{n_1,n_2,\ldots,n_t}) \geq 3$. Thus we only need to exhibit an orientation D of $K_{n_1,n_2,\ldots,n_t}$ such that $\rho_1^*(D) = 3$. Let $V_1, V_2, \ldots, V_t$ be the partite sets of $K_{n_1,n_2,\ldots,n_t}$ such that $|V_i| = n_i$ for $i = 1, 2, \ldots, t$. Let u be a vertex of V_1, let v be a vertex of V_2, and let w be a vertex of V_3. Construct D so that $E(D)$ contains the subset

$$\{(u, v), (v, w), (w, u)\} \cup$$

$$\{(u, x) \mid x \neq w,\ x \in \bigcup_{i=2}^{t} V_i\} \cup \{(v, x) \mid x \neq u,\ x \in V_1\} \cup$$

$$\{(x, w) \mid x \neq u,\ x \in \bigcup_{i \neq 3} V_i\} \cup \{(x, v) \mid x \neq w,\ x \in V_3\}.$$

Then {u, v, w} is a minimum open twin dominating set of D and $\mathrm{dom}_1^*(K_{n_1,n_2,\ldots,n_t}) = 3$. ❑

Notice that Proposition 2.14 also follows as a corollary to Theorem 2.19, where $n_i = 1$ for each $i = 1, 2, \ldots, t$. In the next two results, we present bounds on the lower orientable domination number for other bipartite graphs.

Proposition 2.20 If B is a bipartite graph with $\delta(B) \geq 2$, then $\mathrm{dom}_1^*(B) \geq 4$.

Proof An argument similar to that given for Theorem 2.18 shows that for each valid orientation D of B, every open twin dominating set of D contains at least two vertices from each partite set. So $\rho_1^*(D) \geq 4$ and $\mathrm{dom}_1^*(B) \geq 4$. ❑

The bound in Proposition 2.20 may be quite weak. For example, the cycle C_{2n}, $n \geq 3$, is bipartite and $\mathrm{dom}_1^*(C_{2n}) = 2n > 4$.

Theorem 2.21 If B is a bipartite graph with $\delta(B) \geq 2$, then

$$\mathrm{dom}_1^*(B \times K_2) \leq 2\,\mathrm{dom}_1^*(B).$$

Proof Let D be an orientation of B such that $\rho_1^*(D) = \mathrm{dom}_1^*(B)$, and let S be a minimum open twin dominating set of B. In the graph $B \times K_2$, let V_1 and V_2 be the partite sets in the first copy of B and let V_1' and V_2' be the corresponding partite sets in the second copy of B. Now form an orientation D' of $B \times K_2$ by orienting both copies of B to obtain D, and arbitrarily orienting the edges between the two copies of B. Now let S be a minimum open twin dominating set of D. Then, using the same set S of vertices in the first copy of $B \times K_2$, along with a set S' of corresponding vertices in the second copy of $B \times K_2$, we have an open twin dominating set $S \cup S'$. Thus $\mathrm{dom}_1^*(B \times K_2) \leq \rho_1^*(D') \leq |S \cup S'| = 2|S| = 2\mathrm{dom}_1^*(B)$. ❑

However, there exists a bipartite graph B with $\delta(B) \geq 2$ such that $\mathrm{dom}_1^*(B \times K_2) < 2\mathrm{dom}_1^*(B)$. In particular, let $B = C_4$. Observe that $\mathrm{dom}_1^*(B) = 4$. Consider the orientation D of $B \times K_2 \equiv Q_3$ shown in Figure 17. The set $S = \{1, 2, 3, 4, 5, 6\}$ is an open twin dominating set of D. Thus $\mathrm{dom}_1^*(B \times K_2) \leq \rho_1^*(D) \leq |S| = 6 < 8 = 2\mathrm{dom}_1^*(B)$.

Furthermore, there exists an infinite class of bipartite graphs B for which $\mathrm{dom}_1^*(B \times K_2)$ and $2\mathrm{dom}_1^*(B)$ can be arbitrarily far apart.

Theorem 2.22 For each integer $k \geq 4$, there exists a bipartite graph B such that

$$2\mathrm{dom}_1^*(B) - \mathrm{dom}_1^*(B \times K_2) \geq k.$$

Proof Let $r = \lceil k/2 \rceil$ and let $B = C_{3r}$. Then $\text{dom}_1^*(B) = 3r$. Let $V(B \times K_2) = \{u_1, u_2, ..., u_{3r}\} \cup \{v_1, v_2, ..., v_{3r}\}$ such that $u_1, u_2, ..., u_{3r}, u_1$ is a $3r$-cycle, $v_1, v_2, ..., v_{3r}, v_1$ is a $3r$-cycle, and $u_iv_i \in E(B \times K_2)$ for each $i = 1, 2, ..., 3r$. Construct an orientation D of $B \times K_2$ by letting $u_{3i+1}, u_{3i+2}, v_{3i+2}, v_{3i+1}, u_{3i+1}$ be a 4-cycle of D for each $i = 0, 1, ..., r-1$. Also let D contain the set

$$\{(u_{3i-1}, u_{3i}), (u_{3i}, u_{3i+1}), (v_{3i-1}, v_{3i}), (v_{3i}, v_{3i+1}) \mid i = 1, 2, ..., r\}$$

of arcs. Then $\{u_{3i+1}, u_{3i+2}, v_{3i+2}, v_{3i+1} \mid i = 0, 1, ..., r-1\}$ is an open twin dominating set of D. Thus $\text{dom}_1^*(B \times K_2) \leq \rho_1^*(D) \leq 4r$. Therefore

$$2\text{dom}_1^*(B) - \text{dom}_1^*(B \times K_2) \geq 2 \cdot 3r - 4r = 2r \geq k. \quad \square$$

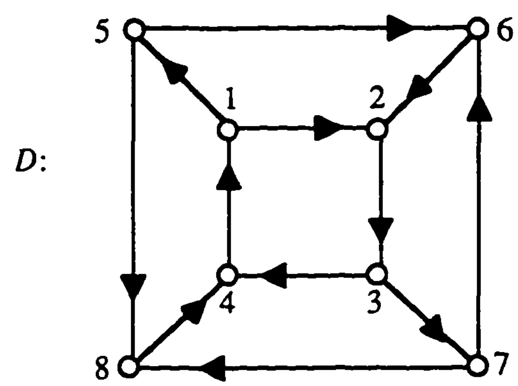

Figure 17. An Orientation of the 3-cube Q_3.

2.4 The Upper Orientable Open Twin Domination Number

We begin this section by presenting an upper bound for $\rho_1^*(D)$ in terms of the size and minimum degree of the digraph D.

Theorem 2.23 Let G be a graph of size m with $\delta(G) \geq 2$. Then for each valid orientation D of G,

$$\rho_1^*(D) \leq \left\lfloor \frac{2m}{\delta(G)} \right\rfloor.$$

Proof Let S be a minimum open twin dominating set of D. Then

$$2m = \sum_{v \in V(G)} \deg_G v \geq \sum_{v \in S} \deg_G v \geq \delta(G)\ |S|.$$

So $\rho_1^*(D) = |S| \leq \frac{2m}{\delta(G)}$, implying that $\rho_1^*(D) \leq \left\lfloor \frac{2m}{\delta(G)} \right\rfloor$. ❑

Combining Corollary 2.16 and an immediate consequence of Theorem 2.23, we obtain the following result.

Corollary 2.24 For every graph G of order n, size m, and minimum degree at least 2,

$$\left\lceil \frac{2n}{\Delta(G)} \right\rceil \leq \mathrm{dom}_1^*(G) \leq \mathrm{DOM}_1^*(G) \leq \left\lfloor \frac{2m}{\delta(G)} \right\rfloor.$$

Both the upper and lower bounds are attained for cycles. However, the next theorem shows that the upper bound is not attained for complete bipartite graphs.

Theorem 2.25 For $m \geq n \geq 2$,

$$\mathrm{DOM}_1^*(K_{m,n}) = 2n.$$

Proof Let V and W be the partite sets of $K_{m,n}$ where $|V| = n$ and $|W| = m$. Let D be any valid orientation of $K_{m,n}$, that is, an orientation of $K_{m,n}$ such that every vertex has both positive indegree and positive outdegree. Clearly the set V openly twin dominates the set W.

Let $S \subsetneq W$ be a set of vertices that is minimal with respect to openly twin dominating the set V, that is, the set S openly twin dominates the set V and for every vertex $w \in S$, the set $S - \{w\}$ does not openly twin dominate V. We show

that $|S| \leq n$. Suppose, to the contrary, that $|S| \geq n+1$. Now, by the minimality of S, we know that for every vertex $w \in S$, there exists a vertex w' in V such that either (a) w' is out-dominated by w, and w' is not out-dominated by any vertex in $S - \{w\}$, or (b) w' is in-dominated by w, and w' is not in-dominated by any vertex in $S - \{w\}$.

We claim that whenever $w \neq x$ is in S, then we have $w' \neq x'$; for if $w' = x'$, then, without loss of generality, w' is out-dominated by w and not by x. Hence x' is in-dominated by x and not by w. However since $|S| \geq n+1 \geq 3$, there exists some vertex $y \in S - \{w, x\}$. Yet y cannot out-dominate w' and y cannot in-dominate x', leaving no orientation for the edge $yx' = yw'$. Thus, as claimed, $w' \neq x'$. Therefore, there is an injective mapping from S to V, implying that $|S| \leq |V| = n$ and producing a contradiction.

Now it follows that $S \cup V$ is an open twin dominating set of D. Thus $\rho_1^*(D) \leq |S \cup V| \leq 2n$. Since this is true for any valid orientation of $K_{m,n}$, it follows that $\mathrm{DOM}_1^*(K_{m,n}) \leq 2n$.

Finally, we construct an orientation D of $K_{m,n}$ such that $\rho_1^*(D) = 2n$. Label the vertices of D so that the partite sets of D are $V = \{v_1, v_2, \dots, v_n\}$ and $W = \{w_1, w_2, \dots, w_m\}$. Let D contain the arcs (w_i, v_i) for $i = 1, 2, \dots, n$ and (w_j, v_n) for $n < j \leq m$. All other edges are oriented from V to W. Then for each $i = 1, 2, \dots, n-1$, the vertex v_i is uniquely out-dominated by w_i, implying that w_i, $1 \leq i \leq n-1$, belongs to every open twin dominating set of D. Also, the vertex v_n is out-dominated by each of the vertices $w_n, w_{n+1}, \dots, w_m$. So at least one of these vertices must belong to any open twin dominating set. Similarly for each $j = 1, 2, \dots, n$, the vertex w_j is uniquely in-dominated by v_j, implying that the set V belongs to every open twin dominating set of D. Hence each open twin dominating set of D contains at least $2n$ vertices. In fact, $S = \{v_i \mid 1 \leq i \leq n\} \cup \{w_i \mid 1 \leq i \leq$

$n\}$ is an open twin dominating set containing $2n$ vertices. Thus S is a minimum open twin dominating set and $\rho_1^*(D) = 2n$. Therefore $\mathrm{DOM}_1^*(K_{m,n}) = 2n$. ❑

We now turn our attention to complete graphs. Although it is clear that $\mathrm{DOM}_1^*(K_3) = 3$ and it is quite easy to verify that $\mathrm{DOM}_1^*(K_4) = 3$, the parameter $\mathrm{DOM}_1^*(K_n)$ is not known, in general. Also, for the complete graph, the bound obtained from Corollary 2.24 is not useful. However, Theorem 2.26 shows that this number, as a function of n, is unbounded.

Theorem 2.26 $\mathrm{DOM}_1^*(K_n)$ is unbounded as $n \to \infty$.

Proof We proceed by a counting argument. Let T_n be the number of labeled tournaments of order n. Since a labeled tournament of order n can be obtained by orienting each edge of K_n, labeled accordingly, in one of two directions, it follows that $T_n = 2^{\binom{n}{2}}$.

Next, let T_n^v denote the number of valid labeled tournaments of order n. We claim that $T_n^v = T_n - 2nT_{n-1} + n(n-1)T_{n-2}$. If a labeled tournament is not valid, then there exists a vertex v such that $\mathrm{id}\, v = 0$ or $\mathrm{od}\, v = 0$. Clearly there cannot exist two vertices v and w such that $\mathrm{id}\, v = 0$ and $\mathrm{id}\, w = 0$ because the edge vw is oriented in one of the two directions. Similarly there cannot exist two vertices v and w such that $\mathrm{od}\, v = 0$ and $\mathrm{od}\, w = 0$. So a labeled tournament that is not valid has at most one vertex with indegree 0 and at most one vertex with outdegree 0. The number of labeled tournaments containing exactly one vertex with indegree 0 is nT_{n-1} since we have n choices for the label of the vertex of indegree 0 and, by removing this vertex, we obtain a labeled tournament of order $n - 1$. Similarly, the number of labeled tournaments containing exactly one vertex with outdegree 0 is nT_{n-1}. The two cases account for $2nT_{n-1}$ labeled tournaments that are not valid. However, it is possible for

a tournament to contain exactly one vertex v with indegree 0 and exactly one vertex w with outdegree 0. Such tournaments are not valid and have been counted twice in the $2nT_{n-1}$ tournaments already described. The number of such labeled tournaments is $n(n-1)T_{n-2}$ since there are n choices for the vertex v, $n-1$ choices for the vertex w, and the remaining $n-2$ vertices form a tournament of order $n-2$. Therefore

$$T_n^v = T_n - 2nT_{n-1} + n(n-1)T_{n-2}.$$

Next, let T_n^i denote the number of labeled tournaments of order n with open twin domination number equal to i, $3 \le i \le n$. Any such labeled tournament T has a minimum open twin dominating set S such that $\langle S \rangle$ is a valid orientation of K_i. Observe that there are $\binom{n}{i}$ ways to choose the vertices of S. Also since $\langle S \rangle$ is a valid tournament, the number of possible orientations for $\langle S \rangle$ is T_i^v. Let T' be the subtournament $\langle V(T) - S \rangle$. Then the number of possible orientations for T' is T_{n-i}. Finally, since S is an open twin dominating set of T, it follows that for every vertex v of T', there is at least one vertex of S adjacent to v and at least one vertex of S adjacent from v. That is, the number of edges directed from S to v is at least 1 and is at most $i-1$. Since this is true for each of the vertices of T', the number of possible orientations of edges between S and $V(T) - S$ is

$$\left(\binom{i}{1} + \binom{i}{2} + \ldots + \binom{i}{i-1}\right)^{n-i}.$$

Therefore,

$$\begin{aligned} T_n^i &\le \binom{n}{i} \cdot T_i^v \cdot T_{n-i} \cdot \left(\binom{i}{1} + \binom{i}{2} + \ldots + \binom{i}{i-1}\right)^{n-i} \\ &= \binom{n}{i} \cdot T_i^v \cdot T_{n-i} \cdot (2^i - 2)^{n-i} \end{aligned}$$

$$= \binom{n}{i} \cdot T_i^v \cdot 2^{\binom{n-i}{2}} \cdot 2^{i(n-i)} \cdot \left(\frac{2^i - 2}{2^i}\right)^{n-i}$$

$$= \binom{n}{i} \cdot \left(2^{\binom{i}{2}} - 2i \cdot 2^{\binom{i-1}{2}} + i(i-1) \cdot 2^{\binom{i-2}{2}}\right) \cdot 2^{\frac{(n-i)(n+i-1)}{2}} \cdot \left(\frac{2^i - 2}{2^i}\right)^{n-i}$$

$$= \binom{n}{i} \cdot \left(2^{\frac{n(n-1)}{2}} - i \cdot 2^{\frac{n^2-n-2i+4}{2}} + i(i-1) \cdot 2^{\frac{n^2-n-4i+6}{2}}\right) \cdot \left(\frac{2^i - 2}{2^i}\right)^{n-i}$$

$$= \binom{n}{i} \cdot \left(2^{\frac{n(n-1)}{2}} - i \cdot 2^{\frac{n(n-1)-2(i-2)}{2}} + i(i-1) \cdot 2^{\frac{n(n-1)-2(2i-3)}{2}}\right) \cdot \left(\frac{2^i - 2}{2^i}\right)^{n-i}$$

$$= \binom{n}{i} \cdot 2^{\binom{n}{2}} \cdot \left(1 - \frac{i}{2^{i-2}} + \frac{i(i-1)}{2^{2i-3}}\right) \cdot \left(\frac{2^i - 2}{2^i}\right)^{n-i}.$$

Also observe that

$$T_n^v = 2^{\binom{n}{2}} - 2n \cdot 2^{\binom{n-1}{2}} + n(n-1) \cdot 2^{\binom{n-2}{2}}$$

$$= 2^{\binom{n}{2}} \cdot \left(1 - \frac{2n}{2^{n-1}} + \frac{n(n-1)}{2^{2n-3}}\right).$$

Now for a fixed integer $k \geq 3$, let $g(n) = T_n^v - \sum_{i=3}^{k} T_n^i$, which counts the number of valid labeled tournaments of order n with open twin domination number exceeding k. Then

$$g(n) = T_n^v - \sum_{i=3}^{k} T_n^i$$

$$\geq 2^{\binom{n}{2}} \cdot \left(1 - \frac{2n}{2^{n-1}} + \frac{n(n-1)}{2^{2n-3}}\right) - \sum_{i=3}^{k} \binom{n}{i} \cdot 2^{\binom{n}{2}} \cdot \left(1 - \frac{i}{2^{i-2}} + \frac{i(i-1)}{2^{2i-3}}\right) \cdot \left(\frac{2^i - 2}{2^i}\right)^{n-i}$$

$$= 2^{\binom{n}{2}} \cdot \left[1 - \frac{n}{2^{n-2}} + \frac{n(n-1)}{2^{2n-3}} - \sum_{i=3}^{k} \binom{n}{i} \cdot \left(1 - \frac{i}{2^{i-2}} + \frac{i(i-1)}{2^{2i-3}}\right) \cdot \left(\frac{2^i - 2}{2^i}\right)^{n-i}\right].$$

Since $\lim_{n\to\infty} \sum_{i=3}^{k} \binom{n}{i} \cdot \left(1 - \frac{i}{2^{i-2}} + \frac{i(i-1)}{2^{2i-3}}\right) \cdot \left(\frac{2^i - 2}{2^i}\right)^{n-i} = 0$, it follows that $\lim_{n\to\infty} g(n) = \infty$. This implies that for each integer $k \geq 3$, there exists a sufficiently large integer n such that for some valid labeled tournament T of order n, $\rho_1^*(T) > k$, that is, $\mathrm{DOM}_1^*(K_n) > k$. Therefore $\mathrm{DOM}_1^*(K_n)$ is unbounded as $n \to \infty$. ❑

We close this chapter by finding the upper orientable open twin domination number for the n-cube and for the Petersen graph. The following lemma will be useful.

Lemma 2.27 Let G be a graph of order n such that $\delta(G) \geq 2$, and let D be a valid orientation of G. Then $\rho_1^*(D) = n$ if and only if every vertex is either (a) adjacent to a vertex of indegree 1 or (b) adjacent from a vertex of outdegree 1.

Proof For the necessity, let v be a vertex of D. Since $\rho_1^*(D) = n$, we know that $V(D) - \{v\}$ is not an open twin dominating set of D. This implies that there is some vertex w that is either not out-dominated by $V(D) - \{v\}$ or not in-dominated by $V(D) - \{v\}$. That is, v is either the unique vertex that out-dominates w or the unique vertex that in-dominates w. In the former case, v is adjacent to w and $\mathrm{id}\, w = 1$, while in the latter, v is adjacent from w and $\mathrm{od}\, w = 1$.

For the sufficiency, again let v be a vertex of D. If (a) holds, then v is adjacent to a vertex w of indegree 1. In this case, the only vertex that out-dominates w is v. Thus v must be included in every open twin dominating set. Similarly, if (b) holds, then v is adjacent from a vertex u of outdegree 1. Hence v is the only vertex that in-dominates u. Again, v must be included in every open twin dominating set. Since v is arbitrary, it follows that every vertex must belong to every open twin dominating set. Therefore $\rho_1^*(D) = n$. ❑

Theorem 2.28 For $n \geq 2$, $\mathrm{DOM}_1^*(Q_n) = 2^n$.

Proof First observe that since $Q_2 = C_4$, we know that $\mathrm{DOM}_1^*(Q_2) = \mathrm{DOM}_1^*(C_4) = 4 = 2^2$.

Next let D_3 be the orientation of Q_3 shown in Figure 18, in which exactly half the vertices have indegree 1 and half the vertices have outdegree 1. By Lemma 2.27, $\rho_1^*(D_3) = 8$, implying that $\mathrm{DOM}_1^*(Q_3) = 2^3$.

D_3:

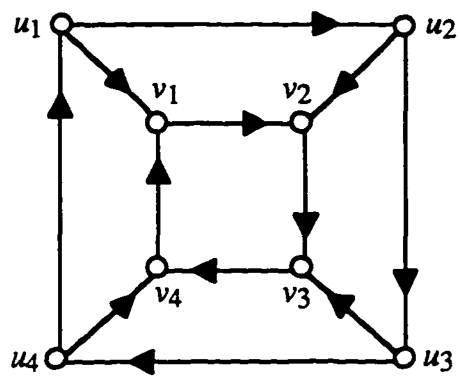

Figure 18. An Orientation D_3 of Q_3 Such That $\rho_1^*(D_3) = 8$.

For $n \geq 4$, we proceed iteratively as follows: Form an orientation D_n of Q_n, using two copies of D_{n-1}. Label the vertices of one copy of D_{n-1} the same as the original, that is, label the vertices in the first copy of D_{n-1} as $u_1, u_2, \ldots, u_{2^{n-2}}, v_1, v_2, \ldots, v_{2^{n-2}}$. Label the second copy with corresponding vertices labeled $u_{2^{n-2}+1}, u_{2^{n-2}+2}, \ldots, u_{2^{n-1}}$ and $v_{2^{n-2}+1}, v_{2^{n-2}+2}, \ldots, v_{2^{n-1}}$. Now add the arcs $(u_i, v_{2^{n-2}+i})$ and $(u_{2^{n-2}+i}, v_i)$ for $i = 1, 2, \ldots, 2^{n-2}$. The resulting digraph D_n is an orientation of Q_n satisfying the hypotheses of Lemma 2.27. Thus $\mathrm{DOM}_1^*(Q_n) = \rho_1^*(D_n) = 2^n$. ❑

For the Petersen graph P, it was shown in [7] that $\mathrm{DOM}(P) = 4$, and in [13] that $\mathrm{DOM}_1(P) = 7$. Furthermore, it was conjectured in [14] that $\mathrm{DOM}_2(P) = \mathrm{DOM}_3(P) = 6$, $\mathrm{DOM}_4(P) = 5$ and $\mathrm{DOM}_k(P)$ does not exist for $k \geq 5$. The next theorem determines $\mathrm{dom}_1^*(P)$ and $\mathrm{DOM}_1^*(P)$.

Theorem 2.29 Let P be the Petersen graph. Then $\mathrm{dom}_1^*(P) = 8$ and $\mathrm{DOM}_1^*(P) = 10$.

Proof By Corollary 2.16, we know that $\mathrm{dom}_1^*(P) \geq \left\lceil \frac{2 \cdot 10}{3} \right\rceil = 7$. We label P as indicated in Figure 19. Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ and $V = \{v_1, v_2, v_3, v_4, v_5\}$.

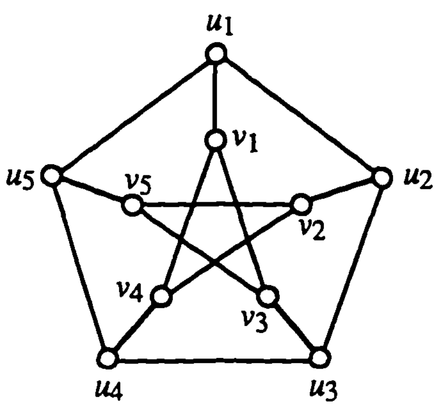

Figure 19. The Petersen Graph.

Let D be any valid orientation of P, and let S be a minimum open twin dominating set of D. Clearly, S contains at least four vertices of either U or V, say the former. We consider these two cases.

Case 1 *The set S contains all five vertices of U.* For each i, $1 \leq i \leq 5$, the vertex v_i of D must be adjacent to some vertex of S and must be adjacent from some vertex of S. In P, the vertex v_i $(1 \leq i \leq 5)$ is adjacent to u_i and the vertices in $S_i = \{v_{i+2}, v_{i+3}\}$, where the subscripts are expressed modulo 5. This implies that

at least one vertex from each of the sets $\{v_3, v_4\}$, $\{v_4, v_5\}$, $\{v_5, v_1\}$, $\{v_1, v_2\}$, $\{v_2, v_3\}$ belongs to S. Hence S contains at least three vertices of V. So $\rho_1^*(D) = |S| \geq 8$.

Case 2 The set S contains exactly four vertices of U. We may assume, without loss of generality, that $u_5 \notin S$. Since the neighbors of u_5 in P are u_1, u_4, and v_5, it follows that each of the two neighbors of these three vertices in $P - u_5$ must belong to S. That is, $\{v_1, v_2, v_3, v_4\} \subseteq S$ and so $|S| \geq 8$.

This verifies that $\text{dom}_1^*(D) \geq 8$. The orientation D_1 of P, shown in Figure 20, has open twin dominating set $S = V(D) - \{v_3, v_5\}$. Since $|S| = 8$, it follows that S is, in fact, a minimum open twin dominating set. Thus $\rho_1^*(D_1) = |S| = 8$, and so $\text{dom}_1^*(P) = 8$.

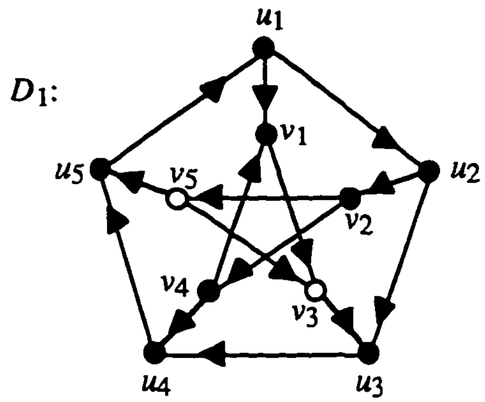

Figure 20. An Orientation D_1 of the Petersen Graph Such That $\rho_1^*(D_1) = 8$.

That the orientation D_2 of P, shown in Figure 21, has open twin domination number 10 follows from Lemma 2.27. Therefore $\text{DOM}_1^*(P) = 10$. ❑

D_2:

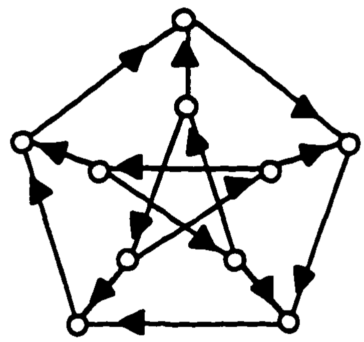

Figure 21. An Orientation D_2 of the Petersen Graph Such That $\rho_1^*(D_2) = 10$.

CHAPTER III

k-STEP TWIN DOMINATION

3.1 k-Step Twin Domination of Digraphs

In this section, we introduce a generalization of open twin domination. Let k be a positive integer. A vertex v in a digraph D is said to *k-step out-dominate* each vertex x for which $d(v, x) = k$ and *k-step in-dominate* each vertex y for which $d(y, v) = k$. A set S of vertices of D is a *k-step twin dominating set* of D if every vertex of D is both k-step in-dominated by some vertex of S and k-step out-dominated by some vertex of S.

For the digraph D of Figure 22, the set $\{v, w, x\}$ is a 2-step twin dominating set of D.

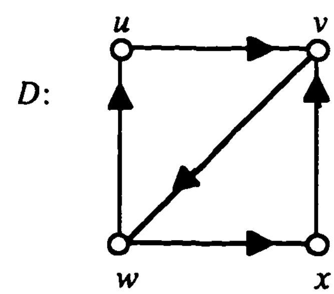

Figure 22. A Digraph D With a 2-step Twin Dominating Set of Cardinality 3.

For a digraph D, a k-step twin dominating set of minimum cardinality is a *minimum k-step twin dominating set*, and the cardinality of such a set is the *k-step twin domination number* $\rho_k^*(D)$ of D.

In the digraph D of Figure 22, observe that u is uniquely 2-step out-dominated by v, and that u is uniquely 2-step in-dominated by w. Furthermore, v is 2-step in-dominated only by u and x. Thus, v, w, and at least one of u and x must belong to every 2-step twin dominating set. Since $\{v, w, x\}$ is, in fact, a 2-step twin dominating set of D, it follows that $\rho_2^*(D) = 3$.

We now present a lower bound for the 2-step twin domination number of an asymmetric digraph.

Proposition 3.1 For every asymmetric digraph D for which $\rho_2^*(D)$ is defined, $\rho_2^*(D) \geq 2$.

Proof Let S be a minimum 2-step twin dominating set of D. Let $u \in S$. Then there exists $v \in S$ such that v 2-step in-dominates u. Since D is asymmetric, we know $u \neq v$. Thus $|S| \geq 2$. ❑

We have seen that $\rho_1^*(D) \geq 3$ for every asymmetric digraph D. For 2-step twin domination, the bound presented in Proposition 3.1 cannot be improved. For example, the digraph D of Figure 23 has minimum 2-step twin dominating set $\{u, v\}$, so $\rho_2^*(D) = 2$.

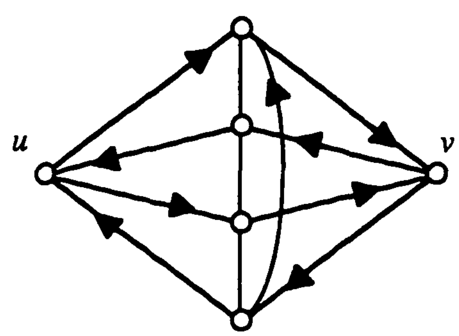

Figure 23. An Asymmetric Digraph D Such That $\rho_2^*(D) = 2$.

The next two results are similar to those stated in Chapter II.

Proposition 3.2 If D is a digraph such that $\rho_k^*(D)$ is defined, then

$$\max\{\rho_k^+(D), \rho_k^-(D)\} \leq \rho_k^*(D) \leq \rho_k^+(D) + \rho_k^-(D).$$

Proof Let S be a minimum k-step twin dominating set of D. Then S is both a k-step in-dominating set of D and a k-step out-dominating set of D. Thus $\rho_k^+(D) \leq \rho_k^*(D)$ and $\rho_k^-(D) \leq \rho_k^*(D)$.

Let S^- be a minimum k-step in-dominating set of D and S^+ be a minimum k-step out-dominating set of D and let $S = S^- \cup S^+$. Then S is a k-step twin dominating set of D. Thus $\rho_k^*(D) \leq |S| \leq |S^-| + |S^+| = \rho_k^-(D) + \rho_k^+(D)$. ❑

Corollary 3.3 If D is a digraph of order n such that $\rho_k^*(D)$ is defined, then

$$\rho_k^-(D) + \rho_k^+(D) \leq 2\rho_k^*(D) \leq 2n.$$

We next determine the k-step twin domination number of cycles.

Proposition 3.4 For integers $n \geq 3$ and k with $1 \leq k \leq n-1$, $\rho_k^*(\vec{C}_n) = n$.

Proof For each vertex v of $\vec{C}_n$, there exists a unique vertex that k-step in-dominates v and a unique vertex that k-step out-dominates v. It follows that every vertex uniquely k-step out-dominates (and uniquely k-step in-dominates) some other vertex. Thus every vertex belongs to every k-step twin dominating set, implying that $\rho_k^*(\vec{C}_n) = n$. ❑

The following corollary is now immediate.

Corollary 3.5 For every integer $n \geq 3$, there exists an integer k with $1 \leq k \leq n-1$ and a digraph D such that $\rho_k^*(D) = n$.

Clearly, for a digraph D, the parameter $\rho_k^*(D)$ is defined if and only if for every vertex v of D, there exists a vertex u such that $d(u, v) = k$ and a vertex w such that $d(v, w) = k$. It follows that if D is a digraph such that $\rho_k^*(D)$ is defined, then $\delta^-(D) \geq 1$ and $\delta^+(D) \geq 1$. However, the converse of this statement is not true, as can be seen from the digraph D of Figure 24. In particular, notice that $\delta^-(D) = 1$ and $\delta^+(D) = 1$, but the vertex z is not 3-step out-dominated by any vertex of D. Thus $\rho_3^*(D)$ is not defined.

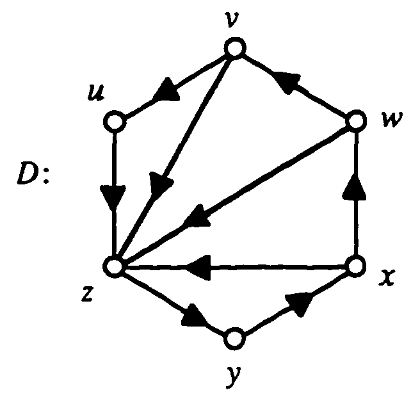

Figure 24. A Digraph D Such That $\delta^-(D) = 1$, $\delta^+(D) = 1$, and $\rho_3^*(D)$ is Not Defined.

Next, we present a necessary and sufficient condition for $\rho_k^*(D)$ to be defined. We first need a few definitions and a lemma. For a digraph D and a vertex v of D, the *in-eccentricity of* v is $e^-(v) = \max\{d(u, v) \mid u \in V(D)\}$ and the *out-eccentricity of* v is $e^+(v) = \max\{d(v, w) \mid w \in V(D)\}$. The *in-radius* $\text{rad}^-(D)$ of D is the minimum of all the in-eccentricities of D and the *out-radius* $\text{rad}^+(D)$ of D is the minimum of all the out-eccentricities of D.

Lemma 3.6 Let D be a digraph with $\delta^-(D) \geq 1$ and $\delta^+(D) \geq 1$. Let v be a vertex of D. Suppose that $e^+(v) = k$ and $e^-(v) = \ell$. Then

(a) for each integer i with $1 \le i \le k$, there exists a vertex u_i such that $d(v, u_i) = i$, and

(b) for each integer j with $1 \le j \le \ell$, there exists a vertex w_j such that $d(w_j, v) = j$.

Proof Since $e^+(v) = k$, there exists a vertex u_k such that $d(v, u_k) = k$. Thus there exists a shortest $v - u_k$ path of length k, say $v, u_1, u_2, ..., u_k$. Clearly $d(v, u_i) \le i$ for each $i = 1, 2, 3, ..., k-1$. And if $d(v, u_i) < i$ for some i, then we would have $d(v, u_k) < k$. Thus $d(v, u_i) = i$ for $i = 1, 2, ..., k$.

Similarly, there exists a vertex w_ℓ such that $d(w_\ell, v) = \ell$. And there exists a shortest $w_\ell - v$ path of length ℓ, say $w_\ell, w_{\ell-1}, ..., w_1, v$. It follows that $d(w_j, v) = j$ for each $j = 1, 2, ..., \ell$. ❑

Theorem 3.7 Let D be a digraph with $\delta^-(D) \ge 1$ and $\delta^+(D) \ge 1$. Then $\rho_k^*(D)$ exists if and only if $\text{rad}^-(D) \ge k$ and $\text{rad}^+(D) \ge k$.

Proof Let v be a vertex of D. Then $e^-(v) \ge \text{rad}^-(D) \ge k$ and $e^+(v) \ge \text{rad}^+(D) \ge k$. By Lemma 3.6, it follows that there exists a vertex u such that $d(v, u) = k$ and a vertex w such that $d(w, v) = k$. Thus v is k-step in-dominated by u and is k-step out-dominated by w. Since this is true for every vertex of D, it follows that there exists a k-step twin dominating set of D. Therefore $\rho_k^*(D)$ exists.

Next let S be a minimum k-step twin dominating set of D such that $|S| = \rho_k^*(D)$. Let v be a vertex of D. Then there exist vertices u and w in S such that $d(v, u) = k$ and $d(w, v) = k$. Hence $e^+(v) \ge d(v, u) = k$ and $e^-(v) \ge d(w, v) = k$. Since this is true for every vertex v of D, it follows that $\text{rad}^-(D) \ge k$ and $\text{rad}^+(D) \ge k$. ❑

3.2 Orientable k-Step Twin Domination of Graphs

For a graph G, the *lower orientable k-step twin domination number* of G and the *upper orientable k-step twin domination number* of G are defined, respectively, as

$$\mathrm{dom}_k^*(G) = \min\{\rho_k^*(D)\}$$

and

$$\mathrm{DOM}_k^*(G) = \max\{\rho_k^*(D)\},$$

where the minimum and maximum, respectively, is taken over all orientations D of G such that $\rho_k^*(D)$ is defined.

We first present a sufficient condition for these parameters to exist.

Theorem 3.8 If G is a 2-edge-connected graph such that $\delta(G) \geq 2$ and $\mathrm{rad}\, G \geq k$, then $\mathrm{dom}_k^*(G)$ and $\mathrm{DOM}_k^*(G)$ exist.

Proof By Theorem 2.A, there exists a strong orientation D of G. Let v be any vertex of D and let w be a vertex of G such that $d_G(v, w) = e(v)$. Since $\mathrm{rad}\, G \geq k$, it follows that $e(v) \geq k$. Since D is strong, D contains a shortest $v - w$ (directed) path, the length of which is necessarily at least $e(v)$. Thus $e^+(v) \geq e(v)$. A similar argument shows that $e^-(v) \geq e(v)$. Therefore $\mathrm{rad}^+(D) \geq \mathrm{rad}\, G \geq k$ and $\mathrm{rad}^-(D) \geq \mathrm{rad}\, G \geq k$, which implies that $\rho_k^*(D)$ is defined. Therefore $\mathrm{dom}_k^*(G)$ and $\mathrm{DOM}_k^*(G)$ exist. ❑

The edge-connectivity stated in Theorem 3.8 is not necessary for $\mathrm{dom}_k^*(G)$ and $\mathrm{DOM}_k^*(G)$ to exist. Consider, for example, the graph G of Figure 25. Notice

that G contains a bridge, yet $\mathrm{dom}_k^*(G)$ and $\mathrm{DOM}_k^*(G)$ exist for $1 \le k \le 4$, as can be seen from the orientation of G in Figure 26.

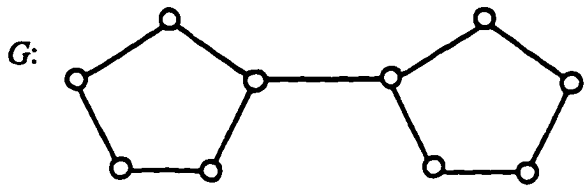

Figure 25. A Graph G That is Not 2-edge-connected.

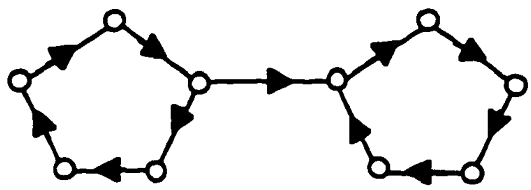

Figure 26. An Orientation of the Graph G Showing That $\mathrm{dom}_k^*(G)$ and $\mathrm{DOM}_k^*(G)$ Exist for $1 \le k \le 4$.

Furthermore, there exists a 2-edge-connected graph G with $\delta(G) \ge 2$ such that $\mathrm{dom}_k^*(G)$ and $\mathrm{DOM}_k^*(G)$ exist yet $\mathrm{rad}\, G < k$, namely $G = C_{k+1}$ $(k \ge 2)$ has these properties.

For a graph G, there exists a relationship between $\mathrm{dom}_k(G)$ and $\mathrm{dom}_k^*(G)$, as is shown next.

Theorem 3.9 If G is a graph such that $\mathrm{dom}_k^*(G)$ is defined, then $\mathrm{dom}_k(G) \le \mathrm{dom}_k^*(G)$.

Proof Let D be an orientation of G such that $\rho_k^*(D)$ exists. Then for every vertex v of D, there exists a vertex u that k-step out-dominates v. Thus $\rho_k(D)$ exists as well. Furthermore, by Proposition 3.2, $\rho_k(D) \le \rho_k^*(D)$. And since $\mathrm{dom}_k(G) = \min\{\rho_k(D) \mid D$ is an orientation of G such that $\rho_k(D)$ exists$\}$ and

$\mathrm{dom}_k^*(G) = \min\{\rho_k^*(D) \mid D$ is an orientation of G such that $\rho_k^*(D)$ exists}, it follows that $\mathrm{dom}_k(G) \le \mathrm{dom}_k^*(G)$. ❑

We next calculate these parameters for cycles.

Proposition 3.10 For integers $n \ge 3$ and k with $1 \le k \le n-1$, $\mathrm{dom}_k^*(C_n) = \mathrm{DOM}_k^*(C_n) = n$.

Proof Let D be an orientation of C_n such that $\rho_k^*(D)$ is defined. Then, necessarily, $\delta^-(D) \ge 1$ and $\delta^+(D) \ge 1$. Consequently, D is a directed n–cycle. By Proposition 3.4, it follows that $\rho_k^*(D) = n$. Thus $\mathrm{dom}_k^*(C_n) = \mathrm{DOM}_k^*(C_n) = n$. ❑

We now determine the lower orientable 2-step twin domination number of complete bipartite graphs.

Theorem 3.11 For integers m and n with $m \ge n \ge 2$, $\mathrm{dom}_2^*(K_{m,n}) = 4$.

Proof Let D be an orientation of $K_{m,n}$ for which $\rho_2^*(D)$ is defined. Let U and V be the partite sets of D, where $|U| = m$ and $|V| = n$. Now let S be a minimum 2-step twin dominating set of D. Observe that the distance of each vertex of U both to and from each vertex of V is odd. Thus no vertex of U can 2-step out-dominate a vertex of V, and vice versa. This implies that $|S \cap U| \ge 1$ and $|S \cap V| \ge 1$. Also, no vertex of U (and similarly V) can 2-step out-dominate or 2-step in-dominate itself. Thus $|S \cap U| \ge 2$ and $|S \cap V| \ge 2$. Therefore $\rho_2^*(D) \ge 4$, implying that $\mathrm{dom}_2^*(K_{m,n}) \ge 4$.

Next, we describe an orientation D of $K_{m,n}$ such that $\rho_2^*(D) = 4$. Let $V(D) = U \cup V$, where $U = \{u_1, u_2, ..., u_m\}$ and $V = \{v_1, v_2, ..., v_n\}$, and let $E(D)$ contain the set

$$\{(u_1, v_1), (v_2, u_1)\} \cup \{(v_1, u_i), (u_i, v_2) \mid 2 \le i \le m\} \cup$$

$$\{(u_1, v_j), (v_j, u_2) \mid 3 \le j \le n\}.$$

Then $S = \{u_1, u_2, v_1, v_2\}$ is a minimum 2-step twin dominating set. This construction is shown for $K_{3,3}$ in Figure 27. Since $|S| = 4$, it follows that $\text{dom}_2^*(K_{m,n}) = 4$. ❑

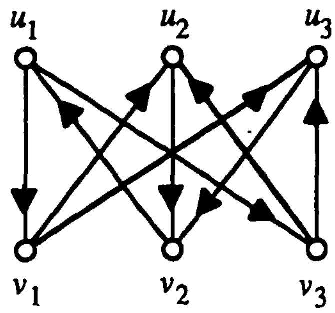

Figure 27. An Example of the Construction in the Proof of Theorem 3.11.

The lower orientable 3-step twin domination number of complete bipartite graphs is also determined.

Theorem 3.12 For integers m and n with $m \ge n \ge 2$, $\text{dom}_3^*(K_{m,n}) = 4$.

Proof Let D be an orientation of $K_{m,n}$ for which $\rho_3^*(D)$ is defined. Let U and V be the partite sets of D, where $|U| = m$ and $|V| = n$. Let S be a minimum 3-step twin dominating set of D. Let $v_1 \in V$ and let $u_1 \in S$ such that u_1 3-step out-dominates v_1. Then $d(u_1, v_1) = 3$ and there exists a directed path u_1, v_2, u_2, v_1 of length 3. Further, since $d(u_1, v_1) = 3$, it follows that (v_1, u_1) is an arc of D. Now there also exists a vertex that 3-step out-dominates v_2 and, clearly, this vertex cannot be u_1. Also since the distance from any vertex of V to the vertex v_2 is even, it follows that the vertex that 3-step out-dominates v_2 must belong to U. This implies that $|S \cap U| \ge 2$. By a similar argument, $|S \cap V| \ge 2$. Therefore $\rho_3^*(D) \ge 4$, implying that $\text{dom}_3^*(K_{m,n}) \ge 4$.

Next, we construct an orientation D of $K_{m,n}$ such that $\rho_3^*(D) = 4$. Let $V(D) = U \cup V$, where $U = \{u_1, u_2, ..., u_m\}$ and $V = \{v_1, v_2, ..., v_n\}$, and let $E(D)$ contain the set

$$\{(v_1, u_2)\} \cup \{(v_2, u_i), (u_i, v_1) \mid i = 1; 3 \le i \le m\} \cup$$
$$\{(u_2, v_j), (v_j, u_1) \mid 2 \le j \le n\}.$$

Then $S = \{u_1, u_2, v_1, v_2\}$ is a minimum 3-step twin dominating set. Since $|S| = 4$, it follows that $\text{dom}_3^*(K_{m,n}) = 4$. ❑

We next investigate k-step twin domination of orientations of complete bipartite graphs where k is odd, $k \ge 5$.

Theorem 3.13 For integers m, n and r, with $m \ge n \ge 2$, and $r \ge 2$, there exists no orientation D of $K_{m,n}$ such that $\rho_{2r+1}^*(D)$ is defined.

Proof Suppose, to the contrary, that D is an orientation of $K_{m,n}$ such that $\rho_{2r+1}^*(D)$ exists. Let S be a minimum $(2r + 1)$-step twin dominating set of D. Then there exist vertices u and v of D such that $u \in S$ and u $(2r + 1)$-step out-dominates v. So there exists a directed path $u = u_1, v_1, u_2, v_2, ..., u_{r+1}, v_{r+1} = v$ of length $2r + 1$ in D. Also, $d(u_1, v_i) = 2i - 1$ for $i = 1, 2, ..., r + 1$, which implies that $(u_1, v_i) \notin E(D)$ for each $i = 2, 3, ..., r + 1$. Thus (v_i, u_1) is an arc of D for each $i = 2, 3, ..., r + 1$. Also $(v_i, u_j) \in E(D)$ whenever $i > j$ (for otherwise $d(u_1, v_i) < 2i - 1$). Furthermore, $(u_j, v_i) \in E(D)$ whenever $i < j, i \ne j - 1$. Now, by construction, we can calculate the distance between each pair of vertices. In particular,

$$d(v_i, v_j) = \begin{cases} 2(j - i) & \text{if } i \le j \\ 2 & \text{if } i > j \end{cases}$$

$$d(u_i, u_j) = \begin{cases} 2(j - i) & \text{if } i \le j \\ 2 & \text{if } i > j \end{cases}$$

$$d(u_i, v_j) = \begin{cases} 2(j - i) + 1 & \text{if } i \le j \\ 3 & \text{if } i = j + 1 \\ 1 & \text{if } i > j \text{ and } i \ne j + 1 \end{cases}$$

$$d(v_i, u_j) = \begin{cases} 2(j - i) - 1 & \text{if } i < j \\ 3 & \text{if } i = j \\ 1 & \text{if } i > j \end{cases}$$

Now notice that $d(u_i, v_r) < 2r + 1$ for each $i = 1, 2, ..., r + 1$. Thus there must exist another vertex u_{r+2} that $(2r + 1)$-step out-dominates v_r. If u_{r+2} is adjacent to v_i for some $i = 1, 2, ..., r + 1$, then

$$\begin{aligned} d(u_{r+2}, v_r) &\le d(u_{r+2}, v_i) + d(v_i, v_r) \\ &= 1 + d(v_i, v_r) \\ &\le \max\{3, 1 + 2(r - i)\} \\ &< 2r + 1, \end{aligned}$$

that is, the vertex u_{r+2} would not $(2r + 1)$-step out-dominate v_r. So $(v_i, u_{r+2}) \in E(D)$ for each $i = 1, 2, ..., r + 1$. Therefore, there must exist a vertex v_{r+2} that is adjacent from u_{r+2}.

Now the vertex v_{r+2} must be $(2r + 1)$-step out-dominated by some vertex. However, $d(u_i, v_{r+2}) = 3$ for each $i = 1, 2, ..., r + 1$. Thus there must exist another

vertex u_{r+3} that $(2r + 1)$-step out-dominates v_{r+2}. Again, if u_{r+3} were adjacent to v_i for some $i = 1, 2, ..., r + 2$, then $d(u_{r+3}, v_i) \leq 3 < 2r + 1$. Thus $(v_i, u_{r+3}) \in E(D)$ for each $i = 1, 2, ..., r + 2$. Again there must exist a vertex v_{r+3} that is adjacent from u_{r+3}. This iterative process produces an infinite graph. Therefore no such orientation D exists. ❑

As a consequence, we have the following.

Corollary 3.14 For integers m, n and k, with $m \geq n \geq 2$ and k odd, $k \geq 5$, $\mathrm{dom}_k^*(K_{m,n})$ and $\mathrm{DOM}_k^*(K_{m,n})$ do not exist.

It is our conjecture that $\mathrm{dom}_k^*(K_{m,n})$ and $\mathrm{DOM}_k^*(K_{m,n})$ exist when k is even, $k \geq 4$. Furthermore, we conjecture that $\mathrm{dom}_k^*(K_{m,n}) = 2k$ for k even, $k \geq 4$.

3.3 k-Step Twin Domination of Tournaments

We now investigate the k-step twin domination numbers of tournaments. We begin with the following observation.

Proposition 3.15 If T is a tournament for which $\rho_2^*(T)$ is defined, then no strong component of T is trivial.

Proof Let $S_1, S_2, ..., S_m$ be the strong components of T where the vertices of S_i are adjacent to the vertices of S_j for $i < j$. Suppose, to the contrary, that S_k contains only one vertex for some k. If $k = 1$, then no vertex 2-step out-dominates the vertex of S_k. If $k > 1$, then, for $i < k$, the vertices of S_i are at distance 1 to the vertex of S_k and again no vertex 2-step out-dominates the vertex of S_k. ❑

We next present a lower bound for the 2-step twin domination number of a tournament.

Theorem 3.16 If T is a tournament for which $\rho_2^*(T)$ is defined, then $\rho_2^*(T) \geq 3$.

Proof Let S be a minimum 2-step twin dominating set of T and let $u \in S$. Then there exists a vertex w of T such that u 2-step out-dominates w. Thus there is a directed path u, v, w of length 2 in T. Since $d(u, w) = 2$, it follows that w is adjacent to u. Now the vertex w must also be 2-step in-dominated by some vertex x of S. Clearly $x \neq u$ (but possibly $x = v$) so that $|S| \geq 2$. Since T is a tournament, either the arc (u, x) or the arc (x, u) belongs to T. If u is adjacent to x, then u is 2-step in-dominated by some vertex y of S and y could be neither x nor u. Similarly, if u is adjacent from x, then u is 2-step out-dominated by some vertex of S other than x. In either case, it follows that $|S| \geq 3$. ❑

Next we present a necessary and sufficient condition for $\rho_2^*(T) = 3$.

Theorem 3.17 For a tournament T, $\rho_2^*(T) = 3$ if and only if T contains a 3-cycle C such that every vertex $x \in V(T) - V(C)$ belongs to a 3-cycle that shares an arc with C.

Proof Let $S = \{u, v, w\}$ be a minimum 2-step twin dominating set of T. Then the vertices of S form a 3-cycle, say $C : u, v, w, u$. Let x be a vertex of T not belonging to C. Then x is 2-step in-dominated by one of the vertices of S, say u. Thus $d(x, u) = 2$ and hence $(u, x) \in E(T)$. Also x is 2-step out-dominated by one of the vertices of S and this vertex cannot be u. So either $d(w, x) = 2$ or $d(v, x) = 2$. This implies that either $(x, w) \in E(T)$ or $(x, v) \in E(T)$. If $(x, w) \in E(T)$, then the 3-cycle u, x, w, u has the desired property. If $(x, w) \notin E(T)$, then, necessarily, the arcs (x, v) and (w, x) belong to T. In this case, the 3-cycle v, w, x, v has the desired property.

For the converse, it is straightforward to verify that $V(C)$ forms a minimum 2-step twin dominating set and $\rho_2^*(T) = 3$. ❑

Using the previous results, we can determine the lower orientable 2-step twin domination number of complete graphs.

Theorem 3.18 For $n \geq 3$, $\mathrm{dom}_2^*(K_n) = 3$.

Proof It follows from Theorem 3.16 that $\mathrm{dom}_2^*(K_n) \geq 3$. We construct a tournament T attaining this lower bound in the following way. Let u, v, and w be three vertices of K_n and let C: u, v, w, u be a 3-cycle of T. Now for each vertex x of $V(K_n) - \{u, v, w\}$, let u, v, x, u be a 3-cycle of T. Any edge not mentioned thus far can be oriented arbitrarily. Then T contains a 3-cycle, namely C, such that every vertex $x \in V(T) - V(C)$ belongs to a 3-cycle that shares an arc with C. So, by Theorem 3.17, it follows that $\rho_2^*(T) = 3$. Therefore $\mathrm{dom}_2^*(K_n) = 3$. ❑

The following theorem will prove to be useful in computing the 2-step twin domination number of a tournament.

Theorem 3.19 Let T be a tournament such that $\rho_2^*(T)$ exists. If $S_1, S_2, \ldots, S_k$ are the strong components of T, then

$$\rho_2^*(T) = \sum_{i=1}^{k} \rho_2^*(S_i).$$

Proof Let U_i be a minimum 2-step twin dominating set of S_i for $i = 1, 2, \ldots, k$. Then, clearly, $U = \bigcup_{i=1}^{k} U_i$ is a 2-step twin dominating set of T. So

$$\rho_2^*(T) \leq |U| = \left|\bigcup_{i=1}^{k} U_i\right| = \sum_{i=1}^{k} |U_i| = \sum_{i=1}^{k} \rho_2^*(S_i). \qquad (1)$$

Next, let S be a minimum 2-step twin dominating set of T, and let $W_i = S \cap V(S_i)$ for $i = 1, 2, ..., k$. Clearly $\rho_2^*(T) = |S| = \sum_{i=1}^{k} |W_i|$. Suppose, to the contrary, that for some i, the set W_i is not a 2-step twin dominating set of S_i. So there exists some vertex $v \in V(S_i)$ such that either v is not 2-step in-dominated by a vertex of W_i or v is not 2-step out-dominated by a vertex of W_i. We consider two cases.

Case 1 *v is not 2-step in-dominated by a vertex of W_i.* Since v is 2-step in-dominated by some vertex of S, there exists a vertex w in W_j $(i \neq j)$ that 2-step in-dominates v. Thus there is a path v, x, w in T_n. Furthermore, $d(v, w) = 2$, implying that $(v, w) \notin E(T)$. So $(w, v) \in E(T)$. But this implies that w belongs to the strong component S_i.

Case 2 *v is not 2-step out-dominated by a vertex of W_i.* Since v is 2-step out-dominated by some vertex of S, there exists a vertex w in W_j $(i \neq j)$ that 2-step out-dominates v. Thus there is a path w, x, v in T. Further $d(w, v) = 2$, implying that $(w, v) \notin E(T)$. So $(v, w) \in E(T)$. But this implies that w belongs to the strong component S_i.

Therefore W_i is a 2-step twin dominating set of S_i, and so $\rho_2^*(S_i) \leq |W_i|$. So

$$\sum_{i=1}^{k} \rho_2^*(S_i) \leq \sum_{i=1}^{k} |W_i| = |S| = \rho_2^*(T). \tag{2}$$

Combining (1) and (2), we obtain the desired result. ❑

If T is a tournament of order $n \geq 3$ for which $\rho_2^*(T)$ is defined, then, of course, $\rho_2^*(T) \leq n$. In Proposition 3.20, a necessary and sufficient condition for equality is given.

Proposition 3.20 Let T be a tournament of order $n \geq 3$. Then $\rho_2^*(T) = n$ if and only if for each vertex v of T, either

(a) there exists a vertex u such that v is the unique vertex for which $d(u, v) = 2$, or

(b) there exists a vertex w such that v is the unique vertex for which $d(v, w) = 2$.

Proof If $\rho_2^*(T) = n$, then for each vertex v of T, the set $V(T) - \{v\}$ is not a 2-step twin dominating set. Thus there is either a vertex of T which is uniquely 2-step out-dominated by v or there is a vertex of T which is uniquely 2-step in-dominated by v. That is, either there exists a vertex u such that v is the unique vertex for which $d(u, v) = 2$, or there exists a vertex w such that v is the unique vertex for which $d(v, w) = 2$.

For the converse, if every vertex v satisfies either condition (a) or (b), then, necessarily, the vertex v must belong to every 2-step twin dominating set. Thus $\rho_2^*(T) = n$. ❑

We now turn our attention to strong tournaments, beginning with those of small order. We first recall the following theorem, due to Moon [20].

Theorem 3.A If T is a strong tournament of order n, then every vertex of T lies on an ℓ-cycle for each ℓ with $3 \leq \ell \leq n$.

Lemma 3.21 If T_4 is the strong tournament of order 4, then $\rho_2^*(T_4) = 3$.

Proof The strong tournament of order 4 is shown in Figure 28. The three shaded vertices form a minimum 2-step twin dominating set. So $\rho_2^*(T_4) = 3$. ❑

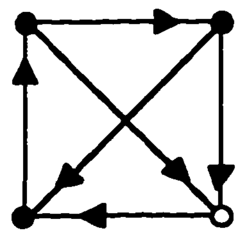

Figure 28. The Strong Tournament of Order 4 Has 2-step Twin Domination Number 3.

Theorem 3.22 If T_5 is a strong tournament of order 5, then $\rho_2^*(T_5) = 3$.

Proof Let T_5 be a strong tournament of order 5 such that $\rho_2^*(T_5)$ exists. Then either $\Delta^+(T_5) = 2$ or $\Delta^+(T_5) = 3$. We consider these two cases.

Case 1 $\Delta^+(T_5) = 2$. In this case, T_5 is the unique 2-regular tournament of order 5, shown in Figure 29. For this tournament, the set $\{u, v, w\}$ is a minimum 2-step twin dominating set. Thus $\rho_2^*(T_5) = 3$.

Case 2 $\Delta^+(T_5) = 3$. Let v be a vertex such that od $v = 3$. Also, by Theorem 3.A, the vertex v lies on a 3-cycle, say u, v, w, u. Let x and y be the other vertices of T_5. So v is adjacent to both x and y. (See Figure 30.)

Subcase 2.1 (x, u) and (y, u) belong to T_5. Then, by Theorem 3.17, the set $\{u, v, w\}$ forms a minimum 2-step twin dominating set of T_5 and $\rho_2^*(T_5) = 3$.

Subcase 2.2 (u, x) and (u, y) belong to T_5. Without loss of generality, assume that (x, y) belongs to T_5. Then, necessarily, the vertex y is adjacent to w. Now regardless of the orientation of the edge wx, the set $\{u, w, y\}$ is a minimum 2-step twin dominating set and $\rho_2^*(T_5) = 3$.

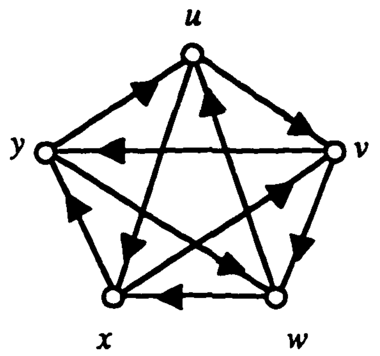

Figure 29. The 2-regular Tournament of Order 5 Has 2-step Twin Domination Number 3.

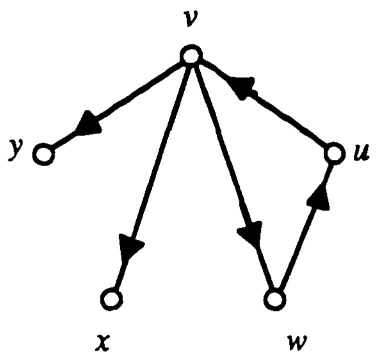

Figure 30. A Step of the Construction in Case 2 of the Proof of Theorem 3.22.

Subcase 2.3 (u, x) and (y, u) belong to T_5. If (x, y) belongs to T_5, then the set $\{u, v, y\}$ is a minimum 2-step twin dominating set of T_5. Thus we assume that y is adjacent to x. Necessarily, it follows that x is adjacent to w. (See Figure 31.)

Now if w is adjacent to y, then the set $\{w, u, x\}$ is a minimum 2-step twin dominating set; while if w is adjacent from y, then the set $\{u, v, w\}$ is a minimum 2-step twin dominating set. Since we have exhausted all possibilities, it follows that $\rho_2^*(T_5) = 3$. ❑

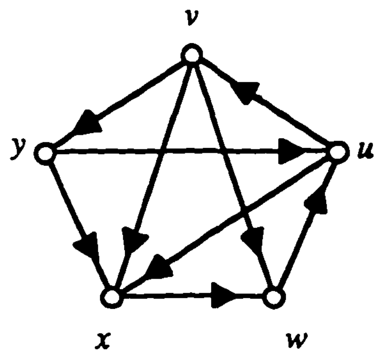

Figure 31. A Step in the Proof of Subcase 2.3 of Theorem 3.22.

Lemma 3.23 If T is a strong tournament of order 6, then $\rho_2^*(T) \leq 4$.

Proof It follows from Theorem 3.A that T contains a 5-cycle $C: u, v, w, x, y, u$. Let z be the vertex of T not belonging to C and let $T' = T - z$. Then, since C belongs to T', it follows that T' is a strong tournament of order 5. By Theorem 3.22, $\rho_2^*(T') = 3$. Without loss of generality, either (1) $\{u, v, w\}$ is a minimum 2-step twin dominating set of T' or (2) $\{u, v, x\}$ is a minimum 2-step twin dominating set of T'. We consider these two cases.

Case 1 *$\{u, v, w\}$ is a minimum 2-step twin dominating set of T'.* Then C': u, v, w, u is a 3-cycle of T'. Also, by Theorem 3.17, the vertex x belongs to a 3-cycle that shares an arc with C' and y belongs to a 3-cycle that shares an arc with C'.

Subcase 1.1 *z is adjacent to each of u, v, and w.* Then either z is adjacent from x, or z is adjacent from y. If z is adjacent from x, then $\{u, v, w, x\}$ is a 2-step twin dominating set of T. If z is adjacent from y, then $\{u, v, w, y\}$ is a 2-step twin dominating set of T. Thus $\rho_2^*(T) \leq 4$.

Subcase 1.2 *z is adjacent from each of u, v, and w.* Then either z is adjacent to x, or z is adjacent to y. In the former case, $\{u, v, w, x\}$ is a 2-step twin dominating set of T, while in the latter case, $\{u, v, w, y\}$ is a 2-step twin dominating set of T. Thus $\rho_2^*(T) \le 4$.

Subcase 1.3 *z is adjacent to exactly two of u, v, and w.* Then, in T, the vertex z belongs to a 3-cycle that shares an arc with C'. Thus $\{u, v, w\}$ is a minimum 2-step twin dominating set of T and $\rho_2^*(T) = 3$.

Subcase 1.4 *z is adjacent to exactly one of u, v, and w.* Again, in T, the vertex z belongs to a 3-cycle that shares an arc with C'. So, as in Subcase 1.3, $\rho_2^*(T) = 3$.

Case 2 *$\{u, v, x\}$ is a minimum 2-step twin dominating set of T'.* Then C': u, v, x, u is a 3-cycle of T'. Also, by Theorem 3.17, the vertex w belongs to a 3-cycle that shares an arc with C' and y belongs to a 3-cycle that shares an arc with C'.

Subcase 2.1 *z is adjacent to each of u, v, and x.* Then either z is adjacent from w, or z is adjacent from y. If z is adjacent from w, then $\{u, v, w, x\}$ is a 2-step twin dominating set of T. If z is adjacent from y, then $\{u, v, x, y\}$ is a 2-step twin dominating set of T. Thus $\rho_2^*(T) \le 4$.

Subcase 2.2 *z is adjacent from each of u, v, and x.* Then either z is adjacent to w, or z is adjacent to y. In the former case, $\{u, v, w, x\}$ is a 2-step twin dominating set of T, while in the latter case, $\{u, v, x, y\}$ is a 2-step twin dominating set of T. Thus $\rho_2^*(T) \le 4$.

Subcase 2.3 *z is adjacent to exactly two of u, v, and x.* Then, in T, the vertex z belongs to a 3-cycle that shares an arc with C'. Thus $\{u, v, x\}$ is a minimum 2-step twin dominating set of T and $\rho_2^*(T) = 3$.

Subcase 2.4 *z is adjacent to exactly one of u, v, and x.* Again, in T, the vertex z belongs to a 3-cycle that shares an arc with C'. So, as in Subcase 2.3, $\rho_2^*(T) = 3$. ❑

We are now prepared to present an upper bound for $\rho_2^*(T)$, where T is a strong tournament of order $n \geq 5$.

Theorem 3.24 If T is a strong tournament of order $n \geq 5$, then $\rho_2^*(T) \leq n - 2$.

Proof We proceed by induction on n. We have seen that the result is true for $n = 5$ and $n = 6$. Assume that every strong tournament of order $n - 1$, for some integer $n \geq 7$, has 2-step twin domination number at most $n - 3$.

Let T be a strong tournament of order n. Since T is strong, we know from Theorem 3.A that T contains an $(n - 1)$-cycle C. Let v be the vertex of T not belonging to C. Consider the tournament $T' = T - v$ of order $n - 1$. Since T' contains an $(n - 1)$-cycle, it follows that T' is hamiltonian and, thus, is strong. So, by the inductive hypothesis, $\rho_2^*(T') \leq n - 3$. Let S be a minimum 2-step twin dominating set of T'. Since T is strong, we know that v lies on a 3-cycle v, u, w, v. Now if $|S| \leq n - 4$, then $S \cup \{u, w\}$ is a 2-step twin dominating set of cardinality at most $n - 2$. That is, in this case, $\rho_2^*(T) \leq n - 2$. So we assume that $|S| = n - 3$.

Let $S^+ = N^+(v) \cap S$ and $S^- = N^-(v) \cap S$. We claim there is some vertex of S^+ that 2-step out-dominates v or there is some vertex of S^- that 2-step in-dominates v. Suppose, to the contrary, that this is not the case. Then, it follows that u is the unique vertex of T that 2-step out-dominates v, and w is the unique vertex of T that 2-step in-dominates v. Thus $S = V(T') - \{u, v, w\}$. Consequently, (a) every vertex of $S^- \cup \{w\}$ is adjacent to every vertex of S^+ and (b) every vertex

of S^- is adjacent to u. Since each vertex of S can only be 2-step in-dominated in T' by w, it follows that (c) w is adjacent to every vertex of S^-. Furthermore, since every vertex of S^+ can only be 2-step out-dominated in T' by u, it follows that (d) every vertex of S^+ is adjacent to u. See Figure 32 for an illustration of the tournament T.

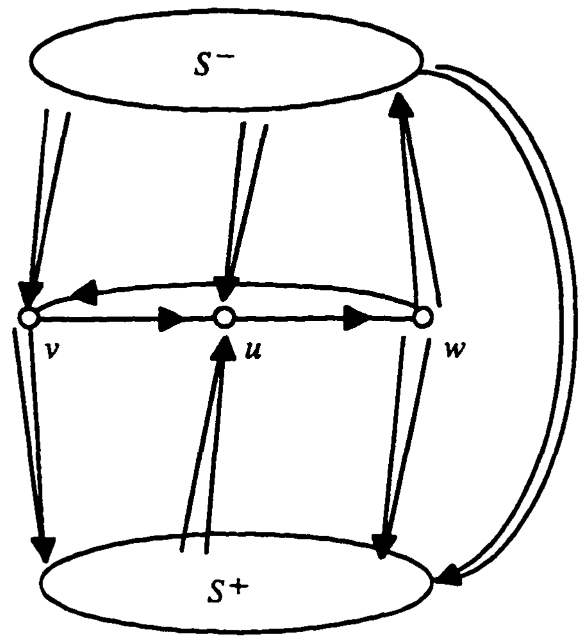

Figure 32. The Tournament Constructed in the Proof of Theorem 3.24.

Now since $|S| = n - 3 \geq 4$ and $S = S^- \cup S^+$, it follows that either $S^- \neq \varnothing$ or $S^+ \neq \varnothing$. We consider these two cases.

Case 1 $S^- \neq \varnothing$. Let $x \in S^-$. Then $\{u, w, x\}$ is a minimum 2-step twin dominating set of T', contradicting $\rho_2^*(T') \geq 4$.

Case 2 $S^+ \neq \varnothing$. Let $y \in S^+$. Then $\{u, w, y\}$ is a minimum 2-step twin dominating set of T', again contradicting $\rho_2^*(T') \geq 4$.

Therefore, there is (a) some vertex of S^+ that 2-step out-dominates v or (b) some vertex of S^- that 2-step in-dominates v. In the first case, the set $S \cup \{w\}$ is a 2-step twin dominating set of T of cardinality at most $n-2$. In the second case, the set $S \cup \{u\}$ is a 2-step twin dominating set of T of cardinality at most $n-2$. That is, $\rho_2^*(T) \leq n-2$. ❑

The next result shows that if T is a tournament of order $n \geq 3$ with $\rho_2^*(T) = n$, then the tournament is unique.

Theorem 3.25 If T is a tournament of order $n \geq 3$ for which $\rho_2^*(T) = n$, then every strong component of T has order 3.

Proof Let $S_1, S_2, ..., S_k$ be the strong components of T. Suppose, to the contrary, that $|V(S_j)| > 3$ for some j, $1 \leq j \leq k$. Then it follows from Lemma 3.21 and Theorem 3.24, that $\rho_2^*(S_j) < |V(S_j)|$. However, by Theorem 3.19,

$$\rho_2^*(T) = \sum_{i=1}^{k} \rho_2^*(S_i) < \sum_{i=1}^{k} |V(S)| = n,$$

which produces a contradiction. ❑

We can now present a formula for the upper orientable 2-step twin domination numbers of complete graphs.

Theorem 3.26 For every integer $n \geq 3$, $\mathrm{DOM}_2^*(K_n) = 3\lfloor n/3 \rfloor$.

Proof We consider three cases.

Case 1 $n \equiv 0 \ (mod\ 3)$. Then $n = 3k$, for some integer $k \geq 1$. Let T_n be the tournament of order n such that every strong component of T_n is a 3-cycle. Let S_1, $S_2, \ldots, S_k$ be the strong components of T_n. See Figure 33.

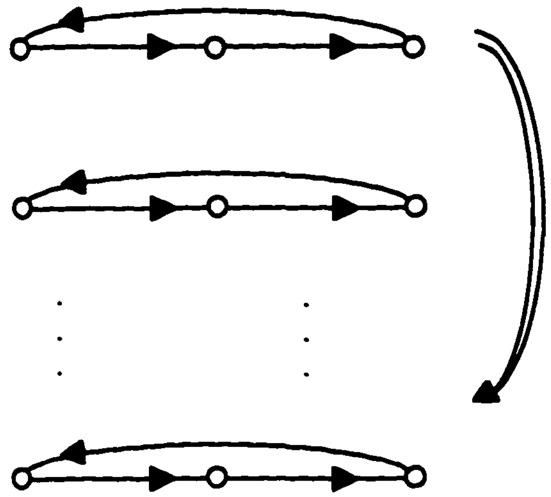

Figure 33. The Tournament T_n of Order $n \equiv 0 \pmod 3$ Such That $\rho_2^*(T_n) = n$.

By Theorem 3.19,

$$\rho_2^*(T_n) = \sum_{i=1}^{k} \rho_2^*(S_i) = 3k = n.$$

Thus $\mathrm{DOM}_2^*(K_n) = n = 3\lfloor n/3 \rfloor$.

Case 2 $n \equiv 1$ (*mod* 3). Let T_n be a tournament such that $\mathrm{DOM}_2^*(K_n) = \rho_2^*(T_n)$. Since $n \equiv 1$ (*mod* 3), there exists a strong component S of T_n such that S has at least four vertices. By Lemma 3.21 and Theorem 3.24, it follows that $\rho_2^*(S) < |V(S)|$. Thus $\rho_2^*(T_n) < n$, that is, $\mathrm{DOM}_2^*(K_n) \leq n - 1$. Next let T be the tournament of order $n = 3k + 1$ formed from one strong component of order 4 and $k - 1$ strong components of order 3. (See Figure 34.) Then $\rho_2^*(T) = 3k = n - 1$, implying that $\mathrm{DOM}_2^*(K_n) = n - 1 = 3\lfloor n/3 \rfloor$.

Case 3 $n \equiv 2$ (*mod* 3). Let T_n be a tournament such that $\mathrm{DOM}_2^*(K_n) = \rho_2^*(T_n)$. Since $n \equiv 2$ (*mod* 3), either there exists a strong component S of T_n such that S

has at least five vertices or there exist two strong components, each containing four vertices. In either case, it follows from Lemma 3.21 and Theorem 3.24, that $\rho_2^*(T_n) \le n-2$. The tournament T of order n formed from one strong component of order 5 and the remaining strong components of order 3 satisfies $\rho_2^*(T_n) = n-2$. Thus $\mathrm{DOM}_2^*(K_n) = n - 2 = 3\lfloor n/3 \rfloor$. ❑

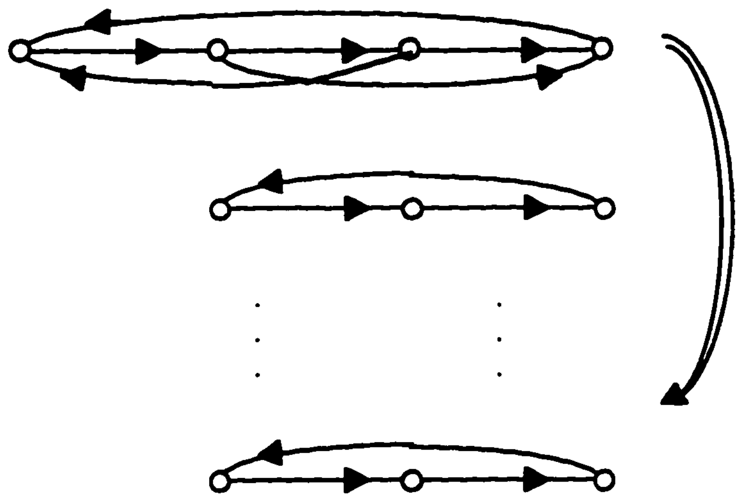

Figure 34. A Tournament T_n of Order $n \equiv 1 \pmod 3$ Such That $\rho_2^*(T_n) = n-1$.

Next we consider the orientable k-step twin domination numbers of complete graphs where $k \ge 3$. We first recall the following result regarding tournaments, originally discovered by Landau [18].

Theorem 3.B Let T be a tournament and v a vertex of maximum outdegree in T. Then $d(v, w) \le 2$ for every vertex w of T.

Theorem 3.27 For integers k and n where $3 \le k \le$ n, $\mathrm{dom}_k^*(K_n)$ and $\mathrm{DOM}_k^*(K_n)$ are not defined.

Proof Let T be a tournament of order n and let v be a vertex of maximum outdegree in T. Then, by Theorem 3.B, for every vertex w of T, we have $d(v, w) \leq 2$, implying that v is not k-step in-dominated by any vertex w of T for $k > 2$. Hence there does not exist a k-step twin dominating set of T for $k > 2$. Therefore $\mathrm{dom}_k^*(K_n)$ and $\mathrm{DOM}_k^*(K_n)$ are not defined for $3 \leq k \leq n$. ❑

CHAPTER IV

OTHER RESULTS AND CONCEPTS INVOLVING TWIN DOMINATION

4.1 (r, s)-Step Domination in Digraphs

A set S of vertices of a digraph D is an *(r, s)-step dominating set of* D if every vertex of D is r-step in-dominated by some vertex of S and is s-step out-dominated by some vertex of S. An (r, s)-step dominating set of minimum cardinality is a *minimum (r, s)-step dominating set* and the cardinality of such a set is the (r, s)-*step domination number* $\rho^*_{r,s}(D)$ of D. Notice that if D' is the converse of a digraph D, then $\rho^*_{r,s}(D) = \rho^*_{s,r}(D')$. Of course, (k, k)-step domination is equivalent to k-step twin domination.

For example, the digraph D of Figure 35 has $(2, 3)$-step domination number $\rho^*_{2,3}(D) = 4$. In fact, the sets $\{u, v, x, w\}$ and $\{u, v, x, y\}$ are the only minimum $(2, 3)$-step dominating sets of D.

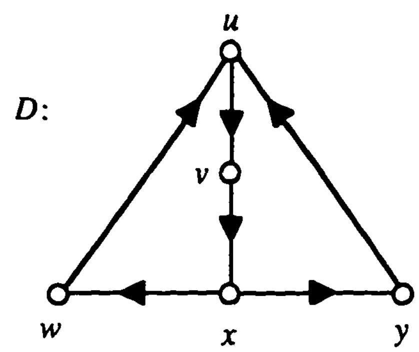

Figure 35. A Digraph D Such That $\rho^*_{2,3}(D) = 4$.

We begin with an observation.

Proposition 4.1 For integers $n \geq 3$, r, and s with $1 \leq r, s \leq n-1$, $\rho^*_{r,s}(\vec{C}_n) = n$.

Proof Let v be any vertex of $\vec{C}_n$. Since $1 \leq r \leq n-1$, there exists a vertex u such that $d(u, v) = r$. Similarly, there is a vertex w such that $d(v, w) = s$. In fact, the vertex u is uniquely r-step in-dominated by v, and w is uniquely s-step out-dominated by v. Thus v must belong to every (r, s)-step dominating set of D. Since this is true for every vertex of D, it follows that $\rho^*_{r,s}(\vec{C}_n) = n$. ❑

A necessary and sufficient condition for the existence of the (r, s)-step domination number is presented next.

Theorem 4.2 Let D be a digraph such that $\delta^-(D) \geq 1$ and $\delta^+(D) \geq 1$. Then $\rho^*_{r,s}(D)$ exists if and only if $\text{rad}^+(D) \geq r$ and $\text{rad}^-(D) \geq s$.

Proof Suppose that $\rho^*_{r,s}(D)$ exists and let S be a minimum (r, s)-step dominating set of D. Let v be a vertex of D. Then v is r-step in-dominated by some vertex of S and v is s-step out-dominated by some vertex of S. Thus there exist vertices u and w in S such that $d(v, u) = r$ and $d(w, v) = s$. Hence $e^+(v) \geq d(v, u) = r$ and $e^-(v) \geq \text{d}(w, v) = s$. That is, for every vertex v of D, $e^+(v) \geq r$ and $e^-(v) \geq s$, which implies that $\text{rad}^+(D) \geq r$ and $\text{rad}^-(D) \geq s$.

For the converse, assume $\text{rad}^+(D) \geq r$ and $\text{rad}^-(D) \geq s$. Let v be any vertex of D. Then $e^+(v) \geq \text{rad}^+(D) \geq r$ and $e^-(v) \geq \text{rad}^-(D) \geq s$. By Lemma 3.6, there exist vertices u and w such that $d(v, u) = r$ and $d(w, v) = s$. Therefore every vertex of D is r-step in-dominated and s-step out-dominated, implying that $\rho^*_{r,s}(D)$ exists. ❑

The following corollary is now immediate.

Corollary 4.3 Let D be a digraph such that $\delta^-(D) \geq 1$ and $\delta^+(D) \geq 1$. If $\rho^*_{r,s}(D)$ exists, then $\rho^*_t(D)$ exists, where $t = \min\{r, s\}$.

Proof Since $\rho^*_{r,s}(D)$ exists, we know $\text{rad}^+(D) \geq r$ and $\text{rad}^-(D) \geq s$. Thus, we have $\text{rad}^+(D) \geq t$ and $\text{rad}^-(D) \geq t$, where $t = \min\{r, s\}$. Therefore, by Theorem 3.7, $\rho^*_t(D)$ exists. ❑

However, it is possible to have a digraph D such that $\rho^*_{r,s}(D)$ exists and exactly one of $\rho^*_r(D)$ and $\rho^*_s(D)$ exists. For the digraph D of Figure 36, $\rho^*_{3,2}(D)$ and $\rho^*_2(D)$ exist, but $\rho^*_3(D)$ does not.

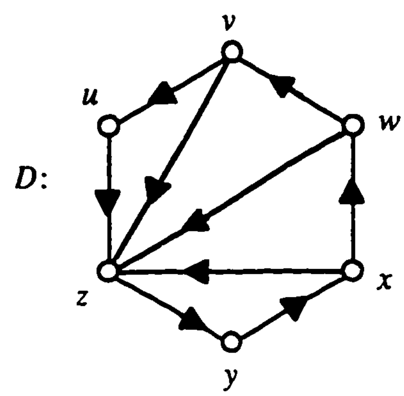

Figure 36. A Digraph D Such That $\rho^*_{3,2}(D)$ and $\rho^*_2(D)$ Exist, but $\rho^*_3(D)$ Does Not Exist.

The next theorem shows that there exists an infinite class of digraphs D such that $\rho^*_{r,s}(D)$ depends on r and s.

Theorem 4.4 For integers r and s with $|r - s| \geq 2$, there exists an infinite class of digraphs D such that $\rho^*_{r,s}(D) = 2 + \max\{r, s\}$.

Proof We assume, without loss of generality, that $r - s \geq 2$. Let D be the digraph of order $r + s + 2$ with $V(D) = \{u_1, u_2, \ldots, u_{r+1}\} \cup \{v_1, v_2, \ldots, v_{s+1}\}$. Let u_1,

$u_2, \ldots, u_{r+1}, u_1$ be an $(r+1)$-cycle of D, let $v_1, v_2, \ldots, v_{s+1}, v_1$ be an $(s+1)$-cycle of D, and let $(u_1, v_1), (v_{s+1}, u_{r+1}) \in E(D)$. Let $S = \{u_1, u_2, \ldots, u_{r+1}, v_1\}$. It is straightforward to show that for each vertex x of S, there exists a vertex of D which is either uniquely r-step in-dominated by x or s-step out-dominated by x. That is, each of the vertices of S must belong to every (r, s)-step dominating set of D. Furthermore, the set S is, indeed, an (r, s)-step dominating set of D. Therefore, $\rho^*_{r,s}(D) = r + 2 = 2 + \max\{r, s\}$. ❑

4.2 Minimal and Efficient Twin Domination in Digraphs

An open twin dominating set S of a digraph D is *minimal* if no proper subset T of S is an open twin dominating set of D. For the digraph D of Figure 37, it is easy to see that $S_1 = \{u, v, x, y\}$ is an open twin dominating set. Furthermore, S_1 is a minimal open twin dominating set since the subdigraph of $\langle S_1 \rangle$ obtained by removing a vertex of S_1 does not have minimum degree at least 2. However, the set $S_2 = \{w, x, y\}$ is an open twin dominating set and thus $\rho_1^*(D) = 3$. Therefore digraphs D exist that contain a minimal open twin dominating set whose cardinality exceeds $\rho_1^*(D)$.

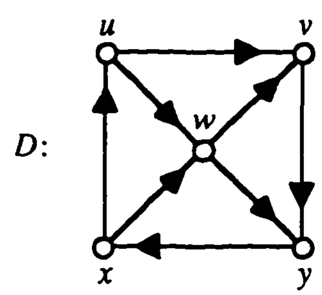

Figure 37. A Digraph D That Contains a Minimal Open Twin Dominating Set of Cardinality Exceeding $\rho_1^*(D)$.

The observation just made leads to the following concept. For a digraph D, the *upper open twin domination number* $\mathrm{P}_1^*(D)$ is defined as the maximum cardinality among the minimal open twin dominating sets of D.

For the digraph D of Figure 37, we know that $\mathrm{P}_1^*(D) \geq 4$. Also $\mathrm{P}_1^*(D) \neq 5$ since $V(D)$ is not a minimal open twin dominating set of D. Thus $\mathrm{P}_1^*(D) = 4$. We show, in general, that $\mathrm{P}_1^*(D) - \rho_1^*(D)$ can be arbitrarily large.

Theorem 4.5 For every positive integer n, there exists a digraph D_n such that

$$\mathrm{P}_1^*(D_n) - \rho_1^*(D_n) \geq n.$$

Proof We construct a digraph D_n of order $n + 6$ with $V(D_n) = U \cup V$, where $U = \{u_1, u_2, u_3\}$ and $V = \{v_1, v_2, \ldots, v_{n+3}\}$. Let u_1, u_2, u_3, u_1 be a directed 3-cycle of D_n, and let $v_1, v_2, \ldots, v_{n+3}, v_1$ be a directed $(n + 3)$-cycle of D_n. We complete the construction by adding the following arcs between U and V:

$$(u_1, v_i) \text{ and } (v_i, u_2), \qquad i \equiv 1 \pmod 3,$$

$$(u_2, v_i) \text{ and } (v_i, u_3), \qquad i \equiv 2 \pmod 3,$$

$$(u_3, v_i) \text{ and } (v_i, u_1), \qquad i \equiv 0 \pmod 3.$$

Figure 38 shows the digraph D_2. It is easy to see that U is an open twin dominating set of cardinality 3. Thus $\rho_1^*(D_n) = 3$. Moreover, V is an open twin dominating set of cardinality $n + 3$. For each proper subset T of V, there exists a vertex v_i of T such that $\mathrm{id}_{\langle T \rangle} v_i = 0$. Thus no other vertex of T out-dominates v_i, implying that T is not an open twin dominating set. Therefore V is a minimal open twin dominating set and thus $\mathrm{P}_1^*(D_n) \geq n + 3$, which implies that

$$\mathrm{P}_1^*(D_n) - \rho_1^*(D_n) \geq n + 3 - 3 = n. \quad \square$$

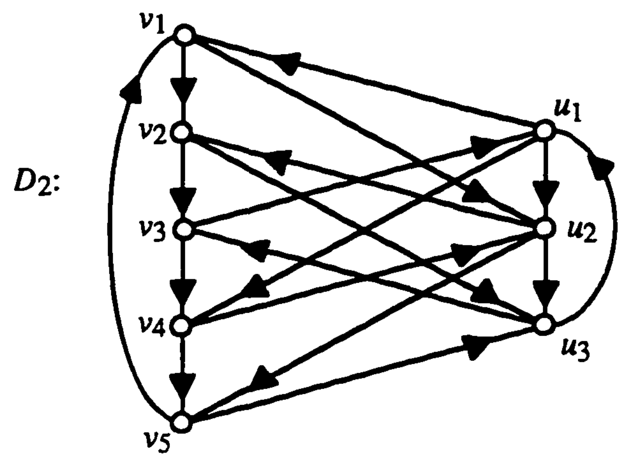

Figure 38. The Construction of the Digraph in the Proof of Theorem 4.5.

If D is a digraph for which $\rho_1^*(D) < P_1^*(D)$ and k is an integer such that $\rho_1^*(D) \leq k \leq P_1^*(D)$, it is natural to ask if there exists a minimal open twin dominating set S of D such that $|S| = k$. Such is not always the case, however, as can be seen from the digraph D_2 of Figure 38. In particular, we know that $\rho_1^*(D_2) = 3$ and $P_1^*(D_2) = 5$. However, we claim that there is no minimal open twin dominating set of cardinality 4 in D_2. To see this, let $V = \{v_1, v_2, v_3, v_4, v_5\}$ and let $U = \{u_1, u_2, u_3\}$. A minimal open twin dominating set S with $|S| = 4$ contains either (a) three vertices from V and one vertex from U or (b) two vertices from V and two vertices from U. Moreover, the subdigraph $\langle S \rangle$ must have minimum indegree at least 1 and minimum outdegree at least 1. There are only seven sets satisfying these properties. However, for each of these seven sets, Table 3 lists a vertex of V that is neither in-dominated nor out-dominated.

A result similar to Theorem 1.A can be stated for minimal open twin dominating sets of digraphs.

Table 3

A Step in Verifying That the Digraph of Figure 38 Has No Minimal Open Twin Dominating Set of Cardinality 4

S	vertex neither in- nor out-dominated by S
$\{v_1, v_2, v_3, u_1\}$	v_4 is not in-dominated
$\{v_2, v_3, v_4, u_2\}$	v_1 is not out-dominated
$\{v_3, v_4, v_5, u_3\}$	v_1 is not in-dominated
$\{v_1, v_2, u_1, u_3\}$	v_4 is not in-dominated
$\{v_2, v_3, u_1, u_2\}$	v_5 is not in-dominated
$\{v_3, v_4, u_2, u_3\}$	v_1 is not out-dominated
$\{v_4, v_5, u_1, u_3\}$	v_1 is not in-dominated

Theorem 4.6 An open twin dominating set S of a digraph D is a minimal open twin dominating set of D if and only if every vertex v of S satisfies at least one of the following two properties:

(a) there exists a vertex u in $V(D) - S$ such that $N^+(u) \cap S = \{v\}$,

(b) there exists a vertex w in $V(D) - S$ such that $N^-(w) \cap S = \{v\}$.

Proof Let S be a minimal open twin dominating set of D and let $v \in S$. Then the set $S - \{v\}$ is not an open twin dominating set of D. Thus either some vertex of D is not openly in-dominated by $S - \{v\}$ or some vertex of D is not openly out-dominated by $S - \{v\}$. Thus either (a) there exists a vertex u such that the only vertex of S that openly in-dominates u is v or (b) there exists a vertex w such that the only vertex of S that openly out-dominates w is v. That is, either there exists a

vertex u in $V(D) - S$ such that $N^+(u) \cap S = \{v\}$ or there exists a vertex w in $V(D) - S$ such that $N^-(w) \cap S = \{v\}$.

For the converse, assume every vertex v of S satisfies at least one of the two properties. Let S' be a proper subset of S. Then there exists a vertex v in $S - S'$. By assumption, the vertex v satisfies at least one of properties (a) and (b). If v satisfies property (a), then there exists a vertex u in $V(D) - S$ such that $N^+(u) \cap S = \{v\}$. Thus $N^+(u) \cap S' = \varnothing$ which implies that u is not openly in-dominated by any vertex of S'. Similarly, if v satisfies property (b), then there exists a vertex w in $V(D) - S$ that is not openly out-dominated by any vertex of S'. In either case, the set S' is not an open twin dominating set of D. Therefore S is a minimal open twin dominating set of D. ❑

An analogous result is stated for minimal k-step twin dominating sets. We omit the proof.

Theorem 4.7 A k-step twin dominating set S of a digraph D is a minimal k-step twin dominating set of D if and only if every vertex v of S satisfies at least one of the following two properties:

(a) there exists a vertex u in $V(D) - S$ such that $N_k^+(u) \cap S = \{v\}$,

(b) there exists a vertex w in $V(D) - S$ such that $N_k^-(w) \cap S = \{v\}$.

Theorem 1.B is closely related to Theorem 1.A; in fact, it follows as a corollary. However, no result similar to Theorem 1.B exists for minimal k-step twin dominating sets. The digraph D of Figure 39 has minimal k-step (where $k = 1$ or $k = 2$) twin dominating set $S = \{u, v, w\}$. However, the set $V(D) - S$ is not a k-step twin dominating set of D.

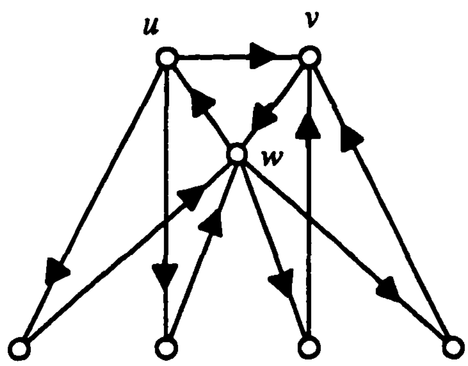

Figure 39. A Digraph Illustrating That Theorem 1.B Does Not Generalize for Minimal k-step Twin Dominating Sets.

Domination was extended to efficient domination in Bange, Barkauskas, and Slater [1]. If S is an open twin dominating set of a digraph D of order n, then certainly $\sum_{v \in S} \deg v \geq 2n$. We define an open twin dominating set S of a digraph D of order n to be *efficient* if $\sum_{v \in S} \deg v = 2n$, that is, every vertex of D is in-dominated by a unique vertex of S and is also out-dominated by a unique vertex of S.

The entire vertex set of $\vec{C}_n$ is an efficient open twin dominating set of $\vec{C}_n$. Also, in the proof of Theorem 4.6, the set U is an efficient open twin dominating set of the digraph D_n. The digraph of Figure 40 shows that Q_4 has an orientation containing an efficient open twin dominating set, as indicated by the shaded vertices.

However, for some positive integers n, no orientations of Q_n have efficient open twin dominating sets, as can be seen from the next result.

Proposition 4.8 Let $n \geq 4$ be a positive integer such that $n \neq 2^k$ for every positive integer k. Then no valid orientation of Q_n contains an efficient open twin dominating set.

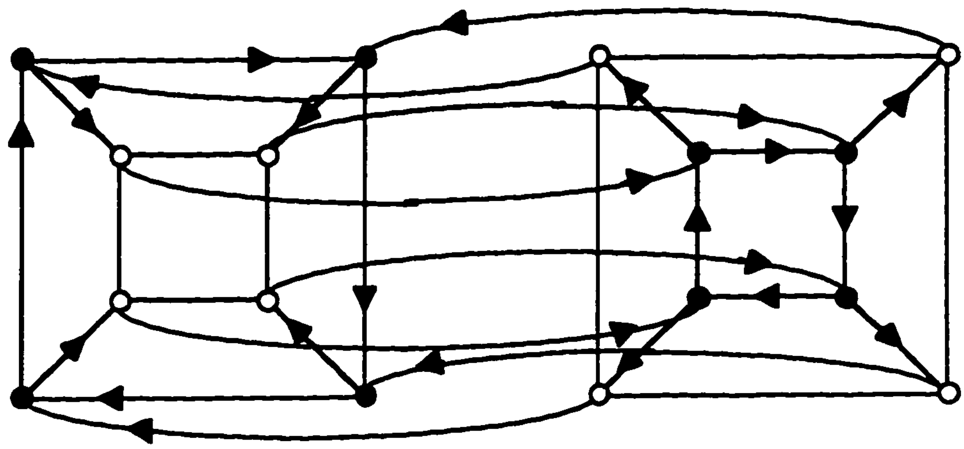

Figure 40. An Orientation of Q_4 Containing an Efficient Open Twin Dominating Set.

Proof The n-cube Q_n has 2^n vertices and is n-regular. Suppose, to the contrary, that some valid orientation D of Q_n contains an efficient open twin dominating set S. Then

$$\sum_{v \in S} \deg v = 2 \cdot 2^n = 2^{n+1}.$$

However,

$$\sum_{v \in S} \deg v = n \cdot |S|.$$

Therefore

$$n \cdot |S| = 2^{n+1} \quad \text{or} \quad |S| = \frac{2^{n+1}}{n},$$

implying that $n = 2^k$ for some positive integer k, contradicting our assumption. ❑

A similar argument shows that for $K_{m,n} \neq K_{2,2}$ $(m, n \geq 2)$, no orientation of $K_{m,n}$ contains an efficient open twin dominating set. Also, no orientation of the

Petersen graph contains an efficient open twin dominating set. Indeed, if G is a cubic graph of order n and size m having such an orientation with an efficient open twin dominating set S, then $2m = 3n$ and $3|S| = 2n$. Consequently, in this case, $n \equiv 0 \pmod 6$. We now show that there exists an infinite class of cubic graphs having such an orientation.

Proposition 4.9 There exist infinitely many cubic graphs having an orientation that contains an efficient open twin dominating set.

Proof For $n \equiv 0 \pmod 6$, let G_n be the 3-regular graph of order n formed by joining opposite vertices of C_n. We show that there exists an orientation D_n of G_n such that D_n has an efficient open twin dominating set. Suppose that $n = 6k$. Let G_n consist of the n-cycle $v_1, v_2, ..., v_n, v_1$ together with the edges $v_i v_{3k+i}$ for $i = 1, 2, ..., 3k$. The orientation D_n is constructed by first forming k directed 4-cycles, namely, $v_{3i+1}, v_{3i+2}, v_{3k+3i+2}, v_{3k+3i+1}, v_{3i+1}$, for each $i = 0, 1, ..., k-1$. Next, for $j = 1, 2, ..., 2k$, let (v_{3j-1}, v_{3j}) and (v_{3j}, v_{3j+1}) belong to $E(D_n)$, where $v_{n+1} = v_1$. Any other edges not mentioned thus far can be oriented arbitrarily. The construction is shown for $n = 12$ in Figure 41. It is straightforward to verify that $S = \{v_{3i+1}, v_{3i+2}, v_{3k+3i+1}, v_{3k+3i+2} \mid i = 0, 1, ..., k-1\}$ is an open twin dominating set. Also observe that $|S| = 4k$ and since G_n is 3-regular, it follows that $\sum_{v \in S} \deg v = 3|S| = 12k = 2n$. Therefore S is an efficient open twin dominating set of D_n. ❑

Not every cubic graph of order $n \equiv 0 \pmod 6$ has an orientation containing an efficient open twin dominating set. For example, the graph G of Figure 42 is a cubic graph of order 12. Thus an efficient open twin dominating set would have cardinality 8. Now the only subgraph H of G which has order 8 and minimum

degree at least 2 is the subgraph induced by the shaded vertices in Figure 42. However, it is impossible, regardless of the orientation of the edges of G, for these eight vertices to form an open twin dominating set.

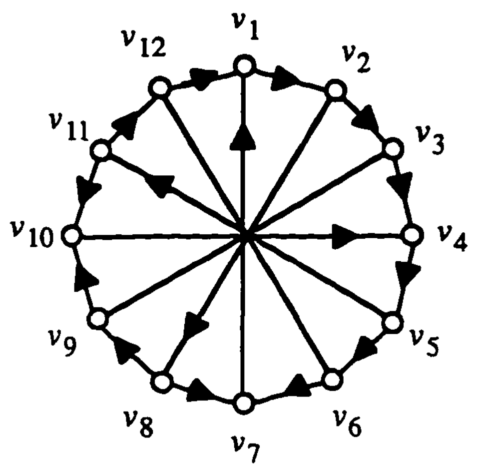

Figure 41. The Construction in the Proof of Proposition 4.9.

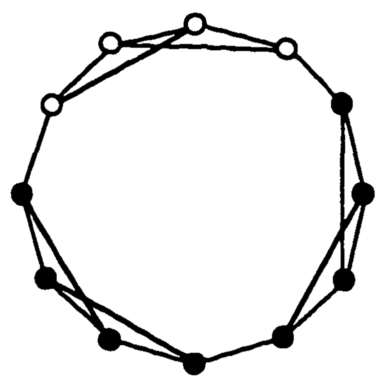

Figure 42. A Cubic Graph of Order 12 Such That No Orientation Contains an Efficient Open Twin Dominating Set.

Upper and lower bounds on the cardinality of an efficient open twin dominating set are presented next.

Theorem 4.10 If D is a digraph of order n containing an efficient open twin dominating set S, then

$$\frac{2n}{\Delta^+(D) + \Delta^-(D)} \le |S| \le \frac{2n}{\delta^+(D) + \delta^-(D)}.$$

Proof Since S is an effiecient open twin dominating set of D, it follows that $\sum_{v \in S} \deg v = 2n$. Also

$$\sum_{v \in S} \deg v \le |S|(\Delta^+(D) + \Delta^-(D)) \text{ and } \sum_{v \in S} \deg v \ge |S|(\delta^+(D) + \delta^-(D)).$$

Therefore $|S| \ge \dfrac{2n}{\Delta^+(D) + \Delta^-(D)}$ and $|S| \le \dfrac{2n}{\delta^+(D) + \delta^-(D)}$. ❑

The bounds in Theorem 4.10 are sharp in the sense that they are attained for directed cycles of order n. Upper and lower bounds for the cardinality of an efficient open twin dominating set can also be found in terms of the order and size of a digraph.

Theorem 4.11 If D is an asymmetric digraph of order n and size m containing an efficient open twin dominating set of cardinality k, then

$$2n - m \le k \le n - \sqrt{2(m - n)}.$$

Proof Let S be an efficient open twin dominating set of D having cardinality k. Then every vertex of S is adjacent from exactly one vertex of S and is adjacent to exactly one vertex of S. Consequently, the subdigraph $\langle S \rangle$ is a union of directed cycles, and thus $\langle S \rangle$ has k arcs. Also, for every vertex v not belonging to S, there exists exactly one arc directed from some vertex of S to v and exactly one arc directed from v to some vertex of S. Furthermore, since D is asymmetric, the subdigraph $\langle V(D) - S \rangle$ contains at most $\binom{n-k}{2}$ arcs. Therefore, we have

$$k + 2(n - k) \le m \le k + 2(n - k) + \binom{n-k}{2}.$$

The lower bound gives us $k \geq 2n - m$. The upper bound is equivalent to

$$k^2 - (1 + 2n)k + n^2 + 3n - 2m \geq 0.$$

So either

$$k \geq \frac{1 + 2n + \sqrt{8m - 8n + 1}}{2} \geq \frac{2n + 2\sqrt{2(m - n)}}{2} = n + \sqrt{2(m - n)}$$

or

$$k \leq \frac{1 + 2n - \sqrt{8m - 8n + 1}}{2} \leq n - \sqrt{2(m - n)}\ .$$

However, since $k \leq n$, we cannot have $k \geq n + \sqrt{2(m - n)}$ unless D is a union of directed cycles. Thus $k \leq n - \sqrt{2(m - n)}$. ❑

Both the upper and lower bounds in Theorem 4.11 are sharp since they are attained for directed n-cycles.

4.3 Some Nordhaus-Gaddum-type Results

Typically, the bounds presented in Nordhaus-Gaddum-type results focus on bounds for the sum and product of parameters for a graph and its complement. Since we have concentrated on asymmetric digraphs, we consider these bounds for a digraph D and its converse D'. We first present a result of this type for the open out-domination number of a digraph and its converse.

Theorem 4.12 If D is a digraph of order n such that $\rho_1^+(D)$ and $\rho_1^+(D')$ are defined, then

$$\frac{n}{\Delta^+(D)} + \frac{n}{\Delta^-(D)} \leq \rho_1^+(D) + \rho_1^+(D') \leq \frac{n}{\delta^+(D)} + \frac{n}{\delta^-(D)}$$

and

$$\frac{n^2}{\Delta^+(D)\,\Delta^-(D)} \le \rho_1^+(D)\,\rho_1^+(D') \le \frac{n^2}{\delta^+(D)\,\delta^-(D)}\,.$$

Proof Let S be a minimum open out-dominating set of D. Then every vertex of S openly out-dominates at most $\Delta^+(D)$ vertices. Furthermore, every vertex of D is openly out-dominated by some vertex of S. So

$$n \le \sum_{v \in S} \text{od } v \le \sum_{v \in S} \Delta^+(D) = |S|\,\Delta^+(D).$$

Thus $\rho_1^+(D) \ge \dfrac{n}{\Delta^+(D)}$. Similarly, we have $\rho_1^+(D') \ge \dfrac{n}{\Delta^+(D')} = \dfrac{n}{\Delta^-(D)}$. Therefore,

$$\rho_1^+(D) + \rho_1^+(D') \ge \frac{n}{\Delta^+(D)} + \frac{n}{\Delta^-(D)} \text{ and } \rho_1^+(D)\,\rho_1^+(D') \ge \frac{n^2}{\Delta^+(D)\,\Delta^-(D)}\,,$$

verifying the lower bounds.

To establish the upper bounds, observe that

$$|S|\,\delta^+(D) \le \sum_{v \in S} \text{od } v \le \sum_{v \in V(D)} \text{od } v = n.$$

Thus $\rho_1^+(D) \le \dfrac{n}{\delta^+(D)}$ and $\rho_1^+(D') \le \dfrac{n}{\delta^+(D')} = \dfrac{n}{\delta^-(D)}$. Therefore

$$\rho_1^+(D) + \rho_1^+(D') \le \frac{n}{\delta^+(D)} + \frac{n}{\delta^-(D)} \text{ and } \rho_1^+(D)\,\rho_1^+(D') \le \frac{n^2}{\delta^+(D)\,\delta^-(D)}\,. \quad \square$$

As a corollary, we have the following bounds for the open in-domination number of a digraph and its converse as well.

Corollary 4.13 If D is a digraph of order n such that $\rho_1^-(D)$ and $\rho_1^-(D')$ are defined, then

$$\frac{n}{\Delta^+(D)} + \frac{n}{\Delta^-(D)} \le \rho_1^-(D) + \rho_1^-(D') \le \frac{n}{\delta^+(D)} + \frac{n}{\delta^-(D)}$$

and

$$\frac{n^2}{\Delta^+(D)\,\Delta^-(D)} \le \rho_1^-(D)\,\rho_1^-(D') \le \frac{n^2}{\delta^+(D)\,\delta^-(D)}.$$

Proof By Proposition 2.1, we know that $\rho_1^-(D) = \rho_1^+(D')$ and $\rho_1^-(D') = \rho_1^+(D)$. The result now follows immediately from Theorem 4.12. ❑

The bounds in Theorem 4.12 and Corollary 4.13 cannot be improved since the directed cycles of order n attain each of the bounds. As another corollary, we have the following bounds for the open twin domination number of a digraph.

Corollary 4.14 If D is a digraph of order n such that $\rho_1^*(D)$ is defined, then

$$\frac{1}{2}\left(\frac{n}{\Delta^+(D)} + \frac{n}{\Delta^-(D)}\right) \le \rho_1^*(D) \le \frac{n}{\Delta^+(D)} + \frac{n}{\Delta^-(D)}.$$

Proof We know that $\rho_1^*(D) \le \rho_1^+(D) + \rho_1^-(D) \le \dfrac{n}{\Delta^+(D)} + \dfrac{n}{\Delta^-(D)}$. Also, by Corollary 2.7,

$$2\rho_1^*(D) \ge \rho_1^+(D) + \rho_1^-(D)$$

or

$$\rho_1^*(D) \ge \frac{1}{2}(\rho_1^+(D) + \rho_1^-(D)) \ge \frac{1}{2}\left(\frac{n}{\Delta^+(D)} + \frac{n}{\Delta^-(D)}\right). \quad ❑$$

The lower bound in Corollary 4.14 is sharp since the lower bound is attained for directed cycles of order n. The upper bound is sharp as well as can be seen from the next theorem.

Theorem 4.15 There exists an infinite class of digraphs D such that

$$\rho_1^*(D) = \frac{n}{\Delta^+(D)} + \frac{n}{\Delta^-(D)}.$$

Proof Let $k \equiv 0 \pmod 6$ be an integer and let $n = 2k$. Let D be a digraph such that $V(D) = \{u_1, u_2, ..., u_k\} \cup \{v_1, v_2, ..., v_k\}$ and let $E(D)$ be defined by the following directed cycles:

$$v_1, v_2, ..., v_k, v_1$$
$$v_1, v_3, v_5, ..., v_{k-1}, v_1$$
$$v_2, v_4, v_6, ..., v_k, v_2$$
$$v_1, v_4, v_7, ..., v_{k-2}, v_1$$
$$v_2, v_5, v_8, ..., v_{k-1}, v_2$$
$$v_3, v_6, v_9, ..., v_k, v_3,$$

as well as the set of directed 3-cycles $\{v_i, v_{i+1}, u_i, v_i \mid i = 1, 2, ..., n\}$, where the subscripts are expressed modulo n. Observe that $\Delta^+(D) = \Delta^-(D) = 4$. Also it is straightforward to show that a minimum open twin dominating set of D is the set $\{v_1, v_2, ..., v_k\}$. Thus $\rho_1^*(D) = k = \frac{2k}{4} + \frac{2k}{4}$. ❑

4.4 Universal and Planetary Twin Domination of Digraphs

Let D be a digraph of order n with $V(D) = \{v_1, v_2, ..., v_n\}$. An ordered pair (s_1, s_2) of two sequences s_1: $k_1, k_2, ..., k_n$ and s_2: $\ell_1, \ell_2, ..., \ell_n$ of positive integers is a *universal twin dominating sequence pair for* D if every vertex of D is k_i-step in-dominated by v_i for some i, $1 \le i \le n$, and is ℓ_j-step out-dominated by v_j for some j, $1 \le j \le n$. As can be seen from Table 4, the digraph D of Figure 43 has universal twin dominating sequence pair s_1: 3, 1, 1, 2, 2 and s_2: 2, 1, 4, 2, 3.

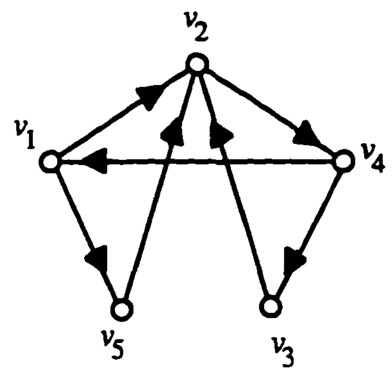

Figure 43. A Digraph D With Universal Twin Dominating Sequence Pair s_1: 3, 1, 1, 2, 2 and s_2: 2, 1, 4, 2, 3.

Table 4

Verification That the Sequence Pair s_1: 3, 1, 1, 2, 2 and s_2: 2, 1, 4, 2, 3
Forms a Universal Twin Dominating Sequence Pair
for the Digraph of Figure 43

vertex	is k_i-step in-dominated	by v_i	is ℓ_j-step out-dominated	by v_j
v_1	1	v_2	3	v_5
v_2	1	v_3	2	v_4
v_3	2	v_5	2	v_1
v_4	3	v_1	3	v_5
v_5	3	v_1	4	v_3

The *universal value of* D *for the sequence pair* (s_1, s_2) is

$$\sum_{i=1}^{n} |N^-_{k_i}(v_i)| + \sum_{j=1}^{n} |N^+_{\ell_j}(v_j)|,$$

which counts the number of times each vertex is k_i-step in-dominated by v_i and ℓ_j-step out-dominated by v_j. The digraph of Figure 43 has universal value 15. Notice that the pair of constant sequences s_1: 1, 1, ..., 1 and s_2: 1, 1, ..., 1 of length n form a universal twin dominating sequence pair for any digraph D with $\delta^-(D) \geq 1$ and $\delta^+(D) \geq 1$. Furthermore, in this case, the universal value of D is

$$\sum_{i=1}^{n} |N^-(v_i)| + \sum_{j=1}^{n} |N^+(v_j)| = \sum_{i=1}^{n} \text{id } v_i + \sum_{j=1}^{n} \text{od } v_j = 2n.$$

The pair of constant sequences s_1: 1, 1, ..., 1 and s_2: 1, 1, ..., 1 of length n is not necessarily the only universal twin dominating sequence pair that yields universal value $2n$. For instance, if D is a directed cycle of order n, then every universal twin dominating sequence pair has universal value $2n$. A digraph D is a *constant universal digraph* if every universal twin dominating sequence pair has the same universal value. For instance, a directed n-cycle is a constant universal digraph since for integers k and ℓ $(1 \leq k, \ell \leq n-1)$ belonging to a universal twin dominating sequence pair, $|N_k^-(v)| = 1$ and $|N_\ell^+(v)| = 1$ for every vertex v of D.

For a digraph D of order n, a pair (s_1, s_2) of two sequences $s_1: k_1, k_2, ..., k_r$ $(r \leq n)$ and $s_2: \ell_1, \ell_2, ..., \ell_t$ $(t \leq n)$ of positive integers is a *planetary twin dominating sequence pair* for D if D contains distinct vertices $v_1, v_2, ..., v_m$ $(m = \max\{r, t\})$ such that every vertex of D is k_i-step in-dominated by v_i for some i, $1 \leq i \leq m$ and is ℓ_j-step out-dominated by v_j for some j, $1 \leq j \leq m$. For the digraph D of Figure 43, the sequences s_1: 1, 1, 2, 2 and s_2: 1, 2, 2 form a planetary twin dominating sequence pair, as can be seen from Table 5.

The *planetary twin domination number* $\text{pl}^*(D)$ of a digraph D is the minimum integer $m = r + t$ such that a sequence pair (s_1, s_2), where s_1 has length r and s_2 has length t, forms a planetary twin dominating sequence pair for D.

Proposition 4.16 For every digraph D such that $\rho_i^*(D)$ is defined, $\mathrm{pl}^*(D) \leq 2\rho_i^*(D)$ for each i.

Table 5

Verification That the Sequence Pair s_1: 1, 1, 2, 2 and s_2: 1, 2, 2
Forms a Planetary Twin Dominating Sequence Pair
for the Digraph of Figure 43

vertex	is k_i-step in-dominated	by v_i	is ℓ_j-step out-dominated	by v_j
v_1	1	v_2	2	v_2
v_2	2	v_3	1	v_1
v_3	1	v_1	2	v_3
v_4	1	v_2	2	v_2
v_5	2	v_4	1	v_1

Proof Notice that, for a digraph D such that $\rho_i^*(D)$ is defined, the constant sequences $s_1 : i, i, \ldots, i$ and s_2: $i, i, \ldots, i$, each of length $\rho_i^*(D)$ form a planetary twin dominating sequence pair. Thus $\mathrm{pl}^*(D) \leq 2\rho_i^*(D)$ for each i. ❑

Proposition 4.17 For every digraph D such that $\rho_{r,s}^*(D)$ is defined, $\mathrm{pl}^*(D) \leq 2\rho_{r,s}^*(D)$.

Proof Notice that, for a digraph D such that $\rho_{r,s}^*(D)$ is defined, the constant sequences $s_1 : r, r, \ldots, r$ and s_2: $s, s, \ldots, s$, each of length $\rho_{r,s}^*(D)$ form a planetary twin dominating sequence pair. Thus $\mathrm{pl}^*(D) \leq 2\rho_{r,s}^*(D)$. ❑

CHAPTER V

PROBLEMS FOR FUTURE STUDY

5.1 Some Open Questions Regarding Open Twin Domination

In Proposition 2.13, we showed that for a graph G with $\delta(G) \geq 2$, there is a direct relationship between the parameters $\mathrm{dom}_1(G)$ and $\mathrm{dom}_1^*(G)$, namely, that $\mathrm{dom}_1(G) \leq \mathrm{dom}_1^*(G)$. It also follows from results in Chapter 2 that $\mathrm{DOM}_1^*(G) \geq \mathrm{DOM}_1(G)$ for classes of graphs such as cycles, complete bipartite graphs and the Petersen graph. Does this inequality hold in general or does there exist a graph G such that $\mathrm{DOM}_1^*(G) < \mathrm{DOM}_1(G)$?

In Section 2.3, several bounds for the lower orientable open twin domination numbers of bipartite graphs were discussed. Can these bounds be improved? For instance, for a bipartite graph B, can upper and lower bounds for $\mathrm{dom}_1^*(B)$ be found in terms of $\delta(B)$?

Using a counting argument, we were able to show in Theorem 2.26 that $\mathrm{DOM}_1^*(K_n)$ is unbounded. It seems that finding a formula for $\mathrm{DOM}_1^*(K_n)$ is difficult. Is it possible to determine the rate of growth of $\mathrm{DOM}_1^*(K_n)$. For instance, is $\mathrm{DOM}_1^*(K_n)$ nondecreasing? Is $\mathrm{DOM}_1^*(K_n)$ bounded by some function of n? Similar results were shown for the parameters $\mathrm{DOM}(K_n)$ in [7] and for $\mathrm{DOM}_1(K_n)$ in [13].

Also in Section 2.4, we determined a formula for the upper orientable open twin domination number for the n-cube Q_n. However, $\mathrm{dom}_1^*(Q_n)$ has not been determined. It follows from Corollary 2.16 that $\mathrm{dom}_1^*(Q_n) \geq \lceil 2^{n+1}/n \rceil$. Furthermore, for $n = 2, 3, 4$, orientations of Q_n exist that attain this lower bound.

5.2 Some Open Questions Regarding k-Step Twin Domination

Many questions regarding the k-step twin domination number of a digraph remain unanswered. For example, it was shown in Proposition 3.1 that $\rho_2^*(D) \geq 2$ for every digraph D and that this bound is sharp. For $k \geq 3$, it certainly is true that $\rho_k^*(D) \geq 2$. Is this lower bound sharp? Is it possible to find bounds on $\rho_k^*(D)$ in terms of k $(k \geq 2)$? Also, which triples a, b, c of integers are attainable for $\rho_k^-(D)$, $\rho_k^+(D)$, $\rho_k^*(D)$?

Another area of investigation is the relationship among the values $\rho_k^*(D)$ for various values of k. For instance, does there exist a digraph D such that $\rho_1^*(D) < \rho_2^*(D)$? More generally, for positive integers i and j, do there exist digraphs D_1 and D_2 such that $\rho_i^*(D_1) > \rho_j^*(D_1)$ and $\rho_j^*(D_2) > \rho_i^*(D_2)$? Also, does there exist a digraph D such that $\rho_k^*(D)$ is defined for many values of k and the numbers $\rho_k^*(D)$ are distinct for many k? That is, for every positive integer t, does there exist a digraph D such that $|\{\rho_k^*(D) \mid k \text{ is a positive integer}\}| \geq t$?

In Chapter II, it was determined that a necessary and sufficient condition for the orientable open twin domination numbers of a graph G to exist is for the graph to have minimum degree at least 2. Theorem 3.8 gave a sufficient condition for the orientable k-step twin domination numbers to exist. What other conditions can be found in this case? For instance, is it true that if G is a connected graph with girth $g(G) \geq k + 1$, then $\mathrm{dom}_k^*(G)$ and $\mathrm{DOM}_k^*(G)$ exist? Can a characterization of connected graphs G for which $\mathrm{dom}_k^*(G)$ and $\mathrm{DOM}_k^*(G)$ exist $(k \geq 2)$ be found?

Chartrand, VanderJagt, and Yue [7] showed that for a given graph G and an integer c such that $\mathrm{dom}(G) \leq c \leq \mathrm{DOM}(G)$, there exists an orientation D of G such that $\gamma(D) = c$. An analogous result was proved in [4] for orientable twin domination. A similar type of Intermediate Value Theorem was proved for orientable open

domination of complete graphs in [13]. Moreover, the conjecture was made that this type of result holds in general for orientable open domination. Does there exist an Intermediate Value Theorem for orientable k-step twin domination?

In Theorem 2.18, it was determined that $\mathrm{dom}_1^*(K_{m,n}) = 4$. And in Section 3.2, the lower orientable k-step twin domination number of complete bipartite graphs was determined for $k = 2$ and $k = 3$. Furthermore, we found that $\mathrm{dom}_k^*(K_{m,n})$ does not exist when k is odd, $k \geq 5$. It is our conjecture that $\mathrm{dom}_k^*(K_{m,n}) = 2k$ when k is even, $k \geq 4$.

For the Petersen graph P, the numbers $\mathrm{DOM}(P)$ and $\mathrm{DOM}_1(P)$have been determined. Furthermore, conjectures have been made for $\mathrm{DOM}_k(P)$ where $k \geq 2$. Also $\mathrm{dom}_1^*(P)$ and $\mathrm{DOM}_1^*(P)$ were determined in Chapter II. For $k \geq 2$, can $\mathrm{dom}_k^*(P)$ and $\mathrm{DOM}_k^*(P)$ be determined?

5.3 Open Questions Concerning Generalizations of Twin Domination

In Chapter IV, minimal and efficient open twin domination were studied. There are several open questions in this topic. Certainly these concepts can be extended to minimal and efficient k-step twin domination. Also, regarding Proposition 4.8, for each $n = 2^k$, does there exist an orientation of Q_n containing an efficient open twin dominating set? This has been verified as true for $k = 1$ and $k =$ 2. Another type of question is the following: Does there exist a digraph D with a partition $\{S_1, S_2, ..., S_k\}$ of $V(D)$ into efficient open twin dominating sets? If so, what is the maximum such k?

We could also consider measuring how close a digraph is with respect to containing an efficient open twin dominating set. For example, define the open twin domination excess $ex(D)$ of D as

$$ex(D) = \frac{\min \sum_{v \in S} \deg v}{2n} - 1$$

over all open twin dominating sets S of D. Then $\mathrm{ex}(D) = 0$ if and only if D contains an efficient open twin dominating set. How large can $ex(D)$ be? Can $ex(D) = 2$? If yes, for each rational number r, $1 < r < 2$, does there exist a digraph D such that $ex(D) = r$?

If S is a minimum open twin dominating set of a digraph D, then every vertex of D is dominated (either in- or out-) at least twice and at most $|S|$ times. We could measure the nearness of D to containing an efficient open twin dominating set by defining $\mu(D)$ to be the maximum number of times a vertex v of D is dominated (in- or out-) by a vertex of S, over all minimum open twin dominating sets S of D. Then $\mu(D) = 2$ if and only if D contains an efficient open twin dominating set. How large can $\mu(D)$ be?

In Section 4.3, some Nordhaus-Gaddum-type results were determined for a digraph and its converse. Is it possible to find some Nordhaus-Gaddum type results for a digraph and its complement?

In Section 4.4, universal and planetary twin domination were introduced. It is clear that if, for every vertex v of a digraph D, $|N_i^-(v)| = c$ and $|N_j^+(v)| = d$ for all integers i, $1 \le i \le e^-(v)$, and j, $1 \le j \le e^+(v)$, and some fixed integers c and d, then D is a constant universal digraph. Is the converse of this statement true? Also what values are possible for the universal value of a digraph D? Finally, can bounds be determined for the planetary twin domination number of a digraph?

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