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A Graph Theoretic Study of the Similarity of Discrete Structures

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A GRAPH THEORETIC STUDY OF THE SIMILARITY OF DISCRETE STRUCTURES

by

Heather D. Gavlas

A Dissertation
Submitted to the
Faculty of The Graduate College
in partial fulfillment of the
requirements for the
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Western Michigan University
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A basic problem in drug design consists of finding a compound that satisfies a spectrum of biological and chemical properties. Although drug design problems are central to pharmaceutical research, statisticians have yet to become involved in this area as these problems are viewed statistically as optimization problems. Before statistical optimization procedures can be defined on these spaces whose points are structures and not vectors, very basic mathematical notions of distance must be defined. Graphs have been used as mathematical models to represent the bonding arrangements of molecules for quite some time. In fact, if some of the methodology used by statisticians for optimization problems could be extended to the metric space consisting of the set of all graphs, with a fixed number of vertices and a fixed number of edges, together with a metric on this set, then some of the problems in drug design might become more accessible.

Various metrics have been studied and some of the results are presented in Chapter I. In this dissertation, we define a metric in terms of some prescribed graph $H$. For certain choices of the graph $H$, this metric is a special case of the previously studied metrics in Chapter I. In Chapter II, conditions are described for certain graphs $H$ that allow us to determine those pairs of graphs for which this metric is defined. Further properties of these metric spaces are studied in Chapter III by means of a graphical interpretation.
Another important problem in the area of mathematical chemistry is the determination of a maximum common substructure shared by two molecular compounds. A special type of commonality that two or more graphs share is called a greatest common subgraph. These concepts have been studied extensively, and some of the results are presented in Chapter I. A certain restriction, inherent to drug design, is imposed on these common subgraphs in Chapter IV, namely, that they preserve distance. These concepts are also applied to trees in Chapter IV. Another type of common subgraph, relative to this distance constraint, is studied in Chapter V.
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To Ryan and Joey
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CHAPTER I
INTRODUCTION

1.1 Introduction

A basic problem in drug design consists of finding a compound that satisfies a spectrum of biological and chemical properties. Although drug design problems are central to pharmaceutical research, statisticians have yet to become involved in this area. A major reason for this is that these problems are viewed statistically as optimization problems, and standard statistical optimization methods are based on Euclidean space or vector representation. Here the formal representations are labeled graphs and/or three-dimensional atomic configurations. So, the statistician is understandably blocked when confronted with a data set consisting of molecular structures. Before statistical optimization procedures can be defined on these spaces whose points are structures and not vectors, very basic mathematical notions of distance must be defined. This dissertation explores some of these notions when the points are labeled graphs in Chapters II and III.

Graphs have been used as mathematical models to represent the bonding arrangements of molecules for quite some time (see Harary [14], for example). A graph $G$ consists of a set $V(G)$ of elements called vertices and a set $E(G)$ of elements called edges, where each edge is an unordered pair of distinct vertices. So when a graph models a molecule, the vertices denote the atoms and the edges denote the bonds. However, in the molecule the atoms are distinguished by their type (hydrogen, carbon, chlorine, etc.) and the bonds by their type (single, double, aromatic, etc.). Thus when representing a molecule by a graph, we may wish to "color" or "label" the
vertices and edges. A graph representation of a molecular structure in which the vertices are colored by atom type and the edges by bond type is called a chemical graph (see [23], for example).

The set \( G \) of all graphs of a fixed order and a fixed size together with a metric \( d \) on \( G \) is a metric space. In fact, if some of the methodology used by statisticians for optimization problems could be extended to this metric space, then some of the problems in drug design might become more accessible. For example, when chemical graphs are used to model a chemical reaction, it is often important to determine the maximum commonalities between these molecular structures or, in fact, to determine the greatest common subgraph of some set of graphs.

In this dissertation, one area that we study is common subgraphs under distance constraints. This type of constraint arises naturally in drug reception/interaction designs. Drugs interact with proteins, where the proteins are made up of sequences (or chains) of amino acids that form certain patterns (like a helix, for example). The drug needs to interact or "fit" into the protein in a specified way. Drugs are chemical compounds that can be represented as chemical graphs. Also, these chemical graphs can be embedded in three dimensions, where we might interpret the distance between two vertices (or atoms) as the Euclidean distance between these two points in \( \mathbb{R}^3 \). Unfortunately, this distance is subject to the particular embedding of the graph in \( \mathbb{R}^3 \), and there are many distinct embeddings of the same graph in \( \mathbb{R}^3 \). Thus, if we measure the distance between two vertices as the length of a shortest path (in the graph) between them, then this distance places constraints on the Euclidean distance between the two vertices when the graph is embedded in \( \mathbb{R}^3 \).

In the area of drug reception/interaction problems, we may have one drug that interacts well with a certain protein and another drug does that not interact well with this same protein. A natural question arises: In what structural way are these two drugs the
same? An analogous question in graph theory then becomes: What is the greatest common subgraph of the two graphs representing these drugs subject to the distance constraint? This topic will be the object of study in Chapters IV and V.

1.2 A Few Metrics for Graphs

In [25] Vizing proposed the following question: For which graphs $G_1$ and $G_2$ of order $p$, does there exist a graph $G$ of order $p + 1$ such that each of $G_1$ and $G_2$ is an induced subgraph of $G$? For each positive integer $p$, let $G_p$ denote the set of all nonisomorphic graphs of order $p$. Zelinka [26] defines a distance $\delta$ on $G_p$ as follows: For graphs $G_1$ and $G_2$ of $G_p$, define $\delta(G_1, G_2)$ to be the least positive integer $k$ such that there exists a graph $G$ of order $p + k$ containing the graphs $G_1$ and $G_2$ as induced subgraphs. Thus Vizing's question becomes: For which graphs $G_1$ and $G_2$ of $G_p$ is it true that $\delta(G_1, G_2) = 1$? In [26] it was shown that $\delta(G_1, G_2) = k$ for graphs $G_1$ and $G_2$ of $G_p$ if and only if there exists a graph $H$ of order at least $p - k$ such that $H$ is an induced subgraph of each of $G_1$ and $G_2$. This distance $\delta$ on $G_p$ is a metric [26] and also $\delta(G_1, G_2) = \delta(G_1, G_2)$ for every pair $G_1, G_2$ of graphs of $G_p$. The graph $D(G_p)$ is defined in [26] as that graph whose vertex set is $G_p$ and two vertices $G_1$ and $G_2$ of $D(G_p)$ are adjacent if and only if $\delta(G_1, G_2) = 1$. Since $\delta(K_p, \overline{K}_p) = p - 1$, it follows that $\text{diam } D(G_p) = p - 1$. For each pair $G_1, G_2$ of graphs of $G_p$, where neither of $G_1$ and $G_2$ is complete nor empty, we have either $K_2$ or $\overline{K}_2$ is an induced subgraph of each of $G_1$ and $G_2$, and thus $\delta(G_1, G_2) \leq p - 2$.

The analogous concepts are studied for trees in [27]. For each positive integer $p$, let $\mathcal{T}_p$ denote the set of all nonisomorphic trees of order $p$. For trees $T_1$ and $T_2$ of $\mathcal{T}_p$, define $\delta_T(T_1, T_2)$ as the least positive integer $k$ such that there exists a tree $T$ of order $p + k$, where each of $T_1$ and $T_2$ is a subtree of $T$. This distance $\delta_T$ is

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different from the distance $\delta$. In fact, there exist trees $T_1$ and $T_2$ of order $p$ for which $\delta(T_1, T_2) < \delta_r(T_1, T_2)$, namely when $T_1$ is a path and $T_2$ is a star. As before, $\delta_r$ is a metric on $\mathcal{T}_p$ (see [27]). Also in [27], $\delta_r(T_1, T_2) = k$ for trees $T_1$ and $T_2$ of $\mathcal{T}_p$ if and only if there exists a tree $T$ of order at least $p - k$ such that $T$ is a subtree of each of $T_1$ and $T_2$. Next define $D(\mathcal{T}_p)$ to be that graph whose vertex set is $\mathcal{T}_p$ and two vertices $T_1$ and $T_2$ of $D(\mathcal{T}_p)$ are adjacent if and only if $\delta_r(T_1, T_2) = 1$. It turns out that the distance from $T_1$ to $T_2$ in $D(\mathcal{T}_p)$ is exactly $\delta_r(T_1, T_2)$ and furthermore that $\text{diam} D(\mathcal{T}_p) = p - 3$. Also in [27], an upper bound for the radius of $D(\mathcal{T}_p)$ is given, and it is conjectured that this upper bound is exact.

Baláž, Koča, Kvasnička, and Sekanina [1] define a distance between graphs using the greatest common induced subgraph. This distance was stimulated by their studies on the mathematical modeling of organic chemistry and the problem of measuring the similarity of two graphs. Let $G_1$ and $G_2$ be two graphs. A greatest common induced subgraph of $G_1$ and $G_2$ is a graph $G$ without isolated vertices of maximum size such that $G$ is an induced subgraph of both $G_1$ and $G_2$. A back-track searching algorithm for the construction of a greatest common induced subgraph of two graphs has been given by McGregor in [22]. A distance $d(G_1, G_2)$ between the graphs $G_1$ and $G_2$ is given in [1] by

$$d(G_1, G_2) = |E(G_1)| + |E(G_2)| - 2|E(G)| + |V(G_1)| - |V(G_2)|.$$

So for graphs with the same order and the same size, this distance is the number of edges that cannot be matched in the construction of a greatest common induced subgraph of $G_1$ and $G_2$. This distance is a metric on the set of all graphs as seen in [1]. For graphs $G_1$ and $G_2$ with the same order, the construction of a greatest common induced subgraph in [1] can be realized by a 1-1 mapping $\phi$ from $V(G_1)$ onto $V(G_2)$. Thus if $A_1$ and $A_2$ are the adjacency matrices for $G_1$ and $G_2$.
respectively, then $\phi$ can be realized by a permutation $P$ of the elements $\{1, 2, \ldots, n\}$ and thus a second alternative distance is given [1] by

$$d(G_1, G_2) = \min |A_1 - P^T A_2 P|,$$

where this minimum is taken over all permutations $P$ of $\{1, 2, \ldots, n\}$ and $|A| = \sum_{i<j} |a_{ij}|$ is the Hamming (linear) norm of a symmetric matrix $A$. This relation is nothing more than the determination of the chemical distance between two graphs representing molecular structure formulas (see [1]).

A new distance function was defined in [9] for graphs with the same order and same size. Let $G_1$ and $G_2$ be two graphs of the same order and same size. The graph $G_2$ can be obtained from $G_1$ by an edge rotation if $G_1$ contains distinct vertices $u, v,$ and $w$ such that $uv \in E(G_1)$, $uw \notin E(G_1)$ and $G_2 = G_1 - uv + uw$. So, $G_2$ can be obtained from $G_1$ by "rotating" the edge $uv$ of $G_1$ into the edge $uw$ of $G_2$. In Figure 1.1, the graph $G_2$ can be obtained from $G_1$ by an edge rotation since $G_2 = G_1 - xy + xz$.

![Figure 1.1](image)

**Figure 1.1** The Graph $G_2$ Can Be Obtained From $G_1$ by an Edge Rotation.

Clearly, a graph $G_2$ can be obtained from $G_1$ by an edge rotation if and only if $G_1$ can be obtained from $G_2$ by an edge rotation. More generally, a graph $G_1$ can be $r$-transformed into a graph $G_2$ if there exists a sequence $G_1 = H_0, H_1, \ldots, H_n = G_2$ ($n \geq 0$) of graphs such that for $i = 1, 2, \ldots, n$, the graph $H_i$ can be obtained from the graph $H_{i-1}$ by an edge rotation. The relation "can be $r$-transformed into" is
an equivalence relation on the set of all graphs, and furthermore if \( G_1 \) can be \( r \)-transformed into \( G_2 \), then \( G_1 \) and \( G_2 \) must have the same order and same size. In fact, it is shown in [9] that the converse of the previous statement is also true.

**Theorem A** Let \( G_1 \) and \( G_2 \) be two graphs. Then \( G_1 \) can be \( r \)-transformed into \( G_2 \) if and only if \( G_1 \) and \( G_2 \) have the same order and same size.

For graphs \( G_1 \) and \( G_2 \) with the same order and same size, the *edge rotation distance* \( d_r(G_1, G_2) \) between \( G_1 \) and \( G_2 \) is defined in [9] as the smallest nonnegative integer \( n \) for which there exists a sequence \( G_1 = H_0, H_1, \ldots, H_n = G_2 \) of graphs such that for \( i = 1, 2, \ldots, n \), the graph \( H_i \) can be obtained from \( H_{i-1} \) by an edge rotation. Thus, by Theorem A, this distance is well-defined and is a metric on the set of all graphs of a fixed order and fixed size. In [9] it is shown that \( d_r(G_1, G_2) = d_r(\overline{G_1}, \overline{G_2}) \), and that for every positive integer \( n \), there exist graphs \( G_1 \) and \( G_2 \) such that \( d_r(G_1, G_2) = n \). In order to determine an upper bound on the edge rotation distance between graphs, we require the following definition. For graphs \( G_1 \) and \( G_2 \), a *greatest common subgraph* of \( G_1 \) and \( G_2 \) is a graph \( G \) of maximum size without isolated vertices that is a subgraph of both \( G_1 \) and \( G_2 \). Thus in [9] we see that for graphs \( G_1 \) and \( G_2 \) of order \( p \) and size \( q \) with greatest common subgraph \( G \) of size \( s \),

\[
d_r(G_1, G_2) \leq 2(q - s).
\]

Furthermore, this bound is sharp, as seen in [9].

In [13] and [17], it is independently shown that \( q - s \) is a lower bound for \( d_r(G_1, G_2) \). Also, it is shown in [13] that this lower bound is attainable as well. In fact, in [13] if \( G_1 \) and \( G_2 \) are 2-regular graphs of order \( p \), then

\[
d_r(G_1, G_2) = p - s,
\]
where $s$ is the size of a greatest common subgraph of $G_1$ and $G_2$. It is also shown in [13] that if $G_1$ and $G_2$ are two connected graphs of order $p$ and size $q$, then

$$d_r(G_1, G_2) \leq 2q - p.$$  

Zelinka gives some comparisons in [28] of the metrics $\delta$, $\delta_T$, and $d_r$ and shows that for two graphs $G_1$ and $G_2$ of the same order and same size,

$$\delta(G_1, G_2) \leq d_r(G_1, G_2).$$

Furthermore, in [28] the edge rotation distance can be arbitrarily larger than the distance $\delta$, that is, for each positive integer $n$, there exist graphs $G_1$ and $G_2$ such that

$$d_r(G_1, G_2) - \delta(G_1, G_2) = n.$$  

Also in [28], we see that for two trees $T_1$ and $T_2$ of the same order, the distance $\delta_T(T_1, T_2)$ is an upper bound for $d_r(T_1, T_2)$, and that these two parameters can be arbitrarily far apart, that is, for every positive integer $n$, there exist nonisomorphic trees $T_1$ and $T_2$ such that

$$\delta_T(T_1, T_2) - d_r(T_1, T_2) = n.$$  

Zelinka [28] has also determined the edge rotation distance between certain pairs of trees, namely, for a tree $T$ of order $n$,

(i) $d_r(T, K_{1,n-1}) = n - 1 - \Delta(T)$,

(ii) $d_r(T, P_n) \leq n - 1 - \text{diam } T$, and

(iii) $d_r(P_n, K_{1,n-1}) = n - 3$.

The bound in (ii) is further investigated in [2]. For a tree $T$, let $\text{end}(T)$ denote the number of end-vertices of $T$. It is shown in [2] that for a tree $T$ of order $n$,

$$d_r(T, P_n) = \text{end}(T) - 2.$$  

In [13] the edge rotation distance of trees is considered. For trees $T_1$ and $T_2$, we say that $T_2$ can be obtained from $T_1$ by a tree rotation if $T_2$ can be obtained from $T_1$ by an edge rotation. Observe that this edge rotation does not disconnect the
The tree rotation distance \( d_r(T_1, T_2) \) between two trees \( T_1 \) and \( T_2 \) of the same order is defined as the minimum number of tree rotations required to transform \( T_1 \) into \( T_2 \). It is shown in [13] that the tree distance is bounded above by twice the edge rotation distance, that is, for trees \( T_1 \) and \( T_2 \) of the same order,

\[
d_r(T_1, T_2) \leq 2d_e(T_1, T_2).
\]

As before, let \( T'_p \) denote the set of all nonisomorphic trees of order \( p \). Let \( D_T(T'_p) \) denote that graph whose vertices are the trees of \( T'_p \) and two vertices \( T_1 \) and \( T_2 \) are adjacent if and only if \( d_r(T_1, T_2) = 1 \). Then in [13], it is shown that

(i) \( \Delta(D_T(T'_p)) \leq p(p - 3) \) for all \( p \geq 4 \),

(ii) \( \text{diam } D_T(T'_p) \leq p - 3 \) for all \( p \geq 3 \), and

(iii) \( \text{rad } D_T(T'_p) \leq p - o(p) \).

Johnson [18] defined a metric on the set of all graphs with applications to medicinal chemistry. Before and after working its biological effect, a drug or compound will undergo a series of complex and diverse interactions, many of which are poorly understood. In fact, Johnson says in [18], "It is widely held that this effect is elicited by the drug interacting with a macromolecule in a steric specific manner often linked to the fit of a key (drug) in a lock (receptor site of the macromolecule)." He goes on to say that the receptor site and the drug must be similar to each other in a complementary way and drugs that are similar to each other, that is, having related groups in corresponding positions, are expected to behave in similar ways. However, some unexpected problems can occur. For example, some compounds bind to the receptor site without producing the desired activity while others may never reach the receptor site because of intervening interactions. Also slight modifications of a drug can alter its activity remarkably. Thus some important questions in medicinal chemistry include: (a) finding compounds in a set of easily accessible compounds that are similar
to some specified compound, and (b) given two structurally diverse compounds, to
determine what they have in common.

For a graph $G$, Johnson [18] defines the cardinality $|G|$ of $G$ as $|V(G)| + |
E(G)|$, and defines a metric $d$ on the set of all graphs as follows: for graphs $G_1$
and $G_2$ with greatest common subgraph $G$, define

$$d(G_1, G_2) = |G_1| + |G_2| - 2|G|.$$ 

It is not difficult to see that this distance is exactly the one given in [1]. In fact, if $G_1$
and $G_2$ are two graphs of order $p$ and size $q$ having a greatest common subgraph of
size $s$, then $d(G_1, G_2) = 2q - 2s$, and hence this distance gives the upper bound for
the edge rotation distance between $G_1$ and $G_2$.

Johnson [17] defines another metric on the set of all connected graphs of a fixed
order and fixed size that is based on a more restricted version of an edge rotation. Let
$G_1$ and $G_2$ be two graphs of the same order and same size. The graph $G_2$ can be
obtained from the graph $G_1$ by an edge slide if there exists distinct vertices $u, v, \text{and}
w$ such that $uv, vw \in E(G_1)$, $uw \notin E(G_1)$ and $G_2 = G_1 - uv + uw$. Thus the graph
$G_2$ can be obtained from the graph $G_1$ by "sliding" the edge $uv$ along the edge $vw$
and into the edge $uw$. Recall that in Figure 1.1, the graph $G_2 = G_1 - xy + xz$ was
obtained from the graph $G_1$ by an edge rotation and, in fact, since $yz \notin E(G_1)$, we
see that $G_2$ can be obtained from $G_1$ by an edge slide as well.

More generally a graph $G_1$ can be $s$-transformed into a graph $G_2$ if there
exists a sequence $G_1 = H_0, H_1, \ldots, H_n = G_2$ ($n \geq 0$) of graphs such that for $i = 1,2, \ldots, n$, the graph $H_i$ can be obtained from the graph $H_{i-1}$ by an edge slide. In
[17] it is shown that edge slides preserve connectedness, that is, if $G_1$ is connected
and $G_2$ can be obtained from $G_1$ by an edge slide, then $G_2$ is connected. The
converse of the previous statement is also true, as was noted in [17].
**Theorem B** If $G_1$ and $G_2$ are two connected graphs of the same order and same size, then $G_1$ can be $s$-transformed into $G_2$.

As a consequence of Theorem B, two graphs of the same order and same size can be $s$-transformed one into the other if and only if they have the same number of components, and corresponding components have the same order and same size. The *edge slide distance* $d_s(G_1, G_2)$ between two graphs $G_1$ and $G_2$ having the same number of components and corresponding components having the same order and same size is defined as the smallest nonnegative integer $n$ for which there exists a sequence $G_1 = H_0, H_1, \ldots, H_n = G_2$ of graphs such that for $i = 1, 2, \ldots, n$, the graph $H_i$ can be obtained from the graph $H_{i-1}$ by an edge slide. This distance is a metric, and since an edge slide is an edge rotation, it immediately follows that $d_r(G_1, G_2) \leq d_s(G_1, G_2)$ for every pair of graphs for which the edge slide distance is defined. Jarrett has shown in [15] that these parameters can be chosen arbitrarily, that is, for positive integers $m$ and $n$ with $m \leq n$, there exist graphs $G_1$ and $G_2$ such that $d_r(G_1, G_2) = m$ while $d_s(G_1, G_2) = n$. In [2] the edge slide distance is determined for various pairs of graphs. For a graph $G$, the *girth* $g(G)$ of $G$ is the length of a shortest cycle.

For every tree $T$ and every connected unicyclic graph $U$ of order $n$,

(i) $d_s(T, P_n) = (n - 1) - \text{diam } T$,

(ii) $d_s(T, K_{1,n-1}) = n - 1 - \Delta(T)$, and

(iii) $d_s(U, C_n) = n - g(U)$.

Another metric for graphs was introduced in [2]. For graphs $G_1$ and $G_2$ of the same order and same size, the graph $G_2$ can be obtained from $G_1$ by an *edge move* if there exist (not necessarily distinct) vertices $u, v, w,$ and $x$ such that $uv \in E(G_1)$, $wx \notin E(G_1)$ and $G_2 = G_1 - uv + wx$. Thus an edge move is an unrestricted transfer of an edge from one graph to another. Clearly, if $G_2$ can be obtained from
G_1 by an edge move, say \( G_2 = G_1 - uv + wx \), and \( u, v, w, \) and \( x \) are not distinct (say \( u = w \)), then \( G_2 \) can be obtained from \( G_1 \) by an edge rotation. Furthermore if \( v \) and \( x \) are adjacent, then \( G_2 \) can be obtained from \( G_1 \) by an edge slide. More generally, a graph \( G_1 \) can be \( m \)-transformed into a graph \( G_2 \) if there exists a sequence \( G_1 = H_0, H_1, \ldots, H_n = G_2 \) \((n \geq 0)\) of graphs such that for \( i = 1, 2, \ldots, n, \) the graph \( H_i \) can be obtained from \( H_{i-1} \) by an edge move. Thus if \( G_1 \) and \( G_2 \) are two graphs of the same order and same size, then \( G_1 \) can be \( m \)-transformed into \( G_2 \). The edge move distance \( d_m(G_1, G_2) \) between two graphs \( G_1 \) and \( G_2 \) having the same order and same size is defined as the minimum number of edge moves required to \( m \)-transform \( G_1 \) into \( G_2 \). This distance is a metric and for two graphs \( G_1 \) and \( G_2 \) for which \( d_s(G_1, G_2) \) is defined, 
\[
d_m(G_1, G_2) \leq d_r(G_1, G_2) \leq d_s(G_1, G_2).
\]
Also in [2], it is shown that if \( d_m(G_1, G_2) = 1 \) for two graphs \( G_1 \) and \( G_2 \), then \( d_r(G_1, G_2) \leq 2 \). Although \( d_m(G_1, G_2) \leq d_r(G_1, G_2) \leq d_s(G_1, G_2) \) for every pair \( G_1, G_2 \) of graphs for which \( d_s(G_1, G_2) \) is defined, there exist graphs for which equality holds. For every tree \( T \) of order \( n \), it is shown in [2] that \( d_m(T, K_{1,n-1}) = d_r(T, K_{1,n-1}) = d_s(T, K_{1,n-1}) = n - 1 - \Delta(T) \). In fact, the edge move distance between every two graphs of the same order and same size is determined in [2]. For graphs \( G_1 \) and \( G_2 \) of order \( p \) and size \( q \), where the size of a greatest common subgraph of \( G_1 \) and \( G_2 \) is \( s \), then \( d_m(G_1, G_2) = q - s \).

We have already noted conditions under which an edge move is also an edge rotation or an edge slide. Suppose that \( G_2 \) can be obtained from \( G_1 \) by an edge move, say \( G_2 = G_1 - uv + wx \), where \( u, v, w, \) and \( x \) are distinct. Then this transformation is neither an edge rotation nor an edge slide and is called an edge jump in [4]. More formally, a graph \( G_2 \) can be obtained from a graph \( G_1 \) by an edge jump \( \ldots \).
if $G_1$ contains distinct vertices $u, v, w,$ and $x$ such that $uv \in E(G_1)$, $wx \in E(G_1)$ and $G_2 = G_1 - uv + wx$. Recall that for the graphs $G_1$ and $G_2$ of Figure 1.1, the graph $G_2$ can be obtained from $G_1$ by an edge rotation, an edge slide, or an edge move. It turns out, however, that $G_2$ cannot be obtained from $G_1$ by an edge jump. Consider the graphs $G_1$ and $G_2$ of Figure 1.2. Since $G_2 = G_1 - uv + wx$ for distinct vertices $u, v, w,$ and $x$, it follows that $G_2$ can be obtained from $G_1$ by an edge jump.

![Figure 1.2 The Graph $G_2$ Can Be Obtained From $G_1$ by an Edge Jump.](image)

For graphs $G_1$ and $G_2$ of the same order and same size, if $G_2$ can be obtained from $G_1$ by a sequence of edge jumps, then $G_1$ can be $j$-transformed into $G_2$. We have already seen conditions under which two graphs of the same order and same size can be $m$-transformed, $r$-transformed, or $s$-transformed one into the other. In [4] conditions are given under which two graphs may be $j$-transformed one into the other.

**Theorem C** If $G_1$ and $G_2$ are two graphs of the same order (at least 5) and same size, then $G_1$ can be $j$-transformed into $G_2$.

We have already defined the edge rotation, edge slide, and edge move distance and hence it is natural to define the jump distance between two graphs. For graphs $G_1$ and $G_2$ of order $p \geq 5$ and size $q$, the *jump distance* $d_j(G_1, G_2)$ is defined as the
minimum number of edge jumps needed to \( j \)-transform \( G_1 \) into \( G_2 \). Clearly, this distance is well-defined and is a metric on the space of all graphs of a fixed order (at least 5) and fixed size. Also the move distance and jump distance are related in the following way: if \( d_m(G_1, G_2) = 1 \) for two graphs \( G_1 \) and \( G_2 \) of order at least 5, and size \( q \), then \( d_j(G_1, G_2) \leq 2 \). Hence, if the size of the greatest common subgraph of \( G_1 \) and \( G_2 \) is \( s \), then we have seen that \( d_m(G_1, G_2) = q - s \) and thus \( d_j(G_1, G_2) \leq 2(q - s) \). The rotation and jump distances are related in [4] where it is shown that each distance is at most twice the other, that is, for every two graphs \( G_1 \) and \( G_2 \) of the same order (at least 5) and same size,

\[
 d_r(G_1, G_2) \leq 2 d_j(G_1, G_2) \quad \text{and} \quad d_j(G_1, G_2) \leq 2 d_r(G_1, G_2)
\]

or

\[
 \frac{1}{2} d_j(G_1, G_2) \leq d_r(G_1, G_2) \leq 2 d_j(G_1, G_2)
\]

and hence

\[
 \frac{1}{2} d_r(G_1, G_2) \leq d_j(G_1, G_2) \leq 2 d_r(G_1, G_2).
\]

These bounds are sharp as seen in [4], where it is shown that for positive integers \( a \) and \( b \) with \( a/2 \leq b \leq 2a \), there exist graphs \( G_1 \) and \( G_2 \) of the same order and same size such that \( d_j(G_1, G_2) = a \) and \( d_r(G_1, G_2) = b \).

We have already seen that metrics defined on graphs may be applied to problems in chemistry (see [1] or [18] for example). Such applications suggest the problem of selecting an appropriate metric. The selection cannot be based on topological properties because each of our metrics induces the discrete topology on its respective domain. However, these metrics can be differentiated. Let \( S \) denote a fixed set of graphs. The \textit{distance graph} \( D_d(S) \) with respect to a metric \( d \) on \( S \) is defined as that graph with vertex set \( S \) such that vertices \( G_1 \) and \( G_2 \) are adjacent in \( D_d(S) \) if and only if \( d(G_1, G_2) = 1 \). If \( d = d_r \), the rotation distance, then \( D_d(S) \) is denoted by \( D_r(S) \) and called the \textit{edge rotation distance graph}. Similarly if \( d = d_s \) or \( d = d_j \), then
$D_d(S)$ is denoted by $D_s(S)$ or $D_j(S)$ and called the edge slide distance graph or jump distance graph, respectively. This question is first considered in [2] and [17], where it is shown in [2] that there exists a set $S$ of graphs for which $D_s(S) = D_j(S) = K_n$.

Every graph is an edge slide distance graph as seen in [3], that is, for every graph $G$, there exists a set $S$ of graphs of the same order and same size such that $D_s(S) = G$. Also in [3] many classes of graphs are known to be edge rotation graphs including complete graphs, cycles, paths, unions and cartesian products of edge rotation graphs, line graphs, and trees. Indeed, it is conjectured in [3] that every graph is an edge rotation distance graph. In [13] it is shown that $K_{3,3}$ is an edge rotation distance graph and that for each positive integer $n$, the graph $K_{n,2}$ is an edge rotation distance graph. Jarrett extends this result in [15] and shows that all complete bipartite graphs are edge rotation graphs. Many graphs are known to be jump distance graphs as well. In [4], it is shown that complete graphs, complete multipartite graphs, trees, cycles, and complements of line graphs are all jump distance graphs and conjectured that every graph is a jump distance graph.

1.3 Some Results on Greatest Common Subgraphs and Other Related Concepts

A greatest common subgraph of a set $G = \{G_1, G_2, \ldots, G_n\}$ ($n \geq 2$) of graphs of the same size is a graph $G$ of maximum size without isolated vertices that is a subgraph of each $G_i$ ($1 \leq i \leq n$). The set of all greatest common subgraphs of $G$ is denoted by $gcs(G)$ or $gcs(G_1, G_2, \ldots, G_n)$. We have already seen how the notion of a greatest common subgraph can be used to describe an upper bound for the edge rotation distance between two graphs. In fact, determining the size of a greatest common subgraph of two graphs with the same order and same size is equivalent to determining the edge move distance between them. Since graphs represent discrete structures in a natural way, greatest common subgraphs represent maximum common
substructures. In [5], four areas where this concept arises are (1) in the context of algorithmically perceiving the structural features that are preserved in a chemical reaction, (2) determining maximum commonalities between molecular structures, (3) developing a metric for studying the relationships between molecular structures and chemical properties, and (4) applying the concept of distance between graphs to object recognition.

In the theory of greatest common subgraphs, a natural question arises: Which graphs are greatest common subgraphs? This question was answered in [10], where it is shown that for every graph $G$ of size $q$ without isolated vertices, there exist graphs $G_1$ and $G_2$ of size $q + 1$ such that $G$ is the unique greatest common subgraph of $G_1$ and $G_2$. Further, this result has been extended and it is shown [10] that for every graph $G$ with isolated vertices, there exist graphs $G_1, G_2,$ and $G_3$ such that $gcs(G_1, G_2, G_3) = \{G\}$. However, not every pair $H_1, H_2$ of graphs of equal size can be the set of greatest common subgraphs. In [10], it was shown that for every pair $G_1, G_2$ of graphs of equal size, $gcs(G_1, G_2) \neq \{K_{1,6}, K_4\}$. Although one cannot always specify the set of greatest common subgraphs, the number of graphs in $G$ and the number of graphs in the set of greatest common subgraphs can be specified. In [10], it is shown that for every pair $m, n$ of integers with $m \geq 2$ and $n \geq 1$, there exists a set $G$ of $m$ (pairwise nonisomorphic) graphs of equal size such that $|gcs G| = n$.

We have already noted that for every graph $G$ without isolated vertices, there exist graphs $G_1$ and $G_2$ such that $gcs(G_1, G_2) = \{G\}$ and, further, there exist graphs $G_1, G_2,$ and $G_3$ such that $gcs(G_1, G_2, G_3) = \{G\}$. This suggests the following question: For a given graph $G$ without isolated vertices and a given integer $n \geq 2$, does there exist a set $G = \{G_1, G_2, \ldots, G_n\}$ of $n$ graphs of equal size such that $gcs G = \{G\}$? Certainly, if $n$ is large, then the graphs $G_1, G_2, \ldots, G_n$ of $G$
must have large size as well. For a graph $G$ without isolated vertices, the greatest common subgraph index or gcs index of $G$, denoted $i(G)$, is the least positive integer $q_0$ such that for any integer $q > q_0$ and any set $G = \{G_1, G_2, \ldots, G_n\}$ ($n \geq 2$) of graphs of size $q$ for which $G \in \text{gcs } G$, it follows that $|\text{gcs } G| > 1$, that is, the graphs $G_1, G_2, \ldots, G_n$ have at least two greatest common subgraphs. If no such $q_0$ exists, then $i(G) = \infty$.

A lower bound for the gcs index was established in [20], where it is shown that if $G$ is a noncomplete graph of order $p$ without isolated vertices, then $i(G) \geq \left( \frac{p}{2} \right)$. This bound is sharp but can be improved if the graph has no end-vertices [20], that is, if $G$ is a graph of order $p$ for which $\delta(G) \geq 2$, then $i(G) \geq \left( \frac{p+1}{2} \right)$. Furthermore, this bound is sharp as well. There do exist graphs for which the gcs index is finite. In fact, in [8] it is shown that $i(K_3) = 6$ and $i(P_4) = 6$. It is also known [20] that there exist graphs of arbitrarily large (but finite) gcs index. A necessary condition is known [20] as well for a graph to have infinite gcs index, namely, if $G$ is a graph with a vertex $v$ of maximum degree such that no component of $G - v$ is isomorphic to $K_2$, then $i(G) = \infty$. As a consequence of this result, complete graphs of order $n$ ($\neq 3$), complete bipartite graphs, cycles of order at least 4, and paths of order $n$ ($\neq 4$) all have infinite gcs index. Another consequence of this result is that all 2-connected graphs of order at least 4 have infinite gcs index. It is well known that for each fixed positive integer $n$, almost every graph is $n$-connected, and hence almost every graph has infinite gcs index. For graphs with finite gcs index, it is conjectured in [20] that if $G$ is a graph for which $i(G)$ is finite, then $i(G) = \left( \frac{n}{2} \right)$ for some integer $n \geq 4$.

Greatest common subgraphs with specified properties have also been studied. We have already noted that for every graph $G$ without isolated vertices, there exist graphs $G_1$ and $G_2$ of equal size such that $G$ is the unique greatest common subgraph of $G_1$ and $G_2$. In the proof [10] of this result, one of $G_1$ and $G_2$ is
disconnected. In [6] it was shown that $G, G_1, \text{ and } G_2$ can be required to be connected, that is, for every connected noncomplete graph $G$, there exist connected graphs $G_1$ and $G_2$ such that $\text{gcs}(G_1, G_2) = \{G\}$.

A greatest common induced subgraph of two graphs $G_1$ and $G_2$ of equal size is a graph $G$ of maximum size that is an induced subgraph of each of $G_1$ and $G_2$. The set of all greatest common induced subgraphs of $G_1$ and $G_2$ is denoted by $\text{gcis}(G_1, G_2)$. It turns out that every graph without isolated vertices is the unique greatest common induced subgraph of two nonisomorphic graphs of the same size. Furthermore, in [6], if $G$ is a nontrivial connected graph, then there exist connected graphs $G_1$ and $G_2$ of equal size such that $\text{gcs}(G_1, G_2) = \{G\}$. These concepts have also been studied for digraphs in [6].

Thus far we have been considering special cases of a more general type of problem: For a given graph $G$ with a specified graphical property $P$, do there exist graphs $G_1$ and $G_2$ of equal size with property $P$ such that $G$ is the unique greatest common subgraph of $G_1$ and $G_2$? We now consider various properties $P$. For example if $P$ is the property "2-connected", then the following is true (see [8]). For a 2-connected graph $G$ of order $p$, where $G \neq K_p$ ($p \geq 3$) and $G \neq K_p - e$ ($p \geq 4$) for some edge $e$ of $K_p$, there exist 2-connected graphs $G_1$ and $G_2$ of equal size such that $\text{gcs}(G_1, G_2) = \{G\}$. The situation in general is not known for $n$-connected graphs. However, for $n$-chromatic graphs ($n \geq 2$), the situation is much different. In [8], if $G$ is an $n$-chromatic graph ($n \geq 2$) without isolated vertices, then there exist $n$-chromatic graphs $G_1$ and $G_2$ of equal size such that $\text{gcs}(G_1, G_2) = \{G\}$.

We now consider the case when $P$ is the property "is a tree". First, for an integer $t \geq 2$, let $D(t)$ denote that tree obtained by connecting the centers of two copies of $K_{1,t}$ by a path of length 3. Not every tree is the unique greatest common subgraph of two nonisomorphic trees. In fact, it is shown in [11] that for a nontrivial
tree $T$, there exist trees $T_1$ and $T_2$ of equal size such that $\text{gcs}(T_1, T_2) = \{T\}$ if and only if $T$ is not a path of order 2, 4, 5, ... and $T \not\equiv D(t)$ for $t \geq 2$. The situation is much different for greatest common induced subgraphs, where in [11] it is shown that every tree of order at least 3 is the unique greatest common subgraph of two nonisomorphic trees of the same order. The properties "connected outerplanar" and "connected planar" have also been studied (see [5]).
CHAPTER II

H-CONNECTED GRAPHS

2.1 Introduction

Let \( G_1 \) and \( G_2 \) be two graphs of the same order and same size such that \( V(G_1) = V(G_2) \), and let \( H \) be a connected graph of order at least 3. Two subgraphs \( H_1 \) and \( H_2 \) of \( G_1 \) and \( G_2 \), respectively, are \( H \)-adjacent if \( H_1 \cong H_2 \cong H \) and \( H_1 \) and \( H_2 \) share some but not all edges, that is, \( E(H_1) \cap E(H_2) \neq \emptyset \) and \( E(H_2) - E(H_1) \neq \emptyset \) (so also \( E(H_1) - E(H_2) \neq \emptyset \)). The graphs \( G_1 \) and \( G_2 \) are themselves \( H \)-adjacent if \( G_1 \) and \( G_2 \) contain \( H \)-adjacent subgraphs \( H_1 \) and \( H_2 \), respectively, such that \( E(H_2) - E(H_1) \subseteq E(G_1) \) and \( G_2 = G_1 - E(H_1) + E(H_2) \). A \( G_1-G_2 \) \( H \)-walk is a sequence \( G_1 = F_0, F_1, \ldots, F_k = G_2 \) of graphs of the same order and same size such that \( F_i \) is \( H \)-adjacent to \( F_{i+1} \) for \( i = 0, 1, \ldots, k - 1 \). A \( G_1-G_2 \) \( H \)-walk \( G_1 = F_0, F_1, \ldots, F_k = G_2 \) in which the graphs \( F_0, F_1, \ldots, F_k \) are distinct is called a \( G_1-G_2 \) \( H \)-path. The graphs \( G_1 \) and \( G_2 \) are \( H \)-connected if there exists a \( G_1-G_2 \) \( H \)-path. The relation \( H \)-connectedness is an equivalence relation on the set of all graphs of the same order and same size.

Let \( H = P_3 \). In Figure 2.1, the path \( H_1: u, v, w \) of \( G_1 \) is \( H \)-adjacent to the path \( H_2: v, w, u \) of \( G_2 \), and in fact, since \( G_2 = G_1 - E(H_1) + E(H_2) \), the graphs \( G_1 \) and \( G_2 \) are \( H \)-adjacent. The path \( H'_2: w, x, y \) of the graph \( G_2 \) is \( H \)-adjacent to the path \( H'_3: x, w, y \) of the graph \( G_3 \), shown in Figure 2.2, and thus since \( G_3 = G_2 - E(H'_2) + E(H'_3) \), the graphs \( G_2 \) and \( G_3 \) are \( H \)-adjacent. Clearly, \( G_1 \) is not \( H \)-adjacent to \( G_3 \) but since \( G_1, G_2, G_3 \) is an \( H \)-path from \( G_1 \) to \( G_3 \), the graph \( G_1 \) is \( H \)-connected to \( G_3 \).
2.2 \( P_3 \)-Connected Graphs

Suppose again that \( H = P_3 \) and that \( G_1 \) and \( G_2 \) are \( H \)-adjacent graphs. Then \( G_1 \) contains a copy \( H_1 \) of \( P_3 \), say \( u, v, w \) and \( G_2 \) contains a copy \( H_2 \) of \( P_3 \) with \( E(H_1) \cap E(H_2) \neq \emptyset \) and \( E(H_2) - E(H_1) \subseteq E(G_1) \). Since \( E(H_1) \subseteq E(G_1) \) and \( H_1 \) has exactly two edges, it follows that \( H_1 \) and \( H_2 \) have exactly one edge in common, say \( uv \), and \( H_2 \) contains exactly one edge that is not present in \( G_1 \). So \( H_2 \): \( u, v, z \) or \( H_2 \): \( z, u, v \) for some \( z \in V(G_2) \). Thus (1) \( G_2 = G_1 - vw + vz \) or (2) \( G_2 = G_1 - vw + uz \), where \( u, v, w, \) and \( z \) are not necessarily distinct. The graph \( G_2 \) is said to be obtained from \( G_1 \) by an edge move if \( G_1 \) contains (not necessarily distinct) vertices \( u, v, w, \) and \( x \) such that \( uv \in E(G_1) \), \( wx \notin E(G_1) \), and \( G_2 = G_1 - uv + wx \). Thus, if the graphs \( G_1 \) and \( G_2 \) are \( P_3 \)-adjacent, then \( G_2 \) can be obtained from \( G_1 \) by an edge move. It was shown in [2] that for every two
graphs of the same order and same size, each can be transformed into the other by a sequence of edge moves.

The graph $G_2$ is said to be obtained from $G_1$ by an *edge rotation* if $G_1$ contains distinct vertices $u, v,$ and $w$ such that $uv \in E(G_1)$, $uw \notin E(G_1)$ and $G_2 = G_1 - uv + uw$. Thus if the graphs $G_1$ and $G_2$ are $P_3$-adjacent and $G_2 = G_1 - vw + vz$, which is case (1), then $G_2$ can be obtained from $G_1$ by an edge rotation. In fact, in this case, $P_3$-adjacency is more restrictive than edge-rotation since $P_3$-adjacency requires the presence of another edge incident to $v$. In [9] it was shown that for every two graphs of the same order and same size, each can be transformed into the other by a sequence of edge rotations. The graph $G_2$ is obtained from $G_1$ by an *edge jump* if $G_1$ contains four distinct vertices $u, v, w,$ and $x$ such that $uv \in E(G_1)$, $wx \notin E(G_2)$ and $G_2 = G_1 - uv + wx$. Thus if $G_1$ and $G_2$ are $P_3$-adjacent with $G_2 = G_1 - uv + wx$ and $u, v, w,$ and $x$ are distinct vertices as in case (2), then $G_2$ is obtained from $G_1$ by an edge jump. In [4] it was shown that every two graphs of the same order (at least 5) and same size can be transformed into one another by a sequence of edge jumps. We now present a corresponding result for $P_3$-adjacency.

**Theorem 2.1** Let $G_1$ and $G_2$ be two graphs of the same order and same size such that $G_2 = G_1 - uv + wx$ for (not necessarily distinct) vertices $u, v, w,$ and $x$, where the edges $uv$ and $wx$ belong to components of order at least 3 of $G_1$ and $G_2$, respectively. Then $G_1$ and $G_2$ are $P_3$-connected.

**Proof** We consider two cases, according to whether the vertices $u, v, w,$ and $x$ are distinct.
Case 1 Suppose that the vertices \( u, v, w, \) and \( x \) are not distinct, say \( v = w \). Now since the edge \( uv \) belongs to a component of order at least 3 of \( G \), it follows that there exists an edge adjacent to \( uv \); so either \( \deg_{G} u > 1 \) or \( \deg_{G} v > 1 \). Suppose first that \( \deg_{G} v > 1 \). Then there exists a vertex \( y \) (\( \neq u \)) of \( G \) adjacent to \( v \), and hence \( G_2 = G_1 - \{uv, vy\} + \{vy, vx\} \). Therefore \( G_1 \) and \( G_2 \) are \( P_3 \)-adjacent.

Thus we assume that \( \deg_{G_1} v = 1 \). Hence \( \deg_{G_1} u > 1 \) and \( \deg_{G_2} v = 1 \). Since \( uv \) and \( vx \) belong to a component of order at least 3 in \( G_1 \) and \( G_2 \), respectively, and \( \deg_{G_2} v = 1 \), it follows that there exist vertices \( z \) and \( y \) (distinct from \( v \)) such that \( uz \in E(G_1) \) and \( xy \in E(G_2) \). First, suppose that \( z = x \). Then \( G_2 = G_1 - \{xu, uv\} + \{vx, xu\} \); so \( G_1 \) is \( P_3 \)-adjacent to \( G_2 \). Thus we may assume that \( z \neq x \) and that \( u \) is not adjacent to \( x \). If \( y = z \), then let \( F_1 = G_1 - \{vu, uz\} + \{ux, uz\} \); so \( F_1 \) and \( G_1 \) are \( P_3 \)-adjacent. Next \( G_2 = F_1 - \{ux, xz\} + \{vx, xz\} \); so \( F_1 \) is \( P_3 \)-adjacent to \( G_2 \), and thus \( G_1 \) and \( G_2 \) are \( P_3 \)-connected. This situation is shown in Figure 2.3, where the graphs \( H_0, H_1, \) and \( H_2 \) are subgraphs of \( G_1, F_1, \) and \( G_2 \), respectively. Therefore, we may now assume that every vertex \( z \) adjacent to \( u \) is not adjacent to \( x \). Let \( F_1 = G_1 - \{vu, uz\} + \{uz, zx\} \); so \( G_1 \) and \( F_1 \) are \( P_3 \)-adjacent. Then \( G_2 = F_1 - \{zx, xy\} + \{vx, xy\} \); so \( G_2 \) is \( P_3 \)-adjacent to \( F_1 \), and thus \( G_1 \) and \( G_2 \) are \( P_3 \)-connected. This transformation is shown in Figure 2.3, where the graphs \( J_0, J_1, \) and \( J_2 \) are subgraphs of \( G_1, F_1, \) and \( G_2 \), respectively.
Case 2 Suppose that the vertices $u, v, w, \text{ and } x$ are distinct. Suppose first that one of the edges $uw, ux, vw, \text{ and } vx$ is not present in $G_1$ (and hence in $G_2$), say $e \not\in E(G_1)$, where $e \in \{uw, ux, vw, vx\}$. Then by Case 1, $G_1$ is $P_3$-connected to $F_1 = G_1 - uv + e$, and $F_1$ is $P_3$-connected to $G_2 = F_1 - e + wx$. Hence $G_1$ and $G_2$ are $P_3$-connected, and therefore we may assume that all of the edges $uw, ux, vw, \text{ and } vx$ are present in $G_1$. Then $G_2 = G_1 - \{vu, uw\} + \{uw, wx\}$; so $G_1$ and $G_2$ are $P_3$-adjacent. □

The distance between an edge $e = uv$ and a subgraph $H$ of a graph $G$ is defined by $d(e, H) = \min \{d(u, x), d(v, x) \mid x \in V(H)\}$. We have seen conditions under which two graphs that differ in exactly one edge are $P_3$-connected. We now determine conditions under which any two graphs are $P_3$-connected.

**Theorem 2.2** Let $G_1$ and $G_2$ be two graphs of the same order and the same size. Then $G_1$ and $G_2$ are $P_3$-connected if and only if each of $G_1$ and $G_2$ contains a subgraph isomorphic to $P_3$. 

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Proof Let $H: u, v, w$ denote a common subgraph of $G_1$ and $G_2$ that is isomorphic to $P_3$. Suppose that $G_1$ has $s$ edges not belonging to $G_2$. We label these edges so that the edges $e_1, e_2, \ldots, e_k (0 \leq k \leq s)$ are adjacent to an edge of $H$, the edges $e_{k+1}, e_{k+2}, \ldots, e_{\ell} (k \leq \ell \leq s)$ are at distance 1 from $H$, and $e_{\ell+1}, e_{\ell+2}, \ldots, e_s$ are the remaining edges. It may be that there are no edges adjacent to an edge of $H$ or at distance 1 from $H$, and in that case, we take $k = 0$ or $\ell = 0$ as needed. Let $e_i = u_i v_i$ for $i = \ell + 1, \ell + 2, \ldots, s$, and observe that by the choice of $k$ and $\ell$, none of $e_{\ell+1}, e_{\ell+2}, \ldots, e_s$ is adjacent to $uv$ or $vw$, and thus the edges $u_{\ell+1}v, u_{\ell+2}v, \ldots, uv_s$ are not present in $G_1$. We now define the graphs $H_0, H_1, \ldots, H_2(s-\ell)$ recursively. Let $H_0 = G_1$, and for $i = 1, 2, \ldots, s-\ell$, let

$$H_{2i-1} = H_{2(i-1)} - \{uv, vw\} + \{uv, vu_{\ell+i}\}$$

and let

$$H_{2i} = H_{2i-1} - \{v_{\ell+i}u_{\ell+i}, u_{\ell+i}v\} + \{u_{\ell+i}v, vw\}.$$ 

So, for $i = 0, 1, \ldots, 2(s-\ell) - 1$, the graph $H_i$ is $P_3$-adjacent to $H_{i+1}$. Thus $G_1, H_1, H_2, \ldots, H_2(s-\ell) = F_1$ is a $P_3$-path from $G_1$ to $F_1$; so $G_1$ is $P_3$-connected to $F_1$. Observe that $F_1$ differs from $G_2$ by the edges $e_1, e_2, \ldots, e_{\ell}, vu_{\ell+1}, vu_{\ell+2}, \ldots, vu_s$.

The graph $G_2$ has $s$ edges not belonging to $G_1$. Label these edges so that the edges $f_1, f_2, \ldots, f_m (0 \leq m \leq s)$ are adjacent to an edge of $H$, the edges $f_{m+1}, f_{m+2}, \ldots, f_n (m \leq n \leq s)$ are at distance 1 from $H$, and $f_{n+1}, f_{n+2}, \ldots, f_s$ are the remaining edges. Let $f_i = w_i x_i$ for $i = n + 1, n + 2, \ldots, s$. The graph $F_2$ is obtained from $G_2$ exactly as $F_1$ was obtained from $G_1$; that is, define $J_0 = G_2$ and for $i = 1, 2, \ldots, s-m$, define

$$J_{2i-1} = J_{2(i-1)} - \{uv, vw\} + \{uv, vw_{m+i}\}$$

and
\[ J_{2i} = J_{2i-1} - \{x_{m+i}w_{m+i}, w_{m+i}v\} + \{w_{m+i}v, vw\}. \]

As before, for \( i = 0, 1, \ldots, 2(s-m) - 1 \), the graph \( J_i \) is \( P_3 \)-adjacent to \( J_{i+1} \). Then \( G_2 = J_0, J_1, \ldots, J_{2(s-m)} = F_2 \) is a \( P_3 \)-path from \( G_2 \) to \( F_2 \); so \( G_2 \) is \( P_3 \)-connected to \( F_2 \). Therefore \( F_2 \) differs from \( G_1 \) by the edges \( f_1, f_2, \ldots, f_n, vw_{n+1}, vw_{n+2}, \ldots, vw_\ell \). Consequently, \( G_1 \) is \( P_3 \)-connected to \( F_1 \), and \( F_2 \) is \( P_3 \)-connected to \( G_2 \). Hence, if \( F_1 \) and \( F_2 \) are \( P_3 \)-connected, then \( G_1 \) and \( G_2 \) are \( P_3 \)-connected. It remains to show that \( F_1 \) and \( F_2 \) are \( P_3 \)-connected.

Let \( H_1 \) be a greatest common subgraph of \( F_1 \) and \( F_2 \) containing \( H \). So \( F_1 \) contains the edges \( a_1, a_2, \ldots, a_r \) not belonging to \( F_2 \), and \( F_2 \) contains the edges \( b_1, b_2, \ldots, b_r \) not belonging to \( F_1 \), where

\[ \{a_1, a_2, \ldots, a_r\} \subseteq \{e_1, e_2, \ldots, e_\ell, vu_\ell+1, vu_\ell+2, \ldots, vu_\delta\} \]

and

\[ \{b_1, b_2, \ldots, b_r\} \subseteq \{f_1, f_2, \ldots, f_\ell, vw_{n+1}, vw_{n+2}, \ldots, vw_\ell\}. \]

Also, each of \( a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_r \) lies on a path of length 2. Furthermore, by the construction of \( F_1 \) and \( F_2 \), each of \( a_i \) and \( b_i \) \((1 \leq i \leq r)\) is adjacent to an edge of \( H \) (and hence of \( H_1 \)) or at distance 1 from \( H \) (and thus from \( H_1 \)). Label the edges \( a_1, a_2, \ldots, a_r \) so that \( a_1, a_2, \ldots, a_c \) \((0 \leq c \leq r)\) are at distance 1 from \( H \) and \( a_{c+1}, a_{c+2}, \ldots, a_r \) are adjacent to an edge of \( H \). Next label the edges \( b_1, b_2, \ldots, b_r \) so that \( b_1, b_2, \ldots, b_d \) \((0 \leq d \leq r)\) are adjacent to an edge of \( H \) and \( b_{d+1}, b_{d+2}, \ldots, b_r \) are at distance 1 from \( H \). Each \( a_i \) \((1 \leq i \leq c)\) must be adjacent to some \( a_j \) for \( c + 1 \leq j \leq r \) and, similarly, each \( b_j \) \((d + 1 \leq j \leq r)\) must be adjacent to some \( b_i \) for \( 1 \leq i \leq d \). Thus we must remove \( a_1, a_2, \ldots, a_c \) from \( F_1 \) before removing \( a_{c+1}, a_{c+2}, \ldots, a_r \) and the edges \( b_1, b_2, \ldots, b_d \) must be present before the edges \( b_{d+1}, b_{d+2}, \ldots, b_r \) can be added. Let \( J_0 = F_1 \) and then for \( i = 1, 2, \ldots, r \), the
graph \( J_r = J_{r-1} - a_r + b_r \) is \( P_3 \)-connected to \( J_{r-1} \) by Theorem 2.1. Since \( F_2 = J_r \), we have that \( F_1 \) is \( P_3 \)-connected to \( F_2 \), and hence \( G_1 \) is \( P_3 \)-connected to \( G_2 \).

Finally, assume that \( G_1 \) is \( P_3 \)-connected to \( G_2 \). Then there exists a \( G_1 \)-\( G_2 \) \( P_3 \)-path, say \( G_1 = F_0, F_1, F_2, \ldots, F_n = G_2 \). Since \( G_1 \) is \( P_3 \)-adjacent to \( F_1 \), the graph \( F_1 = G_1 - E(H_1) + E(H_2) \), where \( H_1 \cong H_2 \cong P_3 \) and \( E(H_1) \subseteq E(G_1) \). Thus, \( G_1 \) contains a subgraph, namely \( H_1 \), that is isomorphic to \( P_3 \). Similarly, since \( F_{n-1} \) is \( P_3 \)-adjacent to \( G_2 \), it follows that \( G_2 \) contains a subgraph that is isomorphic to \( P_3 \).

\[ \square \]

2.3 \( P_4 \)-Connected Graphs

Next we show that every two connected graphs of the same order and same size are \( P_4 \)-connected. First, the following lemma will be useful.

**Lemma 2.3** Let \( G \) be a connected graph containing a connected graph \( H \) as a subgraph. Then every edge of \( G \) belongs to a subgraph that is isomorphic to \( H \) if and only if \( H \) is \( K_2 \), \( P_3 \), or \( P_4 \).

**Proof** Suppose that every edge of \( G \) belongs to a subgraph of \( G \) that is isomorphic to \( H \). We show that \( H = K_2 \), \( P_3 \), or \( P_4 \). Suppose, to the contrary, that \( \Delta(H) \geq 3 \) and let \( k \) be the diameter of \( H \). Obtain the graph \( G \) from \( H \) by joining a path of length \( k + 1 \) to a vertex of \( H \). Then \( G \) contains \( H \) as a subgraph, yet clearly not every edge of \( G \) belongs to a subgraph isomorphic to \( H \), producing a contradiction. Thus \( \Delta(H) \leq 2 \), and \( H \) is a cycle or a path. Suppose first that \( H \) is a cycle, say \( H = C_n \) for some positive integer \( n \geq 3 \). Let \( G \) be obtained from \( H \) by joining a new vertex \( x \) to a vertex \( v \) of \( H \). As before, \( G \) contains \( H \) as a subgraph yet the edge \( xv \) of \( G \) does not belong to a subgraph isomorphic to \( H \), producing a
contradiction. Hence $H$ is a path, say $H = P_n$ for some integer $n \geq 2$. Suppose, to the contrary, that $n \geq 5$. Consider the graph $G$ obtained from the path $P: v_1, v_2, \ldots, v_n$ by joining a new vertex $x$ to $v_3$. So $G$ contains $P_n$ as subgraph yet the edge $xv_3$ does not belong to a subgraph of $G$ isomorphic to $P_n$, producing a contradiction. Hence $n \leq 4$, and $H$ is $K_2$, $P_3$, or $P_4$.

The converse is clearly true if $H = K_2$ or $H = P_3$. Thus, suppose that $H = P_4$, and let $G$ be a graph containing $P_4$ as a subgraph, say $P: x_0, x_1, x_2, x_3$ is a path of length 3 in $G$. Clearly, every edge of $P$ lies on a path of length 3. Next, let $e = uv$ be an edge of $G$ that does not belong to $P$. At most one of $u$ and $v$ can lie on $P$, suppose first that $u$ lies on $P$. So without loss of generality, $u = x_0$ or $u = x_1$. Then $v, u = x_0, x_1, x_2$ or $v, u = x_1, x_2, x_3$ is a path of length 3 containing the edge $uv$. Finally, suppose that neither $u$ nor $v$ lies on $P$. Let $Q: u_0, u_1, \ldots, u_n$ ($n \geq 1$) be a shortest path from the edge $uv$ to $P$. Without loss of generality, assume that $u_0 = u$ and $u_n = x_0$ or $u_n = x_1$. Then $v, u = u_0, u_1, \ldots, u_n = x_0, x_1$ or $v, u = u_0, u_1, \ldots, u_n = x_1, x_0$ is a path of length at least 3, and hence $e$ lies on a path of length 3. □

**Theorem 2.4** Let $G_1$ and $G_2$ be connected graphs of the same order and same size, each of which has a subgraph isomorphic to $P_4$. If $G_2 = G_1 - uv + wx$ for (not necessarily distinct) vertices $u, v, w,$ and $x$ of $G_1$, then $G_1$ and $G_2$ are $P_4$-connected.

**Proof** We consider two cases, depending on whether the vertices $u, v, w,$ and $x$ are distinct.
Case 1  Suppose that the vertices \( u, v, w, \) and \( x \) are not distinct, say \( v = w. \) Since
\( G_1 \) is connected, there exists a shortest \( u-x \) path \( P: u = u_0, u_1, \ldots, u_{n-1}, u_n = x. \)
Next, we proceed depending on whether \( u_1 = v. \)

Subcase 1.1  Suppose that \( u_1 = v, \) that is, suppose that the path \( P \) contains the edge \( uv. \) Since \( vx \) is not an edge of \( G_1, \) it follows that \( n \geq 3. \) Suppose first that \( n \geq 4. \)
Then \( G_2 = G_1 - \{uv, vu_2, u_2u_3\} + \{xv, vu_2, u_2u_3\} \) and \( G_1 \) is \( P_4 \)-adjacent to \( G_2. \)
Now suppose that \( n = 3, \) that is, suppose that \( u, v, u_2 = y, x \) is a shortest \( u-x \) path.
Then in \( G_2, \) the edges \( vx, xy, yv \) form a triangle. Since \( G_2 \) contains a path of length 3 (a copy of \( P_4), \) by Lemma 2.3 the edge \( vx \) must lie on a path of length 3.
Thus since \( uv \notin E(G_2) \) and a shortest path from \( u \) to \( x \) has length 3, there exists a vertex \( z \) of \( G_2 \) such that \( z \) is adjacent to at least one of \( v, x, \) and \( y \) in \( G_2. \)
Suppose first that \( z \) is adjacent to \( y. \) Then \( G_2 = G_1 - \{uv, vy, yz\} + \{vx, vy, yv\}, \)
and \( G_1 \) is \( P_4 \)-adjacent to \( G_2. \) Hence, in what follows, we assume that \( z \) is not
adjacent to \( y. \) Next suppose that \( z \) is adjacent to \( v. \) Let \( F_1 = G_1 - \{zv, vy, yx\} + \{zv, vx, xy\}; \) so \( F_1 \) is \( P_4 \)-adjacent to \( G_2. \) Then let \( F_2 = F_1 - \{uv, vx, xy\} + \{vx, xy, yz\}; \) so \( F_2 \) is \( P_4 \)-adjacent to \( F_1. \) Thus \( G_2 = F_2 - \{vz, zy, vx\} + \{zv, vy, xy\}; \)
so \( G_2 \) and \( F_2 \) are \( P_4 \)-adjacent, and \( G_1, F_1, F_2, G_2 \) is a \( G_1-G_2 \) \( P_4 \)-path. This
situation is represented in Figure 2.4, where \( H_0, H_1, H_2, \) and \( H_3 \) are subgraphs of
\( G_1, F_1, F_2, \) and \( G_2, \) respectively.

Finally, suppose that \( z \) is adjacent only to \( x. \) Let \( F_1 = G_1 - \{zx, xy, vy\} + \{zx, xv, vy\}; \) so \( F_1 \) is \( P_4 \)-adjacent to \( G_1. \) Then let \( F_2 = F_1 - \{uv, vx, xz\} + \{vx, xz, zy\}; \) so \( F_2 \) is \( P_4 \)-adjacent to \( F_1. \) Hence \( G_2 = F_2 - \{vy, yz, zx\} + \{vy, yx, xz\}; \)
so \( G_2 \) is \( P_4 \)-adjacent to \( F_2, \) and \( G_1, F_1, F_2, G_2 \) is a \( G_1-G_2 \) \( P_4 \)-path. This
transformation is represented in Figure 2.5, where the graphs \( H_0, H_1, H_2, \) and \( H_3 \)
are subgraphs of \( G_1, F_1, F_2, \) and \( G_2, \) respectively.
Subcase 1.2 Suppose that a shortest $u$-$x$ path $P$: $u = u_0, u_1, \ldots, u_{n-1}, u_n = x$ does not contain $v$. Suppose first that $n \geq 3$. Let $F_1 = G_1 - \{vu, uu_1, u_1u_2\} + \{u_2u_1, u_1u, u_1x\}$ so that $F_1$ is $P_4$-adjacent to $G_1$. Then $G_2 = F_1 - \{ux, xu_{n-1}, u_{n-1}u_{n-2}\} + \{vx, xu_{n-1}, u_{n-1}u_{n-2}\}$, and $G_2$ and $F_1$ are $P_4$-adjacent. Thus $G_1$ and $G_2$ are $P_4$-connected. Next suppose that $n = 2$. Then $P$: $u, u_1, x$ and $G_2 = G_1 - \{vu, uu_1,$
$G_1$ and $G_2$ are $P_4$-adjacent. Finally, suppose that $n = 0$, that is, $ux \in E(G_1)$. If $\deg_{G_1} v \geq 2$, then there exists a vertex $z$ of $G_1$ adjacent to $v$ and $G_2 = G_1 - \{zv, vu, ux\} + \{zv, vx, xu\}$; so $G_1$ and $G_2$ are $P_4$-adjacent. Thus we assume that $\deg_{G_1} v = 1$. Next suppose that $\deg_{G_1} u \geq 3$ and $\deg_{G_1} x \geq 2$. Then there exist vertices $y$ and $z$ of $G_1$ such that $uy, xz \in E(G_1)$. Let $F_1 = G_1 - \{yu, ux, xz\} + \{yu, ux, xv\}$; so $F_1$ and $G_1$ are $P_4$-adjacent. Let $F_2 = F_1 - \{yu, uv, vx\} + \{uy, yv, vx\}$; so $F_1$ and $F_2$ are $P_4$-adjacent. Then $G_2 = F_2 - \{vy, yu, ux\} + \{yu, ux, xz\}$; so $F_2$ and $G_2$ are $P_4$-adjacent, and $G_1, F_1, F_2, G_2$ is a $P_4$-path. This situation is represented in Figure 2.6, where the graphs $H_0, H_1, H_2,$ and $H_3$ are subgraphs of $G_1, F_1, F_2,$ and $G_2$, respectively.

Thus, we may now assume that $\deg_{G_1} v = 1$ and that $\deg_{G_1} u = 2$ or $\deg_{G_1} x = 1$. Suppose first that $\deg_{G_1} u = 2$. Then a path of length 3 containing the edge $uv$ must also contain the edge $ux$, and there must exist a vertex $z$ adjacent to $x$ in $G_1$. Then $\deg_{G_2} u = \deg_{G_2} v = 1$. Since the edge $vx$ must also lie on a path of length 3, we may assume, without loss of generality, there exists a vertex $y$ of $G_2$ adjacent to $z$. Let $F_1 = G_1 - \{ux, xz, zy\} + \{vx, xz, zy\}$; so $G_1$ and $F_1$ are $P_4$-adjacent. Let $F_2 = F_1 - \{uv, vx, xz\} + \{uy, vx, xz\}$; so $F_1$ and $F_2$ are $P_4$-adjacent. Then $G_2 = F_2 - \{xz, zy, vy\} + \{ux, xz, zy\}$; so $F_2$ and $G_2$ are $P_4$-adjacent, and $G_1, F_1, F_2, G_2$ is a $P_4$-path. This transformation is represented in Figure 2.6, where the graphs $I_0, I_1, I_2,$ and $I_3$ are subgraphs of $G_1, F_1, F_2,$ and $G_2$, respectively.

Finally, suppose that $\deg_{G_1} x = 1$. As before, the edge $uv$ must lie on a path of length 3. Since $\deg_{G_1} v = \deg_{G_1} x = 1$, it follows that the edges $uv$ and $ux$ are not on a common path of length 3 and thus there exist vertices $y$ and $z$ such that $v, u, y, z$ is a path in $G_1$. Let $F_1 = G_1 - \{xu, uy, yz\} + \{vx, xu, uy\}$; so $F_1$ and $G_1$
are $P_4$-adjacent. Then let $F_2 = F_1 - \{vx, xu, uy\} + \{uy, yx, xv\}$; so $F_1$ and $F_2$ are $P_4$-adjacent. Finally $G_2 = F_2 - \{vu, uy, yx\} + \{xu, uy, yz\}$; so $F_2$ and $G_2$ are $P_4$-adjacent, and $G_1, F_1, F_2, G_2$ is a $P_4$-path. This situation is shown in Figure 2.6, where the graphs $J_0, J_1, J_2$, and $J_3$ are subgraphs of $G_1, F_1, F_2$, and $G_2$, respectively.

Figure 2.6 A Transformation of $G_1$ Into $G_2$ When $u$ Is Adjacent to $x$.

**Case 2** Assume that the vertices $u, v, w,$ and $x$ are distinct. Suppose, first, that one of the edges $uw, ux, vw,$ or $vx$ is not present in $G_1$, say $e$ is such an edge. By Case 1, the graph $G_1$ is $P_4$-connected to $F_1 = G_1 - uv + e$, and thus $F_1$ is $P_4$-connected to $G_2 = F_1 - e + wx$. Therefore $G_1$ and $G_2$ are $P_4$-connected. Hence all of $uw, ux, vw,$ and $vx$ must be present in $G_1$. Then $G_2 = G_1 - \{wu, uv, vx\} + \{uw, wx, xv\}$, and the graphs $G_1$ and $G_2$ are $P_4$-adjacent. □
The next theorem describes conditions under which two graphs of the same order and same size are $P_4$-connected. Although the proof is similar to the proof of Theorem 2.2, we include it for completeness.

**Theorem 2.5** Let $G_1$ and $G_2$ be two graphs of the same order and the same size. Then $G_1$ is $P_4$-connected to $G_2$ if and only if each of $G_1$ and $G_2$ contains a subgraph isomorphic to $P_4$.

**Proof** Let $H: u, v, w, x$ denote a common subgraph of $G_1$ and $G_2$ that is isomorphic to $P_3$. Suppose that $G_1$ has $n$ edges not belonging to $G_2$. We label these edges so that the edges $e_1, e_2, \ldots, e_k$ ($0 \leq k \leq n$) are adjacent to an edge of $H$ (if any such edges exist), the edges $e_{k+1}, e_{k+2}, \ldots, e_\ell$ ($k \leq \ell \leq n$) are at distance 1 from $H$ (if any such edges exist), and $e_{\ell+1}, e_{\ell+2}, \ldots, e_n$ are the remaining edges. Let $e_i = uv_i$ for $i = \ell + 1, \ell + 2, \ldots, n$ and observe that by the choice of $k$ and $\ell$, each of $e_{\ell+1}, e_{\ell+2}, \ldots, e_n$ is not adjacent to $uv, vw, or wx$. Next define $H_0 = G_1$, and for $i = 1, 2, \ldots, n - \ell$, define

$$H_{2i-1} = H_{2(i-1)} - \{uv, vw, wx\} + \{u_{\ell+1}, v_{\ell+1}, uv, vw\}$$

and

$$H_{2i} = H_{2i-1} - \{v_{\ell+1}, u_{\ell+1}, u_{\ell+1}v, vw\} + \{u_{\ell+1}, uv, vw\}.$$ 

Thus for $i = 0, 1, \ldots, 2(n - \ell) - 1$ the graphs $H_i$ and $H_{i+1}$ are $P_4$-adjacent. Let $F_1 = H_{2(n-\ell)}$, and thus $G_1 = H_0, H_1, \ldots, H_{2(n-\ell)} = F_1$ is a $G_1-F_1$ $P_4$-path. So, $F_1$ differs from $G_2$ by the edges $e_1, e_2, \ldots, e_\ell, uu_{\ell+1}, uu_{\ell+2}, \ldots, uu_n$.

Next $G_2$ has $n$ edges not belonging to $G_1$. Label these edges so that the edges $f_1, f_2, \ldots, f_s$ ($0 \leq s \leq n$) are adjacent to an edge of $H$, the edges $f_{s+1}, f_{s+2}, \ldots, f_t$ ($s \leq t \leq n$) are at distance 1 from $H$, and $f_{t+1}, f_{t+2}, \ldots, f_n$ are the remaining edges. Let $f_i = v \chi_i$ for $i = t + 1, t + 2, \ldots, n$. Next define $J_0 = G_2$, and for $i = 1, 2, \ldots, n - t$, define
\[ J_{2i-1} = J_{2(i-1)} - \{ uv, vw, wx \} + \{ w_{t+i}, v, w, w \} \]

and

\[ J_{2i} = J_{2i-1} - \{ x_{t+i}w_{t+i}, w_{t+i}, vw \} + \{ w_{t+i}, uv, vw \}. \]

Thus for \( i = 0, 1, \ldots, 2(n-t)-1 \), the graphs \( J_i \) and \( J_{i+1} \) are \( P_4 \)-adjacent. Let \( F_2 = J_{2(n-t)} \), and thus \( G_2 = J_0, J_1, \ldots, J_{2(n-t)} = F_2 \) is a \( P_4 \)-path; so \( G_2 \) and \( F_2 \) are \( P_4 \)-connected. Therefore \( F_2 \) differs from \( G_1 \) by the edges \( f_1, f_2, \ldots, f_t, uw_{t+1}, uw_{t+2}, \ldots, uw_n \).

Let \( H_1 \) be a greatest common subgraph of \( F_1 \) and \( F_2 \) containing \( H \). Furthermore, any edge of \( F_1 \) not belonging to \( F_2 \) is either at distance 1 from \( H \) or is adjacent to an edge of \( H \). Thus \( F_1 \) and \( F_2 \) differ only by edges in the same component as \( H \). Let \( a_1, a_2, \ldots, a_r \) denote the edges of \( F_1 \) not belonging to \( F_2 \). Assume that the edges are labeled so that \( a_1, a_2, \ldots, a_c \) (\( 0 \leq c \leq r \)) are at distance 1 from \( H \) and \( a_{c+1}, a_{c+2}, \ldots, a_r \) are adjacent to an edge of \( H \). Next, let \( b_1, b_2, \ldots, b_r \) denote the edges of \( F_2 \) not belonging to \( F_1 \) and assume that these edges are labeled so that \( b_1, b_2, \ldots, b_d \) (\( 0 \leq d \leq r \)) are adjacent to an edge of \( H \) and \( b_{d+1}, b_{d+2}, \ldots, b_r \) are at distance 1 from an edge of \( H \). Each \( a_i \) (\( 1 \leq i \leq c \)) must be adjacent to some \( a_j \) (\( c+1 \leq j \leq r \)) and similarly each \( b_k \) (\( d+1 \leq k \leq r \)) must be adjacent to some \( b_\ell \) (\( 1 \leq \ell \leq d \)). Thus we must remove the edges \( a_1, a_2, \ldots, a_c \) from \( F_1 \) before removing the edges \( a_{c+1}, a_{c+2}, \ldots, a_r \) and the edges \( b_1, b_2, \ldots, b_d \) must be present in \( F_2 \) before the edges \( b_{d+1}, b_{d+2}, \ldots, b_r \) can be added. Let \( J_0 = F_1 \), and for \( i = 1, 2, \ldots, r \), the graph \( J_i = J_{i-1} - a_i + b_i \) is \( P_4 \)-connected to \( J_{i-1} \) by Theorem 2.4. Since \( F_2 = J_r \), it follows that \( F_1 \) is \( P_4 \)-connected to \( F_2 \), and hence \( G_1 \) is \( P_4 \)-connected to \( G_2 \).

Finally, assume that \( G_1 \) is \( P_4 \)-connected to \( G_2 \). Then there exists a \( G_1-G_2 \) \( P_4 \)-path, say \( G_1 = F_0, F_1, \ldots, F_n = G_2 \). Since \( G_1 \) is \( P_4 \)-adjacent to \( F_1 \), there exist
subgraphs $H$ and $H'$ of $G_1$ and $F_1$, respectively, such that $F_1 = G_1 - E(H) + E(H')$ where $H \equiv H' \equiv P_4$. Thus $G_1$ contains a subgraph isomorphic to $P_4$. Similarly, since $F_{n-1}$ is $P_4$-adjacent to $G_2$, it follows that $G_2$ contains a subgraph isomorphic to $P_4$. □

2.4 Other $H$-Connected Graphs

We have seen that if $H$ is $P_3$ or $P_4$, then every two graphs of the same order and same size containing $H$ as a subgraph are $H$-connected. Although we cannot answer the question in general for $H = P_5$, we can show that every two trees of diameter at least 4 are $H$-connected. In fact, we now show that every tree of diameter $d$ is $P_k$-connected to a path for each integer $k$ with $3 \leq k \leq d$. Thus as a corollary, we have that every two trees of diameter at least $k$ are $P_k$-connected.

**Theorem 2.6** If $T$ is a tree of order $n$ and diameter $d$, then $T$ is $P_k$-connected to $P_n$ for each positive integer $k$ with $3 \leq k \leq d$.

**Proof** Let $k$ be an integer with $3 \leq k \leq d$, and let $P: v_0, v_1, \ldots, v_d$ be a longest path in $T$. Suppose that $v_{\ell}$ $(1 \leq \ell \leq d - 1)$ is a vertex of maximum degree on $P$. If $d = n - 1$, then $T = P_n$. Thus we assume that $d < n - 1$. Let $w$ be a vertex not on $P$ such that the $v_{\ell}$-$w$ path $Q$ contains exactly one vertex of $P$, namely $v_{\ell}$. Let $Q: v_{\ell} = u_0, u_1, \ldots, u_m = w$ $(m \geq 1)$. We consider two cases, depending on whether either $m + \ell$ or $m + d - \ell$ is at least $k - 1$.

**Case 1** Suppose that $m + \ell$ or $m + d - \ell$ is at least $k - 1$, say $m + \ell \geq k - 1$. If $m \geq k - 2$, then let $T_1 = T - \{v_{\ell - 1}, v_{\ell}, u_1, u_2, u_3, \ldots, u_{k-3}u_{k-2}\} + \{v_{\ell - 1}u_{k-2}, u_{k-2}u_{k-3}, u_{k-3}u_{k-4}, \ldots, u_2u_1, u_1v_{\ell}\}$. So $T_1$ is $P_k$-adjacent to $T$, and diam $T_1 = d + k - 2 > d$. Next if $m < k - 2$, then let $i = (k - 1) - m$ and let
\( T_1 = T - \{ v_{\ell-1}v_{\ell-1}v_{\ell-i+1}, v_{\ell-i+1}v_{\ell-i+2}, \ldots, v_{\ell-i+1}v_{\ell}, v_{\ell}u_1, u_1u_2, u_2u_3, \ldots, u_{m-1}w \} \\
+ \{ v_{\ell}u_1, u_1u_2, u_2u_3, \ldots, u_{m-1}w, wv_{\ell-1}, v_{\ell-1}v_{\ell-2}, v_{\ell-2}v_{\ell-3}, \ldots, v_{\ell-i+1}v_{\ell-i} \}. \\
\)

So \( T_1 \) is \( P_k \)-adjacent to \( T \), where \( \text{diam } T_1 = d + m \geq d \). Thus we may continue in this manner, replacing \( T \) with \( T_1 \) until we obtain a tree that is not a path and \( m \) and \( \ell \) cannot be chosen so that either \( m + \ell \) or \( m + d - \ell \) is at least \( k-1 \).

**Case 2** Suppose that \( m + \ell < k-1 \) and \( m + d - \ell < k-1 \). Let

\( T_1 = T - \{ v_0v_1, v_1v_2, \ldots, v_{k-1}v_k \} + \{ wv_{\ell-1}, v_{\ell-1}v_{\ell-2}, v_{\ell-2}v_{\ell-3}, \ldots, v_1v_0, \\
v_0v_{\ell+1}, v_{\ell+1}v_{\ell+2}, v_{\ell+2}v_{\ell+3}, \ldots, v_{k-1}v_k \}. \\
\)

Then \( T_1 \) is \( P_k \)-connected to \( T \), and \( d_{T_1}(v_{\ell}, v_d) = m + d \) so that \( \text{diam } T_1 > d \). Thus we may continue in this manner until \( T_1 \) is a path. □

Hence we have the following.

**Corollary 2.7** Let \( T_1 \) and \( T_2 \) be trees of order \( p \) and let \( d = \min\{ \text{diam } T_1, \text{diam } T_2 \} \). Then \( T_1 \) is \( P_k \)-connected to \( T_2 \) for each positive integer \( k \) with \( 3 \leq k \leq d \).

We have seen that if \( H = P_3 \) or \( H = P_4 \), then every pair of graphs of the same order and same size containing \( H \) as a subgraph are \( H \)-connected. It turns out, however, that if \( H = K_3 \), then not every pair of graphs of the same order and same size containing \( H \) as a subgraph are \( H \)-connected. First, observe that if \( G_1 \) and \( G_2 \) are \( K_3 \)-adjacent, then there exist subgraphs \( H_1 \) and \( H_2 \) of \( G_1 \) and \( G_2 \), respectively, where \( H_1 \equiv H_2 \equiv K_3 \), \( G_2 = G_1 - E(H_1) + E(H_2) \), and \( E(H_2) - E(H_1) \subseteq E(\overline{G_1}) \).

Since any two edges of a triangle uniquely determine the third edge, \( H_1 \) and \( H_2 \) have exactly one common edge. Let \( V(H_1) = \{ u, v, x \} \) and \( V(H_2) = \{ u, v, w \} \), where \( u, v, w, \) and \( x \) are distinct vertices of \( G_1 \). Then \( \deg_{G_1} x = \deg_{G_2} x + 2 \) and \( \deg_{G_1} w = \).
$\text{deg}_{G_2} v - 2$ while $\text{deg}_{G_1} v = \text{deg}_{G_2} v$ for every other vertex $v$ of $G_1$. Thus a necessary condition for two graphs to be $K_3$-adjacent, and hence $K_3$-connected, is that they must have the same number of odd vertices. It is not known, however, if this condition is also sufficient.

Consider the graphs $G_1$, $G_2$, $G_3$, and $G_4$ shown in Figure 2.7. Since $G_4$ has four odd vertices while each of $G_1$, $G_2$, and $G_3$ has two odd vertices, it follows that $G_4$ is not $K_3$-connected to any of $G_1$, $G_2$, and $G_3$. Next $G_2 = G_1 - \{v_1v_2, v_2v_3, v_3v_1\} + \{v_2v_5, v_5v_3, v_3v_5\}$ so that $G_1$ is $K_3$-adjacent to $G_2$. Finally, $G_3 = G_2 - \{v_5v_2, v_2v_3, v_3v_5\} + \{v_5v_2, v_2v_4, v_4v_5\}$ so that $G_2$ is $K_3$-adjacent to $G_3$. Hence $G_1$ is $K_3$-connected to $G_3$ as well. Since $G_1$, $G_2$, $G_3$, and $G_4$ are the only $(5, 5)$-graphs containing a triangle, there are two equivalence classes for $(5, 5)$-graphs with a triangle under the relation $K_3$-connectedness, namely $\{G_1, G_2, G_3\}$ and $\{G_4\}$.

So for two graphs $G_1$ and $G_2$ of the same order and same size, a necessary condition for $G_1$ to be $K_3$-connected to $G_2$ is that $G_1$ must have the same number of odd vertices as $G_2$. However, this condition is not sufficient. Let $G_1 = 3K_3$ and let $G_2 = K_3 \cup C_6$. Then $G_1$ and $G_2$ have the same order, same size, and same number of odd vertices, yet clearly $G_1$ is not $K_3$-connected to $G_2$. Although this
condition is not always sufficient, there is a large class of graphs for which it is necessary.

**Theorem 2.8** Let $H$ be an $r$-regular graph where $r$ is an even positive integer and let $G_1$ and $G_2$ be two graphs. If $G_1$ is $H$-connected to $G_2$, then $G_1$ and $G_2$ have the same number of odd vertices.

**Proof** Suppose first that $G_1$ and $G_2$ are $H$-adjacent. Then there exist subgraphs $H_1$ and $H_2$ of $G_1$ and $G_2$, respectively, such that $H_1 \cong H_2 \cong H$ and $G_2 = G_1 - E(H_1) + E(H_2)$. Clearly if $v$ is not a vertex of $H_1$ or of $H_2$, then $\deg_{G_1} v = \deg_{G_2} v$. Next if $v$ is a vertex of $H_1$ but not a vertex of $H_2$, then $\deg_{G_1} v = \deg_{G_2} v + r$, while if $v$ is a vertex of $H_2$ but not a vertex of $H_1$, then $\deg_{G_1} v = \deg_{G_2} v - r$. Since $r$ is even, $\deg_{G_1} v$ and $\deg_{G_2} v$ are of the same parity. Finally, if $v$ is a vertex of both $H_1$ and $H_2$, then $\deg_{G_1} v = \deg_{G_2} v$. Hence, if $G_1$ and $G_2$ are $H$-adjacent, then $G_1$ and $G_2$ have the same number of odd vertices. Consequently, if $G_1$ and $G_2$ are $H$-connected, then $G_1$ and $G_2$ have the same number of odd vertices. □

We now consider $H$-adjacency when $H = K_{1,3}$. Consider the graphs $G_1$ and $G_2$ of Figure 2.8. Now

$$K_{1,4} \equiv G_1 - \{v_1v_0, v_1v_2, v_1v_4\} + \{v_2v_0, v_2v_1, v_2v_4\}$$

so that $G_1$ is $H$-adjacent to $K_{1,4}$. Also,

$$K_{1,5} \equiv G_2 - \{v_1v_0, v_1v_2, v_1v_5\} + \{v_2v_0, v_2v_1, v_2v_5\}$$

so that $G_2$ is $H$-adjacent to $K_{1,5}$. In fact, we show that every tree of order $p$ is $K_{1,3}$-connected to the star $K_{1,p-1}$.
Theorem 2.9 If $T$ is a tree of order $p$ that is not a path, then $T$ is $K_{1,3}$-connected to $K_{1,p-1}$.

Proof Let $H = K_{1,3}$, and let $P: v_0, v_1, \ldots, v_d$ be a longest path through a vertex of maximum degree. Hence $\text{diam } T \geq d$. If $d = 2$, then $T = K_{1,p-1}$. Thus we assume that $d \geq 3$. We consider two cases.

Case 1 Suppose that $d = 3$. Assume, without loss of generality, that $\deg v_1 = \Delta(T)$. We show, in fact, that the $\text{diam } T = 3$ as well. Each vertex of $T$ adjacent to $v_2$, other than $v_1$, must be an end-vertex; for otherwise, there exists a path of length at least 4 through $v_1$. Similarly, each vertex of $T$ adjacent to $v_1$, other than $v_2$, must be an end-vertex. Thus $\text{diam } T = 3$. If $\deg v_1 = 3$, then $\deg v_2 \leq 3$, and thus $T = G_1$ or $T = G_2$, where $G_1$ and $G_2$ are shown in Figure 2.8. As we have previously seen, both $G_1$ and $G_2$ are $H$-adjacent to $K_{1,4}$ and $K_{1,5}$, respectively. Therefore, we may assume $\deg v_1 > 3$. Thus there exist end-vertices $w_1$ and $w_2$, different from $v_0$, adjacent to $v_1$. Then $F_1 = T - \{v_1w_1, v_1w_2, v_1v_0\} + \{w_1v_1, w_1v_2, w_1v_3\}$ is $H$-adjacent to $T$. Next, $F_2 = F_1 - \{v_2v_1, v_2w_1, v_2v_3\} + \{w_1v_2, w_1v_0, w_1w_2\}$ is $H$-adjacent to $F_1$. Finally, $F_3 = F_2 - \{w_1v_0, w_1v_1, w_1v_2\} + \{v_1w_1, v_1v_0, v_1v_2\}$ is $H$-adjacent to $F_2$, and $T' = F_3 - \{w_1v_1, w_1w_2, w_1v_3\} + \{v_1w_1, v_1v_2, v_1v_3\}$ is $H$-adjacent to $F_3$. This situation is shown in Figure 2.9, where the graphs $H_0, H_1,$
$H_2$, $H_3$, and $H_4$ are subgraphs of $T, F_1, F_2, F_3$, and $T'$, respectively. Now, $T, F_1, F_2, F_3, T'$ is an $H$-walk; so $T$ is $H$-connected to $T'$. Furthermore $\deg_T v_1 < \deg_{T'} v_1$. Thus, we continue in this manner until $\deg v_1 = p - 1$. Hence $T$ is $H$-connected to $K_{1,p-1}$.

Figure 2.9 A $K_{1,3}$-Walk From $T$ to $T'$ When $d = 3$.

Case 2 Suppose that $d > 3$. Now there exists an integer $k (1 \leq k \leq d - 1)$ such that $\deg v_k = \Delta(T)$. Let $w$ be a vertex of $T$ different from $v_{k-1}$ and $v_{k+1}$ that is adjacent to $v_k$. We consider two subcases.

Subcase 2.1 Suppose that we can label the vertices of $P$ so that $k \geq 3$, where $\deg v_k = \Delta(T)$. Then $F_1 = T - \{v_k w, v_k v_{k-1}, v_k v_{k+1}\} + \{wv_k, vw_0, wv_1\}$ is $H$-adjacent to $T$. Next, $F_2 = F_1 - \{v_1 v_0, v_1 v_2, v_1 w\} + \{wv_1, wv_{k-1}, wv_{k+1}\}$ is $H$-adjacent to $F_1$. Finally, $F_3 = F_2 - \{wv_k, vw_0, vw_1\} + \{v_kw, v_kv_0, v_kv_1\}$ is $H$-adjacent to $F_2$, and $T' = F_3 - \{wv_{k-1}, wv_{k+1}\} + \{v_kw, v_kv_{k-1}, v_kv_{k+1}\}$ is $H$-adjacent to $F_3$. This situation is illustrated in Figure 2.10, where the graphs $H_0, H_1, H_2, H_3$, and $H_4$ are subgraphs of $T, F_1, F_2, F_3$, and $T'$, respectively. Thus $T, F_1, F_2, F_3, T'$ is a $H$-
walk (that contains a $H$-path); so $T'$ is $H$-connected to $T$. Furthermore, $\Delta(T') > \Delta(T)$. If $T'$ has diameter 3, then by Case 1, $T'$ is $H$-connected to $K_{1,p-1}$. Otherwise, we proceed with $T'$ as above (that is, let $T = T'$) until we obtain a tree with diameter 3.

**Subcase 2.2** Suppose that we cannot label the vertices of $P$ so that $k \geq 3$. Thus $k = 2$ and $d = 4$. Let $x_1, x_2, \ldots, x_m$ denote the vertices of $T$ not belonging to $P$ that are adjacent to $v_2$. Suppose first that $\deg x_i \geq 2$ for some $i \ (1 \leq i \leq m)$. Let $y_{i,1}$, $y_{i,2}$, $\ldots$, $y_{i,n_i}$ denote the $n_i$ vertices of $T$ that are different from $v_2$ and adjacent to $x_i$. We now show that $\deg y_{i,j} = 1$ for each $i, j$ with $1 \leq i \leq m$ and $1 \leq j \leq n_i$.

Suppose, to the contrary, that $\deg y_{i,j} > 1$ for some $i$ and $j$ with $1 \leq i \leq m$ and $1 \leq j \leq n_i$, say $z \neq x_i$ is adjacent to $y_{i,j}$. Then the path $z, y_{i,j}, x_1, x_2, v_1, v_0$ is a path in $T$ that can be re-labeled so that $k \geq 3$, producing a contradiction. Thus it follows that each $y_{i,j}$ ($1 \leq j \leq n_i$) is an end-vertex. Let $T = F_0$, and for $k = 1, 2, \ldots, \lfloor n_i / 2 \rfloor$, let $F_i = F_{i-1} - \{x_i v_2, x_i y_{i,2k-1}, x_i y_{i,2k}\} + \{v_2 x_i, v_2 y_{i,2k-1}, v_2 y_{i,2k}\}$; so $F_{i-1}$ and $F_i$ are $H$-adjacent. If $n_i = 2a + 1$, then let $F_{a+1} = F_a - \{v_2 y_1, v_2 x_i, v_2 v_3\} + \{x_i y_1, x_i v_2, x_i v_3\}$; next let $F_{a+2} = F_{a+1} - \{x_i y_{i,n_i}, x_i v_1, x_i v_2\} + \{v_2 y_{i,n_i} v_2 v_1, v_2 x_i\}$ and finally let $F_{a+3} = F_{a+2} - \{v_2 v_1, v_2 y_{i,n_i} v_2 x_i\} + \{v_2 v_1, v_2 y_{i,n_i} v_2 y_{i,n_i}\}$. So let $T' = F_{a+3}$ in the case that $n_i$ is odd or let $T' = F_{\lfloor n_i / 2 \rfloor}$. Now $T = F_0, F_1, \ldots, T'$ is an $H$-walk from $T$ to $T'$; so $T$ is $H$-connected to $T'$. Next exactly one of the graphs $H_1$ and $H_2$ shown in Figure 2.11 is a subgraph of $T'$. In fact, $H_1$ is a subgraph of $T'$ if $n_i$ is even, while $H_2$ is a subgraph of $T'$ if $n_i$ is odd.
Figure 2.10 An $H$-Walk From $T$ to $T'$ When $d > 4$ and $k \geq 3$.

Figure 2.11 A Subgraph of $T'$. 

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So in $T'$, the vertices $y_{i,1}, y_{i,2}, \ldots, y_{i,n_i}$ are now adjacent to $v_2$, and possibly $x_i$ is now adjacent to $v_3$. We continue in this manner until $\deg x_i = 1$ for each $i = 1, 2, \ldots, m$. Let $T''$ denote the resulting tree. So, in $T''$, any vertex not belonging to $P$ is an end-vertex. Relabel the vertices adjacent to $v_2$ in $T''$ and not belonging to $P$ by $z_1, z_2, \ldots, z_b$. Let $T'' = F_0$, and for $i = 1, 2, \ldots, \lfloor b / 2 \rfloor$, let $F_i = F_{i-1} - \{v_2v_3, v_2z_{2i-1}, v_2z_{2i}\} + \{v_3v_2, v_3z_{2i-1}, v_3z_{2i}\}$; so $F_{i-1}$ and $F_i$ are $H$-adjacent. If $b = 2c$, then let $T''' = F_c - \{v_3w, v_3v_2, v_3v_4\} + \{v_3w, v_3v_1, v_3v_4\}$, where $w$ is any vertex adjacent to $v_3$ not belonging to $P$; otherwise $b = 2c + 1$, so let $T''' = F_c - \{v_2z_b, v_2v_3, v_2v_1\} + \{v_3v_1, v_3v_2, v_3z_b\}$. In $T'''$, the vertex $v_2$ is an end-vertex as is every other vertex of $T'''$ not belonging to $P$; so $\text{diam}(T''') = 3$. Thus by Case 1, $T'''$ is $H$-connected to $K_{1,p-1}$; so $T$ is $H$-connected to $K_{1,p-1}$. \hfill \Box

As a consequence of Theorem 2.9, we have the following.

**Corollary 2.10** Let $T_1$ and $T_2$ be trees of order $p$. Then $T_1$ is $K_{1,3}$-connected to $T_2$ if and only if $\Delta(T_1) \geq 3$ and $\Delta(T_2) \geq 3$.

**Proof** Let $H = K_{1,3}$. Suppose first that $T_1$ is $H$-connected to $T_2$. Then there exists a $T_1-T_2$ $H$-path $T_1 = F_0, F_1, \ldots, F_k = T_2$. Since $T_1$ is $H$-adjacent to $F_1$, it follows that $T_1$ has a vertex of degree at least 3 (since $H$ does) and hence $\Delta(T_1) \geq 3$. Similarly, since $F_{k-1}$ is $H$-adjacent to $T_2$, it follows that $\Delta(T_1) \geq 3$.

For the converse, suppose that $\Delta(T_1) \geq 3$ and $\Delta(T_2) \geq 3$. By Theorem 2.9, the tree $T_1$ is $H$-connected to $K_{1,p-1}$ and $T_2$ is $H$-connected to $K_{1,p-1}$. Hence $T_1$ and $T_2$ are $H$-connected. \hfill \Box

We now turn from trees to hamiltonian graphs.
Theorem 2.11  If $G_1$ and $G_2$ are two nonisomorphic hamiltonian graphs of the same order and same size, then $G_1$ is $K_{1,3}$-connected to $G_2$.

Proof  Without loss of generality, let $C: v_1, v_2, \ldots, v_p, v_1$ be a hamiltonian cycle in both $G_1$ and $G_2$. If $p = 4$, then $G_1 = G_2$, and the result follows. Thus we may assume that $p \geq 5$. Since $G_1$ and $G_2$ are not isomorphic, there exist chords $v_i v_j$ and $v_k v_\ell$ such that $v_i v_j \in E(G_1) - E(G_2)$ and $v_k v_\ell \in E(G_2) - E(G_1)$. Without loss of generality, we may assume $i < j$, $k < \ell$, and $i < k$. Assume first that $G_2 = G_1 - v_i v_j + v_k v_\ell$.

Next, suppose that $v_i v_j$ and $v_k v_\ell$ are adjacent, that is, $j = k$. Since $i < j < \ell$ and $v_i v_j$ and $v_k v_\ell$ are chords, it follows that $i \neq j - 1$ and $\ell \neq j + 1$. Therefore, $G_2 = G_1 - \{v_i v_j, v_j v_{j-1}, v_j v_{j+1}\} + \{v_k v_\ell, v_j v_{j-1}, v_j v_{j+1}\};$ so $G_1$ and $G_2$ are $K_{1,3}$-adjacent. Thus we may assume that $i, j, k,$ and $\ell$ are distinct. If any one of the edges $v_i v_k, v_j v_k, v_i v_\ell$, and $v_j v_\ell$ is not present in $G_1$, say $v_i v_k$, then let $F_1 = G_1 - \{v_i v_k, v_i v_j, v_j v_\ell, v_\ell v_{\ell - 1}\}$; so $F_1$ and $G_1$ are $K_{1,3}$-adjacent. Hence $G_2 = F_1 - \{v_i v_{i+1}, v_i v_{i+1}, v_i v_k\};$ so $F_2$ and $F_1$ are $K_{1,3}$-adjacent. Thus $G_1$ and $G_2$ are $K_{1,3}$-connected. Therefore, $v_i v_k, v_j v_k, v_i v_\ell, v_j v_\ell \in E(G_1).$ Suppose first that $j \neq \ell - 1$ and that $i \neq \ell + 1 \pmod p$. Let $F_1 = G_1 - \{v_i v_{i+1}, v_i v_j, v_j v_{\ell - 1}\} + \{v_\ell v_{i+1}, v_\ell v_{\ell - 1}, v_\ell v_k\};$ so $F_1$ and $G_1$ are $K_{1,3}$-adjacent. Then either $v_k \neq v_{j-1}$ or $v_k \neq v_{j+1}$, say $v_k \neq v$, where $v \in N(v_j)$. Hence $G_2 = F_1 - \{v_j v_k, v_j v_\ell, v_\ell v_{\ell - 1}\}$; so $F_1$ and $G_2$ are $K_{1,3}$-adjacent, and $G_1$ and $G_2$ are $K_{1,3}$-connected. Thus we assume that $j = \ell - 1$ and that $i \equiv \ell + 1 \pmod p$. Hence $i = 1, \ell = p, j = p - 1, v_1 v_j = v_1 v_{p-1}$, and $v_k v_\ell = v_k v_p$. Since $p \geq 5$, there exists a vertex $v \in N(v_k)$ such that $v$ lies between $v_1$ and $v_k$ on $C$, or $v$ lies between $v_k$ and $v_{p-1}$ on $C$. Let $F_1 = G_1 - \{v_1 v_j, v_{k+1}, v_k v_p\};$ so $F_1$ and $G_1$ are $K_{1,3}$-adjacent. Then let $G_2 = F_1 -$.
\{v_{p-1}v_p, v_{p-1}'v_1, v_{p-1}v_{p-2}\} + \{v_{p-1}v_p, v_{p-1}'v_k, v_{p-1}v_{p-2}\}; so \ F_1 \ is \ K_{1,3}\text{-adjacent to} \ G_2. \ Hence \ G_1 \ and \ G_2 \ are \ K_{1,3}\text{-connected.}

So if \ G_1 \ and \ G_2 \ are two nonisomorphic hamiltonian graphs of the same order and same size such that \ G_2 = G_1 - e + f \ for two edges \ e \ and \ f, \ then \ G_1 \ and \ G_2 \ are \ K_{1,3}\text{-connected.} \ Thus if \ G_1 \ and \ G_2 \ are two nonisomorphic hamiltonian graphs of the same order (at least 5) and the same size, \ G_1 \ and \ G_2 \ are \ K_{1,3}\text{-connected.} \ \square

Thus every two trees of the same order and with maximum degree at least 3 are \ K_{1,3}\text{-connected} as are every two hamiltonian graphs of the same order, same size, and with maximum degree at least 3. For graphs that are not trees and not hamiltonian, it remains to be determined which pairs of these graphs are \ K_{1,3}\text{-connected.}
CHAPTER III

H-DISTANCE AND H-DISTANCE GRAPHS

3.1 Introduction

For graphs \( G_1 \) and \( G_2 \) of the same order and same size, \( G_2 \) is said to be obtained from \( G_1 \) by an edge rotation if \( G_1 \) contains distinct vertices \( u, v, \) and \( w \) such that \( uv \in E(G_1), uw \notin E(G_1), \) and \( G_2 = G_1 - uv + uw. \) Recall that it was shown in [9] that every pair of graphs of the same order and same size can be transformed into one another by a sequence of edge rotations. The edge rotation distance or, more simply, the \( r \)-distance \( d_r(G_1, G_2) \) between \( G_1 \) and \( G_2 \) is the smallest nonnegative integer \( n \) for which there exists a sequence \( G_1 = F_0, F_1, \ldots, F_n = G_2 \) of graphs such that \( F_i \) can be obtained from \( F_{i-1} \) by an edge rotation for \( i = 1, 2, \ldots, n. \) Some properties of edge rotation distance were established in [9], where it was shown that \( d_r(G_1, G_2) = d_r(\overline{G_1}, \overline{G_2}) \), and that for every nonnegative integer \( n, \) there exist graphs \( G_1 \) and \( G_2 \) such that \( d_r(G_1, G_2) = n. \) For nonempty graphs \( G_1 \) and \( G_2, \) recall that a greatest common subgraph of \( G_1 \) and \( G_2 \) is any graph \( G \) of maximum size without isolated vertices that is a subgraph of \( G_1 \) and \( G_2. \) It was also shown in [9] that for two graphs \( G_1 \) and \( G_2 \) of order \( p \) and size \( q, \) where \( s \) is the size of a greatest common subgraph, \( d_r(G_1, G_2) \) is bounded above by \( 2(q - s). \)

Another metric on a space of graphs is given by the edge slide, which is considered in [17]. A graph \( G_2 \) can be obtained from \( G_1 \) by an edge slide if \( G_1 \) contains distinct vertices \( u, v, \) and \( w \) such that \( uv, vw \in E(G_1), uw \notin E(G_1), \) and \( G_2 = G_1 - uv + uw. \) In [17] it was shown that the edge slide preserves connectedness, and that a graph \( G_1 \) can be obtained from a graph \( G_2 \) by a sequence
of edge slides if and only if $G_1$ and $G_2$ have the same number of components and corresponding components of $G_1$ and $G_2$ have the same order and same size. The edge slide distance or the $s$-distance $d_s(G_1, G_2)$ between two graphs $G_1$ and $G_2$ having the same number of components, where corresponding components have the same order and same size, is defined as the smallest nonnegative integer $n$ for which there exists a sequence $G_1 = F_0, F_1, \ldots, F_n = G_2$ of graphs such that, for $i = 1, 2, \ldots, n$, the graph $F_i$ can be obtained from $F_{i-1}$ by an edge slide. Since an edge slide is an edge rotation, it follows that $d_r(G_1, G_2) \leq d_s(G_1, G_2)$ for every pair $G_1, G_2$ of graphs for which $d_s(G_1, G_2)$ is defined. In [15] it was shown that for every pair $m, n$ of positive integers with $m \leq n$, there exist graphs $G_1$ and $G_2$ such that $d_r(G_1, G_2) = m$ while $d_s(G_1, G_2) = n$. We now consider the analogous concepts for $H$-adjacency.

Let $G_1$ and $G_2$ be two graphs of the same order and same size such that $V(G_1) = V(G_2)$, and let $H$ be a connected graph of order at least 3. In Chapter II, two subgraphs $H_1$ and $H_2$ of $G_1$ and $G_2$, respectively, are defined to be $H$-adjacent if $H_1 \equiv H_2 \equiv H$, and $H_1$ and $H_2$ share some but not all edges, that is, $E(H_1) \cap E(H_2) \neq \emptyset$ and $E(H_2) - E(H_1) \neq \emptyset$ (so also $E(H_1) - E(H_2) \neq \emptyset$). Also in Chapter II, the graphs $G_1$ and $G_2$ are themselves defined to be $H$-adjacent if $G_1$ and $G_2$ contain $H$-adjacent subgraphs $H_1$ and $H_2$, respectively, such that $E(H_2) - E(H_1) \subseteq E(G_1)$ and $G_2 = G_1 - E(H_1) + E(H_2)$. A $G_1$-$G_2$ $H$-walk is defined in Chapter II as a sequence $G_1 = F_0, F_1, \ldots, F_k = G_2$ of graphs of the same order and same size such that $F_i$ is $H$-adjacent to $F_{i+1}$ for $i = 0, 1, \ldots, k - 1$. Thus, a $G_1$-$G_2$ $H$-walk $G_1 = F_0, F_1, \ldots, F_k = G_2$ in which the graphs $F_0, F_1, \ldots, F_k$ are distinct is a $G_1$-$G_2$ $H$-path, and the length of the $G_1$-$G_2$ $H$-path is the integer $k$. We define the $H$-distance $d_H(G_1, G_2)$ from $G_1$ to $G_2$ as the length of a shortest $G_1$-$G_2$ $H$-path.
$H$-path. Hence, $H$-distance is a metric on the space of all graphs of a fixed order and a fixed size for which this distance is defined. In fact, it is not clear for a given graph $H$, when this distance is defined. If $H = P_3$ or $H = P_4$, then by Theorems 2.2 and 2.5, this distance is defined for every pair of graphs containing $H$ as a subgraph.

We now show that for every connected graph $H$ of order at least 3 and for every positive integer $n$, there exist graphs whose $H$-distance is $n$.

**Theorem 3.1** For a connected graph $H$ (of order at least 3) and a positive integer $n$, there exist graphs $F_n$ and $G_n$ such that $d_H(F_n, G_n) = n$.

**Proof** Suppose first that $n = 1$. Let $F_1$ be obtained from the graph $G$ shown in Figure 3.1 by identifying a vertex $x$ of $H$ with minimum degree with the vertex $v_0$ of $G$, and let $G_1$ be obtained from $G$ by identifying the vertex $x$ with $v_1$. Clearly, $G_1 \not\equiv F_1$; so $d_H(F_1, G_1) \geq 1$. Also, since $F_1$ is $H$-adjacent to $G_1$, we have that $d_H(F_1, G_1) = 1$. When $H = C_4$, the graphs $F_1$ and $G_1$ are shown in Figure 3.1.

![Figure 3.1 The Graphs $G$, $F_1$, and $G_1$ When $H = C_4$.](image-url)
Next let $n > 1$ be a positive integer. Let $F_n = nF_1$, and let $G_n = nG_1$. Now each component of $F_n$ is $H$-adjacent to each component of $G_n$, so that we may transform $F_n$ into $G_n$ one component at a time. Thus, $d_H(F_n, G_n) \leq n$. We show that $d_H(F_n, G_n) = n$. Since no component of $F_n$ is isomorphic to a component of $G_n$ and the components of $F_n$ and $G_n$ all have the same order and same size, it follows that each component of $F_n$ must have some of its edges moved. Since $H$ is connected, it follows that $H$-adjacency can only move edges from one component at a time. Thus, $d_H(F_n, G_n) \geq n$ and therefore $d_H(F_n, G_n) = n$. □

In the proof of the previous theorem, the graphs $F_n$ and $G_n$ are disconnected. We now show that for a given connected graph $H$ and a positive integer $n$, there exist connected graphs for which the $H$-distance between them is $n$.

**Theorem 3.2** For a connected graph $H$ of order at least 3 and a positive integer $n$, there exist connected graphs $F_n$ and $G_n$ such that $d_H(F_n, G_n) = n$.

**Proof** For $n = 1$, the proof of Theorem 3.1 gives connected graphs $F_1$ and $G_1$ for which $d_H(F_1, G_1) = 1$. Thus we may assume that $n \geq 2$. Let $x$ denote a vertex of $H$ of minimum degree. Next let $P: v_0, v_1, v_2, \ldots, v_n$ be a path of length $n$, and let $H_1, H_2, \ldots, H_{n+1}$ be copies of $H$, where the vertex $x_i (1 \leq i \leq n + 1)$ of $H_i$ corresponds to the vertex $x$ of $H$. Finally, let $F_n$ be obtained from $P$ by identifying the vertex $x_i (1 \leq i \leq n + 1)$ of $H_i$ with the vertex $v_0$ of $P$, and let $G_n$ be obtained from $P$ by identifying the vertex $x_i (1 \leq i \leq n + 1)$ of $H_i$ with the vertex $v_{i-1}$ of $P$. When $H$ is the graph $G$ of Figure 3.1, the graphs $F_3$ and $G_3$ are shown in Figure 3.2.

For $i = 1, 2, \ldots, n$, let $y_{i,1}, y_{i,2}, \ldots, y_{i,k}$ denote the $k$ vertices of $H_i$ adjacent to $x_i$. Let $F_n = J_0$, and for $i = 1, 2, \ldots, n$, define
Since \( H_i = \{y_{i,1}v_0, y_{i,2}v_0, \ldots, y_{i,k}v_0\} \) and \( H_i = \{y_{i,1}v_0, y_{i,2}v_0, \ldots, y_{i,k}v_0\} \neq \emptyset \), for each \( i = 1, 2, \ldots, n \), it follows that \( J_{i-1} \) is \( H \)-adjacent to \( J_i \). Thus, \( F_n = J_0, J_1, \ldots, J_n = G_n \) is an \( F_n - G_n \) \( H \)-path. Therefore, \( J_1 \) is \( H \)-adjacent to \( J_1 \). Thus, \( F_n = J_0, J_1, \ldots, J_n = G_n \) is an \( F_n - G_n \) \( H \)-path. Therefore, \( d_H(F_n, G_n) \leq n \). We show that \( d_H(F_n, G_n) \geq n \). Observe that in \( F_n \), there are at least \( n + 1 \) copies of \( H \) having exactly one vertex in common, namely \( v_0 \), while the vertex set of \( G_n \) can be partitioned into \( n + 1 \) sets, where each set induces a copy of \( H \). Therefore, \( v_0 \) can only belong to one copy of \( H \) in \( G_n \), and hence \( d_H(F_n, G_n) \geq n \). Thus \( d_H(F_n, G_n) = n \). □

![Figure 3.2 The Graphs \( F_3 \) and \( G_3 \) When \( H \) Is the Graph \( G \) of Figure 3.1.](image)

### 3.2 \( H \)-Distance Graphs

Let \( S \) denote a set of graphs of a fixed order and fixed size. The edge rotation distance graph \( D_r(S) \) of \( S \) is defined as that graph with vertex set \( S \) such that the vertices \( G_1 \) and \( G_2 \) are adjacent in \( D_r(S) \) if and only if the graph \( G_2 \) can be obtained from the graph \( G_1 \) by an edge rotation. Many classes of graphs have been shown to be edge rotation distance graphs in \([3, 13, 15]\), including complete graphs, trees, cycles, complete bipartite graphs and all line graphs, and no graph has been found that is not an edge rotation distance graph. In fact, it is conjectured in \([3]\) that every graph is an edge rotation distance graph.
For a set $S$ of graphs of the same order (at least 5) and same size, the jump distance graph $D_j(S)$ of $S$ is that graph whose vertices are the graphs of $S$, and where vertices $G_1$ and $G_2$ in $S$ are adjacent in $D_j(S)$ if and only if the graph $G_2$ can be obtained from the graph $G_1$ by a single edge-jump. In [4] it was shown that complete graphs, complete multipartite graphs, trees, cycles, and the complements of line graphs are jump distance graphs, and it is conjectured that every graph is jump distance graph. We now turn our attention to the analogous concept for $H$-adjacency.

Let $S$ denote a set of graphs of the same order and same size. For a connected graph $H$, the $H$-distance graph $D_H(S)$ of $S$ is that graph whose vertices are the graphs of $S$, and where two vertices $G_j$ and $G_2$ are adjacent in $D_H(S)$ if and only if the graphs $G_j$ and $G_2$ are $H$-adjacent. A graph $G$ is an $H$-distance graph if there exists a set $S$ of graphs of the same order and same size such that $D_H(S) = G$. We begin by showing that paths are $P_3$-distance graphs.

**Theorem 3.3** For every integer $n \geq 2$, the path $P_n$ is a $P_3$-distance graph.

**Proof** Let $H = P_3$, and let $P: v_1, v_2, \ldots, v_{2n}$ denote a path of length $2n - 1$. Let $v_0$ be a new vertex, and for $i = 1, 2, \ldots, n$, let $G_i = P + v_0v_i$. (See Figure 3.3 for the case $n = 4$.) Now the graphs $G_1, G_2, \ldots, G_n$ are pairwise nonisomorphic, and for $i = 2, \ldots, n$, the graph $G_i = G_{i-1} - \{v_{i-1}v_i, v_i\} + \{v_i, v_0\}$ so that $G_i$ and $G_{i-1}$ are $H$-adjacent. Clearly for $i = 2, \ldots, n - 1, G_i$ is $H$-adjacent only to $G_{i-1}$ and $G_{i+1}$. Thus $D_H(G_1, G_2, \ldots, G_n) = P_n$. □

Next, we show that all cycles are $P_3$-distance graphs. First, the following observation will be useful. If $G_1$ and $G_2$ are two $P_3$-adjacent graphs, then we have seen that $G_2 = G_1 - e + f$ for edges $e$ and $f$ of $G_1$ and $G_2$, respectively. Further,
in $G_1 + f$, the edges $e$ and $f$ are either adjacent or are adjacent to a common edge of $G_1$ (and $G_2$). Thus $d(e,f) = 0$ or $d(e,f) = 1$.

Figure 3.3 A Set of Graphs Such That $D_{P_3}(\{G_1, G_2, G_3, G_4\}) = P_4$.

**Theorem 3.4** For every integer $n \geq 3$, the cycle $C_n$ is a $P_3$-distance graph.

**Proof** Let $H = P_3$. Next, let $n \geq 3$ be an integer, and let $P : v_1, v_2, \ldots, v_{2n}$ be a path of length $2n - 1$ and $T : w_1, w_2, w_3, w_1$ be a cycle of length 3. Obtain the graph $G$ from $P$ and $T$ by identifying the vertex $v_1$ of $P$ with the vertex $w_1$ of $T$. For $i = 1, 2, \ldots, n$, let $G_i = G + v_1 v_{n+i}$. The case $n = 4$ is shown in Figure 3.4. Thus each graph $G_i$ has a 3-cycle and an $(n + 1)$-cycle and, furthermore, the distance between the 3-cycle and the $(n + 1)$-cycle is $i - 1$. Therefore, the graphs $G_1, G_2, \ldots, G_n$ are pairwise nonisomorphic.

Now for $i = 1, 2, \ldots, n - 1$, the graph $G_{i+1} = G_i - \{v_i v_{n+i}, v_{i+1} v_{n+i+1}\}$; so $G_{i+1}$ is $H$-adjacent to $G_i$. Also, $G_1 = G_n - \{v_n v_{2n}, v_{n+1} v_{n+1}\}$; so $G_1$ is $H$-adjacent to $G_n$. Let $S = \{G_1, G_2, \ldots, G_n\}$. Since $G_i (1 \leq i \leq n)$ is $H$-adjacent to $G_{i+1}$, where all indices are taken modulo $n$, it follows that $C_n$ is a subgraph of $D_H(S)$. 

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Figure 3.4 A Set of Graphs Such That $D_{P_3}((G_1, G_2, G_3, G_4)) = C_4$.

To see that $D_H(S) = C_n$, for fixed integers $i$ and $j$ with $1 \leq i < j \leq n$, $j - i \neq 1$, and $j - i \neq n - 1$, let $F$ be that graph obtained from $G$ by adding the edges $v_i v_{n+i}$ and $v_j v_{n+j}$. Then $d_F(v_i v_{n+i}, v_j v_{n+j}) \geq 2$, and therefore the graphs $G_i$ and $G_j$ are not $H$-adjacent. Hence $D_H(S) = C_n$. □

Complete graphs are also $P_3$-distance graphs, as the next theorem shows. For every integer $n \geq 3$, denote the wheel of order $n$ by $W_n$, where $W_n = C_n + K_1$.

**Theorem 3.5** For every postive integer $n$, the complete graph $K_n$ is a $P_3$-distance graph.

**Proof** Let $H = P_3$. For $n = 1$, let $S$ consist of only one graph, and then $D_H(S) = K_1$. Next for $n = 2$, the graph $K_2$ is a $P_3$-distance graph by Theorem 3.3. Thus we may assume $n \geq 3$. Let $v_1, v_2, \ldots, v_{2n-2}, v_1$ denote the $(2n - 2)$-cycle of $W_{2n-2}$ and the vertex $v$ will denote the unique vertex of $W_{2n-2}$ having degree $2n - 2$. Next let $G$ denote that graph obtained from $W_{2n-2}$ by joining a new vertex $v_0$ to $v_1$. For $i = 1, 2, \ldots, n$, let $G_i = G - v_1$. The case $n = 4$ is shown in Figure 3.5. Now the
graph $G_1$ has $2n - 2$ vertices of degree 3, while each of the graphs $G_2, G_3, \ldots, G_n$ has $2n - 3$ vertices of degree 3. Next the graph $G_i$ $(1 \leq i \leq n)$ has exactly one 4-cycle at distance $i - 1$ from the vertex $v_0$. Therefore, the graphs $G_1, G_2, \ldots, G_n$ are pairwise nonisomorphic. Since $G_j = G_i - \{vv_{2n-2}, vv_j\} + \{vv_{2n-2}, vv_i\}$ for $1 \leq i < j \leq n$, the graphs $G_1, G_2, \ldots, G_n$ are pairwise $H$-adjacent. Thus if $S = \{G_1, G_2, \ldots, G_n\}$, then $D_H(S) = K_n$. □

![Figure 3.5 A Set of Graphs Such That $D_{P_3}(\{G_1, G_2, G_3, G_4\}) = K_4$.](image)

We now show that every graph is a $P_3$-distance graph.

**Theorem 3.6** Every graph is a $P_3$-distance graph.

**Proof** Let $H = P_3$, and let $G$ be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$. Next let $F$ denote that graph obtained from $G$ by joining, exactly $2i$ $(1 \leq i \leq n)$ new vertices to $v_i$. Let $G_i$ $(1 \leq i \leq n)$ be that graph obtained from $F$ by joining a new vertex $x$ to $v_i$. Thus, the number of end-vertices of $G_i$ adjacent to $v_j$ is $2j$ if $j \neq i$ and $2i + 1$ if $j = i$. The graphs $G_1, G_2, \ldots, G_n$ are pairwise nonisomorphic, and if $v_iv_j$ is an edge of $G$, then $G_j = G_i - \{vx, v_iv_j\} + \{v_iv_j, v_jx\}$; so $G_i$ is $H$-adjacent to $G_j$. Let $i$ and $j$ be distinct integers with $1 \leq i, j \leq n$. Now if $G_i$ and $G_j$ are $H$-adjacent, then $G_j = G_i - e + f$ for edges $e$ and $f$ of $G_i$ and $G_j$, respectively. Also, in the
graph $G_i-e=G_j-f$, there must be an edge $e'$ adjacent to $e$ in $G_i$ and adjacent to $f$ in $G_j$. Thus, since $G_i-xv_i=G_j-xv_j$, the graphs $G_i$ and $G_j$ are $H$-adjacent if and only if the edge $v_iv_j$ is present in $G$. Therefore, $D_H(G_1, G_2, \ldots, G_n) = G$. □

The construction described in the proof of Theorem 3.6 is the construction given in [3] which shows that every graph is an edge slide distance graph. Recall that for two graphs $G_1$ and $G_2$ of the same order and same size, the graph $G_1$ can be transformed into $G_2$ by an edge slide if $G_1$ contains distinct vertices $u$, $v$, and $w$ such that $uv \in E(G_1)$, $uw \not\in E(G_1)$, $vw \in E(G_1)$, and $G_2 = G_1 - uv + uw$. Let $S$ be a set of graphs of the same order and same size. The edge slide distance graph $D_S(S)$ is defined as that graph with vertex set $S$ such that two vertices $G_1$ and $G_2$ of $D_S(S)$ are adjacent if and only if the graph $G_1$ can be transformed into the graph $G_2$ by an edge slide. Next we show that we can extend Theorem 3.6 to show that every graph is, in fact, a $P_n$-distance graph for $n \geq 3$.

**Theorem 3.7** For an integer $n \geq 3$, every graph is a $P_n$-distance graph.

**Proof** Let $H = P_n$, and let $G$ be a graph with $V(G) = \{v_1, v_2, \ldots, v_p\}$. Next let $F$ be that graph obtained from $G$ by identifying an end-vertex of $2i$ ($1 \leq i \leq p$) paths of length $n - 2$ with $v_i$. (The graph $F$ is shown in Figure 3.6 when $n = 4$ and $G = K_4$.) For $i = 1, 2, \ldots, p$, let $G_i$ be that graph obtained from $F$ by identifying the end-vertex $x_{n-1}$ of the path $x_1, x_2, \ldots, x_{n-1}$ (of length $n - 2$) with $v_i$. For each graph $G_i$ ($1 \leq i \leq p$), the number of paths of length $n - 1$ starting at a vertex $v_j$ of $G$ and containing no other vertices of $G$ is $2j$ if $j \neq i$ and $2i + 1$ if $j = i$. Clearly, the graphs $G_1, G_2, \ldots, G_p$ are pairwise nonisomorphic. Let $i$ and $j$ be distinct integers with $1 \leq i, j \leq p$. Then the graphs $G_i$ and $G_j$ differ in exactly $n - 1$ edges. Thus $G_i$ and $G_j$ are $H$-adjacent if and only if there exists an edge $e$ of $G_i - \{x_1, x_2,$
$x_2 x_3, \ldots, x_{n-2} v_i = G_j - \{x_1 x_2, x_2 x_3, \ldots, x_{n-2} v_j\}$ such that $e$ is adjacent to $x_{n-2} v_i$ in $G_i$ and adjacent to $x_{n-2} v_j$ in $G_j$. Therefore $G_i$ and $G_j$ are $H$-adjacent if and only if $v_i v_j \in E(G)$. Hence $D_H([G_1, G_2, \ldots, G_p]) = G$. □

Figure 3.6 The Graph $F$ When $G = K_4$ and $n = 4$.

In fact, we can extend Theorem 3.7 to show that every graph is an $H$-distance graph if $\delta(H) = 1$.

**Theorem 3.8** Let $H$ be a graph with $\delta(H) = 1$. Then every graph is an $H$-distance graph.

**Proof** Let $H$ be a graph with $\delta(H) = 1$ and size $q$. Then there exists an end-vertex $v$ of $H$. Let $u$ be the unique vertex of $H$ adjacent to $v$. Next let $G$ be a graph with $V(G) = \{v_1, v_2, \ldots, v_p\}$, and let $F$ be that graph obtained from $G$ by identifying for $i = 1, 2, \ldots, p$, the vertex $u$ in $2i$ copies of $H - v$ with the vertex $v_i$ in $G$. For $i = 1, 2, \ldots, p$, let $G_i$ be that graph obtained from $F$ by identifying the vertex $u$ in the graph $H' = H - v$ with the vertex $v_i$. Clearly, the graphs $G_1, G_2, \ldots, G_p$ are pairwise nonisomorphic. Let $i$ and $j$ be distinct integers with $1 \leq i, j \leq p$. Then the graphs $G_i$ and $G_j$ differ in exactly $q - 1$ edges or in a copy of $H - v$. We show that $G_i$ and $G_j$ are $H$-adjacent if and only if $v_i v_j \in E(G)$. 

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Let $F_i$ be a copy of $H - v$ in $G_i$ containing exactly one vertex of $G$, namely $v_i$, and let $F_j$ be a copy of $H - v$ in $G_j$ containing exactly one vertex of $G$, namely $v_j$. Furthermore, we may assume that $F_i$ and $F_j$ are identical, that is, $V(F_i) = V(F_j)$ and $E(F_i) = E(F_j)$. Suppose that $v_iv_j \in E(G)$. Since $F_i + v_i v_j \equiv F_j + v_j v_i \equiv H$ and $G_j = G_i - E(F_i) + E(F_j)$, it follows that $G_i$ is $H$-adjacent to $G_j$. Finally, suppose that $G_i$ is $H$-adjacent to $G_j$. Since $G_i$ and $G_j$ differ in exactly one copy of $H - v$ at $v_i$ and $v_j$ in $G_i$ and $G_j$, respectively, we may assume, without loss of generality, that this copy of $H - v$ is $F_i$ in $G_i$ and $F_j$ in $G_j$. Since $F_i$ and $F_j$ have size $q - 1$ and $G_i$ is $H$-adjacent to $G_j$, there must exist an edge $e$ of $G_i - E(F_i) = G_j - E(F_j)$ such that $F_i + e \equiv F_j + e \equiv H$. Furthermore, since $F_i \equiv H - v$ where $\deg v = 1$, it follows that $e$ is incident with exactly one vertex of $F_i$ and exactly one vertex of $F_j$. Next, since $\langle V(F_i) \rangle = F_i$ in $G_i$ and $\langle V(F_j) \rangle = F_j$ in $G_j$, it must be that $e$ is incident with $v_i$ in $F_i$ and $v_j$ in $F_j$. Therefore $e = v_i v_j$ and $v_i v_j \in E(G)$. □

Thus, by Theorem 3.8, we know that for a given tree $T$, every graph is a $T$-distance graph. We now turn our attention to $K_3$-distance graphs and begin by showing that complete graphs are $K_3$-distance graphs.

**Theorem 3.9** For every positive integer $n$, the complete graph $K_n$ is a $K_3$-distance graph.

**Proof** Let $H = K_3$. Next let $P: v_1, v_2, \ldots, v_n$ be a path of length $n - 1$, and let $C: w_1, w_2, w_3, w_4, w_1$ be a 4-cycle. The graph $G$ is obtained by identifying the vertex $v_n$ of $P$ with $w_1$ of $C$. For $i = 1, 2, \ldots, n$, let $G_i$ be that graph obtained from $G$ by adding new vertices $x$ and $y$ and the edges $xy, yv_i$, and $v_ix$. Since $G_j = G_i - \{xy, yv_i, v_ix\} + \{xy, yv_j, v_jx\}$, for $1 \leq i < j \leq n$, it follows that $d_H(G_i, G_j) = 1$. Thus $D_H(\{G_1, G_2, \ldots, G_n\}) = K_n$. □
Using a construction similar to the one given in [3] which shows that every cycle is an edge rotation distance graph, we now show that every cycle is a $K_3$-distance graph.

**Theorem 3.10** For every integer $n \geq 3$, the cycle $C_n$ is a $K_3$-distance graph.

**Proof** Let $H = K_3$ and let $C: v_1, v_2, \ldots, v_{2n+2}, v_1$ be a $(2n + 2)$-cycle. For $i = 1, 2, \ldots, n$, let $F_i$ be obtained from $C$ by joining a new vertex $v$ to $v_i$ and $v_{i+2}$ and adding the edge $v_i v_{i+2}$. For $n = 4$, the graphs $F_1, F_2, F_3,$ and $F_4$ are shown in Figure 3.7.

![Figure 3.7 The Graphs $F_1, F_2, F_3,$ and $F_4$.](image)

Next, for $i = 1, 2, \ldots, n - 1$, let $G_i = F_i \cup F_{i+1}$, and let $G_n = F_n \cup F_1$. Clearly, the graphs $G_i$ and $G_j$, for $1 \leq i < j \leq n$, differ in exactly two edges when $j = i + 1$ or when $i = 1$ and $j = n$, and differ in four edges otherwise. Thus since $d_H(G_i, G_{i+1}) = 1$ for $1 \leq i \leq n - 1$ and $d_H(G_n, G_1) = 1$, it follows that $D_H(\{G_1, G_2, \ldots, G_n\}) = C_n$. □
For a given connected graph $H$ of order at least 3, an induced subgraph of an $H$-distance graph is also an $H$-distance graph. Therefore, we have the following corollary to Theorem 3.10.

**Corollary 3.11** For every integer $n \geq 2$, the path $P_n$ is a $K_3$-distance graph.

At this point, an observation will be helpful. Let $G_1$ and $G_2$ be $K_3$-adjacent graphs. Then there are subgraphs $H_1$ and $H_2$ of $G_1$ and $G_2$, respectively, such that $H_1 \equiv H_2 \equiv K_3$, $E(H_1) \cap E(H_2) \neq \emptyset$, and $E(H_1) - E(H_2) \neq \emptyset$. Since any two edges of a triangle uniquely determine the third edge, it must be that $|E(H_1) \cap E(H_2)| = 1$ or $\{uv\} = E(H_1) \cap E(H_2)$ for vertices $u$ and $v$ of $G_1$ (and hence of $G_2$). Thus there exists a vertex $w$ of $H_1$ and a vertex $x$ of $H_2$ such that $w$ and $x$ are distinct vertices. So, $|E(H_2) - E(H_1)| = 2$, and hence $G_2 = G_1 - \{uw, vw\} + \{ux, vx\}$. The following lemma will be useful in establishing that a number of large classes of graphs are $K_3$-distance graphs.

**Lemma 3.12** Let $G_1$ and $G_2$ be graphs of the same order $n$ and same size. Then $d_{K_3}(G_1, G_2) = 1$ if and only if $d_{K_3}(G_1 + K_1, G_2 + K_1) = 1$.

**Proof** Let $H = K_3$. Clearly, if $d_H(G_1, G_2) = 1$, then $d_H(G_1 + K_1, G_2 + K_1) = 1$. Next suppose that $d_H(G_1 + K_1, G_2 + K_1) = 1$. Let $n = |V(G_1)|$. Since $G_1 + K_1$ is $H$-adjacent to $G_2 + K_1$, there exist vertices $u, v, w$, and $x$ of $G_1 + K_1$ such that $uv \in E(G_1 + K_1)$, and $G_2 + K_1 = (G_1 + K_1) - \{uw, vw\} + \{ux, vx\}$. Hence $ux, vx \in E(G_1 + K_1)$ and $uw, vw \notin E(G_2 + K_1)$. Therefore, in $G_1 + K_1$, the vertices $u$ and $v$ have degree less than $n$ and in $G_2 + K_1$, the vertices $x$ and $w$ have degree less than $n$. Thus $u, v, w, x$ must be vertices of $G_1$, and hence $ux, vx \in E(G_2)$ and $uw, vw \in E(G_1)$. So $G_2 = G_1 - \{uw, vw\} + \{ux, vx\}$. Since $uv \in E(G_1)$, it follows that $d_H(G_1, G_2) = 1$. □
We can now show that the cartesian product of two $K_3$-distance graphs is a $K_3$-distance graph.

**Theorem 3.13** For two $K_3$-distance graphs $G_1$ and $G_2$, the graph $G_1 \times G_2$ is a $K_3$-distance graph.

**Proof** Let $H = K_3$, and let $S$ and $T$ be sets of graphs for which $D_H(S) = G_1$ and $D_H(T) = G_2$. By Lemma 3.12, we may assume that $S$ and $T$ are disjoint and that each graph in $S \cup T$ is 3-connected. Assume that $S = \{G_u \mid u \in V(G_1)\}$ with $d_{H}(G_u, G_w) = 1$ if and only if $uw \in E(G_1)$. Similarly, $T = \{F_v \mid v \in V(G_2)\}$. We show that $G_1 \times G_2 \equiv D_{H}(\{G_u \cup F_v \mid u \in V(G_1), v \in V(G_2)\})$.

Since the graphs $G_u$ and $F_v$ are 3-connected, it follows that $d_{H}(G_u \cup F_v, G_u' \cup F_{v'}) = 1$ if and only if either (1) $G_u = G_u'$ and $d_{H}(F_v, F_{v'}) = 1$ or (2) $H_v = \bar{H}_v'$ and $d_{H}(G_u, G_{u'}) = 1$. Thus $d_{H}(G_u \cup F_v, G_u' \cup F_{v'}) = 1$ if and only if (1) $u = u'$ and $vv' \in E(G_2)$ or (2) $v = v'$ and $uu' \in E(G_1)$, and the result follows.

□

Suppose that $G_1$ and $G_2$ are $K_3$-distance graphs. The graph obtained by identifying a vertex $u$ of $G_1$ with a vertex $v$ of $G_2$ is an induced subgraph of $G_1 \times G_2$. Therefore, we have the corollary to Theorem 3.13.

**Corollary 3.14** For $K_3$-distance graphs $G_1$ and $G_2$, the graph obtained from $G_1$ and $G_2$ by identifying a vertex $u$ of $G_1$ with a vertex $v$ of $G_2$ is also a $K_3$-distance graph.
If the blocks of a connected graph $G$ are $K_3$-distance graphs, then from repeated applications of Corollary 3.14 to the blocks of $G$, we have that $G$ is also a $K_3$-distance graph. Consequently, we have the following corollary.

**Corollary 3.15** Every tree is a $K_3$-distance graph.

We now show that complete bipartite graphs are also $K_3$-distance graphs.

**Theorem 3.16** For integers $m$ and $n$ with $3 \leq m \leq n$, the complete bipartite graph $K_{m,n}$ is a $K_3$-distance graph.

**Proof** Let $H = K_3$, let $P: v_0, v_1, \ldots, v_{n+m}$ be a path, and let $C: u_1, u_2, u_3, u_4, u_1$ be a 4-cycle. Next, let $G$ be that graph obtained by identifying the vertex $v_{n+m}$ of $P$ with the vertex $u_1$ of $C$ and adding $n + m$ vertices $w_1, w_2, \ldots, w_{n+m}$ where for each $i = 1, 2, \ldots, n + m$, the vertex $w_i$ is joined to $v_{i-1}$ and $v_i$. For $i = 1, 2, \ldots, m$, define $G_i$ to be that graph obtained from $G$ by joining a new vertex $x$ to $v_i$. For $m = 2$ and $n = 3$, the graphs $G_1$ and $G_2$ are shown in Figure 3.8. The graphs $G_1$, $G_2$, $\ldots$, $G_m$ have the same order (namely, $2(n+m) + 5$) and same size (namely, $3(n+m) + 5$). Furthermore, in each graph $G_i$ ($1 \leq i \leq m$), there is an end-vertex, namely $x$, at distance $n + m - i$ from the unique 4-cycle of $G_i$. Therefore, the graphs $G_1$, $G_2$, $\ldots$, $G_m$ are pairwise nonisomorphic. Also, for $i \neq j$, the graphs $G_i$ and $G_j$ differ in exactly one edge, namely $xv_i$ in $G_i$ and $xv_j$ in $G_j$, where this edge does not lie on a triangle. Thus $G_i$ and $G_j$ are not $K_3$-adjacent.

Next, we define the $n$ graphs $F_1, F_2, \ldots, F_n$ such that for distinct integers $i$ and $j$ with $1 \leq i < j \leq n$, we have that $d_H(F_i, F_j) > 1$, while for each pair $i, j$ of integers with $1 \leq i \leq m$ and $1 \leq j \leq n$, we have that $d_H(G_i, F_j) = 1$. For each $i = 1, 2, \ldots, n$, define
For $m = 2$ and $n = 3$, the graphs $F_1$, $F_2$, and $F_3$ are shown in Figure 3.9. The graph $F_i$ ($1 \leq i \leq n$) has a cycle of length $2 + i$, and thus the graphs $F_1, F_2, \ldots, F_n$ are pairwise nonisomorphic.

![Figure 3.8 The Graphs $G_1$ and $G_2$.](image)

To see that $d_H(G_i, F_j) = 1$ for each pair $i, j$ of integers with $1 \leq i \leq m$ and $1 \leq j \leq n$, observe that

$$F_j = G_i - \{v_{m+1} v_{m+i} w_{m+i} v_{m+i+1} \} + \{v_{m+i+1} v_0 w_{m+i} v_0 \}.$$

so $G_i$ is $H$-adjacent to $F_j$. We now show for distinct integers $i$ and $j$ with $1 \leq i < j \leq n$ that $d_H(F_i, F_j) > 1$. For $i = 1, 2, \ldots, n$, let $F_{i,1}$ denote the component of $F_i$ containing the unique end-vertex of $F_i$, and let $F_{i,2}$ denote the other component of $F_i$. Next, for distinct integers $i$ and $j$ ($1 \leq i < j \leq n$), the graph $F_{j,1}$ has at least one more triangle than $F_{i,1}$, and $F_{i,2}$ has at least one more triangle than $F_{j,2}$. Thus the graph $F_i$ differs from the graph $F_j$ in at least four edges so that $d_H(F_i, F_j) > 1$. Therefore, $D_H(\{G_1, G_2, \ldots, G_m, F_1, F_2, \ldots, F_n\}) = K_{m,n}$. □
We have seen that if $H$ is a connected graph with $\delta(H) = 1$, then every graph is an $H$-distance graph. Also, many graphs are known to be $K_3$-distance graphs, such as complete graphs, complete bipartite graphs, cycles, paths, cartesian products of $K_3$-distance graphs, and trees. However, we know of no graph that is not a $K_3$-distance graph. In fact, we state the following conjecture.
**Conjecture** Every graph is a $K_3$-distance graph.

We now show that every graph that is an edge rotation distance graph is also a $K_3$-distance graph.

**Theorem 3.17** If $G$ is an edge rotation distance graph, then $G$ is a $K_3$-distance graph.

**Proof** Let $H = K_3$ and let $G$ be an edge rotation distance graph of order $p$. Then there exists a set $\{G_1, G_2, \ldots, G_p\}$ of graphs of the same order and same size, say $m$, such that $D_r(G_1, G_2, \ldots, G_p) \equiv G$. Let $e_{i_1}, e_{i_2}, \ldots, e_{i_m}$ ($1 \leq i \leq p$) denote the $m$ edges of $G_i$. For $i = 1, 2, \ldots, p$, let $F_i$ be that graph obtained from $G_i$ by adding $m$ new vertices $x_{i_1, 1}, x_{i_2, 1}, \ldots, x_{i_m, 1}$ where $x_{i_j}$ ($1 \leq j \leq m$) is joined to the two vertices of $G_i$ incident with the edge $e_{i_j}$. We show that $D_H(F_1, F_2, \ldots, F_p) \equiv G$. Let $V(G) = \{v_1, v_2, \ldots, v_p\}$. Since $D_r(G_1, G_2, \ldots, G_p) \equiv G$, it follows that if $v_iv_j$ is an edge of $G$, then $G_i$ can be obtained from $G_j$ by an edge rotation. Thus $G_i = G_j - uv + uw$ for distinct vertices $u, v,$ and $w$ of $G_j$. Now $uv = e_{j,k}$ and $uw = e_{i, \ell}$ for some $k$ and $\ell$ with $1 \leq k, \ell \leq m$. Hence $G_i - e_{i, \ell} = G_j - e_{j, k}$ so that $F_i - e_{i, \ell} - ux_{i, \ell} - wx_{i, \ell} = F_j - e_{j, k} - ux_{j, \ell} - vx_{j, \ell}$. Next since $u, w,$ and $x_{i, \ell}$ form a triangle in $F_i$ and since $u, v, x_{j,k}$ form a triangle in $F_j$, we have that $F_i = F_j - \{e_{j,k}, ux_{j,k}, vx_{j,k}\} + \{ux_{j,k}, x_{j,k}, uw\}$, and $F_i$ and $F_j$ are $H$-adjacent. Therefore, $G$ is a subgraph of $D_H(F_1, F_2, \ldots, F_p)$.

To see that $D_H(F_1, F_2, \ldots, F_p) \equiv G$, suppose that $v_{i_1}v_{i_2}$ is not an edge of $G$. Then the graph $G_{i_1}$ cannot be obtained from the graph $G_{i_2}$ by an edge rotation. Assume, without loss of generality, that $V(G_{i_1}) = V(G_{i_2})$. Let $s$ denote the size of a greatest common subgraph of $G_{i_1}$ and $G_{i_2}$. Suppose first that $s \leq m - 2$. Then $G_{i_1}$

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and $G_\ell$ differ in at least two edges, that is, $G_k$ contains at least two edges that are not present in $G_\ell$, while $G_\ell$ contains at least two edges that are not present in $G_k$. Hence $F_k$ contains two triangles that are not present in $F_\ell$, while $F_\ell$ contains two triangles that are not present in $F_k$. Therefore, $F_k$ and $F_\ell$ are not $H$-adjacent.

Finally, if $s = m - 1$, then $G_k$ and $G_\ell$ differ in exactly one edge. So $G_k - e = G_\ell - f$ for edges $e$ and $f$ of $G_k$ and $G_\ell$, respectively. Since $G_k$ cannot be obtained from $G_\ell$ by an edge rotation, it follows that $e$ and $f$ are not adjacent. Thus $e = uv$ and $f = wz$ for distinct vertices $u$, $v$, $w$, and $z$ of $G_k$. Hence $F_k$ and $F_\ell$ differ in one triangle, where the triangle in $F_k$ is induced by the vertices $u$, $v$, and $x$ and $x$ corresponds to $x_{k,i}$ for some $i (1 \leq i \leq m)$, and where the triangle in $F_\ell$ is induced by the vertices $w$, $z$, and $x$ and here $x$ corresponds to $x_{\ell,j}$ for some $j (1 \leq j \leq m)$. Hence $F_k$ and $F_\ell$ are not $H$-adjacent. Thus

$$D_H((F_1, F_2, \ldots, F_p)) \equiv G. \quad \Box$$

Thus, if the conjecture that every graph is an edge rotation graph is true, then the conjecture that every graph is a $K_3$-distance graph is true as well.
CHAPTER IV

GREATES COMMON DISTANCE-PRESERVING SUBGRAPHS

4.1 Introduction and Examples

Recall that a greatest common subgraph of two nonisomorphic graphs $G_1$ and $G_2$ is defined as a graph of maximum size without isolated vertices that is a subgraph of both $G_1$ and $G_2$. Chartrand, Saba, and Zou [10] proved that for every graph $G$ without isolated vertices, there exist nonisomorphic graphs $G_1$ and $G_2$ of equal size such that $G$ is the unique greatest common subgraph of $G_1$ and $G_2$. In the proof of this result, one of $G_1$ and $G_2$ is disconnected, regardless of whether $G$ is connected. However, in [6] Chartrand, Johnson, and Oellermann proved that for every connected graph $G$ that is not complete, there exist connected graphs $G_1$ and $G_2$ of equal size such that $G$ is the unique greatest common subgraph of $G_1$ and $G_2$. This concept has also been studied in [5, 6, 8, 10, 11, 12, 20, 21].

We now turn our attention to another common subgraph of two connected graphs. For a subgraph $H$ of a graph $G$, we say that $H$ is distance-preserving if $d_H(u, v) = d_G(u, v)$ for every pair $u, v$ of vertices of $H$. A greatest common distance-preserving subgraph of two connected graphs $G_1$ and $G_2$ is a graph $G$ of maximum size such that $G$ is a distance-preserving subgraph of $G_1$ and $G_2$. This definition certainly implies that every distance-preserving subgraph is connected. The size of a greatest common distance-preserving subgraph is called the gds size, and the set of all such subgraphs is denoted by gds($G_1, G_2$). To illustrate these concepts, we determine the greatest common distance-preserving subgraphs of $P_n$ and $C_n$, the path and cycle on $n$ vertices, respectively.
Consider a greatest common distance-preserving subgraph of $P_6$ and $C_6$. Since the distance between every pair of distinct vertices of $C_6$ is at most 3, it follows that $P_4$ is the unique greatest common distance-preserving subgraph of $P_6$ and $C_6$. In fact, since the distance between every pair of distinct vertices of $C_n$ is at most $\left\lfloor \frac{n}{2} \right\rfloor$, the path $P_{\left\lfloor \frac{n}{2} \right\rfloor + 1}$ is the greatest common distance-preserving subgraph of $C_n$ and $P_n$. Thus the gds size of $P_n$ and $C_n$ is $\left\lfloor \frac{n}{2} \right\rfloor$ and $\text{gds}(P_n, C_n) = \{P_{\left\lfloor \frac{n}{2} \right\rfloor + 1}\}$.

Next we show that distance-preserving subgraphs are induced subgraphs.

**Lemma 4.1** Let $H$ be a distance-preserving subgraph of a connected graph $G$. Then $H$ is an induced subgraph of $G$.

**Proof** Let $H$ be a distance-preserving subgraph of $G$. We show that $\langle V(H) \rangle = H$. Suppose, to the contrary, that there exists an edge $e$ of $\langle V(H) \rangle$ such that $e$ is not an edge of $H$. So $e = xy$ for some $x, y \in V(H)$. Thus, $d_G(x, y) = 1$ while $d_H(x, y) \geq 2$, producing a contradiction. Therefore $\langle V(H) \rangle = H$ and $H$ is an induced subgraph of $G$. \qed

We now show that greatest common distance-preserving subgraphs need not be unique. Consider the graphs $G_1$ and $G_2$ of size 11, shown in Figure 4.1. Now $C_6$ is a distance-preserving subgraph of $G_1$ and $G_2$ and hence the gds size is at least 6.

![Figure 4.1 Graphs With Two Greatest Common Distance-Preserving Subgraphs.](image-url)
Figure 4.2 shows all connected induced subgraphs of size 7 in $G_2$, namely the graphs $H_1, H_2, \ldots, H_6$. Since each graph $H_i$ ($1 \leq i \leq 6$) is not an induced subgraph of $G_1$, it follows that the gds size is 6. The graphs $F_1, F_2, \ldots, F_7$ of Figure 4.2 are all the connected induced subgraphs of size 6 in $G_2$. Now $F_4, F_6,$ and $F_7$ are not induced subgraphs of $G_1$ while $F_1, F_2, F_3,$ and $F_5$ are induced subgraphs of $G_1$. Since $F_3$ and $F_5$ are not distance-preserving subgraphs of $G_2$, it follows that $\text{gds}(G_1, G_2) = \{F_1, F_2\}$.

![Figure 4.2 Connected Induced Subgraphs of Sizes 6 and 7 in $G_2$.](image-url)
A greatest common subgraph and a greatest common distance-preserving subgraph of two given graphs $G_1$ and $G_2$ are, of course, common subgraphs of $G_1$ and $G_2$. A greatest common distance-preserving subgraph need not be a subgraph of a greatest common subgraph, however, as we now show.

Consider the graphs $G_1$ and $G_2$ of size 10 shown in Figure 4.3. Since $G_1$ is not isomorphic to $G_2$, the size of a greatest common subgraph is at most 9. The graph $G$ of Figure 4.3 is a common subgraph of $G_1$ and $G_2$ having size 9, so $G$ is a greatest common subgraph of $G_1$ and $G_2$. Since removing any edge other than the edge $e$ from $G_2$ does not produce a subgraph of $G_1$, it follows that $G$ is the unique greatest common subgraph of $G_1$ and $G_2$. Since $\text{diam} \ G_2 = 3$ and the graph $H$ of Figure 4.3 is the subgraph of maximum size in $G_1$ with diameter 3, we have that $gds(G_1, G_2) = \{H\}$. So the greatest common distance-preserving subgraph of $G_1$ and $G_2$ is not a subgraph of the greatest common subgraph of $G_1$ and $G_2$.

![Figure 4.3 Graphs Where the Greatest Common Distance-Preserving Subgraph Is Not a Subgraph of the Greatest Common Subgraph.](image)

**A greatest common induced subgraph** of two nonisomorphic graphs $G_1$ and $G_2$ is defined as a graph $G$ of maximum size without isolated vertices that is an induced subgraph of both $G_1$ and $G_2$. Although greatest common distance-
preserving subgraphs are induced subgraphs, it need not be the case that a greatest common distance-preserving subgraph is a subgraph of a greatest common induced subgraph, as we shall now see.

Consider the graphs $G_1$ and $G_2$ shown in Figure 4.4. Clearly, $G_1$ and $G_2$ are nonisomorphic graphs of size 12. Now $G_1 - x \cong G_2 - y$, so the size of a greatest common induced subgraph of $G_1$ and $G_2$ is at least 10. Removing one end-vertex from $G_2$ does not produce a subgraph of $G_1$ and thus the size of a greatest common induced subgraph is at most 10. The induced subgraphs of $G_2$ of size 10 are obtained by removing a vertex of degree 2 or by removing two vertices of degree 1. Since removing any two end-vertices or removing any vertex of degree 2 other than $y$ from $G_2$ does not produce a subgraph of $G_1$, it follows that $G_2 - y$ is the unique greatest common induced subgraph of $G_1$ and $G_2$.

![Graphs G1 and G2 and the Graph G](image)

To determine the greatest common distance-preserving subgraphs of $G_1$ and $G_2$ of Figure 4.4, we consider the diameter of $G_2$. The graph $G$ of Figure 4.4 is the unique induced subgraph of $G_1$ of maximum size with diameter 4. Since $\text{diam } G_2 = 4$, it follows that $\text{gds}(G_1, G_2) = \{G\}$. Since $G$ is not a subgraph of $G_2 - y$, the greatest common distance-preserving subgraph of $G_1$ and $G_2$ is not a subgraph.
of the greatest common induced subgraph. Consequently, a greatest common distance-preserving subgraph need not be a subgraph of a greatest common induced subgraph, although both are induced subgraphs. In fact, we now show that the difference in the sizes of the these two subgraphs can be arbitrarily large.

**Theorem 4.2** For every positive integer \( n \), there exist graphs \( G_1 \) and \( G_2 \) of equal size such that the difference between the size of a greatest common induced subgraph and the size of a greatest common distance-preserving subgraph of \( G_1 \) and \( G_2 \) is \( n \).

**Proof** First, let \( n \) be a positive integer. Let \( F = C_{2n+2} \), say \( F : v_1, v_2, \ldots, v_{2n+2}, v_1 \). The graph \( G_1 \) is obtained from \( F \) by adding the vertices \( w_1, w_2, \ldots, w_{n+1} \) along with the edges \( v_1w_1, w_1w_2, w_2w_3, \ldots, w_nw_{n+1}, w_{n+1}v_{n+2} \); while the graph \( G_2 \) is obtained from \( F \) by adding the vertices \( x_1, x_2, \ldots, x_{n+2} \) and the edges \( v_1x_1, x_1x_2, x_2x_3, \ldots, x_{n+1}x_{n+2} \). The graphs \( G_1 \) and \( G_2 \) are shown in Figure 4.5.

![Figure 4.5](image_url)
Each of $G_1$ and $G_2$ has size $3n + 4$ and $G_1 \neq G_2$. Since $G_1 - w_{n+1} = G_2 - x_{n+1} - x_{n+2}$, the size of a greatest common induced subgraph is at least $3n + 2$. Moreover, since $\delta(G_1) \geq 2$, the size of a greatest common induced subgraph of $G_1$ and $G_2$ is at most $3n + 2$. Therefore, the size of a greatest common induced subgraph is $3n + 2$.

Since $\text{diam } G = n + 1$, it follows that if $H$ is a greatest common distance-preserving subgraph of $G_1$ and $G_2$, then $\text{diam } H \leq n + 1$. Note that the $(2n + 2)$-cycle $v_1, v_2, \ldots, v_{2n+2}, v_1$ is a distance-preserving subgraph of $G_1$ and $G_2$. Thus the size of $H$ is at least $2n + 2$. We now show the size of $H$ is at most $2n + 2$.

Suppose, to the contrary, that $H$ has size $q$ where $q > 2n + 2$. Since $H$ is a connected induced subgraph of $G_1$ and $H \neq G_1$, it follows that either $H$ contains one cycle, of length $2n + 2$, or $H$ is tree. Suppose first that $H$ contains exactly one cycle, so $v_1, v_2, \ldots, v_{2n+2}, v_1$ is the cycle of $H$. Since the size of $H$ is at least $2n + 3$, there exists a vertex $x$ not belonging to the cycle of $H$. Then the distance from $x$ to $v_{n+2}$ is at least $n + 2$ and hence $H$ is not distance-preserving. Thus $H$ is a tree. Since $\Delta(G_1) = 3$, it follows that $H$ is a path or $H$ consists of three paths identified at $v_1$. If $H$ is a path of size at least $2n + 3$, then $\text{diam } H \geq 2n + 2$, producing a contradiction. So, $H$ consists of three paths identified at $v_1$. Let $\ell_1$, $\ell_2$, and $\ell_3$ denote the lengths of these three paths. Since $H$ is a distance-preserving subgraph of $G_1$ and $\text{diam } G_1 = n + 1$, it follows that $\ell_1 + \ell_2 \leq n + 1$ and $\ell_2 + \ell_3 \leq n + 1$ so $\ell_1 + 2\ell_2 + \ell_3 \leq 2n + 2$. But, since the size of $H$ is $\ell_1 + \ell_2 + \ell_3$, we have that $\ell_1 + \ell_2 + \ell_3 > 2n + 2$, producing a contradiction. Thus $H$ has size $2n + 2$. Therefore, the difference between the size of the greatest common induced subgraph and the size of the greatest common distance-preserving subgraph is $(3n + 2) - (2n + 2) = n$. \[\square\]
4.2 Graphs With a Prescribed Greatest Common Distance-Preserving Subgraph

We have already noted that every graph without isolated vertices is the unique greatest common subgraph of some pair of nonisomorphic graphs of equal size. We now show that every connected noncomplete graph is the unique greatest common distance-preserving subgraph of two nonisomorphic connected graphs of equal size.

Theorem 4.3 For each connected graph $G$ of order at least 3, there exist nonisomorphic graphs $G_1$ and $G_2$ of equal size such that $\text{gds}(G_1, G_2) = \{G\}$.

Proof First assume that $G = K_n$, where $n \geq 3$. Let $G_1$ be obtained from $G$ by joining a new vertex $z_1$ to any two vertices of $K_n$. Next let $x$ and $y$ be two vertices of $K_n$ and obtain $G_2$ from $G$ by joining a new vertex $w_1$ to $x$ and a new vertex $w_2$ to $y$. Clearly, $\text{gds}(G_1, G_2) = \{G\}$.

Next, let $G$ be a connected noncomplete graph of size $q$ and let $t$ be the number of triangles in $G$. The construction of the graphs $G_1$ and $G_2$ depends on $t$ and the minimum degree $\delta(G)$ of the vertices of $G$.

Case 1 Suppose that $\delta(G) \geq 3$ or $t = 0$. Since $G$ is not complete, there exist vertices $x$ and $y$ such that $d(x, y) = 2$, say $x, w, y$ is an $x$-$y$ path in $G$. Let $G_1$ be obtained from $G$ by adding a new vertex $z_1$ to $G$ as well as the edges $z_1x$ and $z_1w$. Next let $G_2$ be obtained from $G$ by adding a new vertex $z_2$ to $G$ and the edges $z_2x$ and $z_2y$. Since $G_1$ has $t + 1$ triangles while $G_2$ has $t$ triangles, $G_1$ is not isomorphic to $G_2$. Clearly, $G$ is a distance-preserving subgraph of $G_1$ and $G_2$. Therefore, the gds size is at least $q$.

We now show that the gds size is at most $q$. Suppose first that $\delta(G) \geq 3$. Then the removal of any vertex of $G$ from $G_1$ results in a subgraph $F$ of size at
most \( q - 1 \) and thus \( F \) is not a distance-preserving subgraph of maximum size. Hence \( G \) is the unique greatest common distance-preserving subgraph of \( G_1 \) and \( G_2 \). Next suppose that \( t = 0 \). Then \( G_2 \) has no triangles while \( G_1 \) has one triangle. Therefore each greatest common subgraph has no triangles and hence we must remove at least one of \( x, z_1, \) and \( w \) from \( G_1 \) to produce a greatest common distance-preserving subgraph. Since \( \deg_{G_1} w \geq 3 \), we cannot remove \( w \). If \( \deg_{G_1} x \geq 3 \), then we cannot remove \( x \) either, while if \( \deg_{G_1} x = 2 \), then \( G_1 - x \equiv G_1 - z_1 = G \).

Thus, \( G \) is again the unique greatest common distance-preserving subgraph of \( G_1 \) and \( G_2 \).

**Case 2** Suppose that \( \delta(G) = 2 \) and \( t \geq 1 \). Since \( G \) is not regular, there exist vertices \( x \) and \( y \) of \( G \) such that \( \deg_G x \neq \deg_G y \). Let \( G_1 \) be obtained from \( G \) by joining a new vertex \( z_1 \) to \( x \), and let \( G_2 \) be obtained from \( G \) by joining a new vertex \( z_2 \) to \( y \). Clearly, \( G_1 \neq G_2 \). Thus the gds size is at most \( q \). Now \( G \) is a distance-preserving subgraph of \( G_1 \) and \( G_2 \) and hence \( G \) is a greatest common distance-preserving subgraph. Furthermore, since \( \delta(G) = 2 \), it follows that \( G \) is the unique greatest common distance-preserving subgraph of \( G_1 \) and \( G_2 \).

**Case 3** Suppose that \( \delta(G) = 1 \) and \( t \geq 1 \). For each end-vertex \( v \) of \( G \), let \( m_v \) denote the distance from \( v \) to a nearest triangle of \( G \). Let \( m = \max \{ m_v \} \) over all end-vertices \( v \) of \( G \). Let \( x \) be an end-vertex for which \( m_x = m \), and let \( T \) be a triangle of \( G \) whose distance from \( x \) is \( m \). Define \( G_1 \) as that graph obtained from \( G \) by joining a new vertex \( z_1 \) to \( x \) and define \( G_2 \) as that graph obtained from \( G \) by joining a new vertex \( z_2 \) to the unique vertex \( y \) adjacent to \( x \) in \( G \). Now \( G_1 \) has an end-vertex, namely \( z_1 \), whose distance from \( T \) is \( m + 1 \), while every end-vertex of \( G_2 \) has distance at most \( m \) from \( T \). Consequently, \( G_1 \neq G_2 \). Thus the gds size is at
most $q$. Since $G$ is a distance-preserving subgraph, the gds size is at least $q$. We show that $G$ is the unique greatest common distance-preserving subgraph. Suppose, to the contrary, that there exists a greatest common distance-preserving subgraph $H$ of $G_1$ and $G_2$ with $H \neq G$. Then $H \cong H_1$, where $H_1$ is an induced subgraph of $G_1$. Since $H_1$ has size $q$ and $G_1$ has size $q+1$, it follows that $H_1 = G_1 - w_1$ where $\deg_{G_1} w_1 = 1$. Similarly, there exists an induced subgraph $H_2$ of $G_2$ such that $H_2 \cong H$ and $H_2 = G_2 - w_2$, where $w_2$ is an end-vertex of $G_2$. Since $H \neq G$, it follows that $w_1 \neq z_1$ and $w_2 \neq z_2$. So $z_1 \in V(H_1)$. Thus $H_1$ contains an end-vertex, namely $z_1$, whose distance from the nearest triangle is $m + 1$, while every end-vertex in $H_2$ has distance at most $m$ from any triangle. Therefore $H_1 \neq H_2$, producing a contradiction. Hence $G$ is the unique greatest common distance preserving subgraph of $G_1$ and $G_2$. □

Chartrand, Johnson, and Oellermann [6] showed that for every nontrivial connected graph $G$, there exist nonisomorphic graphs $G_1$ and $G_2$ such that $G$ is the unique greatest common induced subgraph of $G_1$ and $G_2$. In fact, in their proof of this result, the graph $G$ is a distance-preserving subgraph of the graphs $G_1$ and $G_2$ constructed, so $G$ is the unique greatest common distance-preserving subgraph of $G_1$ and $G_2$ as well. Also, in their proof, each of the graphs $G_1$ and $G_2$ contains two edges and one or two vertices not present in $G$ so in some sense the proof just presented is simpler.

We now define the greatest common distance-preserving subgraph of $n$ pairwise nonisomorphic connected graphs $G_1, G_2, \ldots, G_n$ as a graph $G$ of maximum size such that $G$ is a distance-preserving subgraph of $G_i$ for $i = 1, 2, \ldots, n$. As before, we refer to the size of $G$ as the gds size and denote the set of greatest common distance-preserving subgraphs of $G_1, G_2, \ldots, G_n$ by $\text{gds}(G_1, G_2, \ldots, G_n)$. 

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In [10] it was shown that for every graph \( G \) without isolated vertices, there exist pairwise nonisomorphic graphs \( G_1, G_2, \) and \( G_3 \) of equal size such that \( G \) is the unique greatest common subgraph of \( G_1, G_2, \) and \( G_3. \) In a similar manner, we can extend Theorem 4.3 to show that for a given noncomplete connected graph \( G, \) there exist nonisomorphic graphs \( G_1, G_2, \) and \( G_3 \) such that \( G \) is the unique greatest common distance-preserving subgraph of \( G_1, G_2, \) and \( G_3. \)

**Theorem 4.4** For each connected graph \( G \) of order at least 3, there exist nonisomorphic graphs \( G_1, G_2, \) and \( G_3 \) such that \( \text{gds}(G_1, G_2, G_3) = \{G\}. \)

**Proof** First, assume that \( G \equiv K_n \) for \( n \geq 3. \) Let \( x \) and \( y \) be two vertices of \( G. \) Obtain \( G_1 \) from \( G \) by joining a new vertex \( z_1 \) to \( x \) and \( y \) and obtain \( G_2 \) from \( G \) by adding two vertices \( w_1 \) and \( w_2 \) and the edges \( xw_1 \) and \( yw_2. \) Finally, let \( G_3 \) be obtained from \( G \) by adding two vertices \( v_1 \) and \( v_2 \) and the edges \( xv_1 \) and \( v_1v_2. \) Then \( \text{gds}(G_1, G_2, G_3) = \{G\}. \)

Now let \( G \) be a noncomplete connected graph of size \( q \) and let \( t \) be the number of triangles in \( G. \) The construction of the graphs \( G_1, G_2, \) and \( G_3 \) depends on \( t \) and \( \delta(G). \)

**Case 1** Suppose that \( \delta(G) \geq 3 \) or \( t = 0. \) As in the proof of Theorem 4.3, there exist vertices \( x \) and \( y \) such that \( d(x, y) = 2, \) say \( x, w, y \) is an \( x-y \) path in \( G. \) Construct \( G_1 \) and \( G_2 \) as in Case 1 of the proof of Theorem 4.3; that is, let \( G_1 \) be obtained from \( G \) by joining a new vertex \( z_1 \) to \( x \) and \( w \) and let \( G_2 \) be obtained from \( G \) by joining a new vertex \( z_2 \) to \( x \) and \( y. \) Next construct \( G_3 \) from \( G \) by joining two new vertices \( w_1 \) and \( w_2 \) to \( x. \) Then since \( G \) is a distance-preserving subgraph of \( G_1, G_2, \) and \( G_3, \) the gds size is at least \( q. \) Also, each greatest common distance-preserving subgraph of \( G_1, G_2, \) and \( G_3 \) is a distance-preserving subgraph of \( G_1 \)
and $G_2$; so the gds size is at most the gds size of $G_1$ and $G_2$ or $q$. Therefore, since $G$ is the unique greatest common distance-preserving subgraph of $G_1$ and $G_2$, it follows that $\text{gds}(G_1, G_2, G_3) = \{G\}$.

Case 2. Suppose that $\delta(G) = 2$ and $t \geq 1$. As in Case 2 of the proof Theorem 4.3, since $G$ is not regular, there exist vertices $x$ and $y$ of $G$ such that $\text{deg}_G x \neq \text{deg}_G y$. Obtain $G_1$ from $G$ by joining a new vertex $z_1$ to $x$ and obtain $G_2$ from $G$ by joining a new vertex $z_2$ to $y$. Next let $G_3$ be obtained from $G$ by joining two new vertices $w_1$ and $w_2$ to $x$. Clearly, $G$ is a distance-preserving subgraph of $G_1, G_2$, and $G_3$ and since $\text{gds}(G_1, G_2) = \{G\}$, it follows that $\text{gds}(G_1, G_2, G_3) = \{G\}$.

Case 3. Suppose that $\delta(G) = 1$ and $t \geq 1$. Construct $G_1$ and $G_2$ as in Case 3 of the proof of Theorem 4.3. Let $G_3$ be obtained from $G_1$ by joining a new vertex $w_1$ to $z_1$. Then since $\text{gds}(G_1, G_2) = \{G\}$ and $G$ is a distance-preserving subgraph of $G_1, G_2$, and $G_3$, it follows that $\text{gds}(G_1, G_2, G_3) = \{G\}$. □

It is a fact that for every positive integer $n$ and every connected, noncomplete graph $G$ of size $q$, there exist graphs $G_1, G_2, \ldots, G_n$ such that $\text{gds}(G_1, G_2, \ldots, G_n) = \{G\}$. To obtain the graphs $G_1, G_2, \ldots, G_n$, we begin by constructing $G_1$ and $G_2$ as in the proof of Theorem 4.3. To construct the graph $G_i$ for $3 \leq i \leq n$, we join $i - 1$ vertices to the vertex of $G$ labeled $x$ in each case of the proof of Theorem 4.3. Since $G$ is a distance-preserving subgraph of $G_i$ ($1 \leq i \leq n$), the gds size of $G_1, G_2, \ldots, G_n$ is at least $q$. Thus since $\text{gds}(G_1, G_2) = \{G\}$, it follows that $\text{gds}(G_1, G_2, \ldots, G_n) = \{G\}$.

Note that the graphs $G_1$ and $G_2$ constructed in Theorem 4.3 have equal size, while the graphs $G_1, G_2, G_3$ of Theorem 4.4 do not have equal size. This suggests
the following question: For a positive integer \( n \) and a connected, noncomplete graph \( G \), do there exist graphs \( G_1, G_2, \ldots, G_n \) of equal size such that \( \text{gds}(G_1, G_2, \ldots, G_n) = \{G\} \)? Certainly, if \( n \) is large, then the size of each graph \( G_i \) must also be large. In [10] the greatest common subgraph index or gcs index of a graph \( G \) without isolated vertices, denoted by \( \text{gcs}(G) \), is defined as the least positive integer \( q_0 \) such that for any integer \( q > q_0 \) and any collection of graphs \( G_1, G_2, \ldots, G_n, n \geq 2 \), of size \( q \) for which \( G \) is a greatest common subgraph of \( G_1, G_2, \ldots, G_n \), it follows that the graphs \( G_1, G_2, \ldots, G_n \), have another greatest common subgraph, different from \( G \). If no such \( q_0 \) exists, then \( \text{gcs}(G) = \infty \).

Similarly, for a connected graph \( G \), the greatest common distance-preserving index or gds index, denoted by \( \text{gds}(G) \), is the least positive integer \( q_0 \) such that for any integer \( q > q_0 \) and any collection of graphs \( G_1, G_2, \ldots, G_n, n \geq 2 \), of size \( q \) for which \( G \in \text{gds}(G_1, G_2, \ldots, G_n) \), it follows that \( |\text{gds}(G_1, G_2, \ldots, G_n)| > 1 \). If no such integer \( q_0 \) exists, then \( \text{gds}(G) = \infty \).

It was shown in [10] that for integers \( r \geq 1 \) and \( n \geq 3 \),

(a) \( \text{gcs}(K_{1,r}) = \infty \)
(b) \( \text{gcs}(K_n) = \begin{cases} 6 & \text{if } n = 3 \\
\infty & \text{if } n \neq 3 \end{cases} \)
(c) \( \text{gcs}(P_n) = \begin{cases} \infty & \text{if } n \neq 4 \\
6 & \text{if } n = 4. \end{cases} \)

Also, Kubicki [20] has shown that for integers \( r \) and \( s \) with \( r, s \geq 1 \), the gcs index \( \text{gcs}(K_{r,s}) = \infty \). He also gave a sufficient condition in [20] for a graph to have infinite gcs index, namely, if a graph \( G \) contains a vertex \( v \) of maximum degree such that no component of \( G - v \) is isomorphic to \( K_2 \), then \( \text{gcs}(G) = \infty \).

We now show similar results for the gds index.
Theorem 4.5  For positive integers $n, r,$ and $s$ with $n \geq 3$ and $s > 1$,

(a) $i_d(K_n) = \infty$
(b) $i_d(C_n) = \infty$
(c) $i_d(K_{r,s}) = \infty$.
(d) For every connected nonregular graph $G$, the gds index $i_d(G) = \infty$.

Proof  (a) Suppose, to the contrary, that $i_d(K_n) = q_0$. Let $m$ be a positive integer such that \( \binom{m}{2} > q_0 \), and let $q = \binom{n}{2}$. Let $G_1$ be obtained from $K_n$ by joining $q - \binom{n}{2}$ new vertices to a vertex of $K_n$ and let $G_2 = K_m$. Since $\text{diam}(G_2) = 1$, it follows that every distance-preserving subgraph of $G_2$ is complete. Thus $\text{gds}(G_1, G_2)$ is the maximum clique of $G_1$ or $\text{gds}(G_1, G_2) = \{K_n\}$, producing a contradiction.

(b) By part (a), we have that $i_d(C_3) = \infty$. Next for $n \geq 4$, suppose, to the contrary, that $i_d(C_n) = q_0$ for some positive integer $q_0$. Let $x$ and $y$ be two vertices of $C_n$ such that $d(x, y) = 2$, say $x, w, y$ is a $x-y$ path in $C_n$. Next, let $k$ be a positive integer such that $n + 2k > q_0$. Obtain $G_1$ from $C_n$ by joining $k$ new vertices $v_1, v_2, \ldots, v_k$ to $x$ and to $w$. Next obtain $G_2$ from $C_n$ by adding new vertices $z_1, z_2, \ldots, z_{2k-1}$ and the edges $xz_1, z_1z_2, z_2z_3, \ldots, z_{2k-2}z_{2k-1}, z_{2k-1}y$. An example of the graphs $G_1$ and $G_2$ is shown in Figure 4.6 when $n = 4$ and $k = 3$. Since $G_2$ has no triangles and $\Delta(G_2) = 3$, it follows that $\text{gds}(G_1, G_2) = \{C_n\}$, producing a contradiction.

(c) Let $r$ and $s$ be positive integers with $r \leq s$ and $s > 1$. Suppose, to the contrary, that $i_d(K_{r,s}) = q_0$ for some positive integer $q_0$. Let $x$ be a vertex of degree $r$ in $K_{r,s}$ and let $y$ be a vertex of degree $s$ in $K_{r,s}$. Next, let $k$ be a positive integer such that $rs + 2k > q_0$. Obtain $G_1$ from $K_{r,s}$ by joining $k$ vertices $v_1, v_2, \ldots, v_k$ to both $x$ and $y$. Next obtain $G_2$ from $K_{r,s}$ by adding new vertices $z_1, z_2, \ldots, z_{2k}$.
and the edges $xz_1, z_1z_2, z_2z_3, \ldots, z_{2k-1}z_{2k}$. Then $G_1$ and $G_2$ have size $rs + 2k$ and since $G_2$ has no triangles and $\Delta(G_2) \leq s + 1$, it follows that $\text{gds}(G_1, G_2) = \{K_{rs}\}$, producing a contradiction.

Figure 4.6 The Graphs $G_1$ and $G_2$ When $k = 3$ and $n = 4$.

(d) Let $G$ be a graph of size $q$ that is not regular. Then there exist vertices $x$ and $y$ such that $\text{deg} y < \text{deg} x = \Delta(G)$. Next suppose, to the contrary, that $i_d(G) = q_0$ for some positive integer $q_0$. Let $m$ be a positive integer such that $q + m > q_0$. Let $G_1$ be obtained from $G$ by joining new vertices $z_1, z_2, \ldots, z_m$ to $x$ and let $G_2$ be obtained from $G$ by adding the vertices $w_1, w_2, \ldots, w_m$ and the edges $yw_1, w_1w_2, w_2w_3, \ldots, w_{m-1}w_m$. Now $\Delta(G_1) = \Delta(G) + m$ while $\Delta(G_2) = \Delta(G)$. Thus if $H \in \text{gds}(G_1, G_2)$, then $\Delta(H) \leq \Delta(G)$. Also since $G$ is a distance-preserving subgraph of $G_1$ and $G_2$, it follows that $H$ has size at least $q$. Hence if $x$ is not a vertex of $H$, then $H$ has size at most $q - \Delta(G)$ and therefore $x$ is a vertex of $H$. Since $\text{deg}_{G_1} x = \Delta(G) + m$ and $\Delta(G_2) = \Delta(G)$, it follows that to obtain $H$ from $G_1$ we must remove at least $m$ vertices $x_1, x_2, \ldots, x_m$ from $G_1$, where each $x_i$ ($1 \leq i \leq m$) is adjacent to $x$. If $\text{deg} x_i \geq 2$ for any $1 \leq i \leq m$, then the size of $H$ is at most $q - 1$
and thus each \( x_i \) must have degree 1. Therefore \( H = G \) and hence \( \text{gds}(G_1, G_2) = \{G\} \), producing a contradiction. \( \square \)

No connected graph \( G \) is known for which \( i_d(G) \) is finite. If such a connected graph \( G \) exists, then \( G \) must be regular and noncomplete, but neither a cycle nor the graph \( K_{n,n} \) for some positive integer \( n \). We have the following conjecture.

**Conjecture** For every connected graph \( G \), the gds index \( i_d(G) \) is infinite.

### 4.3 Greatest Common Distance-Preserving Trees

In [11] Chartrand and Zou studied trees and greatest common subgraphs. Let \( D(t) \) denote that graph obtained from two stars \( K_{1,t} \) whose central vertices are connected by a path of length 3. Chartrand and Zou [11] proved that not every tree is the unique greatest common subgraph of two nonisomorphic trees of equal size, that is, for a nontrivial tree \( T \), there exist nonisomorphic trees \( T_1 \) and \( T_2 \) such that \( T \) is the unique greatest common subgraph of \( T_1 \) and \( T_2 \) if and only if \( T \neq P_n \) for \( n = 2, 4, 5, \ldots \) and \( T \neq D(t) \) for \( t \geq 2 \). However, they also proved in [11] that every tree of order at least 3 is the unique greatest common induced subgraph of two nonisomorphic trees of equal size.

We now study trees and greatest common distance-preserving subgraphs. Recall that greatest common distance-preserving subgraphs need not be unique. By Theorem 4.3, for every tree \( T \), there exist nonisomorphic graphs \( G_1 \) and \( G_2 \) such that \( \text{gds}(G_1, G_2) = \{T\} \). However, in the proof of Theorem 4.3, the graphs \( G_1 \) and \( G_2 \) are not trees. We now show that for every tree \( T \), there exist nonisomorphic trees \( T_1 \) and \( T_2 \) of equal size such that \( T \) is the unique greatest common distance-preserving subgraph of \( T_1 \) and \( T_2 \).
**Theorem 4.6** For every tree $T$ of order at least 3, there exist nonisomorphic trees $T_1$ and $T_2$ of equal size such that $gds(T_1, T_2) = \{T\}$.

**Proof** Let $T$ have order $p$, where $p \geq 3$. Let $x$ be a vertex of maximum degree in $T$. Let $T_1$ be obtained from $T$ by joining a new vertex $z_1$ to $x$ and let $T_2$ be obtained from $T$ by joining a new vertex $z_2$ to an end-vertex of $T$. Since $p \geq 3$, it follows that $T_1 \neq T_2$. Clearly, $T$ is a distance-preserving subgraph of $T_1$ and $T_2$. Also, since $T_1$ has a vertex of degree $\Delta(T) + 1$, namely $x$, while every vertex of $T_2$ has degree at most $\Delta(T)$, it follows that if $H \in gds(T_1, T_2)$, then $H$ has size $p - 1$ and $H = T_1 - v$, where $v$ is an end-vertex of $T_1$ adjacent to $x$. Thus $T_1 - v = T$ so that $H \cong T$. Hence, $gds(T_1, T_2) = \{T\}$. □

We have previously noted that a greatest common distance-preserving subgraph $H$ of two nonisomorphic graphs $G_1$ and $G_2$ need not be a subgraph of any greatest common subgraph $G$ of $G_1$ and $G_2$. However, if $G_1$ and $G_2$ are trees and $G$ is connected, then the situation is different.

**Theorem 4.7** For trees $T_1$ and $T_2$, a greatest common subgraph $T$ of $T_1$ and $T_2$ is connected if and only if $T$ is a greatest common distance-preserving subgraph.

**Proof** First, suppose that $T$ is the greatest common subgraph of $T_1$ and $T_2$ and that $T$ is connected. Let $x$ and $y$ be vertices of $T$. Then since $T_1$ is a tree, there exists a unique $x$-$y$ path $P_1$ in $T_1$. Since $T$ is connected and $P_1$ is the unique $x$-$y$ path in $T_1$, it must be that $P_1$ is the unique $x$-$y$ path in $T$ so that $d_T(x, y) = d_{T_1}(x, y)$. Similarly, $d_T(x, y) = d_{T_2}(x, y)$. Therefore, $T$ is distance-preserving. Since the gds size is at most the size of a greatest common subgraph, $T$ is a greatest common distance-preserving subgraph.
Next suppose that $T$ is a greatest common subgraph such that $T \in \text{gds}(T_1, T_2)$. Since greatest common distance-preserving subgraphs are connected, it follows that $T$ is connected. □

Since greatest common distance-preserving subgraphs are induced subgraphs, it immediately follows that a greatest common subgraph $T$ of two trees $T_1$ and $T_2$ is connected if and only if $T$ is a greatest common induced subgraph of $T_1$ and $T_2$. Next we have an example of two trees $T_1$ and $T_2$ such that $T_1$ and $T_2$ have two greatest common distance-preserving subgraphs.

Consider the trees $T_1$ and $T_2$ shown in Figure 4.7. Now the trees $F_1$ and $F_2$ shown in Figure 4.7 are the only distance-preserving subgraphs of size 5 of $T_1$ and $T_2$.

Thus the gds size is at least 5. The graphs $J_1$, $J_2$, and $J_3$ of Figure 4.8 are all the connected subgraphs of size 6 in $T_1$. Since none of these graphs is a subgraph of $T_2$, it follows that the gds size is at most 5. Hence, $\text{gds}(T_1, T_2) = \{F_1, F_2\}$. 

![Figure 4.7 Trees With Two Greatest Common Distance-Preserving Subgraphs.](image-url)
In [10], it was shown that for every positive integer \( n \), there exist graphs \( G_n \) and \( H_n \) such that \( G_n \) and \( H_n \) have \( n \) distinct greatest common subgraphs. We now show that there exist pairs of graphs having an arbitrarily large number of greatest common distance-preserving subgraphs. In fact, we show that for every positive integer \( n \geq 2 \), there exist graphs \( G_1 \) and \( G_2 \) having \( n \) greatest common distance-preserving subgraphs, each of which is a tree of size \( 2n + 2 \).

![Figure 4.8 Connected Subgraphs of Size 6 in \( T_1 \).](image)

**Theorem 4.8** For every positive integer \( n \geq 2 \), there exist graphs \( G_1 \) and \( G_2 \) such that \( |\text{gds}(G_1, G_2)| = n \).

**Proof** Let \( n \geq 2 \) be a positive integer and let \( P \) be a path of length \( 2n + 1 \), say \( P: v_1, v_2, \ldots, v_{2n+2} \). Next let \( G_1 \) be obtained from \( P \) by adding the vertices \( x_1, x_2, \ldots, x_n \) and the edges \( x_i v_{i+1} \) for \( i = 1, 2, \ldots, n \). Let \( G_2 \) be obtained from \( P \) by adding the vertices \( y_1, y_2, \ldots, y_n \) and \( w_1, w_2, \ldots, w_{n-2} \) and the edges \( y_i v_{i+1} \) for \( i \) odd \((1 \leq i \leq n)\), \( y_i v_{2n+2-i} \) for \( i \) even \((1 \leq i \leq n)\), and \( y_i w_i, w_{i+2} \) for \( 1 \leq i \leq n - 2 \). An example of the graphs \( G_1 \) and \( G_2 \) is shown in Figure 4.9, where \( n = 5 \).

For \( i = 1, 2, \ldots, n \), let \( H_i \) be that graph obtained from \( P \) by joining a new vertex \( z_i \) to \( v_{i+1} \). Then each \( H_i \) is a distance-preserving subgraph of \( G_1 \) and \( G_2 \), and hence the gds size of \( G_1 \) and \( G_2 \) is at least \( 2n + 2 \). Let \( H \) be a common subgraph of \( G_1 \) and \( G_2 \). Since \( H \) is a subgraph of \( G_1 \), it follows that \( H \) has the general form shown in Figure 4.10. We may also assume, without loss of generality,
that the end-vertices of a longest path in $H$ are labeled $v_i$ and $v_j$ for some $1 \leq i < j \leq n$, and that $H$ contains $k$ ($0 \leq k \leq n$) of the vertices $x_1, x_2, \ldots, x_n$.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figures/fig4-9.png}
\caption{The Graphs $G_1$ and $G_2$ When $n = 5$.}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=0.8\textwidth]{figures/fig4-10.png}
\caption{A Generic Subgraph of $G_1$.}
\end{figure}

Since $H$ is a subgraph of $G_2$ and $i_k + 1 \leq n + 1$, it follows that $(i_m + 1) - (i_i + 1)$ is even for each $i$ and $m$ with $1 \leq i < m \leq k$. Note also that if we label the vertices of $H$ as a subgraph of $G_2$, then each vertex $x_{i_j}$ is labeled $y_{i_j}$. Next for each $i$ ($1 \leq i < n$) and $m$ ($1 \leq m \leq \lfloor n/2 \rfloor$ with $i + 2m \leq n$), we have that $d_{G_1}(x_i, x_{i+2m}) = 2m + 2$ while $d_{G_2}(y_i, y_{i+2m}) = 2m$. Thus if $H$ is a distance-preserving subgraph of $G_1$ and $G_2$, it must be that $k = 0$ or $k = 1$. If $H$ is a greatest common
distance-preserving subgraph of $G_1$ and $G_2$, then $H$ has size at least $2n + 2$ and contains at most one of the vertices $x_1, x_2, \ldots, x_n$. Hence $H \equiv H_i$ for some $i$ with $1 \leq i \leq n$. □

Before continuing, we require additional terminology and notation. For a graph $G$, the automorphism group $\text{Aut}(G)$ is the group of automorphisms of $G$. Each automorphism $\alpha$ of $\text{Aut}(G)$ permutes the vertices of $G$. In fact, the group $\text{Aut}(G)$ partitions $V(G)$ into orbits where vertices $x$ and $y$ of $G$ belong to the same orbit if and only if there exists an automorphism $\alpha$ of $\text{Aut}(G)$ such that $\alpha(x) = y$. For positive integers $m$ and $n$, the double star $S_{m,n}$ is that tree of order $m + n$ with exactly two vertices that are not end-vertices, one of degree $m$ and one of degree $n$. For a connected graph $G$ with a subgraph $H$, the distance from a vertex $x$ of $G$ to $H$, denoted by $d(x, H)$, is the length of a shortest path from $x$ to a vertex of $H$, i.e., $d(x, H) = \min\{d(x, y) \mid y \in V(H)\}$. So $d(x, H) = 0$ if and only if $x$ is a vertex of $H$.

We have seen that there exist two trees having exactly two distinct greatest common distance-preserving subgraphs. Chartrand, Saba, and Zou showed in [10] that for every two graphs $G_1$ and $G_2$ of equal size, $\{K_1,6, K_4\}$ is not the set of greatest common subraphs of $G_1$ and $G_2$. We show that if $G_1$ and $G_2$ are two trees with $K_{1,r} \in \text{gds}(G_1, G_2)$ for some positive integer $r$, then $\text{gds}(G_1, G_2) = \{K_{1,r}\}$. We begin by giving conditions under which there exist graphs $G_1$ and $G_2$ having at least two greatest common distance-preserving subgraphs, one of which contains exactly two orbits. Observe that if $T$ is a tree having exactly two orbits, then one orbit contains the end-vertices of $T$ while the other orbit contains all the other vertices, necessarily all of the same degree. Thus $T = S_{1,r}$ or $T = S_{m,m}$ for positive integers $m$ and $r$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Theorem 4.9 Let \( G_1 \) and \( G_2 \) be two trees having at least two greatest common distance-preserving subgraphs \( T_1 \) and \( T_2 \). If \( T_1 \) has exactly two orbits, then \( T_1 \) and \( T_2 \) are double stars, with \( T_1 \equiv S_{m,m} \) for some positive integer \( m \).

Proof Suppose that the tree \( T_1 \) has order \( p \) and contains exactly the two orbits \( O_1 \) and \( O_2 \). Then one of these orbits contains the end-vertices of \( T_1 \) and the other orbit contains all other vertices, say \( O_2 \) is the orbit containing the end-vertices of \( T_1 \). Next let \( w_1 \in V(G_1) - V(T_1) \) such that \( d_{G_1}(w_1, T_1) = 1 \) and let \( w_2 \in V(G_2) - V(T_1) \) such that \( d_{G_2}(w_2, T_1) = 1 \). Since \( G_1 \) and \( G_2 \) are trees, it follows that each of \( w_1 \) and \( w_2 \) is adjacent to a unique vertex of \( T_1 \) in \( G_1 \) or \( G_2 \), respectively. First observe that \( w_1 \) and \( w_2 \) cannot be adjacent to vertices of \( T_1 \) belonging to the same orbit of \( T_1 \); for otherwise \( (V(T_1) \cup \{w_1\}) \) is a subgraph of \( G_1 \) isomorphic to the subgraph \( (V(T_1) \cup \{w_2\}) \) of \( G_2 \), contradicting the fact that the gds size is \( p - 1 \).

Thus we may assume that \( w_1 \) and \( w_2 \) are adjacent to vertices of \( T_1 \) belonging to distinct orbits. Assume, without loss of generality, that \( w_1 \) is adjacent to a vertex \( v \) of \( O_1 \) and that \( w_2 \) is adjacent to some vertex, say \( v_1 \), of \( O_2 \). Also, we may assume that each vertex \( x \) of \( G_1 \) with \( d_{G_1}(x, T_1) = 1 \) is adjacent to a vertex of \( O_1 \) and that each vertex \( y \) of \( G_2 \) with \( d_{G_2}(y, T_1) = 1 \) is adjacent to a vertex of \( O_2 \), for otherwise we are in the situation described above. Among all the vertices of \( G_1 \) at distance 1 from \( T_1 \), choose \( w_1 \) so that \( \deg w_1 \) is maximum.

Suppose first that \( \deg w_1 \geq 2 \). Let \( w \in V(G_1) - V(T_1) \) be adjacent to \( w_1 \). Then there exists \( v_2 \in O_2 \) such that the subgraph \( T = (V(T_1 - v_2) \cup \{w_1, w\}) \) of \( G_1 \) is isomorphic to the subgraph \( (V(T_1) \cup \{w_2\}) \) of \( G_2 \). So \( T \) is a distance-preserving subgraph of \( G_1 \) and \( G_2 \) having size \( p \), producing a contradiction. Therefore \( \deg w_1 = 1 \) and all vertices in \( V(G_1) - V(T_1) \) are end-vertices, adjacent only to vertices of \( O_1 \). If \( |O_1| = 1 \), then \( \text{diam } G_1 = 2 \) and hence \( G_1 \) is a star.
Thus each of $T_1$ and $T_2$ is a star. Since $T_1$ and $T_2$ have the same size, we have that $T_1 \equiv T_2$, producing a contradiction. If $|O_1| = 2$, then diam $T_1 = 3$ and hence $T_1 \equiv S_{m,m}$ for some positive integer $m$ with $2m = p$. Thus since all vertices of $G_1$ not belonging to $T_1$ are end-vertices, we have that $G_1 \equiv S_{k,\ell}$ for positive integers $k$ and $\ell$, with $k + \ell > p$. Since $T_2$ is a subgraph of $G_1$, it must be that $T_2 \equiv S_{r,t}$ for positive integers $r$ and $t$ with $r + t = p$. □

As the next example shows, Theorem 4.9 is best possible in the sense that there exist two graphs $G_1$ and $G_2$ such that $\text{gds}(G_1, G_2) = \{S_{m,m}, S_{k,\ell}\}$, where $k$, $\ell$ and $m$ are distinct positive integers such that $k + \ell = 2m$. Let $k$, $\ell$, and $m$ be positive integers such that $k + \ell = 2m$. Without loss of generality, we may assume that $k \leq m \leq \ell$. Let $G_1 \equiv S_{m,\ell}$ and let $G_2$ be obtained from $S_{m,m}$ and $S_{k,\ell}$ by identifying an end-vertex of $S_{m,m}$ with an end-vertex of $S_{k,\ell}$. This situation is shown in Figure 4.11 for $k = 3$, $\ell = 5$, and $m = 4$. Now all connected subgraphs of $G_1$ are double stars and hence since $S_{m,m}$ and $S_{k,\ell}$ are the double stars of $G_2$ of maximum size, it follows that $\text{gds}(G_1, G_2) = \{S_{m,m}, S_{k,\ell}\}$.

Next, we have the following corollary.

**Corollary 4.10** There do not exist two trees $G_1$ and $G_2$ such that $G_1$ and $G_2$ have at least two greatest common distance-preserving subgraphs $T_1$ and $T_2$ with $T_1 \equiv K_{1,r}$ for some positive integer $r$.

**Proof** Suppose, to the contrary, that there exist two trees $G_1$ and $G_2$ such that $\{T_1, T_2\} \subseteq \text{gds}(G_1, G_2)$ and $T_1 \equiv K_{1,r}$ for some positive integer $r$. Since $T_1$ and $T_2$ are distinct trees, it follows that $r \geq 3$. Then $T_1$ has two orbits and hence by
Theorem 4.9, it must be that $T_1 \equiv S_{m,m}$ for some positive integer $m$, contradicting the fact that $T_1 \equiv K_{1,r}$. 

As a corollary to Corollary 4.10, we have the following.

**Corollary 4.11** If there exist trees $G_1$ and $G_2$ with $K_{1,r} \in \text{gds}(G_1, G_2)$ for some positive integer $r$, then $\text{gds}(G_1, G_2) = \{K_{1,r}\}$.
CHAPTER V

GREATEST COMMON LOCALLY-PRESERVING SUBGRAPHS

5.1 Introduction

In Chapter IV we defined a subgraph $H$ of a graph $G$ to be distance-preserving if $d_H(u, v) = d_G(u, v)$ for every pair $u, v$ of vertices of $H$. Also in Chapter IV, we defined a greatest common distance-preserving subgraph of two connected graphs $G_1$ and $G_2$ as a graph of maximum size that is a distance-preserving subgraph of $G_1$ and $G_2$. This definition clearly implies that every distance-preserving subgraph is connected. We now turn our attention to another type of distance-preserving subgraph of two graphs. For a subgraph $H$ without isolated vertices of a graph $G$, we say that $H$ is locally distance-preserving or simply locally-preserving if $d_H(u, v) = d_G(u, v)$ for every pair $u, v$ of vertices belonging to the same component of $H$. So a distance-preserving subgraph is locally-preserving, but not conversely. A greatest common locally-preserving subgraph of two graphs $G_1$ and $G_2$ is a graph $H$, without isolated vertices, of maximum size such that $H$ is a locally-preserving subgraph of $G_1$ and $G_2$. The size of a greatest common locally-preserving subgraph is called the gls size and the set of all such subgraphs is denoted by $\text{gls}(G_1, G_2)$.

To illustrate these concepts, consider the greatest common locally-preserving subgraphs of $P_n$ and $C_n$. Although $P_n$ is a subgraph of $C_n$, the path $P_n$ is not a locally-preserving subgraph since $\text{diam} P_n = n - 1$ while $\text{diam} C_n = \lfloor n/2 \rfloor$. Thus the gls size is at most $n - 2$. Now if $n$ is even, say $n = 2k$ for a positive integer $k$, then $\text{gls}(P_{2k}, C_{2k}) = \{P_{k+1} \cup P_{k-1}, P_k \cup P_k\}$; while if $n$ is odd, say $n = 2k + 1$, then...
gls\(P_{2k+1}, C_{2k+1}\) = \(P_{k+1} \cup P_k\). In Chapter IV we saw that \(P_{\lfloor n/2 \rfloor + 1}\) is the unique greatest common distance-preserving subgraph of \(P_n\) and \(C_n\). Thus a greatest common locally-preserving subgraph can be different from a greatest common distance-preserving subgraph.

Recall that distance-preserving subgraphs are induced subgraphs. A similar statement is true for locally-preserving subgraphs.

**Lemma 5.1** Let \(H\) be a locally-preserving subgraph of a graph \(G\). Then each component of \(H\) is an induced subgraph of \(G\).

**Proof** Let \(H_1, H_2, \ldots, H_n (n \geq 1)\) denote the components of \(H\). We show that in \(G\), the subgraph \(\langle V(H_1) \rangle\) is in fact \(H_1\). Suppose, to the contrary, that there exists an edge \(e\) of \(\langle V(H_1) \rangle\) such that \(e\) is not an edge of \(H_1\). Then \(e = uv\) for some \(u, v \in V(H_1)\). Thus \(d_G(u, v) = 1\) while \(d_H(u, v) \geq 2\) and since \(u\) and \(v\) belong to the same component of \(H\), we have a contradiction. Hence \(\langle V(H_1) \rangle = H_1\) and \(H_1\) is an induced subgraph of \(G\). Similarly, \(H_2, H_3, \ldots, H_n\) are induced subgraphs of \(G\). \(\Box\)

Although the components of a greatest common locally-preserving subgraph are induced subgraphs, no such component is necessarily a greatest common distance-preserving subgraph, as the next example shows.

Consider the graphs \(G_1\) and \(G_2\) of size 8, shown in Figure 5.1. Since \(G_1\) is not an induced subgraph of \(G_2\), it follows that the gls size is at most 7. Next since \(C_3 \cup C_4\) is a locally-preserving subgraph of \(G_1\) and \(G_2\), we have that the gls size is 7. Since the edge \(e\) is the only bridge of \(G_1\), it follows that any subgraph of size 7 different from \(C_3 \cup C_4\) is connected. Since each component of a locally-preserving subgraph must be an induced subgraph of \(G_1\), it follows that \(gls(G_1, G_2) = (C_3 \cup \ldots \).
The graph $G$, shown in Figure 5.1, is a distance-preserving subgraph of $G_1$ and $G_2$; so the gds size is at least 6. Since $\delta(G_1) = 2$ and greatest common distance-preserving subgraphs are induced subgraphs, it follows that the gds size is at most 6. Now removing any vertex of degree 2 from $G_1$ other than $x$ or $y$ does not produce an induced subgraph of $G_2$ and hence $\text{gds}(G_1, G_2) = \{G\}$. Since $G$ is not a subgraph of $C_3 \cup C_4$, we have that a greatest common distance-preserving subgraph need not be a subgraph of a greatest common locally-preserving subgraph.

![Graphs](image)

**Figure 5.1** Graphs Where the Greatest Common Distance-Preserving Subgraph Is Not a Subgraph of the Greatest Common Locally-Preserving Subgraph.

Since every distance-preserving subgraph is also a locally-preserving subgraph, the gds size is at most the gls size. Next since distance-preserving subgraphs are induced subgraphs, it follows that the gds size is at most the size of a greatest common induced subgraph. For the graphs $G_1$ and $G_2$ of Figure 5.1, we saw that the gds size is 6 while the gls size is 7. Since $\delta(G_1) = 2$, the graph $G_1$ has size 8, and the gds size is 6, it follows that the size of a greatest common induced subgraph is 6. Thus it is possible for the gls size to be larger than the size of a greatest...
common induced subgraph. It is also possible for the gls size to be smaller than the size of a greatest common induced subgraph, as the next example shows.

Consider the graphs $G_1$ and $G_2$ of Figure 5.2. Since $\delta(G_1) = 2$ and $G_1$ has size 13, it follows that the size of a greatest common induced subgraph of $G_1$ and $G_2$ is at most 11. Next since $G_1 - w_2 = G_2 - x_2 - y_3$, we have that the size of a greatest common induced subgraph is 11. Now $G_1$ is 2-edge-connected, so the removal of a single edge from $G_1$ does not produce a subgraph in which each component is an induced subgraph of $G_1$. Hence the gls size is at most 11. Next since diam $G_1 = 4$ and $C_8$ is the largest connected subgraph of $G_2$ with diameter at most 4, it follows that the gds size is 8. Therefore, the size of the largest component of a greatest common locally-preserving subgraph of $G_1$ and $G_2$ is at most 8. Also, $C_8 \cup 2K_2$ is a locally-preserving subgraph of $G_1$ and $G_2$ and hence the gls size is at least 10.

We now show the gls size is exactly 10. Suppose, to the contrary, that the gls size is 11 and let $H \in \text{gls}(G_1, G_2)$. Since $G_1$ is 2-connected and $\Delta(G_2) = 3$, it follows that the removal of any vertex of $G_1$ gives a connected subgraph of size at least 10 and since the gds size is 8, this subgraph is not locally-preserving. So, we must remove two nonadjacent edges from $G_1$ to produce $H$. Furthermore, the removal of these two edges from $G_1$ must result in a disconnected graph. Since $G_2$ has no 9-cycle, the removal of these edges must destroy the 9-cycles of $G_1$. Thus we must remove two nonadjacent edges of $uw_1, w_1w_2, w_2w_3, w_3w_4, w_4v$. Hence the 8-cycle $u, u_1, u_2, u_3, v, v_1, v_2, v_3, u$ of $G_1$ is an 8-cycle of $H$. We have already noted that the size of the largest component of $H$ is at most 8 and hence $H = G_1 - uw_1 - vw_4$. Since $G_2$ has exactly one 8-cycle, it follows that $H = G_2 - ux_1 - vy_1$. But $G_1 - uw_1 - vw_4 \neq G_2 - ux_1 - vy_1$, producing a contradiction. Therefore, the gls size is 10 while the size of a greatest common induced subgraph is 11.
We have already seen that a greatest common distance-preserving subgraph need not be a subgraph of a greatest common locally-preserving subgraph. In fact, the components of a greatest common locally-preserving subgraph need not be subgraphs of a greatest common distance-preserving subgraph. Of course, if $K_2$ or $P_3$ is a component of a greatest common locally-preserving subgraph, then trivially these components are subgraphs of a greatest common distance-preserving subgraph.

Consider the graphs $G_1$ and $G_2$ of Figure 5.3. First, we determine $\text{gls}(G_1, G_2)$. Let $H \in \text{gls}(G_1, G_2)$. Since $G_2$ has no triangles, it follows that $u_3$, $u_4$, and $u_5$ cannot belong to the same component of $H$. Hence at least two of the three edges $u_3u_4$, $u_4u_5$, $u_5u_3$ are not present in $H$. Therefore, $H$ is a subgraph of at least one of the graphs $H_1$, $H_2$, and $H_3$ of size 7, shown in Figure 5.4, where $H_1 = G_2 - u_3u_5 - u_4u_5$, $H_2 = G_2 - u_3u_5 - u_3u_4$, and $H_3 = G_2 - u_3u_4 - u_4u_5$. Since the diameter of each of $H_1$, $H_2$, and $H_3$ is at least 4, and since $\text{diam} G_2 = 3$, it follows that $H$ is a proper subgraph of at least one of $H_1$, $H_2$, and $H_3$ and hence $H$ has size at most 6. In fact, since the graph $2P_4$ is not a locally-preserving subgraph of $G_2$, we have that
$H_1 - u_1u_6$ and $H_1 - u_1u_2$ are the only locally-preserving subgraphs of size 6 and hence  
$\text{gls}(G_1, G_2) = \{H_1 - u_1u_6, H_1 - u_1u_2\}$.

Figure 5.3 Graphs Where Each Component of a Greatest Common Locally-Preserving Subgraph Is Not a Subgraph of Some Greatest Common Distance-Preserving Subgraph.

Figure 5.4 Graphs Obtained From $G_1$ by Removing Two of the Edges $u_3u_4, u_4u_5, u_5u_3$.

Since each greatest common locally-preserving subgraph of $G_1$ and $G_2$ is disconnected, the gds size is at most 5. Also, since $C_5$ is a distance-preserving
subgraph of \( G_1 \) and \( G_2 \), it follows that the gds size is 5. Let \( G \in \text{gds}(G_1, G_2) \).

Then the diameter of \( G \) is at most 3. Since \( G_2 \) has no triangles, it follows that \( G \) has no triangles. Therefore \( G \) is a proper subgraph of at least one of \( H_1, H_2, \) and \( H_3 \). Since all connected subgraphs of \( H_1 \) and \( H_2 \) with diameter at most 4 have size at most 4, it follows that \( G \) is a subgraph of \( H_3 \); in fact, \( G \) is an induced subgraph of the nontrivial component of \( H_3 \). Hence \( G \) is obtained from this nontrivial component by removing a vertex of degree 2 or two vertices of degree 1. Each of \( H_3 - u_1 - u_4, H_3 - u_2 - u_4, \) and \( H_3 - u_5 - u_4 \) has diameter at least 4 and hence \( G = H_3 - u_4 - u_7 - u_8 \). Thus \( \text{gds}(G_1, G_2) = \{C_5\} \), so each component of a greatest common locally-preserving subgraph of \( G_1 \) and \( G_2 \) need not be a subgraph of some greatest common distance-preserving subgraph of \( G_1 \) and \( G_2 \).

5.2 Properties of Greatest Common Locally-Preserving Subgraphs

We begin by showing that a greatest common locally-preserving subgraph can have an arbitrarily large number of components.

**Theorem 5.2** For every integer \( n \geq 2 \), there exist graphs \( G_1 \) and \( G_2 \) of equal size such that a greatest common locally-preserving subgraph of \( G_1 \) and \( G_2 \) has \( n \) components.

**Proof** Let \( n \geq 2 \) be an integer. Next let \( F_1 = K_{1,n} \) where the vertices of \( F_1 \) are labeled \( u, u_1, u_2, \ldots, u_n \) and \( \deg u = n \), and let \( F_2 = K_{1,n} \) where the vertices of \( F_2 \) are labeled \( v, v_1, v_2, \ldots, v_n \) and \( \deg v = n \). We construct \( G_1 \) from \( F_1 \cup F_2 \) by adding the edges \( u_1v_1, u_2v_2, \ldots, u_nv_n \). Let \( G_2 \) be a path of length \( 3n \). The graphs \( G_1 \) and \( G_2 \) are shown in Figure 5.5 when \( n = 3 \).
We show the gls size of $G_1$ and $G_2$ is $n + 2$. Suppose, to the contrary, that the gls size is at least $n + 3$ and let $H \in \text{gls}(G_1, G_2)$. Since $H$ is a subgraph of $G_2$, it follows that every component of $H$ is a path of length at most $\text{diam} G_1 = 3$. Next, if $u$ and $v$ are not vertices of $H$, then the size of $H$ is at most $n$ and hence $u$ or $v$ is a vertex of $H$, say $u$. Now $\deg_H u \leq 2$; so if $v$ is not a vertex of $H$, then the size of $H$ is at most $n + 2$, producing a contradiction. Thus both $u$ and $v$ are vertices of $H$. Furthermore, since $H$ has size $n + 3$, we have that $3 \leq \deg_H u + \deg_H v \leq 4$, say, without loss of generality, that $\deg_H u = 2$.

Suppose first that $u$ and $v$ do not belong to the same component of $H$, say $u$ belongs to component $H_1$ of $H$ and $v$ belongs to component $H_2$ of $H$. Next suppose that $\deg_H u = 2$ and $\deg_H v = 1$, say $u_i$ and $u_j$ ($1 \leq i, j \leq n$) are adjacent to $u$ in $H$. Since $\text{diam} H_1 \leq 3$, it follows that at most one of the edges $u_i v_i$ and $u_j v_j$ is an edge of $H$. Thus at most $n - 1$ of the edges $u_1 v_1, u_2 v_2, \ldots, u_n v_n$ are present in $H$ so that $H$ has size at most $n + 2$, producing a contradiction. Now suppose that $\deg_H u = \deg_H v = 2$, say $u_i$ and $u_j$ ($1 \leq i, j \leq n$) are adjacent to $u$ in $H$ and $v_k$ and $v_\ell$ ($1 \leq k, \ell \leq n$) are adjacent to $v$ in $H$, where at least two of $i, j, k, \ell$ are distinct. Then at most two of the edges $u_i v_i, u_j v_j, u_k v_k, u_\ell v_\ell$ are edges of $H$. 

![Figure 5.5 Graphs Where a Greatest Common Locally-Preserving Subgraph Has Three Components.](image-url)
Therefore at most \( n - 2 \) of the edges \( u_1v_1, u_2v_2, \ldots, u_nv_n \) are present in \( H \) and again the size of \( H \) is at most \( n + 2 \), producing a contradiction. Finally suppose that \( u \) and \( v \) belong to the same component of \( H \), say \( u \) and \( v \) belong to component \( H_1 \) of \( H \). Then since \( \deg_H u + \deg_H v \geq 3 \), it follows that the size of \( H_1 \) must be at least 4. Since \( H_1 \) is a path, we have that \( \text{diam} H_1 \geq 4 \), producing a contradiction. Thus the gls size is at most \( n + 2 \). Since \( P_4 \cup (n - 1)K_2 \) is a locally-preserving subgraph of \( G_1 \) and \( G_2 \), we have that the gls size is \( n + 2 \) and \( P_4 \cup (n - 1)K_2 \in \text{gls}(G_1, G_2) \). Also, \( P_4 \cup (n - 1)K_2 \) has \( n \) components and hence a greatest common locally-preserving subgraph of \( G_1 \) and \( G_2 \) has \( n \) components. □

We now show that the gls size can be arbitrarily larger than the gds size.

**Theorem 5.3** For every positive integer \( n \), there exist graphs \( G_1 \) and \( G_2 \) of equal size such that the difference between the gls size and the gds size is \( n \).

**Proof** For a positive integer \( n \), let \( G_1 \) be constructed from the \((2n + 2)\)-cycle \( v_1, v_2, \ldots, v_{2n+2}, v_1 \) by adding the vertices \( w_1, w_2, \ldots, w_{n+1} \) and the edges \( v_1w_1, w_1w_2, \ldots, w_{n+1}v_{n+2} \) and let \( G_2 \) be obtained from the \((2n + 2)\)-cycle by adding the vertices \( x_1, x_2, \ldots, x_{n+2} \) and the edges \( v_1x_1, x_1x_2, x_2x_3, \ldots, x_{n+1}x_{n+2} \).

The graphs \( G_1 \) and \( G_2 \) of size \( 3n + 4 \) are the graphs constructed in the proof of Theorem 2.2 (see also Figure 2.5) where it was shown that the gds size of \( G_1 \) and \( G_2 \) is \( 2n + 2 \).

Thus, it remains to determine the gls size of \( G_1 \) and \( G_2 \). Observe that \( G_1 \) has no bridges and hence the removal of a single edge from \( G_1 \) does not produce a subgraph where each component is an induced subgraph of \( G_1 \). Therefore the gls size is at most \( 3n + 2 \). Next, \( H = G_1 - v_1w_1 - v_{n+2}w_{n+1} = G_2 - x_{n+1} - x_{n+2} - v_1x_1 \) is a
locally-preserving subgraph of $G_1$ and $G_2$ and hence the gls size is $3n + 2$. Thus
the difference between the gls size and the gds size is $(3n + 2) - (2n + 2) = n$. \Box

We have already seen that a greatest common distance-preserving subgraph need not be a component of a greatest common locally-preserving subgraph and that the gls size can be arbitrarily larger than the gds size. In fact, we now show that the gds size can be arbitrarily larger than the size of a largest component of a greatest common locally-preserving subgraph.

**Theorem 5.4** For every integer $n \geq 2$, there exist graphs $G_1$ and $G_2$ of equal size such that the difference between the gds size of $G_1$ and $G_2$ and the size of the largest component of a greatest common locally-preserving subgraph of $G_1$ and $G_2$ is $n$.

**Proof** Let $n \geq 2$ be an integer. Next let $P$ be a path of length $n$, say $P: v_1, v_2, \ldots, v_{n+1}$, and let $C$ be a cycle of length $n + 1$, say $C: u_1, u_2, \ldots, u_{n+1}, u_1$. The graph $G_1$ is constructed from $C \cup P$ by adding the vertices $w_1, w_2$ and the edges $u_1v_1, w_1v_n, w_2v_n$; while the graph $G_2$ is constructed from $C \cup P$ by adding the vertices $z_1, z_2$ and the edges $u_1v_1, z_1v_2, z_2v_2$. The graphs $G_1$ and $G_2$ are shown in Figure 5.6 when $n = 3$. First, we determine the gds size. Now $(C \cup P) + u_1v_1$ is a distance-preserving subgraph of $G_1$ and $G_2$ and hence the gds size is at least $2n + 2$. We show that the gds size is $2n + 2$. Suppose, the contrary, that the gds size is $2n + 3$ and let $H \in \text{gds}(G_1, G_2)$. Then since $G_1$ has size $2n + 4$ and $H$ is an induced subgraph of $G_1$, it must be that $H$ is obtained from $G_1$ by removing an end-vertex. Hence assume, without loss of generality, that $H = G_1 - w_1$. Then $H$ is not a subgraph of $G_2$, producing a contradiction. Therefore the gds size is $2n + 2$. 

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Finally, we determine $\text{gls}(G_1, G_2)$. Now $G_1 - u_1v_1 = G_2 - u_1v_1$, and thus the gls size is $2n + 3$ and $G_1 - u_1v_1 \in \text{gls}(G_1, G_2)$. In fact, $G_1 - u_1v_1$ consists of two components, with sizes $n + 1$ and $n + 2$. Let $F \in \text{gls}(G_1, G_2)$. Then $F$ must be obtained from $G_1$ by the removal of a single edge. Furthermore, since the gds size is $2n + 2$, this edge must be a bridge. So, $F = G_1 - e$, where $e$ is one of the edges $u_1v_1, v_1v_2, v_2v_3, \ldots, v_nv_{n+1}, w_1v_n, w_2v_n$. Since none of the graphs $G_1 - v_1v_2, G_1 - v_2v_3, \ldots, G_1 - v_nv_{n+1}, G_1 - w_1v_n, G_1 - w_2v_n$ is a subgraph of $G_2$, it follows that $\text{gls}(G_1, G_2) = \{G_1 - u_1v_1\}$ and hence the difference between the gds size and the size of the largest component of the greatest common locally-preserving subgraph of $G_1$ and $G_2$ is $(2n + 2) - (n + 2) = n$. □

In Chapter IV, it was shown that every (connected) graph is the unique greatest common distance-preserving subgraph of two nonisomorphic connected graphs of equal size. We now show that every graph is a greatest common locally-preserving subgraph of two nonisomorphic graphs of equal size.

**Theorem 5.5** For every graph $G$, there exist graphs $G_1$ and $G_2$ of equal size such that $G \in \text{gls}(G_1, G_2)$. 

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Figure 5.6 The Graphs $G_1$ and $G_2$ When $n = 3$. 

![Graphs G1 and G2](image.png)
Proof Let $G$ be a graph of size $q$ and suppose first that $G$ is not vertex-transitive. Then there exist vertices $x$ and $y$ of $G$ such that for every automorphism $\phi$ of $G$, we have that $\phi(x) \neq y$ and $\phi(y) \neq x$. Obtain the graph $G_1$ from $G$ by joining a new vertex $v_1$ to $x$ and obtain the graph $G_2$ from $G$ by joining a new vertex $v_2$ to $y$. Clearly, $G_1 \neq G_2$, and $G_1$ and $G_2$ both have size $q + 1$. Therefore, the gls size is at most $q$. Hence, since $G$ is a locally-preserving subgraph of $G_1$ and $G_2$ of size $q$, it follows that $G \in \text{gls}(G_1, G_2)$.

Finally, suppose that $G$ is vertex-transitive. If $G$ is 1-regular, then $G = nK_2$ for some positive integer $n$. Now for $G_1 = (n - 1)K_2 \cup K_{1,3}$ and $G_2 = (n - 1)K_2 \cup C_3$, we have that $\text{gls}(G_1, G_2) = nK_2$. Next suppose that $G$ is $k$-regular (and vertex-transitive) where $k \geq 2$. Let $G_1$ be obtained from $G$ by joining two new vertices to any vertex of $G$ and let $G_2$ be obtained from $G$ by joining a new vertex to a pair of adjacent vertices. Then $G_1$ and $G_2$ both have size $q + 2$ and $G_1 \neq G_2$. Since $\delta(G_2) = 2$ and since $G_2$ has no bridges, it follows that the gls size of $G_1$ and $G_2$ is at most $q$. Next, since $G$ is a locally-preserving subgraph of $G_1$ and $G_2$, it follows that the gls size is $q$ and $G \in \text{gls}(G_1, G_2)$.

Next, we show that every graph with exactly two 2-edge-connected components is the unique greatest locally-preserving subgraph of two nonisomorphic connected graphs.

Theorem 5.6 For every two 2-edge-connected graphs $H_1$ and $H_2$, there exist graphs $G_1$ and $G_2$ such that $\text{gls}(G_1, G_2) = \{H_1 \cup H_2\}$.

Proof Let $q$ denote the size of $H = H_1 \cup H_2$. Suppose first that both $H_1$ and $H_2$ are vertex-transitive. Let $x_1$ and $x_2$ be two adjacent vertices of $H_1$, and let $y_1$ and $y_2$ be two adjacent vertices of $H_2$. Let $G_1 = H + x_1y_1 + x_2y_1$ and let $G_2 = H + \ldots$
Since $H_1$ and $H_2$ are 2-edge-connected, so too are $G_1$ and $G_2$. So the removal of a single edge from $G_1$ or $G_2$ does not produce an induced subgraph and therefore the gls size is at most $q$. Since $H$ is a locally-preserving subgraph of $G_1$ and $G_2$, it follows that the gls size is $q$ and that $H \in \text{gls}(G_1, G_2)$. We now show, in fact, that $\text{gls}(G_1, G_2) = \{H\}$. Let $F \in \text{gls}(G_1, G_2)$. Since $H_1$ is vertex-transitive, every vertex of $H_1$ lies on the same number, say $t$, of triangles. Similarly, every vertex of $H_2$ lies on the same number, say $s$, of triangles. Thus in $G_1$, the vertices $x_1$ and $x_2$ lie on $t + 1$ triangles and $y_1$ lies on $s + 1$ triangles, while every vertex of $G_2$ lies on $t$ or $s$ triangles. Therefore, to obtain $F$ from $G_1$, we must remove two edges $e_1$ and $e_2$, where each of $e_1$ and $e_2$ lies on a triangle with at least one of $x_1, x_2$, and $y_1$. Assume, without loss of generality, that $e_1$ and $x_1$ lie on a triangle and that $e_2$ and $y_1$ lie on a triangle. If $e_1 \neq x_1y_1$ and $e_2 \neq x_2y_1$, then $e_1$ must be an edge of $H_1$ and $e_2$ must be an edge of $H_2$. Since each of $H_1$ and $H_2$ is 2-edge-connected, it follows that $F$ is connected and hence $F$ is not an induced subgraph of $G_1$. Thus $e_1 = x_1y_1$ or $e_2 = x_2y_1$, say $e_1 = x_1y_1$. If $e_2 \neq x_2y_1$, then $F$ is connected and hence $e_2 = x_2y_1$. Therefore $F = H_1 \cup H_2$, so $\text{gls}(G_1, G_2) = \{H_1 \cup H_2\}$.

Finally, suppose that one of $H_1$ and $H_2$ is not vertex-transitive, say $H_1$. So there exist vertices $x_1$ and $x_2$ of $H_1$ that do not belong to the same orbit of $H_1$. Let $y$ be a vertex of $H_2$. Next let $G_1 = H + x_1y$ and $G_2 = H + x_2y$. Then $G_1 \neq G_2$ and $H$ is a locally-preserving subgraph of $G_1$ and $G_2$. Since each of $H_1$ and $H_2$ are 2-edge-connected, the removal of any single edge of $H_1$ or $H_2$ from $G_1$ leaves a connected subgraph that is not an induced subgraph of $G_1$. Hence $\text{gls}(G_1, G_2) = \{H_1 \cup H_2\}$. □
Next we show that for \( n \)-edge-connected graphs \( H_1, H_2, \ldots, H_n \) (\( n \geq 2 \)), at least one of which is not vertex-transitive, there exist graphs \( G_1 \) and \( G_2 \) such that \( H_1 \cup H_2 \cup \ldots \cup H_n \) is the greatest common locally-preserving subgraph of \( G_1 \) and \( G_2 \).

**Theorem 5.7** For \( n \)-edge-connected graphs \( H_1, H_2, \ldots, H_n \), where \( n \geq 2 \) is a positive integer and at least one \( H_i \) is not vertex-transitive, there exist graphs \( G_1 \) and \( G_2 \) such that \( \text{gls}(G_1, G_2) = \{H_1 \cup H_2 \cup \ldots \cup H_n\} \).

**Proof** Let \( H_1, H_2, \ldots, H_n \) be \( n \)-edge-connected graphs where \( H_1 \) is not vertex-transitive. Then there exist vertices \( x_1 \) and \( y_1 \) of \( H_1 \) such that \( x_1 \) and \( y_1 \) belong to different orbits. Let \( H = H_1 \cup H_2 \cup \ldots \cup H_n \) have size \( q \) and let \( x_i \) be a vertex of \( H_i \) for \( i = 2, \ldots, n \). Let \( G_1 = H + x_1x_2 + x_2x_3 + \ldots + x_{n-1}x_n \) and let \( G_2 = H + y_1x_2 + y_1x_3 + \ldots + y_1x_n \). We begin by showing that the gls size is \( q \). Let \( F \in \text{gls}(G_1, G_2) \). Since \( H \) is a locally-preserving subgraph of \( G_1 \) and \( G_2 \), it follows that \( F \) has size at least \( q \). Also, since each \( H_i \) is \( n \)-edge-connected and \( |V(H)| = |V(G_1)| = |V(G_2)| \), it follows that \( |V(F)| = |V(G_1)| \). Thus \( F \) is obtained from \( G_1 \) by removing the edges \( e_1, e_2, \ldots, e_k \) where \( k \leq n - 1 \). If any edge \( e_i \) belongs to \( H_j \) for some \( j \) (\( 1 \leq j \leq n \)), then since \( H_j \) is \( n \)-edge-connected, \( F \) has a component that is not an induced subgraph of \( G_1 \). Therefore, \( e_1, e_2, \ldots, e_k \) are not edges of \( H \).

Hence if \( F \) has size at least \( q + 1 \), then the graph \( J = H + y_1x_j \) for some \( j \) (\( 2 \leq j \leq n \)) is a subgraph of \( F \). But \( J \) is not a subgraph of \( G_1 \) and hence \( F \) has size \( q \).

Furthermore, it follows that \( k = n - 1 \) and \( F = H_1 \cup H_2 \cup \ldots \cup H_n \). \( \square \)
REFERENCES


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