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A Graph Theoretic Study of the Similarity of Discrete Structures

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A GRAPH THEORETIC STUDY OF THE SIMILARITY OF DISCRETE STRUCTURES

by

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A basic problem in drug design consists of finding a compound that satisfies a spectrum of biological and chemical properties. Although drug design problems are central to pharmaceutical research, statisticians have yet to become involved in this area as these problems are viewed statistically as optimization problems. Before statistical optimization procedures can be defined on these spaces whose points are structures and not vectors, very basic mathematical notions of distance must be defined. Graphs have been used as mathematical models to represent the bonding arrangements of molecules for quite some time. In fact, if some of the methodology used by statisticians for optimization problems could be extended to the metric space consisting of the set of all graphs, with a fixed number of vertices and a fixed number of edges, together with a metric on this set, then some of the problems in drug design might become more accessible.

Various metrics have been studied and some of the results are presented in Chapter I. In this dissertation, we define a metric in terms of some prescribed graph H. For certain choices of the graph H, this metric is a special case of the previously studied metrics in Chapter I. In Chapter II, conditions are described for certain graphs H that allow us to determine those pairs of graphs for which this metric is defined. Further properties of these metric spaces are studied in Chapter III by means of a graphical interpretation.

Another important problem in the area of mathematical chemistry is the determination of a maximum common substructure shared by two molecular compounds. A special type of commonality that two or more graphs share is called a greatest common subgraph. These concepts have been studied extensively, and some of the results are presented in Chapter I. A certain restriction, inherent to drug design, is imposed on these common subgraphs in Chapter IV, namely, that they preserve distance. These concepts are also applied to trees in Chapter IV. Another type of common subgraph, relative to this distance constraint, is studied in Chapter V.

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To Ryan and Joey

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CHAPTER I

INTRODUCTION

1.1 Introduction

A basic problem in drug design consists of finding a compound that satisfies a spectrum of biological and chemical properties. Although drug design problems are central to pharmaceutical research, statisticians have yet to become involved in this area. A major reason for this is that these problems are viewed statistically as optimization problems, and standard statistical optimization methods are based on Euclidean space or vector representation. Here the formal representations are labeled graphs and/or three-dimensional atomic configurations. So, the statistician is understandably blocked when confronted with a data set consisting of molecular structures. Before statistical optimization procedures can be defined on these spaces whose points are structures and not vectors, very basic mathematical notions of distance must be defined. This dissertation explores some of these notions when the points are labeled graphs in Chapters II and III.

Graphs have been used as mathematical models to represent the bonding arrangements of molecules for quite some time (see Harary [14], for example). A *graph* G consists of a set $V(G)$ of elements called *vertices* and a set $E(G)$ of elements called *edges*, where each edge is an unordered pair of distinct vertices. So when a graph models a molecule, the vertices denote the atoms and the edges denote the bonds. However, in the molecule the atoms are distinguished by their type (hydrogen, carbon, chlorine, etc.) and the bonds by their type (single, double, aromatic, etc.). Thus when representing a molecule by a graph, we may wish to "color" or "label" the

vertices and edges. A graph representation of a molecular structure in which the vertices are colored by atom type and the edges by bond type is called a *chemical graph* (see [23], for example).

The set G of all graphs of a fixed order and a fixed size together with a metric d on G is a metric space. In fact, if some of the methodology used by statisticians for optimization problems could be extended to this metric space, then some of the problems in drug design might become more accessible. For example, when chemical graphs are used to model a chemical reaction, it is often important to determine the maximum commonalities between these molecular structures or, in fact, to determine the greatest common subgraph of some set of graphs.

In this dissertation, one area that we study is common subgraphs under distance constraints. This type of constraint arises naturally in drug reception/interaction designs. Drugs interact with proteins, where the proteins are made up of sequences (or chains) of amino acids that form certain patterns (like a helix, for example). The drug needs to interact or "fit" into the protein in a specified way. Drugs are chemical compounds that can be represented as chemical graphs. Also, these chemical graphs can be embedded in three dimensions, where we might interpret the distance between two vertices (or atoms) as the Euclidean distance between these two points in $\mathbb{R}^3$. Unfortunately, this distance is subject to the particular embedding of the graph in $\mathbb{R}^3$, and there are many distinct embeddings of the same graph in $\mathbb{R}^3$. Thus, if we measure the distance between two vertices as the length of a shortest path (in the graph) between them, then this distance places constraints on the Euclidean distance between the two vertices when the graph is embedded in $\mathbb{R}^3$.

In the area of drug reception/interaction problems, we may have one drug that interacts well with a certain protein and another drug does that not interact well with this same protein. A natural question arises: In what structural way are these two drugs the

same? An analogous question in graph theory then becomes: What is the greatest common subgraph of the two graphs representing these drugs subject to the distance constraint? This topic will be the object of study in Chapters IV and V.

1.2 A Few Metrics for Graphs

In [25] Vizing proposed the following question: For which graphs G_1 and G_2 of order p, does there exist a graph G of order $p+1$ such that each of G_1 and G_2 is an induced subgraph of G? For each positive integer p, let G_p denote the set of all nonisomorphic graphs of order p. Zelinka [26] defines a distance δ on G_p as follows: For graphs G_1 and G_2 of G_p, define $\delta(G_1, G_2)$ to be the least positive integer k such that there exists a graph G of order $p+k$ containing the graphs G_1 and G_2 as induced subgraphs. Thus Vizing's question becomes: For which graphs G_1 and G_2 of G_p is it true that $\delta(G_1, G_2) = 1$? In [26] it was shown that $\delta(G_1, G_2) = k$ for graphs G_1 and G_2 of G_p if and only if there exists a graph H of order at least $p-k$ such that H is an induced subgraph of each of G_1 and G_2. This distance δ on G_p is a metric [26] and also $\delta(G_1, G_2) = \delta(\overline{G_1}, \overline{G_2})$ for every pair G_1, G_2 of graphs of G_p. The graph $D(G_p)$ is defined in [26] as that graph whose vertex set is G_p and two vertices G_1 and G_2 of $D(G_p)$ are adjacent if and only if $\delta(G_1, G_2) = 1$. Since $\delta(K_p, \overline{K_p}) = p-1$, it follows that diam $D(G_p) = p-1$. For each pair G_1, G_2 of graphs of G_p, where neither of G_1 and G_2 is complete nor empty, we have either K_2 or $\overline{K_2}$ is an induced subgraph of each of G_1 and G_2, and thus $\delta(G_1, G_2) \leq p-2$.

The analogous concepts are studied for trees in [27]. For each positive integer p, let $\mathcal{T}_p$ denote the set of all nonisomorphic trees of order p. For trees T_1 and T_2 of $\mathcal{T}_p$, define $\delta_T(T_1, T_2)$ as the least positive integer k such that there exists a tree T of order $p+k$, where each of T_1 and T_2 is a subtree of T. This distance δ_T is

different from the distance δ. In fact, there exist trees T_1 and T_2 of order p for which $\delta(T_1, T_2) < \delta_T(T_1, T_2)$, namely when T_1 is a path and T_2 is a star. As before, δ_T is a metric on $\mathcal{T}_p$ (see [27]). Also in [27], $\delta_T(T_1, T_2) = k$ for trees T_1 and T_2 of $\mathcal{T}_p$ if and only if there exists a tree T of order at least $p - k$ such that T is a subtree of each of T_1 and T_2. Next define $D(\mathcal{T}_p)$ to be that graph whose vertex set is $\mathcal{T}_p$ and two vertices T_1 and T_2 of $D(\mathcal{T}_p)$ are adjacent if and only if $\delta_T(T_1, T_2) = 1$. It turns out that the distance from T_1 to T_2 in $D(\mathcal{T}_p)$ is exactly $\delta_T(T_1, T_2)$ and furthermore that diam $D(\mathcal{T}_p) = p - 3$. Also in [27], an upper bound for the radius of $D(\mathcal{T}_p)$ is given, and it is conjectured that this upper bound is exact.

Baláž, Koča, Kvasnička, and Sekanina [1] define a distance between graphs using the greatest common induced subgraph. This distance was stimulated by their studies on the mathematical modeling of organic chemistry and the problem of measuring the similarity of two graphs. Let G_1 and G_2 be two graphs. A *greatest common induced subgraph* of G_1 and G_2 is a graph G without isolated vertices of maximum size such that G is an induced subgraph of both G_1 and G_2. A back-track searching algorithm for the construction of a greatest common induced subgraph of two graphs has been given by McGregor in [22]. A distance $d(G_1, G_2)$ between the graphs G_1 and G_2 is given in [1] by

$$d(G_1, G_2) = |E(G_1)| + |E(G_2)| - 2|E(G)| + \big||V(G_1)| - |V(G_2)|\big|.$$

So for graphs with the same order and the same size, this distance is the number of edges that cannot be matched in the construction of a greatest common induced subgraph of G_1 and G_2. This distance is a metric on the set of all graphs as seen in [1]. For graphs G_1 and G_2 with the same order, the construction of a greatest common induced subgraph in [1] can be realized by a 1-1 mapping ϕ from $V(G_1)$ onto $V(G_2)$. Thus if A_1 and A_2 are the adjacency matrices for G_1 and G_2

respectively, then ϕ can be realized by a permutation P of the elements $\{1, 2, \ldots, n\}$ and thus a second alternative distance is given [1] by

$$d(G_1, G_2) = \min |A_1 - P^T A_2 P|,$$

where this minimum is taken over all permutations P of $\{1, 2, \ldots, n\}$ and $|A| = \sum_{i \le j} |a_{ij}|$ is the Hamming (linear) norm of a symmetric matrix A. This relation is nothing more than the determination of the chemical distance between two graphs representing molecular structure formulas (see [1]).

A new distance function was defined in [9] for graphs with the same order and same size. Let G_1 and G_2 be two graphs of the same order and same size. The graph G_2 can be obtained from G_1 by an *edge rotation* if G_1 contains distinct vertices u, v, and w such that $uv \in E(G_1)$, $uw \notin E(G_1)$ and $G_2 = G_1 - uv + uw$. So, G_2 can be obtained from G_1 by "rotating" the edge uv of G_1 into the edge uw of G_2. In Figure 1.1, the graph G_2 can be obtained from G_1 by an edge rotation since $G_2 = G_1 - xy + xz$.

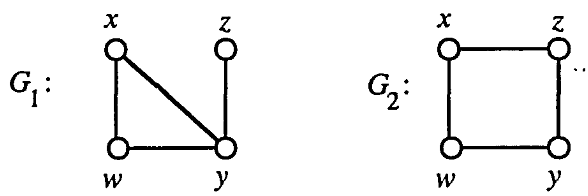

Figure 1.1 The Graph G_2 Can Be Obtained From G_1 by an Edge Rotation.

Clearly, a graph G_2 can be obtained from G_1 by an edge rotation if and only if G_1 can be obtained from G_2 by an edge rotation. More generally, a graph G_1 can be *r-transformed* into a graph G_2 if there exists a sequence $G_1 = H_0, H_1, \ldots, H_n = G_2$ $(n \ge 0)$ of graphs such that for $i = 1, 2, \ldots, n$, the graph H_i can be obtained from the graph H_{i-1} by an edge rotation. The relation "can be r-transformed into" is

an equivalence relation on the set of all graphs, and furthermore if G_1 can be r-transformed into G_2, then G_1 and G_2 must have the same order and same size. In fact, it is shown in [9] that the converse of the previous statement is also true.

Theorem A Let G_1 and G_2 be two graphs. Then G_1 can be r-transformed into G_2 if and only if G_1 and G_2 have the same order and same size.

For graphs G_1 and G_2 with the same order and same size, the *edge rotation distance* $d_r(G_1, G_2)$ between G_1 and G_2 is defined in [9] as the smallest nonnegative integer n for which there exists a sequence $G_1 = H_0, H_1, \dots, H_n = G_2$ of graphs such that for $i = 1, 2, \dots, n$, the graph H_i can be obtained from H_{i-1} by an edge rotation. Thus, by Theorem A, this distance is well-defined and is a metric on the set of all graphs of a fixed order and fixed size. In [9] it is shown that $d_r(G_1, G_2) = d_r(\overline{G}_1, \overline{G}_2)$, and that for every positive integer n, there exist graphs G_1 and G_2 such that $d_r(G_1, G_2) = n$. In order to determine an upper bound on the edge rotation distance between graphs, we require the following definition. For graphs G_1 and G_2, a *greatest common subgraph* of G_1 and G_2 is a graph G of maximum size without isolated vertices that is a subgraph of both G_1 and G_2. Thus in [9] we see that for graphs G_1 and G_2 of order p and size q with greatest common subgraph G of size s,

$$d_r(G_1, G_2) \leq 2(q - s).$$

Furthermore, this bound is sharp, as seen in [9].

In [13] and [17], it is independently shown that $q - s$ is a lower bound for $d_r(G_1, G_2)$. Also, it is shown in [13] that this lower bound is attainable as well. In fact, in [13] if G_1 and G_2 are 2–regular graphs of order p, then

$$d_r(G_1, G_2) = p - s,$$

where s is the size of a greatest common subgraph of G_1 and G_2. It is also shown in [13] that if G_1 and G_2 are two connected graphs of order p and size q, then

$$d_r(G_1, G_2) \leq 2q - p.$$

Zelinka gives some comparisons in [28] of the metrics δ, δ_T, and d_r and shows that for two graphs G_1 and G_2 of the same order and same size,

$$\delta(G_1, G_2) \leq d_r(G_1, G_2).$$

Furthermore, in [28] the edge rotation distance can be arbitrarily larger than the distance δ, that is, for each positive integer n, there exist graphs G_1 and G_2 such that

$$d_r(G_1, G_2) - \delta(G_1, G_2) = n.$$

Also in [28], we see that for two trees T_1 and T_2 of the same order, the distance $\delta_T(T_1, T_2)$ is an upper bound for $d_r(T_1, T_2)$, and that these two parameters can be arbitrarily far apart, that is, for every positive integer n, there exist nonisomorphic trees T_1 and T_2 such that

$$\delta_T(T_1, T_2) - d_r(T_1, T_2) = n.$$

Zelinka [28] has also determined the edge rotation distance between certain pairs of trees, namely, for a tree T of order n,

(i) $d_r(T, K_{1,n-1}) = n - 1 - \Delta(T)$,

(ii) $d_r(T, P_n) \leq n - 1 - \text{diam } T$, and

(iii) $d_r(P_n, K_{1,n-1}) = n - 3$.

The bound in (ii) is further investigated in [2]. For a tree T, let $\text{end}(T)$ denote the number of end-vertices of T. It is shown in [2] that for a tree T of order n,

$$d_r(T, P_n) = \text{end}(T) - 2.$$

In [13] the edge rotation distance of trees is considered. For trees T_1 and T_2, we say that T_2 can be obtained from T_1 by a *tree rotation* if T_2 can be obtained from T_1 by an edge rotation. Observe that this edge rotation does not disconnect the

tree. The *tree rotation distance* $d_T(T_1, T_2)$ between two trees T_1 and T_2 of the same order is defined as the minimum number of tree rotations required to transform T_1 into T_2. It is shown in [13] that the tree distance is bounded above by twice the edge rotation distance, that is, for trees T_1 and T_2 of the same order,

$$d_T(T_1, T_2) \le 2d_r(T_1, T_2).$$

As before, let $\mathcal{T}_p$ denote the set of all nonisomorphic trees of order p. Let $D_T(\mathcal{T}_p)$ denote that graph whose vertices are the trees of $\mathcal{T}_p$ and two vertices T_1 and T_2 are adjacent if and only if $d_T(T_1, T_2) = 1$. Then in [13], it is shown that

(i) $\Delta(D_T(\mathcal{T}_p)) \le p(p-3)$ for all $p \ge 4$,

(ii) diam $D_T(\mathcal{T}_p) \le p - 3$ for all $p \ge 3$, and

(iii) rad $D_T(\mathcal{T}_p) \le p - o(p)$.

Johnson [18] defined a metric on the set of all graphs with applications to medicinal chemistry. Before and after working its biological effect, a drug or compound will undergo a series of complex and diverse interactions, many of which are poorly understood. In fact, Johnson says in [18], "It is widely held that this effect is elicited by the drug interacting with a macromolecule in a steric specific manner often linked to the fit of a key (drug) in a lock (receptor site of the macromolecule)." He goes on to say that the receptor site and the drug must be similar to each other in a complementary way and drugs that are similar to each other, that is, having related groups in corresponding positions, are expected to behave in similar ways. However, some unexpected problems can occur. For example, some compounds bind to the receptor site without producing the desired activity while others may never reach the receptor site because of intervening interactions. Also slight modifications of a drug can alter its activity remarkably. Thus some important questions in medicinal chemistry include: (a) finding compounds in a set of easily accessible compounds that are similar

to some specified compound, and (b) given two structurally diverse compounds, to determine what they have in common.

For a graph G, Johnson [18] defines the *cardinality* $|G|$ of G as $|V(G)| + |E(G)|$, and defines a metric d on the set of all graphs as follows: for graphs G_1 and G_2 with greatest common subgraph G, define

$$d(G_1, G_2) = |G_1| + |G_2| - 2|G|.$$

It is not difficult to see that this distance is exactly the one given in [1]. In fact, if G_1 and G_2 are two graphs of order p and size q having a greatest common subgraph of size s, then $d(G_1, G_2) = 2q - 2s$, and hence this distance gives the upper bound for the edge rotation distance between G_1 and G_2.

Johnson [17] defines another metric on the set of all connected graphs of a fixed order and fixed size that is based on a more restricted version of an edge rotation. Let G_1 and G_2 be two graphs of the same order and same size. The graph G_2 can be obtained from the graph G_1 by an *edge slide* if there exists distinct vertices u, v, and w such that $uv, vw \in E(G_1)$, $uw \notin E(G_1)$ and $G_2 = G_1 - uv + uw$. Thus the graph G_2 can be obtained from the graph G_1 by "sliding" the edge uv along the edge vw and into the edge uw. Recall that in Figure 1.1, the graph $G_2 = G_1 - xy + xz$ was obtained from the graph G_1 by an edge rotation and, in fact, since $yz \in E(G_1)$, we see that G_2 can be obtained from G_1 by an edge slide as well.

More generally a graph G_1 can be *s-transformed* into a graph G_2 if there exists a sequence $G_1 = H_0, H_1, \ldots, H_n = G_2$ $(n \geq 0)$ of graphs such that for $i = 1, 2, \ldots, n$, the graph H_i can be obtained from the graph H_{i-1} by an edge slide. In [17] it is shown that edge slides preserve connectedness, that is, if G_1 is connected and G_2 can be obtained from G_1 by an edge slide, then G_2 is connected. The converse of the previous statement is also true, as was noted in [17].

Theorem B If G_1 and G_2 are two connected graphs of the same order and same size, then G_1 can be s-transformed into G_2.

As a consequence of Theorem B, two graphs of the same order and same size can be s-transformed one into the other if and only if they have the same number of components, and corresponding components have the same order and same size. The *edge slide distance* $d_s(G_1, G_2)$ between two graphs G_1 and G_2 having the same number of components and corresponding components having the same order and same size is defined as the smallest nonnegative integer n for which there exists a sequence $G_1 = H_0, H_1, \dots, H_n = G_2$ of graphs such that for $i = 1, 2, \dots, n$, the graph H_i can be obtained from the graph H_{i-1} by an edge slide. This distance is a metric, and since an edge slide is an edge rotation, it immediately follows that $d_r(G_1, G_2) \le d_s(G_1, G_2)$ for every pair of graphs for which the edge slide distance is defined. Jarrett has shown in [15] that these parameters can be chosen arbitrarily, that is, for positive integers m and n with $m \le n$, there exist graphs G_1 and G_2 such that $d_r(G_1, G_2) = m$ while $d_s(G_1, G_2) = n$. In [2] the edge slide distance is determined for various pairs of graphs. For a graph G, the *girth* $g(G)$ of G is the length of a shortest cycle. For every tree T and every connected unicyclic graph U of order n,

(i) $d_s(T, P_n) = (n-1) - \operatorname{diam} T$,

(ii) $d_s(T, K_{1,n-1}) = n - 1 - \Delta(T)$, and

(iii) $d_s(U, C_n) = n - g(U)$.

Another metric for graphs was introduced in [2]. For graphs G_1 and G_2 of the same order and same size, the graph G_2 can be obtained from G_1 by an *edge move* if there exist (not necessarily distinct) vertices u, v, w, and x such that $uv \in E(G_1)$, $wx \notin E(G_1)$ and $G_2 = G_1 - uv + wx$. Thus an edge move is an unrestricted transfer of an edge from one graph to another. Clearly, if G_2 can be obtained from

G_1 by an edge move, say $G_2 = G_1 - uv + wx$, and u, v, w, and x are not distinct (say $u = w$), then G_2 can be obtained from G_1 by an edge rotation. Furthermore if v and x are adjacent, then G_2 can be obtained from G_1 by an edge slide. More generally, a graph G_1 can be *m-transformed* into a graph G_2 if there exists a sequence $G_1 = H_0, H_1, \dots, H_n = G_2$ $(n \geq 0)$ of graphs such that for $i = 1, 2, \dots, n$, the graph H_i can be obtained from H_{i-1} by an edge move. Thus if G_1 and G_2 are two graphs of the same order and same size, then G_1 can be m-transformed into G_2. The *edge move distance* $d_m(G_1, G_2)$ between two graphs G_1 and G_2 having the same order and same size is defined as the minimum number of edge moves required to m-transform G_1 into G_2. This distance is a metric and for two graphs G_1 and G_2 for which $d_s(G_1, G_2)$ is defined,

$$d_m(G_1, G_2) \leq d_r(G_1, G_2) \leq d_s(G_1, G_2).$$

Also in [2], it is shown that if $d_m(G_1, G_2) = 1$ for two graphs G_1 and G_2, then $d_r(G_1, G_2) \leq 2$. Although $d_m(G_1, G_2) \leq d_r(G_1, G_2) \leq d_s(G_1, G_2)$ for every pair G_1, G_2 of graphs for which $d_s(G_1, G_2)$ is defined, there exist graphs for which equality holds. For every tree T of order n, it is shown in [2] that $d_m(T, K_{1,n-1}) = d_r(T, K_{1,n-1}) = d_s(T, K_{1,n-1}) = n - 1 - \Delta(T)$. In fact, the edge move distance between every two graphs of the same order and same size is determined in [2]. For graphs G_1 and G_2 of order p and size q, where the size of a greatest common subgraph of G_1 and G_2 is s, then $d_m(G_1, G_2) = q - s$.

We have already noted conditions under which an edge move is also an edge rotation or an edge slide. Suppose that G_2 can be obtained from G_1 by an edge move, say $G_2 = G_1 - uv + wx$, where u, v, w, and x are distinct. Then this transformation is neither an edge rotation nor an edge slide and is called an edge jump in [4]. More formally, a graph G_2 can be obtained from a graph G_1 by an *edge jump*

if G_1 contains distinct vertices u, v, w, and x such that $uv \in E(G_1)$, $wx \notin E(G_1)$ and $G_2 = G_1 - uv + wx$. Recall that for the graphs G_1 and G_2 of Figure 1.1, the graph G_2 can be obtained from G_1 by an edge rotation, an edge slide, or an edge move. It turns out, however, that G_2 cannot be obtained from G_1 by an edge jump. Consider the graphs G_1 and G_2 of Figure 1.2. Since $G_2 = G_1 - uv + wx$ for distinct vertices u, v, w, and x, it follows that G_2 can be obtained from G_1 by an edge jump.

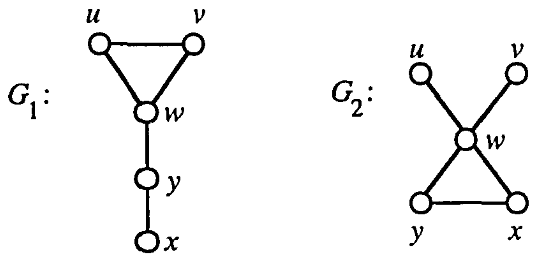

Figure 1.2 The Graph G_2 Can Be Obtained From G_1 by an Edge Jump.

For graphs G_1 and G_2 of the same order and same size, if G_2 can be obtained from G_1 by a sequence of edge jumps, then G_1 can be *j-transformed* into G_2. We have already seen conditions under which two graphs of the same order and same size can be m-transformed, r-transformed, or s-transformed one into the other. In [4] conditions are given under which two graphs may be j-transformed one into the other.

Theorem C If G_1 and G_2 are two graphs of the same order (at least 5) and same size, then G_1 can be j-transformed into G_2.

We have already defined the edge rotation, edge slide, and edge move distance and hence it is natural to define the jump distance between two graphs. For graphs G_1 and G_2 of order $p \geq 5$ and size q, the *jump distance* $d_j(G_1, G_2)$ is defined as the

minimum number of edge jumps needed to j-transform G_1 into G_2. Clearly, this distance is well-defined and is a metric on the space of all graphs of a fixed order (at least 5) and fixed size. Also the move distance and jump distance are related in the following way: if $d_m(G_1, G_2) = 1$ for two graphs G_1 and G_2 of order at least 5, and size q, then $d_j(G_1, G_2) \leq 2$. Hence, if the size of the greatest common subgraph of G_1 and G_2 is s, then we have seen that $d_m(G_1, G_2) = q - s$ and thus $d_j(G_1, G_2) \leq 2(q - s)$. The rotation and jump distances are related in [4] where it is shown that each distance is at most twice the other, that is, for every two graphs G_1 and G_2 of the same order (at least 5) and same size,

$$d_r(G_1, G_2) \leq 2\, d_j(G_1, G_2) \quad \text{and} \quad d_j(G_1, G_2) \leq 2\, d_r(G_1, G_2)$$

or

$$\tfrac{1}{2}\, d_j(G_1, G_2) \leq d_r(G_1, G_2) \leq 2\, d_j(G_1, G_2)$$

and hence

$$\tfrac{1}{2}\, d_r(G_1, G_2) \leq d_j(G_1, G_2) \leq 2\, d_r(G_1, G_2).$$

These bounds are sharp as seen in [4], where it is shown that for positive integers a and b with $a/2 \leq b \leq 2a$, there exist graphs G_1 and G_2 of the same order and same size such that $d_j(G_1, G_2) = a$ and $d_r(G_1, G_2) = b$.

We have already seen that metrics defined on graphs may be applied to problems in chemistry (see [1] or [18] for example). Such applications suggest the problem of selecting an appropriate metric. The selection cannot be based on topological properties because each of our metrics induces the discrete topology on its respective domain. However, these metrics can be differentiated. Let $\mathcal{S}$ denote a fixed set of graphs. The *distance graph* $D_d(\mathcal{S})$ with respect to a metric d on $\mathcal{S}$ is defined as that graph with vertex set $\mathcal{S}$ such that vertices G_1 and G_2 are adjacent in $D_d(\mathcal{S})$ if and only if $d(G_1, G_2) = 1$. If $d = d_r$, the rotation distance, then $D_d(\mathcal{S})$ is denoted by $D_r(\mathcal{S})$ and called the *edge rotation distance graph*. Similarly if $d = d_s$ or $d = d_j$, then

$D_d(\mathcal{S})$ is denoted by $D_s(\mathcal{S})$ or $D_j(\mathcal{S})$ and called the *edge slide distance graph* or *jump distance graph*, respectively. This question is first considered in [2] and [17], where it is shown in [2] that there exists a set $\mathcal{S}$ of graphs for which $D_r(\mathcal{S}) = D_s(\mathcal{S}) = K_n$.

Every graph is an edge slide distance graph as seen in [3], that is, for every graph G, there exists a set $\mathcal{S}$ of graphs of the same order and same size such that $D_s(\mathcal{S}) = G$. Also in [3] many classes of graphs are known to be edge rotation graphs including complete graphs, cycles, paths, unions and cartesian products of edge rotation graphs, line graphs, and trees. Indeed, it is conjectured in [3] that every graph is an edge rotation distance graph. In [13] it is shown that $K_{3,3}$ is an edge rotation distance graph and that for each positive integer n, the graph $K_{n,2}$ is an edge rotation distance graph. Jarrett extends this result in [15] and shows that all complete bipartite graphs are edge rotation graphs. Many graphs are known to be jump distance graphs as well. In [4], it is shown that complete graphs, complete multipartite graphs, trees, cycles, and complements of line graphs are all jump distance graphs and conjectured that every graph is a jump distance graph.

1.3 Some Results on Greatest Common Subgraphs and Other Related Concepts

A *greatest common subgraph* of a set $\mathcal{G} = \{G_1, G_2, \ldots, G_n\}$ $(n \geq 2)$ of graphs of the same size is a graph G of maximum size without isolated vertices that is a subgraph of each G_i $(1 \leq i \leq n)$. The set of all greatest common subgraphs of $\mathcal{G}$ is denoted by gcs $\mathcal{G}$ or gcs$(G_1, G_2, \ldots, G_n)$. We have already seen how the notion of a greatest common subgraph can be used to describe an upper bound for the edge rotation distance between two graphs. In fact, determining the size of a greatest common subgraph of two graphs with the same order and same size is equivalent to determining the edge move distance between them. Since graphs represent discrete structures in a natural way, greatest common subgraphs represent maximum common

substructures. In [5], four areas where this concept arises are (1) in the context of algorithmically perceiving the structural features that are preserved in a chemical reaction, (2) determining maximum commonalities between molecular structures, (3) developing a metric for studying the relationships between molecular structures and chemical properties, and (4) applying the concept of distance between graphs to object recognition.

In the theory of greatest common subgraphs, a natural question arises: Which graphs are greatest common subgraphs? This question was answered in [10], where it is shown that for every graph G of size q without isolated vertices, there exist graphs G_1 and G_2 of size $q + 1$ such that G is the unique greatest common subgraph of G_1 and G_2. Further, this result has been extended and it is shown [10] that for every graph G with isolated vertices, there exist graphs G_1, G_2, and G_3 such that $\text{gcs}(G_1, G_2, G_3) = \{G\}$. However, not every pair H_1, H_2 of graphs of equal size can be the set of greatest common subgraphs. In [10], it was shown that for every pair G_1, G_2 of graphs of equal size, $\text{gcs}(G_1, G_2) \neq \{K_{1,6}, K_4\}$. Although one cannot always specify the set of greatest common subgraphs, the number of graphs in $\mathcal{G}$ and the number of graphs in the set of greatest common subgraphs can be specified. In [10], it is shown that for every pair m, n of integers with $m \geq 2$ and $n \geq 1$, there exists a set $\mathcal{G}$ of m (pairwise nonisomorphic) graphs of equal size such that $|\text{gcs}\,\mathcal{G}| = n$.

We have already noted that for every graph G without isolated vertices, there exist graphs G_1 and G_2 such that $\text{gcs}(G_1, G_2) = \{G\}$ and, further, there exist graphs G_1, G_2, and G_3 such that $\text{gcs}(G_1, G_2, G_3) = \{G\}$. This suggests the following question: For a given graph G without isolated vertices and a given integer $n \geq 2$, does there exist a set $\mathcal{G} = \{G_1, G_2, \ldots, G_n\}$ of n graphs of equal size such that $\text{gcs}\,\mathcal{G} = \{G\}$? Certainly, if n is large, then the graphs $G_1, G_2, \ldots, G_n$ of $\mathcal{G}$

must have large size as well. For a graph G without isolated vertices, the *greatest common subgraph index* or *gcs index* of G, denoted $i(G)$, is the least positive integer q_0 such that for any integer $q > q_0$ and any set $G = \{G_1, G_2, \ldots, G_n\}$ $(n \geq 2)$ of graphs of size q for which $G \in$ gcs G, it follows that $|\text{gcs } G| > 1$, that is, the graphs $G_1, G_2, \ldots, G_n$ have at least two greatest common subgraphs. If no such q_0 exists, then $i(G) = \infty$.

A lower bound for the gcs index was established in [20], where it is shown that if G is a noncomplete graph of order p without isolated vertices, then $i(G) \geq \binom{p}{2}$. This bound is sharp but can be improved if the graph has no end-vertices [20], that is, if G is a graph of order p for which $\delta(G) \geq 2$, then $i(G) \geq \binom{p+1}{2}$. Furthermore, this bound is sharp as well. There do exist graphs for which the gcs index is finite. In fact, in [8] it is shown that $i(K_3) = 6$ and $i(P_4) = 6$. It is also known [20] that there exist graphs of arbitrarily large (but finite) gcs index. A necessary condition is known [20] as well for a graph to have infinite gcs index, namely, if G is a graph with a vertex v of maximum degree such that no component of $G - v$ is isomorphic to K_2, then $i(G) = \infty$. As a consequence of this result, complete graphs of order n $(\neq 3)$, complete bipartite graphs, cycles of order at least 4, and paths of order n $(\neq 4)$ all have infinite gcs index. Another consequence of this result is that all 2–connected graphs of order at least 4 have infinite gcs index. It is well known that for each fixed positive integer n, almost every graph is n–connected, and hence almost every graph has infinite gcs index. For graphs with finite gcs index, it is conjectured in [20] that if G is a graph for which $i(G)$ is finite, then $i(G) = \binom{n}{2}$ for some integer $n \geq 4$.

Greatest common subgraphs with specified properties have also been studied. We have already noted that for every graph G without isolated vertices, there exist graphs G_1 and G_2 of equal size such that G is the unique greatest common subgraph of G_1 and G_2. In the proof [10] of this result, one of G_1 and G_2 is

disconnected. In [6] it was shown that G, G_1, and G_2 can be required to be connected, that is, for every connected noncomplete graph G, there exist connected graphs G_1 and G_2 such that $\text{gcs}(G_1, G_2) = \{G\}$.

A *greatest common induced subgraph* of two graphs G_1 and G_2 of equal size is a graph G of maximum size that is an induced subgraph of each of G_1 and G_2. The set of all greatest common induced subgraphs of G_1 and G_2 is denoted by $\text{gcis}(G_1, G_2)$. It turns out that *every* graph without isolated vertices is the unique greatest common induced subgraph of two nonisomorphic graphs of the same size. Furthermore, in [6], if G is a nontrivial connected graph, then there exist connected graphs G_1 and G_2 of equal size such that $\text{gcis}(G_1, G_2) = \{G\}$. These concepts have also been studied for digraphs in [6].

Thus far we have been considering special cases of a more general type of problem: For a given graph G with a specified graphical property P, do there exist graphs G_1 and G_2 of equal size with property P such that G is the unique greatest common subgraph of G_1 and G_2? We now consider various properties P. For example if P is the property "2-connected", then the following is true (see [8]). For a 2-connected graph G of order p, where $G \ncong K_p$ $(p \geq 3)$ and $G \ncong K_p - e$ $(p \geq 4)$ for some edge e of K_p, there exist 2–connected graphs G_1 and G_2 of equal size such that $\text{gcs}(G_1, G_2) = \{G\}$. The situation in general is not known for n-connected graphs. However, for n-chromatic graphs $(n \geq 2)$, the situation is much different. In [8], if G is an n-chromatic graph $(n \geq 2)$ without isolated vertices, then there exist n-chromatic graphs G_1 and G_2 of equal size such that $\text{gcs}(G_1, G_2) = \{G\}$.

We now consider the case when P is the property "is a tree". First, for an integer $t \geq 2$, let $D(t)$ denote that tree obtained by connecting the centers of two copies of $K_{1,t}$ by a path of length 3. Not every tree is the unique greatest common subgraph of two nonismorphic trees. In fact, it is shown in [11] that for a nontrivial

tree T, there exist trees T_1 and T_2 of equal size such that $\text{gcs}(T_1, T_2) = \{T\}$ if and only if T is not a path of order 2, 4, 5, ... and $T \ncong D(t)$ for $t \geq 2$. The situation is much different for greatest common induced subgraphs, where in [11] it is shown that every tree of order at least 3 is the unique greatest common subgraph of two nonisomorphic trees of the same order. The properties "connected outerplanar" and "connected planar" have also been studied (see [5]).

CHAPTER II

H-CONNECTED GRAPHS

2.1 Introduction

Let G_1 and G_2 be two graphs of the same order and same size such that $V(G_1) = V(G_2)$, and let H be a connected graph of order at least 3. Two subgraphs H_1 and H_2 of G_1 and G_2, respectively, are *H-adjacent* if $H_1 \cong H_2 \cong H$ and H_1 and H_2 share some but not all edges, that is, $E(H_1) \cap E(H_2) \neq \varnothing$ and $E(H_2) - E(H_1) \neq \varnothing$ (so also $E(H_1) - E(H_2) \neq \varnothing$). The *graphs* G_1 and G_2 *are* themselves *H-adjacent* if G_1 and G_2 contain H-adjacent subgraphs H_1 and H_2, respectively, such that $E(H_2) - E(H_1) \subseteq E(\overline{G_1})$ and $G_2 = G_1 - E(H_1) + E(H_2)$. A G_1-G_2 *H-walk* is a sequence $G_1 = F_0, F_1, \ldots, F_k = G_2$ of graphs of the same order and same size such that F_i is H-adjacent to F_{i+1} for $i = 0, 1, \ldots, k-1$. A G_1-G_2 H-walk $G_1 = F_0, F_1, \ldots, F_k = G_2$ in which the graphs $F_0, F_1, \ldots, F_k$ are distinct is called a G_1-G_2 *H-path.* The graphs G_1 and G_2 are *H-connected* if there exists a G_1-G_2 H-path. The relation H-connectedness is an equivalence relation on the set of all graphs of the same order and same size.

Let $H = P_3$. In Figure 2.1, the path H_1: u, v, w of G_1 is H-adjacent to the path H_2: v, w, u of G_2, and in fact, since $G_2 = G_1 - E(H_1) + E(H_2)$, the graphs G_1 and G_2 are H-adjacent. The path H_2': w, x, y of the graph G_2 is H-adjacent to the path H_3: x, w, y of the graph G_3, shown in Figure 2.2, and thus since $G_3 = G_2 - E(H_2') + E(H_3)$, the graphs G_2 and G_3 are H-adjacent. Clearly, G_1 is not H-adjacent to G_3 but since G_1, G_2, G_3 is an H-path from G_1 to G_3, the graph G_1 is H-connected to G_3.

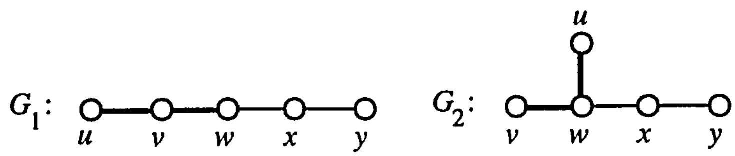

Figure 2.1 Two P_3-Adjacent Graphs.

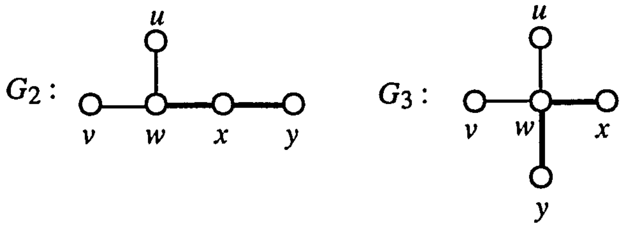

Figure 2.2 Two Other P_3-Adjacent Graphs.

2.2 P_3-Connected Graphs

Suppose again that $H = P_3$ and that G_1 and G_2 are H-adjacent graphs. Then G_1 contains a copy H_1 of P_3, say H_1: u, v, w and G_2 contains a copy H_2 of P_3 with $E(H_1) \cap E(H_2) \neq \emptyset$ and $E(H_2) - E(H_1) \subseteq E(\overline{G_1})$. Since $E(H_1) \subseteq E(G_1)$ and H_1 has exactly two edges, it follows that H_1 and H_2 have exactly one edge in common, say uv, and H_2 contains exactly one edge that is not present in G_1. So H_2: u, v, z or H_2: z, u, v for some $z \in V(G_2)$. Thus (1) $G_2 = G_1 - vw + vz$ or (2) $G_2 = G_1 - vw + uz$, where u, v, w, and z are not necessarily distinct. The graph G_2 is said to be obtained from G_1 by an *edge move* if G_1 contains (not necessarily distinct) vertices u, v, w, and x such that $uv \in E(G_1)$, $wx \notin E(G_1)$, and $G_2 = G_1 - uv + wx$. Thus, if the graphs G_1 and G_2 are P_3-adjacent, then G_2 can be obtained from G_1 by an edge move. It was shown in [2] that for every two

graphs of the same order and same size, each can be transformed into the other by a sequence of edge moves.

The graph G_2 is said to be obtained from G_1 by an *edge rotation* if G_1 contains distinct vertices $u, v,$ and w such that $uv \in E(G_1)$, $uw \notin E(G_1)$ and $G_2 = G_1 - uv + uw$. Thus if the graphs G_1 and G_2 are P_3-adjacent and $G_2 = G_1 - vw + vz$, which is case (1), then G_2 can be obtained from G_1 by an edge rotation. In fact, in this case, P_3-adjacency is more restrictive than edge-rotation since P_3-adjacency requires the presence of another edge incident to v. In [9] it was shown that for every two graphs of the same order and same size, each can be transformed into the other by a sequence of edge rotations. The graph G_2 is obtained from G_1 by an *edge jump* if G_1 contains four distinct vertices $u, v, w,$ and x such that $uv \in E(G_1)$, $wx \notin E(G_2)$ and $G_2 = G_1 - uv + wx$. Thus if G_1 and G_2 are P_3-adjacent with $G_2 = G_1 - uv + wx$ and $u, v, w,$ and x are distinct vertices as in case (2), then G_2 is obtained from G_1 by an edge jump. In [4] it was shown that every two graphs of the same order (at least 5) and same size can be transformed into one another by a sequence of edge jumps. We now present a corresponding result for P_3-adjacency.

Theorem 2.1 Let G_1 and G_2 be two graphs of the same order and same size such that $G_2 = G_1 - uv + wx$ for (not necessarily distinct) vertices $u, v, w,$ and x, where the edges uv and wx belong to components of order at least 3 of G_1 and G_2, respectively. Then G_1 and G_2 are P_3-connected.

Proof We consider two cases, according to whether the vertices $u, v, w,$ and x are distinct.

Case 1 *Suppose that the vertices* u, v, w, *and* x *are not distinct, say* $v = w$. Now since the edge uv belongs to a component of order at least 3 of G_1, it follows that there exists an edge adjacent to uv; so either $\deg_{G_1} u > 1$ or $\deg_{G_1} v > 1$. Suppose first that $\deg_{G_1} v > 1$. Then there exists a vertex y $(\neq u)$ of G_1 adjacent to v, and hence $G_2 = G_1 - \{uv, vy\} + \{yv, vx\}$. Therefore G_1 and G_2 are P_3-adjacent.

Thus we assume that $\deg_{G_1} v = 1$. Hence $\deg_{G_1} u > 1$ and $\deg_{G_2} v = 1$. Since uv and vx belong to a component of order at least 3 in G_1 and G_2, respectively, and $\deg_{G_2} v = 1$, it follows that there exist vertices z and y (distinct from v) such that $uz \in E(G_1)$ and $xy \in E(G_2)$. First, suppose that $z = x$. Then $G_2 = G_1 - \{xu, uv\} + \{vx, xu\}$; so G_1 is P_3-adjacent to G_2. Thus we may assume that $z \neq x$ and that u is not adjacent to x. If $y = z$, then let $F_1 = G_1 - \{vu, uz\} + \{xu, uz\}$; so F_1 and G_1 are P_3-adjacent. Next $G_2 = F_1 - \{ux, xz\} + \{vx, xz\}$; so F_1 is P_3-adjacent to G_2, and thus G_1 and G_2 are P_3-connected. This situation is shown in Figure 2.3, where the graphs H_0, H_1, and H_2 are subgraphs of G_1, F_1, and G_2, respectively. Therefore, we may now assume that every vertex z adjacent to u is not adjacent to x. Let $F_1 = G_1 - \{vu, uz\} + \{uz, zx\}$; so G_1 and F_1 are P_3-adjacent. Then $G_2 = F_1 - \{zx, xy\} + \{vx, xy\}$; so G_2 is P_3-adjacent to F_1, and thus G_1 and G_2 are P_3-connected. This transformation is shown in Figure 2.3, where the graphs J_0, J_1, and J_2 are subgraphs of G_1, F_1, and G_2, respectively.

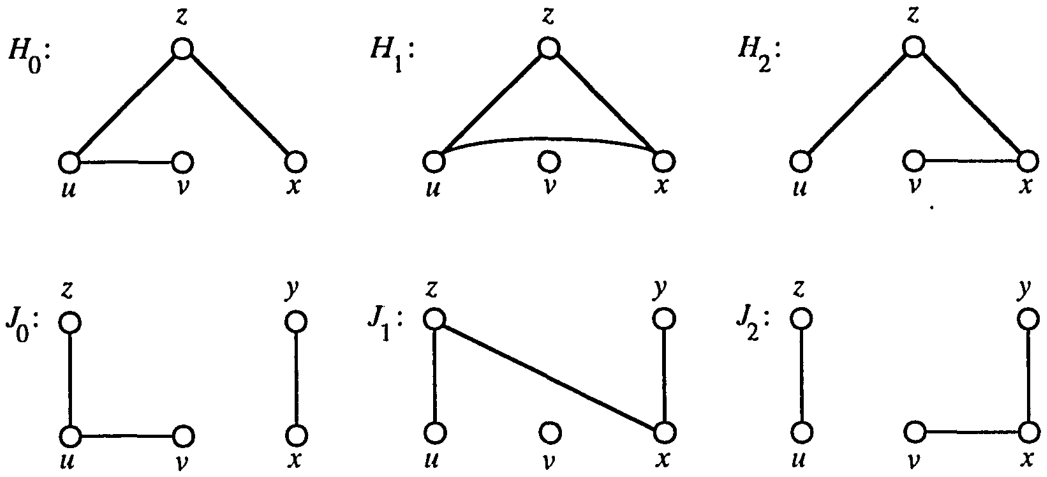

Figure 2.3 A P_3-Path From G_1 to G_2 Where $\deg_{G_1} u \geq 2$ and $\deg_{G_1} v = 1$.

Case 2 *Suppose that the vertices* u, v, w, *and* x *are distinct.* Suppose first that one of the edges uw, ux, vw, and vx is not present in G_1 (and hence in G_2), say $e \notin E(G_1)$, where $e \in \{uw, ux, vw, vx\}$. Then by Case 1, G_1 is P_3-connected to $F_1 = G_1 - uv + e$, and F_1 is P_3-connected to $G_2 = F_1 - e + wx$. Hence G_1 and G_2 are P_3-connected, and therefore we may assume that all of the edges uw, ux, vw, and vx are present in G_1. Then $G_2 = G_1 - \{vu, uw\} + \{uw, wx\}$; so G_1 and G_2 are P_3-adjacent. ❑

The *distance between an edge* $e = uv$ *and a subgraph* H of a graph G is defined by $d(e, H) = \min \{d(u, x), d(v, x) \mid x \in V(H)\}$. We have seen conditions under which two graphs that differ in exactly one edge are P_3-connected. We now determine conditions under which any two graphs are P_3-connected.

Theorem 2.2 Let G_1 and G_2 be two graphs of the same order and the same size. Then G_1 and G_2 are P_3-connected if and only if each of G_1 and G_2 contains a subgraph isomorphic to P_3.

Proof Let H: u, v, w denote a common subgraph of G_1 and G_2 that is isomorphic to P_3. Suppose that G_1 has s edges not belonging to G_2. We label these edges so that the edges $e_1, e_2, \ldots, e_k$ $(0 \le k \le s)$ are adjacent to an edge of H, the edges $e_{k+1}, e_{k+2}, \ldots, e_\ell$ $(k \le \ell \le s)$ are at distance 1 from H, and $e_{\ell+1}$, $e_{\ell+2}, \ldots, e_s$ are the remaining edges. It may be that there are no edges adjacent to an edge of H or at distance 1 from H, and in that case, we take $k = 0$ or $\ell = 0$ as needed. Let $e_i = u_i v_i$ for $i = \ell + 1, \ell + 2, \ldots, s$, and observe that by the choice of k and ℓ, none of $e_{\ell+1}, e_{\ell+2}, \ldots, e_s$ is adjacent to uv or vw, and thus the edges $u_{\ell+1}v, u_{\ell+2}v, \ldots, u_s v$ are not present in G_1. We now define the graphs H_0, H_1, $\ldots, H_{2(s-\ell)}$ recursively. Let $H_0 = G_1$, and for $i = 1, 2, \ldots, s - \ell$, let

$$H_{2i-1} = H_{2(i-1)} - \{uv, vw\} + \{uv, vu_{\ell+i}\}$$

and let

$$H_{2i} = H_{2i-1} - \{v_{\ell+i}u_{\ell+i}, u_{\ell+i}v\} + \{u_{\ell+i}v, vw\}.$$

So, for $i = 0, 1, \ldots, 2(s - \ell) - 1$, the graph H_i is P_3-adjacent to H_{i+1}. Thus G_1, $H_1, H_2, \ldots, H_{2(s-\ell)} = F_1$ is a P_3-path from G_1 to F_1; so G_1 is P_3-connected to F_1. Observe that F_1 differs from G_2 by the edges $e_1, e_2, \ldots, e_\ell, vu_{\ell+1}, vu_{\ell+2}, \ldots, vu_s$.

The graph G_2 has s edges not belonging to G_1. Label these edges so that the edges $f_1, f_2, \ldots, f_m$ $(0 \le m \le s)$ are adjacent to an edge of H, the edges f_{m+1}, $f_{m+2}, \ldots, f_n$ $(m \le n \le s)$ are at distance 1 from H, and $f_{n+1}, f_{n+2}, \ldots, f_s$ are the remaining edges. Let $f_i = w_i x_i$ for $i = n + 1, n + 2, \ldots, s$. The graph F_2 is obtained from G_2 exactly as F_1 was obtained from G_1; that is, define $J_0 = G_2$ and for $i = 1, 2, \ldots, s - m$, define

$$J_{2i-1} = J_{2(i-1)} - \{uv, vw\} + \{uv, vw_{m+i}\}$$

and

$$J_{2i} = J_{2i-1} - \{x_{m+i}w_{m+i}, w_{m+i}v\} + \{w_{m+i}v, vw\}.$$

As before, for $i = 0, 1, \dots, 2(s-m)-1$, the graph J_i is P_3-adjacent to J_{i+1}. Then $G_2 = J_0, J_1, \dots, J_{2(s-m)} = F_2$ is a P_3-path from G_2 to F_2; so G_2 is P_3-connected to F_2. Therefore F_2 differs from G_1 by the edges $f_1, f_2, \dots, f_n, vw_{n+1}, vw_{n+2}, \dots, vw_s$. Consequently, G_1 is P_3-connected to F_1, and F_2 is P_3-connected to G_2. Hence, if F_1 and F_2 are P_3-connected, then G_1 and G_2 are P_3-connected. It remains to show that F_1 and F_2 are P_3-connected.

Let H_1 be a greatest common subgraph of F_1 and F_2 containing H. So F_1 contains the edges $a_1, a_2, \dots, a_r$ not belonging to F_2, and F_2 contains the edges $b_1, b_2, \dots, b_r$ not belonging to F_1, where

$$\{a_1, a_2, \dots, a_r\} \subseteq \{e_1, e_2, \dots, e_\ell, vu_{\ell+1}, vu_{\ell+2}, \dots, vu_s\}$$

and

$$\{b_1, b_2, \dots, b_r\} \subseteq \{f_1, f_2, \dots, f_n, vw_{n+1}, vw_{n+2}, \dots, vw_s\}.$$

Also, each of $a_1, a_2, \dots, a_r, b_1, b_2, \dots, b_r$ lies on a path of length 2. Furthermore, by the construction of F_1 and F_2, each of a_i and b_i $(1 \le i \le r)$ is adjacent to an edge of H (and hence of H_1) or at distance 1 from H (and thus from H_1). Label the edges $a_1, a_2, \dots, a_r$ so that $a_1, a_2, \dots, a_c$ $(0 \le c \le r)$ are at distance 1 from H and $a_{c+1}, a_{c+2}, \dots, a_r$ are adjacent to an edge of H. Next label the edges $b_1, b_2, \dots, b_r$ so that $b_1, b_2, \dots, b_d$ $(0 \le d \le r)$ are adjacent to an edge of H and $b_{d+1}, b_{d+2}, \dots, b_r$ are at distance 1 from H. Each a_i $(1 \le i \le c)$ must be adjacent to some a_j for $c+1 \le j \le r$ and, similarly, each b_j $(d+1 \le j \le r)$ must be adjacent to some b_i for $1 \le i \le d$. Thus we must remove $a_1, a_2, \dots, a_c$ from F_1 before removing $a_{c+1}, a_{c+2}, \dots, a_r$, and the edges $b_1, b_2, \dots, b_d$ must be present before the edges $b_{d+1}, b_{d+2}, \dots, b_r$ can be added. Let $J_0 = F_1$ and then for $i = 1, 2, \dots, r$, the

graph $J_i = J_{i-1} - a_i + b_i$ is P_3-connected to J_{i-1} by Theorem 2.1. Since $F_2 = J_r$, we have that F_1 is P_3-connected to F_2, and hence G_1 is P_3-connected to G_2.

Finally, assume that G_1 is P_3-connected to G_2. Then there exists a G_1-G_2 P_3-path, say $G_1 = F_0, F_1, F_2, \ldots, F_n = G_2$. Since G_1 is P_3-adjacent to F_1, the graph $F_1 = G_1 - E(H_1) + E(H_2)$, where $H_1 \cong H_2 \cong P_3$ and $E(H_1) \subseteq E(G_1)$. Thus, G_1 contains a subgraph, namely H_1, that is isomorphic to P_3. Similarly, since F_{n-1} is P_3-adjacent to G_2, it follows that G_2 contains a subgraph that is isomorphic to P_3.

❑

2.3 P_4-Connected Graphs

Next we show that every two connected graphs of the same order and same size are P_4-connected. First, the following lemma will be useful.

Lemma 2.3 Let G be a connected graph containing a connected graph H as a subgraph. Then every edge of G belongs to a subgraph that is isomorphic to H if and only if H is K_2, P_3, or P_4.

Proof Suppose that every edge of G belongs to a subgraph of G that is isomorphic to H. We show that $H = K_2$, P_3, or P_4. Suppose, to the contrary, that $\Delta(H) \geq 3$ and let k be the diameter of H. Obtain the graph G from H by joining a path of length $k + 1$ to a vertex of H. Then G contains H as a subgraph, yet clearly not every edge of G belongs to a subgraph isomorphic to H, producing a contradiction. Thus $\Delta(H) \leq 2$, and H is a cycle or a path. Suppose first that H is a cycle, say $H = C_n$ for some positive integer $n \geq 3$. Let G be obtained from H by joining a new vertex x to a vertex v of H. As before, G contains H as a subgraph yet the edge xv of G does not belong to a subgraph isomorphic to H, producing a

contradiction. Hence H is a path, say $H = P_n$ for some integer $n \geq 2$. Suppose, to the contrary, that $n \geq 5$. Consider the graph G obtained from the path P: $v_1, v_2, \ldots, v_n$ by joining a new vertex x to v_3. So G contains P_n as subgraph yet the edge xv_3 does not belong to a subgraph of G isomorphic to P_n, producing a contradiction. Hence $n \leq 4$, and H is K_2, P_3, or P_4.

The converse is clearly true if $H = K_2$ or $H = P_3$. Thus, suppose that $H = P_4$, and let G be a graph containing P_4 as a subgraph, say P: x_0, x_1, x_2, x_3 is a path of length 3 in G. Clearly, every edge of P lies on a path of length 3. Next, let $e = uv$ be an edge of G that does not belong to P. At most one of u and v can lie on P, suppose first that u lies on P. So without loss of generality, $u = x_0$ or $u = x_1$. Then $v, u = x_0, x_1, x_2$ or $v, u = x_1, x_2, x_3$ is a path of length 3 containing the edge uv. Finally, suppose that neither u nor v lies on P. Let Q: $u_0, u_1, \ldots, u_n$ $(n \geq 1)$ be a shortest path from the edge uv to P. Without loss of generality, assume that $u_0 = u$ and $u_n = x_0$ or $u_n = x_1$. Then $v, u = u_0, u_1, \ldots, u_n = x_0, x_1$ or $v, u = u_0, u_1, \ldots, u_n = x_1, x_0$ is a path of length at least 3, and hence e lies on a path of length 3. ❑

Theorem 2.4 Let G_1 and G_2 be connected graphs of the same order and same size, each of which has a subgraph isomorphic to P_4. If $G_2 = G_1 - uv + wx$ for (not necessarily distinct) vertices u, v, w, and x of G_1, then G_1 and G_2 are P_4-connected.

Proof We consider two cases, depending on whether the vertices u, v, w, and x are distinct.

Case 1 *Suppose that the vertices* u, v, w, *and* x *are not distinct, say* $v = w$. Since G_1 is connected, there exists a shortest u-x path P: $u = u_0, u_1, \dots, u_{n-1}, u_n = x$. Next, we proceed depending on whether $u_1 = v$.

Subcase 1.1 *Suppose that* $u_1 = v$, *that is, suppose that the path* P *contains the edge* uv. Since vx is not an edge of G_1, it follows that $n \geq 3$. Suppose first that $n \geq 4$. Then $G_2 = G_1 - \{uv, vu_2, u_2u_3\} + \{xv, vu_2, u_2u_3\}$ and G_1 is P_4-adjacent to G_2. Now suppose that $n = 3$, that is, suppose that $u, v, u_2 = y, x$ is a shortest u-x path. Then in G_2, the edges vx, xy, yv form a triangle. Since G_2 contains a path of length 3 (a copy of P_4), by Lemma 2.3 the edge vx must lie on a path of length 3. Thus since $uv \notin E(G_2)$ and a shortest path from u to x has length 3, there exists a vertex z of G_2 such that z is adjacent to at least one of v, x, and y in G_2. Suppose first that z is adjacent to y. Then $G_2 = G_1 - \{uv, vy, yz\} + \{xv, vy, yz\}$, and G_1 is P_4-adjacent to G_2. Hence, in what follows, we assume that z is not adjacent to y. Next suppose that z is adjacent to v. Let $F_1 = G_1 - \{zv, vy, yx\} + \{zv, vx, xy\}$; so F_1 is P_4-adjacent to G_2. Then let $F_2 = F_1 - \{uv, vx, xy\} + \{vx, xy, yz\}$; so F_2 is P_4-adjacent to F_1. Thus $G_2 = F_2 - \{vz, zy, yx\} + \{zv, vy, yx\}$; so G_2 and F_2 are P_4-adjacent, and G_1, F_1, F_2, G_2 is a G_1-G_2 P_4-path. This situation is represented in Figure 2.4, where H_0, H_1, H_2, and H_3 are subgraphs of G_1, F_1, F_2, and G_2, respectively.

Finally, suppose that z is adjacent only to x. Let $F_1 = G_1 - \{zx, xy, yv\} + \{zx, xv, vy\}$; so F_1 is P_4-adjacent to G_1. Then let $F_2 = F_1 - \{uv, vx, xz\} + \{vx, xz, zy\}$; so F_2 is P_4-adjacent to F_1. Hence $G_2 = F_2 - \{vy, yz, zx\} + \{vy, yx, xz\}$; so G_2 is P_4-adjacent to F_2, and G_1, F_1, F_2, G_2 is a G_1-G_2 P_4-path. This transformation is represented in Figure 2.5, where the graphs H_0, H_1, H_2, and H_3 are subgraphs of G_1, F_1, F_2, and G_2, respectively.

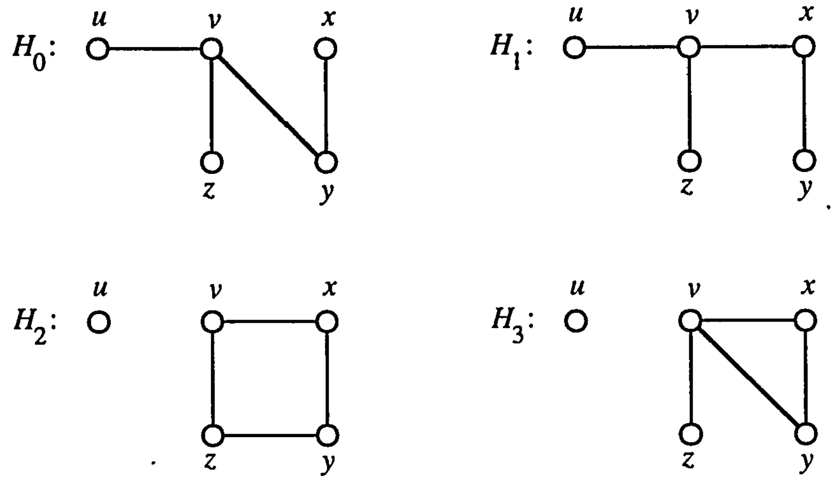

Figure 2.4 A Transformation of G_1 Into G_2 When z Is Adjacent to y.

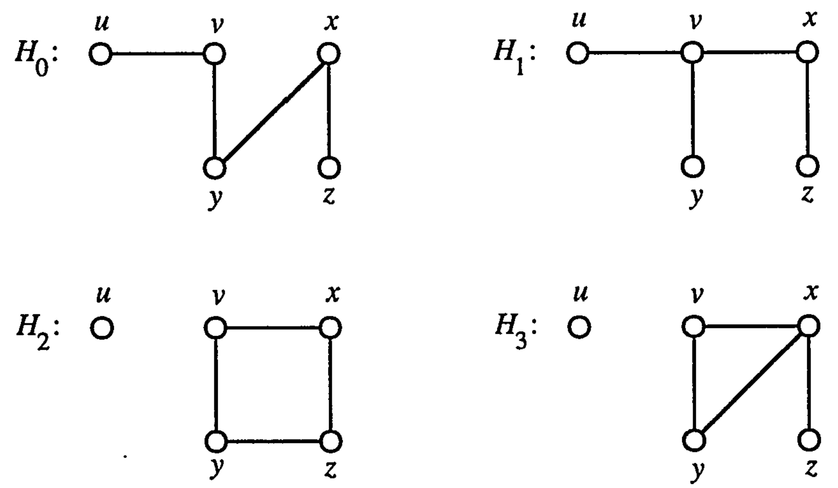

Figure 2.5 A Transformation of G_1 Into G_2 When z Is Adjacent to x.

Subcase 1.2 *Suppose that a shortest u-x path P: $u = u_0, u_1, \ldots, u_{n-1}, u_n = x$ does not contain v.* Suppose first that $n \geq 3$. Let $F_1 = G_1 - \{vu, uu_1, u_1u_2\} + \{u_2u_1, u_1u, ux\}$ so that F_1 is P_4-adjacent to G_1. Then $G_2 = F_1 - \{ux, xu_{n-1}, u_{n-1}u_{n-2}\} + \{vx, xu_{n-1}, u_{n-1}u_{n-2}\}$, and G_2 and F_1 are P_4-adjacent. Thus G_1 and G_2 are P_4-connected. Next suppose that $n = 2$. Then P: u, u_1, x and $G_2 = G_1 - \{vu, uu_1,$

$u_1x\} + \{uu_1, u_1x, xv\}$; so G_2 and G_1 are P_4-adjacent. Finally, suppose that $n = 0$, that is, $ux \in E(G_1)$. If $\deg_{G_1} v \geq 2$, then there exists a vertex z of G_1 adjacent to v and $G_2 = G_1 - \{zv, vu, ux\} + \{zv, vx, xu\}$; so G_1 and G_2 are P_4-adjacent. Thus we assume that $\deg_{G_1} v = 1$. Next suppose that $\deg_{G_1} u \geq 3$ and $\deg_{G_1} x \geq 2$. Then there exist vertices y and z of G_1 such that $uy, xz \in E(G_1)$. Let $F_1 = G_1 - \{yu, ux, xz\} + \{yu, ux, xv\}$; so F_1 and G_1 are P_4-adjacent. Let $F_2 = F_1 - \{yu, uv, vx\} + \{uy, yv, vx\}$; so F_1 and F_2 are P_4-adjacent. Then $G_2 = F_2 - \{vy, yu, ux\} + \{yu, ux, xz\}$; so F_2 and G_2 are P_4-adjacent, and G_1, F_1, F_2, G_2 is a P_4-path. This situation is represented in Figure 2.6, where the graphs H_0, H_1, H_2, and H_3 are subgraphs of G_1, F_1, F_2, and G_2, respectively.

Thus, we may now assume that $\deg_{G_1} v = 1$ and that $\deg_{G_1} u = 2$ or $\deg_{G_1} x = 1$. Suppose first that $\deg_{G_1} u = 2$. Then a path of length 3 containing the edge uv must also contain the edge ux, and there must exist a vertex z adjacent to x in G_1. Then $\deg_{G_2} u = \deg_{G_2} v = 1$. Since the edge vx must also lie on a path of length 3, we may assume, without loss of generality, there exists a vertex y of G_2 adjacent to z. Let $F_1 = G_1 - \{ux, xz, zy\} + \{vx, xz, zy\}$; so G_1 and F_1 are P_4-adjacent. Let $F_2 = F_1 - \{uv, vx, xz\} + \{yv, vx, xz\}$; so F_1 and F_2 are P_4-adjacent. Then $G_2 = F_2 - \{xz, zy, yv\} + \{ux, xz, zy\}$; so F_2 and G_2 are P_4-adjacent, and G_1, F_1, F_2, G_2 is a P_4-path. This transformation is represented in Figure 2.6, where the graphs I_0, I_1, I_2, and I_3 are subgraphs of G_1, F_1, F_2, and G_2, respectively.

Finally, suppose that $\deg_{G_1} x = 1$. As before, the edge uv must lie on a path of length 3. Since $\deg_{G_1} v = \deg_{G_1} x = 1$, it follows that the edges uv and ux are not on a common path of length 3 and thus there exist vertices y and z such that v, u, y, z is a path in G_1. Let $F_1 = G_1 - \{xu, uy, yz\} + \{vx, xu, uy\}$; so F_1 and G_1

are P_4-adjacent. Then let $F_2 = F_1 - \{vx, xu, uy\} + \{uy, yx, xv\}$; so F_1 and F_2 are P_4-adjacent. Finally $G_2 = F_2 - \{vu, uy, yx\} + \{xu, uy, yz\}$; so F_2 and G_2 are P_4-adjacent, and G_1, F_1, F_2, G_2 is a P_4-path. This situation is shown in Figure 2.6, where the graphs J_0, J_1, J_2, and J_3 are subgraphs of G_1, F_1, F_2, and G_2, respectively.

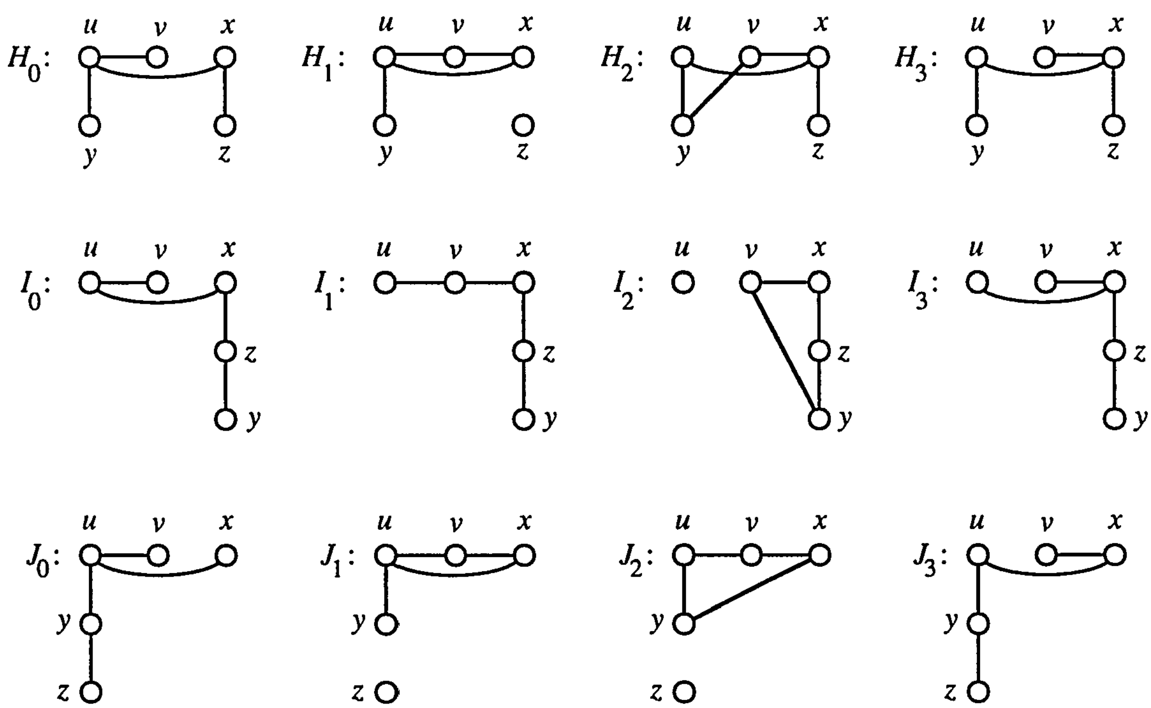

Figure 2.6 A Transformation of G_1 Into G_2 When u Is Adjacent to x.

Case 2 *Assume that the vertices u, v, w, and x are distinct.* Suppose, first, that one of the edges uw, ux, vw, or vx is not present in G_1, say e is such an edge. By Case 1, the graph G_1 is P_4-connected to $F_1 = G_1 - uv + e$, and thus F_1 is P_4-connected to $G_2 = F_1 - e + wx$. Therefore G_1 and G_2 are P_4-connected. Hence all of uw, ux, vw, and vx must be present in G_1. Then $G_2 = G_1 - \{wu, uv, vx\} + \{uw, wx, xv\}$, and the graphs G_1 and G_2 are P_4-adjacent. ❑

The next theorem describes conditions under which two graphs of the same order and same size are P_4-connected. Although the proof is similar to the proof of Theorem 2.2, we include it for completeness.

Thoerem 2.5 Let G_1 and G_2 be two graphs of the same order and the same size. Then G_1 is P_4-connected to G_2 if and only if each of G_1 and G_2 contains a subgraph isomorphic to P_4.

Proof Let H: u, v, w, x denote a common subgraph of G_1 and G_2 that is isomorphic to P_3. Suppose that G_1 has n edges not belonging to G_2. We label these edges so that the edges $e_1, e_2, \ldots, e_k$ $(0 \le k \le n)$ are adjacent to an edge of H (if any such edges exist), the edges $e_{k+1}, e_{k+2}, \ldots, e_\ell$ $(k \le \ell \le n)$ are at distance 1 from H (if any such edges exist), and $e_{\ell+1}, e_{\ell+2}, \ldots, e_n$ are the remaining edges. Let $e_i = u_i v_i$ for $i = \ell + 1, \ell + 2, \ldots, n$ and observe that by the choice of k and ℓ, each of $e_{\ell+1}, e_{\ell+2}, \ldots, e_n$ is not adjacent to uv, vw, or wx. Next define $H_0 = G_1$, and for $i = 1, 2, \ldots, n - \ell$, define

$$H_{2i-1} = H_{2(i-1)} - \{uv, vw, wx\} + \{u_{\ell+i}v, vw, wx\}$$

and

$$H_{2i} = H_{2i-1} - \{v_{\ell+i}u_{\ell+i}, u_{\ell+i}v, vw\} + \{u_{\ell+i}u, uv, vw\}.$$

Thus for $i = 0, 1, \ldots, 2(n - \ell) - 1$ the graphs H_i and H_{i+1} are P_4-adjacent. Let $F_1 = H_{2(n-\ell)}$, and thus $G_1 = H_0, H_1, \ldots, H_{2(n-\ell)} = F_1$ is a G_1-F_1 P_4-path. So, F_1 differs from G_2 by the edges $e_1, e_2, \ldots, e_\ell, uu_{\ell+1}, uu_{\ell+2}, \ldots, uu_n$.

Next G_2 has n edges not belonging to G_1. Label these edges so that the edges $f_1, f_2, \ldots, f_s$ $(0 \le s \le n)$ are adjacent to an edge of H, the edges f_{s+1}, $f_{s+2}, \ldots, f_t$ $(s \le t \le n)$ are at distance 1 from H, and $f_{t+1}, f_{t+2}, \ldots, f_n$ are the remaining edges. Let $f_i = w_i x_i$ for $i = t + 1, t + 2, \ldots, n$. Next define $J_0 = G_2$, and for $i = 1, 2, \ldots, n - t$, define

$$J_{2i-1} = J_{2(i-1)} - \{uv, vw, wx\} + \{w_{\ell+i}v, vw, wx\}$$

and

$$J_{2i} = J_{2i-1} - \{x_{t+i}w_{t+i}, w_{t+i}v, vw\} + \{w_{t+i}u, uv, vw\}.$$

Thus for $i = 0, 1, \dots, 2(n-t) - 1$, the graphs J_i and J_{i+1} are P_4-adjacent. Let $F_2 = J_{2(n-t)}$, and thus $G_2 = J_0, J_1, \dots, J_{2(n-t)} = F_2$ is a P_4-path; so G_2 and F_2 are P_4-connected. Therefore F_2 differs from G_1 by the edges $f_1, f_2, \dots, f_t, uw_{t+1}, uw_{t+2}, \dots, uw_n$.

Let H_1 be a greatest common subgraph of F_1 and F_2 containing H. Furthermore, any edge of F_1 not belonging to F_2 is either at distance 1 from H or is adjacent to an edge of H. Thus F_1 and F_2 differ only by edges in the same component as H. Let $a_1, a_2, \dots, a_r$ denote the edges of F_1 not belonging to F_2. Assume that the edges are labeled so that $a_1, a_2, \dots, a_c$ $(0 \le c \le r)$ are at distance 1 from H and $a_{c+1}, a_{c+2}, \dots, a_r$ are adjacent to an edge of H. Next let $b_1, b_2, \dots, b_r$ denote the edges of F_2 not belonging to F_1 and assume that these edges are labeled so that $b_1, b_2, \dots, b_d$ $(0 \le d \le r)$ are adjacent to an edge of H and $b_{d+1}, b_{d+2}, \dots, b_r$ are at distance 1 from an edge of H. Each a_i $(1 \le i \le c)$ must be adjacent to some a_j $(c + 1 \le j \le r)$ and similarly each b_k $(d + 1 \le k \le r)$ must be adjacent to some b_ℓ $(1 \le \ell \le d)$. Thus we must remove the edges $a_1, a_2, \dots, a_c$ from F_1 before removing the edges $a_{c+1}, a_{c+2}, \dots, a_r$, and the edges $b_1, b_2, \dots, b_d$ must be present in F_2 before the edges $b_{d+1}, b_{d+2}, \dots, b_r$ can be added. Let $J_0 = F_1$, and for $i = 1, 2, \dots, r$, the graph $J_i = J_{i-1} - a_i + b_i$ is P_4-connected to J_{i-1} by Theorem 2.4. Since $F_2 = J_r$, it follows that F_1 is P_4-connected to F_2, and hence G_1 is P_4-connected to G_2.

Finally, assume that G_1 is P_4-connected to G_2. Then there exists a G_1-G_2 P_4-path, say $G_1 = F_0, F_1, \dots, F_n = G_2$. Since G_1 is P_4-adjacent to F_1, there exist

subgraphs H and H' of G_1 and F_1, respectively, such that $F_1 = G_1 - E(H) + E(H')$ where $H \cong H' \cong P_4$. Thus G_1 contains a subgraph isomorphic to P_4. Similarly, since F_{n-1} is P_4-adjacent to G_2, it follows that G_2 contains a subgraph isomorphic to P_4. ❑

2.4 Other H-Connected Graphs

We have seen that if H is P_3 or P_4, then every two graphs of the same order and same size containing H as a subgraph are H-connected. Although we cannot answer the question in general for $H = P_5$, we can show that every two trees of diameter at least 4 are H-connected. In fact, we now show that every tree of diameter d is P_k-connected to a path for each integer k with $3 \le k \le d$. Thus as a corollary, we have that every two trees of diameter at least k are P_k-connected.

Theorem 2.6 If T is a tree of order n and diameter d, then T is P_k-connected to P_n for each positive integer k with $3 \le k \le d$.

Proof Let k be an integer with $3 \le k \le d$, and let P: $v_0, v_1, \ldots, v_d$ be a longest path in T. Suppose that v_ℓ $(1 \le \ell \le d-1)$ is a vertex of maximum degree on P. If $d = n-1$, then $T = P_n$. Thus we assume that $d < n-1$. Let w be a vertex not on P such that the v_ℓ-w path Q contains exactly one vertex of P, namely v_ℓ. Let Q: $v_\ell = u_0, u_1, \ldots, u_m = w$ $(m \ge 1)$. We consider two cases, depending on whether either $m + \ell$ or $m + d - \ell$ is at least $k-1$.

Case 1 *Suppose that* $m + \ell$ *or* $m + d - \ell$ *is at least* $k-1$, *say* $m + \ell \ge k-1$. If $m \ge k-2$, then let $T_1 = T - \{v_{\ell-1}v_\ell, v_\ell u_1, u_1u_2, u_2u_3, \ldots, u_{k-3}u_{k-2}\} + \{v_{\ell-1}u_{k-2}, u_{k-2}u_{k-3}, u_{k-3}u_{k-4}, \ldots, u_2u_1, u_1v_\ell\}$. So T_1 is P_k-adjacent to T, and diam $T_1 = d + k - 2 > d$. Next if $m < k-2$, then let $i = (k-1) - m$ and let

$$T_1 = T - \{v_{\ell-i}v_{\ell-i+1}, v_{\ell-i+1}v_{\ell-i+2}, \dots, v_{\ell-1}v_\ell, v_\ell u_1, u_1u_2, u_2u_3, \dots, u_{m-1}w\}$$
$$+ \{v_\ell u_1, u_1u_2, u_2u_3, \dots, u_{m-1}w, wv_{\ell-1}, v_{\ell-1}v_{\ell-2}, v_{\ell-2}v_{\ell-3}, \dots, v_{\ell-i+1}v_{\ell-i}\}.$$

So T_1 is P_k-adjacent to T, where $\operatorname{diam} T_1 = d + m \geq d$. Thus we may continue in this manner, replacing T with T_1 until we obtain a tree that is not a path and m and ℓ cannot be chosen so that either $m + \ell$ or $m + d - \ell$ is at least $k - 1$.

Case 2 *Suppose that* $m + \ell < k - 1$ *and* $m + d - \ell < k - 1$. Let

$$T_1 = T - \{v_0v_1, v_1v_2, \dots, v_{k-1}v_k\} + \{wv_{\ell-1}, v_{\ell-1}v_{\ell-2}, v_{\ell-2}v_{\ell-3}, \dots, v_1v_0,$$
$$v_0v_{\ell+1}, v_{\ell+1}v_{\ell+2}, v_{\ell+2}v_{\ell+3}, \dots, v_{k-1}v_k\}.$$

Then T_1 is P_k-connected to T, and $d_{T_1}(v_\ell, v_d) = m + d$ so that $\operatorname{diam} T_1 >$ d. Thus we may continue in this manner until T_1 is a path. ❑

Hence we have the following.

Corollary 2.7 Let T_1 and T_2 be trees of order p and let $d = \min\{\operatorname{diam} T_1, \operatorname{diam} T_2\}$. Then T_1 is P_k-connected to T_2 for each positive integer k with $3 \leq k \leq d$.

We have seen that if $H = P_3$ or $H = P_4$, then every pair of graphs of the same order and same size containing H as a subgraph are H-connected. It turns out, however, that if $H = K_3$, then not every pair of graphs of the same order and same size containing H as a subgraph are H-connected. First, observe that if G_1 and G_2 are K_3-adjacent, then there exist subgraphs H_1 and H_2 of G_1 and G_2, respectively, where $H_1 \cong H_2 \cong K_3$, $G_2 = G_1 - E(H_1) + E(H_2)$, and $E(H_2) - E(H_1) \subseteq E(\overline{G_1})$. Since any two edges of a triangle uniquely determine the third edge, H_1 and H_2 have exactly one common edge. Let $V(H_1) = \{u, v, x\}$ and $V(H_2) = \{u, v, w\}$, where u, v, w, and x are distinct vertices of G_1. Then $\deg_{G_1} x = \deg_{G_2} x + 2$ and $\deg_{G_1} w =$

$\deg_{G_2} w - 2$ while $\deg_{G_1} v = \deg_{G_2} v$ for every other vertex v of G_1. Thus a necessary condition for two graphs to be K_3-adjacent, and hence K_3-connected, is that they must have the same number of odd vertices. It is not known, however, if this condition is also sufficient.

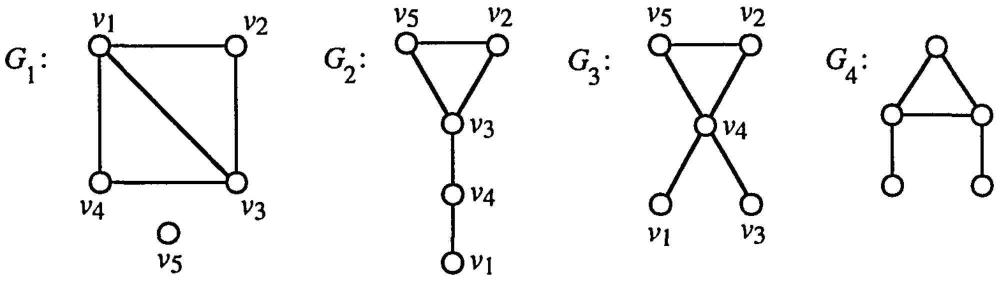

Figure 2.7 Graphs That Are Not K_3-Connected.

Consider the graphs G_1, G_2, G_3, and G_4 shown in Figure 2.7. Since G_4 has four odd vertices while each of G_1, G_2, and G_3 has two odd vertices, it follows that G_4 is not K_3-connected to any of G_1, G_2, and G_3. Next $G_2 = G_1 - \{v_1v_2, v_2v_3, v_3v_1\} + \{v_2v_3, v_3v_5, v_5v_2\}$ so that G_1 is K_3-adjacent to G_2. Finally, $G_3 = G_2 - \{v_5v_2, v_2v_3, v_3v_5\} + \{v_5v_2, v_2v_4, v_4v_5\}$ so that G_2 is K_3-adjacent to G_3. Hence G_1 is K_3-connected to G_3 as well. Since G_1, G_2, G_3, and G_4 are the only $(5, 5)$-graphs containing a triangle, there are two equivalence classes for $(5, 5)$-graphs with a triangle under the relation K_3-connectedness, namely $\{G_1, G_2, G_3\}$ and $\{G_4\}$.

So for two graphs G_1 and G_2 of the same order and same size, a necessary condition for G_1 to be K_3-connected to G_2 is that G_1 must have the same number of odd vertices as G_2. However, this condition is not sufficient. Let $G_1 = 3K_3$ and let $G_2 = K_3 \cup C_6$. Then G_1 and G_2 have the same order, same size, and same number of odd vertices, yet clearly G_1 is not K_3-connected to G_2. Although this

condition is not always sufficient, there is a large class of graphs for which it is necessary.

Theorem 2.8 Let H be an r-regular graph where r is an even positive integer and let G_1 and G_2 be two graphs. If G_1 is H-connected to G_2, then G_1 and G_2 have the same number of odd vertices.

Proof Suppose first that G_1 and G_2 are H-adjacent. Then there exist subgraphs H_1 and H_2 of G_1 and G_2, respectively, such that $H_1 \cong H_2 \cong H$ and $G_2 = G_1 - E(H_1) + E(H_2)$. Clearly if v is not a vertex of H_1 or of H_2, then $\deg_{G_1} v = \deg_{G_2} v$. Next if v is a vertex of H_1 but not a vertex of H_2, then $\deg_{G_1} v = \deg_{G_2} v + r$, while if v is a vertex of H_2 but not a vertex of H_1, then $\deg_{G_1} v = \deg_{G_2} v - r$. Since r is even, $\deg_{G_1} v$ and $\deg_{G_2} v$ are of the same parity. Finally, if v is a vertex of both H_1 and H_2, then $\deg_{G_1} v = \deg_{G_2} v$. Hence, if G_1 and G_2 are H-adjacent, then G_1 and G_2 have the same number of odd vertices. Consequently, if G_1 and G_2 are H-connected, then G_1 and G_2 have the same number of odd vertices. ❑

We now consider H-adjacency when $H = K_{1,3}$. Consider the graphs G_1 and G_2 of Figure 2.8. Now

$$K_{1,4} \cong G_1 - \{v_1v_0, v_1v_2, v_1v_4\} + \{v_2v_0, v_2v_1, v_2v_4\}$$

so that G_1 is H-adjacent to $K_{1,4}$. Also,

$$K_{1,5} \cong G_2 - \{v_1v_0, v_1v_2, v_1v_5\} + \{v_2v_0, v_2v_1, v_2v_5\}$$

so that G_2 is H-adjacent to $K_{1,5}$. In fact, we show that every tree of order p is $K_{1,3}$-connected to the star $K_{1,p-1}$.

Figure 2.8 Graphs That Are $K_{1,3}$-Connected to Stars.

Theorem 2.9 If T is a tree of order p that is not a path, then T is $K_{1,3}$-connected to $K_{1,p-1}$.

Proof Let $H = K_{1,3}$, and let P: $v_0, v_1, \ldots, v_d$ be a longest path through a vertex of maximum degree. Hence diam $T \geq d$. If $d = 2$, then $T = K_{1,p-1}$. Thus we assume that $d \geq 3$. We consider two cases.

Case 1 *Suppose that* $d = 3$. Assume, without loss of generality, that deg $v_1 = \Delta(T)$. We show, in fact, that the diam $T = 3$ as well. Each vertex of T adjacent to v_2, other than v_1, must be an end-vertex; for otherwise, there exists a path of length at least 4 through v_1. Similarly, each vertex of T adjacent to v_1, other than v_2, must be an end-vertex. Thus diam $T = 3$. If deg $v_1 = 3$, then deg $v_2 \leq 3$, and thus $T = G_1$ or $T = G_2$, where G_1 and G_2 are shown in Figure 2.8. As we have previously seen, both G_1 and G_2 are H-adjacent to $K_{1,4}$ and $K_{1,5}$, respectively. Therefore, we may assume deg $v_1 > 3$. Thus there exist end-vertices w_1 and w_2, different from v_0, adjacent to v_1. Then $F_1 = T - \{v_1w_1, v_1w_2, v_1v_0\} + \{w_1v_1, w_1v_2, w_1v_3\}$ is H-adjacent to T. Next, $F_2 = F_1 - \{v_2v_1, v_2w_1, v_2v_3\} + \{w_1v_2, w_1v_0, w_1w_2\}$ is H-adjacent to F_1. Finally, $F_3 = F_2 - \{w_1v_0, w_1v_1, w_1v_2\} + \{v_1w_1, v_1v_0, v_1v_2\}$ is H-adjacent to F_2, and $T' = F_3 - \{w_1v_1, w_1w_2, w_1v_3\} + \{v_1w_1, v_1v_2, v_1v_3\}$ is H-adjacent to F_3. This situation is shown in Figure 2.9, where the graphs H_0, H_1,

H_2, H_3, and H_4 are subgraphs of T, F_1, F_2, F_3, and T', respectively. Now, T, F_1, F_2, F_3, T' is an H-walk; so T is H-connected to T'. Furthermore $\deg_T v_1 < \deg_{T'} v_1$. Thus, we continue in this manner until $\deg v_1 = p-1$. Hence T is H-connected to $K_{1,p-1}$.

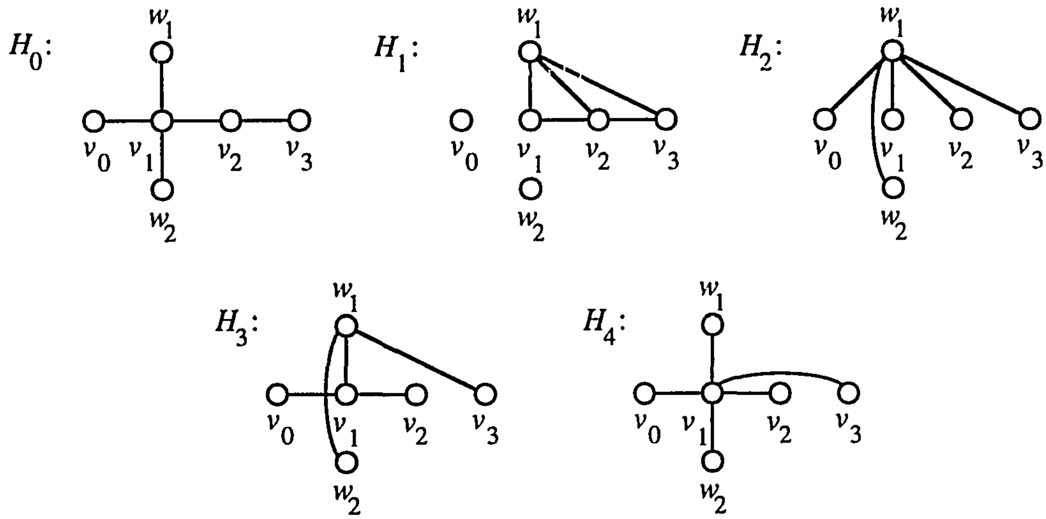

Figure 2.9 A $K_{1,3}$-Walk From T to T' When $d = 3$.

Case 2 *Suppose that* $d > 3$. Now there exists an integer k $(1 \le k \le d-1)$ such that $\deg v_k = \Delta(T)$. Let w be a vertex of T different from v_{k-1} and v_{k+1} that is adjacent to v_k. We consider two subcases.

Subcase 2.1 *Suppose that we can label the vertices of* P *so that* $k \ge 3$, *where* $\deg v_k = \Delta(T)$. Then $F_1 = T - \{v_k w, v_k v_{k-1}, v_k v_{k+1}\} + \{w v_k, w v_0, w v_1\}$ is H-adjacent to T. Next, $F_2 = F_1 - \{v_1 v_0, v_1 v_2, v_1 w\} + \{w v_1, w v_{k-1}, w v_{k+1}\}$ is H-adjacent to F_1. Finally, $F_3 = F_2 - \{w v_k, w v_0, w v_1\} + \{v_k w, v_k v_0, v_k v_1\}$ is H-adjacent to F_2, and $T' = F_3 - \{w v_k, w v_{k-1}, w v_{k+1}\} + \{v_k w, v_k v_{k-1}, v_k v_{k+1}\}$ is H-adjacent to F_3. This situation is illustrated in Figure 2.10, where the graphs H_0, H_1, H_2, H_3, and H_4 are subgraphs of T, F_1, F_2, F_3, and T', respectively. Thus T, F_1, F_2, F_3, T' is a H-

walk (that contains a H-path); so T' is H-connected to T. Furthermore, $\Delta(T') > \Delta(T)$. If T' has diameter 3, then by Case 1, T' is H-connected to $K_{1,p-1}$. Otherwise, we proceed with T' as above (that is, let $T = T'$) until we obtain a tree with diameter 3.

Subcase 2.2 *Suppose that we cannot label the vertices of P so that $k \geq 3$.* Thus $k = 2$ and $d = 4$. Let $x_1, x_2, \ldots, x_m$ denote the vertices of T not belonging to P that are adjacent to v_2. Suppose first that $\deg x_i \geq 2$ for some i $(1 \leq i \leq m)$. Let $y_{i,1}$, $y_{i,2}, \ldots, y_{i,n_i}$ denote the n_i vertices of T that are different from v_2 and adjacent to x_i. We now show that $\deg y_{i,j} = 1$ for each i, j with $1 \leq i \leq m$ and $1 \leq j \leq n_i$. Suppose, to the contrary, that $\deg y_{i,j} > 1$ for some i and j with $1 \leq i \leq m$ and $1 \leq j \leq n_i$, say $z \neq x_i$ is adjacent to $y_{i,j}$. Then the path $z, y_{i,j}, x_1, v_2, v_1, v_0$ is a path in T that can be relabeled so that $k \geq 3$, producing a contradiction. Thus it follows that each $y_{i,j}$ $(1 \leq j \leq n_i)$ is an end-vertex. Let $T = F_0$, and for $k = 1, 2, \ldots, \lfloor n_i / 2 \rfloor$, let $F_i = F_{i-1} - \{x_iv_2, x_iy_{i,2k-1}, x_iy_{i,2k}\} + \{v_2x_i, v_2y_{i,2k-1}, v_2y_{i,2k}\}$; so F_{i-1} and F_i are H-adjacent. If $n_i = 2a + 1$, then let $F_{a+1} = F_a - \{v_2v_1, v_2x_i, v_2v_3\} + \{x_iv_1, x_iv_2, x_iv_3\}$; next let $F_{a+2} = F_{a+1} - \{x_iy_{i,n_i}, x_iv_1, x_iv_2\} + \{v_2y_{i,n_i} v_2v_1, v_2x_i\}$ and finally let $F_{a+3} = F_{a+2} - \{v_2v_1, v_2y_{i,n_i}, v_2x_i\} + \{v_2v_1, v_2y_{i,n_i}, v_2v_3\}$. So let $T' = F_{a+3}$ in the case that n is odd or let $T' = F_{\lfloor n_i/2 \rfloor}$. Now $T = F_0, F_1, \ldots, T'$ is an H-walk from T to T'; so T is H-connected to T'. Next exactly one of the graphs H_1 and H_2 shown in Figure 2.11 is a subgraph of T'. In fact, H_1 is a subgraph of T' if n_i is even, while H_2 is a subgraph of T' is n_i is odd.

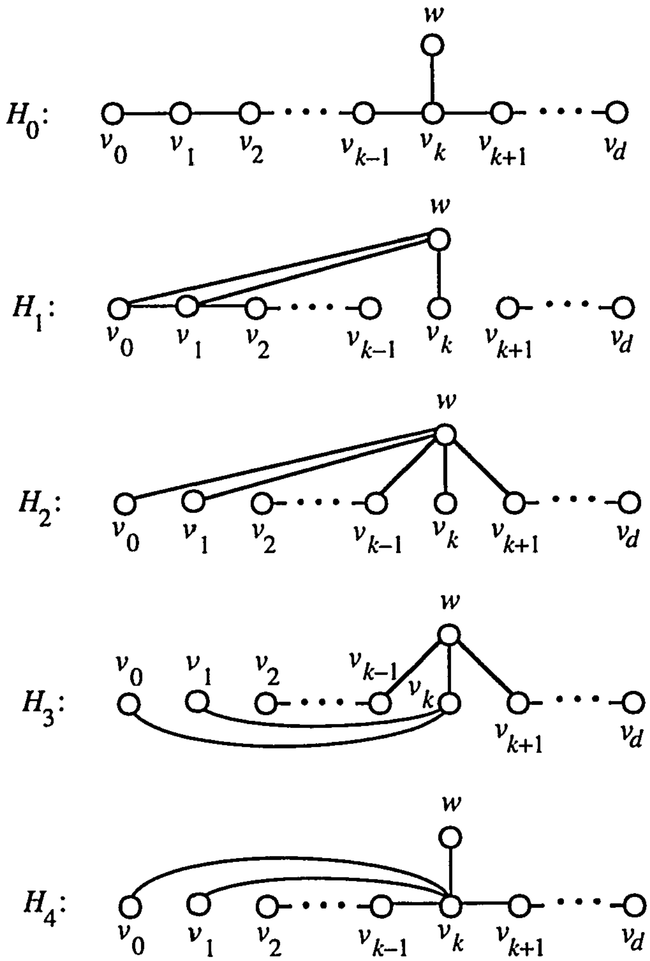

Figure 2.10 An H-Walk From T to T' When $d > 4$ and $k \geq 3$.

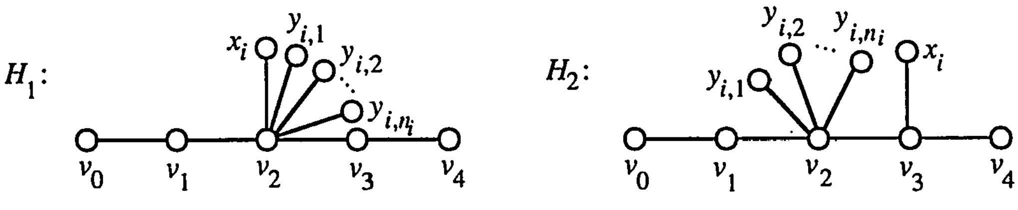

Figure 2.11 A Subgraph of T'.

So in T', the vertices $y_{i,1}, y_{i,2}, \dots, y_{i,n_i}$ are now adjacent to v_2, and possibly x_i is now adjacent to v_3. We continue in this manner until $\deg x_i = 1$ for each $i = 1, 2, \dots, m$. Let T'' denote the resulting tree. So, in T'', any vertex not belonging to P is an end-vertex. Relabel the vertices adjacent to v_2 in T'' and not belonging to P by $z_1, z_2, \dots, z_b$. Let $T'' = F_0$, and for $i = 1, 2, \dots, \lfloor b/2 \rfloor$, let $F_i = F_{i-1} - \{v_2v_3, v_2z_{2i-1}, v_2z_{2i}\} + \{v_3v_2, v_3z_{2i-1}, v_3z_{2i}\}$; so F_{i-1} and F_i are H-adjacent. If $b = 2c$, then let $T''' = F_c - \{v_3w, v_3v_2, v_3v_4\} + \{v_3w, v_3v_1, v_3v_4\}$, where w is any vertex adjacent to v_3 not belonging to P; otherwise $b = 2c + 1$, so let $T''' = F_c - \{v_2z_b, v_2v_3, v_2v_1\} + \{v_3v_1, v_3v_2, v_3z_b\}$. In T''', the vertex v_2 is an end-vertex as is every other vertex of T''' not belonging to P; so $\operatorname{diam}(T''') = 3$. Thus by Case 1, T''' is H-connected to $K_{1,p-1}$; so T is H-connected to $K_{1,p-1}$. ❑

As a consequence of Theorem 2.9, we have the following.

Corollary 2.10 Let T_1 and T_2 be trees of order p. Then T_1 is $K_{1,3}$-connected to T_2 if and only if $\Delta(T_1) \ge 3$ and $\Delta(T_2) \ge 3$.

Proof Let $H = K_{1,3}$. Suppose first that T_1 is H-connected to T_2. Then there exists a T_1-T_2 H-path $T_1 = F_0, F_1, \dots, F_k = T_2$. Since T_1 is H-adjacent to F_1, it follows that T_1 has a vertex of degree at least 3 (since H does) and hence $\Delta(T_1) \ge 3$. Similarly, since F_{k-1} is H-adjacent to T_2, it follows that $\Delta(T_1) \ge 3$.

For the converse, suppose that $\Delta(T_1) \ge 3$ and $\Delta(T_2) \ge 3$. By Theorem 2.9, the tree T_1 is H-connected to $K_{1,p-1}$ and T_2 is H-connected to $K_{1,p-1}$. Hence T_1 and T_2 are H-connected. ❑

We now turn from trees to hamiltonian graphs.

Theorem 2.11 If G_1 and G_2 are two nonisomorphic hamiltonian graphs of the same order and same size, then G_1 is $K_{1,3}$-connected to G_2.

Proof Without loss of generality, let C: $v_1, v_2, \ldots, v_p, v_1$ be a hamiltonian cycle in both G_1 and G_2. If $p = 4$, then $G_1 = G_2$, and the result follows. Thus we may assume that $p \geq 5$. Since G_1 and G_2 are not isomorphic, there exist chords v_iv_j and v_kv_ℓ such that $v_iv_j \in E(G_1) - E(G_2)$ and $v_kv_\ell \in E(G_2) - E(G_1)$. Without loss of generality, we may assume $i < j$, $k < \ell$, and $i < k$. Assume first that $G_2 = G_1 - v_iv_j + v_kv_\ell$.

Next, suppose that v_iv_j and v_kv_ℓ are adjacent, that is, $j = k$. Since $i < j < \ell$ and v_iv_j and v_jv_ℓ are chords, it follows that $i \neq j - 1$ and $\ell \neq j + 1$. Therefore, $G_2 = G_1 - \{v_jv_i, v_jv_{j-1}, v_jv_{j+1}\} + \{v_jv_\ell, v_jv_{j-1}, v_jv_{j+1}\}$; so G_1 and G_2 are $K_{1,3}$-adjacent. Thus we may assume that i, j, k, and ℓ are distinct. If any one of the edges $v_iv_k, v_jv_k, v_iv_\ell$, and v_jv_ℓ is not present in G_1, say v_iv_k, then let $F_1 = G_1 - \{v_iv_{i-1}, v_iv_{i+1}, v_iv_j\} + \{v_iv_{i-1}, v_iv_{i+1}, v_iv_k\}$; so F_1 and G_1 are $K_{1,3}$-adjacent. Hence $G_2 = F_1 - \{v_kv_{k-1}, v_kv_{k+1}, v_kv_i\} + \{v_kv_{i-1}, v_kv_{k+1}, v_kv_\ell\}$; so G_2 and F_1 are $K_{1,3}$-adjacent. Thus G_1 and G_2 are $K_{1,3}$-connected. Therefore, $v_iv_k, v_jv_k, v_iv_\ell, v_jv_\ell \in E(G_1)$. Suppose first that $j \neq \ell - 1$ and that $i \not\equiv \ell + 1 \pmod{p}$. Let $F_1 = G_1 - \{v_\ell v_i, v_\ell v_j, v_\ell v_{\ell-1}\} + \{v_\ell v_i, v_\ell v_{\ell-1}, v_\ell v_k\}$; so F_1 and G_1 are $K_{1,3}$-adjacent. Then either $v_k \neq v_{j-1}$ or $v_k \neq v_{j+1}$, say $v_k \neq v$, where $v \in N(v_j)$. Hence $G_2 = F_1 - \{v_iv_j, v_jv_k, v_jv\} + \{v_jv_k, v_jv_\ell, v_jv\}$; so F_1 and G_2 are $K_{1,3}$-adjacent, and G_1 and G_2 are $K_{1,3}$-connected. Thus we assume that $j = \ell - 1$ and that $i \equiv \ell + 1 \pmod{p}$. Hence $i = 1$, $\ell = p$, $j = p - 1$, $v_iv_j = v_1v_{p-1}$, and $v_kv_\ell = v_kv_p$. Since $p \geq 5$, there exists a vertex $v \in N(v_k)$ such that v lies between v_1 and v_k on C, or v lies between v_k and v_{p-1} on C. Let $F_1 = G_1 - \{v_kv_1, v_kv_{p-1}, v_kv\} + \{v_kv, v_kv_p, v_kv_1\}$; so F_1 and G_1 are $K_{1,3}$-adjacent. Then let $G_2 = F_1 -$

$\{v_{p-1}v_p, v_{p-1}v_1, v_{p-1}v_{p-2}\} + \{v_{p-1}v_p, v_{p-1}v_k, v_{p-1}v_{p-2}\}$; so F_1 is $K_{1,3}$-adjacent to G_2. Hence G_1 and G_2 are $K_{1,3}$-connected.

So if G_1 and G_2 are two nonisomorphic hamiltonian graphs of the same order and same size such that $G_2 = G_1 - e + \mathrm{f}$ for two edges e and f, then G_1 and G_2 are $K_{1,3}$-connected. Thus if G_1 and G_2 are two nonisomorphic hamiltonian graphs of the same order (at least 5) and the same size, G_1 and G_2 are $K_{1,3}$-connected. ❑

Thus every two trees of the same order and with maximum degree at least 3 are $K_{1,3}$-connected as are every two hamiltonian graphs of the same order, same size, and with maximum degree at least 3. For graphs that are not trees and not hamiltonian, it remains to be determined which pairs of these graphs are $K_{1,3}$-connected.

CHAPTER III

H-DISTANCE AND H-DISTANCE GRAPHS

3.1 Introduction

For graphs G_1 and G_2 of the same order and same size, G_2 is said to be obtained from G_1 by an *edge rotation* if G_1 contains distinct vertices u, v, and w such that $uv \in E(G_1)$, $uw \notin E(G_1)$, and $G_2 = G_1 - uv + uw$. Recall that it was shown in [9] that every pair of graphs of the same order and same size can be transformed into one another by a sequence of edge rotations. The *edge rotation distance* or, more simply, the *r-distance* $d_r(G_1, G_2)$ between G_1 and G_2 is the smallest nonnegative integer n for which there exists a sequence $G_1 = F_0, F_1, \dots, F_n = G_2$ of graphs such that F_i can be obtained from F_{i-1} by an edge rotation for $i = 1, 2, \dots, n$. Some properties of edge rotation distance were established in [9], where it was shown that $d_r(G_1, G_2) = d_r(\bar{G}_1, \bar{G}_2)$, and that for every nonnegative integer n, there exist graphs G_1 and G_2 such that $d_r(G_1, G_2) = n$. For nonempty graphs G_1 and G_2, recall that a *greatest common subgraph* of G_1 and G_2 is any graph G of maximum size without isolated vertices that is a subgraph of G_1 and G_2. It was also shown in [9] that for two graphs G_1 and G_2 of order p and size q, where s is the size of a greatest common subgraph, $d_r(G_1, G_2)$ is bounded above by $2(q - s)$.

Another metric on a space of graphs is given by the edge slide, which is considered in [17]. A graph G_2 can be obtained from G_1 by an *edge slide* if G_1 contains distinct vertices u, v, and w such that $uv, vw \in E(G_1)$, $uw \notin E(G_1)$, and $G_2 = G_1 - uv + uw$. In [17] it was shown that the edge slide preserves connectedness, and that a graph G_1 can be obtained from a graph G_2 by a sequence

of edge slides if and only if G_1 and G_2 have the same number of components and corresponding components of G_1 and G_2 have the same order and same size. The *edge slide distance* or the *s-distance* $d_s(G_1, G_2)$ between two graphs G_1 and G_2 having the same number of components, where corresponding components have the same order and same size, is defined as the smallest nonnegative integer n for which there exists a sequence $G_1 = F_0, F_1, \dots, F_n = G_2$ of graphs such that, for $i = 1, 2, \dots, n$, the graph F_i can be obtained from F_{i-1} by an edge slide. Since an edge slide is an edge rotation, it follows that $d_r(G_1, G_2) \leq d_s(G_1, G_2)$ for every pair G_1, G_2 of graphs for which $d_s(G_1, G_2)$ is defined. In [15] it was shown that for every pair m, n of positive integers with $m \leq n$, there exist graphs G_1 and G_2 such that $d_r(G_1, G_2) = m$ while $d_s(G_1, G_2) = n$. We now consider the analogous concepts for H-adjacency.

Let G_1 and G_2 be two graphs of the same order and same size such that $V(G_1) = V(G_2)$, and let H be a connected graph of order at least 3. In Chapter II, two subgraphs H_1 and H_2 of G_1 and G_2, respectively, are defined to be H-adjacent if $H_1 \cong H_2 \cong H$, and H_1 and H_2 share some but not all edges, that is, $E(H_1) \cap E(H_2) \neq \emptyset$ and $E(H_2) - E(H_1) \neq \emptyset$ (so also $E(H_1) - E(H_2) \neq \emptyset$). Also in Chapter II, the graphs G_1 and G_2 are themselves defined to be H-adjacent if G_1 and G_2 contain H-adjacent subgraphs H_1 and H_2, respectively, such that $E(H_2) - E(H_1) \subseteq E(\overline{G}_1)$ and $G_2 = G_1 - E(H_1) + E(H_2)$. A G_1-G_2 H-walk is defined in Chapter II as a sequence $G_1 = F_0, F_1, \dots, F_k = G_2$ of graphs of the same order and same size such that F_i is H-adjacent to F_{i+1} for $i = 0, 1, \dots, k-1$. Thus, a G_1-G_2 H-walk $G_1 = F_0, F_1, \dots, F_k = G_2$ in which the graphs $F_0, F_1, \dots, F_k$ are distinct is a G_1-G_2 H-path, and the *length* of the G_1-G_2 H-path is the integer k. We define the *H-distance* $d_H(G_1, G_2)$ from G_1 to G_2 as the length of a shortest G_1-G_2

H-path. Hence, H-distance is a metric on the space of all graphs of a fixed order and a fixed size for which this distance is defined. In fact, it is not clear for a given graph H, when this distance is defined. If $H = P_3$ or $H = P_4$, then by Theorems 2.2 and 2.5, this distance is defined for every pair of graphs containing H as a subgraph.

We now show that for every connected graph H of order at least 3 and for every positive integer n, there exist graphs whose H-distance is n.

Theorem 3.1 For a connected graph H (of order at least 3) and a positive integer n, there exist graphs F_n and G_n such that $d_H(F_n, G_n) = n$.

Proof Suppose first that $n = 1$. Let F_1 be obtained from the graph G shown in Figure 3.1 by identifying a vertex x of H with minimum degree with the vertex v_0 of G, and let G_1 be obtained from G by identifying the vertex x with v_1. Clearly, $G_1 \not\cong F_1$; so $d_H(F_1, G_1) \geq 1$. Also, since F_1 is H-adjacent to G_1, we have that $d_H(F_1, G_1) = 1$. When $H = C_4$, the graphs F_1 and G_1 are shown in Figure 3.1.

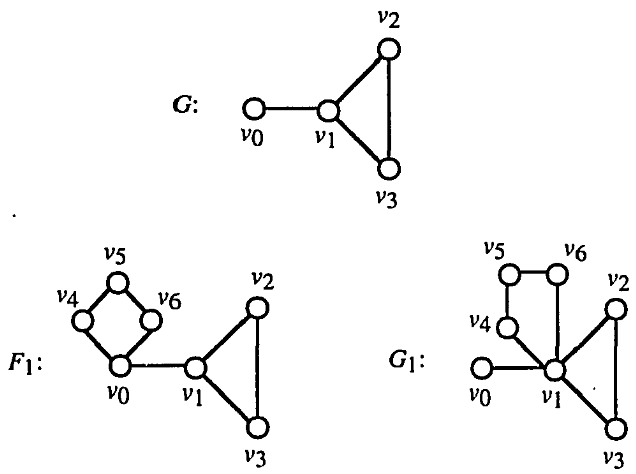

Figure 3.1 The Graphs G, F_1, and G_1 When $H = C_4$.

Next let $n > 1$ be a positive integer. Let $F_n = nF_1$, and let $G_n = nG_1$. Now each component of F_n is H-adjacent to each component of G_n, so that we may transform F_n into G_n one component at a time. Thus, $d_H(F_n, G_n) \le n$. We show that $d_H(F_n, G_n) = n$. Since no component of F_n is isomorphic to a component of G_n and the components of F_n and G_n all have the same order and same size, it follows that each component of F_n must have some of its edges moved. Since H is connected, it follows that H-adjacency can only move edges from one component at a time. Thus, $d_H(F_n, G_n) \ge n$ and therefore $d_H(F_n, G_n) = n$. ❑

In the proof of the previous theorem, the graphs F_n and G_n are disconnected. We now show that for a given connected graph H and a positive integer n, there exist connected graphs for which the H-distance between them is n.

Theorem 3.2 For a connected graph H of order at least 3 and a positive integer n, there exist connected graphs F_n and G_n such that $d_H(F_n, G_n) = n$.

Proof For $n = 1$, the proof of Theorem 3.1 gives connected graphs F_1 and G_1 for which $d_H(F_1, G_1) = 1$. Thus we may assume that $n \ge 2$. Let x denote a vertex of H of minimum degree. Next let P: $v_0, v_1, v_2, \dots, v_n$ be a path of length n, and let $H_1, H_2, \dots, H_{n+1}$ be copies of H, where the vertex x_i $(1 \le i \le n+1)$ of H_i corresponds to the vertex x of H. Finally, let F_n be obtained from P by identifying the vertex x_i $(1 \le i \le n+1)$ of H_i with the vertex v_0 of P, and let G_n be obtained from P by identifying the vertex x_i $(1 \le i \le n+1)$ of H_i with the vertex v_{i-1} of P. When H is the graph G of Figure 3.1, the graphs F_3 and G_3 are shown in Figure 3.2.

For $i = 1, 2, \dots, n$, let $y_{i,1}, y_{i,2}, \dots, y_{i,k}$ denote the k vertices of H_i adjacent to x_i. Let $F_n = J_0$, and for $i = 1, 2, \dots, n$, define

$$J_i = J_{i-1} - \{y_{i,1}v_0, y_{i,2}v_0, \dots, y_{i,k}v_0\} + \{y_{i,1}v_i, y_{i,2}v_i, \dots, y_{i,k}v_i\}.$$

Since $H_i - \{y_{i,1}v_0, y_{i,2}v_0, \dots, y_{i,k}v_0\} + \{y_{i,1}v_i, y_{i,2}v_i, \dots, y_{i,k}v_i\} \cong H$ and $H_i - \{y_{i,1}v_0, y_{i,2}v_0, \dots, y_{i,k}v_0\} \neq \varnothing$, for each $i = 1, 2, \dots, n$, it follows that J_{i-1} is and

H-adjacent to J_i. Thus, $F_n = J_0, J_1, \dots, J_n = G_n$ is an F_n–G_n H-path. Therefore,$_{i-1}$ is

H-adjacent to J_i. Thus, $F_n = J_0, J_1, \dots, J_n = G_n$ is an F_n–G_n H-path. Therefore,
$\mathrm{d}_H(F_n, G_n) \le n$. We show that $\mathrm{d}_H(F_n, G_n) \ge n$. Observe that in F_n, there are at least $n + 1$ copies of H having exactly one vertex in common, namely v_0, while the vertex set of G_n can be partioned into $n + 1$ sets, where each set induces a copy of H. Therefore, v_0 can only belong to one copy of H in G_n, and hence $\mathrm{d}_H(F_n, G_n) \ge n$. Thus $\mathrm{d}_H(F_n, G_n) = n$. ❑

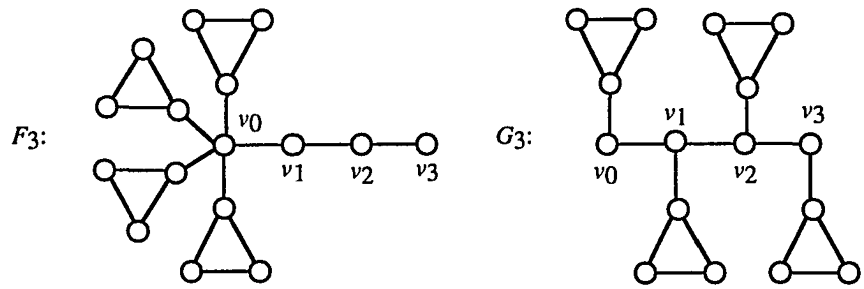

Figure 3.2 The Graphs F_3 and G_3 When H Is the Graph G of Figure 3.1.

3.2 H-Distance Graphs

Let S denote a set of graphs of a fixed order and fixed size. The *edge rotation distance graph $D_r(S)$ of S* is defined as that graph with vertex set S such that the vertices G_1 and G_2 are adjacent in $D_r(S)$ if and only if the graph G_2 can be obtained from the graph G_1 by an edge rotation. Many classes of graphs have been shown to be edge rotation distance graphs in [3, 13, 15], including complete graphs, trees, cycles, complete bipartite graphs and all line graphs, and no graph has been found that is not an edge rotation distance graph. In fact, it is conjectured in [3] that every graph is an edge rotation distance graph.

For a set S of graphs of the same order (at least 5) and same size, the *jump distance graph* $D_j(S)$ *of* S is that graph whose vertices are the graphs of S, and where vertices G_1 and G_2 in S are adjacent in $D_j(S)$ if and only if the graph G_2 can be obtained from the graph G_1 by a single edge-jump. In [4] it was shown that complete graphs, complete multipartite graphs, trees, cycles, and the complements of line graphs are jump distance graphs, and it is conjectured that every graph is jump distance graph. We now turn our attention to the analogous concept for H-adjacency.

Let S denote a set of graphs of the same order and same size. For a connected graph H, the *H-distance graph* $D_H(S)$ *of* S is that graph whose vertices are the graphs of S, and where the vertices G_1 and G_2 are adjacent in $D_H(S)$ if and only if the graphs G_1 and G_2 are H-adjacent. A graph G is an *H-distance graph* if there exists a set S of graphs of the same order and same size such that $D_H(S) = G$. We begin by showing that paths are P_3-distance graphs.

Theorem 3.3 For every integer $n \geq 2$, the path P_n is a P_3-distance graph.

Proof Let $H = P_3$, and let P: $v_1, v_2, \dots, v_{2n}$ denote a path of length $2n - 1$. Let v_0 be a new vertex, and for $i = 1, 2, \dots, n$, let $G_i = P + v_0v_i$. (See Figure 3.3 for the case $n = 4$.) Now the graphs $G_1, G_2, \dots, G_n$ are pairwise nonisomorphic, and for $i = 2, \dots, n$, the graph $G_i = G_{i-1} - \{v_0v_{i-1}, v_{i-1}v_i\} + \{v_{i-1}v_i, v_0v_i\}$ so that G_i and G_{i-1} are H-adjacent. Clearly for $i = 2, \dots, n-1$, G_i is H-adjacent only to G_{i-1} and G_{i+1}. Thus $D_H(\{G_1, G_2, \dots, G_n\}) = P_n$. ❑

Next, we show that all cycles are P_3-distance graphs. First, the following observation will be useful. If G_1 and G_2 are two P_3-adjacent graphs, then we have seen that $G_2 = G_1 - e + f$ for edges e and f of G_1 and G_2, respectively. Further,

in $G_1 + f$, the edges e and f are either adjacent or are adjacent to a common edge of G_1 (and G_2). Thus $d(e,f) = 0$ or $d(e,f) = 1$.

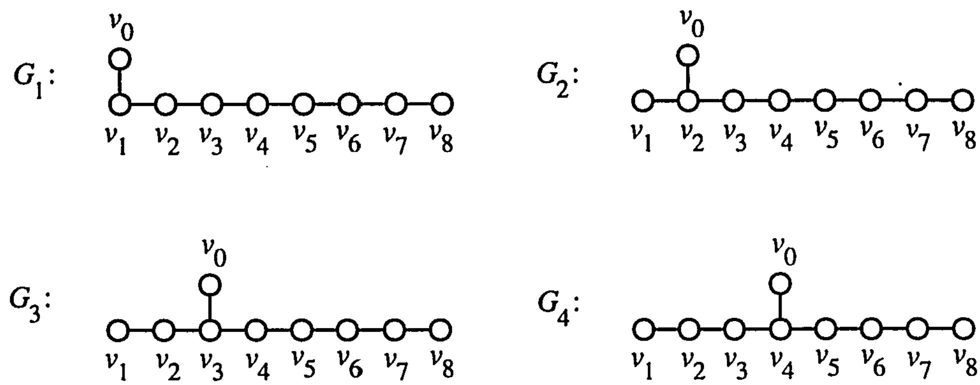

Figure 3.3 A Set of Graphs Such That $D_{P_3}(\{G_1, G_2, G_3, G_4\}) = P_4$.

Theorem 3.4 For every integer $n \geq 3$, the cycle C_n is a P_3-distance graph.

Proof Let $H = P_3$. Next, let $n \geq 3$ be an integer, and let P: $v_1, v_2, \ldots, v_{2n}$ be a path of length $2n - 1$ and T: w_1, w_2, w_3, w_1 be a cycle of length 3. Obtain the graph G from P and T by identifying the vertex v_1 of P with the vertex w_1 of T. For $i = 1, 2, \ldots, n$, let $G_i = G + v_iv_{n+i}$. The case $n = 4$ is shown in Figure 3.4. Thus each graph G_i has a 3-cycle and an $(n + 1)$-cycle and, furthermore, the distance between the 3-cycle and the $(n + 1)$-cycle is $i - 1$. Therefore, the graphs $G_1, G_2, \ldots, G_n$ are pairwise nonisomorphic.

Now for $i = 1, 2, \ldots, n - 1$, the graph $G_{i+1} = G_i - \{v_iv_{n+i}, v_iv_{i+1}\} + \{v_iv_{i+1}, v_{i+1}v_{n+i+1}\}$; so G_{i+1} is H-adjacent to G_i. Also, $G_1 = G_n - \{v_nv_{2n}, v_nv_{n+1}\} + \{v_nv_{n+1}, v_{n+1}v_1\}$; so G_1 is H-adjacent to G_n. Let $S = \{G_1, G_2, \ldots, G_n\}$. Since G_i $(1 \leq i \leq n)$ is H-adjacent to G_{i+1}, where all indices are taken modulo n, it follows that C_n is a subgraph of $D_H(S)$.

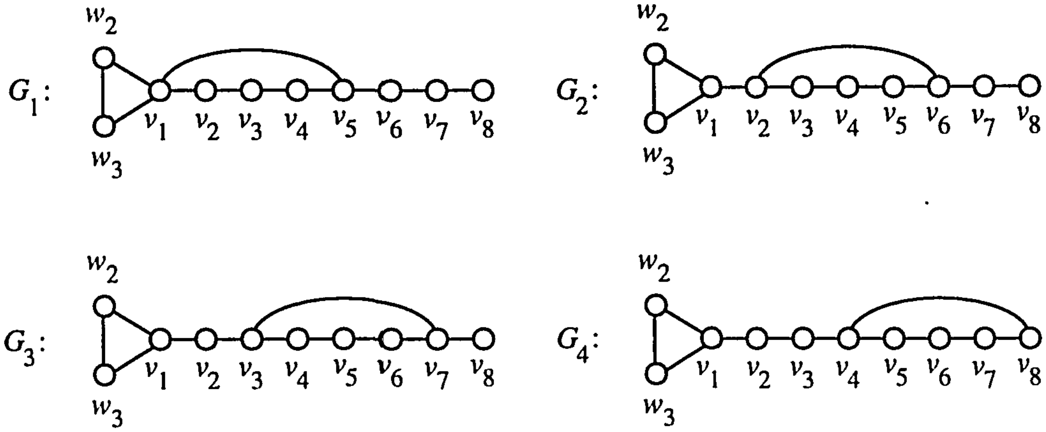

Figure 3.4 A Set of Graphs Such That $D_{P_3}(\{G_1, G_2, G_3, G_4\}) = C_4$.

To see that $D_H(S) = C_n$, for fixed integers i and j with $1 \le i < j \le n$, $j - i \ne 1$, and $j - i \ne n - 1$, let F be that graph obtained from G by adding the edges v_iv_{n+i} and v_jv_{n+j}. Then $d_F(v_iv_{n+i}, v_jv_{n+j}) \ge 2$, and therefore the graphs G_i and G_j are not H-adjacent. Hence $D_H(S) = C_n$. ❑

Complete graphs are also P_3-distance graphs, as the next theorem shows. For every integer $n \ge 3$, denote the *wheel* of order n by W_n, where $W_n = C_n + K_1$.

Theorem 3.5 For every postitive integer n, the complete graph K_n is a P_3-distance graph.

Proof Let $H = P_3$. For $n = 1$, let S consist of only one graph, and then $D_H(S) = K_1$. Next for $n = 2$, the graph K_2 is a P_3-distance graph by Theorem 3.3. Thus we may assume $n \ge 3$. Let $v_1, v_2, \dots, v_{2n-2}, v_1$ denote the $(2n-2)$-cycle of W_{2n-2} and the vertex v will denote the unique vertex of W_{2n-2} having degree $2n-2$. Next let G denote that graph obtained from W_{2n-2} by joining a new vertex v_0 to v_1. For $i = 1, 2, \dots, n$, let $G_i = G - vv_i$. The case $n = 4$ is shown in Figure 3.5. Now the

graph G_1 has $2n-2$ vertices of degree 3, while each of the graphs $G_2, G_3, \ldots, G_n$ has $2n-3$ vertices of degree 3. Next the graph G_i $(1 \le i \le n)$ has exactly one 4-cycle at distance $i-1$ from the vertex v_0. Therefore, the graphs $G_1, G_2, \ldots, G_n$ are pairwise nonisomorphic. Since $G_j = G_i - \{vv_{2n-2}, vv_j\} + \{vv_{2n-2}, vv_i\}$ for $1 \le i < j \le n$, the graphs $G_1, G_2, \ldots, G_n$ are pairwise H-adjacent. Thus if $S = \{G_1, G_2, \ldots, G_n\}$, then $D_H(S) = K_n$. ❑

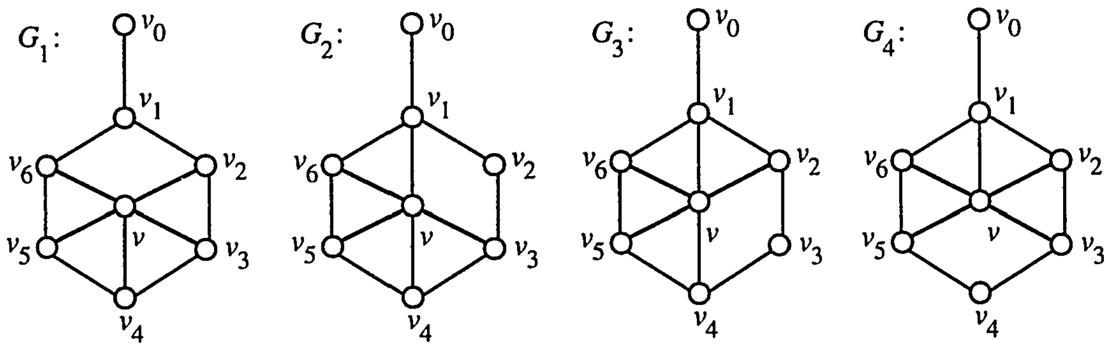

Figure 3.5 A Set of Graphs Such That $D_{P_3}(\{G_1, G_2, G_3, G_4\}) = K_4$.

We now show that every graph is a P_3-distance graph.

Theorem 3.6 Every graph is a P_3-distance graph.

Proof Let $H = P_3$, and let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$. Next let F denote that graph obtained from G by joining, exactly $2i$ $(1 \le i \le n)$ new vertices to v_i. Let G_i $(1 \le i \le n)$ be that graph obtained from F by joining a new vertex x to v_i. Thus, the number of end-vertices of G_i adjacent to v_j is $2j$ if $j \ne i$ and $2i+1$ if $j = i$. The graphs $G_1, G_2, \ldots, G_n$ are pairwise nonisomorphic, and if v_iv_j is an edge of G, then $G_j = G_i - \{xv_i, v_iv_j\} + \{v_iv_j, v_jx\}$; so G_i is H-adjacent to G_j. Let i and j be distinct integers with $1 \le i, j \le n$. Now if G_i and G_j are H–adjacent, then $G_j = G_i - e + f$ for edges e and f of G_i and G_j, respectively. Also, in the

graph $G_i - e = G_j - f$, there must be an edge e' adjacent to e in G_i and adjacent to f in G_j. Thus, since $G_i - xv_i = G_j - xv_j$, the graphs G_i and G_j are H-adjacent if and only if the edge v_iv_j is present in G. Therefore, $D_H(\{G_1, G_2, \dots, G_n\}) = G$. ❑

The construction described in the proof of Theorem 3.6 is the construction given in [3] which shows that every graph is an *edge slide distance graph.* Recall that for two graphs G_1 and G_2 of the same order and same size, the graph G_1 *can be transformed into* G_2 *by an edge slide* if G_1 contains distinct vertices u, v, and w such that $uv \in E(G_1)$, $uw \notin E(G_1)$, $vw \in E(G_1)$, and $G_2 = G_1 - uv + uw$. Let S be a set of graphs of the same order and same size. The *edge slide distance graph* $D_s(S)$ is defined as that graph with vertex set S such that two vertices G_1 and G_2 of $D_s(S)$ are adjacent if and only if the graph G_1 can be transformed into the graph G_2 by an edge slide. Next we show that we can extend Theorem 3.6 to show that every graph is, in fact, a P_n-distance graph for $n \geq 3$.

Theorem 3.7 For an integer $n \geq 3$, every graph is a P_n-distance graph.

Proof Let $H = P_n$, and let G be a graph with $V(G) = \{v_1, v_2, \dots, v_p\}$. Next let F be that graph obtained from G by identifying an end-vertex of $2i$ $(1 \leq i \leq p)$ paths of length $n-2$ with v_i. (The graph F is shown in Figure 3.6 when $n = 4$ and $G = K_4$.) For $i = 1, 2, \dots, p$, let G_i be that graph obtained from F by identifying the end-vertex x_{n-1} of the path $x_1, x_2, \dots, x_{n-1}$ (of length $n-2$) with v_i. For each graph G_i $(1 \leq i \leq p)$, the number of paths of length $n-1$ starting at a vertex v_j of G and containing no other vertices of G is $2j$ if $j \neq i$ and $2i + 1$ if $j = i$. Clearly, the graphs $G_1, G_2, \dots, G_p$ are pairwise nonisomorphic. Let i and j be distinct integers with $1 \leq i, j \leq p$. Then the graphs G_i and G_j differ in exactly $n-1$ edges. Thus G_i and G_j are H-adjacent if and only if there exists an edge e of $G_i - \{x_1x_2,$

$x_2x_3, \dots, x_{n-2}v_i\} = G_j - \{x_1x_2, x_2x_3, \dots, x_{n-2}v_j\}$ such that e is adjacent to $x_{n-2}v_i$ in G_i and adjacent to $x_{n-2}v_j$ in G_j. Therefore G_i and G_j are H-adjacent if and only if $v_iv_j \in E(G)$. Hence $D_H(\{G_1, G_2, \dots, G_p\}) = G$. ❑

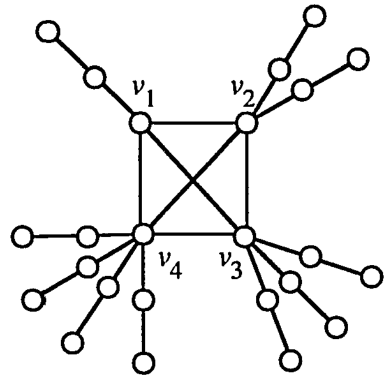

Figure 3.6 The Graph F When $G = K_4$ and $n = 4$.

In fact, we can extend Theorem 3.7 to show that every graph is an H-distance graph if $\delta(H) = 1$.

Theorem 3.8 Let H be a graph with $\delta(H) = 1$. Then every graph is an H-distance graph.

Proof Let H be a graph with $\delta(H) = 1$ and size q. Then there exists an end-vertex v of H. Let u be the unique vertex of H adjacent to v. Next let G be a graph with $V(G) = \{v_1, v_2, \dots, v_p\}$, and let F be that graph obtained from G by identifying for $i = 1, 2, \dots, p$, the vertex u in $2i$ copies of $H - v$ with the vertex v_i in G. For $i = 1, 2, \dots, p$, let G_i be that graph obtained from F by identifying the vertex u in the graph $H' = H - v$ with the vertex v_i. Clearly, the graphs $G_1, G_2, \dots, G_p$ are pairwise nonisomorphic. Let i and j be distinct integers with $1 \le i, j \le p$. Then the graphs G_i and G_j differ in exactly $q - 1$ edges or in a copy of $H - v$. We show that G_i and G_j are H-adjacent if and only if $v_iv_j \in E(G)$.

Let F_i be a copy of $H - v$ in G_i containing exactly one vertex of G, namely v_i, and let F_j be a copy of $H - v$ in G_j containing exactly one vertex of G, namely v_j. Furthermore, we may assume that F_i and F_j are identical, that is, $V(F_i) = V(F_j)$ and $E(F_i) = E(F_j)$. Suppose that $v_iv_j \in E(G)$. Since $F_i + v_iv_j \cong F_j + v_jv_i \cong H$ and $G_j = G_i - E(F_i) + E(F_j)$, it follows that G_i is H-adjacent to G_j. Finally, suppose that G_i is H-adjacent to G_j. Since G_i and G_j differ in exactly one copy of $H - v$ at v_i and v_j in G_i and G_j, respectively, we may assume, without loss of generality, that this copy of $H - v$ is F_i in G_i and F_j in G_j. Since F_i and F_j have size $q - 1$ and G_i is H-adjacent to G_j, there must exist an edge e of $G_i - E(F_i) = G_j - E(F_j)$ such that $F_i + e \cong F_j + e \cong H$. Furthermore, since $F_i \cong H - v$ where $\deg v = 1$, it follows that e is incident with exactly one vertex of F_i and exactly one vertex of F_j. Next, since $\langle V(F_i)\rangle = F_i$ in G_i and $\langle V(F_j)\rangle = F_j$ in G_j, it must be that e is incident with v_i in F_i and v_j in F_j. Therefore $e = v_iv_j$ and $v_iv_j \in E(G)$. ❑

Thus, by Theorem 3.8, we know that for a given tree T, every graph is a T-distance graph. We now turn our attention to K_3-distance graphs and begin by showing that complete graphs are K_3-distance graphs.

Theorem 3.9 For every positive integer n, the complete graph K_n is a K_3-distance graph.

Proof Let $H = K_3$. Next let P: $v_1, v_2, \ldots, v_n$ be a path of length $n - 1$, and let C: w_1, w_2, w_3, w_4, w_1 be a 4-cycle. The graph G is obtained by identifying the vertex v_n of P with w_1 of C. For $i = 1, 2, \ldots, n$, let G_i be that graph obtained from G by adding new vertices x and y and the edges xy, yv_i, and v_ix. Since $G_j = G_i - \{xy, yv_i, v_ix\} + \{xy, yv_j, v_jx\}$, for $1 \le i < j \le n$, it follows that $d_H(G_i, G_j) = 1$. Thus $D_H(\{G_1, G_2, \ldots, G_n\}) = K_n$. ❑

Using a construction similar to the one given in [3] which shows that every cycle is an edge rotation distance graph, we now show that every cycle is a K_3-distance graph.

Theorem 3.10 For every integer $n \geq 3$, the cycle C_n is a K_3-distance graph.

Proof Let $H = K_3$ and let C: $v_1, v_2, \ldots, v_{2n+2}, v_1$ be a $(2n + 2)$-cycle. For $i = 1, 2, \ldots, n$, let F_i be obtained from C by joining a new vertex v to v_1 and v_{i+2} and adding the edge $v_1 v_{i+2}$. For $n = 4$, the graphs F_1, F_2, F_3, and F_4 are shown in Figure 3.7.

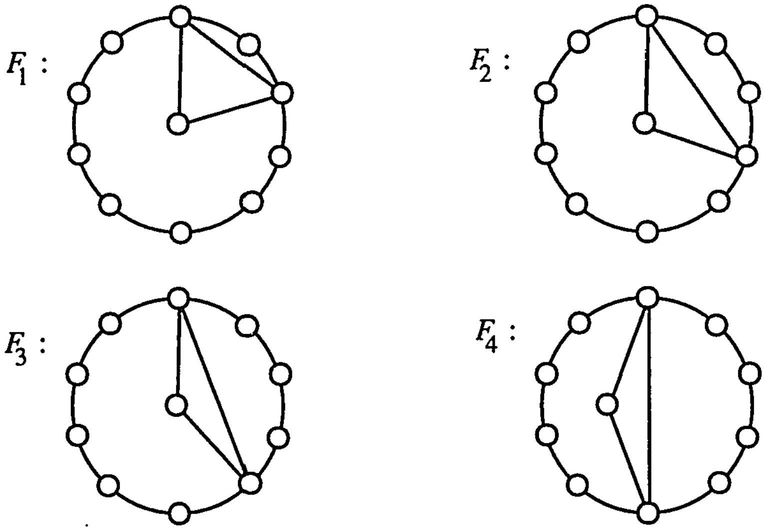

Figure 3.7 The Graphs F_1, F_2, F_3, and F_4.

Next, for $i = 1, 2, \ldots, n-1$, let $G_i = F_i \cup F_{i+1}$, and let $G_n = F_n \cup F_1$. Clearly, the graphs G_i and G_j, for $1 \leq i < j \leq n$, differ in exactly two edges when $j = i + 1$ or when $i = 1$ and $j = n$, and differ in four edges otherwise. Thus since $d_H(G_i, G_{i+1}) = 1$ for $1 \leq i \leq n-1$ and $d_H(G_n, G_1) = 1$, it follows that $D_H(\{G_1, G_2, \ldots, G_n\}) = C_n$. ❑

For a given connected graph H of order at least 3, an induced subgraph of an H-distance graph is also an H-distance graph. Therefore, we have the following corollary to Theorem 3.10.

Corollary 3.11 For every integer $n \geq 2$, the path P_n is a K_3-distance graph.

At this point, an observation will be helpful. Let G_1 and G_2 be K_3-adjacent graphs. Then there are subgraphs H_1 and H_2 of G_1 and G_2, respectively, such that $H_1 \cong H_2 \cong K_3$, $E(H_1) \cap E(H_2) \neq \emptyset$, and $E(H_2) - E(H_1) \neq \emptyset$. Since any two edges of a triangle uniquely determine the third edge, it must be that $|E(H_1) \cap E(H_2)| = 1$ or $\{uv\} = E(H_1) \cap E(H_2)$ for vertices u and v of G_1 (and hence of G_2). Thus there exists a vertex w of H_1 and a vertex x of H_2 such that w and x are distinct vertices. So, $|E(H_2) - E(H_1)| = 2$, and hence $G_2 = G_1 - \{uw, vw\} + \{ux, vx\}$. The following lemma will be useful in establishing that a number of large classes of graphs are K_3-distance graphs.

Lemma 3.12 Let G_1 and G_2 be graphs of the same order n and same size. Then $d_{K_3}(G_1, G_2) = 1$ if and only if $d_{K_3}(G_1 + K_1, G_2 + K_1) = 1$.

Proof Let $H = K_3$. Clearly, if $d_H(G_1, G_2) = 1$, then $d_H(G_1 + K_1, G_2 + K_1) = 1$. Next suppose that $d_H(G_1 + K_1, G_2 + K_1) = 1$. Let $n = |V(G_1)|$. Since $G_1 + K_1$ is H-adjacent to $G_2 + K_1$, there exist vertices u, v, w, and x of $G_1 + K_1$ such that $uv \in E(G_1 + K_1)$, and $G_2 + K_1 = (G_1 + K_1) - \{uw, vw\} + \{ux, vx\}$. Hence ux, $vx \notin E(G_1 + K_1)$ and $uw, vw \notin E(G_2 + K_1)$. Therefore, in $G_1 + K_1$, the vertices u and v have degree less than n and in $G_2 + K_1$, the vertices x and w have degree less than n. Thus u, v, w, x must be vertices of G_1, and hence $ux, vx \in E(G_2)$ and uw, $vw \in E(G_1)$. So $G_2 = G_1 - \{uw, vw\} + \{ux, vx\}$. Since $uv \in E(G_1)$, it follows that $d_H(G_1, G_2) = 1$. ❑

We can now show that the cartesian product of two K_3-distance graphs is a K_3-distance graph.

Theorem 3.13 For two K_3-distance graphs G_1 and G_2, the graph $G_1 \times G_2$ is a K_3-distance graph.

Proof Let $H = K_3$, and let S and T be sets of graphs for which $D_H(S) \cong G_1$ and $D_H(T) \cong G_2$. By Lemma 3.12, we may assume that S and T are disjoint and that each graph in $S \cup T$ is 3-connected. Assume that $S = \{G_u \mid u \in V(G_1)\}$ with $d_H(G_u, G_w) = 1$ if and only if $uw \in E(G_1)$. Similarly, $T = \{F_v \mid v \in V(G_2)\}$. We show that

$$G_1 \times G_2 \cong D_H(\{G_u \cup F_v \mid u \in V(G_1), v \in V(G_2)\}).$$

Since the graphs G_u and F_v are 3-connected, it follows that $d_H(G_u \cup F_v, G_{u'} \cup F_{v'}) = 1$ if and only if either (1) $G_u = G_{u'}$ and $d_H(F_v, F_{v'}) = 1$ or (2) $H_v = H_{v'}$ and $d_H(G_u, G_{u'}) = 1$. Thus $d_H(G_u \cup F_v, G_{u'} \cup F_{v'}) = 1$ if and only if (1) $u = u'$ and $vv' \in E(G_2)$ or (2) $v = v'$ and $uu' \in E(G_1)$, and the result follows. ❑

Suppose that G_1 and G_2 are K_3-distance graphs. The graph obtained by identifying a vertex u of G_1 with a vertex v of G_2 is an induced subgraph of $G_1 \times G_2$. Therefore, we have the corollary to Theorem 3.13.

Corollary 3.14 For K_3-distance graphs G_1 and G_2, the graph obtained from G_1 and G_2 by identifying a vertex u of G_1 with a vertex v of G_2 is also a K_3-distance graph.

If the blocks of a connected graph G are K_3-distance graphs, then from repeated applications of Corollary 3.14 to the blocks of G, we have that G is also a K_3-distance graph. Consequently, we have the following corollary.

Corollary 3.15 Every tree is a K_3-distance graph.

We now show that complete bipartite graphs are also K_3-distance graphs.

Theorem 3.16 For integers m and n with $3 \leq m \leq n$, the complete bipartite graph $K_{m,n}$ is a K_3-distance graph.

Proof Let $H = K_3$, let P: $v_0, v_1, \ldots, v_{n+m}$ be a path, and let C: u_1, u_2, u_3, u_4, u_1 be a 4-cycle. Next, let G be that graph obtained by identifying the vertex v_{n+m} of P with the vertex u_1 of C and adding $n + m$ vertices $w_1, w_2, \ldots, w_{n+m}$, where for each $i = 1, 2, \ldots, n + m$, the vertex w_i is joined to v_{i-1} and v_i. For $i = 1, 2, \ldots, m$, define G_i to be that graph obtained from G by joining a new vertex x to v_i. For $m = 2$ and $n = 3$, the graphs G_1 and G_2 are shown in Figure 3.8. The graphs $G_1, G_2, \ldots, G_m$ have the same order (namely, $2(n + m) + 5$) and same size (namely, $3(n + m) + 5$). Furthermore, in each graph G_i $(1 \leq i \leq m)$, there is an end-vertex, namely x, at distance $n + m - i$ from the unique 4-cycle of G_i. Therefore, the graphs $G_1, G_2, \ldots, G_m$ are pairwise nonisomorphic. Also, for $i \neq j$, the graphs G_i and G_j differ in exactly one edge, namely xv_i in G_i and xv_j in G_j, where this edge does not lie on a triangle. Thus G_i and G_j are not K_3-adjacent.

Next, we define the n graphs $F_1, F_2, \ldots, F_n$ such that for distinct integers i and j with $1 \leq i < j \leq n$, we have that $d_H(F_i, F_j) > 1$, while for each pair i, j of integers with $1 \leq i \leq m$ and $1 \leq j \leq n$, we have that $d_H(G_i, F_j) = 1$. For each $i = 1, 2, \ldots, n$, define

$$F_i = G_1 - \{v_{m+i-1}v_{m+i}, w_{m+i}v_{m+i}\} + \{v_{m+i-1}v_0, w_{m+i}v_0\}.$$

For $m = 2$ and $n = 3$, the graphs F_1, F_2, and F_3 are shown in Figure 3.9. The graph F_i $(1 \le i \le n)$ has a cycle of length $2 + i$, and thus the graphs $F_1, F_2, \dots, F_n$ are pairwise nonisomorphic.

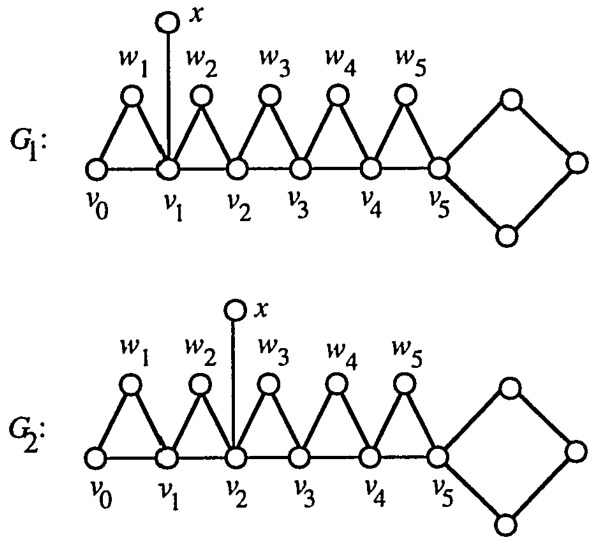

Figure 3.8 The Graphs G_1 and G_2.

To see that $d_H(G_i, F_j) = 1$ for each pair i, j of integers with $1 \le i \le m$ and $1 \le j \le n$, observe that

$$F_j = G_i - \{v_{m+j-1}v_{m+j}, v_{m+j}w_{m+j}, w_{m+j}v_{m+j-1}\} + \{v_{m+j}w_{m+j}, w_{m+j}v_0, v_0v_{m+j}\}$$

so G_i is H-adjacent to F_j. We now show for distinct integers i and j with $1 \le i < j \le n$ that $d_H(F_i, F_j) > 1$. For $i = 1, 2, \dots, n$, let $F_{i,1}$ denote the component of F_i containing the unique end-vertex of F_i, and let $F_{i,2}$ denote the other component of F_i. Next, for distinct integers i and j $(1 \le i < j \le n)$, the graph $F_{j,1}$ has at least one more triangle than $F_{i,1}$, and $F_{i,2}$ has at least one more triangle than $F_{j,2}$. Thus the graph F_i differs from the graph F_j in at least four edges so that $d_H(F_i, F_j) > 1$. Therefore, $D_H(\{G_1, G_2, \dots, G_m, F_1, F_2, \dots, F_n\}) = K_{m,n}$. ❑

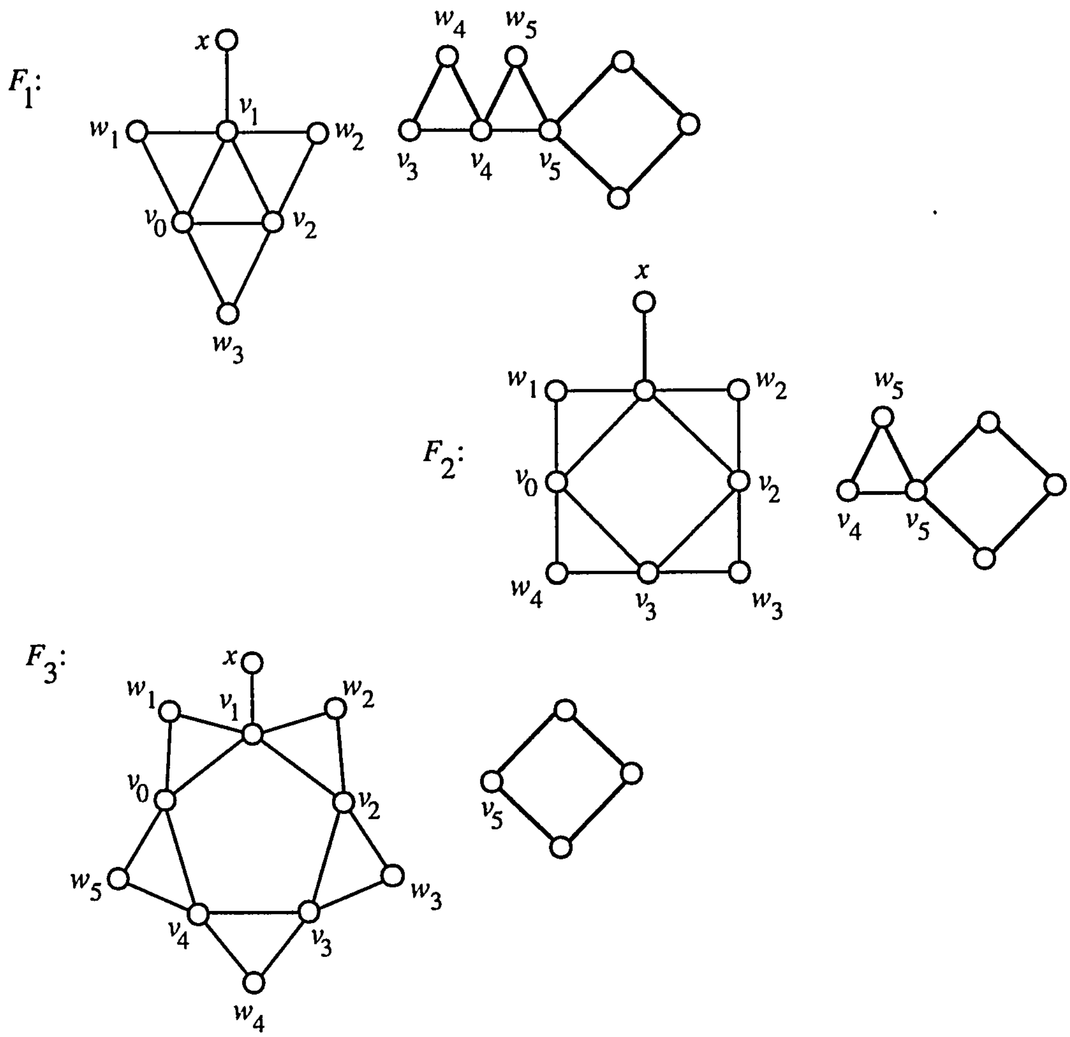

Figure 3.9 The Graphs F_1, F_2, and F_3.

We have seen that if H is a connected graph with $\delta(H) = 1$, then every graph is an H-distance graph. Also, many graphs are known to be K_3-distance graphs, such as complete graphs, complete bipartite graphs, cycles, paths, cartesian products of K_3-distance graphs, and trees. However, we know of no graph that is not a K_3-distance graph. In fact, we state the following conjecture.

Conjecture Every graph is a K_3-distance graph.

We now show that every graph that is an edge rotation distance graph is also a K_3-distance graph.

Theorem 3.17 If G is an edge rotation distance graph, then G is a K_3-distance graph.

Proof Let $H = K_3$ and let G be an edge rotation distance graph of order p. Then there exists a set $\{G_1, G_2, \ldots, G_p\}$ of graphs of the same order and same size, say m, such that $D_r(\{G_1, G_2, \ldots, G_p\}) \cong G$. Let $e_{i,1}, e_{i,2}, \ldots, e_{i,m}$ $(1 \le i \le p)$ denote the m edges of G_i. For $i = 1, 2, \ldots, p$, let F_i be that graph obtained from G_i by adding m new vertices $x_{i,1}, x_{i,2}, \ldots, x_{i,m}$, where $x_{i,j}$ $(1 \le j \le m)$ is joined to the two vertices of G_i incident with the edge $e_{i,j}$. We show that $D_H(\{F_1, F_2, \ldots, F_p\}) \cong G$. Let $V(G) = \{v_1, v_2, \ldots, v_p\}$. Since $D_r(\{G_1, G_2, \ldots, G_p\}) \cong G$, it follows that if v_iv_j is an edge of G, then G_i can be obtained from G_j by an edge rotation. Thus $G_i = G_j - uv + uw$ for distinct vertices u, v, and w of G_j. Now $uv = e_{j,k}$ and $uw = e_{i,\ell}$ for some k and ℓ with $1 \le k, \ell \le m$. Hence $G_i - e_{i,\ell} = G_j - e_{j,k}$ so that $F_i - e_{i,\ell} - ux_{i,\ell} - wx_{i,\ell} = F_j - e_{j,\ell} - ux_{j,\ell} - vx_{j,\ell}$. Next since u, w, and $x_{i,\ell}$ form a triangle in F_i and since u, v, $x_{j,k}$ form a triangle in F_j, we have that

$$F_i = F_j - \{e_{j,k}, ux_{j,k}, vx_{j,k}\} + \{ux_{j,k}, x_{j,k}w, uw\},$$

and F_i and F_j are H-adjacent. Therefore, G is a subgraph of $D_H(\{F_1, F_2, \ldots, F_p\})$.

To see that $D_H(\{F_1, F_2, \ldots, F_p\}) \cong G$, suppose that v_kv_ℓ is not an edge of G. Then the graph G_k cannot be obtained from the graph G_ℓ by an edge rotation. Assume, without loss of generality, that $V(G_k) = V(G_\ell)$. Let s denote the size of a greatest common subgraph of G_k and G_ℓ. Suppose first that $s \le m - 2$. Then G_k

and G_ℓ differ in at least two edges, that is, G_k contains at least two edges that are not present in G_ℓ, while G_ℓ contains at least two edges that are not present in G_k. Hence F_k contains two triangles that are not present in F_ℓ, while F_ℓ contains two triangles that are not present in F_k. Therefore, F_k and F_ℓ are not H-adjacent. Finally, if $s = m - 1$, then G_k and G_ℓ differ in exactly one edge. So $G_k - e = G_\ell - f$ for edges e and f of G_k and G_ℓ, respectively. Since G_k cannot be obtained from G_ℓ by an edge rotation, it follows that e and f are not adjacent. Thus $e = uv$ and $f = wz$ for distinct vertices u, v, w, and z of G_k. Hence F_k and F_ℓ differ in one triangle, where the triangle in F_k is induced by the vertices u, v, and x and x corresponds to $x_{k,i}$ for some i $(1 \le i \le m)$, and where the triangle in F_ℓ is induced by the vertices w, z, and x and here x corresponds to $x_{\ell,j}$ for some j $(1 \le j \le m)$. Hence F_k and F_ℓ are not H-adjacent. Thus

$$D_H(\{F_1, F_2, \ldots, F_p\}) \cong G. \quad \square$$

Thus, if the conjecture that every graph is an edge rotation graph is true, then the conjecture that every graph is a K_3-distance graph is true as well.

CHAPTER IV

GREATEST COMMON DISTANCE–PRESERVING SUBGRAPHS

4.1 Introduction and Examples

Recall that a *greatest common subgraph* of two nonisomorphic graphs G_1 and G_2 is defined as a graph of maximum size without isolated vertices that is a subgraph of both G_1 and G_2. Chartrand, Saba, and Zou [10] proved that for every graph G without isolated vertices, there exist nonisomorphic graphs G_1 and G_2 of equal size such that G is the unique greatest common subgraph of G_1 and G_2. In the proof of this result, one of G_1 and G_2 is disconnected, regardless of whether G is connected. However, in [6] Chartrand, Johnson, and Oellermann proved that for every connected graph G that is not complete, there exist connected graphs G_1 and G_2 of equal size such that G is the unique greatest common subgraph of G_1 and G_2. This concept has also been studied in [5, 6, 8, 10, 11, 12, 20, 21].

We now turn our attention to another common subgraph of two connected graphs. For a subgraph H of a graph G, we say that H is *distance–preserving* if $d_H(u, v) = d_G(u, v)$ for every pair u, v of vertices of H. A ***greatest common distance–preserving subgraph*** of two connected graphs G_1 and G_2 is a graph G of maximum size such that G is a distance–preserving subgraph of G_1 and G_2. This definition certainly implies that every distance–preserving subgraph is connected. The size of a greatest common distance–preserving subgraph is called the *gds size*, and the set of all such subgraphs is denoted by $\mathrm{gds}(G_1, G_2)$. To illustrate these concepts, we determine the greatest common distance–preserving subgraphs of P_n and C_n, the path and cycle on n vertices, respectively.

Consider a greatest common distance–preserving subgraph of P_6 and C_6. Since the distance between every pair of distinct vertices of C_6 is at most 3, it follows that P_4 is the unique greatest common distance–preserving subgraph of P_6 and C_6. In fact, since the distance between every pair of distinct vertices of C_n is at most $\lfloor n/2 \rfloor$, the path $P_{\lfloor n/2 \rfloor + 1}$ is the greatest common distance–preserving subgraph of C_n and P_n. Thus the gds size of P_n and C_n is $\lfloor n/2 \rfloor$ and $\text{gds}(P_n, C_n) = \{P_{\lfloor n/2 \rfloor + 1}\}$.

Next we show that distance–preserving subgraphs are induced subgraphs.

Lemma 4.1 Let H be a distance–preserving subgraph of a connected graph G. Then H is an induced subgraph of G.

Proof Let H be a distance–preserving subgraph of G. We show that $\langle V(H) \rangle = H$. Suppose, to the contrary, that there exists an edge e of $\langle V(H) \rangle$ such that e is not an edge of H. So $e = xy$ for some $x, y \in V(H)$. Thus, $d_G(x, y) = 1$ while $d_H(x, y) \geq 2$, producing a contradiction. Therefore $\langle V(H) \rangle = H$ and H is an induced subgraph of G. ❑

We now show that greatest common distance–preserving subgraphs need not be unique. Consider the graphs G_1 and G_2 of size 11, shown in Figure 4.1. Now C_6 is a distance–preserving subgraph of G_1 and G_2 and hence the gds size is at least 6.

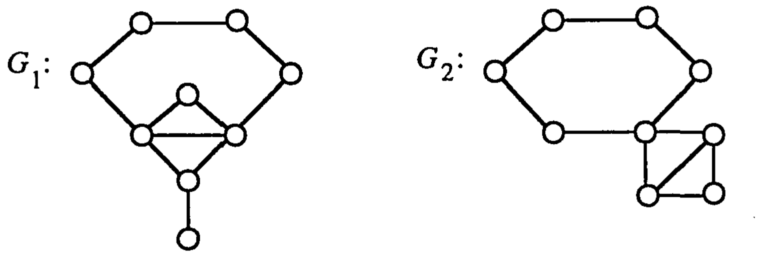

Figure 4.1 Graphs With Two Greatest Common Distance–Preserving Subgraphs.

Figure 4.2 shows all connected induced subgraphs of size 7 in G_2, namely the graphs $H_1, H_2, \dots, H_6$. Since each graph H_i $(1 \le i \le 6)$ is not an induced subgraph of G_1, it follows that the gds size is 6. The graphs $F_1, F_2, \dots, F_7$ of Figure 4.2 are all the connected induced subgraphs of size 6 in G_2. Now F_4, F_6, and F_7 are not induced subgraphs of G_1 while F_1, F_2, F_3, and F_5 are induced subgraphs of G_1. Since F_3 and F_5 are not distance–preserving subgraphs of G_1, it follows that $\mathrm{gds}(G_1, G_2) = \{F_1, F_2\}$.

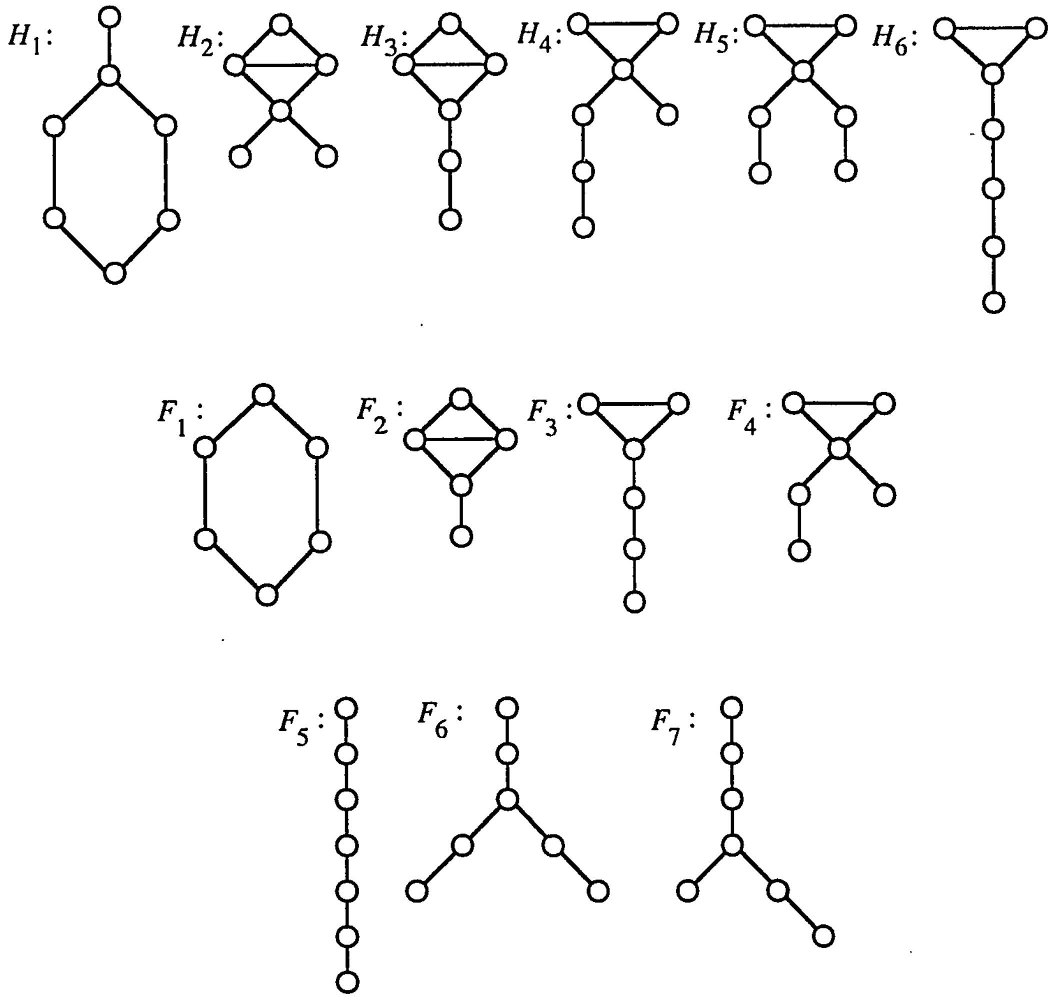

Figure 4.2 Connected Induced Subgraphs of Sizes 6 and 7 in G_2.

A greatest common subgraph and a greatest common distance–preserving subgraph of two given graphs G_1 and G_2 are, of course, common subgraphs of G_1 and G_2. A greatest common distance–preserving subgraph need not be a subgraph of a greatest common subgraph, however, as we now show.

Consider the graphs G_1 and G_2 of size 10 shown in Figure 4.3. Since G_1 is not isomorphic to G_2, the size of a greatest common subgraph is at most 9. The graph G of Figure 4.3 is a common subgraph of G_1 and G_2 having size 9, so G is a greatest common subgraph of G_1 and G_2. Since removing any edge other than the edge e from G_2 does not produce a subgraph of G_1, it follows that G is the unique greatest common subgraph of G_1 and G_2. Since diam $G_2 = 3$ and the graph H of Figure 4.3 is the subgraph of maximum size in G_1 with diameter 3, we have that $\mathrm{gds}(G_1, G_2) = \{H\}$. So the greatest common distance–preserving subgraph of G_1 and G_2 is not a subgraph of the greatest common subgraph of G_1 and G_2.

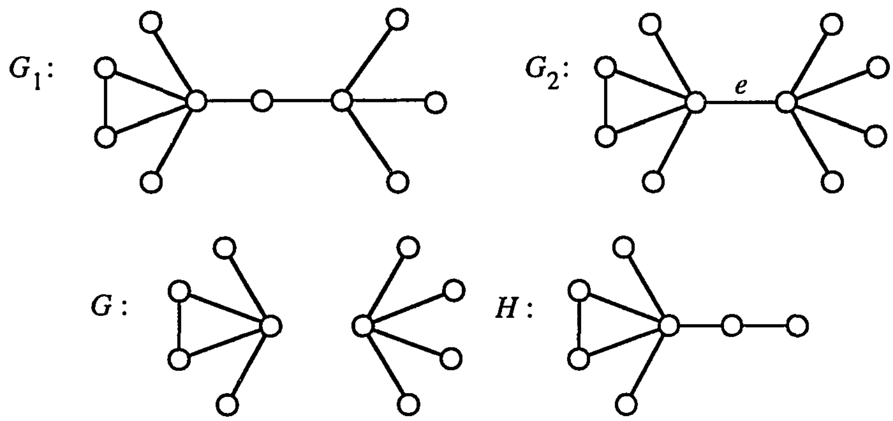

Figure 4.3 Graphs Where the Greatest Common Distance–Preserving Subgraph Is Not a Subgraph of the Greatest Common Subgraph.

A *greatest common induced subgraph* of two nonisomorphic graphs G_1 and G_2 is defined as a graph G of maximum size without isolated vertices that is an induced subgraph of both G_1 and G_2. Although greatest common distance–

preserving subgraphs are induced subgraphs, it need not be the case that a greatest common distance–preserving subgraph is a subgraph of a greatest common induced subgraph, as we shall now see.

Consider the graphs G_1 and G_2 shown in Figure 4.4. Clearly, G_1 and G_2 are nonisomorphic graphs of size 12. Now $G_1 - x \cong G_2 - y$, so the size of a greatest common induced subgraph of G_1 and G_2 is at least 10. Removing one end–vertex from G_2 does not produce a subgraph of G_1 and thus the size of a greatest common induced subgraph is at most 10. The induced subgraphs of G_2 of size 10 are obtained by removing a vertex of degree 2 or by removing two vertices of degree 1. Since removing any two end–vertices or removing any vertex of degree 2 other than y from G_2 does not produce a subgraph of G_1, it follows that $G_2 - y$ is the unique greatest common induced subgraph of G_1 and G_2.

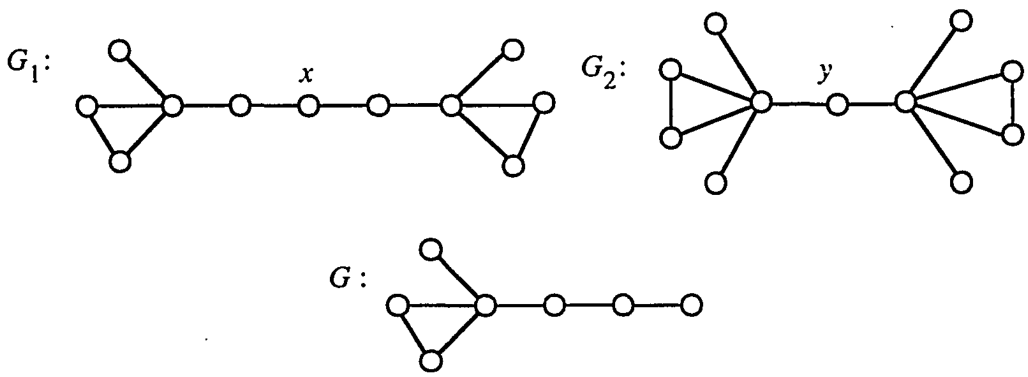

Figure 4.4 The Graphs G_1 and G_2 and the Graph G.

To determine the greatest common distance–preserving subgraphs of G_1 and G_2 of Figure 4.4, we consider the diameter of G_2. The graph G of Figure 4.4 is the unique induced subgraph of G_1 of maximum size with diameter 4. Since diam $G_2 = 4$, it follows that $\mathrm{gds}(G_1, G_2) = \{G\}$. Since G is not a subgraph of $G_2 - y$, the greatest common distance–preserving subgraph of G_1 and G_2 is not a subgraph

of the greatest common induced subgraph. Consequently, a greatest common distance–preserving subgraph need not be a subgraph of a greatest common induced subgraph, although both are induced subgraphs. In fact, we now show that the difference in the sizes of the these two subgraphs can be arbitrarily large.

Theorem 4.2 For every positive integer n, there exist graphs G_1 and G_2 of equal size such that the difference between the size of a greatest common induced subgraph and the size of a greatest common distance–preserving subgraph of G_1 and G_2 is n.

Proof First, let n be a positive integer. Let $F = C_{2n+2}$, say F: $v_1, v_2, \dots, v_{2n+2}, v_1$. The graph G_1 is obtained from F by adding the vertices $w_1, w_2, \dots, w_{n+1}$ along with the edges $v_1w_1, w_1w_2, w_2w_3, \dots, w_nw_{n+1}, w_{n+1}v_{n+2}$; while the graph G_2 is obtained from F by adding the vertices $x_1, x_2, \dots, x_{n+2}$ and the edges v_1x_1, $x_1x_2, x_2x_3, \dots, x_{n+1}x_{n+2}$. The graphs G_1 and G_2 are shown in Figure 4.5.

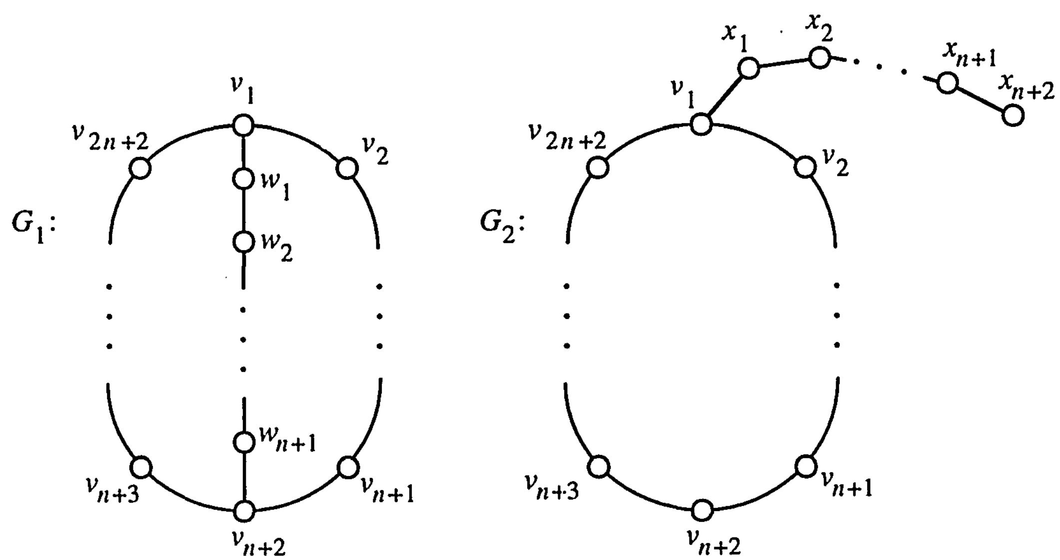

Figure 4.5 Graphs Where the Difference Between the Size of a Greatest Common Subgraph and the Size of a Greatest Common Distance–Preserving Subgraph Is n.

Each of G_1 and G_2 has size $3n+4$ and $G_1 \neq G_2$. Since $G_1 - w_{n+1} = G_2 - x_{n+1} - x_{n+2}$, the size of a greatest common induced subgraph is at least $3n + 2$. Moreover, since $\delta(G_1) \geq 2$, the size of a greatest common induced subgraph of G_1 and G_2 is at most $3n + 2$. Therefore, the size of a greatest common induced subgraph is $3n + 2$.

Since $\operatorname{diam} G = n + 1$, it follows that if H is a greatest common distance–preserving subgraph of G_1 and G_2, then $\operatorname{diam} H \leq n + 1$. Note that the $(2n + 2)$–cycle $v_1, v_2, \ldots, v_{2n+2}, v_1$ is a distance–preserving subgraph of G_1 and G_2. Thus the size of H is at least $2n + 2$. We now show the size of H is at most $2n + 2$. Suppose, to the contrary, that H has size q where $q > 2n + 2$. Since H is a connected induced subgraph of G_1 and $H \neq G_1$, it follows that either H contains one cycle, of length $2n + 2$, or H is tree. Suppose first that H contains exactly one cycle, so $v_1, v_2, \ldots, v_{2n+2}, v_1$ is the cycle of H. Since the size of H is at least $2n + 3$, there exists a vertex x not belonging to the cycle of H. Then the distance from x to v_{n+2} is at least $n+2$ and hence H is not distance–preserving. Thus H is a tree. Since $\Delta(G_1) = 3$, it follows that H is a path or H consists of three paths identified at v_1. If H is a path of size at least $2n + 3$, then $\operatorname{diam} H \geq 2n + 2$, producing a contradiction. So, H consists of three paths identified at v_1. Let ℓ_1, ℓ_2, and ℓ_3 denote the lengths of these three paths. Since H is a distance–preserving subgraph of G_1 and $\operatorname{diam} G_1 = n + 1$, it follows that $\ell_1 + \ell_2 \leq n + 1$ and $\ell_2 + \ell_3 \leq n + 1$ so $\ell_1 + 2\ell_2 + \ell_3 \leq 2n + 2$. But, since the size of H is $\ell_1 + \ell_2 + \ell_3$, we have that $\ell_1 + \ell_2 + \ell_3 > 2n + 2$, producing a contradiction. Thus H has size $2n + 2$. Therefore, the difference between the size of the greatest common induced subgraph and the size of the greatest common distance–preserving subgraph is $(3n + 2) - (2n + 2) = n$. ❑

4.2 Graphs With a Prescribed Greatest Common Distance–Preserving Subgraph

We have already noted that every graph without isolated vertices is the unique greatest common subgraph of some pair of nonisomorphic graphs of equal size. We now show that every connected noncomplete graph is the unique greatest common distance–preserving subgraph of two nonisomorphic connected graphs of equal size.

Theorem 4.3 For each connected graph G of order at least 3, there exist nonisomorphic graphs G_1 and G_2 of equal size such that $\text{gds}(G_1, G_2) = \{G\}$.

Proof First assume that $G \cong K_n$, where $n \geq 3$. Let G_1 be obtained from G by joining a new vertex z_1 to any two vertices of K_n. Next let x and y be two vertices of K_n and obtain G_2 from G by joining a new vertex w_1 to x and a new vertex w_2 to y. Clearly, $\text{gds}(G_1, G_2) = \{G\}$.

Next, let G be a connected noncomplete graph of size q and let t be the number of triangles in G. The construction of the graphs G_1 and G_2 depends on t and the minimum degree $\delta(G)$ of the vertices of G.

Case 1 *Suppose that* $\delta(G) \geq 3$ *or* $t = 0$. Since G is not complete, there exist vertices x and y such that $d(x, y) = 2$, say x, w, y is an x–y path in G. Let G_1 be obtained from G by adding a new vertex z_1 to G as well as the edges z_1x and z_1w. Next let G_2 be obtained from G by adding a new vertex z_2 to G and the edges z_2x and z_2y. Since G_1 has $t+1$ triangles while G_2 has t triangles, G_1 is not isomorphic to G_2. Clearly, G is a distance–preserving subgraph of G_1 and G_2. Therefore, the gds size is at least q.

We now show that the gds size is at most q. Suppose first that $\delta(G) \geq 3$. Then the removal of any vertex of G from G_1 results in a subgraph F of size at

most $q-1$ and thus F is not a distance–preserving subgraph of maximum size. Hence G is the unique greatest common distance–preserving subgraph of G_1 and G_2. Next suppose that $t = 0$. Then G_2 has no triangles while G_1 has one triangle. Therefore each greatest common subgraph has no triangles and hence we must remove at least one of x, z_1, and w from G_1 to produce a greatest common distance–preserving subgraph. Since $\deg_{G_1} w \geq 3$, we cannot remove w. If $\deg_{G_1} x \geq 3$, then we cannot remove x either, while if $\deg_{G_1} x = 2$, then $G_1 - x \cong G_1 - z_1 = G$. Thus, G is again the unique greatest common distance–preserving subgraph of G_1 and G_2.

Case 2 *Suppose that* $\delta(G) = 2$ *and* $t \geq 1$. Since G is not regular, there exist vertices x and y of G such that $\deg_G x \neq \deg_G y$. Let G_1 be obtained from G by joining a new vertex z_1 to x, and let G_2 be obtained from G by joining a new vertex z_2 to y. Clearly, $G_1 \not\cong G_2$. Thus the gds size is at most q. Now G is a distance–preserving subgraph of G_1 and G_2 and hence G is a greatest common distance–preserving subgraph. Furthermore, since $\delta(G) = 2$, it follows that G is the unique greatest common distance–preserving subgraph of G_1 and G_2.

Case 3 *Suppose that* $\delta(G) = 1$ *and* $t \geq 1$. For each end–vertex v of G, let m_v denote the distance from v to a nearest triangle of G. Let $m = \max\ \{m_v\}$ over all end–vertices v of G. Let x be an end–vertex for which $m_x = m$, and let T be a triangle of G whose distance from x is m. Define G_1 as that graph obtained from G by joining a new vertex z_1 to x and define G_2 as that graph obtained from G by joining a new vertex z_2 to the unique vertex y adjacent to x in G. Now G_1 has an end–vertex, namely z_1, whose distance from T is $m+1$, while every end–vertex of G_2 has distance at most m from T. Consequently, $G_1 \not\cong G_2$. Thus the gds size is at

most q. Since G is a distance–preserving subgraph, the gds size is at least q. We show that G is the unique greatest common distance–preserving subgraph. Suppose, to the contrary, that there exists a greatest common distance–preserving subgraph H of G_1 and G_2 with $H \not\cong G$. Then $H \cong H_1$, where H_1 is an induced subgraph of G_1. Since H_1 has size q and G_1 has size $q+1$, it follows that $H_1 = G_1 - w_1$ where $\deg_{G_1} w_1 = 1$. Similarly, there exists an induced subgraph H_2 of G_2 such that $H_2 \cong H$ and $H_2 = G_2 - w_2$, where w_2 is an end–vertex of G_2. Since $H \not\cong G$, it follows that $w_1 \neq z_1$ and $w_2 \neq z_2$. So $z_1 \in V(H_1)$. Thus H_1 contains an end–vertex, namely z_1, whose distance from the nearest triangle is $m + 1$, while every end–vertex in H_2 has distance at most m from any triangle. Therefore $H_1 \not\cong H_2$, producing a contradiction. Hence G is the unique greatest common distance preserving subgraph of G_1 and G_2. ❑

Chartrand, Johnson, and Oellermann [6] showed that for every nontrivial connected graph G, there exist nonisomorphic graphs G_1 and G_2 such that G is the unique greatest common induced subgraph of G_1 and G_2. In fact, in their proof of this result, the graph G is a distance–preserving subgraph of the graphs G_1 and G_2 constructed, so G is the unique greatest common distance–preserving subgraph of G_1 and G_2 as well. Also, in their proof, each of the graphs G_1 and G_2 contains two edges and one or two vertices not present in G so in some sense the proof just presented is simpler.

We now define the *greatest common distance–preserving subgraph* of n pairwise nonisomorphic connected graphs $G_1, G_2, \ldots, G_n$ as a graph G of maximum size such that G is a distance–preserving subgraph of G_i for $i = 1, 2, \ldots,$ n. As before, we refer to tthe size of G as the *gds size* and denote the set of greatest common distance–preserving subgraphs of $G_1, G_2, \ldots, G_n$ by gds($G_1, G_2, \ldots,$

G_n). In [10] it was shown that for every graph G without isolated vertices, there exist pairwise nonisomorphic graphs G_1, G_2, and G_3 of equal size such that G is the unique greatest common subgraph of G_1, G_2, and G_3. In a similar manner, we can extend Theorem 4.3 to show that for a given noncomplete connected graph G, there exist nonisomorphic graphs G_1, G_2, and G_3 such that G is the unique greatest common distance–preserving subgraph of G_1, G_2, and G_3.

Theorem 4.4 For each connected graph G of order at least 3, there exist nonisomorphic graphs G_1, G_2, and G_3 such that $\text{gds}(G_1, G_2, G_3) = \{G\}$.

Proof First, assume that $G \cong K_n$ for $n \geq 3$. Let x and y be two vertices of G. Obtain G_1 from G by joining a new vertex z_1 to x and y and obtain G_2 from G by adding two vertices w_1 and w_2 and the edges xw_1 and yw_2. Finally, let G_3 be obtained from G by adding two vertices v_1 and v_2 and the edges xv_1 and v_1v_2. Then $\text{gds}(G_1, G_2, G_3) = \{G\}$.

Now let G be a noncomplete connected graph of size q and let t be the number of triangles in G. The construction of the graphs G_1, G_2, and G_3 depends on t and $\delta(G)$.

Case 1 *Suppose that* $\delta(G) \geq 3$ *or* $t = 0$. As in the proof of Theorem 4.3, there exist vertices x and y such that $d(x, y) = 2$, say x, w, y is an x–y path in G. Construct G_1 and G_2 as in Case 1 of the proof of Theorem 4.3; that is, let G_1 be obtained from G by joining a new vertex z_1 to x and w and let G_2 be obtained from G by joining a new vertex z_2 to x and y. Next construct G_3 from G by joining two new vertices w_1 and w_2 to x. Then since G is a distance–preserving subgraph of G_1, G_2, and G_3, the gds size is at least q. Also, each greatest common distance–preserving subgraph of G_1, G_2, and G_3 is a distance–preserving subgraph of G_1

and G_2; so the gds size is at most the gds size of G_1 and G_2 or q. Therefore, since G is the unique greatest common distance–preserving subgraph of G_1 and G_2, it follows that $\text{gds}(G_1, G_2, G_3) = \{G\}$.

Case 2 *Suppose that* $\delta(G) = 2$ *and* $t \geq 1$. As in Case 2 of the proof Theorem 4.3, since G is not regular, there exist vertices x and y of G such that $\deg_G x \neq \deg_G y$. Obtain G_1 from G by joining a new vertex z_1 to x and obtain G_2 from G by joining a new vertex z_2 to y. Next let G_3 be obtained from G by joining two new vertices w_1 and w_2 to x. Clearly, G is a distance–preserving subgraph of $G_1, G_2,$ and G_3 and since $\text{gds}(G_1, G_2) = \{G\}$, it follows that $\text{gds}(G_1, G_2, G_3) = \{G\}$.

Case 3 *Suppose that* $\delta(G) = 1$ *and* $t \geq 1$. Construct G_1 and G_2 as in Case 3 of the proof of Theorem 4.3. Let G_3 be obtained from G_1 by joining a new vertex w_1 to z_1. Then since $\text{gds}(G_1, G_2) = \{G\}$ and G is a distance–preserving subgraph of $G_1, G_2,$ and G_3, it follows that $\text{gds}(G_1, G_2, G_3) = \{G\}$. ❑

It is a fact that for every positive integer n and every connected, noncomplete graph G of size q, there exist graphs $G_1, G_2, \ldots, G_n$ such that $\text{gds}(G_1, G_2, \ldots, G_n) = \{G\}$. To obtain the graphs $G_1, G_2, \ldots, G_n$, we begin by constructing G_1 and G_2 as in the proof of Theorem 4.3. To construct the graph G_i for $3 \leq i \leq n$, we join $i - 1$ vertices to the vertex of G labeled x in each case of the proof of Theorem 4.3. Since G is a distance-preserving subgraph of G_i $(1 \leq i \leq n)$, the gds size of $G_1, G_2, \ldots, G_n$ is at least q. Thus since $\text{gds}(G_1, G_2) = \{G\}$, it follows that $\text{gds}(G_1, G_2, \ldots, G_n) = \{G\}$.

Note that the graphs G_1 and G_2 constructed in Theorem 4.3 have equal size, while the graphs G_1, G_2, G_3 of Theorem 4.4 do not have equal size. This suggests

the following question: For a positive integer n and a connected, noncomplete graph G, do there exist graphs $G_1, G_2, \ldots, G_n$ of equal size such that $\mathrm{gds}(G_1, G_2, \ldots, G_n) = \{G\}$? Certainly, if n is large, then the size of each graph G_i must also be large. In [10] the *greatest common subgraph index* or *gcs index* of a graph G without isolated vertices, denoted by $i(G)$, is defined as the least positive integer q_0 such that for any integer $q > q_0$ and any collection of graphs $G_1, G_2, \ldots, G_n$, $n \geq 2$, of size q for which G is a greatest common subgraph of $G_1, G_2, \ldots, G_n$, it follows that the graphs $G_1, G_2, \ldots, G_n$ have another greatest common subgraph, different from G. If no such q_0 exists, then $i(G) = \infty$.

Similarly, for a connected graph G, the *greatest common distance-preserving index* or *gds index*, denoted by $i_d(G)$, is the least positive integer q_0 such that for any integer $q > q_0$ and any collection of graphs $G_1, G_2, \ldots, G_n$, $n \geq 2$, of size q for which $G \in \mathrm{gds}(G_1, G_2, \ldots, G_n)$, it follows that $|\mathrm{gds}(G_1, G_2, \ldots, G_n)| > 1$. If no such integer q_0 exists, then $i_d(G) = \infty$.

It was shown in [10] that for integers $r \geq 1$ and $n \geq 3$,

(a) $i(K_{1,r}) = \infty$

(b) $i(K_n) = \begin{cases} 6 & \text{if } n = 3 \\ \infty & \text{if } n \neq 3 \end{cases}$

(c) $i(P_n) = \begin{cases} \infty & \text{if } n \neq 4 \\ 6 & \text{if } n = 4. \end{cases}$

Also, Kubicki [20] has shown that for integers r and s with $r, s \geq 1$, the gcs index $i(K_{r,s}) = \infty$. He also gave a sufficient condition in [20] for a graph to have infinite gcs index, namely, if a graph G contains a vertex v of maximum degree such that no component of $G - v$ is isomorphic to K_2, then $i(G) = \infty$.

We now show similar results for the gds index.

Theorem 4.5 For positive integers n, r, and s with $n \geq 3$ and $s > 1$,

(a) $i_d(K_n) = \infty$

(b) $i_d(C_n) = \infty$

(c) $i_d(K_{r,s}) = \infty$.

(d) For every connected nonregular graph G, the gds index $i_d(G) = \infty$.

Proof (a) Suppose, to the contrary, that $i_d(K_n) = q_0$. Let m be a positive integer such that $\binom{m}{2} > q_0$, and let $q = \binom{m}{2}$. Let G_1 be obtained from K_n by joining $q - \binom{n}{2}$ new vertices to a vertex of K_n and let $G_2 = K_m$. Since $\text{diam}(G_2) = 1$, it follows that every distance–preserving subgraph of G_2 is complete. Thus $\text{gds}(G_1, G_2)$ is the maximum clique of G_1 or $\text{gds}(G_1, G_2) = \{K_n\}$, producing a contradiction.

(b) By part (a), we have that $i_d(C_3) = \infty$. Next for $n \geq 4$, suppose, to the contrary, that $i_d(C_n) = q_0$ for some positive integer q_0. Let x and y be two vertices of C_n such that $d(x, y) = 2$, say x, w, y is a x-y path in C_n. Next, let k be a positive integer such that $n + 2k > q_0$. Obtain G_1 from C_n by joining k new vertices $v_1, v_2, \ldots, v_k$ to x and to w. Next obtain G_2 from C_n by adding new vertices $z_1, z_2, \ldots, z_{2k-1}$ and the edges $xz_1, z_1z_2, z_2z_3, \ldots, z_{2k-2}z_{2k-1}, z_{2k-1}y$. An example of the graphs G_1 and G_2 is shown in Figure 4.6 when $n = 4$ and $k = 3$. Since G_2 has no triangles and $\Delta(G_2) = 3$, it follows that $\text{gds}(G_1, G_2) = \{C_n\}$, producing a contradiction.

(c) Let r and s be positive integers with $r \leq s$ and $s > 1$. Suppose, to the contrary, that $i_d(K_{r,s}) = q_0$ for some positive integer q_0. Let x be a vertex of degree r in $K_{r,s}$ and let y be a vertex of degree s in $K_{r,s}$. Next, let k be a positive integer such that $rs + 2k > q_0$. Obtain G_1 from $K_{r,s}$ by joining k vertices $v_1, v_2, \ldots, v_k$ to both x and y. Next obtain G_2 from $K_{r,s}$ by adding new vertices $z_1, z_2, \ldots, z_{2k}$

and the edges $xz_1, z_1z_2, z_2z_3, \ldots, z_{2k-1}z_{2k}$. Then G_1 and G_2 have size $rs + 2k$ and since G_2 has no triangles and $\Delta(G_2) \le s + 1$, it follows that $\text{gds}(G_1, G_2) = \{K_{r,s}\}$, producing a contradiction.

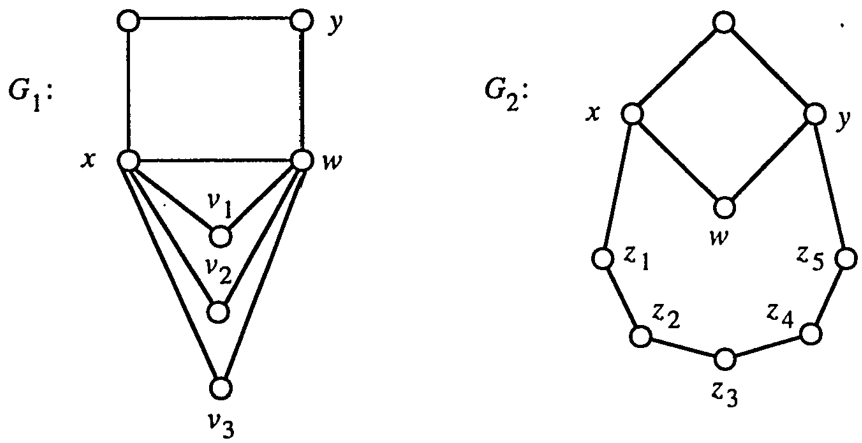

Figure 4.6 The Graphs G_1 and G_2 When $k = 3$ and $n = 4$.

(d) Let G be a graph of size q that is not regular. Then there exist vertices x and y such that $\deg y < \deg x = \Delta(G)$. Next suppose, to the contrary, that $i_d(G) = q_0$ for some positive integer q_0. Let m be a positive integer such that $q + m > q_0$. Let G_1 be obtained from G by joining new vertices $z_1, z_2, \ldots, z_m$ to x and let G_2 be obtained from G by adding the vertices $w_1, w_2, \ldots, w_m$ and the edges $yw_1, w_1w_2, w_2w_3, \ldots, w_{m-1}w_m$. Now $\Delta(G_1) = \Delta(G) + m$ while $\Delta(G_2) = \Delta(G)$. Thus if $H \in \text{gds}(G_1, G_2)$, then $\Delta(H) \le \Delta(G)$. Also since G is a distance–preserving subgraph of G_1 and G_2, it follows that H has size at least q. Hence if x is not a vertex of H, then H has size at most $q - \Delta(G)$ and therefore x is a vertex of H. Since $\deg_{G_1} x = \Delta(G) + m$ and $\Delta(G_2) = \Delta(G)$, it follows that to obtain H from G_1 we must remove at least m vertices $x_1, x_2, \ldots, x_m$ from G_1, where each x_i $(1 \le i \le m)$ is adjacent to x. If $\deg x_i \ge 2$ for any $1 \le i \le m$, then the size of H is at most $q - 1$

and thus each x_i must have degree 1. Therefore $H = G$ and hence $\text{gds}(G_1, G_2) = \{G\}$, producing a contradiction. ❑

No connected graph G is known for which $i_d(G)$ is finite. If such a connected graph G exists, then G must be regular and noncomplete, but neither a cycle nor the graph $K_{n,n}$ for some positive integer n. We have the following conjecture.

Conjecture For every connected graph G, the gds index $i_d(G)$ is infinite.

4.3 Greatest Common Distance–Preserving Trees

In [11] Chartrand and Zou studied trees and greatest common subgraphs. Let $D(t)$ denote that graph obtained from two stars $K_{1,t}$ whose central vertices are connected by a path of length 3. Chartrand and Zou [11] proved that not every tree is the unique greatest common subgraph of two nonisomorphic trees of equal size, that is, for a nontrivial tree T, there exist nonisomorphic trees T_1 and T_2 such that T is the unique greatest common subgraph of T_1 and T_2 if and only if $T \neq P_n$ for $n = 2, 4, 5, \ldots$ and $T \neq D(t)$ for $t \geq 2$. However, they also proved in [11] that every tree of order at least 3 is the unique greatest common induced subgraph of two nonisomorphic trees of equal size.

We now study trees and greatest common distance-preserving subgraphs. Recall that greatest common distance–preserving subgraphs need not be unique. By Theorem 4.3, for every tree T, there exist nonisomorphic graphs G_1 and G_2 such that $\text{gds}(G_1, G_2) = \{T\}$. However, in the proof of Theorem 4.3, the graphs G_1 and G_2 are not trees. We now show that for every tree T, there exist nonisomorphic trees T_1 and T_2 of equal size such that T is the unique greatest common distance–preserving subgraph of T_1 and T_2.

Theorem 4.6 For every tree T of order at least 3, there exist nonisomorphic trees T_1 and T_2 of equal size such that $\text{gds}(T_1, T_2) = \{T\}$.

Proof Let T have order p, where $p \geq 3$. Let x be a vertex of maximum degree in T. Let T_1 be obtained from T by joining a new vertex z_1 to x and let T_2 be obtained from T by joining a new vertex z_2 to an end–vertex of T. Since $p \geq 3$, it follows that $T_1 \neq T_2$. Clearly, T is a distance–preserving subgraph of T_1 and T_2. Also, since T_1 has a vertex of degree $\Delta(T) + 1$, namely x, while every vertex of T_2 has degree at most $\Delta(T)$, it follows that if $H \in \text{gds}(T_1, T_2)$, then H has size $p - 1$ and $H = T_1 - v$, where v is an end–vertex of T_1 adjacent to x. Thus $T_1 - v = T$ so that $H \cong T$. Hence, $\text{gds}(T_1, T_2) = \{T\}$. ❑

We have previously noted that a greatest common distance–preserving subgraph H of two nonisomorphic graphs G_1 and G_2 need not be a subgraph of any greatest common subgraph G of G_1 and G_2. However, if G_1 and G_2 are trees and G is connected, then the situation is different.

Theorem 4.7 For trees T_1 and T_2, a greatest common subgraph T of T_1 and T_2 is connected if and only if T is a greatest common distance–preserving subgraph.

Proof First, suppose that T is the greatest common subgraph of T_1 and T_2 and that T is connected. Let x and y be vertices of T. Then since T_1 is a tree, there exists a unique x–y path P_1 in T_1. Since T is connected and P_1 is the unique x-y path in T_1, it must be that P_1 is the unique x–y path in T so that $d_T(x, y) = d_{T_1}(x\ y)$. Similarly, $d_T(x, y) = d_{T_2}(x\ y)$. Therefore, T is distance–preserving. Since the gds size is at most the size of a greatest common subgraph, T is a greatest common distance–preserving subgraph.

Next suppose that T is a greatest common subgraph such that $T \in$ gds(T_1, T_2). Since greatest common distance–preserving subgraphs are connected, it follows that T is connected. ❑

Since greatest common distance–preserving subgraphs are induced subgraphs, it immediately follows that a greatest common subgraph T of two trees T_1 and T_2 is connected if and only if T is a greatest common induced subgraph of T_1 and T_2. Next we have an example of two trees T_1 and T_2 such that T_1 and T_2 have two greatest common distance–preserving subgraphs.

Consider the trees T_1 and T_2 shown in Figure 4.7. Now the trees F_1 and F_2 shown in Figure 4.7 are the only distance–preserving subgraphs of size 5 of T_1 and T_2.

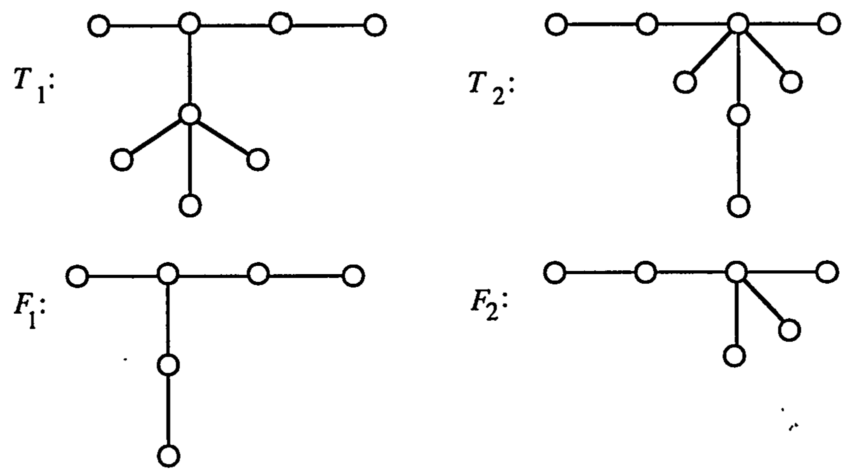

Figure 4.7 Trees With Two Greatest Common Distance–Preserving Subgraphs.

Thus the gds size is at least 5. The graphs J_1, J_2, and J_3 of Figure 4.8 are all the connected subgraphs of size 6 in T_1. Since none of these graphs is a subgraph of T_2, it follows that the gds size is at most 5. Hence, gds(T_1, T_2) = $\{F_1, F_2\}$.

In [10], it was shown that for every positive integer n, there exist graphs G_n and H_n such that G_n and H_n have n distinct greatest common subgraphs. We now show that there exist pairs of graphs having an arbitrarily large number of greatest common distance–preserving subgraphs. In fact, we show that for every positive integer $n \geq 2$, there exist graphs G_1 and G_2 having n greatest common distance–preserving subgraphs, each of which is a tree of size $2n + 2$.

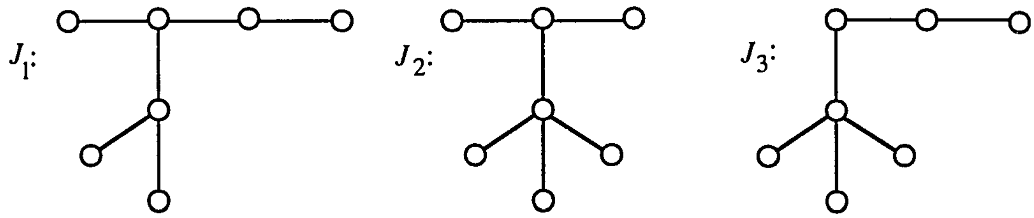

Figure 4.8 Connected Subgraphs of Size 6 in T_1.

Theorem 4.8 For every positive integer $n \geq 2$, there exist graphs G_1 and G_2 such that $|\text{gds}(G_1, G_2)| = n$.

Proof Let $n \geq 2$ be a positive integer and let P be a path of length $2n + 1$, say P: $v_1, v_2, \ldots, v_{2n+2}$. Next let G_1 be obtained from P by adding the vertices x_1, $x_2, \ldots, x_n$ and the edges x_iv_{i+1} for $i = 1, 2, \ldots, n$. Let G_2 be obtained from P by adding the vertices $y_1, y_2, \ldots, y_n$ and $w_1, w_2, \ldots, w_{n-2}$ and the edges y_iv_{i+1} for i odd $(1 \leq i \leq n)$, y_iv_{2n+2-i} for i even $(1 \leq i \leq n)$, and y_iw_i, w_iy_{i+2} for $1 \leq i \leq n - 2$. An example of the graphs G_1 and G_2 is shown in Figure 4.9, where $n = 5$.

For $i = 1, 2, \ldots, n$, let H_i be that graph obtained from P by joining a new vertex z_i to v_{i+1}. Then each H_i is a distance–preserving subgraph of G_1 and G_2, and hence the gds size of G_1 and G_2 is at least $2n + 2$. Let H be a common subgraph of G_1 and G_2. Since H is a subgraph of G_1, it follows that H has the general form shown in Figure 4.10. We may also assume, without loss of generality,

that the end-vertices of a longest path in H are labeled v_i and v_j for some $1 \le i < j \le n$, and that H contains k $(0 \le k \le n)$ of the vertices $x_1, x_2, \dots, x_n$.

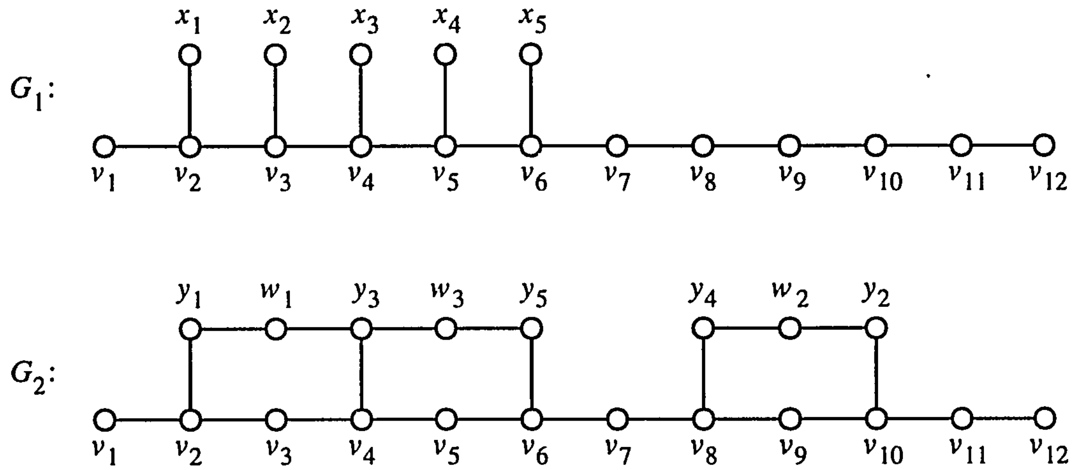

Figure 4.9 The Graphs G_1 and G_2 When $n = 5$.

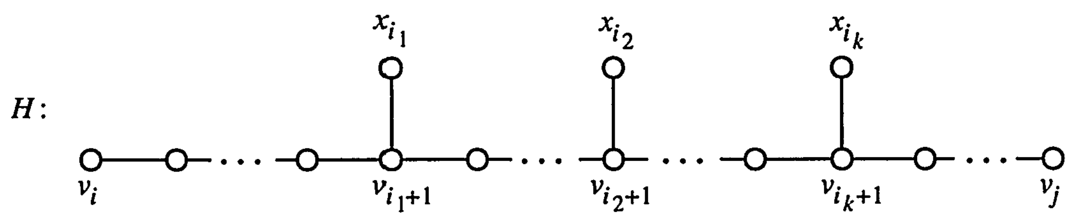

Figure 4.10 A Generic Subgraph of G_1.

Since H is a subgraph of G_2 and $i_k + 1 \le n + 1$, it follows that $(i_m + 1) - (i_l + 1)$ is even for each l and m with $1 \le l < m \le k$. Note also that if we label the vertices of H as a subgraph of G_2, then each vertex x_{i_l} is labeled y_{i_l}. Next for each i $(1 \le i < n)$ and m $(1 \le m \le \lfloor n/2 \rfloor$ with $i + 2m \le n)$, we have that $d_{G_1}(x_i, x_{i+2m}) = 2m + 2$ while $d_{G_2}(y_i, y_{i+2m}) = 2m$. Thus if H is a distance-preserving subgraph of G_1 and G_2, it must be that $k = 0$ or $k = 1$. If H is a greatest common

distance–preserving subgraph of G_1 and G_2, then H has size at least $2n+2$ and contains at most one of the vertices $x_1, x_2, \ldots, x_n$. Hence $H \cong H_i$ for some i with $1 \le i \le n$. ❑

Before continuing, we require additional terminology and notation. For a graph G, the *automorphism group* Aut(G) is the group of automorphisms of G. Each automorphism α of Aut(G) permutes the vertices of G. In fact, the group Aut(G) partitions $V(G)$ into *orbits* where vertices x and y of G belong to the same orbit if and only if there exists an automorphism α of Aut(G) such that $\alpha(x) = y$. For positive integers m and n, the *double star* $S_{m,n}$ is that tree of order $m+n$ with exactly two vertices that are not end-vertices, one of degree m and one of degree n. For a connected graph G with a subgraph H, the *distance* from a vertex x of G to H, denoted by $d(x, H)$, is the length of a shortest path from x to a vertex of H, i.e., $d(x, H) = \min\{d(x, y) \mid y \in V(H)\}$. So $d(x, H) = 0$ if and only if x is a vertex of H.

We have seen that there exist two trees having exactly two distinct greatest common distance-preserving subgaphs. Chartrand, Saba, and Zou showed in [10] that for every two graphs G_1 and G_2 of equal size, $\{K_{1,6}, K_4\}$ is not the set of greatest common subraphs of G_1 and G_2. We show that if G_1 and G_2 are two trees with $K_{1,r} \in \mathrm{gds}(G_1, G_2)$ for some positive integer r, then $\mathrm{gds}(G_1, G_2) = \{K_{1,r}\}$. We begin by giving conditions under which there exist graphs G_1 and G_2 having at least two greatest common distance–preserving subgraphs, one of which contains exactly two orbits. Observe that if T is a tree having exactly two orbits, then one orbit contains the end-vertices of T while the other orbit contains all the other vertices, necessarily all of the same degree. Thus $T = S_{1,r}$ or $T = S_{m,m}$ for positive integers m and r.

Theorem 4.9 Let G_1 and G_2 be two trees having at least two greatest common distance–preserving subgraphs T_1 and T_2. If T_1 has exactly two orbits, then T_1 and T_2 are double stars, with $T_1 \cong S_{m,m}$ for some positive integer m.

Proof Suppose that the tree T_1 has order p and contains exactly the two orbits O_1 and O_2. Then one of these orbits contains the end-vertices of T_1 and the other orbit contains all other vertices, say O_2 is the orbit containing the end-vertices of T_1. Next let $w_1 \in V(G_1) - V(T_1)$ such that $d_{G_1}(w_1, T_1) = 1$ and let $w_2 \in V(G_2) - V(T_1)$ such that $d_{G_2}(w_2, T_1) = 1$. Since G_1 and G_2 are trees, it follows that each of w_1 and w_2 is adjacent to a unique vertex of T_1 in G_1 or G_2, respectively. First observe that w_1 and w_2 cannot be adjacent to vertices of T_1 belonging to the same orbit of T_1; for otherwise $\langle V(T_1) \cup \{w_1\}\rangle$ is a subgraph of G_1 isomorphic to the subgraph $\langle V(T_1) \cup \{w_2\}\rangle$ of G_2, contradicting the fact that the gds size is $p - 1$. Thus we may assume that w_1 and w_2 are adjacent to vertices of T_1 belonging to distinct orbits. Assume, without loss of generality, that w_1 is adjacent to a vertex v of O_1 and that w_2 is adjacent to some vertex, say v_1, of O_2. Also, we may assume that each vertex x of G_1 with $d_{G_1}(x, T_1) = 1$ is adjacent to a vertex of O_1 and that each vertex y of G_2 with $d_{G_2}(y, T_1) = 1$ is adjacent to a vertex of O_2, for otherwise we are in the situation described above. Among all the vertices of G_1 at distance 1 from T_1, choose w_1 so that $\deg w_1$ is maximum.

Suppose first that $\deg w_1 \geq 2$. Let $w \in V(G_1) - V(T_1)$ be adjacent to w_1. Then there exists $v_2 \in O_2$ such that the subgraph $T = \langle V(T_1 - v_2) \cup \{w_1, w\}\rangle$ of G_1 is isomorphic to the subgraph $\langle V(T_1) \cup \{w_2\}\rangle$ of G_2. So T is a distance–preserving subgraph of G_1 and G_2 having size p, producing a contradiction. Therefore $\deg w_1 = 1$ and all vertices in $V(G_1) - V(T_1)$ are end-vertices, adjacent only to vertices of O_1. If $|O_1| = 1$, then $\operatorname{diam} G_1 = 2$ and hence G_1 is a star.

Thus each of T_1 and T_2 is a star. Since T_1 and T_2 have the same size, we have that $T_1 \cong T_2$, producing a contradiction. If $|O_1| = 2$, then $\text{diam } T_1 = 3$ and hence $T_1 \cong S_{m,m}$ for some positive integer m with $2m = p$. Thus since all vertices of G_1 not belonging to T_1 are end-vertices, we have that $G_1 \cong S_{k,\ell}$ for positive integers k and ℓ, with $k + \ell > p$. Since T_2 is a subgraph of G_1, it must be that $T_2 \cong S_{r,t}$ for positive integers r and t with $r + t = p$. ❑

As the next example shows, Theorem 4.9 is best possible in the sense that there exist two graphs G_1 and G_2 such that $\text{gds}(G_1, G_2) = \{S_{m,m}, S_{k,\ell}\}$, where k, ℓ and m are distinct positive integers such that $k + \ell = 2m$. Let k, ℓ, and m be positive integers such that $k + \ell = 2m$. Without loss of generality, we may assume that $k \leq m \leq \ell$. Let $G_1 \cong S_{m,\ell}$ and let G_2 be obtained from $S_{m,m}$ and $S_{k,\ell}$ by identifying an end-vertex of $S_{m,m}$ with an end-vertex of $S_{k,\ell}$. This situation is shown in Figure 4.11 for $k = 3$, $\ell = 5$, and $m = 4$. Now all connected subgraphs of G_1 are double stars and hence since $S_{m,m}$ and $S_{k,\ell}$ are the double stars of G_2 of maximum size, it follows that $\text{gds}(G_1, G_2) = \{S_{m,m}, S_{k,\ell}\}$.

Next, we have the following corollary.

Corollary 4.10 There do not exist two trees G_1 and G_2 such that G_1 and G_2 have at least two greatest common distance-preserving subgraphs T_1 and T_2 with $T_1 \cong K_{1,r}$ for some positive integer r.

Proof Suppose, to the contrary, that there exist two trees G_1 and G_2 such that $\{T_1, T_2\} \subseteq \text{gds}(G_1, G_2)$ and $T_1 \cong K_{1,r}$ for some positive integer r. Since T_1 and T_2 are distinct trees, it follows that $r \geq 3$. Then T_1 has two orbits and hence by

Theorem 4.9, it must be that $T_1 \cong S_{m,m}$ for some positive integer m, contradicting the fact that $T_1 \cong K_{1,r}$. ❑

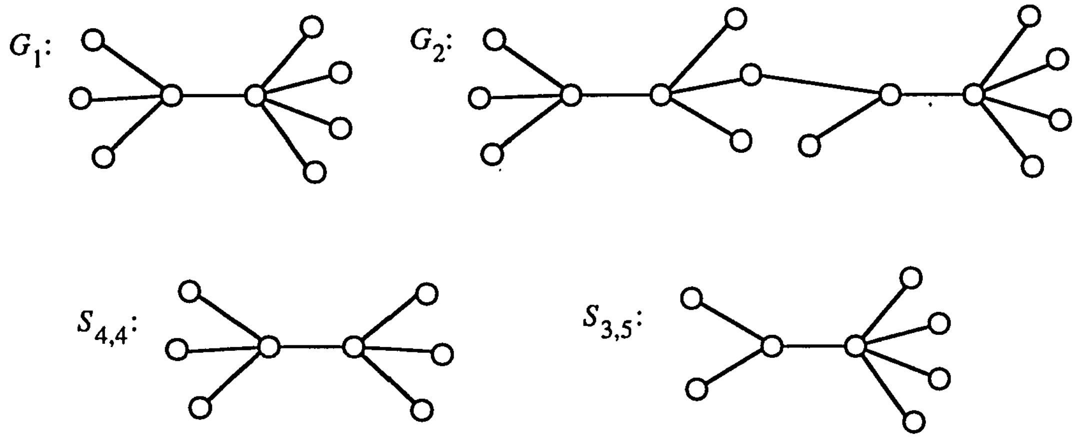

Figure 4.11 The Graphs G_1, G_2, $S_{4,4}$, and $S_{3,5}$ When $k = 3$, $\ell = 5$, and $m = 4$.

As a corollary to Corollary 4.10, we have the following.

Corollary 4.11 If there exist trees G_1 and G_2 with $K_{1,r} \in \text{gds}(G_1, G_2)$ for some positive integer r, then $\text{gds}(G_1, G_2) = \{K_{1,r}\}$.

CHAPTER V

GREATEST COMMON LOCALLY–PRESERVING SUBGRAPHS

5.1 Introduction

In Chapter IV we defined a subgraph H of a graph G to be distance–preserving if $d_H(u, v) = d_G(u, v)$ for every pair u, v of vertices of H. Also in Chapter IV, we defined a greatest common distance–preserving subgraph of two connected graphs G_1 and G_2 as a graph of maximum size that is a distance–preserving subgraph of G_1 and G_2. This definition clearly implies that every distance–preserving subgraph is connected. We now turn our attention to another type of distance–preserving subgraph of two graphs. For a subgraph H without isolated vertices of a graph G, we say that H is *locally distance–preserving* or simply *locally–preserving* if $d_H(u, v) = d_G(u, v)$ for every pair u, v of vertices belonging to the same component of H. So a distance–preserving subgraph is locally–preserving, but not conversely. A *greatest common locally–preserving subgraph* of two graphs G_1 and G_2 is a graph H, without isolated vertices, of maximum size such that H is a locally–preserving subgraph of G_1 and G_2. The size of a greatest common locally–preserving subgraph is called the *gls size* and the set of all such subgraphs is denoted by $\mathrm{gls}(G_1, G_2)$.

To illustrate these concepts, consider the greatest common locally–preserving subgraphs of P_n and C_n. Although P_n is a subgraph of C_n, the path P_n is not a locally–preserving subgraph since $\operatorname{diam} P_n = n - 1$ while $\operatorname{diam} C_n = \lfloor n/2 \rfloor$. Thus the gls size is at most $n - 2$. Now if n is even, say $n = 2k$ for a positive integer k, then $\mathrm{gls}(P_{2k}, C_{2k}) = \{P_{k+1} \cup P_{k-1}, P_k \cup P_k\}$; while if n is odd, say $n = 2k + 1$, then

$\text{gls}(P_{2k+1}, C_{2k+1}) = \{P_{k+1} \cup P_k\}$. In Chapter IV we saw that $P_{\lfloor n/2 \rfloor+1}$ is the unique greatest common distance–preserving subgraph of P_n and C_n. Thus a greatest common locally–preserving subgraph can be different from a greatest common distance–preserving subgraph.

Recall that distance–preserving subgraphs are induced subgraphs. A similar statement is true for locally–preserving subgraphs.

Lemma 5.1 Let H be a locally–preserving subgraph of a graph G. Then each component of H is an induced subgraph of G.

Proof Let $H_1, H_2, \dots, H_n$ $(n \geq 1)$ denote the components of H. We show that in G, the subgraph $\langle V(H_1) \rangle$ is in fact H_1. Suppose, to the contrary, that there exists an edge e of $\langle V(H_1) \rangle$ such that e is not an edge of H_1. Then $e = uv$ for some $u, v \in V(H_1)$. Thus $d_G(u, v) = 1$ while $d_H(u, v) \geq 2$ and since u and v belong to the same component of H, we have a contradiction. Hence $\langle V(H_1) \rangle = H_1$ and H_1 is an induced subgraph of G. Similarly, $H_2, H_3, \dots, H_n$ are induced subgraphs of G. ❑

Although the components of a greatest common locally–preserving subgraph are induced subgraphs, no such component is necessarily a greatest common distance–preserving subgraph, as the next example shows.

Consider the graphs G_1 and G_2 of size 8, shown in Figure 5.1. Since G_1 is not an induced subgraph of G_2, it follows that the gls size is at most 7. Next since $C_3 \cup C_4$ is a locally–preserving subgraph of G_1 and G_2, we have that the gls size is 7. Since the edge e is the only bridge of G_1, it follows that any subgraph of size 7 different from $C_3 \cup C_4$ is connected. Since each component of a locally–preserving subgraph must be an induced subgraph of G_1, it follows that $\text{gls}(G_1, G_2) = \{C_3 \cup$

$C_4\}$. The graph G, shown in Figure 5.1, is a distance–preserving subgraph of G_1 and G_2; so the gds size is at least 6. Since $\delta(G_1) = 2$ and greatest common distance–preserving subgraphs are induced subgraphs, it follows that the gds size is at most 6. Now removing any vertex of degree 2 from G_1 other than x or y does not produce an induced subgraph of G_2 and hence $\text{gds}(G_1, G_2) = \{G\}$. Since G is not a subgraph of $C_3 \cup C_4$, we have that a greatest common distance–preserving subgraph need not be a subgraph of a greatest common locally–preserving subgraph.

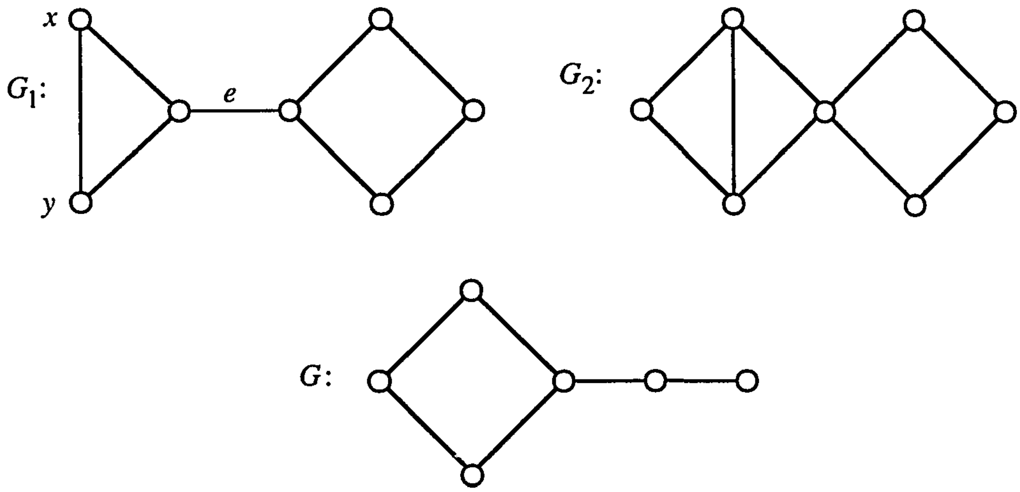

Figure 5.1 Graphs Where the Greatest Common Distance–Preserving Subgraph Is Not a Subgraph of the Greatest Common Locally–Preserving Subgraph.

Since every distance–preserving subgraph is also a locally–preserving subgraph, the gds size is at most the gls size. Next since distance–preserving subgraphs are induced subgraphs, it follows that the gds size is at most the size of a greatest common induced subgraph. For the graphs G_1 and G_2 of Figure 5.1, we saw that the gds size is 6 while the gls size is 7. Since $\delta(G_1) = 2$, the graph G_1 has size 8, and the gds size is 6, it follows that the size of a greatest common induced subgraph is 6. Thus it is possible for the gls size to be larger than the size of a greatest

common induced subgraph. It is also possible for the gls size to be smaller than the size of a greatest common induced subgraph, as the next example shows.

Consider the graphs G_1 and G_2 of Figure 5.2. Since $\delta(G_1) = 2$ and G_1 has size 13, it follows that the size of a greatest common induced subgraph of G_1 and G_2 is at most 11. Next since $G_1 - w_2 = G_2 - x_2 - y_3$, we have that the size of a greatest common induced subgraph is 11. Now G_1 is 2-edge-connected, so the removal of a single edge from G_1 does not produce a subgraph in which each component is an induced subgraph of G_1. Hence the gls size is at most 11. Next since diam $G_1 = 4$ and C_8 is the largest connected subgraph of G_2 with diameter at most 4, it follows that the gds size is 8. Therefore, the size of the largest component of a greatest common locally–preserving subgraph of G_1 and G_2 is at most 8. Also, $C_8 \cup 2K_2$ is a locally–preserving subgraph of G_1 and G_2 and hence the gls size is at least 10.

We now show the gls size is exactly 10. Suppose, to the contrary, that the gls size is 11 and let $H \in \text{gls}(G_1, G_2)$. Since G_1 is 2-connected and $\Delta(G_2) = 3$, it follows that the removal of any vertex of G_1 gives a connected subgraph of size at least 10 and since the gds size is 8, this subgraph is not locally–preserving. So, we must remove two nonadjacent edges from G_1 to produce H. Furthermore, the removal of these two edges from G_1 must result in a disconnected graph. Since G_2 has no 9-cycle, the removal of these edges must destroy the 9-cycles of G_1. Thus we must remove two nonadjacent edges of $uw_1, w_1w_2, w_2w_3, w_3w_4, w_4v$. Hence the 8-cycle $u, u_1, u_2, u_3, v, v_1, v_2, v_3, u$ of G_1 is an 8-cycle of H. We have already noted that the size of the largest component of H is at most 8 and hence $H = G_1 - uw_1 - vw_4$. Since G_2 has exactly one 8-cycle, it follows that $H = G_2 - ux_1 - vy_1$. But $G_1 - uw_1 - vw_4 \neq G_2 - ux_1 - vy_1$, producing a contradiction. Therefore, the gls size is 10 while the size of a greatest common induced subgraph is 11.

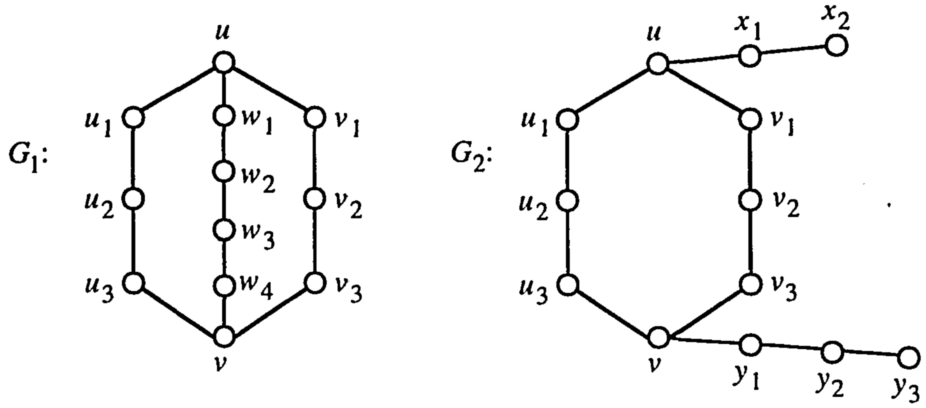

Figure 5.2 Graphs Where the Size of the Greatest Common Induced Subgraph Is Larger Than the gls Size.

We have already seen that a greatest common distance–preserving subgraph need not be a subgraph of a greatest common locally–preserving subgraph. In fact, the components of a greatest common locally–preserving subgraph need not be subgraphs of a greatest common distance–preserving subgraph. Of course, if K_2 or P_3 is a component of a greatest common locally–preserving subgraph, then trivially these components are subgraphs of a greatest common distance–preserving subgraph. Consider the graphs G_1 and G_2 of Figure 5.3. First, we determine $\text{gls}(G_1, G_2)$.

Let $H \in \text{gls}(G_1, G_2)$. Since G_2 has no triangles, it follows that u_3, u_4, and u_5 cannot belong to the same component of H. Hence at least two of the three edges u_3u_4, u_4u_5, u_5u_3 are not present in H. Therefore, H is a subgraph of at least one of the graphs H_1, H_2, and H_3 of size 7, shown in Figure 5.4, where $H_1 = G_2 - u_3u_5 - u_4u_5$, $H_2 = G_2 - u_3u_5 - u_3u_4$, and $H_3 = G_2 - u_3u_4 - u_4u_5$. Since the diameter of each of H_1, H_2, and H_3 is at least 4 and since $\text{diam}\, G_2 = 3$, it follows that H is a proper subgraph of at least one of H_1, H_2, and H_3 and hence H has size at most 6. In fact, since the graph $2P_4$ is not a locally-preserving subgraph of G_2, we have that

$H_1 - u_1u_6$ and $H_1 - u_1u_2$ are the only locally-preserving subgraphs of size 6 and hence $\mathrm{gls}(G_1, G_2) = \{H_1 - u_1u_6, H_1 - u_1u_2\}$.

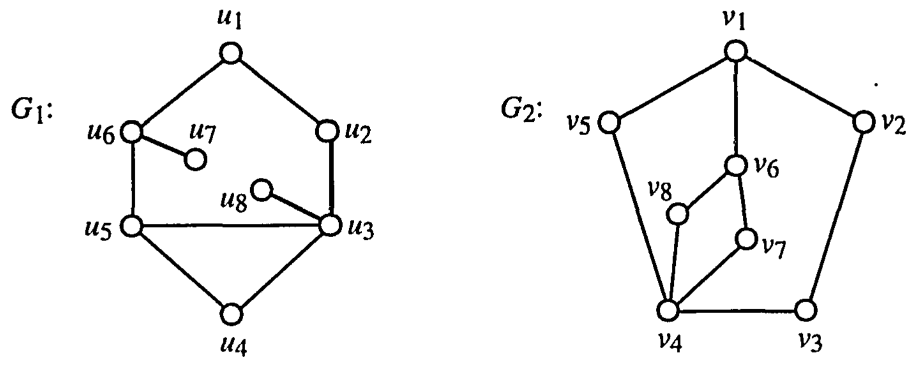

Figure 5.3 Graphs Where Each Component of a Greatest Common Locally–Preserving Subgraph Is Not a Subgraph of Some Greatest Common Distance–Preserving Subgraph.

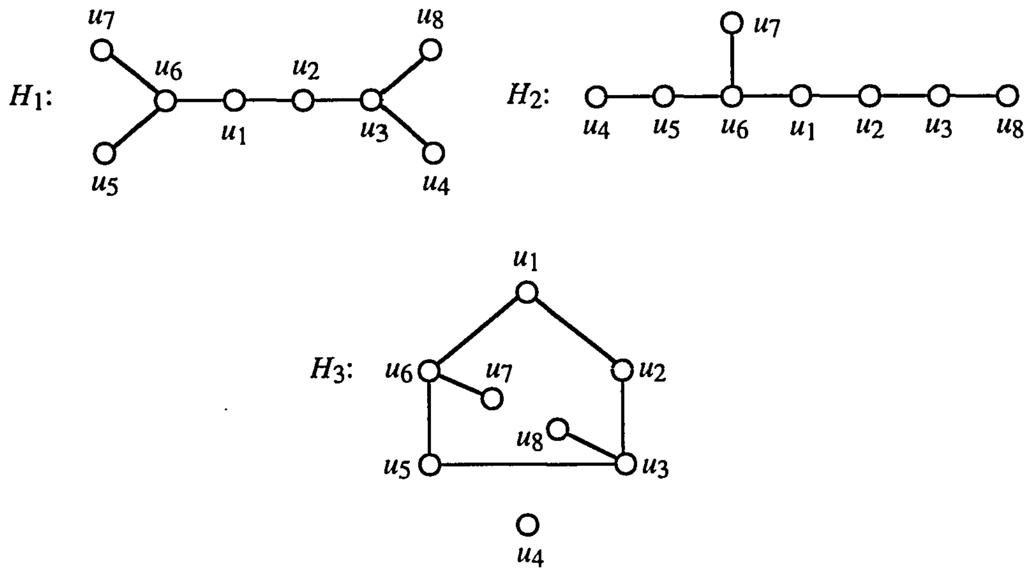

Figure 5.4 Graphs Obtained From G_1 by Removing Two of the Edges u_3u_4, u_4u_5, u_5u_3.

Since each greatest common locally–preserving subgraph of G_1 and G_2 is disconnected, the gds size is at most 5. Also, since C_5 is a distance–preserving

subgraph of G_1 and G_2, it follows that the gds size is 5. Let $G \in \text{gds}(G_1, G_2)$. Then the diameter of G is at most 3. Since G_2 has no triangles, it follows that G has no triangles. Therefore G is a proper subgraph of at least one of H_1, H_2, and H_3. Since all connected subgraphs of H_1 and H_2 with diameter at most 3 have size at most 4, it follows that G is a subgraph of H_3; in fact, G is an induced subgraph of the nontrivial component of H_3. Hence G is obtained from this nontrivial component by removing a vertex of degree 2 or two vertices of degree 1. Each of $H_3 - u_1 - u_4$, $H_3 - u_2 - u_4$, and $H_3 - u_5 - u_4$ has diameter at least 4 and hence $G = H_3 - u_4 - u_7 - u_8$. Thus $\text{gds}(G_1, G_2) = \{C_5\}$, so each component of a greatest common locally–preserving subgraph of G_1 and G_2 need not be a subgraph of some greatest common distance–preserving subgraph of G_1 and G_2.

5.2 Properties of Greatest Common Locally–Preserving Subgraphs

We begin by showing that a greatest common locally–preserving subgraph can have an arbitrarily large number of components.

Theorem 5.2 For every integer $n \geq 2$, there exist graphs G_1 and G_2 of equal size such that a greatest common locally–preserving subgraph of G_1 and G_2 has n components.

Proof Let $n \geq 2$ be an integer. Next let $F_1 = K_{1,n}$ where the vertices of F_1 are labeled $u, u_1, u_2, \ldots, u_n$ and $\deg u = n$, and let $F_2 = K_{1,n}$ where the vertices of F_2 are labeled $v, v_1, v_2, \ldots, v_n$ and $\deg v = n$. We construct G_1 from $F_1 \cup F_2$ by adding the edges $u_1v_1, u_2v_2, \ldots, u_nv_n$. Let G_2 be a path of length $3n$. The graphs G_1 and G_2 are shown in Figure 5.5 when $n = 3$.

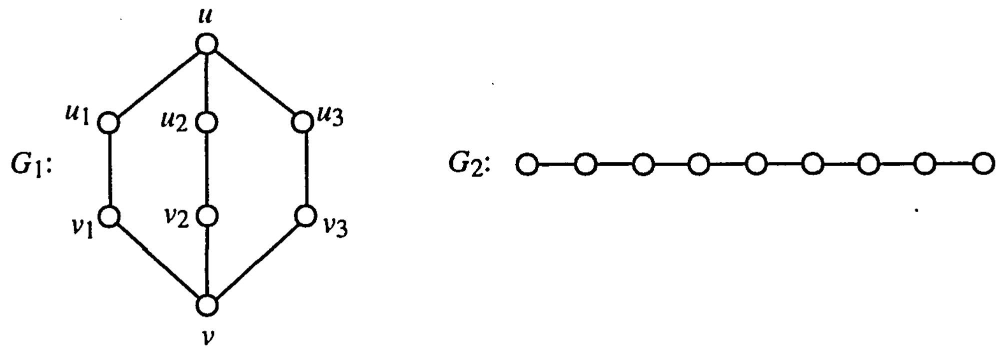

Figure 5.5 Graphs Where a Greatest Common Locally–Preserving Subgraph Has Three Components.

We show the gls size of G_1 and G_2 is $n+2$. Suppose, to the contrary, that the gls size is at least $n+3$ and let $H \in \text{gls}(G_1, G_2)$. Since H is a subgraph of G_2, it follows that every component of H is a path of length at most $\text{diam}\, G_1 = 3$. Next, if u and v are not vertices of H, then the size of H is at most n and hence u or v is a vertex of H, say u. Now $\deg_H u \leq 2$; so if v is not a vertex of H, then the size of H is at most $n+2$, producing a contradiction. Thus both u and v are vertices of H. Furthermore, since H has size $n+3$, we have that $3 \leq \deg_H u + \deg_H v \leq 4$, say, without loss of generality, that $\deg_H u = 2$.

Suppose first that u and v do not belong to the same component of H, say u belongs to component H_1 of H and v belongs to component H_2 of H. Next suppose that $\deg_H u = 2$ and $\deg_H v = 1$, say u_i and u_j $(1 \leq i, j \leq n)$ are adjacent to u in H. Since $\text{diam}\, H_1 \leq 3$, it follows that at most one of the edges $u_i v_i$ and $u_j v_j$ is an edge of H. Thus at most $n-1$ of the edges $u_1 v_1, u_2 v_2, \ldots, u_n v_n$ are present in H so that H has size at most $n+2$, producing a contradicition. Now suppose that $\deg_H u = \deg_H v = 2$, say u_i and u_j $(1 \leq i, j \leq n)$ are adjacent to u in H and v_k and v_ℓ $(1 \leq k, \ell \leq n)$ are adjacent to v in H, where at least two of i, j, k, ℓ are distinct. Then at most two of the edges $u_i v_i, u_j v_j, u_k v_k, u_\ell v_\ell$ are edges of H.

Therefore at most $n-2$ of the edges $u_1v_1, u_2v_2, \dots, u_nv_n$ are present in H and again the size of H is at most $n+2$, producing a contradiction. Finally suppose that u and v belong to the same component of H, say u and v belong to component H_1 of H. Then since $\deg_H u + \deg_H v \geq 3$, it follows that the size of H_1 must be at least 4. Since H_1 is a path, we have that $\text{diam } H_1 \geq 4$, producing a contradiction. Thus the gls size is at most $n+2$. Since $P_4 \cup (n-1)K_2$ is a locally–preserving subgraph of G_1 and G_2, we have that the gls size is $n+2$ and $P_4 \cup (n-1)K_2 \in \text{gls}(G_1, G_2)$. Also, $P_4 \cup (n-1)K_2$ has n components and hence a greatest common locally–preserving subgraph of G_1 and G_2 has n components. ❑

We now show that the gls size can be arbitrarily larger than the gds size.

Theorem 5.3 For every positive integer n, there exist graphs G_1 and G_2 of equal size such that the difference between the gls size and the gds size is n.

Proof For a positive integer n, let G_1 be constructed from the $(2n+2)$-cycle $v_1, v_2, \dots, v_{2n+2}, v_1$ by adding the vertices $w_1, w_2, \dots, w_{n+1}$ and the edges $v_1w_1, w_1w_2, \dots, w_nw_{n+1}, w_{n+1}v_{n+2}$ and let G_2 be obtained from the $(2n+2)$-cycle by adding the vertices $x_1, x_2, \dots, x_{n+2}$ and the edges $v_1x_1, x_1x_2, x_2x_3, \dots, x_{n+1}x_{n+2}$. The graphs G_1 and G_2 of size $3n+4$ are the graphs constructed in the proof of Theorem 2.2 (see also Figure 2.5) where it was shown that the gds size of G_1 and G_2 is $2n+2$.

Thus, it remains to determine the gls size of G_1 and G_2. Observe that G_1 has no bridges and hence the removal of a single edge from G_1 does not produce a subgraph where each component is an induced subgraph of G_1. Therefore the gls size is at most $3n+2$. Next, $H = G_1 - v_1w_1 - v_{n+2}w_{n+1} = G_2 - x_{n+1} - x_{n+2} - v_1x_1$ is a

locally–preserving subgraph of G_1 and G_2 and hence the gls size is $3n + 2$. Thus the difference between the gls size and the gds size is $(3n + 2) - (2n + 2) = n$. ❑

We have already seen that a greatest common distance–preserving subgraph need not be a component of a greatest common locally–preserving subgraph and that the gls size can be arbitrarily larger than the gds size. In fact, we now show that the gds size can be arbitrarily larger than the size of a largest component of a greatest common locally–preserving subgraph.

Theorem 5.4 For every integer $n \geq 2$, there exist graphs G_1 and G_2 of equal size such that the difference between the gds size of G_1 and G_2 and the size of the largest component of a greatest common locally–preserving subgraph of G_1 and G_2 is n.

Proof Let $n \geq 2$ be an integer. Next let P be a path of length n, say P: $v_1, v_2, \ldots, v_{n+1}$, and let C be a cycle of length $n + 1$, say C: $u_1, u_2, \ldots, u_{n+1}, u_1$. The graph G_1 is constructed from $C \cup P$ by adding the vertices w_1, w_2 and the edges u_1v_1, w_1v_n, w_2v_n; while the graph G_2 is constructed from $C \cup P$ by adding the vertices z_1, z_2 and the edges u_1v_1, z_1v_2, z_2v_2. The graphs G_1 and G_2 are shown in Figure 5.6 when $n = 3$. First, we determine the gds size. Now $(C \cup P) + u_1v_1$ is a distance–preserving subgraph of G_1 and G_2 and hence the gds size is at least $2n + 2$. We show that the gds size is $2n + 2$. Suppose, the contrary, that the gds size is $2n + 3$ and let $H \in \text{gds}(G_1, G_2)$. Then since G_1 has size $2n + 4$ and H is an induced subgraph of G_1, it must be that H is obtained from G_1 by removing an end-vertex. Hence assume, without loss of generality, that $H = G_1 - w_1$. Then H is not a subgraph of G_2, producing a contradiction. Therefore the gds size is $2n + 2$.

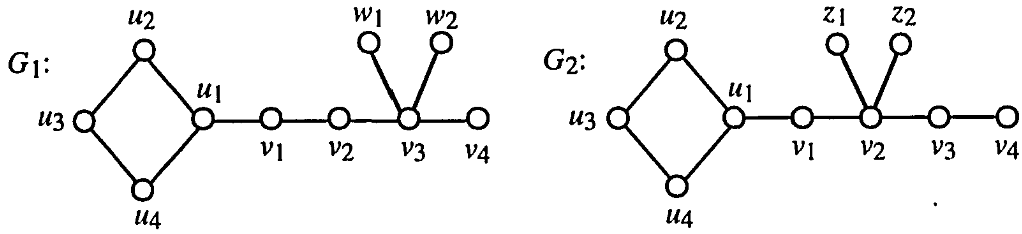

Figure 5.6 The Graphs G_1 and G_2 When $n = 3$.

Finally, we determine $\text{gls}(G_1, G_2)$. Now $G_1 - u_1v_1 = G_2 - u_1v_1$, and thus the gls size is $2n+3$ and $G_1 - u_1v_1 \in \text{gls}(G_1, G_2)$. In fact, $G_1 - u_1v_1$ consists of two components, with sizes $n+1$ and $n+2$. Let $F \in \text{gls}(G_1, G_2)$. Then F must be obtained from G_1 by the removal of a single edge. Furthermore, since the gds size is $2n+2$, this edge must be a bridge. So, $F = G_1 - e$, where e is one of the edges $u_1v_1, v_1v_2, v_2v_3, \ldots, v_nv_{n+1}, w_1v_n, w_2v_n$. Since none of the graphs $G_1 - v_1v_2$, $G_1 - v_2v_3, \ldots, G_1 - v_nv_{n+1}, G_1 - w_1v_n, G_1 - w_2v_n$ is a subgraph of G_2, it follows that $\text{gls}(G_1, G_2) = \{G_1 - u_1v_1\}$ and hence the difference between the gds size and the size of the largest component of the greatest common locally-preserving subgraph of G_1 and G_2 is $(2n+2) - (n+2) = n$. ❑

In Chapter IV, it was shown that every (connected) graph is the unique greatest common distance-preserving subgraph of two nonisomorphic connected graphs of equal size. We now show that every graph is a greatest common locally-preserving subgraph of two nonisomorphic graphs of equal size.

Theorem 5.5 For every graph G, there exist graphs G_1 and G_2 of equal size such that $G \in \text{gls}(G_1, G_2)$.

Proof Let G be a graph of size q and suppose first that G is not vertex-transitive. Then there exist vertices x and y of G such that for every automorphism ϕ of G, we have that $\phi(x) \neq y$ and $\phi(y) \neq x$. Obtain the graph G_1 from G by joining a new vertex v_1 to x and obtain the graph G_2 from G by joining a new vertex v_2 to y. Clearly, $G_1 \not\cong G_2$, and G_1 and G_2 both have size $q + 1$. Therefore, the gls size is at most q. Hence, since G is a locally-preserving subgraph of G_1 and G_2 of size q, it follows that $G \in \text{gls}(G_1, G_2)$.

Finally, suppose that G is vertex-transitive. If G is 1-regular, then $G = nK_2$ for some positive integer n. Now for $G_1 = (n-1)K_2 \cup K_{1,3}$ and $G_2 = (n-1)K_2 \cup C_3$, we have that $\text{gls}(G_1, G_2) = nK_2$. Next suppose that G is k-regular (and vertex-transitive) where $k \geq 2$. Let G_1 be obtained from G by joining two new vertices to any vertex of G and let G_2 be obtained from G by joining a new vertex to a pair of adjacent vertices. Then G_1 and G_2 both have size $q + 2$ and $G_1 \not\cong G_2$. Since $\delta(G_2) = 2$ and since G_2 has no bridges, it follows that the gls size of G_1 and G_2 is at most q. Next, since G is a locally-preserving subgraph of G_1 and G_2, it follows that the gls size is q and $G \in \text{gls}(G_1, G_2)$. ❑

Next, we show that every graph with exactly two 2-edge-connected components is the unique greatest locally–preserving subgraph of two nonisomorphic connected graphs.

Theorem 5.6 For every two 2-edge-connected graphs H_1 and H_2, there exist graphs G_1 and G_2 such that $\text{gls}(G_1, G_2) = \{H_1 \cup H_2\}$.

Proof Let q denote the size of $H = H_1 \cup H_2$. Suppose first that both H_1 and H_2 are vertex-transitive. Let x_1 and x_2 be two adjacent vertices of H_1, and let y_1 and y_2 be two adjacent vertices of H_2. Let $G_1 = H + x_1y_1 + x_2y_1$ and let $G_2 = H +$

$x_1y_1 + x_2y_2$. Since H_1 and H_2 are 2-edge-connected, so too are G_1 and G_2. So the removal of a single edge from G_1 or G_2 does not produce an induced subgraph and therefore the gls size is at most q. Since H is a locally–preserving subgraph of G_1 and G_2, it follows that the gls size is q and that $H \in \text{gls}(G_1, G_2)$. We now show, in fact, that $\text{gls}(G_1, G_2) = \{H\}$. Let $F \in \text{gls}(G_1, G_2)$. Since H_1 is vertex-transitive, every vertex of H_1 lies on the same number, say t, of triangles. Similarly, every vertex of H_2 lies on the same number, say s, of triangles. Thus in G_1, the vertices x_1 and x_2 lie on $t+1$ triangles and y_1 lies on $s+1$ triangles, while every vertex of G_2 lies on t or s triangles. Therefore, to obtain F from G_1, we must remove two edges e_1 and e_2, where each of e_1 and e_2 lies on a triangle with at least one of x_1, x_2, and y_1. Assume, without loss of generality, that e_1 and x_1 lie on a triangle and that e_2 and y_1 lie on a triangle. If $e_1 \neq x_1y_1$ and $e_2 \neq x_2y_1$, then e_1 must be an edge of H_1 and e_2 must be an edge of H_2. Since each of H_1 and H_2 is 2-edge-connected, it follows that F is connected and hence F is not an induced subgraph of G_1. Thus $e_1 = x_1y_1$ or $e_2 = x_2y_1$, say $e_1 = x_1y_1$. If $e_2 \neq x_2y_1$, then F is connected and hence $e_2 = x_2y_1$. Therefore $F = H_1 \cup H_2$, so $\text{gls}(G_1, G_2) = \{H_1 \cup H_2\}$.

Finally, suppose that one of H_1 and H_2 is not vertex-transitive, say H_1. So there exist vertices x_1 and x_2 of H_1 that do not belong to the same orbit of H_1. Let y be a vertex of H_2. Next let $G_1 = H + x_1y$ and $G_2 = H + x_2y$. Then $G_1 \neq G_2$ and H is a locally–preserving subgraph of G_1 and G_2. Since each of H_1 and H_2 are 2-edge-connected, the removal of any single edge of H_1 or H_2 from G_1 leaves a connected subgraph that is not an induced subgraph of G_1. Hence $\text{gls}(G_1, G_2) = \{H_1 \cup H_2\}$. ❑

Next we show that for n-edge-connected graphs $H_1, H_2, \ldots, H_n$ $(n \geq 2)$, at least one of which is not vertex-transitive, there exist graphs G_1 and G_2 such that $H_1 \cup H_2 \cup \ldots \cup H_n$ is the greatest common locally–preserving subgraph of G_1 and G_2.

Theorem 5.7 For n-edge-connected graphs $H_1, H_2, \ldots, H_n$, where $n \geq 2$ is a positive integer and at least one H_i is not vertex-transitive, there exist graphs G_1 and G_2 such that $\text{gls}(G_1, G_2) = \{H_1 \cup H_2 \cup \ldots \cup H_n\}$.

Proof Let $H_1, H_2, \ldots, H_n$ be n-edge-connected graphs where H_1 is not vertex-transitive. Then there exist vertices x_1 and y_1 of H_1 such that x_1 and y_1 belong to different orbits. Let $H = H_1 \cup H_2 \cup \ldots \cup H_n$ have size q and let x_i be a vertex of H_i for $i = 2, \ldots, n$. Let $G_1 = H + x_1x_2 + x_2x_3 + \ldots + x_{n-1}x_n$ and let $G_2 = H + y_1x_2 + y_1x_3 + \ldots + y_1x_n$. We begin by showing that the gls size is q. Let $F \in \text{gls}(G_1, G_2)$. Since H is a locally–preserving subgraph of G_1 and G_2, it follows that F has size at least q. Also, since each H_i is n-edge-connected and $|V(H)| = |V(G_1)| = |V(G_2)|$, it follows that $|V(F)| = |V(G_1)|$. Thus F is obtained from G_1 by removing the edges $e_1, e_2, \ldots, e_k$ where $k \leq n-1$. If any edge e_i belongs to H_j for some j $(1 \leq j \leq n)$, then since H_j is n-edge-connected, F has a component that is not an induced subgraph of G_1. Therefore, $e_1, e_2, \ldots, e_k$ are not edges of H. Hence if F has size at least $q + 1$, then the graph $J = H + y_1x_j$ for some j $(2 \leq j \leq n)$ is a subgraph of F. But J is not a subgraph of G_1 and hence F has size q. Furthermore, it follows that $k = n - 1$ and $F = H_1 \cup H_2 \cup \ldots \cup H_n$. ❑

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