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Isospectral Graphs and the Expander Coefficient

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ISOSPECTRAL GRAPHS AND THE EXPANDER COEFFICIENT

by

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ISOSPECTRAL GRAPHS AND THE EXPANDER COEFFICIENT

Ian Campbell Walters Jr., Ph.D.

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The expander coefficient of a graph is a parameter that is utilized to quantify the rate at which information is spread throughout a graph. The eigenvalues of the Laplacian of a graph provide a bound for the expander coefficient of the graph. In this dissertation, we construct many pairs of isospectral graphs with different expander coefficients.

In Chapter I, we define the problem and present some preliminary definitions. We then introduce two constructions that are related to graph composition and that may be employed to produce cospectral and isospectral graphs.

In Chapter II, we investigate the connectivity of and distance in the graphs formed by the constructions of Chapter I.

In Chapter III, we construct four pairs of infinite sequences of isospectral graphs. For each pair of infinite sequences, it is demonstrated that the expander coefficients of the corresponding graphs of the sequences are different. We determine the limit of each sequence of each pair, and see that these limits need not be equal.

In Chapter IV, the Folkman graph is used to construct a pair of isospectral graphs that more accurately model networks. The expander coefficients for the graphs are then shown to be unequal.

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Ian Campbell Walters Jr.

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CHAPTER I

INTRODUCTION

1.1 Definitions

Graphs have been used effectively to model discrete situations, such as people and friendship, intersections and roads, or information networks. When modeling a network, one might be curious as to how to quantify the rate at which information is spread throughout the network. One such measure is the expander coefficient. In some lecture notes, F.R.K. Chung gave one definition of the expander coefficient, and provided an eigenvalue bound for the value of the expander coefficient ([3]). In his paper in the *Notices of the American Mathematical Society* [1], Frederick Bien gave another definition of the expander coefficient, and also presented some results relating the spectrum of the Laplacian of a graph to the expander coefficient of the graph. He then suggested that there was no reason to believe the spectrum of the Laplacian of a graph should uniquely determine the expander coefficient of the graph, but had no examples to demonstrate that it did not. Actually, there are two versions of this question. Must cospectral graphs have the same expander coefficient? Bien gave an example of two graphs that are cospectral and have unequal expander coefficients. Must isospectral graphs have the same expander coefficient? He did not answer this question. We answer both of these questions negatively for the definition of the expander

coefficient given by Chung. We present two constructions that yield cospectral graphs, discuss some of the properties of the constructed graphs, and then employ the constructions to form isospectral expander graphs with unequal expander coefficients.

The *spectrum* of a graph G is the collection of eigenvalues (including multiplicities) of the adjacency matrix of the graph. Two graphs G and H are said to be *cospectral* if the spectra of the two graphs are identical. The *degree matrix* of a graph G, deg(G), is the diagonal matrix diag(deg(v_1),...deg(v_p)). Graphs G and H are said to be *isospectral* if the eigenvalues of deg(G) – A(G) and deg(H) – A(H) are identical. The matrix deg(G) – A(G) is called the *Laplacian* of the graph G. The graphs in Figure 1.1 are the smallest pair of connected cospectral graphs, and have as thier characteristic polynomial $(x - 1)^2(x + 1)(x^2 - x^2 - 5x + 1)$.

Figure 1.1

We wish to examine a pair of constructions that yield cospectral graphs. These constructions are generalizations of composition graphs. If G and H are two graphs, and the vertex set of G is $\{v_1,...v_p\}$, the *composition of G with H*, G[H], is the graph formed by associating one

copy of the graph H with each vertex of the graph G, and whenever $v_i v_j$ is an edge in G, each vertex of the i^{th} copy of H is joined to every vertex in the j^{th} copy of H. A copy of H is referred to as a *module*. The copy of the graph H that is associated with the vertex u of G is referred to as the *u-module*. For the graphs G and H in Figure 1.2, the graph G[H] is formed.

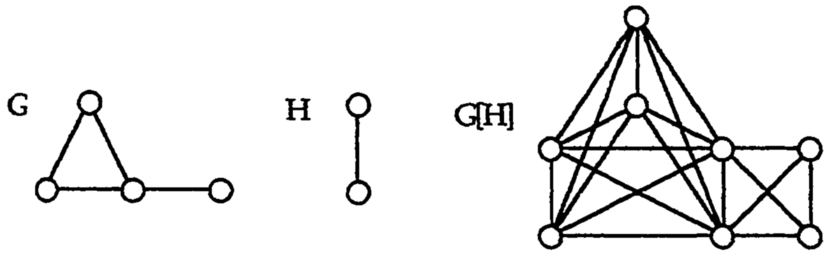


Figure 1.2

We define a new construction that generalizes composition graphs. Let G be a graph and let A and B be a partition of the vertices of G into two disjoint sets, and let H and K be two graphs. The *2-set composition of G with H and K*, denoted G(A,H;K), is the graph formed by associating one copy of the graph H with each vertex of the set A and one copy of the graph K with each vertex of the set B, and whenever the edge uv is present in the graph G, we join each vertex of the u-module to every vertex of the v-module. The set A uniquely determines the set B, which is why the set B is not used in the notation. When G is a bipartite graph, it is natural to select the sets that partition G to be the partite sets of G. This construction is illustrated in Figure 1.3.

The second construction, which is called *bipartite composition*, is similar to 2-set composition. Let G be a bipartite graph with partite sets G_1

and G_2, H be a bipartite graph with partite sets H_1 and H_2, and let K be a bipartite graph with partite sets K_1 and K_2. The bipartite composition of G with H and K, denoted $G^*(G_1,H,H_1;K,K_2)$, is the graph formed by associating a copy of the graph H with each vertex of G_1 and a copy of the graph K with each vertex of G_2. Whenever the edge uv is present in the graph G, with u from the partite G_1, we join each vertex of the partite set H_1 (respectively H_2) of the u-module with every vertex of the partite set K_2 (respectively K_1) of the v-module. The notation $G^*(G_1,H,H_1;K,K_2)$ identifies the partite set G_1 of G, the graph associated with the vertices of G_1, the graph associated with the vertices of G_2, and one of the pairs of the

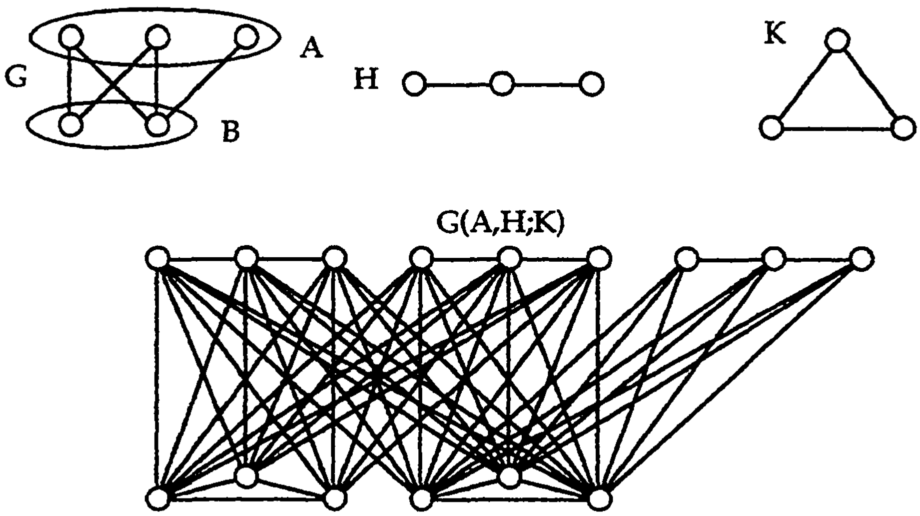


Figure 1.3

partite sets of H and K that are joined in the construction. The other information is then uniquely determined, and implied. A partite set of a module in $G^*(G_1,H,H_1;K,K_2)$ is referred to as a *half-module*. Clearly, $G^*(G_1,H,H_1;K,K_2)$ is a bipartite graph, with one partite set consisting of all

the vertices of the partite set H_1 of each H-module of $G^*(G_1,H,H_1;K,K_2)$ and all the vertices of the partite set K_1 of each K-module of $G^*(G_1,H,H_1;K,K_2)$, and the other consisting of all the vertices of the partite set H_2 of each H-module of $G^*(G_1,H,H_1;K,K_2)$ and all the vertices of the partite set K_2 of each K-module of $G^*(G_1,H,H_1;K,K_2)$. This construction is illustrated in Figure 1.4. In this figure, the vertices u, v, and w are repeated top and bottom, as if drawn on a cylinder.

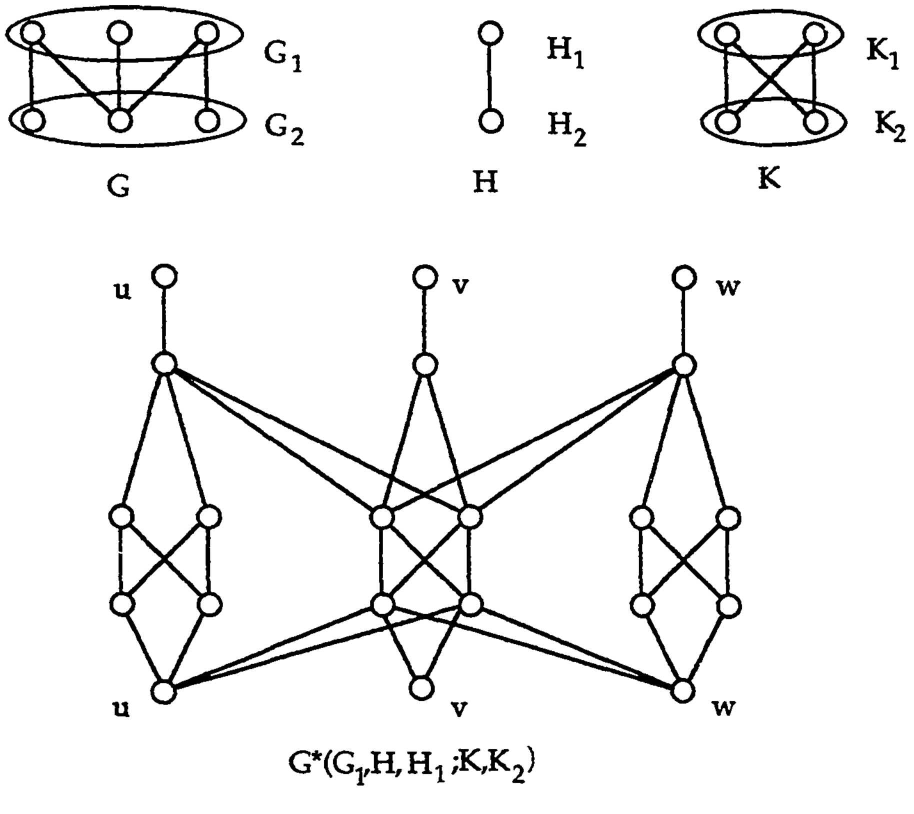


Figure 1.4

In different sources, and sometimes in the same source, there have

been different definitions of an expander graph and of the expander coefficient (see [1],[3],[6]). We define an (n,k,c)-expander graph to be a k-regular graph on n vertices such that for every subset S of the vertex set of G, V(G), the inequality

$$|B(S)| \geq \frac{c}{n}(n - |S|)\,|S|$$

is satisfied, and B(S), the *boundary* of the set S, is $\{v \in V(G)-S \mid uv \in E(G) \text{ and } u \in S\}$. The largest c for which the inequality is true for all subsets S of V(G) is the expander coefficient of G, that is, c = expan(G). We choose this definition because the formula is a discrete form of the differential equation

$$\frac{dy}{dx} = aN(P - N),$$

which represents the problem of an infection spreading through a population P. Solving for c in the inequality yields

$$c \leq \frac{n|B(S)|}{|S|(n-|S|)}.$$

If we let $Ex(S) = \frac{n|B(S)|}{|S|(n-|S|)}$, finding expan(G) is equivalent to finding the smallest value of Ex(S) over all subsets S of V(G). The first graph in Figure 1.5 demonstrates a (14,3,2/7) - expander graph. The second graph in Figure 1.5 shows a (20,4,20/21) - expander graph. In each figure, a set S that minimizes the value of Ex(S) is shown as consisting of striped vertices. For all other graph theory terms, see [2].

1.2 Preliminary Results

We wish to demonstrate that the compositions defined in Section 1.1 can be used to produce cospectral graphs. To do this, Newton's recurrence may be employed to evaluate the coefficients of the characteristic polynomial of the adjacency matrix A(G) of a graph from the moments of A(G) ([4],[6]). The k^{th} moment of A(G) may be viewed as the sum of the eigenvalues of $[A(G)]^k$, which, interpreted graphically, is the sum of the number of closed walks of length k in the graph G. So, if two graphs G and H have the same number of closed walks of length k for all values of k, then A(H) and A(G) have the same moments. It happens that this guarantees A(H) and A(G) also have the same characteristic polynomial.

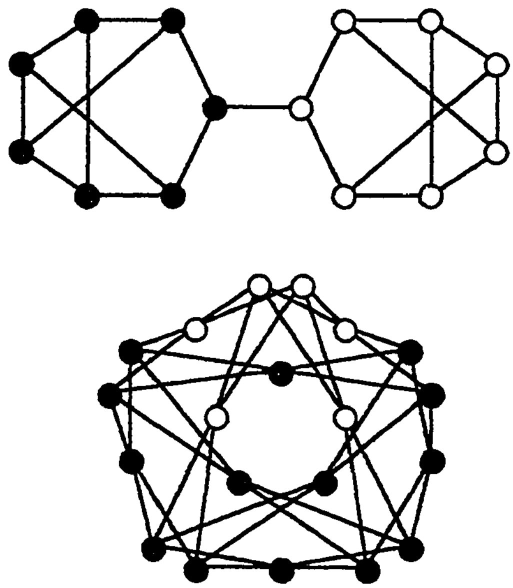


Figure 1.5

Therefore, the graphs G and H are cospectral if and only if the number of closed walks of length k, for all values of k, is the same for each graph G and H [6]. To determine if two graphs are cospectral , we only need to demonstrate that there is a one-to-one correspondence between the closed walks of each length of each of the graphs. We employ this process to demonstrate that the two constructions produce cospectral graphs, under certain loose restrictions.

Before we can prove the theorems concerning the cospectrality of the composition graphs, we must first identify what a closed walk looks like in the composition graphs. Let G be a bipartite graph with partite sets $A = \{a_1, a_2, \ldots a_n\}$ and $B = \{b_1, b_2, \ldots b_m\}$ and let H and K be two arbitrary graphs. The walks in the graph G(A,H;K) fall into two categories, those walks that lie entirely in one module, and those walks that visit more than one module. Let W be such a walk. Then W may be written

$$W = [a_0, X_0]\, e_0\, [b_1, Y_1]\, e_1 \ldots e_{q-1}\, [b_q, Y_q]\, e_q\,,$$

where X_i is a walk in the a_i-module, Y_j is a walk in the b_j-module and e_k is the edge joining the terminal vertex of the walk in the k^{th} module and the initial vertex of the $(k+1)^{st}$ module. We are now ready to prove the desired results.

Theorem 1.1 Let G be a bipartite graph with partite sets A and B, and let H and K be any two graphs. Then G(A,H;K) and G(A,K;H) are cospectral if

(i) $|A| = |B|$ or

(ii) H and K are cospectral.

Proof Let $G_1 = G(A,H;K)$ and $G_2 = G(A,K;H)$. For any integer k, we merely

present a one-to-one correspondence from the closed walks of length k in G_1 to the closed walks of length k in G_2. Let W be a walk of length k in the graph G_1. There are two cases.

Case 1. The walk W is entirely contained in a single module. Without loss of generality, let W be a walk in the H-module associated with the vertex u of G. If the graphs H and K are cospectral, then there is a walk corresponding to W in the graph K. In G_2, the u-module is a copy of K. So there is a walk corresponding to W in G_2, namely the walk in K that resides in the u-module. If $|A| = |B|$, then there is a one-to-one correspondence between the vertices of the partite set A of G and the partite set B of G. So there is a walk identical to W in the graph G_2 in the H-module associated with the vertex that corresponds to u in the partite set B. So there is a one-to-one correspondence between the closed walks of length k that are entirely contained in a single module in G_1 and those that are entirely contained in a single module of G_2.

Case 2. The walk W visits more than one module. Then, as we described earlier, the walk W may be written

$$W = [a_0,X_0]\, e_0\, [b_1,Y_1]\, e_1 \ldots e_{q-1}\, [b_q,Y_q]\, e_q\, .$$

Let φ be the mapping from the closed walks of length k that visit more than one module in G_1 to the closed walks of length k that visit more than one module in G_2 defined by

$$\varphi\,(W) = [a_0,Y_1]\, f_0\, [b_1,X_2]\, f_1 \ldots f_{q-1}\, [b_q,X_0]\, f_q\, .$$

This is a walk in G_2, because the A-modules are now copies of K and the

B-modules are copies of H. The edges f_j exist and are unique, since the vertices a_i and b_{i+1} are adjacent in G and, by the definition of G_2, the terminal vertex of the walk in H (respectively K) is adjacent to the initial vertex of the walk in K (respectively H). The image of W is found by shifting the walks X_i or Y_i one module left in the sequence. Also, any walk W' of G_2 that visits more than one module may be written

$$[a_0,Y_1]\, f_0\, [b_1,X_2]\, f_1 \dots f_{q-1}\, [b_q,X_0]\, f_q .$$

The walk in G_1

$$[a_0,X_0]\, e_0\, [b_1,Y_1]\, e_1 \dots e_{q-1}\, [b_q,Y_q]\, e_q$$

is the preimage of the walk W' in G_1. Thus, the map φ defined above is one-to-one and onto, giving the desired correspondence between the closed walks of length k that visit more than one module in G_1 and G_2.

Since there is a one-to-one correspondence from the closed walks of any length k in the graph G_1 to the walks of length k in the graph G_2, the graphs G_1 and G_2 are cospectral. ❑

Under some loose restrictions, a similar argument is valid in proving that the bipartite composition produces cospectral graphs.

Theorem 1.2 The graphs $G_1 = G^*(A,H,H_1;B,K,K_2)$ and $G_2 = G^*(A,K,K_1;B,H,H_2)$ are cospectral if

(i) $|A| = |B|$ or

(ii) H and K are cospectral.

Proof For any integer k, we merely present a one-to-one correspondence

from the closed walks of length k in G_1 to the closed walks of length k in G_2. Let W be a walk of length k in the graph G_1. There are two cases.

Case 1. The walk W is entirely contained in a single module. Without loss of generality, let W be a walk in the H-module associated with the vertex u of G. If the graphs H and K are cospectral, then there is a walk corresponding to W in the graph K. In G_2, the u-module is a copy of K. So there is a walk corresponding to W in G_2, namely the walk in K that resides in the u-module. If $|A| = |B|$, then there is a one-to-one correspondence between the vertices of the partite set A of G and the partite set B of G. So there is a walk identical to W in the graph G_2 in the H-module associated with the vertex that corresponds to u in the partite set B. So there is a one-to-one correspondence between the closed walks of length k that are entirely contained in a single module in G_1 and those that are entirely contained in a single module of G_2.

Case 2. Any other closed walk W of G_1 must visit more than module. So W may be written

$$W = [a_0;X_0]\, e_0\, [b_1;Y_1]\, e_1\, [a_2;X_2]\, e_2 \ldots e_{q-1}\ [b_q;Y_q]\, e_q .$$

Let φ be the mapping that sends closed walks of length k that visit more than one module in G_1 to closed walks of length k that visit more than one module in G_2 given by

$$\varphi\,(W) = [a_0;Y_1]\, f_0\, [b_1;X_2]\, f_1\, [a_2;Y_3]\, f_2 \ldots f_{q-1}\ [b_{q-1};X_0]\, f_q .$$

It is clear that the identical walks X_i and Y_j exist in the graph G_2 in the copies of the graphs H and K associated with the vertices v_{i+1} and u_{j+1}

respectively. Again, each f_i is uniquely determined, if it exists. We must simply demonstrate the existence of the edges that join the last vertex of a given subwalk to the first vertex of the following subwalk. Let X_i and Y_{i+1} be consecutive subwalks of φ (W) associated with the vertices b_{i-1} and a_i in G_2. Then X_i and Y_{i+1} are consecutive subwalks in W associated with the vertices a_i and b_{i+1} in G_1. Let x be the last vertex of the subwalk X_i and y be the first vertex of the subwalk Y_{i+1}. Then x and y are in different partite sets, since xy is an edge in G_1. Without loss of generality, let x be in the partite set A. So x is in the copy of H_1 associated with the vertex u_i and y is in the copy of K_2 associated with the vertex v_{i+1}. Since there is an edge from the last vertex of the subwalk Y_i to the first vertex of the subwalk X_{i+1}, v_i and u_{i+1} are adjacent in G_2. By the definition of G_2, x and y are adjacent in G_2 and therefore the edge f_i exists in the graph G_2. Since this was done without loss of generality, each edge f_i of the walk φ(W) is present in the graph G_2. So φ is one-to-one. Also, any walk W' of G_2 that visits more than one module may be written

$$[a_0,Y_1]\, f_0\, [b_1,X_2]\, f_1 \ldots f_{q-1}\, [b_q,X_0]\, f_q\,.$$

The walk in G_1

$$[a_0,X_0]\, e_0\, [b_1,Y_1]\, e_1 \ldots e_{q-1}\, [b_q,Y_q]\, e_q$$

is the preimage of the walk W' in G_1. Thus, the map φ defined above is one-to-one and onto, giving the desired correspondence between the closed walks of length k that visit more than one module in G_1 and G_2,

and therefore the graphs G_1 and G_2 are cospectral. ❑

Since we are also interested in constructing isospectral graphs, we present a sufficient condition for two graphs to be isospectral. In order to do this, we must first present a well known result from linear algebra.

Theorem 1.3 Let A be an nxn matrix with eigenvalues λ_i, $1 \le i \le n$. Then the matrix $rI_n - A$ has eigenvalues $r - \lambda_i$, $1 \le i \le n$.

Proof Let λ_i be an eigenvalue of A and let x be the corresponding eigenvector, and let $M = rI_n - A$. Then $Mx = (rI_n - A)x = rx - Ax = (r - \lambda_i)x$. So $r - \lambda_i$ is an eigenvalue of $rI_n - A$. Since this was done for an arbitrary eigenvalue of A, the eigenvalues of $rI_n - A$ are $r - \lambda_i$, $1 \le i \le n$. ❑

Theorem 1.4 Let G and H be cospectral k-regular graphs. Then G and H are isospectral graphs.

Proof Since G and H are k-regular, $\deg(G) = \deg(H) = kI_n$, where n is the order of the graphs G and H. Let λ_i, $1 \le i \le n$, be the eigenvalues of G and H. By Theorem 1.3, the spectrum of $\deg(G) - A(G)$ and $\deg(H) - A(H)$ is $k - \lambda_i$, $1 \le i \le n$. So G and H are isospectral. ❑

Before we use these constructions to build expander graphs, we wish to determine some parameters of 2-set composition graphs and bipartite composition graphs. With Theorems 1.1, 1.2 and 1.4, we will then be able to answer the question of whether isospectral graphs must have equal expander coefficients.

CHAPTER II

SOME PARAMETERS OF 2-SET COMPOSITION AND BIPARTITE COMPOSITION GRAPHS

2.1 Connectivity of 2-set Composition and Bipartite Composition

In Chapter I, we defined two constructions, 2-set composition and bipartite composition. In this chapter, we will examine some of the parameters of the constructed graphs. In particular, in order to examine the expander coefficients of the graphs that will be constructed, we will need the connectivity of the graphs. We will also look at distance in the two constructions.

We want to address the connectivity of graphs constructed by 2-set composition and bipartite composition. Since 2-set composition is a special case of generalized composition, we will find the connectivity of generalized composition graphs instead.

Let G be a graph with vertex set $V(G) = \{x_1,x_2,...,x_p\}$ and let $H_1,H_2,...,H_p$ be a collection of graphs. The generalized composition $G(H_1,H_2,...,H_p)$ is the graph $H_1 \cup H_2 \cup ... \cup H_p$ along with the edges joining each vertex in one copy of the graph H_i with all the vertices of the graph H_j whenever the edge x_ix_j is present in the graph G. Given a set A of the vertices of $G(H_1,H_2,...,H_p)$, we define the projection $\pi(G(H_1,H_2,...,H_p) - A)$ to be the vertex induced subgraph of G with vertex set $\{x_i \in G : \exists\ v \in V(H_i),\ v \notin A\}$. In other words, the vertex x_i is present in

the graph $\pi(G(H_1,H_2,...,H_p) - A)$ if any vertex of the graph H_i remains after removing the set A. We now present a necessary condition for $G(H_1,H_2,...,H_p) - A$ to be connected.

Lemma 2.1 Let $A \subseteq V(G(H_1,H_2,...,H_p))$. If $G(H_1,H_2,...,H_p) - A$ is connected, then $\pi(G(H_1,H_2,...,H_p) - A)$ is connected.

Proof Let $G^* = G(H_1,H_2,...,H_p) - A$, and assume G^* is connected. If $|V(\pi(G^*)| = 1$, then $\pi(G^*)$ is connected. Assume $|V(\pi(G^*)| \geq 2$ and let u and v be vertices of $\pi(G^*)$. It will be demonstrated that there is a u-v path in G^*. If u=v then the path is the trivial path. So assume that $u \neq v$. Then $u = x_i$ and $v = x_j$ for some $i,j \in \{1,2,..,p\}$. So there must be vertices $y_i \in V(H_i)$ and $y_j \in V(H_j)$ such that both y_i and y_j are present in G^*. Since G^* is connected, there is a y_i-y_j path in G^*, say P: $y_i\ x_1\ x_2\ ...\ x_{n-1}\ y_j$. Let v_k be the vertex in G corresponding to the module in which x_k lies. By the definition of $G(H_1,H_2,...,H_p)$, each edge $v_kv_{k+1} \in E(G)$ for m= 1,2,...,k–1 . So the vertices v_m form a u-v walk (provided we allow loops for the moment) W in $\pi(G^*)$. Since W is a u-v walk in $\pi(G^*)$, W contains a u-v path in $\pi(G^*)$. Since this was done for any two vertices in $\pi(G^*)$, $\pi(G^*)$ is connected. ❑

We now present a sufficient condition for $G(H_1,H_2,...,H_p) - A$ to be connected.

Lemma 2.2 Let $A \subseteq V(G(H_1,H_2,...,H_p))$. If $\pi(G(H_1,H_2,...,H_p) - A)$ is connected and

(1) H_i is complete for each i, or

(2) $|V(\pi(G(H_1,H_2,...,H_p) - A)| \geq 2$,

then $G(H_1,H_2,...,H_p) - A$ is connected.

Proof Let $G^* = G(H_1,H_2,...,H_p) - A$ and assume $\pi(G^*)$ is connected. Suppose that $|V(\pi(G^*)| = 1$. Then all the vertices of G^* belong to a single module, say H_i. So the connectivity of G^* is guaranteed if H_i is complete. Now suppose that $|V(\pi(G^*))| \geq 2$, and let u and v be two vertices of G^*. Suppose u and v are both from the same module H_i. Since the vertex x_i of $\pi(G^*)$ that corresponds to the module H_i is not isolated, there is some vertex w_j of the graph $\pi(G^*)$ that is adjacent to x_i. By the definition of $\pi(G^*)$, there is some vertex y of the module H_j present in the graph G^*. Since the vertices x_i and w_j are adjacent in G, the edge uw and the edge wv are both edges of G^*. So there is a u-v path in G^*. Now suppose u is a vertex of the module H_i and v is a vertex of the module H_j. Then the vertices x_i and x_j associated with the modules H_i and H_j are in the vertex set of $\pi(G^*)$. Since $\pi(G^*)$ is connected, there is an x_i-x_j path in $\pi(G^*)$, say P: $x_i\ w_1 \ldots w_{n-1}\ x_j$. For these vertices to be in the vertex set of $\pi(G^*)$, there is at least one vertex, y_k, of the graph H_k present in G^* for each $k = 1,2,...,n-1$. By the definition of $G(H_1,H_2,...,H_p)$, each edge $y_k y_{k+1}$ is in the edge set of G^*. So the vertices y_k form a u-v path in G^*. Since the vertices u and v are arbitrary, there is a path between any two vertices of G^*. Therefore, G^* is connected. ❑

By combining these two results, the following corollary is obtained.

Corollary 2.3 Let $A \subseteq V(G(H_1,H_2,\ldots,H_p))$ and $|V(\pi(G(H_1,H_2,\ldots,H_p) - A))| \geq 2$. Then $\pi(G(H_1,H_2,\ldots,H_p) - A)$ is connected if and only if $G(H_1,H_2,\ldots,H_p) - A$ is connected.

Suppose that S is a minimal cut set of the graph G, and let A be the subset of the vertices of $G(H_1,H_2,\ldots,H_p)$ that belong to modules associated with the vertices of S. It is clear that the graph $G(H_1,H_2,\ldots,H_p) - A$ is disconnected. Let $k_1 = \min_S \sum_{v_i \in S} |H_i|$, with the minimum taken over all minimal cut sets S of G. Now let A_i be the set of all vertices of $G(H_1,H_2,\ldots,H_p)$ that do not belong to the module H_i, along with a cut set of H_i of minimum order. It is clear that the graph $G(H_1,H_2,\ldots,H_p) - A_i$ is disconnected. The order of such a set A_i is $|A_i| = |V(G(H_1,H_2,\ldots,H_p))| - |H_i| + k(H_i)$. Let k_2 be the minimum taken over all i of $|A_i|$.

Theorem 2.4 The connectivity of $G(H_1,H_2,\ldots,H_p)$ is the minimum of k_1 and k_2.

Proof Let $k(G(H_1,H_2,\ldots,H_p))$ be the connectivity of $G(H_1,H_2,\ldots,H_p)$. The definitions of k_1 and k_2 show that $k(G(H_1,H_2,\ldots,H_p)) \leq \min(k_1,k_2)$. Let A be a cut set of $G(H_1,H_2,\ldots,H_p)$. Then either $|V(\pi(G(H_1,H_2,\ldots,H_p) - A))| = 1$ or $|V(\pi(G(H_1,H_2,\ldots,H_p) - A))| \geq 2$. If $|V(\pi(G(H_1,H_2,\ldots,H_p)-A))| \geq 2$, then $\pi(G(H_1,H_2,\ldots,H_p)-A)$ must be disconnected, by Corollary 2.3. This occurs only if $V(G) - \pi(G(H_1,H_2,\ldots,H_p)-A)$ is a cut set of G. But $V(G) - \pi(G(H_1,H_2,\ldots,H_p)-A)$ is a cut set of G if and only if it contains a minimal cut set of G. So $|V(G(H_1,H_2,\ldots,H_p)-A))| \geq k_1 \geq \min(k_1,k_2)$. If, on the other hand $|V(\pi(G(H_1,H_2,\ldots,H_p)-A))| = 1$, all the vertices of all but one module belong to the set A, along with a cut set of that module. So

$|G(H_1,H_2,\ldots,H_p) - A| \geq k_2 \geq \min(k_1,k_2)$. So the connectivity of $G(H_1,H_2,\ldots,H_p)$ is $\min(k_1,k_2)$. □

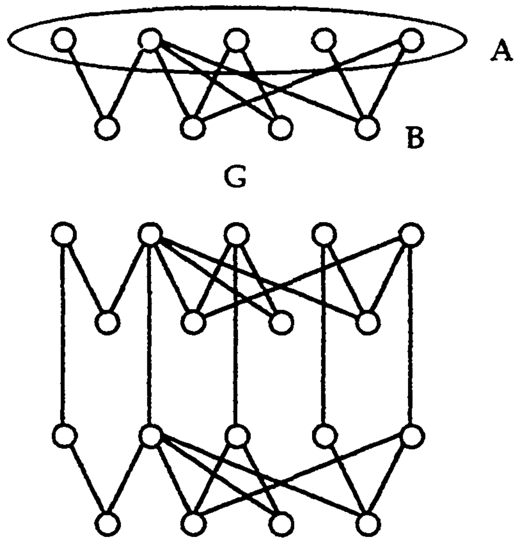
Knowing the connectivity of generalized composition graphs allows us to determine the connectivity of 2-set composition graphs. The difference is that there are only two graphs used in the composition, which permits a simpler formula. So let G be a graph and let A be a subset of V(G), and let H and K be two arbitrary graphs with orders p_1 and p_2.

Corollary 2.5 Let S be a minimal cut set of G, and let a be the number of vertices of the minimal cut set that belong to the set A, and $b = |V(G)| - a$. Define $k_1 = \min_S (ap_1 + bp_2)$, with the minimum taken over the minimal cut sets S of the graph G and define $k_2 = \min(\,|V(G(A,H;K))| - p_1 + k(H),\; |V(G(A,H;K))| - p_2 + k(K)\,)$. Then $k(G(A,H;K)) = \min(k_1,k_2)$.

Next, we wish to discuss the connectivity of the bipartite composition. Unfortunately, the composition does not lend itself as well to inspection. However, every bipartite composition graph with one of the modules being a complete bipartite graph has a spanning subgraph isomorphic to a 2-set composition graph. Utilizing this, we can establish a lower bound on the connectivity of a bipartite composition graph. First, we must define a process that yields the subgraph of a bipartite composition graph that is isomorphic to a 2-set composition graph.

Let G be a bipartite graph with partite sets A and B. We define the graph $G^*K_2(A)$ to be the graph $GxK_2 - \{\, b_{i,1}b_{i,2} : b \in B\}$. This is just two

copies of the graph G, along with an edge joining corresponding vertices of the partite set A. An example of this construction is illustrated in Figure 2.1.



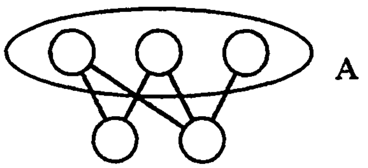
$G^*K_2\,(A)$

Figure 2.1

Lemma 2.6 Let the subset C of the vertex set of G^*K_2 (A) consist of the vertices from each copy of the graph G in the graph G^*K_2 (A) that belong to the set A of G. Then the graphs $G^*K_2(A)(C,nK_1;mK_1)$ and $G^*(A,K_{n,n},nK_1;2mK_1,mK_1)$ are isomorphic.

Proof The notation in this lemma is complicated and not enlightening. Instead, Figures 2.2 and 2.3 demonstrate the fact that these two graphs are isomorphic. ❑

Corollary 2.7 If the graph H is a complete bipartite graph, then the connectivity of the bipartite composition graph $G^*(A,H,H_1;K,K_2)$ is the same or greater than the connectivity of the graph $G^*K_2(A)(C,K_n;mK_1)$.



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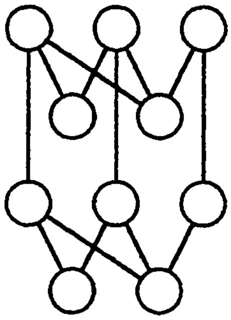
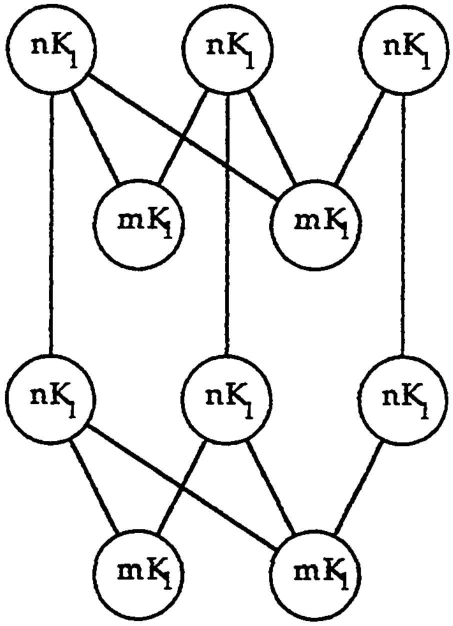
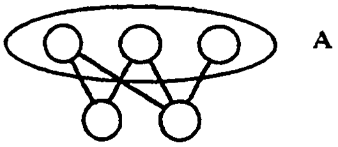


$G^*K_2(A)$

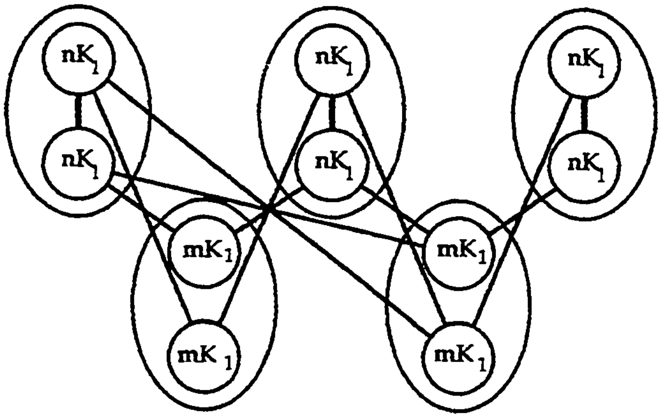


$G^*K_2(A)$ $(C,nK_1;mK_1)$

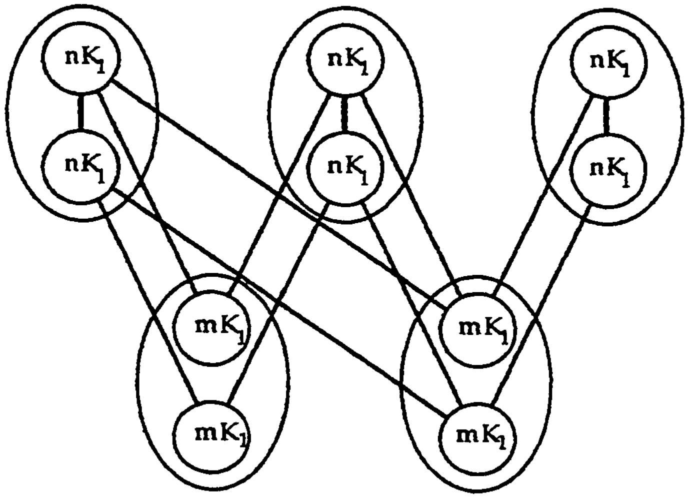
Figure 2.2



G



$G(A, K_{n,n}, nK_1; 2mK_1, mK_1)$



$G(A, K_{n,n}, nK_1; 2mK_1, mK_1)$

Figure 2.3

2.2 Distance in 2-set Composition and Bipartite Composition Graphs

Now we turn our attention to distance in the composition graphs. We will first look at 2-set composition graphs, again as a special case of generalized composition, and then examine bipartite composition.

Lemma 2.8 Let G be a connected graph of order $p(G) \geq 2$ and vertex set $V(G) = \{u_1, u_2, \ldots, u_p\}$. Let $H_1, H_2, \ldots, H_p$ be graphs. Let $G^* = G(H_1, H_2, \ldots, H_p)$. Let v be a vertex from the module u_i in the graph G^* and let w be a vertex from the module u_j, $i \neq j$, in the graph G^*. Then $d_{G^*}(v,w) = d_G(u_i,u_j)$.

Proof Let $P = u_i\ u_2 \ldots u_{n-1}\ u_j$ be a shortest u_i-u_j path in the graph G. Let v_k be a vertex from the module u_k. Then $P' = v\ v_2 \ldots v_{n-1}\ w$ is a v-w path in the graph G^* of length $d_G(u_i,u_j)$. So $d_{G^*}(v,w) \leq d_G(u_i,u_j)$. Assume there is a v-w path in G^* such that $d_{G^*}(v,w) < d_G(u_i,u_j)$, say $P^* = v\ v_2 \ldots v_{m-1}\ w$. Let u_k be the module containing v_k. Then $W = u_i\ u_2 \ldots u_{n-1}\ u_j$ is a v-w walk in the graph G with $n < d_G(u_i,u_j)$. This is a contradiction. So there is no v-w path in the graph G^* with length less than $d_G(u_i,u_j)$. Therefore $d_{G^*}(v,w) = d_G(u_i,u_j)$. ❑

Lemma 2.9 Let G be a connected graph of order $p(G) \geq 2$ and vertex set $V(G) = \{u_1, u_2, \ldots, u_p\}$. Let $H_1, H_2, \ldots, H_p$ be graphs. Let $G^* = G(H_1, H_2, \ldots, H_p)$. Let v and w be vertices from the module u_i in the graph G^*. Then $d_{G^*}(v,w) = 1$ or $d_{G^*}(v,w) = 2$.

Proof If the edge vw is present in the graph H_i, then the edge vw is present in the graph G^*, and $d_{G^*}(v,w) = 1$. Suppose the edge vw is not present in the graph H_i. Since G is connected, there is a vertex x in the

graph G so that the edge xu_i is present in the graph G. By the definition of G^*, v and w are adjacent to each vertex in the module x. Let y be a vertex from the module x. Then P = v y w is a v-w path of length two in the graph G^*. Therefore $d_{G^*}(v,w)= 2$. ❑

These results allow us to address some of the parameters of the 2-set composition graphs. Recall that the *eccentricity* of a vertex u, denoted e(u), is defined to be $\max_{v \in V(G)} d(u,v)$. The *radius* of a graph G, denoted rad(G), is $\min_{v \in V(G)} e(v)$. The *center* of a graph G, written C(G), is the subgraph of G that is induced by the vertices of G with eccentricity equal to the radius of G. Using the previous two lemmas, we are able to prove the following results.

Lemma 2.10 Let G be a graph with $rad(G) \geq 2$. Then for every vertex v of the graph $G^* = G(H_1, H_2,..., H_p)$, $e_{G^*}(v) = e_G(u_i)$, when v is any vertex in the module u_i.

Proof Let v be a vertex of the module u_i in the graph G^*. Let $e_{G^*}(u_i) = n$. Then there is a vertex w of G so that $d_G(u_i,w) = n \geq 2$. Let y be any vertex of the module w in the graph G^*. By Lemma 2.8, $d_{G^*}(v,y) = d_G(u_i,w) = n \geq 2$. This implies that $e_{G^*}(v) \geq n$. Suppose $e_{G^*}(v) > n$. Then there is a vertex z in the module u_j in the graph G^* so that $d_{G^*}(v,z) > n$. Again, by Lemma 2.8, we know $d_G(u_i,u_j) = d_{G^*}(v,z) > n$. This contradicts $e_{G^*}(u_i) = n$. Therefore $e_{G^*}(v) = e_G(u_i)$. ❑

Lemma 2.11 Let G be a graph with $rad(G) \geq 2$. Then $C(G(H_1, H_2,..., H_p))$ is the subgraph of $G(H_1, H_2,..., H_p)$ induced by the vertices of the modules associated with the vertices of C(G).

Proof By Lemma 2.10, for each vertex v of $G^* = G(H_1, H_2,..., H_p)$, $e_{G^*}(v) = e_G(u)$, where v is a vertex of the module u in the graph G^*. If u is a vertex of the center of G, then u has minimum eccentricity in G. So no vertex in G^* has an eccentricity less than the eccentricity of u in G. So $e_{G^*}(v)$ is minimum in G^*. So the vertex v is a vertex of $C(G^*)$. If w is a vertex of a module x that is not a vertex of C(G), then $e_{G^*}(w) = e_G(x) > e_G(u) = e_{G^*}(v)$. So w is not a vertex of $C(G^*)$. Therefore, $C(G(H_1, H_2,..., H_p))$ is the subgraph of $G(H_1, H_2,..., H_p)$ induced by the vertices of the modules associated with the vertices of C(G). ❑

We can now construct pairs of graphs with a very interesting property. Given any two graphs, H_1 and H_2, there exists an infinite family of pairs of cospectral graphs, one of which has a center isomorphic to the graph H_1 and the other which has a center isomorphic to the graph H_2.

First we must define the graphs G_k. Let G_k be the bipartite graph with partite sets $A = \{a_1,a_2,...,a_k\}$ and $B = \{b_1,b_2,...,b_k\}$ and edge set $E(G_k) = \{a_1b_i : 1 \leq i \leq k\} \cup \{a_jb_j : 2 \leq j \leq k\}$.The graph G_5 is illustrated in Figure 2.4. Then $e(a_1) = 2$ and $e(v) \geq 3$ for all other vertices v of the graph G. We are now ready for the result.

Theorem 2.12 Given any pair of graphs H_1 and H_2, there are infinitely many pairs of cospectral graphs such that H_1 is isomorphic to the center of

one graph of each pair and H_2 is isomorphic to the center of the other graph of each pair.

Proof Let H_1 and H_2 be any two graphs. For each graph G_k, as defined above, $|A| = |B|$. By Theorem 1.1, the graphs $G_k(A,H_1;H_2)$ and $G_k(A,H_2;H_1)$ are cospectral. Also, by Lemma 2.11, $C(G_k(A,H_1;H_2)) = H_1$ and $C(G_k(A,H_2;H_1)) = H_2$, since $\mathrm{rad}(G_k) \geq 2$, as noted above. ❑

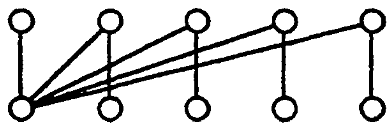


Figure 2.4

We now turn our attention to graphs formed by bipartite composition. However, certain restrictions must be forced upon the graphs used in the composition. For the remainder of the discussion, we will let G be a connected bipartite graph, with partite sets $A = \{a_1,a_2,\ldots,a_r\}$ and $B = \{b_1,b_2,\ldots,b_s\}$, and let H and K be bipartite graphs, with minimum degree greater than zero, and with non-empty partite sets H_1 and H_2, and K_1 and K_2, respectively. We now determine the distance between vertices of the graph $G^*(A,H,H_1;K,K_2)$.

Lemma 2.13 Let u and v be two vertices of the graph $G^*(A,H,H_1;K,K_2)$ that are from the same module and same partite set of $G^*(A,H,H_1;K,K_2)$. Then $d(u,v) = 2$.

Proof Suppose, without loss of generality, that u and v are from the module associated with the vertex x. Since G is connected, there is some

vertex y adjacent to x in the graph G. The module associated with the vertex y has a vertex, say w, that belongs to the other partite set of $G^*(A,H,H_1;K,K_2)$. By the definition of $G^*(A,H,H_1;K,K_2)$, the edges uw and vw exist in the graph $G^*(A,H,H_1;K,K_2)$. So $d(u,v) \leq 2$. Yet the graph $G^*(A,H,H_1;K,K_2)$ is bipartite, and u and v belong to the same partite set, so $d(u,v)$ is even. Therefore $d(u,v) = 2$. ❑

Lemma 2.14 Let u and v be two vertices of the graph $G^*(A,H,H_1;K,K_2)$ that are from the same module, say x, but from different partite sets of the graph $G^*(A,H,H_1;K,K_2)$. Then $d(u,v) = 1$ if u and v are adjacent in the module x, or $d(u,v) = 3$.

Proof If the vertices u and v are adjacent in the module x, then $d(u,v) = 1$ by the definition of $G^*(A,H,H_1;K,K_2)$. So assume that they are not adjacent in the module x. Since the graph G is connected, there is a vertex, y, of the graph G so that x and y are adjacent in G. Since the partite sets of the modules are non-empty, there is a vertex w_1 in the module y that is adjacent to the vertex u in the graph $G^*(A,H,H_1;K,K_2)$. Also, the degree of the vertex w_1 in the module y is not zero, so there is a vertex w_2 of the module y that is adjacent to the vertex w_1. By the definition of the graph $G^*(A,H,H_1;K,K_2)$, the vertices v and w_2 are adjacent. So $d(u,v) \leq 3$. Since this is a bipartite graph, $d(u,v)$ is odd, and is assumed not to be one. So $d(u,v) = 3$. ❑

Before we discuss the distance between two vertices from different modules, we need to define the parity of a vertex in the bipartite composition graphs. Two vertices u and v are said to have the same

parity if (i) they are from the same partite set of the bipartite composition graph and are from modules at an even distance from each other, or (ii) are from different partite sets of the bipartite composition graph and are from modules at an odd distance from each other. The vertices are said to have different parity if (i) they are from different partite sets of the bipartite composition graph and from modules at an even distance from each other, or (ii) they are from the same partite set of the bipartite composition graph and are from modules that are at an odd distance from each other. This simply partitions the vertices into the partite sets of the bipartite composition graph.

Lemma 2.15 Let u and v be two vertices of $G^*(A,H,H_1;K,K_2)$ that are from different modules, say the modules x and y. Then the distance from u to v is $d(u,v) = d_G(x,y)$ if the vertices have the same parity, or $d(u,v) = d_G(x,y) + 1$. if the vertices have different parity.

Proof If the vertices u and v have the same parity, a path of length $d_G(x,y)$ may be found, following the same process as used in Lemma 2.8. However, if u and v have different parity, then a path of length $d_G(x,y)+1$ may be formed, utilizing the path of length $d_G(x,y)$ ending at a vertex w that is adjacent to v in the y module, and then traversing the edge from w to v. ❑

Using these results, we can say something about the eccentricity of graphs formed by bipartite composition.

Lemma 2.16 Let rad(G) ≥ 3, and let G, H and K satisfy the conditions above. Then, for any vertex u of the module x of the graph $G^*(A,H,H_1;K,K_2)$, the eccentricity of u in $G^*(A,H,H_1;K,K_2)$ is $e(u) = e_G(x) + 1$.

Proof Since rad(G) ≥ 3, $e(u) \geq 3$. Let y be a vertex of G so that $d_G(x,y) = e_G(x)$. Then all the vertices of one of the partite sets of the module y are at distance $d_G(x,y) + 1$ from the vertex u. So $e(u) \geq e_G(x) + 1$. If, however, $e(u) > e_G(x) + 1$, then there is some vertex w of the graph $G^*(A,H,H_1;K,K_2)$ so that $d(u,w) > e_G(x) + 1$. So, the module to which w belongs, say the module z, satisfies $d_G(x,z) > d_G(x,y)$. This is a contradiction. So $e(u) = e_G(x) + 1$. ❑

Lemma 2.17 Under the same conditions as in Lemma 2.16, $C(G^*(A,H,H_1;K,K_2))$ is the subgraph of $G^*(A,H,H_1;K,K_2)$ induced by the vertices of the modules associated with the vertices of C(G).

Proof By the same process as used in 2.11, this result is clear. ❑

We are now able to produce a result similar to that of Theorem 2.12. First, we require the graphs H and K to have non empty partite sets and minimum degree greater than zero. Let the graph G_k be k copies of P_4, along with k-1 more edges, joining a vertex v of degree one in the first copy of P_4 and one of the vertices of degree two in each of the other copies of P_4, and let the partite set A of the graph G_k be the partite set containing v. The graph G_5 is pictured in Figure 2.5. For each k, the graph G_k is bipartite with partite sets A and B satisfying $|A| = |B|$.

Theorem 2.18 Let H and K be two bipartite graphs satisfying the conditions set out above. Then there are infinitely many pairs of bipartite graphs so that the graph H is the center of one of the graphs of the pair, and K is the center of the other graph of the pair.

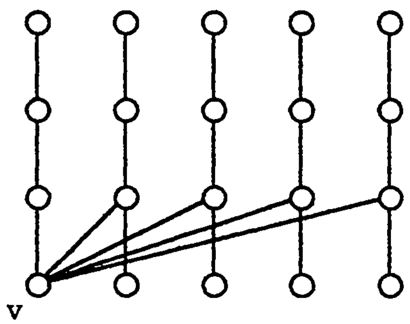


Figure 2.5

Proof Let the graphs G_k be defined as above. Then, by Theorem 1.2, the graphs $G_k^*(A,H,H_1;K,K_2)$ and $G_k^*(A,K,K_2;H,H_1)$ are cospectral. Also, $e_G(v) = 3$, and $e_G(u) > 3$ for every other vertex u of G_k. So $C(G_k) = v$. By lemma 2.17, $C(G_k^*(A,H,H_1;K,K_2)) = H$ and $C(G_k^*(A,K,K_2;H,H_1)) = K$. ❑

CHAPTER III

2-SET AND BIPARTITE COMPOSITION GRAPHS AND THE EXPANDER COEFFICIENT

3.1 Preliminary Results

We wish to construct cospectral pairs of graphs that have different expander coefficients. Using the constructions of Chapter I and the results of Chapter II, we are in a position to begin. We will first need a few preliminary results and a general strategy for computing the expander coefficients. Then the graphs that we need will be constructed and proven to have the properties that we desire.

When evaluating the expander coefficient of a graph G, each nonempty proper subset S of the vertex set of G must satisfy the inequality

$$\operatorname{expan}(G) \geq \operatorname{Ex}(S) = \frac{p|B(S)|}{|S|(p-|S|)}.$$

Practically speaking, we wish to find the smallest value of Ex(S), over all nonempty proper subsets S of the vertex set V(G). As noted before, examining each of the subsets of a large graph would be prohibitive. We employ a strategy to systematically eliminate from consideration as many of these subsets as possible.

In Chapter II, we looked at the connectivity of two-set composition and bipartite composition graphs. We use the connectivity of the graphs

in order to reduce the number of subsets S of the vertex set of a graph G for which we compute Ex(S). In particular, if $|B(S)|$ is smaller than the connectivity of the graph, then S is an ineffective choice. We show that if $|B(S)| < k(G)$, then $|B(S)| = p - |S|$, reducing Ex(S) to $\frac{p}{|S|}$.

Lemma 3.1 Let G be a connected graph and S a subset of V(G). If B(S) is not a cut set of G, then $|B(S)| = p - |S|$.

Proof Since B(S) is not a cut set, G – B(S) is connected. Then there are no vertices in V(G) – S – B(S), and this gives the desired result. ❑

Corollary 3.2 Let G be a connected graph, k(G) be the connectivity of G, and S a subset of V(G). If $|B(S)| < k(G)$, then $|B(S)| = |V(G)| - |S|$.

Corollary 3.3 For any graph G, $\text{expan}(G) \leq \frac{p}{p-1}$.

Proof Any set S so that B(S) is not a cut set has $\text{Ex}(S) = \frac{p}{|S|}$. This is a minimum when $|S| = p - 1$. Since expan(G) is the largest real number so that $\text{expan}(G) \leq \text{Ex}(S)$ for all non-empty proper subsets S of the vertex set of the graph G, $\text{expan}(G) \leq \frac{p}{p-1}$. ❑

Corollary 3.4 Let G be a graph on p vertices and suppose there is a subset S of the vertex set of G so that $\text{Ex}(S) < \frac{p}{p-1}$. Then any set T with a boundary B(T) that is not a cut set satisfies $\text{Ex}(T) > \text{Ex}(S) \geq \text{expan}(G)$.

Proof $\text{Ex}(T) = \frac{p}{|T|} \geq \frac{p}{p-1} > \text{Ex}(S) \geq \text{expan}(G)$, and so $\text{Ex}(T) > \text{Ex}(S) \geq \text{expan}(G)$.

3.2 2-set Composition and the Expander Coefficient

We now present our first family of isospectral graphs with unequal expander coefficients. However, as we take successive pairs in the sequence, the difference between their expanders decreases to zero.

The strategy we use to compute the expander coefficient in the graphs we construct is the same throughout this dissertation. We first pick a likely candidate, S, as a set that might minimize Ex(T). We then show that any other set, S', satisfies Ex(S') ≥ Ex(S) for various reasons.

To construct our first family of pairs of graphs, we employ the graphs G_k of Figure 3.1, with the columns of six vertices repeated 2k times, and joined by 12-cycles, as between the first two columns of six vertices.

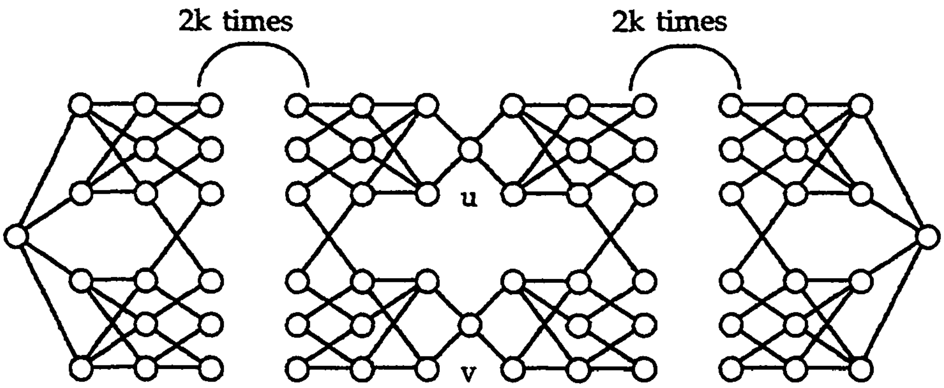


Figure 3.1

The graph G_1 is illustrated in Figure 3.2. We construct 2-set composition graphs from these graphs. We will then have a family of pairs of graphs that are isospectral and, pairwise, have different expander coefficients.

We have chosen these graphs because it is likely that the expander coefficients of these graphs will be what we claim them to be. However, since the connectivity is so small, it is unlikely that these graphs would model realistic networks. Let the modules, H and K, of the 2-set composition graphs be K_5 and $4K_1$, and let the partite set A of the graph G_k be the partite set containing the vertices u and v. The partite sets A and B each have order 22+12k, which guarantees the cospectrality of $G_k(A,4K_1;K_5)$ and $G_k(A,K_5;4K_1)$ by Theorem 1.2. We chose the graphs K_5 and $4K_1$ to insure that the graphs $G_k(A,4K_1;K_5)$ and $G_k(A,K_5;4K_1)$ are each 20-regular. These graphs are then isospectral by Theorem 1.4. The order of these graphs is $9(22 + 12k)$.

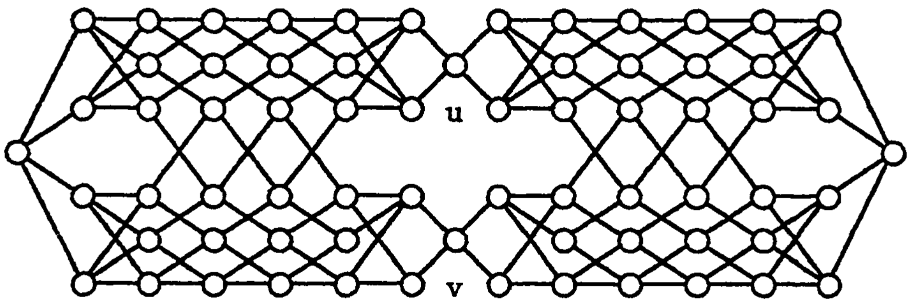


Figure 3.2

Lemma 3.5 The connectivity of the graph $G_k(A,K_5;4K_1)$ is 10.

Proof If the vertices of the modules associated with the vertices u and v in Figure 3.1 are removed from $G_k(A,K_5;4K_1)$, the result is a disconnected graph. So the connectivity of $G_k(A,K_5;4K_1)$ is at most 10. However, any other minimal cut set of G_k consists of at least three vertices. Because each module contains at least four vertices, any other minimal cut set of

$G_k(A,K_5;4K_1)$ must at least contain 12 vertices. Thus the connectivity is 10. ❑

Lemma 3.6 The connectivity of $G_k(A,4K_1;K_5)$ is 8.

Proof If the vertices of the modules associated with the vertices u and v in Figure 3.1 are removed from $G_k(A,4K_1;K_5)$, the result is a disconnected graph. So the connectivity of $G_k(A,4K_1;K_5)$ is at most 8. However, any other minimal cut set of G_k consists of at least three vertices. Because each module contains at least four vertices, any other minimal cut set of $G_k(A,K_5;4K_1)$ must at least contain 12 vertices. Thus the connectivity is 8. ❑

Lemma 3.7 The expander coefficient of $G_k(A,4K_1;K_5)$ is

$$\mathrm{expan}(G_k(A,4K_1;K_5)) = \frac{72(22+12k)}{(95+54k)(103+54k)}.$$

Proof By choosing the set S to be all the vertices of the modules associated with the vertices to the left of the vertices u and v in Figure 3.1, $|B(S)| = 8$ and $|S| = 95+54k$. Ex(S) yields the value above. If there is a subset S' of the graph $G_k(A,4K_1;K_5)$ so that the value of Ex(S') is smaller than Ex(S), then the set S' would satisfy the inequality

$$\frac{9(22+12k)\,|B(S')|}{[9(11+6k)]^2} \leq Ex(S') < \frac{72(22+12k)}{(95+54k)(103+54k)},$$

since $[9(11+6k)]^2 \geq |S'|(p - |S'|)$. Solving this inequality for $|B(S')|$ yields

$$|B(S')| < \frac{8(9)^2(11+6k)^2}{(95+54k)(103+54k)}$$

$$\frac{648(121 + 132k + 36k^2)}{9785 + 10692k + 2416k^2}$$

$$= 8 + \frac{128}{(95+54k)(103+54k)}.$$

Since $|B(S')|$ is an integer, $|B(S')| \leq 8$. By Corollary 3.4, and the fact that $Ex(S) < \frac{p}{p-1}$, $B(S')$ must be a cut set, and so $|B(S')| \geq 8$. Yet the only cut set of order 8 is $B(S)$. So there is no set S' so that $Ex(S') < Ex(S)$. Therefore, $\text{expan}(G_k(A,4K_1;K_5) = Ex(S) = \frac{72(22+12k)}{(95+54k)(103+54k)}$. □

Lemma 3.8 The expander coefficient of $G_k(A,K_5;4K_1)$ is

$$\text{expan}(G_k(A,K_5;4K_1)) = \frac{90(22+12k)}{(94+54k)(104+54k)}.$$

Proof By choosing the set S to be all the vertices of the modules associated with the vertices to the left of the vertices u and v in Figure 3.1, then $|B(S)|=10$ and $|S|=94+54k$. Ex(S) yields the value above. If there is a subset S' of the graph $G_k(A,K_5;4K_1)$ so that the value of Ex(S') is smaller than Ex(S), then the set S' would satisfy the inequality

$$\frac{9(22+12k)\ |B(S')|}{[9(11+6k)]^2} \leq Ex(S') < \frac{90(22+12k)}{(94+54k)(104+54k)},$$

since $[9(11+6k)]^2 \geq |S'|(p - |S'|)$. Solving this inequality for $|B(S')|$ yields

$$|B(S')| < \frac{810(11+6k)^2}{(94+54k)(104+54k)}$$

$$\frac{810(121 + 132k + 36k^2)}{9776 + 10692k + 2416k^2}$$

$$= 10 + \frac{210}{(94+54k)(104+54k)}.$$

By similar arguments as above, the expander coefficient must then be $\frac{90(22+12k)}{(94+54k)(104+54k)}$. □

The results we wished for are direct corollaries of the preceding theorems.

Corollary 3.9 The expander coefficients of the graphs $G_k(A,K_5;4K_1)$ and $G_k(A,4K_1;B,K_5)$ are different.

Corollary 3.10 $\lim_{k\to\infty} \text{expan}(G_k(A,K_5;4K_1)) = \lim_{k\to\infty} \text{expan}(G_k(A,4K_1;K_5)) = 0.$

Our second family of pairs of isospectral graphs is also a sequence of pairs of 2-set composition graphs. But this time successive pairs have expander coefficients that approach different limits.

To construct the graphs of this family, we employ the graph G of Figure 3.3 to construct the pairs of 2-set composition graphs. We form a family of pairs of graphs that are isospectral and, pairwise, have different expander coefficients. For the 2-set composition modules, let the graphs H and K be K_n and mK_1 , and let the partite set A of the graph G be the partite set containing the vertices u and v. The partite sets A and B each have order 22, which guarantees the cospectrality of $G(A,mK_1;K_n)$ and $G(A,K_n;mK_1)$, by Theorem 1.1. To insure isospectrality, the graphs should be regular of the same degree. A vertex from a copy of the graph K_n in the graph $G(A,K_n;mK_1)$ or $G(A,mK_1;K_n)$ is adjacent to all the vertices in four copies of mK_1 and to n-1 vertices of the graph K_n. Such a vertex would have degree $4m + n - 1$. A vertex of a copy of the graph mK_1 is adjacent to each of the vertices of four copies of the graph K_n, so such a vertex

would have degree 4n. For the graphs to be isospectral, these degrees must be equal. So the relation between n and m is $3n = 4m - 1$. This equation has the solutions $n = 4t + 1$ and $m = 3t + 1$, for all positive integers t.

Lemma 3.11 If $n = 4t + 1$ and $m = 3t + 1$, then the graphs $G(A,mK_1;K_n)$ and $G(A,K_n;mK_1)$ are isospectral.

Proof By Theorem 1.4 and the fact that the equations $n = 4t + 1$ and $m = 3t + 1$ force the graphs $G(A,mK_1;K_n)$ and $G(A,K_n;mK_1)$ each to be regular of degree $4n = 16t + 4$, the graphs $G(A,mK_1;K_n)$ and $G(A,K_n;mK_1)$ are isospectral. ❑

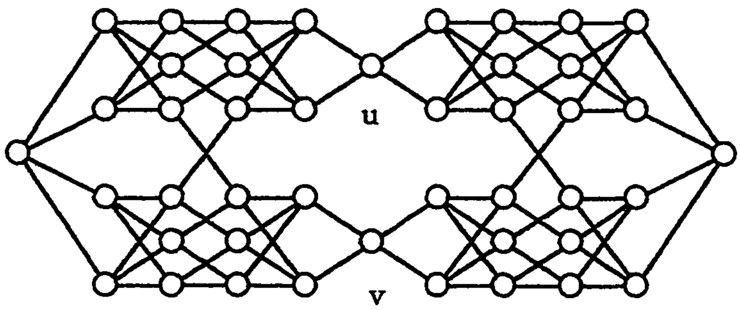


Figure 3.3

Lemma 3.12 The connectivity of $G(A,mK_1;K_n)$ is 2m.

Proof The graph G is 2-connected. The connectivity of $G(A,mK_1;K_n)$ is at most 2m. Any other minimal cut set of G must contain at least three vertices. Since each module contains at least m vertices, any other minimal cut set of $G(A,mK_1;K_n)$ must contain at least 3m vertices. So the connectivity of the graph $G(A,mK_1;K_n)$ is 2m. ❑

Lemma 3.13 The connectivity of $G(A,K_n;mK_1)$ is 2n.

Proof The graph G is two connected. The connectivity of $G(A,K_n;mK_1)$ is at most 2n. Any other minimal cut set of G must contain at least three vertices of G. Since each module contains at least m vertices, any other minimal cut set of $G(A,K_n;B,mK_1)$ must contain at least 3m vertices. But $3m = 9t + 3 > 8t + 2 = 2n$. So the connectivity of $G(A,K_n;mK_1)$ is 2n. □

Theorem 3.14 $\mathrm{Expan}(G(A,K_n;mK_1)) \leq \frac{16(11n)(7n+1)}{(73n+11)(81n+11)}$.

Proof : Let S be the set of vertices of $G(A,K_n;mK_1)$ containing all the vertices of all the modules associated with the vertices of $G(A,K_n;mK_1)$ to the left of the vertices u and v in Figure 3.3. Then $|S| = 11m + 10n$. Since $p(G(A,K_n;mK_1)) = \frac{11(7n+1)}{2}$, we have $\mathrm{Ex}(S) = \frac{16(11n)(7n+1)}{(73n+11)(81n+11)}$. Yet $\mathrm{expan}(G(A,K_n;mK_1)$ is the maximum real number c such that c is less than Ex(T) over all nonempty proper subsets T of the vertex set of $G(A,K_n;mK_1)$. So $\mathrm{expan}(G(A,K_n;mK_1)) \leq \frac{16(11n)(7n+1)}{(73n+11)(81n+11)}$. □

Corollary 3.15 If S is a subset of the vertex set of $G(A,K_n;mK_1)$ and $|B(S)| \geq \frac{32n|S|(p-|S|)}{(73n+11)(83n+11)}$, then $\mathrm{Ex}(S) \geq \frac{16(11n)(7n+1)}{(73n+11)(81n+11)}$.

Proof : Solve the following inequality for $|B(S)|$.

$$\mathrm{Ex}(S) = \frac{p|B(S)|}{|S|(p-|S|)} \geq \frac{16(11n)(7n+1)}{(73n+11)(81n+11)}. \ \square$$

Corollary 3.16 If S is a subset of the vertex set of $G(A,K_n;mK_1)$ and $|B(S)| \geq \frac{242n(7n+1)^2}{(73n+11)(83n+11)}$, then $\mathrm{Ex}(S) \geq \frac{16(11n)(7n+1)}{(73n+11)(81n+11)}$.

Proof : Let $|S| = p/2$ in Corollary 3.15. This value maximizes the denominator of the formula for Ex(S), thus giving the minimum value for Ex(S). ❑

Corollary 3.17 Let S be a subset of the vertex set of $G(A,K_n;mK_1)$. If S contains an uncovered module, that is to say S has vertices that belong to a module M, but no vertices belonging to S are from modules adjacent to M, then $Ex(S) \geq \frac{16(11n)(7n+1)}{(73n+11)(81n+11)}$.

Proof : If S has an uncovered module M, then B(S) contains all the vertices of the modules adjacent to M. Since there are four modules adjacent to M, B(S) contains at least 4m vertices. Yet $4m = 3n + 1 > \frac{242n(7n+1)^2}{(73n+11)(83n+11)}$. By Corollary 3.16, $Ex(S) \geq \frac{16(11n)(7n+1)}{(73n+11)(81n+11)}$. ❑

Lemma 3.18 Suppose that S is a subset of the vertex set of $G(A,K_n;mK_1)$, and that S contains a partial module. Then there is a subset T of the vertex set of $G(A,K_n;mK_1)$ so that $Ex(S) \geq Ex(T)$ and T contains no partial modules.

Proof : By Corollary 3.17, all the partial modules of S are covered. By Corollary 3.15, if $|B(S)| \geq \frac{32n|S|(p-|S|)}{(73n+11)(83n+11)}$, then we do not consider the set S. So assume that $|B(S)| < \frac{32n|S|(p-|S|)}{(73n+11)(83n+11)}$. Let M be the vertices of some partial module of S. Let the set S' be the set of vertices belonging to S along with one vertex, u, of M that is not in S. Since M is a partial module of S, B(S') is a subset of B(S). Since M is a covered module of S, however, the vertex u is not in the set B(S'), but is in the set B(S). So $|B(S')| = |B(S)| - 1$. Also, $|S'| = |S| + 1$. We now show that $Ex(S') \leq Ex(S)$. Suppose this is not the case. Let $b = |B(S)|$ and $s = |S|$. We find

$$Ex(S) - Ex(S') = \frac{pb}{s(p-s)} - \frac{p(b-1)}{(s+1)(p-s-1)} < 0.$$

By solving for b in this inequality, we have

$$\begin{array}{ll} b < \frac{-s(p-s)}{p-2s-1} & \text{when } p-2s-1>0 \\ 0<0 & \text{when } p-2s-1=0 \\ b > \frac{s(p-s)}{2s+1-p} & \text{when } p-2s-1<0. \end{array}$$

The first two cases are clearly contradictions, since $|B(S)|$ is a positive integer. So we may assume that we are in the third case, that $p-2s-1<0$ and that $b > \frac{s(p-s)}{2s+1-p}$. By employing Corollary 3.14, we show that b satisfies the inequality $\frac{32ns(p-s)}{(73n+11)(83n+11)} > b > \frac{s(p-s)}{2s+1-p}$. Solving for s shows that the order of S must be greater than the number of vertices in $G(A,K_n;mK_1)$. This is a contradiction. So we must have $Ex(S') \leq Ex(S)$. If we repeat the process until the new set has no partial modules, we have found a set T that satisfies the conclusion to the lemma. ❑

Following the same process, we arrive at corresponding results for the graph $G(A,mK_1;K_n)$.

Theorem 3.19 $Expan(G(A,mK_1;K_n)) \leq \frac{11(3n+1)(7n+1)}{(37n+1)(40n+1)}$.

Corollary 3.20 If S is a subset of the vertex set of $G(A,mK_1;K_n)$ and $|B(S)| \geq \frac{2(3n+1)|S|(p-|S|)}{(37n+1)(40n+1)}$, then $Ex(S) \geq \frac{11(3n+1)(7n+1)}{(37n+1)(40n+1)}$.

Corollary 3.21 If S is a subset of the vertex set of $G(A,mK_1;K_n)$ and $|B(S)| \geq \frac{121(3n+1)(7n+1)^2}{(37n+1)(40n+1)}$, then $Ex(S) \geq \frac{11(3n+1)(7n+1)}{(37n+1)(40n+1)}$.

Corollary 3.22 Let S be a subset of the vertex set of $G(A,mK_1;K_n)$. If S contains an uncovered module, then $Ex(S) \geq \frac{11(3n+1)(7n+1)}{(37n+1)(40n+1)}$.

Lemma 3.23 Suppose that S is a subset of the vertex set of $G(A,mK_1;K_n)$, and that S contains a partial module. Then there is a subset T of the vertex set of $G(A,mK_1;K_n)$ so that $Ex(S) \geq Ex(T)$ and T contains no partial modules.

We now compute the expander coefficients of the graphs $G(A,K_n;mK_1)$ and $G(A,mK_1;K_n)$.

Theorem 3.24 The expander coefficient of $G(A,K_n;mK_1)$ is $\frac{(16)11n(7n+1)}{(73n+11)(81n+11)}$.

Proof If the set S is chosen to be all vertices of the modules associated with the vertices of G to the left of u and v in Figure 3.3, the boundary of S has cardinality 2n and the number of vertices in S is 11m + 10n. Then Ex(S) yields the given value. If a set S' were to satisfy Ex(S') < Ex(S), then it would also satisfy

$$\frac{p\,|B(S)|}{[p/2]^2} \leq \frac{p\,|B(S')|}{|S|(p-|S|)} = Ex(S') < Ex(S) = \frac{(16)11n(7n+1)}{(73n+11)(81n+11)}.$$

The boundary B(S') would then satisfy the inequality

$$|B(S')| < \frac{11858n^3 + 3388n^2 + 242n}{5913n^2 + 1694n + 121} < 2.006n + .9\,.$$

However, by Lemma 3.18, any set S' that could provide the smallest value of Ex(S) must have no partial modules. So $|B(S')| \leq 2n$. By Corollary 3.4, and that $Ex(S) < \frac{p}{p-1}$, B(S') is a cut set, and so $|B(S')| \geq 2n$. If this set, B(S'),

existed, it would be another cut set of minimum order, which is a contradiction. So the existence of such a set is impossible. Since the denominator was chosen to be the largest possible, there is no subset of vertices of $G(A,K_n;mK_1)$ that results in a value smaller than Ex(S). So the expander coefficient for the graph $G(A,K_n;mK_1)$ is $Ex(S) = \frac{(16)11n(7n+1)}{(73n+11)(81n+11)}$. □

Theorem 3.25 The expander coefficient of $G(A,mK_1;K_n)$ is $\frac{(11)(3n+1)(7n+1)}{(37n+1)(40n+1)}$.

Proof If the set S is chosen to be all vertices of the modules associated with the vertices of G to the left of the vertices u and v in Figure 3.3, the boundary of S has cardinality 2n and the number of vertices in S is 11n + 10m. Then Ex(S) yields the given value. If a set S' were to satisfy Ex(S') < Ex(S), then it would also satisfy

$$\frac{p|B(S)|}{[p/2]^2} \le \frac{p|B(S')|}{|S|(p-|S|)} = Ex(S') < Ex(S) = \frac{(11)(3n+1)(7n+1)}{(37n+1)(40n+1)}.$$

The boundary B(S') would then satisfy the inequality

$$|B(S')| < \frac{121(147n^3+91n^2+17n+1)}{16(1480n^2+77n+1)} < 1.6n + .6 < 3m.$$

However, by Lemma 3.23, any set S' that could provide the smallest value of Ex(S) must have no partial modules. So $|B(S')| \le 2m$. By Corollary 3.4 and that $Ex(S) \le \frac{p}{p-1}$, B(S') must be a cut set, and so $|B(S')| \ge 2m$. If this set existed, it would be another cut set of minimum order, which is a contradiction. So the existence of this set is impossible. Since the denominator was chosen to be the largest possible, there is no subset of

vertices of $G(A,mK_1;K_n)$ so that $Ex(T) < Ex(S)$. So the expander coefficient for the graph $G(A,mK_1;K_n)$ is $Ex(S) = \frac{(11)(3n+1)(7n+1)}{(37n+1)(40n+1)}$. □

The results we wished for are direct corollaries of the preceding theorems.

Corollary 3.26 The expander coefficients of the graphs $G(A,mK_1;K_n)$ and $G(A,K_n;mK_1)$ are different.

Corollary 3.27 $\lim_{n\to\infty} \text{expan}(G(A,K_n;mK_1)) = \frac{1232}{5913}$.

Corollary 3.28 $\lim_{n\to\infty} \text{expan}(G(A,mK_1;K_n)) = \frac{231}{1480}$.

3.3 Bipartite Composition and the Expander Coefficient

Now that we have formed some examples of isospectral graphs with unequal expander coefficients, we turn our attention to the construction of isospectral bipartite graphs with unequal expander coefficients. The process is going to mirror that which was used for the non-bipartite case. We will first construct a sequence of pairs of isospectral bipartite graphs that pairwise have different expander coefficients, but that have equal limits. We then construct another sequence of pairs of isospectral bipartite graphs that pairwise have unequal expander coefficients, but have unequal limits. To construct this family of pairs of graphs, we use the graph G_k in Figure 3.4, with the columns of ten vertices repeated 2k times, and joined by 20-cycles and a matching, as between the first two columns of ten vertices. The graph G_1

is illustrated in Figure 3.5. We use these graphs to build pairs of bipartite composition graphs, namely, the graphs $G_k^*(A,K_{6,6},6K1;10K_1,5K_1)$ and $G_k^*(A,10K_1,5K_1;K_{6,6},6K_1)$. We have chosen these graphs because, again, it is likely that the expander coefficients of these graphs will be what we claim them to be. However, since the connectivity is so small, it is unlikely that these graphs would model realistic networks. It is clear that the partite sets of G_k have 34 + 20k vertices, and so the order of each of $G_k^*(A,K_{6,6},6K1;10K_1,5K_1)$ and $G_k^*(A,10K_1,5K_1;K_{6,6},6K_1)$ is 22(34 + 20k) for each k. We let the partite set A be the set including the vertices u and v of the graph. Also notice that the resulting graphs are 36-regular. Theorems 1.2 and 1.4 guarantee the isospectrality of these two graphs. We now address the expander coefficients of these two graphs. The first results we need are direct consequences of Corollary 2.7.

Lemma 3.29 The connectivity of $G_k^*(A,K_{6,6},6K_1;10K_1,5K_1)$ is 24.

Lemma 3.30 The connectivity of $G_k^*(A,10K_1,5K_1;K_{6,6},6K_1)$ is 20.

Lemma 3.31 The expander coefficient of $G_k^*(A,K_{6,6},6K_1;10K_1,5K_1)$ is

$$\frac{528(34+20k)}{(362+220k)(386+220k)}\cdot$$

Proof Let the set S be the set of all vertices of the modules associated with the vertices to the left of u and v in Figure 3.4. It is easily seen that, for this set S, $|B(S)| = 24$ and $|S| = 12(16+10k) + 10(17+10k)$. By computing Ex(S)

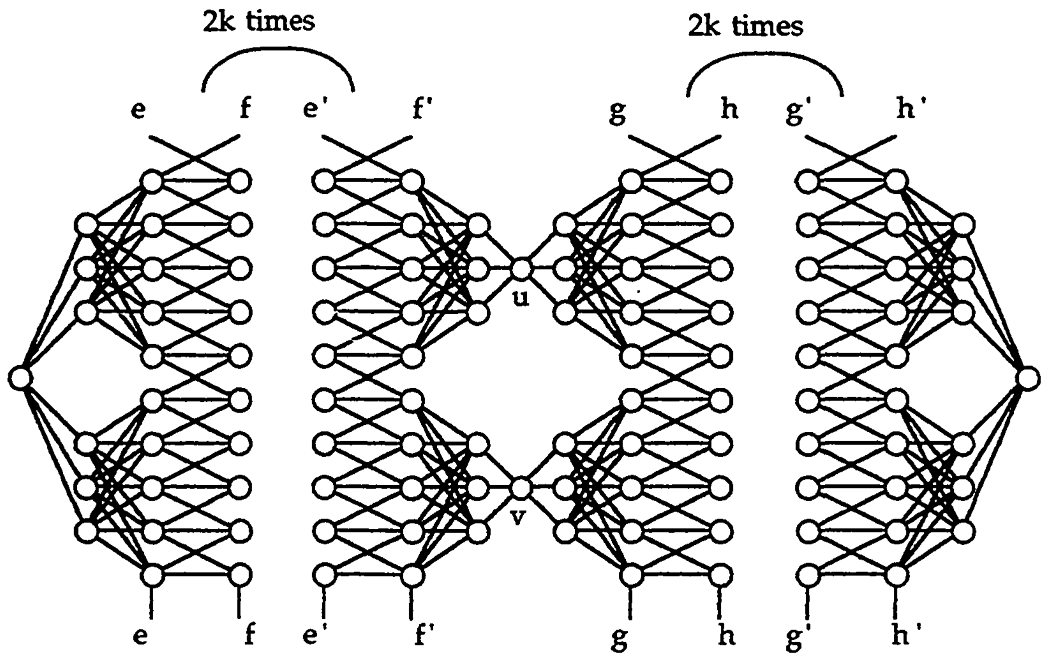


Figure 3.4

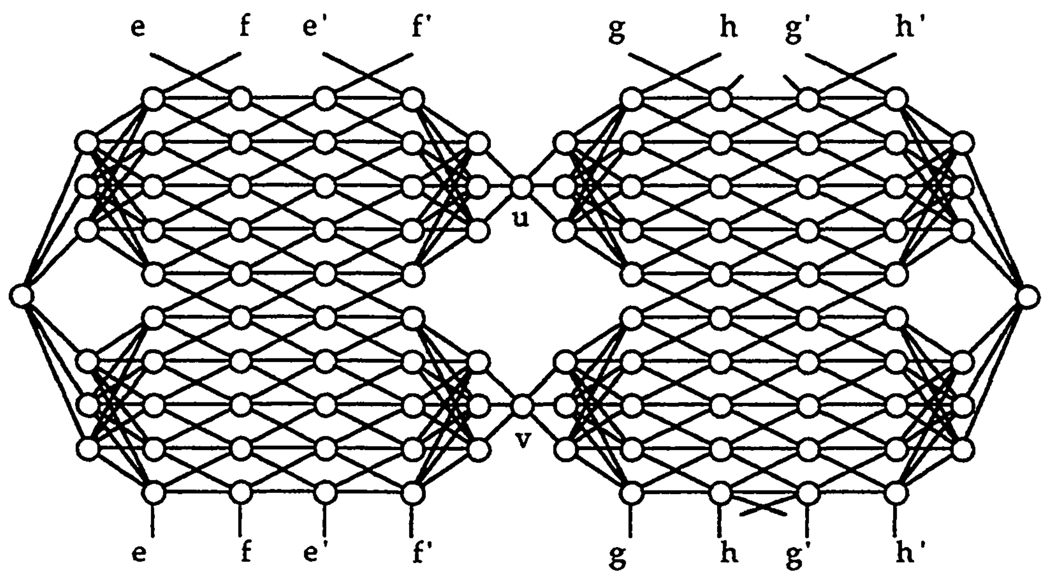


Figure 3.5

for this set, we get the value given above. Now suppose that a subset S' of $G_k^*(A,K_{6,6},6K_1;10K_1,5K_1)$ is chosen so that Ex(S') < Ex(S). The set S' would then satisfy the inequality

$$\frac{22(34+20k)\,|B(S')|}{[11(34+20k)]^2} \leq Ex(S') < \frac{528(34+20k)}{(362+220k)(386+220k)},$$

since $[11(34+20k)]^2 \geq |S'|(p - |S'|)$. Solving this inequality for $|B(S')|$ yields

$$|B(S')| < \frac{2904(34+20k)^2}{(362+220k)(386+220k)}$$

$$= \frac{2904(1156+1360k+400k^2)}{139732+164560k+48400k^2}$$

$$= 24 + \frac{3456}{(362+220k)(386+220k)}.$$

However, $|B(S')|$ must be an integer, so $|B(S')| \leq 24$. By Corollary 3.4, this implies that $|B(S')| = 24$. Yet, $|B(S)|$ is the only cut set of $G_k^*(A,K_{6,6},6K_1;10K_1,5K_1)$ that has 24 vertices. This is a contradiction, so no such set S' exists. ❑

Lemma 3.32 The expander coefficient of $G_k^*(A,10K_1,5K_1;K_{6,6},6K_1)$ is

$$\frac{440(34+20k)}{(364+220k)(384+220k)}.$$

Proof Let the set S be the set of all vertices of the modules associated with the vertices to the left of u and v in Figure 3.5, It is easily seen that, for the set S, $|B(S)| = 20$ and $|S| = 12(17+10k) + 10(16+10k)$. By computing Ex(S) for this set, we get the value given above. Now suppose that a

subset S' of $G_k{}^*(A,10K_1,5K_1;K_{6,6},6K_1)$ is chosen so that Ex(S') < Ex(S). The set S' would then satisfy the inequality

$$\frac{22(34+20k)\ |B(S')|}{[11(34+20k)]^2} \leq Ex(S') < \frac{440(34+20k)}{(364+220k)(384+220k)},$$

since $[11(34+20k)]^2 \geq |S'|(p - |S'|)$. Solving this inequality for $|B(S')|$ yields

$$|B(S')| < \frac{2420(34+20k)^2}{(364+220k)(384+220k)}$$

$$= \frac{2420(1156 + 1360k + 400k^2)}{139776 + 164560k + 48400k^2}$$

$$= 20 + \frac{2000}{(364+220k)(384+220k)}.$$

However, $|B(S')|$ must be an integer, so $|B(S')| \leq 20$. By Corollary 3.4, this implies that $|B(S')| = 20$. Yet, $|B(S)|$ is the only cut set of $G_k{}^*(A,10K_1,5K_1;K_{6,6},6K_1)$ that has 20 vertices. This is a contradiction, so no such set S' exists. ❑

Corollary 3.33 The expander coefficients of $G_k{}^*(A,K_{6,6},6K_1;10K_1,5K_1)$ and $G_k{}^*(A,10K_1,5K_1;K_{6,6},6K_1)$ are different.

Corollary 3.34 As k approaches infinity, the expander coefficients of $G_k{}^*(A,K_{6,6},6K_1;10K_1,5K_1)$ and $G_k{}^*(A,10K_1,5K_1;K_{6,6},6K_1)$ each approach zero.

The second sequence of pairs of isospectral bipartite graphs is formed using the graph G from Figure 3.6 and bipartite composition. Notice that G is 6-regular, and we let the partite set A be the set containing

the vertices u and v. We want to use a complete bipartite graph, $K_{n,n}$, and an empty graph, $2mK_1$, for the modules. In order for $G^*(A,K_{n,n},nK_1;2mK_1,mK_1)$ and $G^*(A,2mK_1,mK_1;K_{n,n},nK_1)$ to be regular of the same degree, we require that $n = \frac{6}{5}m$. This forces the two graphs to be 6n-regular. By Theorems 1.2 and 1.4, the graphs are isospectral. We now apply the same strategy to the evaluation of the expander coefficients of $G^*(A,K_{n,n},nK_1;2mK_1,mK_1)$ and $G^*(A,2mK_1,mK_1;K_{n,n},nK_1)$ as we used in Section 3.2. However, we need a new term. Since a vertex in a module of either of the graphs is only adjacent to half of the vertices in an adjacent module, we use the term *half-module* to describe partial modules. In other words, when we try to fill out the partial modules, we can only work on half-modules, or one partite set of any module, at one time.

Lemma 3.35 The graph $G^*(A,K_{n,n},nK_1;2mK_1,mK_1)$ is 4n-connected.

Proof Since the vertices u and v of the graph G of Figure 3.6 provide a cut set for G, we know that the vertices of the modules associated with u and v are a cut set of $G^*(A,K_{n,n},nK_1;2mK_1,mK_1)$. So the connectivity of $G^*(A,K_{n,n},nK_1;2mK_1,mK_1)$ is at most 4n. If there is a cut set of $G^*(A,K_{n,n},nK_1;2mK_1,mK_1)$ with fewer than 4n vertices, say V, then that set would contain a minimal cut set with fewer than 4n vertices. This set would then be the vertices of modules associated with a cut set of G. Yet any other minimal cut set of G would have three or more vertices, and so the set V is required to contain at least 6m = 5n vertices. So there is no cut set of $G^*(A,K_{n,n},nK_1;2mK_1,mK_1)$ with fewer than 4n vertices. Therefore, the connectivity of $G^*(A,K_{n,n},nK_1;2mK_1,mK_1)$ is 4n. ❑

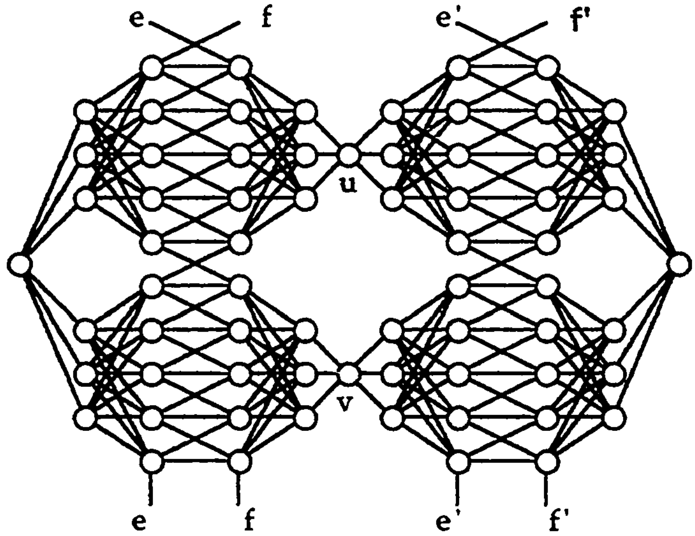


Figure 3.6

Lemma 3.36 The graph $G^*(A,2mK_1,mK_1;K_{n,n},nK_1)$ is 4m-connected.

Proof Since the vertices u and v of the graph G provide a cut set for G, we know that the vertices of the modules associated with u and v are a cut set of $G^*(A,2mK_1,mK_1;K_{n,n},nK_1)$. So the connectivity of $G^*(A,2mK_1,mK_1;K_{n,n},nK_1)$ is at most 4m. If there is a cut set of $G^*(A,2mK_1,mK_1;K_{n,n},nK_1)$ with fewer than 4m vertices, say V, then that set would contain a minimal cut set with fewer than 4m vertices. This set would then be the vertices of modules associated with a cut set of G. Yet any other minimal cut set of G would have three or more vertices, and so the set V is required to contain at least 6m vertices. So there is no cut set of $G^*(A,2mK_1,mK_1;K_{n,n},nK_1)$ with fewer than 4m vertices. Therefore, the connectivity of $G^*(A,2mK_1,mK_1;K_{n,n},nK_1)$ is 4m. ❑

Theorem 3.37 $\text{Expan}(G^*(A,K_{n,n},nK_1;2mK_1,mK_1)) \le \frac{4488}{34933}$.

Proof : Let S be the set of vertices of $G^*(A,K_{n,n},nK_1;2mK_1,mK_1)$ containing all the vertices of the modules associated with the vertices of $G^*(A,K_{n,n},nK_1;2mK_1,mK_1)$ to the left of the vertices u and v in Figure 3.6. Then $|S| = 32n + 34m = \frac{362}{5}m$. Since the order of the graph $G^*(A,K_{n,n},nK_1;2mK_1,mK_1)$ is $\frac{748m}{5}$, we find the value $\text{Ex}(S) = \frac{4488}{34933}$. Yet $\text{expan}(G^*(A,K_{n,n},nK_1;2mK_1,mK_1))$ is the maximum real number c such that c is no more than Ex(T) over all nonempty proper subsets T of the vertex set of $G^*(A,K_{n,n},nK_1;2mK_1,mK_1)$. So $\text{expan}(G^*(A,K_{n,n},nK_1;2mK_1,mK_1)) \le \frac{4488}{34933}$. ❑

Corollary 3.38 Suppose that S is a subset of the vertex set of $G^*(A,K_{n,n},nK_1;2mK_1,mK_1)$, and that $|B(S)| \ge \frac{4488|S|(p-|S|)}{34933p}$, then $\text{Ex}(S) \ge \frac{4488}{34933}$.

Proof: Solve the inequality $\text{Ex}(S) = \frac{p|B(S)|}{|S|(p-|S|)} \ge \frac{4488}{34933}$ for $|B(S)|$. ❑

Corollary 3.39 Suppose that S is a subset of the vertex set of $G^*(A,K_{n,n},nK_1;2mK_1,mK_1)$, and that the boundary, B(S), satisfies the inequality $|B(S)| \ge \frac{(4488)(187)m}{(5)\,34933}$. Then $\text{Ex}(S) \ge \frac{4488}{34933}$.

Proof : Let $|S| = p/2$ in Corollary 3.37. This value maximizes the denominator of the formula for Ex(S), thus giving the minimum value for Ex(S). ❑

Corollary 3.40 Let S be a subset of the vertex set of $G^*(A,K_{n,n},nK_1;2mK_1,mK_1)$. If S contains an uncovered half-module, that

is to say S has vertices that belong to a half-module M, but no vertices belong to S that are from half-modules adjacent to M, then $Ex(S) \geq \frac{4488}{34933}$.

Proof: If S has an uncovered half-module M, then B(S) contains all the vertices of the half-modules adjacent to M. Since there are six half-modules adjacent to M, B(S) contains at least 6m vertices. Yet $6m > \frac{(4488)(187)m}{(5)(34933)}$. By Corollary 3.38, $Ex(S) \geq \frac{4488}{34933}$. □

Lemma 3.41 Suppose that S is a subset of the vertex set of $G^*(A,K_{n,n},nK_1;2mK_1,mK_1)$, and that S contains a partial half-module. Then there is a subset T of the vertex set of $G^*(A,K_{n,n},nK_1;2mK_1,mK_1)$ so that $Ex(S) \geq Ex(T)$ and T contains no partial half-modules.

Proof: By Corollary 3.39, all the partial half-modules of S are covered. By Corollary 3.37, if $|B(S)| \geq \frac{488|S|(p-|S|)}{34933p}$, then we do not consider the set S. So we may assume that $|B(S)| < \frac{4488|S|(p-|S|)}{34933p}$. Let M be the vertices of some partial half-module of S. Let the set S' be the set of vertices belonging to S along with one vertex, u, of M that is not in S. Since M is a partial half-module of S, B(S') is a subset of B(S). Since M is a covered half-module of S, however, the vertex u is not in the set B(S'), but is in the set B(S). So $|B(S')| = |B(S)| - 1$. Also, $|S'| = |S| + 1$. We now show that $Ex(S') \leq Ex(S)$. Suppose this is not the case. Let $b = |B(S)|$ and $s = |S|$. We then have

$$Ex(S) - Ex(S') = \frac{pb}{s(p-s)} - \frac{p(b-1)}{(s+1)(p-s-1)} < 0.$$

By solving for b in this inequality, we have

$$b < \frac{-s(p-s)}{p-2s-1} \qquad \text{when } p - 2s - 1 > 0$$

$$0 < 0 \qquad \text{when } p - 2s - 1 = 0$$
$$b > \frac{s(p-s)}{2s+1-p} \qquad \text{when } p - 2s - 1 < 0.$$

The first two cases are clearly contradictions, since $|B(S)|$ is a positive integer. So we may assume that we are in the third case, that $p - 2s - 1 < 0$ and that $b > \frac{s(p-s)}{2s+1-p}$. Employing Corollary 3.37 shows that $\frac{4488s(p-s)}{34933p} > b > \frac{s(p-s)}{2s+1-p}$. Solving for s shows that the order of S must be greater than the number of vertices in $G^*(A,K_{n,n},nK_1;2mK_1,mK_1)$. This is a contradiction. So we must have $Ex(S') \leq Ex(S)$. If we repeat the process until the new set has no partial half-modules, we have found a set T that satisfies the conclusion to the lemma. ❑

Following the same process, we arrive at corresponding results for the graph $G^*(A,2mK_1,mK_1;K_{n,n},nK_1)$.

Theorem 3.42 $\mathrm{Expan}(G^*(A,2mK_1,mK_1;K_{n,n},nK_1)) \leq \frac{935}{8736}$.

Corollary 3.43 Suppose that S is a subset of the vertex set of $G^*(A,2mK_1,mK_1;K_{n,n},nK_1)$, and that $|B(S)| \geq \frac{935|S|(p-|S|)}{8736p}$. Then $Ex(S) \geq \frac{935}{8736}$.

Corollary 3.44 Suppose that S is a subset of the vertex set of $G^*(A,2mK_1,mK_1;K_{n,n},nK_1)$, and that $|B(S)| \geq \frac{935\,p}{34944}$. Then $Ex(S) \geq \frac{935}{8736}$.

Corollary 3.45 Let S be a subset of the vertex set of $G^*(A,2mK_1,mK_1;K_{n,n},nK_1)$. If S contains an uncovered half-module, then $Ex(S) \geq \frac{935}{8736}$.

Lemma 3.46 Suppose that S is a subset of the vertex set of $G^*(A,2mK_1,mK_1;K_{n,n},nK_1)$, and that S contains a partial half-module. Then there is a subset T of the vertex set of $G^*(A,2mK_1,mK_1;K_{n,n},nK_1)$ so that $Ex(S) \geq Ex(T)$ and T contains no partial half-modules.

We are now prepared to find the expander coefficients of the graphs in question, and to discuss the limits of the expanders as we take pairs of the graphs further in the sequence.

Theorem 3.47 The expander coefficient of $G^*(A,K_{n,n},nK_1;2mK_1,mK_1)$ is $\frac{4488}{34933}$.

Proof Assume there is a set S so that $Ex(S) < \frac{4488}{34933}$. Then we know that $|B(S)| < \frac{4488\,|S|(p-|S|)}{34933\,p} \leq \frac{4488\,[p/2]^2}{34933\,p}$, since $|S|(p-|S|) \leq [p/2]^2$. Solving for $|B(S)|$, we see that $|B(S)| < \frac{3357024m}{698660} < 5n$. Since there can be no partial half-modules, we can say that $|B(S)| \leq 4n$. Also, by Corollary 3.4, $|B(S)| \geq 4n$. So $|B(S)| = 4n$. Yet this implies that the boundary of the set S is the unique minimum cut set of $G^*(A,K_{n,n},nK_1;2mK_1,mK_1)$. So the set in question must be the set of vertices belonging to the modules to the left of the vertices u and v in Figure 3.6. So the expander coefficient of $G^*(A,K_{n,n},nK_1;2mK_1,mK_1)$ is $\frac{4488}{34933}$. □

Theorem 3.48 The expander coefficient of $G^*(A,2mK_1,mK_1;K_{n,n},nK_1)$ is $\frac{935}{8736}$

Proof Assume there is a set S so that $Ex(S) < \frac{935}{8736}$ Then we know that $|B(S)| < \frac{935\,|S|(p-|S|)}{8736\,p} \leq \frac{935\,[p/2]^2}{8736\,p}$, since $[p/2]^2 \geq |S|(p-|S|)$. Solving for $|B(S)|$, we observe that $|B(S)| < \frac{34969}{8736}m < 5m$. Since there can be no

partial half-modules, we can say that $|B(S)| \leq 4m$. Also, by Corollary 3.4, $|B(S)| \geq 4m$. So $|B(S)| = 4m$. Yet this implies that the boundary of the set S is the unique minimum cut set of $G^*(A,2mK_1,mK_1;K_{n,n},nK_1)$. So the set in question must be the set of vertices belonging to the modules to the left of the vertices u and v in Figure 3.6. So the expander coefficient of $G^*(A,2mK_1,mK_1;K_{n,n},nK_1)$ is $\frac{935}{8736}$. ❑

Corollary 3.49 The expander coefficients of $G^*(A,K_{n,n},nK_1;2mK_1,mK_1)$ and $G^*(A,2mK_1,mK_1;K_{n,n},nK_1)$ are different.

Corollary 3.50 The limit of the expander coefficients of $G^*(A,K_{n,n},nK_1;2mK_1,mK_1)$ is $\frac{4488}{34933}$.

Corollary 3.51 The limit of the expander coefficients of $G^*(A,2mK_1,mK_1;K_{n,n},nK_1)$ is $\frac{935}{8736}$.

Corollary 3.52 The limit of the expander coefficients of $G^*(A,K_{n,n},nK_1;2mK_1,mK_1)$ is different from the limit of the expander coefficients of $G^*(A,2mK_1,mK_1;K_{n,n},nK_1)$.

CHAPTER IV

THE FOLKMAN GRAPH, 2-SET COMPOSITION AND THE EXPANDER COEFFICIENT

We have seen four families of pairs of isospectral graphs that have unequal expander coefficients. The graphs used, however, do not model what would be considered good networks. In this chapter, we will form a pair of isospectral graphs that more accurately represent realistic networks, and compute their expander coefficients.

The graph in Figure 4.1 is often referred to as the Folkman graph. It is a 4-regular bipartite graph on twenty vertices, with each partite set containing ten vertices. It also happens to be the smallest edge transitive graph that is not vertex transitive. That it is a regular bipartite graph makes the Folkman graph a candidate to use for 2-set composition resulting in cospectral graphs. Since we are also interested in isospectral graphs, we will need to chose as modules graphs that will result in regular graphs after the composition is performed. We will again use a complete graph, K_n, and an empty graph, mK_1. Since the Folkman graph is 4-regular, each vertex in a K_n module will be adjacent to all the vertices in four mK_1 modules, or 4m vertices, along with n-1 vertices from its own module. Any vertex in an mK_1 module will only be adjacent to the vertices of four K_n modules, or 4n vertices. For the resulting graph to be regular, m and n must satisfy $4m + n - 1 = 4n$, or $4m = 3n + 1$. The

smallest non-trivial solutions to this equation are m = 4 and n = 5. The graphs are then 20-regular and each have order 90.

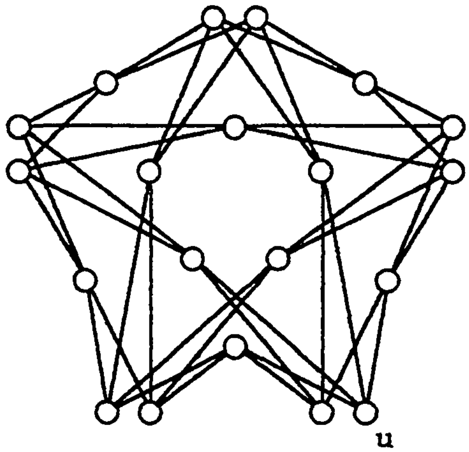


Figure 4.1

Lemma 4.1 Let G be the Folkman graph, and let A be the partite set of G containing the vertex u in Figure 4.1. Then the graphs $G(A,K_5;4K_1)$ and $G(A,4K_1;K_5)$ are isospectral.

Proof By the discussion above, the graphs are both 20-regular. That the graphs are cospectral is a result of Theorem 1.2. That they are isospectral is a result of Theorem 1.4. ❑

Now that we determined that the two graphs are isospectral, we need to compute the expander coefficient of each graph. The arguments are similar to those of Chapter III, but it is more difficult to establish that the expander coefficient is what we claim it to be. We first compute a value for Ex(S) for a certain set S. We then find an upper bound on the order of the boundary of any set that might improve the computed value

of Ex(S). We can then assert that for any subset T of the vertex set of either graph that contains partial modules, there is another set, T', with no partial modules so that Ex(T') ≤ Ex(T). We then examine possible cut sets of the 2-set composition graphs to see that no other set T provides a value of Ex(S) smaller than the value we claim it to be.

Lemma 4.2 The graphs $G(A,K_5;4K_1)$ and $G(A,4K_1;K_5)$ are 16-connected.

Proof It is easily seen that the graph G is 4-connected. If the vertices of the modules associated with the vertices that are shaded in Figure 4.2 are removed from the graph $G(A,K_5;4K_1)$, the result is disconnected graph. So the connectivity of $G(A,K_5;4K_1)$ is at most 16. The minimum number of vertices of the graph $G(A,K_5;4K_1)$ that are associated with four vertices of G is 16. Also, any minimal cut set of G with more than four vertices is associated with more than 16 vertices in the graph $G(A,K_5;4K_1)$. The connectivity of $G(A,K_5;4K_1)$ is 16. Similarly, for the graph $G(A,4K_1;K_5)$, the vertices of the modules associated with the vertices of G shaded in Figure 4.3 is a cut set of $G(A,4K_1;K_5)$ of order 16, and a similar argument shows that the connectivity of $G(A,4K_1;K_5)$ is 16. ❑

We first examine the graph $G(A,K_5;4K_1)$, and then address the graph $G(A,4K_1;K_5)$.

Lemma 4.3 $\mathrm{Expan}(G(A,K_5;4K_1)) \leq \frac{45}{52}$.

Proof Let S be the set of vertices of $G(A,K_5;4K_1)$ containing all the vertices of the modules associated with the vertices of G indicated by Figure 4.4. Then $|S| = 64$, $|B(S)| = 16$, and $p - |S| = 26$. Computing Ex(S)

for this set yields the value above. Yet expan(G(A,K_5;$4K_1$)) is the minimum of Ex(T) taken over all nonempty proper subsets T of the vertex set of G(A,K_5;$4K_1$). So expan(G(A,K_5;$4K_1$)) $\leq \frac{45}{52}$. □

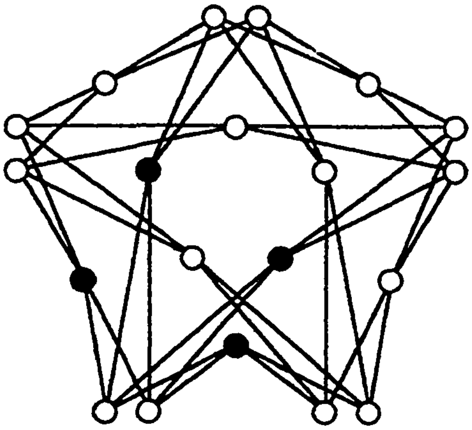


Figure 4.2

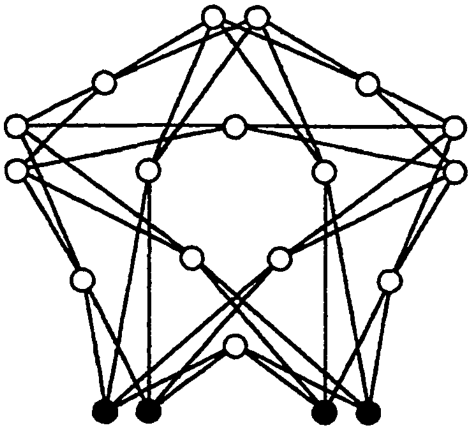


Figure 4.3

Corollary 4.4 If S is a subset of the vertex set of G(A,K_5;$4K_1$) and $|B(S)| \geq \frac{|S|(90-|S|)}{104}$, then $Ex(S) \geq \frac{45}{52}$.

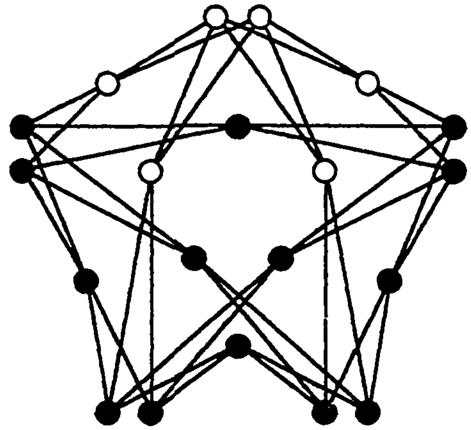


Figure 4.4

Proof Solve the following inequality for $|B(S)|$.

$$\text{Ex}(S) = \frac{90\,|B(S)|}{|S|(90 - |S|)} \geq \frac{90}{104}. \square$$

Corollary 4.5 If S is a subset of the vertex set of $G(A,K_5;4K_1)$ and $|B(S)| \geq 20$, then $\text{Ex}(S) \geq \frac{45}{52}$.

Proof Let $|S| = \frac{p}{2} = 45$ in Corollary 4.4. This value maximizes the denominator of the formula for Ex(S), thus giving the minimum possible value for Ex(S). $\square$

Corollary 4.6 Let S be a subset of the vertex set of $G(A,K_5;4K_1)$. If S contains an uncovered $4K_1$ module, then $\text{Ex}(S) \geq \frac{45}{52}$.

Proof If S contains such an uncovered module, say M, then B(S) contains all the vertices of the modules adjacent to M. Since each of these modules is a copy of K_5, $|B(S)| \geq 20$, and by Corollary 4.7, $\text{Ex}(S) \geq \frac{45}{52}$.

$\square$

Lemma 4.7 If S is a set of vertices in $G(A,K_5;4K_1)$ with more than one partial module, then there is some subset of the vertex set of $G(A,K_5;4K_1)$, say S', containing at most one partial module which satisfies $|S| = |S'|$ and $|B(S')| \leq |B(S)|$.

Proof A process will be defined that, when repeated, will yield the desired set S'. By the hypothesis, there are at least two partial modules, M_1 and M_2, of the set S. Let u be a vertex of M_1 that is not in S and let v be a vertex of M_2 that is in S. Let $S' = S - \{v\} + \{u\}$. We will discuss two cases.

Case 1: One of M_1 and M_2 is covered. It is noted here that, for this discussion, a module that is a copy of K_5 covers itself. Without loss of generality, let M_1 be the covered module. Then B(S') is no larger than B(S), since u is in B(S) but not in B(S'), and v is not in B(S) but may be in B(S'). Also, if M_2 is then empty and there is some empty module M_3 that is then left uncovered, B(S') is smaller still. Repeat this process until M_1 is no longer a partial module, or until S and M_2 share no vertices.

Case 2: Neither M_1 nor M_2 are covered and they are each copies of $4K_1$. Then $|B(S')| \leq |B(S)|$. Repeat until M_1 is no longer partial or until M_2 and S share no vertices.

By repeating this process, a set S' is formed satisfying the conditions set out above, that has at most one partial module. ❑

Corollary 4.8 Let S be a subset of the vertex set of $G(A,K_5;4K_1)$. If S contains an uncovered K_5 module, say M, and B(S) contains all the vertices of some module not adjacent to M, then $Ex(S) \geq \frac{45}{52}$.

Proof If S contains such an uncovered module, then B(S) contains all the vertices of the modules adjacent to M. Since each of these modules is a copy of $4K_1$, $|B(S)| \geq 16$. Yet, B(S) contains some module not adjacent to M. This implies $|B(S)| \geq 16 + 4 = 20$. By Corollary 4.7, $Ex(S) \geq \frac{45}{52}$. ❑

Lemma 4.9 Let S be a subset of the vertex set of $G(A,K_5;4K_1)$ that has a K_5 module that is not covered by some other module. Then $Ex(S) \geq \frac{(90)(16)}{(65)(25)}$.

Proof Suppose S contains an uncovered K_5 module, say M. By Corollary 4.9, B(S) can contain no modules other than those adjacent to M. By inspection of the graph G, it is clear that S must either (i) contain all the vertices associated with the modules indicated by the Figure 4.5, less up to four vertices, since S may have up to one partial module, or (ii) contain all the vertices of one of the modules, and any of the vertices of the other module, indicated by Figure 4.6. These possibilities are due to the fact that the boundary cannot contain all the vertices of any module not adjacent to the uncovered copy of K_5, and that B(S) needs to be a cut set, by Corollary 3.4. In the first case, the smallest numerator possible for Ex(S) is (90)(16), by Corollary 3.4 and the fact that the connectivity of this graph is 16. The largest denominator possible is (65)(25), since the set S would contain all the vertices of the modules indicated in Figure 4.5, less up to four vertices. So $Ex(S) \geq \frac{(90)(16)}{(65)(25)}$ in this case. In the second case,

the numerator is again as small as possible. If any vertex of S belonged to a module other than those indicated by Figure 4.6, the boundary would contain at least one more module, which cannot happen by Corollary 4.9. Also, any vertices of S may belong to the other indicated module, since these two modules have identical neighborhoods. The largest possible denominator is then (80)(10), and $\mathrm{Ex}(S) \geq \frac{(90)(16)}{(10)(80)} > \frac{(90)(16)}{(65)(25)}$ in this case.
❑

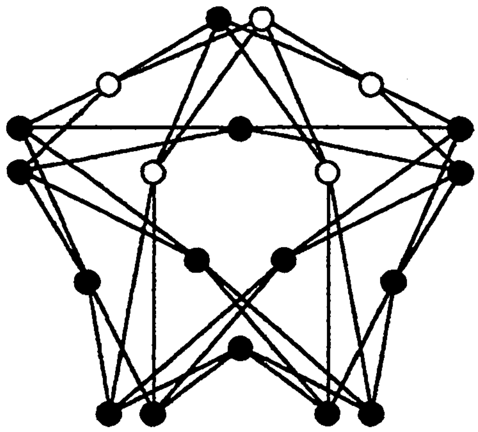


Figure 4.5

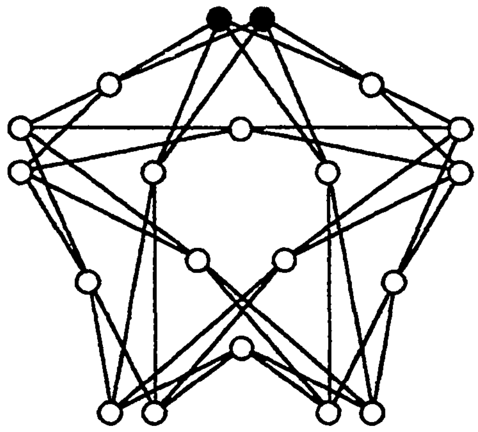


Figure 4.6

Corollary 4.10 If a subset S of the vertex set of $G(A,K_5;4K_1)$ satisfies the conditions of Lemma 4.9, then $Ex(S) \geq \frac{45}{52}$.

Lemma 4.11 For any set S with one partial module M in the graph $G(A,K_5;4K_1)$, there is a set S' with no partial modules such that $Ex(S) \geq Ex(S')$.

Proof: By Lemma 4.7, S has at most one partial module. If it does not have any, we are done. So assume S contains one partial module. By Corollary 4.6 and Corollary 4.10, any set S that might satisfy $Ex(S) \leq \frac{45}{52}$ has no uncovered partial modules. By Corollary 4.4, if $|B(S)| \geq \frac{|S|(90-|S|)}{104}$, then we do not consider the set S. So we may assume that $|B(S)| < \frac{|S|(90-|S|)}{104}$. Let M be the vertices of the partial module of S. Let the set S' be the set of vertices belonging to S along with one vertex, u, of M that is not in S. Since M is a partial module of S, B(S') is a subset of B(S). Since M is a covered module of S, however, the vertex u is not in the set B(S'), but is in the set B(S). So $|B(S')| = |B(S)| - 1$. Also, $|S'| = |S| + 1$. We now show that $Ex(S') \leq Ex(S)$. Suppose this is not the case. Let $b = |B(S)|$ and $s = |S|$. We then have

$$Ex(S) - Ex(S') = \frac{90b}{s(90-s)} - \frac{90(b-1)}{(s+1)(90-s-1)} < 0.$$

By solving for b in this inequality, we have

$$b < \frac{-s(90-s)}{90-2s-1} \qquad \text{when } 89 - 2s > 0$$

$$0 < 0 \qquad \text{when } 89 - 2s = 0$$

$$b > \frac{s(90-s)}{2s+1-90} \qquad \text{when } 89 - 2s < 0.$$

The first two cases are clearly contradictions, since $|B(S)|$ is a positive integer. So we may assume that we are in the third case, that s and b satisfy $89 - 2s < 0$ and that $b > \frac{s(90-s)}{2s-89}$. Employing Corollary 4.4 shows that $\frac{s(90-s)}{104} > b > \frac{s(90-s)}{2s-89}$. Solving for s shows that the order of S must be greater than the number of vertices in $G(A,K_5;4K_1)$. This is a contradiction. So we must have $Ex(S') \leq Ex(S)$. If we repeat the process until the new set has no partial modules, we have found a set T that satisfies the conclusion to the lemma. ❑

Following the same process, we find some corresponding results for the graph $G(A,4K_1;K_5)$. However, some preliminary comments are needed. In a graph H, we may *subdivide an edge uv* by removing the edge uv of H, and adding a vertex w along with the edges uw and vw. The *subdivision graph of H,* denoted SH, is the graph formed by subdividing each edge of H. If we let the set A be the vertices of SK_5 that are the original vertices of the K_5, then $SK_5(A,8K_1;K_5)$ and $G(A,4K_1;K_5)$ are isomorphic. In some of the following results, it is more helpful to view the graph $G(A,4K_1;K_5)$ as the graph $SK_5(A,8K_1;K_5)$.

Lemma 4.12 $\mathrm{Expan}(G(A,4K_1;K_5)) \leq \frac{(16)(90)}{1449}$.

Proof Let S be the set of vertices of $G(A,4K_1;K_5)$ containing all the vertices of the modules associated with the vertices of G indicated by Figure 4.7. Then $|S| = 69$, $|B(S)| = 16$, and $p - |S| = 21$. Computing Ex(S) for this set yields the value above. Yet $\mathrm{expan}(G(A,4K_1;K_5))$ is the minimum of Ex(T) taken over all subsets T of the vertex set of $G(A,4K_1;K_5)$. So $\mathrm{expan}(G(A,4K_1;K_5)) \leq \frac{(16)(90)}{1449}$. ❑

Corollary 4.13 If S is a subset of the vertex set of $G(A,4K_1;K_5)$ and $|B(S)| \geq \frac{16|S|(90-|S|)}{1449}$, then $Ex(S) \geq \frac{(16)(90)}{1449}$.

Proof Solve the following inequality for $|B(S)|$.

$$Ex(S) = \frac{90|B(S)|}{|S|(90-|S|)} \geq \frac{(16)(90)}{1449}. \square$$

Corollary 4.14 If S is a subset of the vertex set of $G(A,4K_1;K_5)$ and $|B(S)| \geq 23$, then $Ex(S) \geq \frac{(16)(90)}{11449}$.

Proof Let $|S| = \frac{p}{2} = 45$ in Corollary 4.6. This value maximizes the denominator of the formula for Ex(S), thus giving the minimum value for Ex(S). $\square$

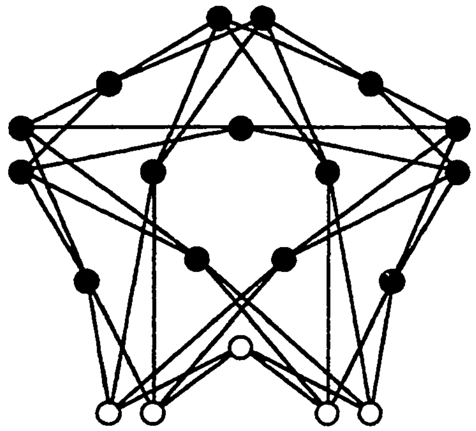


Figure 4.7

Corollary 4.15 Let S be a subset of the vertex set of $SK_5(A,8K_1;K_5)$. If S contains an uncovered $8K_1$ module, and all the vertices of some other module, then $Ex(S) \geq \frac{(16)(90)}{1449}$.

Proof If S contains such an uncovered module, say M, then B(S) contains all the vertices of the modules adjacent to M. Since each of

these modules is a copy of K_5, $|B(S)| \geq 20$. If B(S) also contains all the vertices of some other module, then $|B(S)| \geq 24$, and by Corollary 4.14, $Ex(S) \geq \frac{(16)(90)}{1449}$. ❑

Corollary 4.16 Let S be a subset of the vertex set of $SK_5(A,8K_1;K_5)$. If S contains an uncovered K_5 module, say M, and B(S) contains all the vertices of some two other modules not adjacent to M, then $Ex(S) \geq \frac{(16)(90)}{1449}$.

Proof If S contains such an uncovered module, then B(S) contains all the vertices of the modules adjacent to M. Since each of these modules is a copy of $8K_1$, $|B(S)| \geq 16$. Yet, B(S) contains some two modules not adjacent to M. This implies $|B(S)| \geq 16 + 8 = 23$. By Corollary 4.14, $Ex(S) \geq \frac{(16)(90)}{1449}$. ❑

Lemma 4.17 If S is a set of vertices in $SK_5(A,8K_1;K_5)$ with more than one partial module, then there is some subset of the vertex set of $SK_5(A,8K_1;K_5)$, say S', containing at most one partial module which satisfies $|S| = |S'|$ and $|B(S')| \leq |B(S)|$.

Proof The proof of this lemma is identical to that of Lemma 4.7. ❑

Lemma 4.18 For any set S with one partial module M in the graph $SK_5(A,8K_1;K_5)$, there is a set S' with no partial modules such that $Ex(S) \geq Ex(S')$.

Proof Let M be a covered partial module in $SK_5(A,8K_1,K_5)$. The same argument as used in Lemma 4.11 carries over, with slightly different constants. Now suppose the module M of the set S is an uncovered

partial module. If $|B(S)| \geq 23$, we do not consider this set . So $16 \leq |B(S)| \leq 22$. If M is a copy of K_5, it covers itself and this case was handled earlier. If M is a copy of $8K_1$, then $|B(S)| \geq 20$. Since M is the only partial module, $|B(S)| = 20$. This occurs when the set S consists of all the vertices of the modules of $SK_5(A,8K_1,K_5)$ indicated in Figure 4.8 or Figure 4.9. If S is the set indicated by Figure 4.8, S = M. This implies that $1 \leq |S| \leq 7$. In this case, $Ex(S) \geq \frac{(20)(90)}{(7)(83)}$ and so $Ex(S) \geq \frac{(16)(90)}{(1449)}$. If S is the set indicated by Figure 4.9, $S = V(G) - (B(S) + M - S)$. This implies that $|S| \geq 63$, and $Ex(S) \geq \frac{(20)(90)}{(63)(27)} \geq \frac{(16)(90)}{1449}$. So either $Ex(S) \geq \frac{(16)(90)}{1449}$ or, as in Lemma 4.11, there is another set S' that has no partial modules and Ex(S) $\geq$ Ex(S'). ❑

We are almost to the point where we can compute the expander coefficients of the two graphs $G(A,K_5;4K_1)$ and $G(A,4K_1;K_5)$. Before we do, we need to discuss what minimal cut sets in the Folkman graph look like. As was earlier noted, the Folkman graph is 4-connected. The only cut sets of the Folkman graph of order four are the vertices adjacent to any one vertex. Any minimal cut set with five vertices must be the unique set (up to symmetry) indicated by the shaded vertices of Figure 4.8. Now we compute the expander coefficients of the graphs $G(A,K_5;4K_1)$ and $G(A,4K_1;K_5)$.

Theorem 4.19 The expander coefficient of the graph $G(A,K_5;4K_1)$ is $\frac{45}{52}$.

Proof We already have seen, in Lemma 4.3, that $\text{expan}(G(A,K_5;4K_1)) \leq \frac{90}{104}$. Also, any set S that might satisfy $Ex(S) \leq \frac{90}{104}$ must satisfy $|B(S)| \leq 19$. By Corollary 3.4, B(S) must be a cut set of $G(A,K_5;4K_1)$, which would

contain a minimal cut set of $G(A,K_5;4K_1)$. All minimal cut sets of $G(A,K_5;4K_1)$ are vertices contained in the modules of minimal cut sets of G, as in Theorem 2.4. Any minimal cut set of $G(A,K_5;4K_1)$ with five or more vertices then contains at least 22 vertices. So the boundary of a set S that might satisfy $Ex(S) \le \frac{90}{104}$ cannot have a boundary containing more than four modules. If B(S) is a cut set containing four copies of K_5 then $|B(S)| \ge 20$ and again this case is not considered. The only other subsets of the vertex set of $G(A,K_5;4K_1)$ left under consideration are the vertices of the modules associated with the vertices shaded in the graph of Figure 4.6 or Figure 4.4. The first set, however, has $Ex(S) = \frac{(90)(16)}{(10)(80)} > \frac{90}{104}$. The other set is the set used to establish Lemma 4.3. So $\text{expan}(G(A,K_5;4K_1)) = \frac{90}{104}$. □

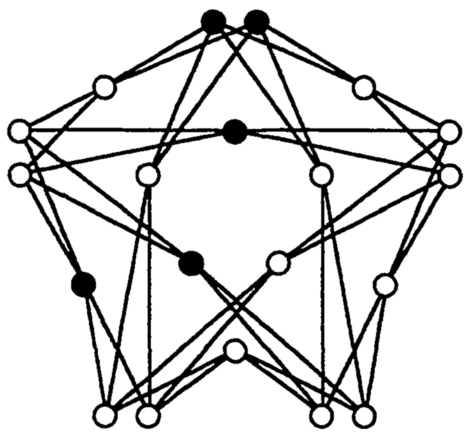


Figure 4.8

Theorem 4.20 The expander coefficient of the graph $G(A,4K_1;K_5)$ is $\frac{(16)(90)}{1449}$.

Proof We already have seen, in Lemma 4.12, that expan(G(A,4K_1;K_5)) $\le \frac{(16)(90)}{1449}$. Also, any set S that might satisfy Ex(S) $\le \frac{90}{104}$ must satisfy $|B(S)| \le 22$. By Corollary 3.4, B(S) must be a cut set of G(A,4K_1;K_5), which would contain a minimal cut set of G(A,K_5;4K_1). All minimal cut sets of G(A,4K_1;K_5) are vertices contained in the modules of minimal cut sets of G, as in Theorem 2.4. Any minimal cut set of G(A,4K_1;K_5) with five or more vertices then contains at least 23 vertices. So the boundary of a set S that might satisfy Ex(S) $\le \frac{(16)(90)}{1449}$ cannot have a boundary containing more than four modules. If B(S) is a cut set containing four copies of K_5 then $|B(S)| = 20$. The only subsets of the vertex set of G(A,4K_1;K_5) in this situation under consideration are the sets consisting of the vertices of the modules associated with the vertices shaded in either graph of Figure 4.12. The values of Ex(S) in these two cases are $\frac{(20)(90)}{(8)(82)}$ and $\frac{(20)(90)}{(62)(28)}$, each of which is larger than $\frac{(16)(90)}{1449}$. The only other subsets of the vertex set of G(A,4K_1;K_5) left under consideration are the vertices of the modules associated with the vertices shaded in either Figure 4.9 or Figure 4.7. The first set, however, has Ex(S) $= \frac{(90)(16)}{(5)(85)} \ge \frac{(16)(90)}{1449}$. The other set is the set used to establish Lemma 4.12. Therefore expan(G(A,4K_1;K_5)) $= \frac{(16)(90)}{1449}$. ❑

In this dissertation, we have defined two compositions that can be used to construct cospectral graphs, and described some of the properties of those graphs. We have also answered the question whether or not isospectral graphs must have equal expander coefficients. We have seen four infinite families of isospectral pairs of graphs with unequal expander

coefficients, and also a pair of isospectral graphs that more realistically represent networks, and that have unequal expander coefficients.

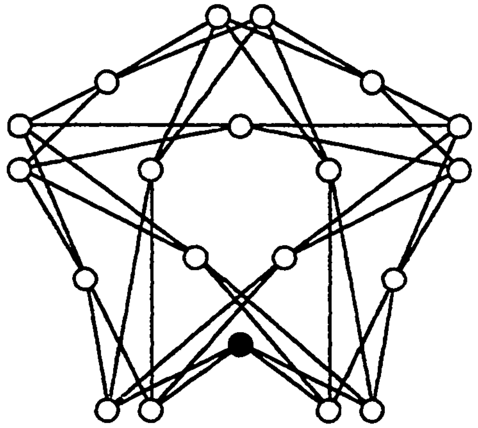


Figure 4.9

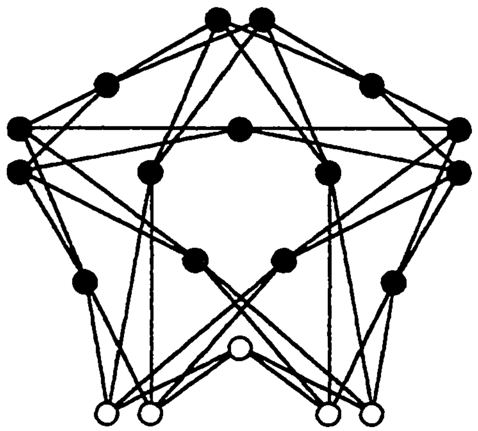


Figure 4.10

There are more questions that might be answered. Can the Folkman graphs be used to form another infinite family of isospectral graphs with unequal expander coefficients? Such a family would be easy

to define, but the computation of the expander coefficients is more difficult. What about the other definitions of the expander coefficient? Can the compositions introduced here be used to answer the same questions? Also, there are many other parameters of graphs that are at least bounded by the spectrum of the graph. How different can those parameters be? The constructions defined here might be useful in exploring these questions.

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