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Symbol Sense and Its Development in Two Computer Algebra System Environments

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SYMBOL SENSE AND ITS DEVELOPMENT IN TWO
COMPUTER ALGEBRA SYSTEM ENVIRONMENTS

by

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Brian A. Keller, Ph.D.
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The purposes of this study were to: (a) explicate the nature of symbol sense, (b) determine the differential effects of two computer algebra system (CAS) environments on students' development of symbol sense, (c) explore differences in symbol sense among students using a CAS and students not using a CAS, and (d) examine students' achievement in calculus with and without the use of a CAS.

Six sections of first-semester calculus at Western Michigan University during the 1992-93 academic year were used in the study. The investigator taught two CAS sections both semesters (n = 41, 34 Fall and n = 35, 34 Winter). Each semester a different professor who was recognized by students and the university for excellence in teaching taught a third comparison section (n = 39 and n = 36). The CAS sections used investigator-developed laboratory materials for MAPLE and THEORIST. The comparison sections followed a traditional lecture/discussion format which included the use of graphics calculators. Data were gathered from pre- and post-treatment administration of an investigator-developed test of symbol sense, from departmental comprehensive final exams, from a laboratory practical, and from 5 video-taped interviews with 4 students from each section.

Quantitative data were analyzed with the pretest as a covariate. The results indicated no significant differences among treatment groups on any of the criterion measures with the exception of a significant difference in favor of the comparison section over the THEORIST section on the Fall comprehensive final exam (p < .04).
The interviews suggested that the CAS students more frequently identified patterns and standard forms, while the traditional students were better able to modify standard procedures. The CAS students used comparative information not presented within the problem to guide judgments and strategy-making more often than the comparison students. For the THEORIST sections, this information was typically graphical in nature. The CAS students more often purposefully used the symbolic transformations of calculus than did the comparison students.

It was concluded that the manner of symbolic manipulation may not be a factor in students' development of symbol sense and that a laboratory approach to calculus may aid in students' broader development of symbol sense.
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Keller, Brian A., Ph.D.
Western Michigan University, 1993

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This work is dedicated to Daniel H. Carpenter, a friend who is missed greatly.

Brian A. Keller
# TABLE OF CONTENTS

**ACKNOWLEDGMENTS** ....................................................................................... ii  
**LIST OF TABLES** ................................................................................................... vi  
**LIST OF FIGURES** .................................................................................................. viii  

## CHAPTER

**I. INTRODUCTION** ............................................................................................. 1
  Climate of Change ................................................................................. 1  
  Number Sense ........................................................................................ 4  
  Symbol Sense ......................................................................................... 8  
  Making Sense of Symbol Sense ............................................................. 11  
    Completing the List of Behaviors Exhibiting Symbol Sense ...... 11  
    Theory-Based Symbol Sense ........................................................ 15  
  Symbol Sense and the Present Study ..................................................... 19

**II. REVIEW OF THE LITERATURE** ................................................................ 22
  Overview ................................................................................................ 22  
  Use of Symbolic Algebra Software ..................................................... 22  
    Use of Computer Algebra Systems .................................................. 23  
    Comments on the Use of Computer Algebra Systems.............. 27  
    Use of Super Calculators ............................................................ 28  
    Use of Calculus-Specific Software ............................................... 29  
  Symbol Sense ......................................................................................... 32  
    Student Intuition for Symbolic Expressions ......................... 32  
    Pattern Recognition Within Symbolic Expressions ............... 35  
    Metacognitive Behaviors on Symbolic Expressions ............. 37
Table of Contents—Continued

CHAPTER

Use of Technology to Create Active Symbolic Expressions .......... 39
Symbol Sense Problems ................................................................. 41
Summary .......................................................................................... 43

III. RESEARCH DESIGN AND METHODOLOGY .............................. 45

Introduction ..................................................................................... 45
Design .............................................................................................. 47
Setting ............................................................................................. 47
Selection of CAS Packages .......................................................... 47
  Laboratory Materials ................................................................. 49
Selection of Subjects ...................................................................... 50
  Laboratory Facilities ................................................................. 51
  Experimental Sections .............................................................. 51
Assessment of Outcomes ............................................................. 52
  Quantitative Measures and Their Administration ................... 53
  Qualitative Interviews and Their Conduct ............................... 57
Null Hypotheses ............................................................................. 63

IV. RESULTS .................................................................................. 65

Introduction ..................................................................................... 65
Pretest .............................................................................................. 66
Examination of Semester Effects ................................................. 67
Analysis of Treatment Effects on Symbol Sense ......................... 67
Analysis of Treatment Effects on Comprehensive Final ............. 69
Analysis of Treatment Effects on Laboratory Practical ............... 71
# Table of Contents—Continued

## CHAPTER

V. INTERVIEWS ................................................................. 74  
  Introduction ................................................................. 74  
  Differences Among Treatments ........................................... 75  
    Identification of Patterns and Forms ............................. 75  
    Modification of Standard Procedures ....................... 76  
    Comparison and Journeys Outside of the Problem Domain .... 77  
    Planned Use of Manipulations ....................................... 78  
  Commonalities Among Treatments ...................................... 79  
    Function Notation ..................................................... 79  
    Creating Examples .................................................... 80  
    Conceptual Understanding ......................................... 81  
    Problem Solving ....................................................... 82  
    Number Sense ......................................................... 82  

VI. SUMMARY AND CONCLUSIONS ........................................... 84  

## APPENDICES

A. Selected Laboratory Materials ............................................ 93  
B. Pretest and Posttest Instruments ......................................... 181  
C. Pretest and Posttest Scoring Rubric .................................... 197  
D. Comprehensive Final Exams and Laboratory Practicals ............ 205  
E. Additional Interview Tasks ............................................... 224  
F. Results of Tests for Homogeneity of Regression .................... 227  
G. Human Subjects Institutional Review Board 
    Research Protocol Clearance ....................................... 231  

BIBLIOGRAPHY ..................................................................... 233  

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LIST OF TABLES

1. Summary of Behaviors Exhibiting Number Sense and Symbol Sense........... 13
2. Initial and Final Enrollment by Section and Semester............................ 50
3. Categorization of Symbol Sense Test Items by Behavior.......................... 55
4. Mean Performance on Symbol Sense Pretest by Treatment and Semester...... 66
5. Analysis of Covariance for Symbol Sense Posttest Scores — Fall .......... 68
6. Analysis of Covariance for Symbol Sense Posttest Scores — Winter .......... 68
7. Mean Posttest Symbol Sense Score by Treatment — Fall........................ 69
8. Mean Posttest Symbol Sense Score by Treatment — Winter...................... 69
9. Analysis of Covariance for Comprehensive
   Final Exam Scores — Fall ....................................................................... 70
10. Analysis of Covariance for Comprehensive
    Final Exam Scores — Winter .................................................................. 70
11. Mean Comprehensive Final Exam Score by Treatment — Fall .................. 72
12. Mean Comprehensive Final Exam Score by Treatment — Winter .............. 72
13. Analysis of Covariance for Laboratory Practical Scores.......................... 73
14. Mean Laboratory Practical Scores by Treatment..................................... 73
15. Results of Test for Semester Effects on Symbol Sense Posttest ............... 228
16. Results of Test for Semester Effects on Comprehensive Final Exam.......... 228
17. Results of Test for Semester Effects on Laboratory Practical ................. 228
18. Results of Test for Homogeneity of Regression of
    Posttest Score on Pretest Score — Fall .................................................. 229
19. Results of Test for Homogeneity of Regression of
    Posttest Score on Pretest Score — Winter ............................................. 229
20. Results of Test for Homogeneity of Regression of Comprehensive
    Final Exam Score on Pretest Score — Fall ............................................ 230
List of Tables—Continued

21. Results of Test for Homogeneity of Regression of Comprehensive Final Exam Score on Pretest Score — Winter ......................................................... 230

22. Results of Test for Homogeneity of Regression of Laboratory Practical Score on Pretest Score ................................................................. 230
LIST OF FIGURES

1. Sample Pictogram Posttest Items ................................................................. 54
2. Interview Question on the Existence of Limits ........................................... 58
3. Interview Question on the Creation of a Discontinuous Function ............... 59
4. Identification of Equivalent Forms of the Definition of the Derivative ......... 59
5. Fourth Task from Third Interview Session .................................................. 60
6. Tasks 1, 2, and 3 from Fourth Interview Session ........................................ 60
7. Tasks 4, 5, and 6 from Fourth Interview Session ......................................... 61
8. Tasks 7 and 8 from Fourth Interview Session ........................................... 61
9. Posttest Question 10 .................................................................................. 76
10. Posttest Question 6 .................................................................................. 77
CHAPTER I

INTRODUCTION

Climate of Change

Undergraduate mathematics education must change to improve the effectiveness of undergraduate mathematics for all students, regardless of background, interest, or intended major (Mathematical Association of America [MAA], 1991; National Research Council [NRC], 1989, 1991; Steen, 1989a; Steen et al., 1990). Specifically, "instruction needs to become an active, constructive process in which students learn to communicate about mathematics, to build mathematical models, and to connect mathematical ideas with the world around them" (Steen, 1992, p. i). Similar goals for instruction of school mathematics exist to encourage students' development of a "sense-making" approach to mathematics (NRC, 1989, 1990; National Council of Teachers of Mathematics [NCTM], 1989). Three of the driving forces for change in the undergraduate curriculum and the school curriculum are: new demands of society, new cognitive theories, and new advances in technology (NCTM, 1989; NRC, 1990, 1991; Steen, 1989a; Steen et al., 1990). Each of these forces has altered our perception of the role of symbols in mathematics education.

Evidence from many sources indicate that the current educational system does not meet the "quantitative literacy" needs of society today and will meet even less the needs of tomorrow (NRC, 1989; Sons, 1992; Steen, 1989) as illustrated by the following passage which appears in Moving Beyond Myths: Revitalizing Undergraduate Mathematics.
At the college and university level—the focus of this report—mathematics forms the core of the quantitative skills needed by our nation's scientific, technical, and managerial work force, including the nation's future mathematics teachers. Yet even this system—the linchpin of mathematics education in the nation—is beset by weaknesses that threaten the health of U.S. science and technology...

Far from achieving its ideal as an agent for social equalization, undergraduate mathematics education as currently practiced bestows uneven benefits on different groups within our society... The result has been a growing polarization of society along the dimension of mathematical power that will, if left unchecked, exacerbate social and economic tensions by widening disparities in opportunities and earning capacities. (NRC, 1991, pp. 1-2)

Some of the reasons given for the importance of quantitative literacy in the education of all students are: (a) the role of mathematics as the language of science and technology, (b) the increased quantification in all areas of study, (c) the general expansion of the language of mathematics in society, and (d) the abstraction and reasoning associated with learning mathematics (NRC, 1989, 1990; Sons, 1992; Steen, 1990a). Symbols play a central role in the language of mathematics and the process of abstraction in mathematics. Therefore, a reasonable component of quantitative literacy is the ability to make sense of symbolic expressions and actions upon symbolic expressions.

As the mathematics needed by society has changed, so too have perspectives on how students learn mathematics. Constructivism is currently a prevalent position in mathematics education (Noddings, 1990; Vergnaud, 1990). From a constructivist perspective, knowledge is not passively received either through the senses or by way of communication, but are products of the individual's conceptual organization of the individual's experiences (von Glasersfeld, 1987, 1990). From this perspective, the goal of mathematics instruction is to help students in learning "how to see the world through a set of quantitative lenses which ... provide a powerful way of making sense of the world, how to reflect on those lenses to create more and more powerful lenses and how to appreciate the role these lenses play in the development of their culture" (Confrey, 1990, pp. 110-11). Symbolic expressions and operations on symbolic
expressions provide a powerful and essential lens for making sense of the world (Goldin, 1987; NRC, 1989). Constructivist principles suggest new perspectives on learning mathematics associated with symbolic expressions. "Constructivists agree that mathematical learning involves the active manipulation of meanings, not just numbers and formulas. They reject the notion that mathematics is learned in a cumulative, linear fashion" (Davis, Maher & Noddings, 1990, p. 187). Constructivists argue that for students to learn mathematics they must be actively engaged in making sense of situations including the use and manipulation of symbolic expressions.

Technology has dramatically altered the practice of mathematics at every level of use (NRC, 1989; Steen et al., 1990) and "can enhance mathematical learning by extending each student's mathematical power" (NRC, 1989, p. 84). Fey (1989) lists many of the technology-oriented changes that are being researched in mathematics education which includes the use of technology to: (a) decrease attention to those aspects of mathematical work that are readily done by machines and increase emphasis on the conceptual thinking and planning required in any tool environment; (b) enhance and extend the current curriculum to mathematical ideas and applications of greater complexity than those accessible to most students via traditional methods; (c) shift teacher roles from expositor and drill-master to tasksetter, counselor, information resource, manager, explainer and fellow student, in which students engage in considerably more self-directed exploratory learning; and (d) provide an environment in which mathematical or real world objects, relations, and operations are represented electronically in ways that permit controlled exploratory manipulation and observation of properties by learners searching to abstract underlying mathematical principles.
Advances in technology have altered the nature of quantitative literacy that is desired. "Innovative instruction based on a new symbiosis of machine calculation and human thinking can shift the balance of learning toward understanding, insight, and mathematical intuition" (NRC, 1989, p. 63) and away from algebraic skills. Technology has not simply removed computational gates to understanding mathematics, but has also provided new avenues for the development of quantitative literacy involving symbolic expressions through new ways of representing mathematics and connecting representations of mathematics.

The changes in society, in perspectives on learning, and in technology have resulted in a new form of quantitative literacy whose importance is evident in the following goals for mathematics education:

1. A primary goal of elementary school mathematics is to develop number sense (NCTM, 1989; NRC, 1989, 1990).

2. A major goal of secondary mathematics should be to develop symbol sense and the continued development of number sense (NRC, 1989, 1990).

3. An essential goal of undergraduate mathematics should be to develop function sense and the continued development of symbol sense (Eisenberg, 1992; NRC, 1989).

Research on each of these goals is currently in progress with most of the effort and discussion focusing on number sense. The following sections describe current perspectives on number sense and symbol sense, describe parallels between number sense and symbol sense, and further develop the concept of symbol sense.

Number Sense

The Curriculum and Evaluation Standards for School Mathematics (NCTM, 1989) recommends that the K-4 curriculum include whole number concepts and skills
so that students can develop number sense and that the grades 5-8 curriculum include continued development of number sense. Number sense is described to be "an intuition about numbers that is drawn from all the varied meanings of number" (NCTM, 1989, p. 39). Howden (1989) describes number sense "as good intuition about numbers and their relationships" (p. 11). The heritage of "quantitative intuition", now commonly referred to as number sense (Sowder, 1988, 1992), is evident in each of these descriptions. Sowder (1988) describes number sense as a "well organized conceptual network that enables a person to relate number and operation properties" (p. 183). Sowder's description of number sense places an emphasis on the cognitive structures involved in number sense. Alternatively, Greeno (1991) describes number sense in a manner that places the cognitive processes at the forefront. Greeno describes number sense as knowing the environment of the domain of numbers and quantities. Greeno states that people with number sense "know where they are in the environment, which things are nearby, which things are easy to reach from where they are, how routes can be combined flexibly to reach other places efficiently, and how to transform the things in the environment to form other things by combinations, separations, and other operations" (p. 185). However, many believe that construction of an operational definition of number sense to guide instruction and assessment is unlikely (Greeno, 1991; Hope, 1989; NCTM, 1991; Sowder, 1988, 1992, in press). Discussion of number sense is commonly done through a description of behaviors exhibiting number sense or through a description of experiences in which the development of number sense flourishes.

In the Curriculum and Evaluation Standards for School Mathematics (NCTM, 1989), "children with good number sense (1) have well-understood number meaning, (2) have developed multiple relationships among numbers, (3) recognize the relative magnitudes of numbers, (4) know the relative effect of operating on numbers, and (5)
develop referents for measures of common objects and situations in their environments" (p. 38). These five categories provide an overview of number sense and suggests an interplay between number sense and each of the following: conceptual understanding, numeration, number magnitude, (mental) computation, and estimation. In addition to elaborating on the relationship of number sense to these five categories, Sowder (in press) synthesized the following list of behaviors exhibiting number sense: (a) ability to compose and decompose numbers; ability to move flexibly among different representations; ability to recognize when one representation is more useful than another; (b) ability to recognize the relative magnitude of numbers; (c) ability to deal with the absolute magnitude of numbers; (d) ability to use benchmarks; (e) ability to link numeration, operation, and relation symbols in meaningful ways; (f) understanding the effects of operations on numbers; (g) ability to perform mental computation through invented strategies that take advantage of numerical and operational properties; (h) being able to use numbers flexibly to estimate numerical answers to computations, and to recognize when an estimate is appropriate; and (i) a disposition toward making sense of numbers.

Sowder also describes several limitations of describing number sense in this fashion. First, a single instance of one of these behaviors does not necessarily mean the student has a well-developed number sense. Second, there are varying levels of number sense. Furthermore, a student's number sense depends on the number system in question.

In the descriptions of number sense presented so far, the interplay between a well-developed conceptual network and number sense is evident, but number sense also involves the manner in which students use their conceptual network in solving problems. Resnik (1989) relates number sense with other forms of higher order reasoning and thinking. She describes number sense as non-algorithmic and yielding
multiple solutions with associated costs and benefits rather than unique solutions. Resnik further suggests that number sense involves uncertainty, self-regulation, and imposing meaning. The behaviors above suggest that two factors of number sense are the extent to which a student possesses a highly connected conceptual network and the extent to which a student can use this network to regulate and monitor a solution process.

Alternatively to describing number sense through behaviors of students, the components of number sense are explored by examining the environment and experiences in which it flourishes. In order to develop number sense, students need an environment and experiences which provide a diverse set of representations, focus on operations and methods of computation, and require interpretation of processes and results (NRC, 1990). Hope (1989) describes the development of number sense using many of the above practices in investigations on calculation, measurement, and estimation. Sowder (in press) discusses how instruction which encourages discussion and exploration of alternative strategies in each of the four areas: numeration, number magnitude, mental computation, and computational estimation will help students to develop number sense. While there is considerable interplay between the development of number sense and the development of numeration, number magnitude, computation, estimation, and measurement which can be taught directly, a common consensus is that number sense can not be taught directly, and that the environments and experiences which promote number sense promotes meaningful mathematics learning in general (Greeno, 1989; Hiebert, 1989; NCTM, 1991; Silver, 1989; Sowder, 1992). For example, in Developing Number Sense (NCTM, 1991), a list of teacher roles are discussed and the manner in which each role can aid the development of number sense. Some of the roles listed are: (a) use process questions and class discussion, (b) focus on student-generated methods of solution, (c) accept
and encourage appropriate computational methods, and (d) provide process-oriented activities. Each of these instructional practices are viewed as encouraging meaningful mathematics learning in general (NCTM, 1989).

Note that many of the roles for the teacher focus on the processes involved. They included encouraging students to develop their own solution techniques using written, mental, approximate, and electronic methods as appropriate. Howden (1989) states that "the calculator is an excellent vehicle for developing number sense, since it allows students to explore freely the ways in which numbers are related" (p. 8). However, much of the research on number sense has involved the use of physical manipulatives and the interplay between the development of number sense and instruction on mental estimation, mental computation, and approximation.

In summary, several key features of number sense have been identified. Number sense involves a well developed conceptual network in which many connections within and between number concepts and operations exist. Number sense involves the use of this conceptual network in self-regulation, monitoring, and strategy-making. The development of number sense requires experience with various forms of computation and representation of numbers. Finally, the environment and experiences which foster the development of number sense are those that foster a sense-making approach to mathematics in general.

Symbol Sense

A notion similar to number sense is symbol sense which involves intuition and insight in dealing with symbolic expressions and algebraic operations. However, the ideas and instructional materials for symbol sense are not as fully developed as for number sense (Fey, 1990). Attempts to clarify symbol sense have begun by the Working Group on Learning/Teaching of Algebra and Quantitative Analysis.
The purpose of this section is to present existing descriptions of symbol sense. Mason (1987) discusses the palpability and imagery of symbols across the spectrum of mathematical notation which he considers as forms of symbol sense. In this study, the domain of symbol sense will be limited to the algebraic expressions that arise in calculus. As with number sense, symbol sense has been described through the behaviors that exhibit symbol sense and through the types of experiences that should foster symbol sense.

Similar to the list of behaviors exhibiting number sense described above, Fey (1990, p. 80) states that "a reasonable set of goals for teaching symbol sense would include at least the following basic themes:" (a) ability to scan an algebraic expression to make rough estimates of the patterns that would emerge in numeric or graphic representation; (b) ability to make informed comparisons of orders of magnitude for functions with rules of the form $n, n^2, n^3, \ldots$, and $a^n$; (c) ability to scan a table of function values or a graph or to interpret verbally stated conditions in order to identify the likely form of an algebraic rule that expresses the appropriate pattern; (d) ability to inspect algebraic operations and predict the form of the result or, as in arithmetic estimation, to inspect the result and judge the likelihood that it has been performed correctly; and (e) ability to determine which of several equivalent forms might be most appropriate for answering particular questions.

Features of this list for symbol sense that differ from the corresponding list for number sense are the specificity of the forms of representations, emphasis on transformation between representations, and the greater number of verbs suggesting interpretation, self-regulation, and monitoring. In Reshaping School Mathematics (NRC, 1990), an emphasis on interpretation and experience with a rich set of representations and operations are presented as central ingredients to the development
of symbol sense. Description of symbol sense through behaviors has similar shortcomings as with number sense. A student may exhibit a behavior without it being reflective of symbol sense. As with number sense, there are varying levels of symbol sense, and symbol sense can depend on the type of algebraic expressions involved.

Interest in symbol sense and research on the development of symbol sense is increasing and is primarily motivated by the interest in instructional uses of technology. Symbol sense deals with an ability to transform information between symbolic representations or between a symbolic representation and another representation. Technology has dramatically increased student access to a variety of representations and means to manipulate representations including symbolic representations, thereby generating many questions related to symbol sense:

1. Can students develop mathematical intuition without performing extensive mathematical manipulations? (Steen, 1989b)

2. What is symbolic computational estimation, how does it develop, and can we build on prior efforts involving numbers? (Shumway, 1992)

3. How is symbol-sense developed? Would talking aloud about feelings/images while doing algebra in public help or hinder the development of symbol sense in students? (Mason, 1987)

4. How does the use of a computer algebra system affect symbol sense? (Page, 1992)

5. What form of software will enhance mathematical learning? (Dubinsky & Tall, 1991)

Fey (1990) states that "promising work from current projects shows how numerical and graphic computer tools can be used effectively to build student intuition about algebraic symbolic forms. Nevertheless, the development of more
general symbol sense remains an important research task on the path to new approaches for developing conceptual and procedural knowledge of quantity." (p. 81).

Making Sense of Symbol Sense

Von Glasersfeld (1987) describes two meanings of "making sense". The first is the organization of a collection of experiences which "permits us to make more or less reliable predictions" (p. 9). The preceding section provided an organization of attempts to characterize symbol sense. Von Glasersfeld's second interpretation of "making sense" involves the forming of expectations concerning the remainder of the piece that we have not yet perceived. This section presents the author's attempt of enhancing the description of symbol sense through the formation of expectations about symbol sense based upon the descriptions of number sense and symbol sense, and based upon current cognitive theories of learning.

Completing the List of Behaviors Exhibiting Symbol Sense

If we view symbol sense as the result of "making sense" of symbolic expressions, then a student's development of symbol sense depends upon the student's organization of experiences with symbolic expressions. That is, symbol sense requires a well-developed conceptual network. Therefore, similar to the description of number sense by Sowder (1988), one description of symbol sense is a well organized conceptual network that enables a person to relate symbolic expressions and operation properties.

When comparing the descriptions of symbol sense with those of number sense, several commonalities begin to emerge (see Table 1). Both number sense and symbol sense involve transformations within and between classes of representations, and as such, experience with multiple representations. As noted earlier, a primary
difference between number sense and symbol sense is the nature of the representations involved. Number sense involves the ability to move flexibly among a variety of representations including concrete representations, such as money and "groupings", and symbolic representations, such as integers, fractions, and decimals. In descriptions of symbol sense, the representations are primarily algebraic expressions, numeric information usually in the form of a table, and graphical representations. Thus another difference between number sense and symbol sense may be where each falls on the concrete-abstract continuum.

Discussion of both number sense and symbol sense include an emphasis on operations. As with number sense, the development of symbol sense is likely to be enhanced in an environment which encourages a variety of computation methods and strategies. Moreover, both number sense and symbol sense involve behaviors typical of higher order thinking such as interpretation, self-monitoring and regulation in environments rich in representations and operations.

The list of behaviors exhibiting symbol sense given in Table 1 includes three categories: attitude, estimation, and monitoring, which were suggested by the behaviors exhibiting number sense. Certainly a characterization of symbol sense ought to be a disposition towards making sense of algebraic expressions. Corresponding behaviors of estimation might include: (a) the ability to estimate quantitative information from algebraic expressions such as deducting that an expression has a maximum, and (b) the ability to recognize when a simplified algebraic expression is appropriate such as the elimination of higher order errors in determining approximations.

The last addition to our description of symbol sense focuses on how symbols are used to guide a reasoning process. Mason (1987) describes the palpability of symbols and how symbols can aid in the monitoring process during problem solving.
Schoenfeld (1987a) gives a similar description of the use of symbols in solving equations, and how the form of symbolic expressions can aid monitoring the solution process. Additional discussion of the role of symbols in guiding the solution process is given by Skemp (1987) and Davis (1984). The extent to which symbolic expressions guide the solution process seems to be a natural characteristic of symbol sense.

Table 1

Summary of Behaviors Exhibiting Number Sense and Symbol Sense

<table>
<thead>
<tr>
<th>Transformation within the representation system</th>
<th>Number Sense</th>
<th>Symbol Sense</th>
</tr>
</thead>
<tbody>
<tr>
<td>Compose and Decompose Numbers.</td>
<td></td>
<td>Inspect operations on algebraic expressions and predict the form of the result.</td>
</tr>
</tbody>
</table>

| Transformation to other representation systems | Move flexibly among different representations. | Scan an algebraic expression and predict patterns that would emerge in numeric or graphic representations. |

| Transformation from other representation systems | Scan a table of function values or a graph or to interpret verbally stated conditions, to identify the likely form of an algebraic rule that expresses the appropriate pattern. |

| Magnitude | Recognize relative magnitude of numbers. | Make informed comparisons of orders of magnitude for functions with rules of the form \( n, n^2, n^3, \ldots, \) and \( a^n \). |

| Magnitude | Deal with the absolute magnitude of numbers. | |

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Table 1 - Continued

<table>
<thead>
<tr>
<th></th>
<th>Number Sense</th>
<th>Symbol Sense</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Estimation</strong></td>
<td>Use numbers flexibly to estimate numeric answers to computations.</td>
<td>Estimate attributes of algebraic expressions.$^a$</td>
</tr>
<tr>
<td></td>
<td>Recognize when an estimate is appropriate.</td>
<td>Recognize when an estimate is appropriate.$^a$</td>
</tr>
<tr>
<td><strong>Monitoring</strong></td>
<td>Link numeration, operation, and relational symbols in meaningful ways.</td>
<td>Use algebraic symbols to guide a reasoning process.$^a$</td>
</tr>
<tr>
<td><strong>Regulation</strong></td>
<td>Use benchmarks - use well known number facts to figure out facts of which one is not so sure.</td>
<td>Inspect and judge the likelihood that a computation has been performed correctly.</td>
</tr>
<tr>
<td><strong>Strategy Making</strong></td>
<td>Recognize when one representation is more useful than another.</td>
<td>Determine which of several equivalent forms might be most appropriate for answering particular questions.</td>
</tr>
<tr>
<td></td>
<td>Perform mental computation through invented strategies that take advantage of numerical and operational properties.</td>
<td></td>
</tr>
<tr>
<td><strong>Attitude</strong></td>
<td>Have a disposition toward making sense of numbers.</td>
<td>Have a disposition toward making sense of symbols.$^a$</td>
</tr>
</tbody>
</table>

**Note.** The elements from column one are based upon Sowder (in press) and the elements not otherwise noted in column two are based upon Fey (1990).

$^a$These items are new.
Theory-Based Symbol Sense

In the remainder of this section we will consider symbol sense in light of contemporary learning theory constructs: conceptual schemas and conceptual environments, and the extent to which each theory emphasizes the components of number sense and symbol sense listed.

The use of schemas in the development of cognitive theories for learning is common throughout contemporary literature. (e.g. Davis, 1984; Dubinsky, 1991; Silver, 1987; Skemp, 1987) A schema is a conceptual structure (Skemp, 1987), a special knowledge representation structure (Davis, 1987), or an interconnected collection of objects and processes (Dubinsky, 1991). Schema aid in sifting information, suggest important information that is essential and possibly missing, and provide defaults for missing secondary information. When a problem situation is encountered, internalized information triggers or resonates one or more conceptual schemas. The degree to which situations involving algebraic expressions consistently resonate desirable schemas can be considered as one form of symbol sense. Since more than one appropriate schema may be resonating at a time, a second form of symbol sense can be considered as the selection of the most appropriate schema. A third form of symbol sense can be considered as the ability to shift to alternative schema once the existing schema is deemed inadequate.

The schema-based model suggests that both syntactic (surface) structures and semantic (deep) structures are involved in symbol sense (Mason, 1987; Skemp, 1987). Syntactic structures provide the intuitions for automated processes, while semantic structures are more likely to be the source of meaningful intuitions. Symbol sense usually requires both a conceptual and symbolic understanding. Consider the
following problem from Chapter II which assesses whether a student can recognize why a computation must be incorrect.

\[ \int_{-1}^{+1} \frac{1}{x^2} \, dx = \left[ -\frac{1}{x} \right]_{-1}^{-1} = -1 - 1 = -2. \]

One possible explanation might be to relate the definite integral given to the area under the curve of a positive valued function. Certainly a student who has not connected the definite integral to the area under the curve may have difficulty in discussing this problem, but so will a student who does not see the positive valued function. If it is known that a student has the desired conceptual and symbolic understanding to solve this problem, then the solution of the problem depends upon a student's symbol sense.

Symbol sense can be viewed as phenomenological generalizations based upon the commonalities of constructed schema and as the manner in which we use these schemas. In this framework, the issue of how symbol sense develops is equivalent to determining how broad and deep schemas are constructed and how schemas are used in cognitive efforts. Central to the development of symbol sense would be the development of the student's conceptual network. Direct instruction on estimation, mental computation, magnitude, and notation should foster the development of symbol sense directly through an enhanced conceptual network. An environment that supports a sense-making perspective should also support symbol sense indirectly through enhanced metacognitive abilities. This view of symbol sense suggests a primary emphasis on the development of the conceptual network.

In attempting to characterize number sense theoretically, Greeno (1991) discusses conceptual environments and number sense as a general condition of knowing within the conceptual environment of numbers and quantities.
The metaphor of a subject-matter domain as a conceptual environment, with knowing the domain as the ability to find and use concepts and principles in the environment as resources, seems to apply well to the domain of numbers and quantities, particularly in its suggestions regarding number sense. ... People with number sense know where they are in the environment, which things are nearby, which things are easy to reach from where they are, and how routes can be combined flexibly to reach other places efficiently. They also know how to transform the things in the environment to form other things by combinations, separations, and other operations (p. 185).

Symbol sense can be described in a similar fashion. With an environment/model view, conceptual reasoning is mainly a mental version of concrete activity (Greeno, 1991). Therefore, a natural question concerns the generalization of a conceptual environment perspective to advanced mathematical learning in which the objects of the environment do not have physical equivalents. From a conceptual environment perspective, "number sense is likely to develop as students engage in activities in which properties of numbers and quantities are salient factors, and where the recognition and inference of properties and relations among numbers and quantities play an instrumental role" (Greeno, 1991, p. 209). The conceptual environment view emphasizes more heavily a metacognitive component to number and symbol sense than the conceptual schema perspective.

It should be noted that while these theories are in competition, neither theory excludes the processes that the other construct considers fundamental (Greeno, 1991). For example both Mason (1987) and Skemp (1987) describe a voyage or journey through a mathematical landscape that is based upon schema. The value of each lies in its ability to emphasize different aspects of symbol sense. Which theory best explains symbol sense may very well depend upon the question about symbol sense that is being asked. For example, some behaviors exhibiting symbol sense are automatized and incorporated into the network itself which the schema model supports more readily, while the environmental model supports explanation of exhibitions of dynamically constructed symbol sense.
Cognitive theories are being developed for how students become competent in working with symbols (Goldin, 1987; Kaput, 1987; Wearne & Hiebert, 1988). Wearne and Hiebert (1988) hypothesize a sequence of five processes which when obtained in succession lead to symbol competence:

1. the Connecting Process, in which individual symbols are connected with referents;
2. the Developing Process in which the symbol manipulation rules are developed from actions on referents;
3. the Elaborating Process, in which the rules are extended to similar but more complex problems;
4. the Routinizing Process, in which the rules are memorized and automatized; and
5. the Building Process, in which the symbols and rules are used as referents to build more abstract symbol systems. (p. 224)

These processes parallel the steps of transition from operational to structural conceptions given by Sfard (1992) which include (a) interiorization, a process performed on already familiar objects; (b) condensation, the process is turned into a more compact, self-contained whole; and then (c) reification, when the process is viewed as a permanent object. Lack of a foundation in the earlier processes is viewed as the primary difficulty to students' mathematical learning in both theories. Based upon the characterizations of symbol sense given, it is likely that a student with strong symbol sense will also be at these later stages in the understanding of algebraic expressions. Other theories (e.g. Dubinsky, 1991; Tall, 1992) address how mathematical concepts are internalized and how processes become cognitive objects. Tall (1992) discusses how symbols can represent both a process and a concept. He suggests that good mathematicians exploit this duality of symbols often without making explicit which component they are using. These theories provide different perspectives on how symbols are used in the learning of mathematics, and so suggest different ways in which symbol sense may be exhibited. Research is needed to determine how instruction based upon these theories interacts with a student's symbol sense.
This section has attempted to use the efforts to understand number sense to enhance perspectives on symbol sense. First, by comparing descriptions of number sense and symbol sense, various components of symbol sense were explored. Symbol sense involves both cognitive and metacognitive abilities, both syntactic and semantic understanding, both horizontal and vertical growth, and both representations and transformations. Second, by comparing behaviors exhibiting number sense with those exhibiting symbol sense, an elaborated list of behaviors has been constructed. Finally, the generalization of cognitive theories for number sense to symbol sense provides further insight into the nature of symbol sense.

Symbol Sense and the Present Study

During the last decade there has been an increasing interest and effort in calculus reform. Some of the reasons for change have been discussed already: changing needs of society, new perspectives on learning, and advances in technology. Also, past failure rates of students are often cited as reason for change.

A brief summary of national data alone provides compelling documentation of the need for change. Annually, about 600,000 students enroll in calculus in four-year colleges and universities. About half of these students are enrolled in mainstream "engineering" calculus; of these 300,000, only 140,000 finish the year with a grade of D or higher. The average section size is 34 students; only 7% of calculus courses use computers; and exam questions require students to "solve, sketch, find, graph, evaluate, determine, and calculate" in a straightforward fashion. (Ferrini-Mundy & Graham, 1991a, p. 635)

This passage also suggests the type of problems which students are being asked to perform. Problems are skills-based, and yet so few students can successfully perform the basic skills which has been the focus of their study. A consistent uncertainty over the role of by-hand manipulation in the learning of mathematics has risen throughout the reform effort.
At the Williamstown conference on the future of college mathematics, Maurer (1983) states that "it may just be that skill and training in symbolic manipulation is closely tied to being successful as a mathematician, or scientist, or engineer, or even to being an astutely analytic businessman" (p. 170). Small, Hosack, and Lane (1986) state that:

Hand computation may be an important aspect of developing an understanding of concepts. In carrying out a computation by hand, a student may gain a feeling for the concepts involved and the relative importance of various parts of the problem computation. (p. 433)

In Calculus for a New Century, Zorn (1987, p. 238), who advocates the use of a symbolic algebra packages, asks "if hand computation builds algebraic intuition- 'symbol sense'-will machine computation destroy it?" Sfard (1992) states that:

Lately, the role of proficiency in executing algorithms may have been slightly underestimated. ... a person must become quite skillful in certain procedures in order to attain a good idea of the abstract objects involved in these manipulations. (p. 78)

In addition to the references presented in the initial section on symbol sense and the citations above, the body of literature questioning the role of by-hand manipulations in learning is extensive (e.g. Heid, 1988; Hosack, 1988; Judson, 1988; Palmiter, 1990; Schoen, 1991; Schrock, 1989; Silver, 1987; D. A. Smith, 1992).

One thread in the discussion of the role of by-hand manipulation in learning is a focus on intermediate forms of algebraic expressions; namely discussing the kinds of experiences students must have with steps in an algebraic solution to generalize the solution process. Maurer (1983) suggests a decreased emphasis on the algebraic rewriting that technology can do, and increased emphasis on identification of intermediate forms. Fey (1983) suggests that student ability to manipulate algebraic expressions needs to cover only the simpler cases of useful operations and those that are needed to give students an adequate understanding of the computer-generated
results. Coxford (1985) emphasizes that even with a computer algebra system, students will need the ability to recognize algebraic forms.

Interest in determining the form of symbolic manipulation experiences which promote mathematical learning has motivated, in part, discussion and development of cognitive technologies (e.g. Child, 1991; Dubinsky & Tall, 1991; Goldenberg, 1988; Pea, 1987; Tall, 1992). Child (1991) discusses various issues on the pedagogically-oriented design of a computer algebra system. Child states that many desirable pedagogical features are missing from computer algebra systems designed for professionals such as MAPLE or MATHEMATICA, and other packages such as THEORIST, DERIVE, and CALCULUS T/L are more pedagogically-oriented. Similar claims have been made by others (e.g. Williams, 1991; R. F. Smith, 1991).

Therefore, a natural direction for research on symbol sense is to investigate different forms of computational environments and experiences, and examine the differences in symbol sense across these environments and experiences. Specifically, the following questions about symbol sense are addressed.

1. What, if any, are the consequences of reduced attention to by-hand manipulation on the development of symbol sense? (Zorn, 1987)

2. What, if any, are the differential effects of computer algebra systems on the development of symbol sense? (Page, 1992; Steen, 1989b)

3. What characteristics of computer algebra systems enhance the development of symbol sense? (Dubinsky & Tall, 1991)
CHAPTER II

REVIEW OF THE LITERATURE

Overview

Mathematicians and mathematics educators involved in the calculus reform movement have raised many questions regarding the use of computer algebra systems in advanced mathematics. Yet very few research studies at the collegiate level have been conducted. "The amount of research in collegiate-level mathematics education is relatively small by modern standards. A dedicated scholar (or masochist) could conceivably read the entire extant literature" (Cipra, 1992, p.167). Becker and Pence (1991) provide a summary of collegiate-level mathematics education research for the years 1975-1989. A more focused summary of the calculus reform effort and relevant research is given by Ferrini-Mundy & Graham (1991a). This chapter will review the investigations involving the use of a symbolic algebra software and the relevant literature relating to the nature and development of symbol sense.

Use of Symbolic Algebra Software

The primary use of symbolic algebra software in collegiate-level mathematics has been in the calculus curriculum. Specific discussion of calculus reform projects is given in Priming the Calculus Pump (Tucker, 1990) and discussion of the uses of a computer algebra system is elaborated in Symbolic Computation in Undergraduate Mathematics Education (Karian, 1992). In an early review of research on computer-enriched instruction which includes the use of the computer as a calculating device, programming tool, and simulator, Kulik and Kulik (1987) concluded that students
generally learned more and with less instructional time in computer-enriched environments. Since this time there have been several studies investigating the use of a variety of symbolic algebra packages in the mathematics classroom. Three different forms of symbolic algebra software have been studied: computer algebra systems, super calculators, and calculus-specific linked/package software.

Use of Computer Algebra Systems

Heid (1984, 1988) investigated the effects of resequencing skills and concepts in a one-semester applied calculus course. Two sections of applied calculus students \((n = 39\) total) used a variety of computer software to analyze concepts of calculus and to perform routine calculations during the first 12 weeks. In addition to software for graphing, fitting data, creating tables, and other demonstration software, MUMATH, a command-oriented symbolic algebra package, was used. Both sections spent the last 3 weeks on skills development. The two sections differed in that the basic algorithms were demonstrated in one of the two sections during the first 12 weeks. Comparison information was gathered from a traditional large lecture class \((n = 100)\). The data included quantitative information in the form of a final exam containing both skills and concept questions, and qualitative information in the form of interviews, student work, and classroom tapes.

The results of the study showed that it is possible for introductory calculus students to understand the concepts of calculus when their primary means of symbolic computation is the use of computer software. Students in the experimental classes were better able than the students in the comparison class to answer conceptually-oriented questions. On the skills portion of a departmental final exam, the experimental sections performed almost as well as the comparison section. Interviews with students suggested that the experimental students spoke more flexibly
on topics of calculus and were more likely to create and use their own language and representations in discussing the concepts of calculus.

Heid provides several data-based hypothesis and questions for future research. She suggests that when students use symbol manipulation programs they use intermediate answers as visual cues on what to do next. The concrete periodic display of results activates parts of the problem solving process. Heid recommends that future research investigate: (a) the role of intermediate forms in student's development of sense of symbolic solutions, (b) the skills or problem-solving procedures that the use of a symbolic manipulation strengthen, (c) the extent to which a compressed skills course results in better student ability to distinguish among rules, and (d) whether students think about what the symbols represent as they are performing skills work. Since this exploratory study in applied calculus, a few studies have investigated the potential of computer algebra systems in engineering calculus and applied calculus.

The use of the computer algebra system MACSYMA in an engineering integral calculus course was investigated by Palmiter (1986, 1991). An experimental group \( (n = 40) \) used MACSYMA to compute integrals during the first five weeks. At the end of the first five weeks, the experimental group was given a conceptual exam. During the last five weeks, the experimental group was taught paper-and-pencil methods for evaluating integrals. At the end of this period students were given a computational exam. A traditional section \( (n = 41) \), taught by a different instructor, was given the same conceptual and computational exams at the end of a ten week term. On both the conceptual and computational exams, the experimental class scored significantly higher \( (p < .001) \) than the traditional section. Palmiter also tracked the progress of those students continuing on into multivariate calculus and sequence/series calculus. The average grade of the traditional section declined from
integral calculus to the fourth quarter calculus while the average grade of the experimental section continued to rise.

Judson (1988) investigated the effects of resequencing skills and applications in an elementary business calculus course. Two experimental sections \((n = 11\) and \(n = 13\)) were taught concepts and applications of derivatives using the computer algebra system MAPLE. Two traditional sections \((n = 10\) and \(n = 14\)) were taught algorithms together with concepts before applications. The investigator and another instructor taught one experimental and one traditional section. The final exam was the primary source of information. Questions on the final exam were classified as skill, concept, or application-oriented.

The results showed no significant differences in achievement between the experimental and traditional groups in any of the three areas: skills, concepts, or applications. However, based upon student evaluations, students found the material more interesting when applications were studied first, and were more motivated to learn about the derivative. Judson lists several consequences of using a computer algebra system. Using MAPLE, students were forced to pay attention to details and gained appreciation for the importance of speaking mathematics correctly. She described how the MAPLE commands seemed to focus on understanding the notation involved. In later work using MAPLE, Judson (1990, 1991) suggested that pattern recognition may be enhanced using a computer algebra system.

Schrock (1989) investigated differences in understanding and computational ability of students in an engineering calculus course. An experimental section \((n = 24)\) used MAPLE for 12 weeks. The investigator taught the experimental section and another control section \((n = 25)\). Another instructor taught a third section \((n = 25)\) which also served as a control section. During the thirteenth week of class, all three sections were given a concept-oriented exam. The results showed a
significant difference ($p < .05$) in favor of the experimental section. In addition, a common final exam was given which was primarily computational. There were no significant differences found in the performance of the three sections. Schrock suggests that by-hand skills should not be eliminated because of symbolic algebra packages, but that students should learn when each method of computation is most appropriate.

In a qualitative study of nine students from a section of engineering calculus using the MATHEMATICA-based materials from the University of Illinois, Crocker (1991, 1993) investigated the concept development and problem-solving skills of students learning calculus in a laboratory environment. Crocker observed that by the middle of the second quarter of calculus, all students were developing a sense of the derivative concept which included a strong connection between the concept of derivative and its relationship to graphs of functions. Differences in the problem-solving responses of students were found to be related to students' ability at the beginning of the semester. In particular, she found that middle or low ability students were more likely to experiment and try varied approaches to problems, while higher level students exhibited the most difficulty often failing to attempt problems or to use multiple strategies.

The next two studies focused on a more specific component of instruction. K. B. Smith (1991) investigated the use of DERIVE in enhancing the instruction of linear programming for two large classes of Calculus for Business Administration and Social Sciences. In one section, the software was used only for demonstration purposes ($n = 70$). Students in a second section ($n = 70$) used the software outside of class. A third group ($n = 70$) was used as a comparison section. The instructional period lasted one month. No significant difference in achievement was found among groups.
Melin-Conejeros (1992) investigated the effects of doing calculus homework assignments in a mathematics laboratory equipped with the computer algebra system DERIVE. Students in the experimental section \((n = 12)\) were assigned homework which was to be done in the computer laboratory. A control group \((n = 16)\) of students completed essentially the same type of homework in the usual paper-and-pencil manner. No significant differences were found between groups on overall achievement, on skills achievement, or on concept achievement. Although interviews suggested the experimental students had a better understanding of selected concepts. If a computer algebra system is to be used in teaching calculus, then Melin-Conejeros recommends that it should not be used for homework only, but should be integrated with all instruction both in and out of class.

Comments on the Use of Computer Algebra Systems

Four of the five long-term studies involving computer algebra systems in calculus (Crocker, 1991, 1993; Heid, 1984, 1988; Palmiter, 1986, 1991; Schrock, 1989) found an improved understanding of the concepts of calculus with no significant loss of computational skill. A fifth study (Judson, 1988) found no differences in either conceptual understanding or computational skill. Each of these studies measured the conceptual growth of students via their solution to concept-based questions. Often these questions required students to translate to or from algebraic expressions. For example, Palmiter (1991) asked students whether the following algebraic equation was true or false and supplied a graph of a function:

\[
\int_{b}^{c} f(x) \, dx = \int_{b}^{e} f(x) \, dx + \int_{e}^{c} f(x) \, dx .
\]
As discussed in Chapter I, a student with a strong connection between the area under a curve and the definite integral should be able to reason through the symbols. But this question could also be answered solely on the symbolic pattern that the definite integral from $a$ to $b$ plus the integral from $b$ to $c$ is the definite integral from $a$ to $c$. Judson (1988) asked students as part of a concept-oriented exam to compute the following limit:

$$\lim_{x \to \infty} \frac{x^2 - 3x}{1 - 4x^2}.$$

How students actually approached this question is not fully described. Students may have assimilated a general pattern regarding end-behavior of a rational function and the role of the highest power in the denominator and numerator. Or students may have applied a standard algorithm of dividing by the highest power of $x$. Additional research is needed to determine what part, if any, of the improved conceptual understanding that was observed in students using a computer algebra system can be explained by improved sense of symbols.

**Use of Super Calculators**

With the continued advances in technology, the power of computer algebra systems are beginning to emerge in the form of hand-held devices such as the HP 28S and HP 48SX. Stout (1991) investigated, in a two-day study, the differences in students' understanding of derivatives with and without use of a HP 28S. Stout found that students using the HP 28S were significantly ($p < .01$) better able to determine the graph of the derivative of a function, given the algebraic and graphic representations of the function. Also students were better able, but not significantly, to determine the graph of a function given a graphical representation of the derivative.
In a more extensive study, Hart (1991) investigated students' representational knowledge and concept image in a super calculator environment. She collected paper-and-pencil responses from 324 students from 12 institutions who were studying calculus using the Oregon State University calculus curriculum materials. Interviews were conducted with 33 students using the experimental material. In addition, information from paper-and-pencil responses and interviews with 30 students from traditional calculus classes was collected.

Because of the nature of the study, no statistical tests were performed. Hart examined the types and manner in which students used various representations to solve problems. Some of Hart's findings were: (a) experimental students showed greater facility with graphical and numerical representations and exhibited better ties among the three representations than traditional students; (b) individual students show definite preferences for certain representations, but different factors influence their choices; (c) grades do not appear to be a good predictor of the quality of the connections among the representations; and (d) students who lack confidence in their symbol manipulation skills appear to use the calculator more readily than those who are confident in their symbol manipulation skills.

Use of Calculus-Specific Software

With the potential benefits of technology to mathematics education, considerable effort is being devoted on designing mathematical software to enhance instruction. As noted in Chapter I, a typical computer algebra system is designed for someone already familiar with the tasks. Other software is continuing to be developed specifically intended for use in calculus instruction. True BASIC Calculus and the Calculus Toolkit are examples of two such programs. They have been used in four studies (Cunningham, 1991; Hamm, 1989; Siler, 1990; Thongyoo, 1989) to
investigate the effects on conceptual understanding of students in a first term calculus course within a microcomputer environment. Using performance on the Calculus Readiness Test of the MAA as a covariate, Thongyoo found no significant differences in achievement on a comprehensive final exam between students taught with the use of the software and those taught by traditional methods. Hamm used experimenter-developed achievement examinations and also found no significant differences in achievement. Siler developed materials to be used in a laboratory setting for investigating the concepts of calculus. He found that students' grasp of fundamental concepts was generally stronger than that of previous students. Siler stated that some of the advantages of the laboratory approach were a greater emphasis on more direct intuitive comprehension of the material, students' active involvement in the learning process, increased opportunity for student collaboration, and the cultivation of careful observation.

Cunningham (1991) investigated the effects of reduced by-hand symbolic manipulation via use of True BASIC Calculus in freshman calculus. He administered a manipulative skills achievement test and a concept-oriented achievement test to students in two sections of first-semester calculus. An experimental section used the software extensively to perform symbolic manipulation. The experimental section was split into two groups. One group used the computer on the conceptual test and not on the manipulative test, and the second group used the computer on the manipulative test and not on the conceptual test. Cunningham found no significant differences between the experimental students who did not use of the computer and the traditional students on either test. However, he found significant differences \( (p < .05) \) in favor of the experimental students using the software on both the manipulative and concept-based tests.
It is beneficial to contrast the results of generic computer algebra systems with those of more specifically engineered calculus software. With the calculus-specific software, a finding of no enhanced conceptual or manipulative achievement was most common while with more general symbolic algebra packages, a finding of enhanced conceptual understanding with no significant loss of skill was most common. Further research is needed to determine if computer algebra systems possess an unsuspected trait which is beneficial to the learning of calculus.

Of the all the studies discussed so far, none have reported any significant differences in favor of the traditional, non-technology enhanced, lecture-based sections. Yet many of the studies have suggested improved conceptual understanding with no considerable decline in skills. Quite often, the experimental section’s use of multiple representations, primarily increased access to graphical representations, is cited as possibly providing the greatest enhancement to the learning process. Several studies have been conducted on the effects of graphics calculators and graphing software on conceptual understand and manipulative skills. Dunham (1993) provides a survey of research on graphing technology. She states:

Computing technology puts the focus on learning rather than manipulation. Access to technology allows students to explore connections between graphs and algebraic representations, to solve complex algebraic problems graphically, to engage in more problem solving activities, and to take a more active role in learning. This leads to a better understanding of the concepts needed for success in calculus and beyond. (p. 98)

Dunham gives several general questions needing further research including:

1. What will be the long term effect of a conceptual graphical development of algebraic and calculus topics?

2. How much hand manipulation is necessary for success in mathematics?
Symbol Sense

Since interest in the nature and development of symbol sense per se is relatively new, there is little specific research literature on symbol sense. Occasionally, reported studies will speak on the intuition of students for symbolic forms or translation skills to and from algebraic representations. This section focuses primarily on studies which provide some insight into symbol sense in calculus. The discussion begins with research on understanding of calculus concepts that address possible issues regarding symbol sense. Symbol sense involves both cognitive and metacognitive abilities and so the discussion will flow from cognitive studies to metacognitive studies. The several studies involving technology already discussed suggests that technology may provide a rich environment which fosters the development of certain aspects of symbol sense. Selected studies involving technology will be presented which have not yet been discussed and speak directly to some form of symbol sense. Finally, a few references are given which provide insight into symbol sense through an analysis of sample problems and students solutions to these problems.

Student Intuition for Symbolic Expressions

Mathematical notation is often created to aid intuition and continues to evolve as all languages do (e.g. Knuth, 1992). A discussion of the role of symbols in mathematics, knowledge, and thinking is beyond the scope of this review. Some interesting discussions are given by Davis (1984), Skemp (1987), Janvier (1987), and Tall (1991). Skemp states that "it is largely by the use of symbols that we achieve voluntary control over our thoughts" (p. 57) and that "a good or bad symbol system can be a great help, or a severe hindrance, in evoking and manipulating the right
concepts in the right relationships" (p. 58). As illustration, Eisenberg (1991) presents the following example involving functional notation.

Seventy percent of beginning calculus students at Ben-Gurion University could not solve the following problem:

If 2 and 4 are the values of \( x \) for which \( f(x) = 0 \), what are the values of \( x \) for which \( f(4x) = 0 \)?

This same problem was rephrased:

Only the values of 2 and 4 go to zero under the function \( f \). What values multiplied by four will go to zero under the function \( f \)?

Seventy percent of the students could answer the problem in this form, and they seemed to have a basic understanding of what they were doing. (Eisenberg, 1991, pp. 150-151).

Eisenberg describes alternative notation for functions which might aid the development of students' intuition. While students' understanding of a mathematical concepts is often the focus of most studies, some information has been produced on students' intuitions that arise from the symbolization used within calculus.

Orton (1983a, 1983b) used clinical interviews to investigate students' understanding of the concepts of differentiation and integration in elementary calculus. Orton selected a diverse group of students (\( n = 110 \)) who were studying mathematics at the school and collegiate level. Students had difficulties with the individual symbols \( dx \) and \( dy \) and the relationship between \( \Delta y / \Delta x \) and \( dy / dx \). The only routine aspect of differentiation posing any difficulty was in handling the derivative of \( y = 2 / x^2 \) in which 18% of the students gave the response \(-4 / x \). Orton considered this to be an executive error, i.e. failure to carry out manipulations, though the principles involved may have been well understood. Results on more conceptual tasks suggests students lacked an "intuitive" understanding of the derivative. Orton concluded that the symbols of differentiation were badly understood by students and states that:
A careful programme of introducing symbols over a period of time is important, but so also is constant revision and reconsideration of the origin of the symbols. Where possible, it seems appropriate to leave the standard symbols until a second stage approach to calculus. (Orton, 1983b, p.244)

A similar recommendation is given by Sfard (1992) who discusses operational and structural forms of understanding the concept of function. Sfard suggests that the development of concepts first would aid the development of symbolic understanding and may lead to greater symbol sense. Heid (1984) and others have shown that concepts and skills could be resequenced through the use of a symbolic manipulator. However, further research is needed to determine if the resequencing of skills and concepts aid the development of symbol sense, and in what ways does the use of a symbolic algebra package by students alter their development of symbol sense.

Ferrini-Mundy and Graham (1991b) present the analysis of one of six first-semester calculus students interviewed over a two-semester period from a traditional large-lecture calculus course. The interviews suggests that graphical contexts and algebraic contexts may function as separate worlds for students. Similar observations were made by Dick (1988) and Monk (1989). Dick noted that students tended to ignore graphical information when judging the reasonableness of the result of a definite integral even when the graph was produced prior to the integration and was visually adjacent to the integration problem. Monk observed that students were able to answer some verbally-stated questions using the graph, but not when given a symbolic form. Many studies involving graphics technology have shown that students using technology have stronger connections among the standard three representations than students not using technology (Dunham, 1993; Leinhard, Zaslavsky & Stein, 1990; Ruthven, 1990). Ability to translate to and from algebraic representations is one form of symbol sense. Further research in needed to examine
how the symbol sense of students who use technology differs from students who manipulate expressions by-hand.

The most famous, or infamous, example of translation difficulties of students is the "student-professor" problem in which nearly half of all college students represent the statement that there are six times as many students as professors by $6S = P$. In a recent study, MacGregor and Stacey (1993) provide both a retrospect of the problem and further analysis. Among their conclusions is that:

Students generally try to make sense of the text of problems, and in doing so they intuitively construct mental images or models. What they then write on the page is an attempt to reproduce the content and form of the model. For statements about compared quantities, intuitive models underlying comprehension are frequently misleading and inappropriate as a basis for direct representation in mathematical code. (p. 230)

The student-professor problem has received considerable notoriety perhaps because of its simplicity and yet surprising results among educated people. In MacGregor and Stacey's study, it was determined that 95% of the students tested had correctly understood additive and multiplicative relationships expressed in words. The error is primarily in translation to symbolic form. If symbol sense includes the ability to identify the likely form of an algebraic rule that expresses verbally stated conditions and a disposition toward making sense of symbolic expressions, then we would expect students with well-developed symbol sense to recognize and correct the reversal error.

Pattern Recognition Within Symbolic Expressions

The ability to perceive patterns in algebraic expressions is another form of symbol sense for which some information is available. Ferrini-Mundy and Graham (1991b) found that calculus students will actively formulate their own theories, build their own connections, and readily construct meaning for problem situations. They
found that students have powerful tendencies to call upon familiar examples and
frequently-used patterns. Yet Wagner (1981) found that some students fail to
conserve equations and functions under transformations of variable. For example,
some students will solve both equations, $7W + 22 = 109$ and $7N + 22 = 109$, when
asked to compare $W$ and $N$. In a latter study, Steinberg, Sleeman, and Ktorza (1991)
examined algebra students' knowledge of equivalence of equations. Students were
presented with pairs of equations, such as $4 + 2x = 16$ and $4 + 2x - 4 = 16 - 4$, and
asked if they were equivalent. Students reasoned either by focusing on the
transformation involved or by calculating solutions to both equations. While these
studies were focusing on algebra, similar difficulties have been described, while not
fully studied, at the collegiate level. Eisenberg (1992) discusses issues of "reverse
path development" and it uses as an assessment tool. Some examples discussed are:
(a) finding $f\left(\frac{2 + 5w}{3}\right)$, given $f\left(\frac{1 + 3x}{2}\right)$, (b) recognizing certain limits as values of
derivatives, and (c) evaluating limiting sums as definite integrals. Another commonly
discussed situation in which students have difficulties finding patterns within
algebraic expressions involves function composition, namely determining $g(x)$, given
$f(g(x))$ and $f(x)$.

Perception of pattern is also an important skill in problem solving. Norman
and Prichard (1991) interviewed six students enrolled in a first-semester calculus
class. Each student was videotaped or audiotaped as they solved a series of calculus
problems across four clinical interviews. Norman and Prichard found that students
who demonstrated facility in generalizing, flexibility of thought, and reversibility of
thinking, tended to have more success in calculus. They discuss some of the
cognitive obstacles associated with students intuitions about mathematical
representations. Furthermore, they argue that some of the cognitive obstacles arise
from the beneficial cognitive processes of flexibility, generalization, and reversibility. For example, students must be able to generalize to make sense of mathematics, yet the process of generalization can create obstacles to the correct interpretation of mathematical language, such as the tendency to view juxtaposition as multiplication in the expression \( \cos(x) \). Norman and Prichard discuss how intuitions on symbols can serve as cognitive obstacles to learning. Yet many researchers have stated the need to develop even more the intuition of students. The nature and kind of desirable intuitions of students need to be determined. Moreover, research is needed on how students develop intuition (Ferrini-Mundy & Graham, 1991a).

**Metacognitive Behaviors on Symbolic Expressions**

Research on the perception of patterns in algebraic expressions at the collegiate level have primarily focused upon expert-novice approaches to problem solving. One of the primary benefits of the analysis of differences in the problem solving approaches of experts and novices is the identification of deeper forms of algebraic skill and understanding which are desired of students (Wenger, 1987). Both Wenger (1987) and Schoenfeld (1987b) discuss the difficulties of students in perceiving patterns of linearity in expressions such as \( x^2y + 2xy + x - y = 2 \) or \( y\sqrt{x} = 1 + 2y\sqrt{1 + x} \) when solving for \( y \). In the second expression, the radical acts as a visual clue for most students to use an "elimination of radicals" algorithm. Kirschner (1989) investigates further the visual syntax of algebra and the dependence of individuals on these visual cues in their syntactic decision making. "Whereas almost all students learn to parse an expression that is visually present, only some develop sound propositional knowledge of syntactic rules" (Kirschner, 1989, p. 286). According to Kirschner, the connections among visual syntactic templates and
semantic understanding is fundamental to competent performance in elementary algebra.

Bookman (1991) investigated the metacognitive behaviors exhibited by nine novices, college freshman, and six experts, graduate students in mathematics, while solving four mathematics problems: (1) a routine problem, (2) a problem with more than one obvious path, (3) a nonroutine problem that involved the use of the same skills as in the routine problem, and (4) a problem with insufficient or contradictory information. Bookman was particularly interested in the extent to which solutions were schema driven. He found that experts possessed a variety of appropriate schemas and used schemas to solve problems, but schema-use did not fully or adequately characterize expertise. Bookman also found that assessment of work was not in and of itself useful. In particular, novices often assessed their work but ignored potentially productive observations.

Shoenfeld (1987a) states that "the willingness to tolerate temporary chaos in the service of long term goals, the eventual simplification of the equation, constitute part of the mathematician's competence" (p. 16). Moreover, the skill of a mathematician in working algebraic problems is not simply a sum of steps which lead to an appropriate conclusion. The work of experts is far more efficient. Research is needed to determine the kind of experiences that are necessary for students not only to see the next step in a solution process, but to see many steps as a unit or 'form' of the solution process. Schoenfeld (1987b) discusses a typical scenario in which almost half of his students in integral calculus failed to recognize the use of substitution to solve the integral \( \int \frac{x}{x^2 - 9} \, dx \), and instead used partial fractions or a trigonometric substitution. The discussion of pattern recognition presented accentuate the appropriateness of some of the traits of symbol sense presented in Chapter I, and also
suggests the complex nature of symbol sense. While symbol sense and metacognition intersect, the above discussion suggests that symbol sense is more than just the union of good metacognition and a well-develop conceptual network.

Use of Technology to Create Active Symbolic Expressions

There have been a few studies using technology in calculus to create new symbolic objects to represent mathematical concepts. With traditional written expressions, students must perform desired transformations as well as know what transformations are available. With new symbolic forms available through technology, the symbolic forms can act upon themselves and thus can be used to turn processes into objects. Also these new symbolic forms often provide intermediate symbolic forms in transition to formal mathematical notation.

Ayers, Davis, Dubinsky, and Lewin (1988) investigated the use of Unix shell scripts and pipes as a method of constructing mathematical concepts as objects. A shell script is a program at the Unix operating system level and a pipe is a method of routing the information that a shell acts upon or chaining scripts together. For this study, the shells scripts were used to create functions as objects and piping to serve as a method of composition of functions. Three sections of an optional first-year college mathematics lab were used for the study. Two sections ($n = 10, 14$) taught by different instructors used Unix shells and pipes when studying the concept of function. A third group ($n = 20$) taught by one of the two instructors used only paper-and-pencil when studying the concept of function. They found that the experimental students were more likely to encapsulate a function as a cognitive object. Furthermore, the experimental sections were more successful on nonroutine questions involving functions. They conjecture that this might have been due to students' generalization of the concept of function beyond algebraic formulas. The results on
more traditional function problems suggests that the computer treatment was non-detrimental to the ability of students to perform the standard manipulations with functions given by algebraic formulas.

Schwingendorf, Hawks, and Beineke (1992) investigated vertical and horizontal growth of students' understanding of functions in first-semester calculus. One experimental section ($n = 36$) used ISETL to construct mathematical concepts as processes by writing their own computer programs. Students were asked "What is a function?" and their descriptions were classified as: prefunction, action, process, correspondence, and dependence. Students' examples of functions were also classified. A second questionnaire asked students which symbolic and graphical forms were representations of functions. Information from a section of traditional students ($n = 20$) was also collected. Schwingendorf, Hawks, and Beineke found that the use of ISETL produced substantial progress in the development of students' understanding and "sense" of the function concept. Specifically, students seemed to possess more connections between graphs and analytic descriptions of functions.

Other studies involving student construction of computer programs to investigate mathematical phenomena suggest similar enhanced learning environments. Dolcetta, Emmer, Falcone, and Vita (1988) found that students had greater opportunity to develop mathematical intuition and to gain basic knowledge more quickly through programming. Tufte (1990a) found that programming helped students in calculus deal with definitions at a more basic level, pay attention to the language of mathematics, and deal with the association between that language and their own conceptions. Furthermore, programming served as a bridge between the gap of natural language and the formal language of mathematics. Tufte (1990b) also found that students who wrote programs to perform some of the processes of calculus were better able to recognize various forms of the definitions of the derivative and
integral, and to relate those forms, as well as algebraic representations of functions, to graphical representations of functions.

**Symbol Sense Problems**

Selden, Mason and Selden (1989) investigated the manner and extent to which average calculus students solve nonroutine problems. A group of seventeen students who had completed the first quarter of calculus with a C volunteered with monetary compensation and prize incentives, and were given a test consisting of five questions (Selden, Mason, & Selden, 1989, p. 48):

1. Find values of \(a\) and \(b\) so that the line \(2x + 3y = a\) is tangent to the graph of \(f(x) = bx^2\) at the point where \(x = 3\).
2. Does \(x^{21} + x^{19} - x^{-1} + 2 = 0\) have any roots between \(-1\) and \(0\)? Why or why not?
3. Let \(f(x) = \begin{cases} ax, & x \leq 1 \\ bx^2 + x + 1, & x > 1 \end{cases}\). Find \(a\) and \(b\) so that \(f\) is differentiable at \(1\).
4. Find at least one solution to the equation \(4x^3 - x^4 = 30\) or explain why no such solution exists.
5. Is there an \(a\) so that \(\lim_{n \to 3} \frac{2x^2 - 3ax + x - a - 1}{x^2 - 2x - 3}\) exists? Explain your answer.

No student got an entire problem correct, and most students made no progress. The seventeen students produced 85 solution attempts with only 6 attempts showing substantial progress toward a correct solution. Observe that problem 2 could be solved by noting that the expression \(x^{21} + x^{19}\) is always greater than \(-2\) and the expression \(-x^{-1}\) is always positive for \(x\) between \(-1\) and \(0\). Therefore the entire expression is always positive for \(x\) between \(-1\) and \(0\), and so there are no roots on this interval. On problem 2, nine students substituted a few numbers for \(x\) and then made a guess. These situations illustrate the interplay between number sense and symbol sense described in Chapter I.
In question 5, since 3 is a root of the denominator of multiplicity 1, the question is equivalent to determining a value of $a$ for which 3 is a root of the numerator. Given the role of $a$ in the expression, this is clearly possible. On question 5, three students used L'Hôpital's rule, and eight students substituted $x = 3$ in the function, found the denominator was 0 and either didn't continue or concluded the limit could not exist because one can not divide by zero. Students spend a great deal of time in secondary and collegiate mathematics factoring quadratics. Do students have a "sense" for when expressions factor? If so, does this "sense" get activated during the solution process? Similar issues about students sense of functions arise in question 4.

On question 4, two students worked the problem in some correct fashion. Four students factored $4x^3 - x^4$ and set each factor equal to 30, and five students substituted a few values for $x$ and guessed. No mention is given of students having access to graphics calculators. Students with graphics calculators tend to graph the function $y = 4x^3 - x^4$ and then trace, if possible, to the desired value. However, when this does not work as in this case, students are often unsure how to proceed. Do students "see" in either the algebraic or graphic representations, a quartic function which opens downward and hence has a maximum? This observation suggests determining the maximum value of the function which turns out to be 27. So no solutions to question 4 can exist. The results on this question suggest that for some students when the standard algorithms do not succeed, there is no alternative direction in which to proceed.

In a discussion of students' sense for functions, Eisenberg (1992) compiled a list of function sense questions and a list of problems that most calculus course graduates will fail similar to those discussed above. Many of these problems require students to have a strong conceptual network and an intuition about mathematical
symbols. The expressed desire by many (e.g. Dreyfus, 1991; Eisenberg, 1992; Tufte, 1988) for students to have the ability to "sense" an error when they see a result such as:

\[
\int_{-1}^{+1} \frac{1}{x^2} dx = \left[ -\frac{1}{x} \right]_{-1}^{+1} = -2
\]

or to realize rather quickly that:

\[
\int_{a}^{b} f(x+k) dx = \int_{a+k}^{b+k} f(x) dx
\]

is not to suggest particular items to teach towards, but as indicators of past "success" and desire for students' development of intuition about symbols. These problems aid in understanding the complex nature of symbol sense, and underscore the appropriateness of the description given in Chapter I.

Summary

Within the research studies themselves and through their synthesis several issues regarding students development of intuition on symbols have surfaced. Several studies (e.g. Heid, 1984; Orton, 1983a, 1983b; Palmier, 1986) and new cognitive theories as discussed in Chapter I (e.g. Wearne & Heibert, 1988; Sfard, 1992, Dubinsky, 1991, Tall, 1991) suggest that concepts can and should be introduced before the symbols and skills associated with the concepts of calculus.

There are several forms of technology that can facilitate the resequencing of concepts and skills. Several research studies have found that the use of a computer algebra system or the use of programming to construct objects out of processes enhanced students' conceptual understanding of calculus when compared with traditional students. However, the use of calculus-specific software has not as
consistently produced similar results. Research is needed on the form of software that is helpful for students learning advanced mathematics.

While some research has focused on students' sense of symbols within calculus (e.g. Orton, 1983a, 1983b; Ferrini-Mundy & Graham, 1991b), further research is needed to: (a) detail the symbol sense that students develop associated with calculus, (b) examine the role of intermediate forms in the development of symbol sense, (c) determine the role of by-hand manipulation in the development of symbol sense, and (d) investigate the differences in students' development of symbol sense in various computational environments.
CHAPTER III
RESEARCH DESIGN AND METHODOLOGY

Introduction

The purposes of this study were to: (a) explicate the nature of symbol sense, (b) determine the differential effects of two computer algebra systems on students' development of symbol sense, (c) explore the differences in symbol sense among students using a computer algebra system and students not using a computer algebra system, and (d) further inform practice by examining students' achievement in calculus with and without the use of computer algebra systems. Chapters I and II elaborated on the nature of symbol sense as described in the literature. Therefore, the appropriateness of the resulting description of symbol sense was one item of interest.

Research has suggested that student use of certain forms of computer software, one of which is a computer algebra system, produces enhanced student conceptual understanding. Advances in technology have provided new representations and methods of manipulating representations for mathematical concepts which may foster symbolic understanding and as such symbol sense. In addition, there is considerable interest in the naturalization of software interfaces for student use.

The existence of computer algebra systems has forced many to consider the role of by-hand manipulation in the learning of mathematics. Research has shown that the use of a computer algebra system can permit the resequencing of skills and concepts. However, considerable concern has surfaced over the loss of possible benefits that by-hand manipulation may have to students' development of symbol
sense. Relatively new symbolic algebra packages have been developed which provide an intermediate step between by-hand manipulation and the use of powerful, professionally-oriented computer algebra systems. These software packages may provide the benefits of using a computer algebra system that have been observed in both research and practice, and maintain the possible benefits of by-hand manipulation. Thus, the differential effects of two forms of computer algebra software and effects of CAS in general on students' development of symbol sense were items of interest. Specifically, this study examined students' symbol sense development in three different calculus environments: laboratory-based using MAPLE, laboratory-based using THEORIST, and a traditional lecture/discussion format.

Given the nature of the research questions, this study has two components. The differential effects of two computer algebra systems lends itself well to a quasi-experimental (Campbell & Stanley, 1963) approach since many of the intervening variables can be controlled. The planned use of the computer algebra systems included group work, resequencing of concepts and skills, and different emphasis on selected topics. These factors, together with other intervening variables such as instructor effect, warrant caution on any comparison of the use of a computer algebra system to traditional methods of instruction solely through quasi-experimental methods (Lester, 1992). Qualitative information gained through interviews with selected students in each instructional method are used to provide further detail on the possible differences in symbol sense of students and on the nature of symbol sense itself.
Design

Setting

Since one purpose of this study was to explore the nature of symbol sense, it was desirable to study students who would be: (a) using previously developed symbol sense for certain algebraic expressions, (b) continuing their symbol sense development for some symbolic expressions, and (c) developing their symbol sense for new symbolic expressions. In first-semester calculus, students are typically working with algebraic expressions with which they have considerable experience, further developing their understanding of functions and function notation, and learning new notations for differentiation and integration. Other courses such as second-semester calculus, which involves techniques of integration, may require more frequent use of symbol sense. The use of a computer algebra system might produce greater effects on the development of symbol sense in such courses. But while techniques of integration may require more pattern recognition and other aspects of symbol sense, little new notation is introduced. Furthermore, there is a research base on the use of a computer algebra system in calculus. Thus, first-semester calculus was selected as the course in which to set the study.

Selection of CAS Packages

There are many software packages (e.g. MATHEMATICA, MATHCAD, DERIVE, ISETL, and MICROCALC) available whose use in mathematics is appropriate (Cunningham & Smith, 1988; Quinn, 1990). The selection of software packages to be used was influenced by the nature of the study. Three main selection criteria were used: (1) the manner in which students manipulate expression, i.e.
command-oriented versus concrete symbol manipulation; (2) the extent to which intermediate steps are present in symbolic manipulation; and (3) the visual nature of the symbols involved. Also there was a desire to have the software packages on the same platform or brand of computer, and to have software of the same general capabilities.

THEORIST is a symbolic algebra package which capitalizes on the Macintosh environment. While not nearly as powerful as many other computer algebra systems, THEORIST has a user-interface which greatly resembles by-hand methods. In addition, expressions are entered and displayed in traditional form, and expressions are manipulated in a manner that "mimics the ways in which humans do mathematics" (R. F. Smith, 1991, p. 1253). A unique feature of THEORIST permits symbols to be "grabbed" and actively moved within algebraic expressions. Each manipulation creates a result which is displayed directly below the original expression. THEORIST was chosen as one of the CAS packages because of these unique characteristics.

MAPLE was chosen as the second computer algebra system. MAPLE is a command-oriented language in which a single command might require several intermediate steps if done by-hand. Expressions are entered in a FORTRAN-like manner, and displayed in mock traditional form. Thus MAPLE and THEORIST differ in all three of the principle categories of interest. MAPLE is also available on the Macintosh. In addition both MAPLE and THEORIST have similar capabilities although THEORIST has by far the better graphics user-interface. It should be noted that other computer algebra systems, e.g. MATHEMATICA, have characteristics similar to those of MAPLE. The final selection of MAPLE was due to a university decision to purchase a site license for MAPLE.
Laboratory Materials

From a constructivist perspective, students should actively investigate the concepts of mathematics with or without the use a computer algebra system. While CAS-based calculus materials exist, most of these materials are supplemental laboratories which assume topics are first presented in class and then further explored in a laboratory setting. Other materials that are CAS-specific were not sufficiently parallel to each other to be used in a quasi-experimental design. Thus the researcher developed parallel materials for each CAS package. These laboratories were based in part upon the works of Beckmann and Sundstrom (1992), James (1991), and Judson (in press). Seven laboratories were created and piloted in sections of first-semester calculus during the Spring and Summer 1992 sessions at Western Michigan University. Based upon the pilots, the materials were modified to provide additional directions on the use of the software and to clarify the intent of questions within the materials. Appendix A contains the revised introductory handout and materials for the first two laboratories in each environment. Subsequent labs in Appendix A alternate between the two computer algebra systems.

Several key features of these materials are: (a) a delayed introduction of notation, (b) increased opportunities for conjectures and exploring patterns, and (c) an emphasis on writing and opportunity for students own language. Typically, a laboratory begins in a familiar concrete setting and moves gradually towards more formal mathematics. New notations for limits, derivatives, and integrals, were delayed until students had partially explored the underlying concepts. The materials were designed to promote active learning of the concepts of calculus, and were not specifically designed with the sole intent to develop symbol sense.
Selection of Subjects

Three of the nine sections of first-semester calculus during the Fall 1992 semester and three of eight sections during the Winter 1993 semester were selected. Initial and final enrollment by section and semester are given in Table 2. The sections were selected to be at 10 am, 11 am, and noon. The investigator taught two of the three sections both semesters, and each semester a different professor who was recognized by students and the university for excellence in teaching taught a third comparison section.

Table 2
Initial and Final Enrollment by Section and Semester

<table>
<thead>
<tr>
<th>Section</th>
<th>Fall</th>
<th>Winter</th>
</tr>
</thead>
<tbody>
<tr>
<td>MAPLE</td>
<td>46 (41)</td>
<td>43 (35)</td>
</tr>
<tr>
<td>THEORIST</td>
<td>40 (34)</td>
<td>42 (34)</td>
</tr>
<tr>
<td>Comparison</td>
<td>46 (39)</td>
<td>44 (36)</td>
</tr>
</tbody>
</table>

Note. The final enrollments in parentheses are after a six week drop/add period.

All sections of first-semester calculus used the text by Zill (1992). Also all students, beginning with the Fall 1992 semester, were required to have a graphics calculator. Graphics calculators had been required in precalculus the previous two years. Students in both comparison sections followed a traditional syllabus and lecture/discussion format.
Laboratory Facilities

During the first portion of the Fall 1992 semester both CAS sections used the Mathematics Department's Macintosh Laboratory consisting of 13 networked Mac II's with math coprocessors. MAPLE, version V, and THEORIST, version 1.11, were installed on a server. The Prescience Corporation generously supplied 21 copies of THEORIST to be used in this study. Both MAPLE and THEORIST were available through three student computer laboratories consisting of 250 Mac IIsi's. Since the university had purchased a site license for MAPLE, each of these machines could be running MAPLE independently, while only 21 copies of THEORIST could be running at any given moment. The Mathematics Department's Lab was open from 1 PM to 8 PM during the week, and from noon to 5 PM on Friday, Saturday, and Sunday. The three university labs together were open 24 hours a day.

Because of unexpected enrollments and incomplete renovations of the Mathematics Department's Lab, the class size required students to work in-class in groups of three or four in the MAPLE section, and groups of two or three in the THEORIST section. The MAPLE section, being the larger of the two CAS sections, was moved in early October to a laboratory in the College of Business which consisted of 50 Mac IIsi's with math coprocessors. By early October, the size of the THEORIST section allowed students to work primarily in groups of two with a few groups of three. During the Winter semester, both CAS sections used the Business College Lab, and students worked in pairs.

Experimental Sections

Each semester, both experimental sections used revised laboratory materials developed by the investigator. Typically, students spent two consecutive class...
periods in the computer lab followed by two class periods in the classroom. Occasionally, between labs and near the end of the semester, the class met more frequently in the classroom.

Students were required to complete and submit the laboratory assignments in groups of two or three. The initial grouping of students was based upon completed survey forms and pretest results. Students were grouped together who indicated similar open blocks of time outside of class, but who had different levels of experience in using a computer and different levels of performance on the pretest. Subsequent regrouping of students, approximately after every two labs, was based upon performance on in-class exams and attendance. Each group was assigned a companion group and encouraged to discuss questions and issues with the companion group before asking the instructor.

Assessment of Outcomes

Since the purposes of this study were to explore the nature of symbol sense and to determine the differential effects of students' use of two computer algebra systems, both qualitative and quantitative information was gathered. It was believed that the qualitative information might (a) suggest further aspects of symbol sense, (b) provide a measure of the appropriateness of the description of symbol sense given in Chapter I, and (c) provide details on the nature of differences in symbol sense among students. The quantitative information was intended to provide a more general measurement of the differences in students' development of symbol sense using a computer algebra system. Sowder (1992) summarizes a conference on assessment of number sense issues in which Resnick "pointed out that it is possible to assign numbers, if that is what is wanted, by using the model of judging Olympic performance" (p. 384). Both the quantitative and qualitative information gathered are
based upon this philosophy. Students' responses to task-based items were examined for the behaviors described in Table 1 of Chapter I and rated for the degree of symbol sense. For example, a student who graphs a linear function using only two points is considered to be exhibiting greater symbol sense, and thus received a higher rating than a student who uses more than two points even though both students might produce a correct graph.

**Quantitative Measures and Their Administration**

**Symbol Sense Pretest and Posttest.** The investigator developed pretest and posttest versions of a written paper-and-pencil instrument whose questions were based upon the behaviors exhibiting symbol sense as described in Chapter I. The tests can be found in Appendix B. The pretest involved notation and concepts from precalculus and the posttest used notation and concepts from precalculus and calculus. The exam consisted of three different types of questions: multiple choice, short answer, and pictogram or fill-in-the-box. Sowder (1992) describes the first two question formats in her discussion of assessment issues of number sense. The third type of question format as shown in Figure 1, is specifically intended to explore the visual nature of symbol sense. Since students are not as familiar with this type of question, the majority of the questions were of a multiple choice or short answer format. The test questions were validated through discussions with another mathematician and mathematics educator.

Given the written nature of the tests, the questions emphasized the cognitive aspects of symbol sense more than the metacognitive aspects. Since the behaviors exhibiting symbol sense as described in Chapter I are not independent, many of the questions deal with multiple aspects of symbol sense. For example, the pictogram questions in Figure 1 involve transformation within symbolic representations. But the
solution of the problems require pattern recognition, strategy-making and other metacognitive behaviors. The questions were developed around the three different transformation abilities. Not all of the questions listed under "transformation within the algebraic system" required algebraic transformations such as pretest item 23 in which students are asked to determine the minimum value of the expression $\sqrt{(x-a)^2 + 3}$. These questions are most likely answered without reference to other representation systems and therefore included in the first category. A summary categorization of questions is given in Table 3.

In items 29 and 30, supply the missing steps and missing information to make the equations true.

| 29. | \[
\frac{\cos^2(x) + 2\sin^2(x)}{\cos^3(x)} = \frac{1+ \sin^2(x)}{\cos^3(x)}
\] |
| 30. | \[
\int 2x(7-3x^2)dx = \frac{(7-3x^2)^{\square}}{\square} + c
\] |

Figure 1. Sample Pictogram Posttest Items.

Several items were parallel between the pretest and posttest and listed as coordinate pairs, they were: (2, 1), (3, 3), (5, 8), (6, 11), (7, 4), (8, 2), (9, 5), (12, 13), (13, 18), (18, 20), (19, 12), (20, 15), (21, 17), (22, 16), (23, 19), (25, 21), (26, 26), (27, 27), and (28, 29). While a natural desire was to have similar forms, the additional content covered necessitated some differences between the pretest and posttest instruments. Posttest items 1, 2, 6, 7, 10, 12, 14, 15, 22, 23, 24, 25, 28, and
30 focus on some aspect of symbol sense that may have been developed with the study of calculus. Noticeably absent from the posttest are questions on transforming from symbolic expressions to verbal descriptions. The pretest item 14 was intended to determine students' ability to recognize useful symbolic forms. Recognizing useful forms of symbolic expressions is a component of posttest items 6, 10, and 29. There are also two additional questions on the posttest requiring interpretation of verbal descriptions.

Table 3
Categorization of Symbol Sense Test Items by Behavior

<table>
<thead>
<tr>
<th>Transformation</th>
<th>Pretest</th>
<th>Posttest</th>
</tr>
</thead>
<tbody>
<tr>
<td>Within the Algebraic System</td>
<td>1, 5, 7, 12, 13, 15, 16, 17, 18, 21, 23, 24, 26-30</td>
<td>4, 7, 8, 9, 10, 13, 17, 18, 19, 20, 22, 23, 25, 26-30</td>
</tr>
<tr>
<td>From Graphical Representations</td>
<td>3, 4, 6, 25</td>
<td>3, 11, 21</td>
</tr>
<tr>
<td>From Verbal Descriptions</td>
<td>2, 9, 19, 20</td>
<td>1, 5, 12, 14, 15, 24</td>
</tr>
<tr>
<td>To Graphical Representations</td>
<td>8, 10, 22</td>
<td>2, 6, 16</td>
</tr>
<tr>
<td>To Verbal Descriptions</td>
<td>11, 14</td>
<td></td>
</tr>
</tbody>
</table>

The tests were piloted during the Spring and Summer 1992 sessions in which students were given 1 1/2 hours to complete the exams with most students finishing within 40-50 minutes. All responses were categorized and selected students were asked to discuss their solutions. A scoring rubric developed on the basis of the pilot trials is provided in Appendix C. Some modifications were made to both the pretest and posttest based upon the spring semester pilot. Using the Cronbach alpha formula
for estimating reliability, the pretest had reliability estimates of $\alpha = .78 \ (n = 29)$ for the spring session and $\alpha = .80 \ (n = 21)$ for the summer session. The posttest had reliability estimates of $\alpha = .72 \ (n = 25)$ and $\alpha = .75 \ (n = 24)$ for the spring and summer sessions respectively. A few additional modifications were made to both instruments based upon the summer session results.

**Comprehensive Final.** Since symbol sense requires a well-developed conceptual network, students' overall performance in calculus was of interest. Scores from a comprehensive multiple-choice departmental final exam were collected. Each semester, the final exam is constructed by a faculty member not currently teaching calculus. The faculty member solicits questions from the current instructors over designated sections, and then selects 30 questions to comprise the final. The Fall 1992 and Winter 1993 comprehensive final exams are provided in a Appendix D.

**Laboratory Practical.** Students in the CAS sections were also given a laboratory practical (Appendix D). The purpose of the laboratory practical was to determine the extent to which students could use the software in solving calculus problems. The lab practicals were identical for both sections.

**Administration.** On the first day of class each semester, a pretest was given after handing out course information. Students transferring into the selected sections were asked to take the pretest on their first day of attendance. During the thirteenth week of classes, a posttest was administered to all students. Students missing this class period completed the exam as soon as possible. The laboratory practical was given each semester to students at arranged times during the fourteenth week. The departmental final exam was given the following Monday.
Students received feedback on the pretest during the first week indicating that review was either required or would be helpful in each of the following categories: algebraic skills, graphs, functions, and trigonometric functions. Students received feedback on the posttest indicating percent correct and a letter grade equivalent using a distribution that was based upon past distributions of course grades in calculus.

Qualitative Interviews and Their Conduct

Five interviews of selected students from each section were used to provide further detail on students' symbol sense. The interviews were task-based and students were instructed to "think aloud as much as possible". Students were also told that the interviews were to determine how they thought about questions, not if they could solve the problem. To encourage students' verbalization of thoughts, students were asked to read the question aloud and periodically asked to express their thoughts aloud. The interview tasks dealt primarily with students' sense of calculus-based manipulations and students' recognition and use of standard or familiar forms. Some of the interview questions were drawn from the pretest and posttest. The interview questions not from the pretest or posttest are provided in Appendix E. The interviews were constructed and planned so that the prerequisite conceptual material and skills would have been covered at least one week in advanced of the interview session.

First Two Interview Sessions. The first two interview sessions covered all questions from the pretest instrument. These sessions provided information about students' thought processes not captured by a written test, provided a setting in which the students could become comfortable with the video-taping, and provided an opportunity to further validate the test instrument and scoring scheme. Students were instructed that their answers need not to match their previous written responses.
**Third Interview Session.** The third interview consisted of four questions. The first question, as shown in Figure 2, was to investigate the symbol sense that students used in conjecturing if a given limit exists. Limits (a) and (d) are determinable if an algebraic transformation is first observed. To predict the non-existence of limit (b) and the existence of limit (e) requires recognition of the dominance of the denominator. The parameter in the numerator was used to shift the problem from a solely algebraic transformation question similar to item (a). Limit (c) depends on students' knowledge of the reciprocal limit.

1. Without actually finding each of the following limits, which of the limits are likely to exist? Explain your reasoning in each case.

   \[
   \begin{align*}
   (a) \quad & \lim_{x \to 4} \frac{x^2 - x - 12}{x - 4} \\
   (b) \quad & \lim_{x \to -1} \frac{x^2 + ax - 3}{(x + 1)^3} \\
   (c) \quad & \lim_{x \to 0} \frac{3x}{\sin(x)} \\
   (d) \quad & \lim_{x \to 0} \frac{1 - \cos^2(x)}{\sin^2(x)} \\
   (e) \quad & \lim_{x \to \infty} \frac{x^4 - 3x}{x^6 - x \cos(x)}
   \end{align*}
   \]

   **Figure 2.** Interview Question on the Existence of Limits.

   The second question, as shown in Figure 3, examines the extent to which a student can determine a symbolic form for a verbally stated condition. While a variety of forms could be used, it was expected that students would use either a piece-wise defined function or a rational function with a denominator having a root at \(x = 2\). Students unable to produce a symbolic expression would be asked if they could produce a graph of a possible function. It was hypothesized that some students have mental pictures of discontinuous functions, but do not have closely tied symbolic expressions.
2. Give an equation of a function which is discontinuous at \( x = 2 \). Explain how you know your function is discontinuous at \( x = 2 \).

Figure 3. Interview Question on the Creation of a Discontinuous Function.

The third question explored students' ability to recognize equivalent forms of the definition of the derivative as shown in Figure 4. A student with a sound understanding of the definition of the derivative could reason through each limit separately or a student could reason through the invariant roles of the symbols. Either behavior was of interest. Before progressing to question 4, students were asked to select and explain one of the six forms in question 3. If a student was uncertain how to begin or failed to introduce the concept of a tangent line or secant line, the student was asked to explain the relationship between derivatives and tangent lines. The student was asked to give a graphical explanation if one was not already given.

3. Which of the following limits can be used to define the derivative of \( f(x) \) at the point \((a, f(a))\)? Explain your reasoning in each case.

(a) \[ \lim_{x \to 0} \frac{f(x) - f(a)}{x - a} \]
(b) \[ \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \]
(c) \[ \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \]
(d) \[ \lim_{a \to b} \frac{f(b) - f(a)}{b - a} \]
(e) \[ \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x} \]
(f) \[ \lim_{h \to 0} \frac{f(a - h) - f(a)}{-h} \]

Figure 4. Identification of Equivalent Forms of the Definition of the Derivative.

Finally the fourth question, as shown in Figure 5, investigated students' symbol sense in a problem-solving domain. The primary interest of this question was if students could see through the symbols to produce additional information about the nature of the functions. The problem reduces to determining which functions have identically zero derivatives.
Fourth Interview Session. The fourth interview focused on students' symbol sense for derivatives that may have developed. The first three questions as shown in Figure 6 were intended to provide differentiation contexts in which to explore students' symbol sense. The first question was intended to provide an initial feeling of success while also examining students' recognition of the composition of functions. Question 2 examines further how students deal with abstract compositions when finding derivatives. The third item was intended to explore students' perception of patterns in derivatives and ability to express patterns as well as dealing with a parameter in the expression.

4. Let \( f(x) \) and \( g(x) \) be two functions. Recall that, in general, the derivative of the product \( f(x)g(x) \) is:

\[
\frac{d}{dx} [f(x) g(x)] = f(x) g'(x) + f'(x) g(x). 
\]

Suppose for a certain function \( g(x) \), the derivative of the product \( f(x)g(x) \) is:

\[
\frac{d}{dx} [f(x) g(x)] = f'(x) g(x). 
\]

What type of a function must \( g(x) \) be in this case?

Figure 5. Fourth Task from Third Interview Session.

1. Find the derivative of \( f(x) = \sin(\cos(3x)) \).

2. Let \( G \) be a differentiable function. What is \( \frac{d}{dx}[G(-x^2)]^2 \)?

3. Given that \( n \) is a positive integer, find a formula for the derivative \( \frac{d^n}{dx^n} x^n \).

Figure 6. Tasks 1, 2, and 3 from Fourth Interview Session.
Questions 4, 5, and 6, as shown in Figure 7, investigated the manner in which students identified characteristics of various functions. Each of these questions can be approached graphically and/or symbolically. Students approaching the problems entirely in one representation system were asked to provide additional evidence which might support their reasoning.

4. Find the absolute extrema for the function $y = (x - 1)(x + 1)$.

5. Determine where the function $g(x) = (x - 2)^4$ is increasing.

6. Determine where the function $f(x) = |1 - x^2|$ is concave up.

Figure 7. Tasks 4, 5, and 6 from Fourth Interview Session.

Question 7, as shown in Figure 8, was intended to investigate students' perception of the forms of resulting derivatives. Item (a) was to provide an initial feeling of success with this type of question, as well as provide a foundation for item (c). Students were encouraged to continue with item (a) until they determined one possible solution. Question 8 investigated students' perception of the notation difference from standard questions and how students would alter their standard algorithms for implicit differentiation.

7. Determine a function whose first derivative is each of the following.
   a. $x^2$
   b. $2x \sin(x) + x^2 \cos(x)$
   c. $72x^2(2x^3 + 5)^5$

8. Given the equation $x^2 + x^2 y^3 + y^4 = 7$, find $\frac{dx}{dy}$.

Figure 8. Tasks 7 and 8 from Fourth Interview Session.
Fifth Interview Session. The last interview covered questions 1, 2, 6, 7, 10, 12, 13, 15, 22, 25, 28, and 30 from the posttest. Most of these questions were the calculus-based questions not appearing on the pretest. As with the first two interview sessions, the fifth interview session was intended to provide information not captured by the written test and to provide further validation of the test instrument and scoring scheme.

Conduct of the Interviews. Four students from each of the calculus sections involved in the study were asked to participate voluntarily in a series of five video-taped interviews. The students were selected based upon their performance on the pretest instrument. From each section, one student was selected at the middle of each of the lower and upper quartiles, and two students were selected at the median score. More students were chosen at the median score under the hypothesis that their symbol sense was most in transition. Students were also chosen to insure a balance of gender. Students were informed that the interviews were to gain an understanding of the stages in which students learn calculus and that there would be no impact upon their grade. Most of the students welcomed the opportunity for individual attention. Only two of the initially selected students who were asked to participate declined, and both students declined due to personal time constraints.

The first interview sessions were scheduled during the end of the second week of classes. The second interview sessions were conducted immediately after the first interview sessions were completed and continued into the fourth week of classes. The third and fourth interview sessions occurred during weeks 6 and 10, respectively. The fifth session was scheduled after the posttest during the fourteenth week of classes.
Null Hypotheses

The posttest scores, the final exam scores, and laboratory practical scores were analyzed based upon pretest scores and treatment group to test the following null hypotheses.

**Hypothesis I.** There are no significant differences in performance on the symbol sense posttest among the three treatments.

**Hypothesis II.** There are no significant differences in performance on a departmental comprehensive final exam among the three treatments.

**Hypothesis III.** There are no significant differences in performance on the laboratory practical between the computer algebra system treatments.

The effects of the use of a computer algebra system on each of the measures may be different across different levels when compared to lecture/discussion based instruction. Therefore, the individual regression lines for each treatment were examined for the null hypothesis of equal slopes among treatment groups. If this null hypothesis is rejected, the specific regression lines will be examined to determine the nature of the interaction. Otherwise, analysis of covariance will be used to determine possible treatment effects for each hypothesis, namely the null hypothesis that each regression line has the same intercept.

Tapes of the interview sessions were analyzed for those behaviors exhibiting symbol sense as described in Chapter I. Since the written tests provide considerable information on the cognitive aspects of symbol sense, special attention was given to possible metacognitive behaviors when interpreting the interviews. The primary role of the interviews was to provide additional insight into the differences in symbol sense of students from each of the treatment sections. All twelve interview students from the fall semester completed the five interview sessions. During the winter
semester, nine of the twelve students completed the five interview sessions. The other three students, two students from the MAPLE section and one from the comparison section, completed three of the five sessions and did not complete the course.
CHAPTER IV

RESULTS

Introduction

This chapter reports the results of the quantitative analysis of the study. It is based on students who completed each of the criterion measures. The first section reports the results on the pretest which is used as a covariate in most of the remaining analyses.

Since the computer environment differed between the Fall and Winter semesters and the characteristics of a typical first-semester calculus student often differ between Fall and Winter semesters, the second section examines possible semester effects. The pretest, posttest, and laboratory practical were identical across the two semesters. However, while the final exam was comparable across semesters, it was not identical. Possible semester effects are examined for each of the posttest, comprehensive final, and lab practical with the pretest score as a covariate. Specifically, the homogeneity of regression by semester is tested. If no semester effects exists, i.e. the regression lines for each treatment across semesters are statistically equivalent, then the data from the two semesters is pooled by treatment in the remaining analyses. However, when a semester effect is determined, the data is analyzed separately by semester.

Each of the null hypotheses for symbol sense posttest scores, comprehensive final exam scores, and laboratory practical is examined by first testing the equality of treatment regression slopes with symbol sense pretest score as a covariate. If the null hypothesis of equal regression slopes is accepted, analysis of covariance is used to
examine treatment effects. If however, the regression slopes are not comparable, then the regressions lines are examined for the nature of the intersections.

Pretest

The number of subjects, mean pretest score and standard deviation by treatment section are given in Table 4. Using the Cronbach alpha formula, the pretest had reliability estimates of $\alpha = .78 \ (n = 146)$ for the Fall semester and $\alpha = .77 \ (n = 96)$ for the Winter semester. These estimates for reliability are comparable to the pilot estimates of $\alpha = .78$ and $\alpha = .80$.

Table 4
Mean Performance on Symbol Sense Pretest by Treatment and Semester

<table>
<thead>
<tr>
<th>Treatment</th>
<th>Fall n</th>
<th>Mean</th>
<th>Std. Dev.</th>
<th>Winter n</th>
<th>Mean</th>
<th>Std. Dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>MAPLE</td>
<td>37</td>
<td>45.4</td>
<td>13.7</td>
<td>30</td>
<td>44.5</td>
<td>13.5</td>
</tr>
<tr>
<td>THEORIST</td>
<td>27</td>
<td>40.8</td>
<td>13.6</td>
<td>31</td>
<td>43.8</td>
<td>16.4</td>
</tr>
<tr>
<td>Comparison</td>
<td>39</td>
<td>46.1</td>
<td>17.6</td>
<td>31</td>
<td>53.2</td>
<td>16.6</td>
</tr>
</tbody>
</table>

Note. The maximum possible score was 120.

Analysis of variance indicates no significant difference ($p = .35$) in Fall mean pretest scores among the treatment groups, but a significant difference ($p = .04$) in Winter mean pretest scores. Bonferroni's method for post hoc comparison among means (Neter, Wasserman & Kutner, 1985, pp. 582-584) was used to determine the existence of pairwise significant differences among means. The difference between
the MAPLE and THEORIST sections was not significant \( (p = 0.99) \). The difference in means between the comparison section and each of the computer algebra system sections was not significant at the .05 level \( (p = .10 \text{ for MAPLE and } p = .06 \text{ for THEORIST}) \).

Examination of Semester Effects

To determine a possible semester effect on each of the posttest, final exam, and laboratory practical, a test for homogeneity of regression by treatment independent of semester was examined for each measure and is presented in Tables 15, 16 and 17 of Appendix F. The analyses indicate a significant effect due to semester for the posttest \( (p = .01) \) and the final \( (p = .02) \), but no significant effect due to semester on the laboratory practical \( (p = .14) \). Since the materials, pretest, laboratory practical, and instructor for the CAS sections were identical for both semesters, a pooled analysis of the laboratory practical was conducted.

Analysis of Treatment Effects on Symbol Sense

**Hypothesis I:** There are no significant differences in performance on the symbol sense posttest among the three treatments.

The results of a test of homogeneity of regression for Fall and Winter semesters are summarized in Tables 18 and 19 of Appendix F. The results indicate that parallel regression lines can be assumed for both semesters. Therefore analysis of covariance was used to test the existence of treatment effects. Based upon Tables 5 and 6, no significant differences \( (p = .25 \text{ for Fall and } p = .31 \text{ for Winter}) \) exist among treatment groups in performance on the symbol sense posttest.
Table 5
Analysis of Covariance for Symbol Sense Posttest Scores — Fall

<table>
<thead>
<tr>
<th>Sources</th>
<th>df</th>
<th>Adj. SS</th>
<th>MS</th>
<th>F</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>Covariate</td>
<td>1</td>
<td>13891.2</td>
<td>13891.2</td>
<td>58.03</td>
<td>0.000</td>
</tr>
<tr>
<td>Treatments</td>
<td>2</td>
<td>681.0</td>
<td>340.5</td>
<td>1.42</td>
<td>0.246</td>
</tr>
<tr>
<td>Error</td>
<td>99</td>
<td>23698.3</td>
<td>239.4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>102</td>
<td>38579.2</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 6
Analysis of Covariance for Symbol Sense Posttest Scores — Winter

<table>
<thead>
<tr>
<th>Sources</th>
<th>df</th>
<th>Adj. SS</th>
<th>MS</th>
<th>F</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>Covariate</td>
<td>1</td>
<td>10280.6</td>
<td>10280.6</td>
<td>74.55</td>
<td>0.000</td>
</tr>
<tr>
<td>Treatments</td>
<td>2</td>
<td>327.5</td>
<td>163.8</td>
<td>1.19</td>
<td>0.310</td>
</tr>
<tr>
<td>Error</td>
<td>88</td>
<td>12135.9</td>
<td>137.9</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>91</td>
<td>24629.9</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Tables 7 and 8 summarize the posttest means of the treatment groups for the Fall and Winter semesters. The reliability estimates for the posttest using the Cronbach alpha formula were $\alpha = 0.82$ ($n = 107$) for the Fall and $\alpha = 0.77$ ($n = 96$) for the Winter. These estimates for reliability are slightly higher than the pilot estimates of $\alpha = .72$ and $\alpha = .75$. The improved reliability estimates are most likely the result of the modifications made to the posttest to clarify some test items.
Table 7
Mean Posttest Symbol Sense Score by Treatment — Fall

<table>
<thead>
<tr>
<th>Treatment</th>
<th>Mean</th>
<th>Std. Dev.</th>
<th>Adj. Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>MAPLE</td>
<td>55.6</td>
<td>18.4</td>
<td>54.8</td>
</tr>
<tr>
<td>THEORIST</td>
<td>48.1</td>
<td>19.5</td>
<td>50.9</td>
</tr>
<tr>
<td>Comparison</td>
<td>50.2</td>
<td>20.2</td>
<td>48.9</td>
</tr>
</tbody>
</table>

Note. The maximum possible score was 120.

Table 8
Mean Posttest Symbol Sense Score by Treatment — Winter

<table>
<thead>
<tr>
<th>Treatment</th>
<th>Mean</th>
<th>Std. Dev.</th>
<th>Adj. Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>MAPLE</td>
<td>44.3</td>
<td>15.4</td>
<td>46.1</td>
</tr>
<tr>
<td>THEORIST</td>
<td>43.1</td>
<td>17.5</td>
<td>45.4</td>
</tr>
<tr>
<td>Comparison</td>
<td>54.0</td>
<td>14.5</td>
<td>49.9</td>
</tr>
</tbody>
</table>

Note. The maximum possible score was 120.

Analysis of Treatment Effects on Comprehensive Final

Hypothesis II. There are no significant differences in performance on a departmental comprehensive final exam among the three treatments. The results of a tests of homogeneity of regression as shown in Tables 20 and 21 indicate that the assumption of equal regression slopes may be assumed. Thus, analysis of covariance was performed on the comprehensive final scores of the three
treatment groups. The results are summarized in Tables 9 and 10, and further described in Tables 11 and 12.

Table 9
Analysis of Covariance for Comprehensive Final Exam Scores — Fall

<table>
<thead>
<tr>
<th>Sources</th>
<th>df</th>
<th>Adj. SS</th>
<th>MS</th>
<th>F</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>Covariate</td>
<td>1</td>
<td>214.34</td>
<td>214.34</td>
<td>11.24</td>
<td>0.001</td>
</tr>
<tr>
<td>Treatments</td>
<td>2</td>
<td>138.26</td>
<td>69.13</td>
<td>3.62</td>
<td>0.030</td>
</tr>
<tr>
<td>Error</td>
<td>99</td>
<td>1888.49</td>
<td>19.08</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>102</td>
<td>2299.65</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 10
Analysis of Covariance for Comprehensive Final Exam Scores — Winter

<table>
<thead>
<tr>
<th>Sources</th>
<th>df</th>
<th>Adj. SS</th>
<th>MS</th>
<th>F</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>Covariate</td>
<td>1</td>
<td>514.05</td>
<td>514.05</td>
<td>29.05</td>
<td>0.000</td>
</tr>
<tr>
<td>Treatments</td>
<td>2</td>
<td>70.55</td>
<td>35.27</td>
<td>1.99</td>
<td>0.142</td>
</tr>
<tr>
<td>Error</td>
<td>88</td>
<td>1557.11</td>
<td>17.69</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>91</td>
<td>2270.96</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The $F$ statistic for equal treatment effects during the Fall has a value of 3.62 with 2 and 99 degrees of freedom ($p = .03$). Therefore, the null hypothesis of no treatment effects on the comprehensive final was rejected at the .05 level. Bonferroni's method for post hoc comparison among adjusted means (Neter, Wasserman & Kutner, 1985, pp. 858-860) was used to determine which of the
pairwise differences among means were significant. The analysis indicates a significant treatment effect in favor of the comparison section over the THEORIST section \( (p = .04) \). However, no significant differences were determined between the MAPLE and THEORIST sections \( (p = 0.11) \) or between the MAPLE and comparison sections \( (p = .94) \).

For the Winter semester, the \( F \) statistic for equal treatment effects has a value of 1.99 with 2 and 88 degrees of freedom \( (p = .14) \). Therefore the null hypothesis of no significant differences on the comprehensive final is not rejected at the .05 level.

Analysis of Treatment Effects on Laboratory Practical

**Hypothesis III.** There are no significant differences in performance on the laboratory practical between the computer algebra system treatments.

The results of a test of homogeneity of regression as shown in Table 22 (Appendix F) indicate that the assumption of equal regression slopes may be assumed. Thus, analysis of covariance was performed on the laboratory practical scores of the two treatment groups.

The results are summarized in Table 13 and further described in Table 14. Based on Table 13, there is no significant difference \( (p = .17) \) in the performance on the laboratory practical by the two computer algebra sections.
### Table 11

Mean Comprehensive Final Exam Score by Treatment — Fall

<table>
<thead>
<tr>
<th>Treatment</th>
<th>Mean</th>
<th>Std. Dev.</th>
<th>Adj. Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>MAPLE</td>
<td>15.6</td>
<td>4.68</td>
<td>15.5</td>
</tr>
<tr>
<td>THEORIST</td>
<td>12.8</td>
<td>4.27</td>
<td>13.1</td>
</tr>
<tr>
<td>Comparison</td>
<td>16.2</td>
<td>4.71</td>
<td>16.0</td>
</tr>
</tbody>
</table>

**Note.** The maximum possible score was 30. The mean score for all Fall students was 15.7 with a standard deviation of 4.66.

### Table 12

Mean Comprehensive Final Exam Score by Treatment — Winter

<table>
<thead>
<tr>
<th>Treatment</th>
<th>Mean</th>
<th>Std. Dev.</th>
<th>Adj. Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>MAPLE</td>
<td>17.4</td>
<td>4.44</td>
<td>17.8</td>
</tr>
<tr>
<td>THEORIST</td>
<td>15.8</td>
<td>5.39</td>
<td>16.3</td>
</tr>
<tr>
<td>Comparison</td>
<td>19.4</td>
<td>4.57</td>
<td>18.5</td>
</tr>
</tbody>
</table>

**Note.** The maximum possible score was 30. The mean score for all Winter students was 16.5 with a standard deviation of 4.92.
Table 13
Analysis of Covariance for Laboratory Practical Scores

<table>
<thead>
<tr>
<th>Sources</th>
<th>df</th>
<th>Adj. SS</th>
<th>MS</th>
<th>F</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>Covariate</td>
<td>1</td>
<td>918.64</td>
<td>918.64</td>
<td>9.35</td>
<td>0.003</td>
</tr>
<tr>
<td>Treatments</td>
<td>1</td>
<td>185.94</td>
<td>185.94</td>
<td>1.89</td>
<td>0.171</td>
</tr>
<tr>
<td>Error</td>
<td>122</td>
<td>11982.56</td>
<td>98.22</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>124</td>
<td>13020.67</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 14
Mean Laboratory Practical Scores by Treatment

<table>
<thead>
<tr>
<th>Treatment</th>
<th>n</th>
<th>Mean</th>
<th>Std. Dev.</th>
<th>Adj. Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>MAPLE</td>
<td>67</td>
<td>39.5</td>
<td>11.31</td>
<td>39.3</td>
</tr>
<tr>
<td>THEORIST</td>
<td>58</td>
<td>41.5</td>
<td>8.84</td>
<td>41.7</td>
</tr>
</tbody>
</table>

Note. The maximum possible score was 60.
CHAPTER V

INTERVIEWS

Introduction

This chapter presents a summary of the interviews with the twelve selected students from the Fall semester. Each student completed the five interview sessions described in Chapter III. In addition to the videotaped interviews, the pretests and posttests of these twelve students were examined for patterns which were common for at least three of the four students per section.

Several differences among treatment groups were determined. The differences identified involved students' perception of patterns and standard forms, students' modification of standard procedures, and students' use of information not contained within the problem statement. Commonalities across treatments included students' generalizations involving function notation, students' ability to create examples, and characteristics of students that interact with symbol sense.

While several tasks were examined for each observation, the discussion focuses on representative task(s) to support the observation. These tasks are reproduced within the section. The other cited tasks can be found in Appendix B for the pretest and posttest items or in Appendix E for the interview items.
Differences Among Treatments

Identification of Patterns and Forms

The interviews suggested that the CAS students perceived certain patterns and forms more readily than the comparison students. In particular, the CAS students were more successful in identifying which functions were the result of or involved the product rule, quotient rule, and chain rule. This trait was slightly stronger for the MAPLE section than for the THEORIST section. For example, almost all students had some initial difficulties with question 7 from the fourth interview session asking them to determine a function whose first derivative was specified. By the posttest, the CAS students interviewed more readily worked through questions 10, 28, and 30 identifying more salient features than the comparison students. For example, on posttest question 10 as shown in Figure 9, the computer algebra students not only examined if the given derivative included a sum, but a sum of two related products and so selected items (b) and (e), but not (a) or (c). Most students did not select item (d), stating either that it was the result of the chain rule or the fact that it was "not a sum" of two factors. When item (f) was not selected, the common reason given was that it should involve a negative somewhere. Interestingly, in the CAS sections, little emphasis was placed on the derivative of a quotient, and while the CAS students required more time than the comparison students in solving question 28 (see Appendix B), they did so more often and with more certainty. The CAS students more frequently identified compositions of functions requiring the chain rule, e.g. questions 1 and 2 from the fourth interview session or question 30 from the posttest. However, the comparison students were more successful at correctly finding an antiderivative involving the chain rule once this fact was identified.
10. Each of the following expressions is the result of taking the derivative of various functions. Which expressions are the direct result of taking the derivative of a product of two functions?

(a) \( x^2 + 2x \sin(x) \)  
(d) \( 6(x^2 + 4x)^3(2x + 4) \)

(b) \( x^2 \cos(x) + 2x \sin(x) \)  
(e) \( 7x^6(x^2 + 2) + x^7(2x) \)

(c) \( 7x^6 + 3x^2 \)  
(f) \( \sin(x) \cos(x) + \cos(x) \sin(x) \)

Figure 9. Posttest Question 10.

The CAS students more frequently associated the difference quotient with the slope formula. By examining the solution methods of students for pretest items 14a and 20, question 3 of session 3, and posttest item 6, the investigator determined that the CAS students more often independently identified the difference quotient as calculating a slope. For example, two THEORIST students when discussing posttest item 6, as illustrated in Figure 10, reasoned through items (a), (b), (d), and (e) using slopes of tangent lines and through items (c) and (f) using "average slopes" or slopes of secant lines. The comparison students were as successful as the CAS students on items (c) and (f), but typically reasoned solely through the values involved getting a positive value divided by a positive value.

Modification of Standard Procedures

The comparison section students performed better than either of the CAS sections on problems requiring a modification or generalization of a standard procedure. For example, on question 8 from interview session 4 (see Figure 8), the CAS students had a difficult time implicitly differentiating with respect to \( y \),
sometimes not even certain where to begin. The comparison students generally started implicit differentiation with respect to $x$ using a $y'$ notation. However when asked what the question stated, the comparison students with some thought modified their solution using a $x'$ notation. When asked to explain the notation, two of the comparison students remarked on the interchangeable roles of variable names.

6. Given the graph of the function $f$ on the right, which of the following statements are true?

(a) $f'(b) > 0$
(b) $f'(b) < f'(a)$
(c) $\frac{f(c) - f(a)}{c - a} < 0$
(d) $f'(a) > 0$
(e) $f'(b) > f'(a)$
(f) $\frac{f(c) - f(a)}{c - a} > 0$

Figure 10. Posttest Question 6.

Comparison and Journeys Outside of the Problem Domain

One of the clear patterns to emerge was the extent to which students introduced comparative information to guide their judgments and strategy-making that was not contained within the problem or generated as a result of standard manipulations. The CAS students more frequently introduced information not specifically stated within the problem. For the THEORIST section, this information was commonly graphical in nature. For example, on posttest question 6 discussed previously, two THEORIST students made a graph of $f'(x)$. A MAPLE student introduced the function $1/x$ in her argument that the limit:

$$\lim_{x \to \infty} \frac{x^4 - 3x}{x^6 - x \cos(x)}$$
would exist. The CAS students were more likely to bound the cosine function when discussing this limit. A THEORIST student uncertain about the general graphical behavior of the expression $100 - x^4$ compared it to the function $100 - x^2$. While a few examples existed of comparison students introducing information not contained in the problem, such information tended not to aid the student in finding a correct solution. For example, one comparison student claimed that the function of posttest question 6 of Figure 10 resembled a cubic, and that the derivative of a cubic was a parabola which she drew lying entirely above the $x$-axis.

**Planned Use of Manipulations**

Somewhat related to the introduction of additional information to guide the reasoning process is the extent to which manipulations are performed with a specific purpose. For example, some students expanded a factored polynomial with a single root believing that the factored form of the polynomial would be different. Other students expanded a polynomial to calculate a value that would be most easily calculated with the factored expression, or in hopes of somehow changing where the expression was zero. While all students seemed somewhat prone to manipulate expressions first and ask questions second, the interviews suggested that the comparison students were more likely to continue this practice with the manipulations associated with calculus. For example, on question 4, 5 and 6 of the fourth interview session (see Appendix E), several comparison students quickly took the first and second derivatives, and then were uncertain what they were to do with this information. None of the CAS students took a derivative without having a subsequent step in mind.

The fourth problem of the third interview session (Figure 5, Chapter III) required students to reason through stated conditions to deduce characteristics of an
unknown function, and then to use these characteristics to deduce a class of functions. The results on this question somewhat reflect the differences among the sections. Most students determined that the unknown function had to have an identically zero derivative. One comparison student reasoned solely on the form of the given derivative. A few students remarked that one possible case was $f(x)$ equal to zero but did not pursue this case. The comparison students seemed to approach this problem more readily than the CAS section students. However, there seemed to be no differences in students' ability to determine that a function with an identically zero derivative is a constant function. For some students, this result was immediate, others guessed a linear function first, and some just required substantial time to reason through the constraints. Once the unknown function was determined to be a constant, the CAS section students more readily explained why the derivative of a constant is zero, usually by referring to a graph of a horizontal line. The comparison students commonly argued via the power rule or responded that it was just a rule.

Commonalities Among Treatments

This section details three of the commonalities that surfaced among all treatment groups. The first involves functions, function notation, and students' growth of symbol sense involving function notation. The second discusses students' ability to create examples. Finally, several characteristics of students that seemed to interact with symbol sense are discussed. These included students' conceptual understanding, problem solving approaches and disposition, and number sense.

Function Notation

Students' work with functions and function notation was tracked through pretest items 5, 8, 9, 12, 13, 15; interview session 3 questions 3 and 4; interview
session 4 questions 1 and 2; and posttest questions 2, 5, 8, 9, 13, and 18. As the semester progressed, students were increasingly over-generalizing properties to function notation. For example, many of the interview students who were able to correctly answer pretest item 12 concerning \( f(x + h) \) responded to posttest question 13 using the simplification \( f(x + h) = f(x) + h \) or \( f(x + h) = f(x) + f(h) \). Almost all of the students initially treated interview questions 1 and 2 from session 4 as a product often writing the sine function without an argument. Students who performed cancellation in the expression \( \cos(x)/x \) on the pretest used similar cancellations on the posttest. Every student interviewed either during the interviews or on the posttest committed an error suggesting an inappropriate generalization involving function notation. Often the error was subtle and did not produce consistent mistakes. For example, one student consistently read \( f(x) \) as "f of x", but \( G(x) \) as "G times x" even when the problem stated \( G \) was a function.

Creating Examples

Several of the interview questions and written test items, e.g. interview session 1 question 2, pretest item 20 or posttest item 15, required students to construct a symbolic expression that satisfied certain constraints. Almost all the interviewed students were troubled at some point by providing information necessary to create an example. On pretest item 20, students were asked to give an example of a line having slope 6. Several students gave a general form \( y = 6x + b \) but were troubled with specifying a particular value for \( b \).

Posttest item 15 in which students were asked to provide a function having a tangent line at \( x = 1 \) of slope 3 proved to be even more difficult. Five of the interviewed students accurately drew a picture involving a quadratic and a tangent line with slope 3 at \( x = 1 \), but were uncertain how to proceed. Five students began by
finding an equation for the tangent line. Interestingly, most of these students used the point \((1, 0)\) or \((0, 0)\) in doing so. When asked if they were finding the function or the tangent line, three of the five students responded the tangent line and were unable to proceed or integrated their linear expression. Two of the five students responded the tangent line, but realized that a line of slope 3 also satisfied the problem. Only two students started the problem by finding a function whose derivative had a value of 3 at \(x = 1\). In both cases, the resulting function was nonlinear.

**Conceptual Understanding**

The interviews suggested that students with a deeper conceptual understanding of calculus exhibited a wider range of symbol sense than those with a surface or procedural understanding. The behaviors exhibiting symbol sense of students with a procedural understanding of calculus were more frequently the cognitive behaviors as describe in Chapter I, whereas students with a richer understanding of calculus exhibited both cognitive and metacognitive behaviors associated with symbol sense.

Three students, one from each section, seemed to have poor conceptual understanding and were relying on procedural understanding throughout the course. While these students were moderately successful on questions such as identifying which derivatives were the results of products, they had a difficult time translating to graphical representations as required on posttest item 6 described previously. They had even greater difficulty with the pictogram questions on the posttest. In contrast, the students who illustrated a broader understanding of calculus were equally successful at translating to graphical representations and working with the pictogram questions.
Problem Solving

The interviews, pretest, and posttest were task-based and provided information regarding the manner in which students approached problem situations. The information suggests that students' problem-solving approaches and disposition towards problem solving interact with the students' symbol sense. For example, many of the students approached several of the pretest items by trying specific values, points, or cases. Students would graph lines by plotting numerous points, determine where a function is positive by testing many values, or test properties such as distributing square roots by trying a few cases. By the end of the semester, very few students used a case or value approach, although many students retained a point approach to graphing and did not utilize traits such as linearity. A disposition towards making sense of symbolic expressions was usually accompanied by a disposition towards problem solving.

Two students, one each in the MAPLE and comparison sections seemed to have a definite disposition toward making sense of problem situations which included making sense of the symbols involved. The comparison student had strong algebraic skills, and the MAPLE student had strong graphical skills. Both students were performing slightly above average, but were not in the top 25% of their respective sections in terms of achievement. Yet both of these students showed a disposition towards symbol sense, problem solving, and making sense far exceeding the other ten interviewed students.

Number Sense

The interviews clearly indicated that symbol sense is dependent on number sense. As noted previously, several students when working with limits involving
factored polynomials expanded the polynomial expecting that the value of the polynomial at the point of interest might change. Other students when confronted with a zero in the denominator rewrote the expression using negative exponents, again hoping that the value of the expression might change. The interplay between number sense and symbol sense was also evident on items such as the exponents pictogram question 29 on the pretest, determining the degree of \((x - 1)^5 - (x + 2)^5\) on both the pretest and posttest, or determining the maximum or minimum of a symbolic expression such as \(\sqrt{3 + 2\sin^2(x)}\). Number sense and symbol sense were clearly intertwined.

One student performed a cancellation in the expression \((a + b) / a\). The student asked the investigator if this step was okay. The student was asked if there was some way he could check. The student tried using integers and determined that it probably was not a correct step. Another student on determining the degree of \((x - 1)^5 - (x + 2)^5\) maintained that "all the x's would cancel", but was perplexed by the difference always being a constant stating that she did not see how this would be possible referring to \((-1)^5 - 2^5\) versus \(10^5 - 13^5\). These examples show that a students' number sense can sometime facilitate symbolic reasoning, and also that a students' number sense can be in conflict with a students' perception of symbolic operations.
CHAPTER VI

SUMMARY AND CONCLUSIONS

The purpose of this study was to explicate the nature of symbol sense and its development in two computer algebra system environments. The first chapter described current perspectives on number sense and symbol sense, and further developed the construct of symbol sense. The remaining text focused on a study of the differential effects of two computer algebra systems on students' development of symbol sense. The symbol sense exhibited by students in six sections of first-semester calculus during the 1992-93 academic year at Western Michigan University was examined in three different computational environments: two computer algebra system environments and a traditional lecture-based environment which included the use of graphics calculators.

The investigator taught two CAS sections both semesters using investigator-developed materials for MAPLE and THEORIST. These two software packages were chosen due to their differences in: (a) the manner in which students manipulate expressions, i.e. command-oriented versus concrete symbol manipulation; (b) the extent to which intermediate steps are present in symbolic manipulation; and (c) the visual nature of the symbols involved. THEORIST has a user interface which resembles by-hand manipulation and form, whereas MAPLE has a programming language, command-oriented interface.

Students in the CAS sections worked together in groups of two or three on laboratory investigations which began with familiar concrete settings and moved to more formal mathematics. Key features of these materials included: (a) a delayed
introduction of notation, (b) increased opportunity for conjectures and exploring patterns, and (c) an emphasis on writing and opportunity for students' own language. Typically, two class days of laboratory work were followed by two days of in-class discussion.

Each semester, a different professor who was recognized by students and the university for excellence in teaching taught a third comparison section which followed a traditional lecture/discussion format. All sections of calculus required the use of a graphics calculator.

Information was gathered on each section from pre- and post-treatment administrations of an investigator-developed test of symbol sense, from departmental comprehensive final exams, from a laboratory practical, and from five video-taped interviews with four students from each section. The quantitative data was analyzed using analysis of covariance with the pretest as a covariate. Differences across semesters were determined to exist for the posttest and final exams, but not for the laboratory practical. Therefore the posttest and final exams were analyzed by semester. A pooled analysis of the laboratory practical was conducted. The qualitative information was examined for the behaviors exhibiting symbol sense as described in Chapter I. The results of both analyses are summarized together below.

Hypothesis I: There are no significant differences in performance on the symbol sense posttest among the three treatments.

This hypothesis was not rejected at the 0.05 level ($p = .25$ for Fall and $p = .31$ for Winter). While the posttest results do not indicate any significant differences among the three treatments, the interviews with students suggest four possible differences among the sections:
1. The CAS students perceived certain patterns and forms more readily than the comparison students particularly in regards to the product rule, chain rule, and associating the difference quotient with the slope formula.

2. The comparison students performed better than the CAS students on problems requiring modification or generalization of standard procedures such as implicit differentiation and multiple indefinite integrals of polynomials.

3. The CAS students more frequently introduced comparative information to guide judgments and strategy-making that was not specifically stated within the problem. For students in the THEORIST section, this information was usually graphical in nature.

4. The CAS students more frequently had a holistic approach to problems involving the concepts of calculus.

The interviews suggested some factors interacting with symbol sense and its development. Students with a deeper conceptual understanding of calculus also exhibited a wider range of symbol sense behaviors than those having primarily a procedural understanding of calculus. Students with a disposition towards problem solving also exhibited a disposition towards making sense of symbolic expressions. Symbol sense and number sense often interact positively and a student's symbol sense is frequently built upon a foundation of number sense. However, a student's number sense can be in conflict with a student's symbol sense.

The interviews and symbol sense tests results suggest that students have considerable difficulty with creating examples requiring the production of additional information. However, students had the most difficult time with the pictogram questions. Students' work on these questions during the interviews supports the findings of Kirschner (1989) that students learn to parse expressions that are visually present, but only some develop sound propositional knowledge of syntactic rules.
However, there are even fewer students for whom symbols reach a level of palpability as described by Mason (1987).

**Hypothesis II:** There are no significant differences in performance on a departmental comprehensive final exam among the three treatments.

The results indicated no significant differences among treatment groups on the comprehensive final exam with the exception of a significant difference \((p < .04)\) in favor of the comparison section over the THEORIST section on the Fall comprehensive final exam. The interviews provided additional details on students' understanding of calculus.

The interviews suggested that the comparison students were better at performing limit computations symbolically. The CAS students more frequently reasoned graphically about derivatives and more often produced a graphical rendering of derivatives than the comparison students. The CAS students used this graphical reasoning to relate the behavior of the first and second derivatives to the concepts of increasing/decreasing and concavity. As noted earlier, the CAS students more frequently associated the difference quotient in the definition of the derivative with the slope formula. Students from each section occasionally considered the concept of an increasing function graphically as originating from the origin outward rather than moving from left to right. The interviews suggested a comparable understanding of the definite integral.

The interviews also suggested observations similar to those of Orton (1983a, 1983b) that students have a difficult time with the individual symbols \(dx\) and \(dy\), and the relationship between \(\Delta y / \Delta x\) and \(dy / dx\). However, students seemed to have an even greater difficulty with the notation \(d / dx\) and \(d^n / dx^n\). Additionally, many students maintained that \(\Delta x\) could not be negative. The interviews indicated that every student was susceptible to over-generalizations as noted by Norman and
Prichard (1991) particularly in regards to function notation. Selden, Mason and Selden (1989) noted that average calculus students rarely used calculus in their solution attempts to nonroutine problems. Students' work during the interviews suggest that students would use a graphical method to determine limits, to find where a function is increasing or decreasing, or to determine concavity even when symbolic reasoning might be more appropriate.

**Hypothesis III:** There are no significant differences in performance on the laboratory practical between the computer algebra system treatments.

The results indicate no significant difference \((p = .17)\) between the treatments in students' ability to use a computer algebra system to solve calculus problems. Based upon classroom observations and student evaluations, students in the THEORIST sections showed greater frustration than students in the MAPLE sections. However, students in the THEORIST sections commented more frequently on usefulness of THEORIST's graphics. In addition, while the MAPLE sections' mean score on the pretest, posttest, and final exam were always greater than those of the corresponding THEORIST sections, the THEORIST mean score on the laboratory practical was greater, but not significantly, than those of the corresponding MAPLE sections. Students in both treatments had a difficult time in syntactically defining a function.

Other classroom observations support several of the remarks made by Judson (1990, 1991). Students using a computer algebra system were forced to pay attention to details and gained appreciation for the importance of speaking mathematics correctly. The use of a computer algebra system seemed to focus student attention on understanding the notation involved. This facet was more evident for students in the command-oriented MAPLE section than for the visually-oriented THEORIST section.
In summary, this study has shown that the reduction of time spent on by-hand manipulation through students' use of a computer algebra system does not negatively effect the students' development of symbol sense or performance on departmental comprehensive final exams. The results suggest that the form of symbolic manipulation may not be a factor in students' development of symbol sense and that a laboratory approach to calculus may aid in students' broader development of symbol sense.

In addition to furthering the development of the construct of symbol sense, the results of this study suggest the following implications for practice:

1. The use of a computer algebra system provides opportunity for increased exploration of the concepts of calculus and a reduced emphasis on skills development. The results of this study and the results of other studies (Crocker, 1991; Heid, 1984; Palmiter, 1986, Schrock, 1989) suggest that a reduced emphasis on by-hand manipulation does not negatively affect students' symbolic skill or symbol sense.

2. The manner of symbolic computation does not seem to be a major factor in the development of symbol sense. Considerable concern has been expressed over the possible deleterious effect on students' development of number sense in classrooms using calculators. A similar concern, as described in Chapter I, has been raised in terms of students' development of symbol sense in CAS environments. Research has shown that the use of a calculator does not inhibit and can be used to enhance students' development of number sense. The results of this study suggest students' development of symbol sense is not adversely affected by instruction using a computer algebra system.

3. Students are equally successful at learning and using MAPLE or THEORIST to solve calculus problems. Whereas students found certain aspects of both software packages frustrating, there were no observed differences between
students using each package with the exception of the THEORIST students' graphical preference. One of the commonly criticized features of computer algebra systems is their deviation away from standard by-hand notation and form. Observations associated with this study and those by Judson (1990, 1991) suggest that these differences in notation force greater attentiveness to notation on the part of students and in so doing may be beneficial to their development of understanding of symbols.

4. Finally, students who are encouraged to explore patterns, make conjectures, and focus on communicating mathematically in a laboratory-based instructional environment exhibit a greater variety of metacognitive behaviors exhibiting symbol sense than those in traditional lecture-based instruction.

The results of this study and the investigator's experience in conducting the investigation and earlier pilot studies suggest the following possible directions for future research:

1. The most notable differences among the sections were the CAS students' introduction of comparative information not contained within the problem and holistic approach to problems. These behaviors suggest a conceptual environment perspective of symbol sense similar to that of Greeno (1991) for number sense. This study focused primarily on the cognitive aspects of symbol sense and the role of a well-developed conceptual network as described in Chapter I in the development of symbol sense. The results of this study suggest the appropriateness of this characterization. However, future research is needed which focuses on the metacognitive aspects of symbol sense.

2. The interviews suggested that the laboratory-based instruction may have fostered a broader development of symbol sense than lecture-based instruction. Other laboratory materials for use of a computer algebra system exist and should be used in replications of this study. In addition, given the longitudinal development of symbol
sense, students' progress using a computer algebra system over several semesters should be examined.

3. The relationship and interplay between number sense and symbol sense, and between problem solving and symbol sense needs to be further explored.

4. Several studies (Ayes, Davis, Dubinsky & Lewin, 1988; Dolcetta, Emmer, Falcone & Vita, 1988; Schwngendorf, Hawks, and Beineke, 1992; Tufte, 1990a) have shown an enhanced understanding of the concept of a function when using technology to produce new ways of representing functions. Students had difficulty with syntactic form for functions within both MAPLE and THEORIST. Further research is needed on various notational forms for functions and the symbol sense that each requires and perhaps fosters. The interviews provided evidence of students' difficulty with the notation of calculus. Research is needed to examine the implications for the development of symbol sense when the standard notation of calculus is delayed as long as possible as has been recommended by Orton (1983a, 1983b), Sfard (1992), and others.

5. Other courses such as second semester calculus involving techniques of integration may require greater symbol sense than first-semester calculus. Palmiter (1986, 1991) found that instruction incorporating a computer algebra system enhanced students' performance on conceptual and computational exams and in subsequent courses. Research is needed to explore students' development of symbol sense in other courses such as integral calculus with and without the use of a computer algebra system.

6. New cognitive technologies (Pea, 1987; Tall, 1992) are being developed which allow students to interact with one representation system and observe the impact within the same or other representation systems. Research will be needed to explore students' development of symbol sense in these new environments.
Enabling students to approach mathematics as a sense-making activity is a central goal of the current reform effort. A fundamental component of students' sense-making ability is their development of symbol sense. Continued investigation of symbol sense and its development offers promise for enhancing our understanding of how students learn mathematics.
Appendix A

Selected Laboratory Materials
The following are selected laboratories used in conjunction with the study. Since the laboratory materials for Maple and Theorist were constructed to be as identical as possible only the introduction and first two labs are provided for both software packages. The remaining five labs are alternately from Theorist and Maple. The labs are reproduced here at 88% of original scale and were originally copied back-to-back. Some minor repagination and line shifting have occurred as a result. In addition, even and odd page headers with page numbers were used. Labs 3 and 6 were handed out in parts. Students had to complete the early parts before receiving the remaining parts. This was done in part to encourage students to finish the labs on a timely basis, but for lab 6 in particular, the last part of the lab might have influenced their work on the first part. This appendix ends with the computer notebooks or worksheets that were used in conjunction with the materials.
Introduction to Theorist

OVERVIEW

This handout is an introduction to the use of Theorist. Theorist is a computer program that performs algebraic manipulations, such as factoring and solving equations, and produces graphical representations of functions and equations. It is meant to be a tool for doing and learning about mathematics. Theorist will perform most of the calculations that you have learned before calculus and most of the calculations you will use in calculus. Part of your responsibility is to ask questions, pose problems, and seek limitations. This includes knowing how and when to use Theorist, how to judge the correctness of the information that Theorist presents to you, and how to approach new and different problems. Much of the information below is only the tip of the iceberg. It is assumed that you will explore the software to a greater depth than will be covered in the handouts.

Before the first meeting in the Haworth College of Business Macintosh Lab (2270 HCOB), you should do the following:

• Read through this introduction.
• Purchase a double-sided, double-density 3 1/2 inch diskette, and initialize it.
• Start Theorist, type in your full name, and print the notebook.

GETTING STARTED

Turn on the computer by pressing the uppermost right key (with a triangle). If you are on a machine with two slots, you will need a system disk corresponding to your machine number which is in the desk in the lab - insert this disk into either slot. If you are on a machine with one slot, you do not need a system disk.

Initializing a disk

Insert your disk into the vacant slot at the front of the machine. When a disk is entered, the computer checks to see if it has been initialized. If it believes that it has not been properly initialized, a new window will appear asking if you wish to do so. Use the mouse to position the arrow on the button Two Sided. Press the mouse button once. The computer will again ask if you really want to erase and format the disk. Click (position the arrow and press the mouse button) on the button ERASE. A third and final window will appear asking you to name your disk. Type in your full name. Now press return and the computer will format your disk. You should have received a label with your disk which you should use - to avoid confusing your disk with those of your classmates.

WARNING: Not all Macintosh drives are the same. If you use a double-sided, double-density disk as recommended, this will not be an issue. But should the computer ask you to initialize a disk which you believe to be already formatted and containing information, click on CANCEL and ask for help.

Use the mouse to position the arrow on top of the Math&Stat Server icon; then press the mouse button twice in rapid succession. The process of positioning the mouse and double clicking the mouse button is commonly called 'opening'.

The new screen shows the various software available on the server. Open the Theorist folder by positioning the arrow on the folder icon and clicking twice as before. There you will see 13 folders. Each machine has its own copy to use. Note there is only one non-shaded folder.
Introduction to *Theorist*

whose name corresponds to the number of your machine. Open this folder as before. Finally, open the copy of *Theorist* that you find there.

You should see:

![Declarations](image)

![?](image)

*Theorist* automatically opens a new workspace or notebook for you to use. *Theorist* uses a `?` to indicate where information is expected. The flashing cursor indicates where information will be entered. The first line, Declarations, stores information that *Theorist* uses - AVOID DELETING THIS LINE IF POSSIBLE. If you delete this line, *Theorist* will forget how you defined variables and the definitions will start to clutter your notebook.

**Warning: Theorist** may during your work occasionally ask questions of you to define the environment in which you are working. For example, if you type \( y = mx + b \), *Theorist* has built in defaults for assuming \( x \) and \( y \) are variables, but would ask if \( m \) and \( b \) are variables, constants, or functions. When this occurs, you will have to decide on what is appropriate. *Theorist* will ask even for some mathematical symbols such as: \( \tan \), \( \cot \), \( e \). In these cases simply select *Predefined* in the given window.

The square on the second line denotes a line that will hold an equation or expression. Type in your name and press return.

*Theorist* also provides a palette of mathematical symbols. There are two forms of the palette called a variable palette and a function palette. The function palette contains most of what you will be using in terms of symbols. To switch between the two palettes click on the large square in the left portion of the window. Try this a few times. Also try clicking on other objects from both palettes.

**SAVING YOUR NOTEBOOK**

To save a notebook in *Theorist*, use the mouse to position the pointer to the *FILE* Menu. Press and hold the mouse button. While holding the button down, move to the entry *SAVE NOTEBOOK*, and then release the mouse button. A new window will appear. You must tell *Theorist* where to save your notebook. Click on *DRIVE* until your disk shows up. Type in a filename to use - if you do not, a default name of 'Untitled' is used. For notebooks you will be turning in, such as this one, your name is required.

**Warning 2: Theorist** may occasionally halt during your work. It is best to periodically save your notebook. You should save your notebook before printing, after a modest amount of changes, and before trying something new.

**EXITING Theorist AND SUBMITTING A NOTEBOOK**

Select the *QUIT* option from the *FILE* menu. You will be asked if you wish to save any changes to the existing notebook. To see the files on your disk, click the mouse button twice on the icon representing your disk. To submit a notebook electronically, find the folder labeled *KELLER* under the *Theorist* folder. Position the windows so that the notebook and the *KELLER* folder are both visible. Position a window by clicking on the top of the window and dragging it to a new location. A window can be resized by clicking on the lower right hand corner of the window. Click and hold on the icon of the file you wish to move. Drag the pointer over to the *KELLER* folder icon and release. This places a copy in my folder and the computer will tell you
Introduction to *Theorist*

that you will be unable to make future changes. If you wish to submit a revised notebook as a replacement, save it with a similar name, but with a '#2' appended. Earlier notebooks will be discarded.

This completes the portion of the assignment to be turned in. You are to read the remainder of this introduction. However, most students find it beneficial to spend a few minutes more trying things in *Theorist*. Those with little or no experience using the Macintosh and the standard editing features may wish to use the *Tour of the Macintosh* disk available from the lab monitor.

Considerable time, effort, and frustration can be saved by learning how to CUT, COPY, and PASTE. Not only will this help with *Theorist*, but is useful for any program on the Macintosh.

**Entering Expressions**

The first skill you must master is entering expressions. There are two ways of entering information: using the keyboard and using the menus and the palette. Most of you will find a mixture that works best for you. The usual binary operations are most easily entered using the keyboard:

\[ + \text{ for addition} \quad \ast \text{ for multiplication} \quad (x \ast y \text{ for } xy) \quad / \text{ for division} \]

\[ - \text{ for subtraction} \quad \text{OR Space Bar for multiplication} \quad \wedge \text{ for exponentiation} \quad (x^\wedge 2 \text{ for } x^2) \]

Other Important Keys:

<table>
<thead>
<tr>
<th>Key</th>
<th>Function</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>return</code></td>
<td>Ends the current expression or equation and starts a new one.</td>
</tr>
<tr>
<td><code>delete</code></td>
<td>Removes any information to the left of the flashing cursor or any highlighted information.</td>
</tr>
<tr>
<td><code>esc</code></td>
<td>This key is useful to end exponentiation, square roots, etc. and to continue on with the rest of the expression.</td>
</tr>
<tr>
<td><code>enter</code></td>
<td>This key will open a new line for you to record comments or notes. The icon will look like a cartoon dialog box.</td>
</tr>
<tr>
<td><code>option</code></td>
<td>This key is pressed and held while pressing another key or mouse button and is used to produce other (Greek) symbols. For example, option p (the option key together with the 'p' key) produces the symbol for pi, π.</td>
</tr>
<tr>
<td>⌘</td>
<td>This key is often called the 'open-apple' or 'command' key. In the Macintosh world it is usually pressed and held while pressing another key. Notice in most menus, a menu option is followed by a corresponding key sequence. For example, the two keys, ⌘ L, together is the same as choosing the 'Copy' menu option under the 'Edit' menu.</td>
</tr>
</tbody>
</table>

You will make mistakes. To correct typing mistakes, do one of the following:

a) Press the delete key repeatedly until you have erased back the mistake.

b) Highlight the entire region containing the error and press delete.
Introduction to *Theorist*

Try typing each of the following. Press return after each one.

\[ x^4 - 5x^3 + 7 \] Entered as "x ^ 4 ESC - 5 x ^ 3 ESC + 7".

\[ \frac{x^3 + 2}{\sin(x)} \] Enter this as \((x^3+2)/\sin(x)\). Even though *Theorist* places the closing parentheses in for you, you should get in the habit of typing it in.

\[ \sqrt{\frac{x-9}{xy}} \] Enter this as \(\text{sqrt(abs((x-9)/(x*y)))}\) or using the function palette for the square root sign and absolute value. Be careful with the product \(x*y\).

\[ \cos(\frac{\pi}{2} x + 2) \] Enter this as "\(\cos ( \pi / 2 \ ESC x + 2)\). Use the option key together with the letter p to get the symbol \(\pi\). The \(\text{esc}\) key will end the denominator.

---

For those who like to explore or who have experience using a computer programming language such as *BASIC* or *Fortran*, find the PREF menu at the top of the screen, click and hold down on it, pull down to FORTRANISH, and release the key. Now try entering the expressions without using the \(\text{esc}\) key.

---

**Manipulating Expressions**

*Theorist* can reduce the amount of time you spend on solving problems by doing most, if not all, the algebra for you. There is *always* a way of using *Theorist* to perform whatever mathematics you would do by hand. Usually there are many ways. Do not be satisfied with the first method you come across if it does not seem efficient. You must demand of yourself to seek out the best use of *Theorist*.

When *Theorist* performs a manipulation, it produces an indented result or conclusion and distinguishes it by using a triangle icon. Try the following:

- Enter \((x + y)^5\), highlight the expression and choose *Expand* from the *Manipulate* menu.

- Enter \(x^4 - 14x^3 - 140x^2 - 162x + 135\), highlight the expression and choose *Factor* from the *Manipulate* menu.

- Enter \(\frac{1}{7}x + \frac{3}{4}x\), highlight the expression and choose *Simplify* from the *Manipulate* menu. Observe what *Theorist* performed. Now highlight the expression and choose *Calculate*.

Another way of changing an expression is to highlight the expression and then choose an operation from the function palette. For example, enter \(x+3\), highlight and then select \(\sqrt{x}\) from the function palette.

---

By now your notebook may be getting quite messy. Periodically you should delete unwanted lines by clicking on the associated icon and then pressing delete. You may select more than one line by pressing shift after the first item has been selected. Delete will then remove all currently highlighted items.
Introduction to Theorist

Graphing Functions

In this course you will be making extensive use of graphs to analyze and solve problems involving functions. Good graphing skills using a computer requires practice and effort. This section describes some of the tools which will be useful in graphing functions.

For this exercise you will graph the function given by the equation \( y = \sin(x) \). To begin graphing a function, enter the function, \( y = \sin(x) \), and then highlight the function. Move the pointer to the GRAPH Menu. Depress the clicker and hold it down. While holding the clicker down move down to the entry "y = f(x)" and then over to the entry "Linear", and then release the clicker.

**MOVING THE GRAPH RIGHT, LEFT, UP, OR DOWN:** Move the pointer into the middle of the graph (notice how it now looks like a little hand) and depress and hold the clicker down and slide the mouse any direction. Release the clicker. What is the result? Experiment!

**ZOOM OUT** (Rocketing out): To view the graph from twice as far away you find the little rocket icon along the right side of the graph and click once. See what happens. Do it again. Again. Also notice the changes in the numerical markings along the x and y axes.

**ZOOM IN** (Unrocketing out): To reverse the rocketing process, press the option key while clicking on the rocket.

**SELECTING A PORTION OF THE GRAPH** (Cutting): To examine a part of the graph close-up, click once (and release) on the knife icon which is located just above the rocket. Now imagine a part of the screen you want to examine close up. Move the pointer to one corner of this target area you want to view; press and hold down the clicker, and while it is depressed, move the mouse around, observing the changing box which will become your window. When you have the box you want, release the clicker.

---

**Good Graphing Technique:** You should not be satisfied with your graph if an area along the top or bottom of your graphing window is unused, or the left or right portion is similarly unused. For instance, if your function's graph is very wide but not very tall (and so does not make use of the full window), you should immediately cut a new window which likewise is very wide but not very tall. To keep in touch with reality, observe the numerical units along the x and y axes.

**IMPROVING THE RESOLUTION OF THE GRAPH:** If a graph appears jagged, click on the picture of the fine grid underneath the rocket a couple of times. Graphs will then take a little longer to be drawn, but the tradeoff will be finer resolution. For the reverse effect, the "coarser-but-speedier" button is just beneath the "finer" button. There are eight resolution settings.

**DELETING A GRAPH:** In the upper left corner of the graph is a tiny square box. Clicking on this square will highlight the graph. Press the delete key.

**SUPERIMPOSING A SECOND FUNCTION:** Theorist will only graph one function at a time of the form \( y = \). To add the graph of \( y = x^2 \), type in \( z = x^2 \), highlight it, and choose ADD LINE PLOT from the GRAPH menu. While the same function may be graphed several times on various ranges, to get two functions on the same graph, they must be assigned to different variables. You can tell what equations are being used by Theorist by the black dot (•) in the icon of the equation.

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Introduction to *Theorist*

Clean Slate: You should periodically remove unwanted expressions by highlighting the ones you wish to remove (holding the shift key down when selecting the second and subsequent expressions), and then pressing delete. Or you can close the current notebook and open a new one using options from the File menu.

Printing: It is a good idea to highlight and delete all unnecessary equations and graphs from the screen before printing; otherwise the printing process can be too lengthy. Make certain that the printer is turned on, and that the select button on the printer is on. Pull the FILE menu down to the PRINT NOTEBOOK entry. Click "best" for the quality of printing, then click the "OK" button.

A FINAL NOTE ON THE USE OF *Theorist* AND LAB REPORTS

The object of this course is to investigate and explore mathematics, and to use *Theorist* as a tool. As with any tool, misuse of the tool will produce imperfect, often undesirable effects. You should have a reason for every action.

In the laboratory reports you will be asked to respond to questions. Use complete sentences and include the question in your response. For example, to answer *as part of a report* the question:

What is the sum of the first 3 positive integers?

You should respond with:

The sum of the first 3 positive integers is 6 since \(1 + 2 + 3\) equals 6.

All reports can be done using the word processing and text features of *Theorist*. However, Microsoft Word and WordPerfect are available for more polished reports including spelling and grammar checkers.

If you use a word processor, you can use COPY AS PICT from the EDIT menu to move any highlighted object from your notebook to your word processing document. Once a graph is copied into a word processing document it will not change and so is secure from future work in your notebook. Therefore, a single document can contain the solutions to all problems. I am willing to help all those interested in learning the preparation of such reports.

Whenever turning in a notebook include the full names of all group members. Use the enter key to include comments in the form of complete sentences. They should inform the reader as to what is going to occur. For example, the first comment in a notebook should briefly describe the problem. Subsequent comments should describe the process by which the problem is solved. A final comment should state the solution to the problem.
Introduction to Maple

OVERVIEW

This handout is an introduction to the use of Maple. Maple is a computer program that performs algebraic manipulations, such as factoring and solving equations, and produces graphical representations of functions and equations. It is meant to be a tool for doing and learning mathematics. Maple will perform most of the calculations that you have learned before calculus and most of the calculations you will use in calculus. Part of your responsibility is to ask questions, pose problems, and seek limitations. This includes knowing how and when to use Maple, how to judge the correctness of the information that Maple presents to you, and how to approach new and different problems. Much of the information below is only the tip of the iceberg. It is assumed that you will explore the software to a greater depth than will be covered in the handouts.

Before the first meeting in the Haworth College of Business Macintosh Lab (2270 HCOB), you should do the following:

• Read through this introduction.
• Purchase a double-sided, double-density 3 1/2 inch diskette, and initialize it.
• Start Maple, type in your hill name, and print the worksheet.

GETTING STARTED

Turn on the computer by pressing the uppermost right key (with a triangle). If you are on a machine with two slots, you will need a system disk corresponding to your machine number which is in the desk in the lab - insert this disk into either slot. If you are on a machine with one slot, you do not need a system disk.

Initializing a disk

Insert your disk into the vacant slot at the front of the machine. When a disk is entered, the computer checks to see if it has been initialized. If it believes that it has not been properly initialized, a new window will appear asking if you wish to do so. Use the mouse to position the arrow on the button Two Sided. Press the mouse button once. The computer will again ask if you really want to erase and format the disk. Click (position the arrow and press the mouse button) on the button ERASE. A third and final window will appear asking you to name your disk. Type in your full name. Now press return and the computer will format your disk. You should have received a label with your disk which you should use - to avoid confusing your disk with those of your classmates.

WARNING: Not all Macintosh drives are the same. If you use a double-sided, double-density disk as recommended, this will not be an issue. But should the computer ask you to initialize a disk which you believe to be already formatted and containing information, click on CANCEL and ask for help.

Use the mouse to position the arrow on top of the Math&Stat Server icon; then press the mouse button twice in rapid succession. The process of positioning the mouse and double clicking the mouse button is commonly called 'opening'.

The new screen shows the various software available on the server. Open the Maple folder by positioning the arrow on the folder icon and clicking twice as before. Finally, open the file called Maple that you find there.
Introduction to Maple

You should see:

Maple automatically opens a new worksheet for you to use. The top of the window deals with word processing features of Maple. These features are similar to those found in Microsoft Word and WordPerfect and can be used to format your notebook. Maple distinguishes between three types of information: input, output, and text. The input for Maple is usually in the form of a command such as expand, solve, or factor, and is usually preceded by a bullet (*). The output or result of a command is usually centered. Finally text or comments can be inserted where needed to describe the problem and solution methods. You can switch between input and text mode by choosing Insert Prompt and Enter Text respectively on the FORMAT menu. Include the names of your group members at the top of any worksheet turned in. Appropriate comments written in proper English sentences are required on your work to be turned in.

SAVING YOUR WORKSHEET

To save a worksheet in Maple, use the mouse to position the pointer to the FILE Menu. Press and hold the mouse button. While holding the button down, move to the entry SAVE, and then release the mouse button. A new window will appear. You must tell Maple where to save your worksheet. Click on DRIVE until your disk shows up. Type in a filename to use - if you do not, a default name of 'Untitled-1' is used. For worksheets you will be turning in, such as this one, your name is required.

Warning: Maple may occasionally halt during your work. It is best to periodically save your worksheet. You should save worksheet before printing, after a modest amount of changes, and before trying something new.

EXITING Maple AND SUBMITTING A WORKSHEET

Select the QUIT option from the FILE menu. You will be asked if you wish to save any changes to the existing worksheet. To see the files on your disk, click the mouse button twice on the icon representing your disk. To submit a worksheet electronically, find the folder labeled KELLER under the Maple folder. Position the windows so that the worksheet and the KELLER folder are both visible. Position a window by clicking on the top of the window and dragging it to a new location. A window can be resized by clicking on the lower right hand corner of the window. Click and hold on the icon of the file you wish to move. Drag the pointer over to the KELLER folder icon and release. This places a copy in my folder and the computer will tell you that you will be unable to make future changes. This is just a warning and is normal. If you wish to submit a revised worksheet as a replacement, save it with a similar name, but with a '#2' appended. Earlier worksheets will be discarded.

This completes the portion of the assignment to be turned in. You are to read the remainder of this introduction. However, most students find it beneficial to spend a few minutes more trying things in Maple. Those with little or no experience using the Macintosh and the standard editing features may wish to use the Tour of the Macintosh disk available from the lab monitor. Considerable time, effort, and frustration can be saved by learning how to CUT, COPY, and PASTE. Not only will this help with Maple, but is useful for any program on the Macintosh.
Introduction to Maple

Entering Expressions

The first skill you must master is entering expressions. Maple expects expressions and equations to be entered in a fashion similar to many computer languages such as BASIC or Fortran. A condensed list of information regarding the use of Maple can be found at the end of this introduction. Below is a list of common things to remember when working in Maple.

• An asterisk (*) is required between all products such as $7*x$ instead of $7x$.
• All input must end with a semicolon or colon.
• No output will be printed if a colon is used.
• When entering a command into Maple, a prompt of a bullet (•) should be present and the Input box on the ruler should be darkened. If this is not the case, then use Insert Prompt from the FORMAT menu.

You will make mistakes. To correct typing mistakes, do one of the following:

a) Press the delete key repeatedly until you have erased back to the mistake.
b) Highlight the entire region containing the error and press delete.

As you advance in your knowledge of the Macintosh, features such as CUT, COPY, and PASTE will increase your productivity immeasurably! Below are examples of entering some expressions. Try entering these into Maple, remembering to press enter after each one.

$x^4 - 5x^3 + 7$ is entered as: $x^4 - 5*x^3 + 7$; $x^3 + 2/\sin(x)$ is entered as: $(x^3+2)/\sin(x)$;

$\sqrt{\frac{x-9}{xy}}$ is entered as: sqrt(abs((x-9)/(x*y)));

$\cos\left(\frac{\pi}{2}x + 2\right)$ is entered as: $\cos(\pi/2\times x + 2)$;

Manipulating Expressions

Maple can reduce the amount of time you spend on solving problems by doing most, if not all, the algebra for you. A list of the more commonly used commands can be found at the end of this introduction. There is always a way of using Maple to perform whatever mathematics you would do by hand. Usually there are many ways. Do not be satisfied with the first method you come across if it does not seem efficient. You must demand of yourself to seek out the best use of Maple.

When Maple performs a manipulation, it produces an indented result. Try the following:

• expand((x + y)^5); [Don't forget to use the enter key.]
• factor(x^4-27*x^3+125*x^2+51*x-630);
• solve("=0, x");
• 1/7*x+1/4*x;
• evalf("); [Note what Maple does with this!]
Introduction to Maple

Naming Expressions and Defining Functions

Any expression which you expect to use frequently should be given a name. Sometimes the name is naturally suggestive. For example, suppose you are working with the parabola \( y = 2x^2 - 8 \). The following commands are typical.

- \( y := 2x^2 - 8; \) Assigns the name \( y \) to the expression \( 2x^2 - 8 \).
- \( \text{subs}(x=1, y); \) Computes the value of \( y \) at \( x = 1 \).
- \( \text{solve}(y=0, x); \) Finds the solutions to the equation \( 2x^2 - 8 = 0 \).
- \( \text{factor}(y); \) Factors the expression \( 2x^2 - 8 \) into \( 2(x - 2)(x + 2) \).

The use of "\( := \)" allows you to save typing. Maple can work with equations. For example the command "\( \text{solve}((2x - y = 2, 7x - 6y = -3)); \)" will solve the system of two equations in two unknowns. To unassign or clear the meaning of any name, use a command similar to "\( y := 'y'; \)" which clears any meaning previously given to the variable \( y \).

In Maple, functions are entered quite differently. To define the function \( f \) given by \( f(x) = 2x^2 - 8 \), use the command "\( f := \text{proc}(x) 2x^2 - 8 \text{ end;} \). Notice this assigns the name \( f \) to the right hand side. The first part "\( \text{proc}(x) \)" specifies that a procedure which depends upon \( x \) is being defined. The procedure is terminated by the end statement. Now the function can be used as you would normally. The expression "\( f(3) \)" is equivalent to the \( 2*3^2 - 8 \). The expression "\( f(a + b) \)" is equivalent to \( 2*(a + b)^2 - 8 \).

Each of the following statements has a distinct meaning in Maple. Part of your job is to determine when each should be used.

\[
y = 6x - 7; \quad y := 6x - 7; \quad y := \text{proc}(x) 6x - 7 \text{ end;}
\]

Graphing

In this course you will be making extensive use of graphs to analyze and solve problems involving functions. Good graphing skills using a computer requires practice and effort. However, it is simple to produce an initial graph in Maple. For example to graph the sine function, enter the command "\( \text{plot} ( \sin (x)); \)". Maple has a default domain for \( x \) of \([-10, 10]\) and chooses a range for the 'y-axis' so that all values are shown. However, sometimes you will wish to specify a domain of the function. For example, the sine function has period \( 2\pi \). To graph it on the interval \([0, 2\pi]\), use the command "\( \text{plot} (\sin(x), x = 0..2*\pi); \)". \( y \) will wish to specify the range for the y-axis also. For example, the command "\( \text{plot}(\sin(x))^2 \)" will produce a graph with \( x \) values ranging from \([-10\) to 10] and \( y \) values ranging from \([-1\) to 1]. To produce a graph in true proportions, use a command similar to "\( \text{plot}\left(\sin(x), x =-5..5, y =-5..5\right)\);" which would produce a graph with both \( x \) and \( y \) values ranging from \([-5\) to 5].

**Good Graphing Technique:** You should not be satisfied with your graph if an area along the top or bottom of your graphing window is unused, or the left or right portion is similarly unused. For instance, if your function's graph is very wide but not very tall (and so does not make use of the full window), you should immediately choose more appropriate ranges. To keep in touch with reality, observe the numerical units along the \( x \) and \( y \) axes.
Introduction to Maple

Clean Slate: You should periodically remove unwanted expressions by highlighting the ones you wish to remove and then pressing delete. Or you can exit Maple and start fresh.

Printing: It is a good idea to highlight and delete all unnecessary equations and graphs from the screen before printing; otherwise the printing process can be too lengthy. Make certain that the printer is turned on, and that the select button on the printer is on. Pull the FILE menu down to the PRINT WORKSHEET entry. Click "best" for the quality of printing, and then click the "OK" button.

Getting help: Maple has extensive information about the commands. There are two ways of getting help on a question.

- Choose Help on the WINDOWS menu, or
- Use the help command such as "help(solve);".

A FINAL NOTE ON THE USE OF Maple AND LAB REPORTS

The object of this course is to investigate-and explore mathematics, and to use Maple as a tool. As with any tool, misuse of the tool will produce imperfect, often undesirable effects. You should have a reason for every action.

In the laboratory reports you will be asked to respond to questions. Use complete sentences and include the question in your response. For example, to answer as part of a report the question:

What is the sum of the first 3 positive integers?

You should respond with:

The sum of the first 3 positive integers is 6 since 1 + 2 + 3 equals 6.

All reports can be done using the word processing and text features of Maple. However, Microsoft Word and WordPerfect are available for more polished reports including spelling and grammar checkers.

After producing a graph to be included either in your worksheet or in a word processing document, use COPY from the EDIT menu. Use PASTE from the EDIT menu to include the picture at the current location of your worksheet. Once a graph is copied, it will not change and so is secure from future work in your worksheet. Therefore, a single document can contain the solutions to all problems. I am willing to help all those interested in learning the preparation of such reports.

Whenever turning in a worksheet include the full names of all group members. Use the text feature of Maple to include comments in the form of complete sentences. They should inform the reader as to what is going to occur. For example, the first comment in a worksheet should briefly describe the problem. Subsequent comments should describe the process by which the problem is solved. A final comment should state the solution to the problem.
Introduction to Maple

Maple Information

Unless otherwise indicated, all commands are in lower case. These two pages contain almost all the relevant information for how to enter input commands in Maple.

**enter**

This key is used to tell Maple to process an input line and can be pressed when the cursor is at any location on the line to be executed. REMEMBER: Input lines must end in a semicolon (;) or colon (:). The enter key has no effect on output or text lines.

**return**

This key is used when you are near to the right margin of a line and wish to continue typing. Input commands may span multiple lines since Maple looks for the semicolon or colon to terminate the input.

**delete**

Removes any information to the left of the flashing cursor or any highlighted information.

+, -, *, /

Addition, subtraction, multiplication, and division respectively.

^, ^2

Exponentiation: The expression $x^2$ is entered as $x^2$.

⌘ &

This key is often called the 'open-apple' or 'command' key. In the Macintosh world it is usually pressed and held while pressing another key. Notice in most menus, a menu option is followed by a corresponding key sequence. For example, the two keys, ⌘ C, together is the same as choosing the 'Copy' menu option under the 'Edit' menu.

; or :

A semicolon or colon must follow every command. If a colon is used, the output of the command is suppressed.

" "

The double quote (single keystroke) refers to the previous display or output.

""

Two double quotes refer to the next to the last display.

Pi, E, I, infinity

Maple denotes the common mathematical symbols π, e = 2.71828..., $i = \sqrt{-1}$, and $\infty$ by Pi, E, I, and infinity respectively. Note the use of uppercase letters.

The following mathematical functions are recognized by Maple:

<table>
<thead>
<tr>
<th>Function</th>
<th>Argument</th>
</tr>
</thead>
<tbody>
<tr>
<td>abs(x)</td>
<td>x</td>
</tr>
<tr>
<td>sqrt(x)</td>
<td>x</td>
</tr>
<tr>
<td>exp(x)</td>
<td>$e^x$</td>
</tr>
<tr>
<td>ln(x)</td>
<td></td>
</tr>
<tr>
<td>sin(x)</td>
<td></td>
</tr>
<tr>
<td>cos(x)</td>
<td></td>
</tr>
<tr>
<td>tan(x)</td>
<td></td>
</tr>
<tr>
<td>cot(x)</td>
<td></td>
</tr>
<tr>
<td>sec(x)</td>
<td></td>
</tr>
<tr>
<td>csc(x)</td>
<td></td>
</tr>
<tr>
<td>arcsin(x)</td>
<td>x</td>
</tr>
<tr>
<td>arccos(x)</td>
<td>x</td>
</tr>
<tr>
<td>arctan(x)</td>
<td>x</td>
</tr>
<tr>
<td>arccot(x)</td>
<td>x</td>
</tr>
<tr>
<td>arccsc(x)</td>
<td>x</td>
</tr>
<tr>
<td>Command</td>
<td>Explanation</td>
</tr>
<tr>
<td>---------</td>
<td>-------------</td>
</tr>
<tr>
<td>help(command);</td>
<td>Accesses the extensive <em>Maple</em> on-line help on the given command.</td>
</tr>
<tr>
<td>quit</td>
<td>Equivalent to choosing <em>Quit</em> off the <em>FILE</em> menu.</td>
</tr>
<tr>
<td>y := expression;</td>
<td>Assigns the name <em>y</em> to the given expression.</td>
</tr>
<tr>
<td>y := 'y';</td>
<td>Unassigns <em>y</em>. Returns <em>y</em> to a variable.</td>
</tr>
<tr>
<td>f := proc(x) expr end;</td>
<td>Defines <em>f</em> to be the function of <em>x</em> given by the expression.</td>
</tr>
<tr>
<td>f := &lt;expr l x&gt;</td>
<td>A shorthand version for defining a function as a single expression.</td>
</tr>
<tr>
<td>expand(expression);</td>
<td>Expands the expression.</td>
</tr>
<tr>
<td>factor(expression);</td>
<td>Factors the expression.</td>
</tr>
<tr>
<td>evalf(expression);</td>
<td>Evaluates the expression to a number.</td>
</tr>
<tr>
<td>evalf(&quot;);</td>
<td>Evaluates the last result as a number.</td>
</tr>
<tr>
<td>plot(expression);</td>
<td>Plots the expression on a default domain [-10, 10].</td>
</tr>
<tr>
<td>plot(expr, x=a..b);</td>
<td>Plots the expression of <em>x</em> on the domain [a, b].</td>
</tr>
<tr>
<td>plot({y1,y2}, x=a..b);</td>
<td>Plots the two expressions <em>y1</em> and <em>y2</em> on the domain [a, b].</td>
</tr>
<tr>
<td>simplify(expression);</td>
<td>Simplifies any expression.</td>
</tr>
<tr>
<td>subs(x=b, expression);</td>
<td>Substitutes the value <em>b</em> for the specified variable <em>x</em> in the expression.</td>
</tr>
<tr>
<td>solve(equation,variable);</td>
<td>Solves the given equation for the specified variable in exact form.</td>
</tr>
<tr>
<td>fsolve(eq, var);</td>
<td>Solves the given equation for the specified variable numerically.</td>
</tr>
<tr>
<td>Caution: <em>Maple</em> sometimes only presents one of the solutions.</td>
<td></td>
</tr>
<tr>
<td>solve({eq1,eq2), {x,y});</td>
<td>Solves the equations for the given variables.</td>
</tr>
<tr>
<td>sum(expression, j=a..b);</td>
<td>Sums the expression over the index values of <em>j</em> = <em>a</em> to <em>j</em> = <em>b</em>.</td>
</tr>
<tr>
<td>limit(expression, x=a);</td>
<td>Computes the two-sided limit of the expression at <em>x</em> = <em>a</em>.</td>
</tr>
<tr>
<td>limit(expr, x=a, dir);</td>
<td>Computes the left- or right-handed limit of the expression at <em>x</em> = <em>a</em>.</td>
</tr>
<tr>
<td>diff(expression, x);</td>
<td>Differentiates the expression with respect to the variable <em>x</em>.</td>
</tr>
<tr>
<td>int(expression, x);</td>
<td>Integrates the expression with respect to the variable <em>x</em>.</td>
</tr>
<tr>
<td>int(expression, x=a..b);</td>
<td>Computes the definite integral of the expression on given range.</td>
</tr>
<tr>
<td>collect(expression, x);</td>
<td>Collects terms in the given expressions according to powers of <em>x</em>.</td>
</tr>
<tr>
<td>denom(expression);</td>
<td>Returns the denominator of the rational expression.</td>
</tr>
<tr>
<td>numer(expression);</td>
<td>Returns the numerator of the rational expression.</td>
</tr>
<tr>
<td>normal(expression);</td>
<td>Simplifies a rational expression.</td>
</tr>
<tr>
<td>simplify(expr, trig);</td>
<td>Simplifies any trigonometric expression.</td>
</tr>
<tr>
<td>simplify(expr, exp);</td>
<td>Simplifies any exponential expression.</td>
</tr>
<tr>
<td>simplify(expr, power);</td>
<td>Simplifies powers, exponentials and logarithms in the expression.</td>
</tr>
<tr>
<td>with(student);</td>
<td>Loads in optional student packages with procedures for calculus.</td>
</tr>
<tr>
<td>************</td>
<td>Useful to clear <em>Maple</em>’s parsing after syntax errors.</td>
</tr>
</tbody>
</table>

**Example for solving an equation:**

```maple
solve(x^2+5*x+7=0,x); Solves the quadratic equation exactly.
s := "; Assigns the name *s* to the list of roots found.
evalf(s[1]); Approximates the first root.
evalf(s[2]); Approximates the second root.
```
Laboratory One: Basic Skills & Review of Precalculus

Names 1. ________________  
2. ________________  
3. ________________

Directions: Be sure to attempt to answer all questions. Some questions are open-ended and there is no single right answer. Read the entire question to know what you'll need to print - to save pages. Be sure to clean up your notebook before printing and to select "best" after choosing PRINT NOTEBOOK from the FILE menu. A labeled graph is a graph with the algebraic form of the function as the title as well as the axes being labeled.

Note: Use ' ^ ' to get exponents and press the esc key to end entering the exponent OR turn on the FORTRANISH option under the PREF menu.

1. Use Theorist to expand \((x - 1)^7\) by entering and highlighting the expressions and then choosing EXPAND from the MANIPULATE menu.

2. Find all real roots of the polynomial \(x^3 - 2x^2 - 123x + 540\) with an accuracy of at least 0.01 by the following methods.

   (a) **Graphically:** Enter the function \(y = x^3 - 2x^2 - 123x + 540\). Graph the function by highlighting it (click on the equation's icon or on the '='), and click and hold on the GRAPH menu; pull down to 'y=f(x)' and then over to Linear.

      Click on the rocket icon to zoom out until you can determine roughly the interval where the roots could lie. Use rocketing out and cutting to determine the roots. How do you know when you have a range where all roots must lie?

      Attach and label graph(s) which you feel show the behavior of this function including all the roots. These graphs DO NOT have to show the root to the desired accuracy.

      Use the knife to select a portion of the graph where you suspect a root to exist. Find all roots to within 0.01 and describe why you know they are accurate to within 0.01.

   (b) **Symbolically:** Highlight the right hand side of your equation by positioning the arrow to just before the first term of the polynomial as follows: Press and hold the mouse button and drag the arrow across the polynomial and then release the mouse button. Choose the Factor command from the Manipulate menu. Interpret the result.
Laboratory One: Basic Skills & Review of Precalculus

3. Find all real roots of the polynomial \( x^6 + 6x^5 - 16x^4 - x^2 - 6x + 17 \) with an accuracy of at least 0.01 by the graphical method on page 1. Recall that Theorist can only have one function named \( y \). Either start a new notebook or to continue in the same notebook, use \( w = \ldots \)

How do you know when you have located all the roots?

Attach and label graphs which you feel show the behavior of this function. You will want more than one for this function. These graphs DO NOT have to show the roots accurately.

4. Find all real solutions to the equation \( \frac{(20-x)^3}{8} = x \) with an accuracy of at least 0.01 by the following methods.

(a) Graphically: Graph the function \( y = \frac{(20-x)^3}{8} \) as before or highlight the equation and press the keys \( \odot \) and G simultaneously. Enter a new function \( z = x \). Highlight the graph (by clicking in the upper left-hand square of the graph) and the equation by holding shift while clicking on the second function. Choose ADD LINE PLOT from the GRAPH menu. Use the rocket and knife as before to determine the coordinates of the intersection points to within 0.01.

Is there a way of determining the solutions graphically by using a single function? If so, explain how. If not, explain why not.

(b) Symbolically: Enter the original equation \( \frac{(20-x)^3}{8} = x \). Select (highlight) the numerator of the left hand side. This is tricky but be patient. Choose EXPAND from the MANIPULATE menu. Highlight the right hand side of the equation. While holding down the \( \odot \) key position the arrow to the right hand side. The arrow should change into a 'hand'. Click and hold the mouse button and drag the hand to the left hand side and release the mouse button. What does this process do?

Solve the resulting expression symbolically as before by factoring.

Highlight portions of an expression. Experiment with dragging the hand even further to the left (all the way to the icon) and release. What is happening?
Laboratory One: Basic Skills & Review of Precalculus

5. Solve the equation \( y = 1.5^a \sin(ax + b) \) for \( a, b, \) and \( c \) by either highlighting and then choosing \textit{ISOLATE} from the \textit{MANIPULATE} menu or by highlighting and dragging as above. Note: You will be asked to define \( b \) and \( c \) by \textit{Theorist}. Are they constants or variables? You decide. It has a default for \( x, y, \) and \( a \).

6. Graph \( y = 2\cos\left(\frac{x}{7}\right) + 2\cos(x) \) and find its domain and range. Experiment with different viewing windows to find the maximum and minimum \( y \)'s. Use the \textit{rocket} and \textit{knife} to improve your picture. If you have too many squiggles for the computer to handle correctly, cut down the size of your window. If your graph is jagged, also try increasing the resolution by clicking on the icon with denser gridlines. Attach a labeled graph and on the \( y \)-axis, shade in the occurring \( y \)-values in the range \textit{by-hand}.

\[
\text{Domain: } \quad \text{Range: }
\]

7. Graph \( y = \sqrt{x^3 - x} \). This function is entered as sqrt(x^3-3) or using the function palette. Determine the domain and range.

\[
\text{Domain: } \quad \text{Range: }
\]

The domain is unusual. Explain why the domain is in two parts. It helps to graph what is inside the square root: \( z = x^3 - x \).

8. Graph the function \( y = \frac{x^3 - 1}{x - 1} \). There is something wrong with this graph. What is it?
Laboratory One: Basic Skills & Review of Precalculus

9. In this course we will be using piece-wise defined functions. Enter the piece-wise function
\[ y = \begin{cases} 
  x^2 + 3 & (x < 0) \\
  \sqrt{x^2 + 1} & (x \geq 0) 
\end{cases} \]
by entering 'y=' and then on the function palette click once on the left curled bracket. Recall the variable and function palettes can be switched by clicking on the square at the left. Using the mouse, position and click on locations where information is missing. Fill in the completed information. Attach a labeled graph of this function showing its behavior. What is wrong with this graph?

10. After you have finished question 9, modify the function to be
\[ y = \begin{cases} 
  x^2 + 3 & (x < 0) \\
  \sqrt{x^2 + 1} & (x < 3) \\
  6 - x & (x \geq 3) 
\end{cases} \]
by positioning and clicking at the right hand side (at the end of the equation). You should see a flashing vertical line. On the function palette click and hold on the square with dashes in it (next to the piece-wise symbol), and move to the symbol with three bold horizontal black dots at the top of the column to the right and release. Modify the function as before. Attach a labeled graph of this function.

11. Theorist has a special format for defining functions and using functional notation. Reproduce the following three lines in Theorist. Explain what you think each line accomplishes. When Theorist prompts you to declare \( f \), be sure to choose that it is a user defined function.

\[ \Box \begin{align*} 
  f(a) &= b^2 \\
  m &= \frac{f(4) - f(1)}{4 - 1} \\
  y &= m(x - 1) + f(1) 
\end{align*} \]

12. Find a real number root accurate to within 0.001 between 1.1 and 2.2 for \( 3\cos(\cos(x)) = 2x \).

13. Explain your solution method to the following problem in sufficient detail so that it could be understood by a classmate.

The process of triangulation via a broadcast signal used by engineers and security agents is equivalent to finding a point \((h,k)\) which is equidistant from three given points. Find a point equidistant from the three points \((1,5)\), \((2,1)\), and \((3,3)\) by the following steps.

What is the distance between \((h,k)\) and \((1,5)\)?
Laboratory One: Basic Skills & Review of Precalculus

What is the distance between \((h,k)\) and \((2,1)\)? _________________________

Equating these two distances yields one equation in two unknowns. Repeat this process for the points \((1,5)\) and \((3,3)\) to get a second equation in two unknowns.

You can (and should) equate the square of the distance to avoid using square roots. Why is this?

_Theorist_ can be used to solve this system of equations symbolically. First _EXPAND_ each equation (involving the distance squared). Move the right hand side of the equation to the left hand side as before. The result should be two linear equations. Solve one linear equation for either \(h\) or \(k\) by highlighting and using the _ISOLATE_ command or by moving it to the far left or right as noted earlier.

Now to substitute this new equation into the other equation, highlight just the equation (not the icon) and the icon of the second equation (by holding the _shift_ key while clicking on the icon). Then choose _SUBSTITUTE_ from the _MANIPULATE_ menu. Continue this process of _EXPANDING, SIMPLIFYING, and SUBSTITUTING_ to determine the solution to these two equations.

What is the desired point?

What is the distance from this point to any of the three points?

The process of triangulation is also equivalent to finding the center of a circle passing through three points. Using your information above, graph a circle using _Theorist_ which passes through the three given points.

Keep a printout of the entire (cleaned up) notebook including the graph with the three points indicated by hand for your records. Include comment lines by pressing _enter_. At the top of the notebook include a description of the problem and the names of the investigators.

Challenge: Modify your solution method to find a point \((h,k)\) which is equidistant from three points \((a,b), (c,d),\) and \((e,f)\).

14. Write a short paragraph describing your experience using _Theorist_. Be sure to indicate anything about _Theorist_ you discovered and any frustrations. Also indicate how you feel about the usefulness of _Theorist_. Be honest; there is only one wrong answer - no answer.
ADVANCED THEORIST

For those of you who wish to explore further on your own or by reading the manual, investigate the following areas:

*FIND ROOT* command in the MANIPULATE menu: This command can be beneficial to determining roots of polynomials and functions. To use it you must graph the function and zoom in to a window with only one crossing of the x-axis. Choose *FIND ROOT*. It will locate the root as accurately as possible (up to 19 digits).

*ANIMATE* command in the GRAPH menu. Enter the two equations \(a = 1\) and \(y = x^2 + ax\). Graph the second equation. Highlight the constant 'a' in the second equation, and then choose *ANIMATE*.

More suggestions will follow with each lab. Use the remaining space to record any notes on the use of Theorist.
Laboratory One: Basic Skills & Review of Precalculus

Laboratory One
Basic Skills & Review of
Precalculus

Names
1.____________________
2.____________________
3.____________________

Directions: Be sure to attempt to answer all questions. Some questions are open-ended and there is no single right answer. Read the entire question so you can plan ahead. You will be turning in your Maple session. Use Maple's text feature to include appropriate comments in your notebook. The separator lines can be turned off or removed at the end to speed up printing. Be sure to clean up your notebook before printing and to select "best" after choosing PRINT from the FILE menu.

Close all graphs (plots) before continuing with additional Maple commands by clicking in the small square in the upper left of the plot window. If you do not, you may run out of memory. You can switch between plots and the current worksheet by using the WINDOWS menu.

Remember to use '^' for exponentiation, to end all commands with a semicolon, and to press the enter key to execute input commands. Refer back to the introduction for more information.

1. (a) Use the expand command to expand \((x - 1)^7\).

(b) Use the subs command to evaluate the result at \(x = \frac{3}{4}\).

2. Find all real roots of the polynomial \(x^3 - 2x^2 - 123x + 540\) with an accuracy of at least 0.01 by the following methods. Recall a root of a polynomial is a value for \(x\) for which the value of the polynomial is zero.

(a) Graphically: Assign the polynomial to the variable \(y\). Use the plot command to plot \(y\) on the default viewing range. Modify the viewing range as needed so that the global behavior of the function is evident. Note intervals where possible roots may lie.

How do you know when you have a range where all roots must lie?

Use additional plot commands to restrict the viewing range until the roots are known to within 0.01. Retain in your worksheet the plot commands showing the global behavior and the ones which show the root to within 0.01.

(b) Symbolically: There are three distinct commands which can be given to find roots of polynomials. The command factor will try to factor the polynomial into rational roots. The command solve will try to determine exact solutions. The command fsolve will try to determine approximate solutions through numeric methods. The success of all three commands depends upon the function, and each should be verified with graphical evidence. Try each of the these commands noting the difference in each.

(c) With \(y\) still assigned as above, try each of the commands factor, solve, and fsolve on \(y+1 = 0\).
3. Find all real roots of the polynomial $x^6 + 6x^5 - 16x^4 - x^2 - 6x + 17$ with an accuracy of at least 0.01 by the graphical method on page 1.

How do you know when you have located all the roots?

4. Find all real solutions to the equation $\frac{(20 - x)^3}{8} = x$ with an accuracy of at least 0.1 by the following methods.

(a) Graphically: Assign the left and right hand side of the equation to two variables. Plot both expressions at the same time. Adjust the viewing range so that the behavior of both functions is shown. Use additional plot commands to determine where the two graphs intersect.

Is there a way of determining the solutions graphically by using a single function? If so, explain how. If not, explain why not.

(b) Symbolically: Use the `fsolve` command on the original equation. You need not retype the expressions, simply refer to them by name.

Use the `solve` command to determine the solutions, and then the `evalf` command to determine the decimal equivalents.

5. The `solve` command does more than just give exact numeric solutions. Use the following steps to rewrite the equation $y = 1.5c \sin(ax + b)$ for $a$, $b$, and $c$.

- $y := 'y';;$ [Clears any value given to $y$]
- $\text{Eq} := y = 1.5^c \sin(a \times x + b);$ 
- $\text{solve(Eq, a)};$
- $\text{solve(Eq, b)};$
- $\text{solve(Eq, c)};$

6. Graph $y = 2\cos(\frac{x}{2}) + 2\cos(x)$ and find its domain and range. To do so, it will help to experiment with different viewing windows to find the maximum and minimum $y$'s. If you have too many squiggles for the computer to handle correctly, cut down the size of your window.

Domain: _______________________ Range: _________________________
Laboratory One: Basic Skills & Review of Precalculus

7. Graph \( y = \sqrt{x^2 - x} \) and determine the domain and range of this function.

   Domain: __________________________  Range: __________________________

   The domain is unusual. Explain why the domain is in two parts.

8. Graph the function \( y = \frac{x^3 - 1}{x - 1} \). There is something wrong with this graph. What is it?

9. Instead of using the substitute command, you can define a function using Maple procedures. This will be helpful when you need to evaluate a function many times. Explain what each of the following lines of Maple commands accomplishes.

   • \( f := \text{proc}(x) \ x^2 \ \text{end;} \)
   • \( f(1); \)
   • \( f(4); \)
   • \( m := (f(4) - f(1)) / (4 - 1); \)
   • \( y := m*(x - 1) + f(1); \)
   • \( \text{plot}(f, y, x = 0..5); \)

10. In this course you will be using piece-wise defined functions. To enter the piece-wise function

\[
    f(x) = \begin{cases} 
        x^2 + 3 & (x < 0) \\
        \sqrt{x^3 + 1} & (x \geq 0) 
    \end{cases}
\]

Use the following Maple command:

   • \( f := \text{proc}(x) \)
     if \( x < 0 \) then \( x^2 + 3 \)
     else \( \sqrt{x^3 + 1} \)
     fi
   end;
   • \( \text{plot}(f, x = -3..3); \)

   What is wrong with this graph?

11. After you have finished question 10, modify the function to be \( f(x) = \begin{cases} 
        x^2 + 3 & (x < 0) \\
        \sqrt{x^3 + 1} & (x \leq 3) \\
        6 - x & (x > 3) 
    \end{cases} \) by changing the line 'else \( \sqrt{x^3 + 1} \)' to the two lines:

   • \( \text{elif } x <= 3 \text{ then } \sqrt{x^3 + 1} \)  ['elif' can be thought of as else if]
   • \( \text{else } 6 - x \)

Plot this function on the interval \([-5, 5]\). Include this graph in your worksheet by using the COPY and PASTE options from the EDIT menu.
12. Find a real number root accurate to within 0.001 between 1.1 and 2.2 for \(3\cos(\cos(x)) = 2x\).

13. Explain your solution method to the following problem in sufficient detail so that it could be understood by a classmate.

The process of triangulation via a broadcast signal used by engineers and security agents is equivalent to finding a point \((h, k)\) which is equidistant from three given points. Find a point equidistant from the three points \((1,5), (2,1),\) and \((3,3)\) by the following steps.

What is the distance between \((h, k)\) and \((1,5)\)?

What is the distance between \((h, k)\) and \((2,1)\)?

Equating these two distances yields one equation in two unknowns. Repeat this process for the points \((1,5)\) and \((3,3)\) to get a second equation in two unknowns.

You can (and should) equate the square of the distance to avoid using square roots. Why is this?

Use Maple to solve this system of equations using a command similar to:

\[
\text{solve((Equation1, Equation2),\{h, k\});}
\]

What is the desired point?

What is the distance from this point to any of the three points?

The process of triangulation is also equivalent to finding the center of a circle passing through three points. Using your information, graph a circle using Maple which passes through the three given points. Copy the graph into your worksheet.

Challenge: Modify your solution method to find a point \((f, t, k)\) which is equidistant from three points \((a, b), (c, d),\) and \((e, f)\).

14. Using the text feature of Maple, write a short paragraph describing your experience using Maple. Be sure to indicate anything about Maple you discovered and any frustrations. Also indicate how you feel about the usefulness of Maple. Include an estimate of the amount of time required to complete this lab. Be honest; there is only one wrong answer - no answer.
Laboratory Two: Local Behavior of Functions

Laboratory Two Names

1. _____________________
2. _____________________
3. _____________________

You will be asked to turn in selected portions of this LAB.

NOTE: You should not expect anything saved on the LAB disks to remain there. They will be periodically cleaned out. ALWAYS SAVE TO YOUR OWN DISK. If uncertain on how to do so, ASK FOR HELP. As a last resort try saving by using the SAVE NOTEBOOK command from the FILE menu.

PART ONE: RATE OF CHANGE - DISTANCE VERSUS TIME

You should record your answers to Part One on a separate piece of paper.

Commuters are often interested in how fast they must travel to get from one place to another, knowing the amount of time available to complete the journey, and the distance to be traveled. This interest increases if the time is shorter than usual, and several police officers are stationed along the route. This part of the investigation explores the relationship between distance and time on large, then smaller intervals of time. In particular, the commuter's point of view of velocity will be compared with that of an officer of the law.

COMMUTER'S POINT OF VIEW

1. How would a commuter determine average velocity for the particular commute given below?

<table>
<thead>
<tr>
<th>Time</th>
<th>Odometer (miles)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial Point</td>
<td>2:30 pm</td>
</tr>
<tr>
<td>Final Point</td>
<td>3:12 pm</td>
</tr>
<tr>
<td>62,137.3</td>
<td>62,172.4</td>
</tr>
</tbody>
</table>

What is the average velocity of the commuter over the time interval given?

2. Sketch a possible graph of the distance versus time function: y = f(t) for the commuter's travel using initial and final times and distances given in the table above. Keep in mind that the commuter must stop at traffic lights, stop signs, etc. and cannot travel at a constant speed from the moment travel begins until arrival at a final destination.

Describe the commuter's velocity according to the graph you've drawn. Your description should include reasons for the commuter's change in velocity at various time intervals including:

---

1 Derived from Exploring Calculus by Charlene E. Beckmann and Theodore A. Sundstrom

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Laboratory Two: Local Behavior of Functions

(1) Are there certain time intervals during which the commuter seems not to be traveling any distance at all? Label these times on your graph. What is the appearance of the graph at such times?
(2) What might be happening on each of those intervals of time where the graph is not rising?

How does the commuter's average velocity, as determined in question 1, relate to the distance versus time graph sketched on page 1?

On the distance versus time graph, draw a line containing the points that indicate the initial and final times and positions. How does the velocity found in question 1 relate to this line?

On the distance versus time graph, what is the slope of the line containing the points (2:30, 62, 137.3) and (3:12, 62, 172.4)?

How does this value compare with the average velocity found in question 1?

3. Suppose \( t = a \) represents the initial time and \( t = b \) the final time over which travel took place.

Label these values on the axes of the graph.

Assuming distance is a function \( f \) of time (distance changes according to time), what do \( f(a) \) and \( f(b) \) represent?

Over the time interval \([a, b]\), where a position function is given by \( y = f(t) \), write an expression for average velocity.

Does this quantity look familiar? What is this quantity in terms of the graph of \( f \)?

The line sketched in question 2 contains the points \((a, f(a)) \) and \((b, f(b))\). If \( y = f(t) \) represents distance traveled, how might the average velocity over a given time interval of a distance versus time function be determined if the function is given graphically?

POLICE OFFICER'S POINT OF VIEW

A police officer patrolling a road over which a commuter must travel has a different point of view from that of the commuter.

4. Describe the police officer's perception of a commuter's velocity as the commuter travels over a stretch of road that the police officer is patrolling.

Compare the police officer's perception of the commuter's velocity with that of the commuter's perception of his or her average velocity. Compare the time intervals of interest to the commuter to those of interest to the police officer.

How might the average velocity of the commuter, from the police officer's point of view, be determined? How might this be done accurately?

5. Suppose that the police officer begins observing the commuter at time \( c \) as marked on the horizontal axis for the distance versus time graph. He observes the commuter's travel for a very short period of time, say \( h \) seconds.

What is the final time?
Laboratory Two: Local Behavior of Functions

What are the distances corresponding to the initial and final observation times?

Write an algebraic expression for the average velocity over this short time interval.

6. The graph at the right depicts a distance vs. time function. Time \( c \) is indicated. For \( h = 4 \) seconds, sketch a line containing \((c, f(c))\) and \((c + 4, f(c + 4))\).

Such lines are called secant lines. Draw secant lines containing \((c, f(c))\) and \((c + h, f(c + h))\) for \( h = 3, 2, 1, \) and 0.1.

As \( h \) decreases in magnitude, how do the secant lines compare with the curve of \( y = f(t) \) on a small interval containing \( c \)?

Over the same time intervals for \( h \) as above, how well would the values of the slopes of successive secant lines approximate the velocity of the commuter at time \( t = c \)?

7. In this course asking "Why?" is very important. Write two questions. The first question should be about anything which you were uncertain on the above questions. Write a second question about the situation which is of interest to you. Neither question has to be answerable by you.

PART TWO: CLOSE UP AND PERSONAL

The remaining portion of this lab should be completed using Theorist. The purpose of this part is to investigate the behavior of a function near a given point. You may record your answers here or on separate paper.

8. Graph the function \( y = x^2 + 1 \). Zoom in (Option Rocket) about the point (1, 2) until the x-axis labels differ at most by 0.01. What does the graph look like?

Determine a second point on the graph within this viewing range. Once you have determined the \( x \) value, you can use Theorist or your calculator to determine the \( y \) value or you may use the grid on the graph if it is such that you can accurately read a second point. Find the equation of the line through these two points.

Second Point: _______________ Slope: _______________

Equation of the line: _______________

Using ADD LINE PLOT graph this line on the zoomed-in graph above. Don't forget Theorist will only plot one function 'y=' so you will have to use a different variable name for the equation of the line in Theorist. How does the graph of the line compare to the graph of the quadratic?
Laboratory Two: Local Behavior of Functions

Zoom out (Rocket) a couple of times. Now how does the line compare to the given quadratic?

For each of the points below, zoom in about the point until the x-axis labels differ at most by 0.01. Find a second point on the graph and complete the table.

<table>
<thead>
<tr>
<th>First Point</th>
<th>Second Point</th>
<th>Slope</th>
<th>First Point</th>
<th>Second Point</th>
<th>Slope</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 2)</td>
<td></td>
<td></td>
<td>(0,</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(2,</td>
<td></td>
<td></td>
<td>(-1,</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(3,</td>
<td></td>
<td></td>
<td>(-2,</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Were there any points for which it was unnecessary to zoom in to know the result? Explain.

By hand, graph the slope versus x-coordinates for the points above. What, if any, relationship exists?

9. In question 8, after zooming in sufficiently close, you should have observed that the function resembled or is closely approximated by a (straight) line. For each of the following functions and corresponding points, determine if a single linear approximation would be appropriate by repeatedly zooming in. If a linear approximation is not appropriate, explain why.

(a) \( y = |x^2 - 4| + 3 \) at \( x = 0 \)? at \( x = 2 \)?
(b) \( y = x^{2/5} \) at \( x = 0 \)? at \( x = 1 \)?
(c) \( y = \frac{\sin(x)}{x} \) at \( x = 0 \)?
(d) \( y = \sin(\frac{1}{x}) \) at \( x = 0 \)? at \( x = 1 \)?
(e) \( y = x\sin(\frac{1}{x}) \) at \( x = 0 \)? at \( x = 0.1 \)?

Observe that the functions in parts (c), (d), and (e) are not defined at \( x = 0 \). For parts (c) and (e), Theorist plots a value anyway. Why do you think this is? Are the values Theorist shows appropriate?
Laboratory Two: Local Behavior of Functions

10. For the function \( y = \begin{cases} -x^2 + 7.8 & (x < 2) \\ x^2 - 9x + 18 & (2 \leq x < 3) \\ c \sin(\pi x) & (x \geq 3) \end{cases} \) investigate the following questions.

(a) Suppose that \( c = 4 \). Can this function be reasonably approximated by a line for \( x \) near 2? Give evidence to support your claim.

(b) Can this function be reasonably approximated by a line for \( x \) near 3?

(c) Answer part (b) if \( c \) were changed to 10, to 3, to 2, to 1, and to 0.

(d) When a function can reasonably be approximated by a line over a small interval containing a point, we sometimes say the function is \emph{locally linear}. Of course the function usually is not linear, but just approximately linear. For what range of values of \( c \) do you feel the graph is not locally linear for \( x \) near 3? For what range of values of \( c \) are you uncertain? Are there any values of \( c \) for which you are absolutely certain \( f \) is locally linear for \( x \) near 3?

**PART THREE: ANOTHER PERSPECTIVE**

In parts one and two, you found a secant line which closely resembled the function "locally" by using two points close together on the graph of the function. In this third part of the lab, you will investigate the behavior of secant lines about a point and how this behavior relates to the function.

11. \emph{Theorist} can aid in the exploration of secant lines. A notebook \emph{SECANT LINES} has been prepared to use in this investigation. First close any currently open notebook by clicking in the small square in the upper left hand corner of the notebook or by choosing \emph{CLOSE NOTEBOOK} from the \emph{FILE} menu. From the \emph{FILE} menu choose \emph{OPEN NOTEBOOK}. A new window will open. Move to the \emph{Calculus-1} folder under the \emph{Theorist} folder by some combination of clicking on the folder at the top of the window and pressing the \emph{DRIVE} button. The specific steps depend on how you started \emph{Theorist} and your machine. Ask for help if you're having difficulties. Then follow the directions at the top of the notebook.

Describe what you are seeing as a result of the animation.

Zoom in a couple of times on the point (1, 0). Repeat the animation process. Do the secant lines appear to be approaching any specific line?
Laboratory Two: Local Behavior of Functions

Instead of varying $b$ from 3 to 1, vary $b$ from -3 to 1. How does this affect the secant lines?

Do the secant lines appear to be approaching any specific line? If so, is it the same line observed before?

In as much detail as possible, describe "m" as calculated by the equation $m = \frac{f(b) - f(a)}{b - a}$ and the line represented by $s = m(x - a) + f(a)$.

12. You may have observed that the value of $b$ determines how far away the second point is from the first point $a$. Let $h$ be the difference of $a$ and $b$. Then $b = a + h$. Modify the notebook by deleting the equation 'b=' and replace $b$ throughout by the expression $a + h$. You'll need to include an equation assigning $h$ a value.

Notice the denominator for the slope can be simplified. What should the denominator be?

To animate now, highlight any one of the $h$'s in the equation for the slope. Proceed as before. On what range must $h$ be animated in order for it to be equivalent to animating $b$ from 3 up to 1?

If we animated $h$ from -0.5 up to 0, what is the corresponding values of $b$? Write your answer in terms of $a$.

Remove the unnecessary comments and modify the other comments appropriately. Save one copy of this new notebook for your group on your disk and one copy in the folder KELLER.

13. What is the relationship between the secant lines under investigation and the average velocity of part one? How does the behavior of the secant lines over increasingly smaller intervals relate to instantaneous velocity?
Laboratory Two: Local Behavior of Functions

14. Using the new notebook you created in question 12, for each of the functions and corresponding points of questions 9 a, b, c and e (not d), zoom in a couple of times to within 0.5 of the desired point. Do not zoom in too closely. By using the \textit{knife} cut away any unused space. To facilitate this, you may want to delete the graph and just graph the function \( y = f(x) \). Then before animating, you will have to graph the secant line using \textit{ADD LINE PLOT}. Animate \( h \) \textit{FROM} 0.5 \textit{UP TO} 0 and \textit{FROM} -0.5 \textit{UP TO} 0. Do the secant lines for both ranges of \( h \) appear to approach the same line? Record your answers below.

(i) \( y = |x^2 - 4| + 3 \) at \( x = a = 0 \) at \( x = a = 2 \)?

(ii) \( y = x^{2/5} \) at \( x = a = 0 \) at \( x = a = 1 \)?

(iii) \( y = \frac{\sin(x)}{x} \) at \( x = a = 0 \)?

(iv) \( y = x \sin(\frac{1}{x}) \) at \( x = a = 0.1 \)?

\[ \text{[For (iv) you'll need to select your own viewing window and FROM value for } h. \]}

15. For a fixed value of \( a \) the equation of the slope of the secant line through \((a, f(a))\) and \((a + h, f(a + h))\) is a function of \( h \) alone. For each of the functions and corresponding points of questions 14 parts (i), (ii), and (iii), obtain a graph of \( m \) versus \( h \) by highlighting the equation in your notebook and graphing before.

- You are only interested in small values of \( h \), namely \(-0.5 < h < 0.5\).
- As such, the graph of the function should show only \( a - 0.5 < x < a + 0.5 \)
- Don’t forget to change the value of \( a \) appropriately.

Print a copy of each case with a graph of the function and a graph of the slope of the secant lines as a function of \( y \). By cleaning up your notebook, this should and will fit on one page. This totals 5 pages of printed notebooks.

You were investigating what happens to the secant lines as the two chosen points got closer together. Near what value of \( h \) on this graph are we interested?

What characteristics does the graph of \( m \) versus \( h \) have near this point when the secant lines did not approach a common line?
Laboratory Two: Local Behavior of Functions

What characteristics does the graph have near this point when the secant lines did approach a common line?

Can the graph of \( m \) versus \( h \) be used to determine the slope of the line which the secant lines approached? If so, when?

---

**ADVANCED THEORIST**

For those of you who wish to explore further (ahead) on your own or by reading the manual, investigate the following areas:

**GRAPHING IMPLICIT EQUATIONS**

It is sometimes desirable to graph all points \((x, y)\) which satisfy a given equation. However, it is sometimes not possible or very easy to solve an equation of two variables for one of the variables. For example, let's say we wish to graph the ellipse \( x^2 + xy + y^2 = 6 \). You can use an advanced feature of Theorist to plot all points satisfying \( x^2 + xy + y^2 - 6 = 0 \) by creating a new function \( z = x^2 + xy + y^2 - 6 \) and choosing ZERO CONTOUR from the submenu for 'z=f(x,y)' on the GRAPH menu. This will plot in two dimensions values of \( x \) and \( y \) for which the value of \( x^2 + xy + y^2 - 6 \) is 0. Note: You can use this to plot circles such as in question 13 of Lab One using a single equation.

*Modifying the Appearance of your graph*

Click on the icon of the graph which looks like a typed page (all horizontal lines). This will open the graph details box. In this area you can change the viewing ranges, the line styles (especially useful when more than one function is plotted), and modify the functions to be plotted.
Laboratory Two
Local Behavior of Functions

LABORATORY REPORT

Use complete sentences. Do not give just the responses to the questions. Rather, include the question in your work. For example, to answer the question:

What is the sum of the first 3 positive integers?

respond with:

The sum of the first 3 positive integers is 6 since 1 + 2 + 3 equals 6.

For this lab, you should turn in the following attached to this cover sheet:

• A short paragraph in your own words explaining the relationship between the three parts of this laboratory in terms of the local behavior of a function.
• Your answer to question 7.
• A summary (no more than 2 paragraphs) of what you found in questions 8 and 9.
• Your answer to the last part of questions 10 and 11.
• A comparison of your answers to questions 9 and 14.
• The ten graphs and all responses to question 15.

Approximate amount of time to complete this lab: ________
Laboratory Two: Local Behavior of Functions

Laboratory Two Names 1. _____________________
Local Behavior of 2. ____________________
Functions 3. ____________________

You will be asked to turn in selected portions of this LAB.

NOTE: You should not expect anything saved on the LAB disks to remain there. They will be periodically cleaned out. ALWAYS SAVE TO YOUR OWN DISK. If uncertain on how to do so, ASK FOR HELP. As a last resort try saving by using the SAVE command from the FILE menu.

PART ONE: RATE OF CHANGE - DISTANCE VERSUS TIME

You should record your answers to Part One on a separate piece of paper.

Commuters are often interested in how fast they must travel to get from one place to another, knowing the amount of time available to complete the journey, and the distance to be traveled. This interest increases if the time is shorter than usual, and several police officers are stationed along the route. This part of the investigation explores the relationship between distance and time on large, then smaller intervals of time. In particular, the commuter's point of view of velocity will be compared with that of an officer of the law.

COMMUTER'S POINT OF VIEW

1. How would a commuter determine average velocity for the particular commute given below?

<table>
<thead>
<tr>
<th>Odometer</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial Point</td>
<td>2:30 pm</td>
</tr>
<tr>
<td>Final Point</td>
<td>3:12 pm</td>
</tr>
</tbody>
</table>

What is the average velocity of the commuter over the time interval given?

2. Sketch a possible graph of the distance versus time function: $y = f(t)$ for the commuter's travel using initial and final times and distances given in the table above. Keep in mind that the commuter must stop at traffic lights, stop signs, etc. and cannot travel at a constant speed from the moment travel begins until arrival at a final destination.

Describe the commuter's velocity according to the graph you've drawn. Your description should include reasons for the commuter's change in velocity at various time intervals including:

---

1Derived from *Exploring Calculus* by Charlene E. Beckmann and Theodore A. Sundstrom

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Laboratory Two: Local Behavior of Functions

(1) Are there certain time intervals during which the commuter seems not to be traveling any distance at all? Label these times on your graph. What is the appearance of the graph at such times?
(2) What might be happening on each of those intervals of time where the graph is not rising?

How does the commuter's average velocity, as determined in question 1, relate to the distance versus time graph sketched on page 1?

On the distance versus time graph, draw a line containing the points that indicate the initial and final times and positions. How does the velocity found in question 1 relate to this line?

3. Suppose $t = a$ represents the initial time and $t = b$ the final time over which travel took place.

Label these values on the axes of the graph.

Assuming distance is a function $f$ of time (distance changes according to time), what do $f(a)$ and $f(b)$ represent?

Over the time interval $[a, b]$, where a position function is given by $y = f(t)$, write an expression for average velocity.

Does this quantity look familiar? What is this quantity in terms of the graph of $f$?

The line sketched in question 2 contains the points $(a, f(a))$ and $(b, f(b))$. If $y = f(t)$ represents distance traveled, how might the average velocity over a given time interval of a distance versus time function be determined if the function is given graphically?

POLICE OFFICER'S POINT OF VIEW

A police officer patrolling a road over which a commuter must travel has a different point of view from that of the commuter.

4. Describe the police officer's perception of a commuter's velocity as the commuter travels over a stretch of road that the police officer is patrolling.

Compare the police officer's perception of the commuter's velocity with that of the commuter's perception of his or her average velocity. Compare the time intervals of interest to the commuter to those of interest to the police officer.

How might the average velocity of the commuter, from the police officer's point of view, be determined? How might this be done accurately?

5. Suppose that the police officer begins observing the commuter at time $c$ as marked on the horizontal axis for the distance versus time graph. He observes the commuter's travel for a very short period of time, say $h$ seconds.

What is the final time?
What are the distances corresponding to the initial and final observation times?

Write an algebraic expression for the average velocity over this short time interval.
6. The graph at the right depicts a distance vs. time function. Time $c$ is indicated. For $h = 4$ seconds, sketch a line containing $(c, f(c))$ and $(c + 4, f(c + 4))$.

Such lines are called secant lines. Draw secant lines containing $(c, f(c))$ and $(c + h, f(c + h))$ for $h = 3, 2, 1, 0.1$.

As $h$ decreases in magnitude, how do the secant lines compare with the curve of $y = f(t)$ on a small interval containing $c$?

Over the same time intervals for $h$ as above, how well would the values of the slopes of successive secant lines approximate the velocity of the commuter at time $t = c$?

7. In this course asking "Why?" is very important. Write two questions. The first question should be about anything which you were uncertain on the above questions. Write a second question about the situation which is of interest to you. Neither question has to be answerable by you.

PART TWO: CLOSE UP AND PERSONAL

The remaining portion of this lab should be completed using Maple. The purpose of this part is to investigate the behavior of a function near a given point. You may record your answers here or on separate paper. For this part of the lab, use a square viewing window centered about the point of interest. That is, make sure the $x$ and $y$ ranges for plot commands have the same width and are centered around the point given.

8. Graph the function $y = x^2 + 1$ about the point (1, 2) within 0.01 - use a viewing range of [0.99, 1.01] for $x$ and [1.99, 2.01] for $y$. What does the graph look like?

Determine a second point on the graph within this viewing range. Once you have determined the $x$ value, use Maple's substitute command to determine the corresponding $y$ value. Find the equation of the line through these two points.

Second Point: _____________________ Slope: _____________________

Equation of the line: _____________________

Plot both the function and the line on the same viewing range as above. How does the graph of the line compare to the graph of the quadratic?
Laboratory Two: Local Behavior of Functions

Graph the function and the line on a larger viewing range of [-5, 5] by [-5, 10]. Now how does the line compare to the given quadratic?

For each of the points below, repeat the previous steps to find a second point near the first point on the graph and the corresponding slope between the two points.

<table>
<thead>
<tr>
<th>First Point</th>
<th>Second Point</th>
<th>Slope</th>
<th>First Point</th>
<th>Second Point</th>
<th>Slope</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 2)</td>
<td>(0, )</td>
<td></td>
<td>(0, )</td>
<td>(-1, )</td>
<td></td>
</tr>
<tr>
<td>(2, )</td>
<td>(-1, )</td>
<td></td>
<td>(-1, )</td>
<td>(-2, )</td>
<td></td>
</tr>
</tbody>
</table>

Were there any points for which it was unnecessary to zoom in to know the result? Explain.

By hand, graph the slope versus x-coordinates for the points above. What, if any, relationship exists?

9. In question 8, after viewing a function sufficiently close, you should have observed that the function resembled or is closely approximated by a (straight) line. For each of the following functions and corresponding points, determine if a single linear approximation would be appropriate by viewing the function with a window centered at the given point. The width of the square viewing range you use will depend upon the function. If a linear approximation is not appropriate, explain why.

(a) $y = |x^2 - 4|+3$ at $x = 0$? at $x = 2$?
(b) $y = x^{2/3}$ at $x = 0$? at $x = 1$?

(c) $y = \frac{\sin(x)}{x}$ at $x = 0$?
(d) $y = \sin\left(\frac{1}{x}\right)$ at $x = 0$? at $x = 1$?

(e) $y = x\sin\left(\frac{1}{x}\right)$ at $x = 0$? at $x = 0.1$?

Observe that the functions in parts (c), (d), and (e) are not defined at $x = 0$. For parts (c) and (e), Maple plots a value anyway. Why do you think this is? Are the values Maple shows appropriate?
Laboratory Two: Local Behavior of Functions

10. For the function
\[ y = \begin{cases} 
-x^2 + 7.8 & (x < 2) \\
x^2 - 9x + 18 & (2 \leq x < 3) \\
c \sin(\pi x) & (x \geq 3) 
\end{cases} \]

investigate the following questions.

(a) Suppose that \( c = 4 \). Can this function be reasonably approximated by a line for \( x \) near 2? Give evidence to support your claim.

(b) Can this function be reasonably approximated by a line for \( x \) near 3?

(c) Answer part (b) if \( c \) were changed to 10, to 3, to 2, to 1, and to 0.

(d) When a function can reasonably be approximated by a line over a small interval containing a point, we sometimes say the function is \textit{locally linear}. Of course the function usually is not linear, but just approximately linear. For what range of values of \( c \) do you feel the graph is not locally linear for \( x \) near 3? For what range of values of \( c \) are you uncertain? Are there any values of \( c \) for which you are absolutely certain it is locally linear for \( x \) near 3?

PART THREE: ANOTHER PERSPECTIVE

In parts one and two, you found a secant line which closely resembled the function "locally" by using two points close together on the graph of the function. In this third part of the lab, you will investigate the behavior of secant lines about a point and how this behavior relates to the function.

11. Below is the graph of the function \( y = x^2 + 1 \). By hand sketch in the secant line for points with coordinates \( x = 1 \) and \( x = 6 \). Repeat with second point having coordinates \( x = 5, 4, 3, 2, -4, -3, -2, -1, 0 \) in that order.

As the second point approaches the first point with coordinate \( x = 1 \), do the secant lines appear to be approaching any specific line?
Laboratory Two: Local Behavior of Functions

You can use Maple to draw these secant lines. Enter the following procedure or open the file called *secants* under the calculus folder and copy the lines from it to your worksheet.

```maple
* secants := proc(f, a, b) local s, c, m;
  s := {f(x)};
  for c from b by -(b - a)/5 to a + (b - a)/5 do
    m := ((f(c) - f(a)) / (c - a));
    s := s union {m * (x - a) + f(a)}
  od:
  plot(s, x = a - (b - a)..a + (b - a));
end;
```

Enter the function \( f(x) = x^2 + 1 \) using a Maple procedure. Verify the previous graph by issuing the two commands:

- `secants(f, 1, 6);` \# Secant lines from the right
- `secants(f, 1, -4);` \# Secant lines from the left

What is the role of the third parameter \( b \) of the procedure 'secants'?

What is the role of the variable \( c \) of the procedure? For \( b = 6 \), what values does \( c \) take on?

In as much detail as possible, describe "\( m \)" as calculated by the equation \( m := (f(c) - f(a)) / (c - a) \).

The variable \( s \) collects expressions to plot. In as much detail as possible, describe the expression "\( m (x - a) + f(a) \)".

12. What is the relationship between the secant lines under investigation and the average velocity of part one? How does the behavior of the secant lines over increasingly smaller intervals relate to instantaneous velocity?
Laboratory Two: Local Behavior of Functions

13. For questions 11 and 12, you investigated secant lines where one of two points was fixed and the other point approached the fixed point. The lines pass throughed the same point but had different slopes. For each function below, define the function as a Maple procedure. Use the 'secant' procedure to examine the behavior of the secant lines from the left and from the right. For each value of $a$ given, use the values $a - 0.5$ and $a + 0.5$ for $b$.

Do the secant lines from the left and from the right appear to approach the same line? Record your answers below.

(i) $y = |x^2 - 4| + 3$ at $x = a = 0$? at $x = a = 2$?

(ii) $y = x^{2/3}$ at $x = a = 0$? at $x = a = 1$?

(iii) $y = \frac{\sin(x)}{x}$ at $x = a = 0$?

(iv) $y = x \sin(\frac{1}{x})$ at $x = a = 0.1$?

[For (iv) you'll need to select more appropriate values for $b$.]

14. Observe that in questions 11-13, the second point used for the secant line could be determined by knowing its distance from the first point with coordinate $x = a$. Write a procedure named $m$ which returns the slope of the secant line when given a function $f$, an initial point $a$, and a distance, $h$, away from $a$.

It may help to answer the following questions. Assume $f(x)$ is a function given to you.

What is the value of the function at $x = a$? __________________.

What is the $x$ value of the second point? ________________.

What is the value of the function at the second point? ___________.

What is the slope of the line passing through these two points?

Define your procedure for the three parameters $f$, $a$, and $h$. It should look like:

- $m := \text{proc } (f, a, h) \text{.....} \text{end;}$
Laboratory Two: Local Behavior of Functions

15. For a fixed value of $a$, the equation of the slope of the secant line through $(a, f(a))$ and $(a + h, f(a + h))$ is a function of $h$ alone. For each of the functions and corresponding points of questions 13 parts (i), (ii), and (iii), obtain a graph of the function $f$ near $a$ and a graph of $m$ versus $h$ by using a command similar to "plot($m(f, 0, h)$, $h = -0.5..0.5$)". Keep in mind:

- You are only interested in small values of $h$, namely $-0.5 < h < 0.5$.
- As such, the graph of the function should show only $a - 0.5 < x < a + 0.5$

Copy both graphs into your worksheet. Print your worksheet when finished with all 10 graphs. Make sure to clean up your worksheet before printing! Answer the following questions by examining the five pairs of graphs.

You were investigating what happens to the secant lines as the two chosen points got closer together. Near what value of $h$ on this graph are we interested?

What characteristics does the graph of $m$ versus $h$ have near this point when the secant lines did not approach a common line?

What characteristics does the graph have near this point when the secant lines did approach a common line?

Can the graph of $m$ versus $h$ be used to determine the slope of the line which the secant lines approached? If so, when?
Laboratory Two
Local Behavior of Functions

LABORATORY REPORT

Use complete sentences. Do not give just the responses to the questions. Rather, include the question in your work. For example, to answer the question:

What is the sum of the first 3 positive integers?

respond with:

The sum of the first 3 positive integers is 6 since $1 + 2 + 3$ equals 6.

For this lab, you should turn in the following attached to this cover sheet:

• A short paragraph in your own words explaining the relationship between the three parts of this laboratory in terms of the local behavior of a function.
• Your answer to question 7.
• A summary (no more than 2 paragraphs) of what you found in questions 8 and 9.
• Your answer to the last part of question 10 and the last two parts of question 11.
• A comparison of your answers to questions 9 and 13.
• The ten graphs and all responses to question 15.

Approximate amount of time to complete this lab: _______
Laboratory Three: Exploring Derivatives of Functions

RECAP

In the last lab you investigated the relationship between average rate of change of a function and the slopes of secant lines by using the difference quotient $\frac{f(a+h) - f(a)}{h}$. By letting the length of the interval get smaller, you explored the relationship between instantaneous rate of change of a function at a point, the 'limit' of the secant lines, and tangents to curves. You discovered that when $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ exists, its value was the same as the slope of the tangent line to $f$ at $a$. When this limit exists, it is called the derivative of $f$ at $a$, and denoted by $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$.

PART ONE

Often the height of a projectile is modeled by a quadratic function of the form $s(t) = s_0 + v_0 t + \frac{1}{2} a t^2$ where $s(t)$ is the displacement at time $t$, $s_0$ is the initial displacement, $v_0$ is the initial velocity and $a$ is the acceleration due to gravity.

1. Suppose a ball is thrown upward with a velocity of 10 meters per second from a height of 1.6 meters. Since the ball is thrown upward, gravity exerts a negative force on the height. So $a = -9.8 \text{ m/s}^2$. Find a function that describes the height of the ball at time $t$. $s(t) = \underline{__________________________}$.

A graph of the function is given below. What checks can you perform to verify your equation for the function?

2. Use the graph to determine the maximum height of the ball and the time at which this occurs.
Laboratory Three: Exploring Derivatives of Functions

3. Determine the time when the ball strikes the ground. Explain how you found this and the accuracy of your answer.

4. By sketching a tangent line and determining its slope, approximate the instantaneous rate of change (velocity) at each of the following times.

   a) \( s'(0.5) = \) _____
   b) \( s'(1.5) = \) _____
   c) \( s'(2.0) = \) _____

5. Since the graph of the function is a parabola opening down, the maximum height of the ball can be determined by finding the vertex of the parabola. However, you can also determine the maximum height of the ball by using the physics of the situation. Label the (approximate) time on your graph when the ball reaches its maximum height by \( a \). What is the velocity of the ball when it reaches its maximum height? Explain.

6. You can use Theorist to determine \( a \) as follows. Using Theorist, first expand and simplify the difference quotient

\[
\frac{s(a + h) - s(a)}{h} = \text{______________________________}
\]

Determine the limit of the difference quotient as \( h \) approaches zero. This will be the derivative of \( s \) at \( a \) or the instantaneous rate of change of \( s \) at \( a \).

\[
s'(a) = \text{______________________________}
\]

Use this information together with your response to item 5 to determine the time when the maximum height occurs.

7. Observe that you were able to determine the instantaneous velocity at \( a \) without knowing the point of interest. Use this result to compute the following.

   a) \( s'(0.5) = \) _____
   b) \( s'(1.5) = \) _____
   c) \( s'(2.0) = \) _____

   How does this compare with what you found in item 4?

8. Use your information from 7 (a) to determine the equation of a tangent line at \( t = 0.5 \).
Laboratory Three: Exploring Derivatives of Functions

PART TWO: (Adapted from Chapter 8 of Exploring Calculus by Beckman and Sundstrom)

It is possible to estimate \( y = f'(x) \) graphically given the graph of \( y = f(x) \). It will help to refer to the graph and values of Part One. Items 9, 10, and 11 provide some landmarks of the graph of \( f \) which will be helpful in sketching \( f' \).

9. Consider values of \( x \) where the slope of the tangent is zero:
   a. Describe characteristics of the graph of a function \( f \) on an interval containing \( x = c \) for which the slope of the tangent through the point \( (c, f(c)) \) is zero. Use terms such as increasing, decreasing, constant.
   b. If \( f'(c) = 0 \), what point is on the graph of \( y = f'(x) \)? ( _______, _______ )

10. Consider intervals where the slopes of the tangents to \( y = f(x) \) are negative:
   a. Describe the appearance of the graph of \( f \) on an interval over which \( f'(x) < 0 \).
   b. If the slope of \( y = f(x) \) is negative on an interval, then \( f'(x) \) __ 0 (Complete with <, >, or =).
      If \( f'(x) < 0 \) on an interval, where does the graph of \( y = f'(x) \) lie with respect to the x-axis?

11. Consider intervals where the slopes of the tangents to \( y = f(x) \) are positive:
   a. Describe the appearance of the graph of \( f \) on an interval over which \( f'(x) > 0 \).
   b. If the slope of \( y = f(x) \) is positive on an interval, then \( f'(x) \) __ 0 (Complete with <, >, or =).
      If \( f'(x) > 0 \) on an interval, where does the graph of \( y = f'(x) \) lie with respect to the x-axis?

12. Below are two pairs of axes. Use the second axes provided to sketch the graph of the derivative of \( y = f'(x) \), where \( f \) is the function given in the first graph. Sketch the derivative of \( y = f(x) \) as follows:
   a. Draw several tangent lines along the curve of \( f \).
   b. Estimate the slope \( f' \) of each tangent, using the grid marks as a guide.
   c. Plot the points \( (x, f'(x)) \) on the second axes.
   d. Connect the points to sketch the graph of \( y = f'(x) \).
13. The graphs of several common functions \( y = f(x) \) are sketched below. Complete the following for each function.
   
a. Sketch the graph of the derivative \( f' \) on the same axes as \( f \), following the suggestions given in item 12. Briefly explain why your graph is sensible.
   
b. Conjecture the form of the equation for the derivative of the given function \( y = f(x) \) (i.e. constant, linear, quadratic, cubic etc.).

   ![Graphs of common functions]

   Constant Function \( f(x) = c \)  
   Linear Function \( f(x) = mx + b \)  
   Quadratic Function \( f(x) = ax^2 + bx + c \)  
   Cubic Function \( f(x) = ax^3 + bx^2 + cx + d \)

14. Given below are the graphs of five functions \( f_1 \) through \( f_5 \). For each of these functions, find from among the functions \( g_1 \) through \( g_{10} \) the derivative of the function. Briefly explain your choice below the graph in each case.

   ![Graphs of functions]

   \( f_1 \)  
   \( f_2 \)
15. Sketched in the figure below is the graph of a function $f$. Suppose another function $g$ has the following properties: $g(-1) = -2$ and $g'(x) = f'(x)$ for all $x$. Sketch the graph of $g$ on the same axes. Briefly explain your results.
Laboratory Three: Exploring Derivatives of Functions

PART THREE

16. Use _Theorist_ and the limit definition of the derivative to verify your conjectures in item 13 (b). For each function,

- enter the function into _Theorist_. Be sure to use the shadow (bold) variables from the variable palette.
- On a new line enter the difference quotient \( \frac{f(x+h)-f(x)}{h} \).
- Highlight this line and the equation of the function (but not the icon) and choose SUBSTITUTE from the manipulate menu. Or highlight the equation of the function, then press the option key and point at the equation of the function and drag it to the icon of the difference quotient.

Recall _Theorist_ will asked you how to define \( f, b, c, d, h, \) and \( m \).
Define \( b, c, d, \) and \( m \) as constants, \( h \) as a variable, and \( f \) as a function.

Example:

\[
\frac{f(x+h)-f(x)}{h} = \frac{(h+x)^2 - x^2}{h} = \frac{h^2 + 2hx}{h} = h + 2x
\]

Reduce the right hand side as much as possible by using EXPAND, SIMPLIFY, and COLLECT. COLLECT asks like factoring by pulling out common variables from sums. You will need to highlight exactly what you want expanded, simplified, or collected (factored).

<table>
<thead>
<tr>
<th>( f(x) )</th>
<th>( f(x+h)-f(x) )</th>
<th>( f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^2 + 1 )</td>
<td>( 2x + h )</td>
<td>( 2x )</td>
</tr>
<tr>
<td>( mx + b )</td>
<td>( m )</td>
<td>( m )</td>
</tr>
<tr>
<td>( ax^2 + bx + c )</td>
<td>( 2x )</td>
<td>( 2x )</td>
</tr>
<tr>
<td>( ax^3 + bx^2 + cx + d )</td>
<td>( c )</td>
<td>( c )</td>
</tr>
<tr>
<td>( \frac{c}{x} ) or ( c \cdot x^{-1} )</td>
<td>( \frac{c}{x} )</td>
<td>( \frac{c}{x} )</td>
</tr>
<tr>
<td>( c \cdot x^{-2} )</td>
<td>( c \cdot x^{-2} )</td>
<td>( c \cdot x^{-2} )</td>
</tr>
<tr>
<td>( x^n )</td>
<td>( XXXXXXXXXXXXXXXXXXXX )</td>
<td>( XXXXXXXXXXXXXXXXXXXX )</td>
</tr>
</tbody>
</table>

For the last function in the table, use the patterns above to complete the last column only.

What generalizations can you make about the derivative of: a constant; the derivative of a constant times a function; the derivative of the sum of two functions; and the derivative of the difference of two functions? Give evidence from the table above.
Laboratory Three: Exploring Derivatives of Functions

17. In item 16, you discovered some methods for finding derivatives mentally. These were suggested by algebraic patterns you produced using the limit definition of a derivative. Sometimes it will be difficult to algebraically determine the derivative of a function using the limit definition. You will next investigate a graphical approach to find the derivative of a function.

In Lab Two, we investigated the relationship between the tangent line and the difference quotient \( f(x + h) - f(x) \) for small values of \( h \). You found that as the values of \( h \) approached zero, the slope of the secant line through the points \((x, f(x))\) and \((x + h, f(x + h))\) approached the slope of the tangent line. Thus one way of approximating the derivative is to choose a small value of \( h \) in the difference quotient. As the value of \( h \) gets smaller, the approximation should get better and better. HOWEVER, due to the bounds on the precision of machine computation, a value too close to zero could result in a worse approximation.

a) Determine an approximation to the value of the derivative of the sine function at \( x = 1.2 \) using a value of \( h = 0.01 \). That is, if \( f(x) = \sin(x) \), approximate \( f'(1.2) \) by calculating the difference quotient above.

b) Rather than compute the difference quotient for lots of values of \( x \) to get an impression of the derivative of \( \sin(x) \), we can use either a graphics calculator or Theorist to graph the difference quotient. Let \( dq \) equal the above difference quotient for the sine function. Obtain a graph of the difference quotient with a value of \( h = 0.01 \). Do this by writing the difference quotient in terms of the sine function, \( x \), and \( h \). Then on a separate line enter the value of \( h \). Then highlight and graph the difference quotient. Using ADD LINE PLOT graph the function \( y = \cos(x) \). How does the graph of the cosine function compare with the graph of the difference quotient?

c) Pick any spot on the graph. Zoom in several times. Now how do the graphs compare?

d) Without changing your zoomed in graph, decrease the values of \( h \). How do the graphs compare?

e) Keeping in mind that the limit of the difference quotient as \( h \) approaches zero is the derivative, write a conjecture about the derivative of the function \( f(x) = \sin(x) \).

\[ f'(x) = \] 

18. Repeat parts (a) – (e) of item 17 for the cosine function. Conjecture a rule for the derivative of the cosine function. Test your hypothesis by graphing your conjecture and the difference quotient.
Laboratory Three: Exploring Derivatives of Functions

19. a) Assume you are traveling at a constant rate on a train to Boston. Starting from the rear of the train you walk towards the front at a rate of 5 mph. If the train is traveling at a rate of 60 mph, how fast are you approaching Boston?

b) Let \( S(t) \) be your distance from Kalamazoo, \( P(t) \) be the train's distance from Kalamazoo measured from the rear of the train, and \( L(t) \) be the distance in the train you have walked. Express your distance from Kalamazoo as a function of \( P(t) \) and \( L(t) \).

c) Using \( S'(t) \) for your velocity from Kalamazoo, express your answer to the part (a) in terms of \( P(t), L(t), P'(t), \) and \( L'(t) \).

d) Repeat parts (a), (b), and (c), but assume instead you start at the front of the train and walk towards the rear. What additional information do you need? Choose a reasonable value for any missing information. What is your resulting expression for \( S'(t) \)?

20. Complete the following table. First, determine by hand the product of the two functions \( f \) and \( g \). Then determine derivative of each function using the rules you developed earlier. If you have difficulty determining the derivative of the product by hand, try using the limit definition in Theorist. As a last resort graph the difference quotient and try to determine a similar function.

<table>
<thead>
<tr>
<th>Function ( f(x) )</th>
<th>g(x)</th>
<th>( (fg)(x) )</th>
<th>( f )</th>
<th>g</th>
<th>( (fg)(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Derivative ( x )</td>
<td></td>
<td>( x^5 )</td>
<td></td>
<td>x</td>
<td>( x^2 )</td>
</tr>
<tr>
<td>1</td>
<td>( 4x^3 )</td>
<td>( 5x^4 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( x^2 )</td>
<td>( x^5 )</td>
<td>( x^3 )</td>
<td></td>
<td>x</td>
<td>( x^7 )</td>
</tr>
<tr>
<td>( x )</td>
<td>( x^2 + x )</td>
<td>( x )</td>
<td></td>
<td>x</td>
<td>( x^3 + x )</td>
</tr>
</tbody>
</table>

Make a conjecture as to the derivative of \( y = f(x)g(x) \) in terms of \( f(x), g(x), f'(x) \) and \( g'(x) \).

21. Using your conjecture, complete the following table as above. After determining the derivative of the product using your conjecture, graph your hypothesized derivative along with the difference quotient. If your derivative is correct, the graphs of these two functions should be similar.

<table>
<thead>
<tr>
<th>Function ( f(x) )</th>
<th>g(x)</th>
<th>( (fg)(x) )</th>
<th>( f )</th>
<th>g</th>
<th>( (fg)(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Derivative ( (x^3 + 1) )</td>
<td>( 2x - 6 )</td>
<td>( (x^3+1)(2x-6) )</td>
<td>x</td>
<td>sin(x)</td>
<td></td>
</tr>
<tr>
<td>( x^2 )</td>
<td>sin(x)</td>
<td>( x )</td>
<td></td>
<td>x</td>
<td>cos(x)</td>
</tr>
</tbody>
</table>
Laboratory Three: Exploring Derivatives of Functions

22. This question is an example of an exam question testing your understanding of the labs done so far.

The graph of the natural logarithm function, \( f(x) = \ln x \), is given.

a) Sketch the graph of its derivative on the same axes. Briefly explain how you determined the graph of the derivative.

b) Accurately estimate the derivative of \( f(x) = \ln x \) for \( x = 2 \). Explain your process.

c) Different from the process chosen in part (b), explain another method that you could use to estimate the derivative of \( f(x) = \ln x \) at \( x = 2 \). Do not use the actual derivative, even if you know it.
Laboratory Four: Further Explorations of Derivatives

Laboratory Four  
Further Explorations  
of Derivatives

Finding Derivatives

In Lab Three, you found several generalizations about the form of a derivative. These generalizations allow you to find derivatives without using the limit of the difference quotient. The rules developed so far are:

- If \( h(x) = c \), then \( h'(x) = 0 \).
- If \( h(x) = cf(x) \), then \( h'(x) = cf'(x) \).
- If \( h(x) = f(x) + g(x) \), then \( h'(x) = f'(x) + g'(x) \).
- If \( h(x) = f(x) \cdot g(x) \), then \( h'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x) \).
- If \( h(x) = \sin(x) \), then \( h'(x) = \cos(x) \).
- If \( h(x) = \cos(x) \), then \( h'(x) = -\sin(x) \).

You are responsible for knowing these rules and how they were developed for the final exam.

There are many ways to find a derivative using Maple. The simplest way is to use the `diff` command. Maple uses the more general notation of the partial derivative from multivariate calculus. As with all Maple commands, additional information on the command `diff` can be obtained by using the command "help(diff);".

To find the derivative of \( y = (x^2 + 3x) \sin(x) \), we can enter

\[
\begin{align*}
\text{diff((x^2+3*x)*sin(x), x); or } dy & := \text{diff((x^2+3*x)*sin(x), x)}. \\
\end{align*}
\]

It will be helpful to assign the derivative to a variable whose name is suggestive such as "dy" for the derivative of \( y \). You should try to determine the form of the derivative mentally whenever possible.

Often you will already have the function defined as a procedure in your notebook. To differentiate a function entered as a procedure in Maple, use a command similar to "diff(f(x), x);".

Second derivatives (the derivative of the derivative) can be found in a similar fashion. For example, the command "diff(diff(x^3,x),x);" yields the second derivative of \( x^3 \) or 6\( x \). The help text describes a way of combining the two `diff` commands into one.

1. Recall that velocity is the instantaneous rate of change (derivative) of the position of an object. Complete the following reasonable generalization:
   
   Acceleration is the instantaneous rate of change (derivative) of the _______ of an object.

In Lab Three, you investigated the relationship between the displacement and the velocity of an object. In this part of the lab you will investigate the relationship between all three functions: displacement, velocity, and acceleration.

In Lab Three, the height of a projectile was modeled by the equation \( s = 1.6 + 10t - 4.9t^2 \).
Laboratory Four: Further Explorations of Derivatives

Determine the velocity and acceleration for this projectile (by hand):

\[ v = s' = \frac{d}{dt} s = \quad \text{and} \quad a = v' = \frac{d}{dt} v = \quad \]

Below is a graph of all three functions. Complete the statements that follow with the words: positive, negative, zero, increasing, decreasing, constant, concave upward, concave downward.

When the displacement is increasing, the velocity is ________________.

When the velocity is negative, the displacement is ________________.

When the acceleration is negative, the velocity is ________________.

When the displacement is zero, the velocity is ________________.

The shape of the displacement curve is ________________.

While your intuition about velocity and acceleration will be an aid, it can be a hindrance and you should not over rely on it. For example, it is possible for the velocity to be increasing and the distance from an origin to be decreasing. Describe a real life situation which would have this behavior. *Hint:* Velocity can be negative as well as positive.

2. One way of remembering the direction a parabola opens is by noting the sign of the squared term in its equation. If the coefficient of the squared term is positive, the parabola opens upward. If the coefficient of the squared term is negative, the parabola opens downward.

Let \( y = ax^2 + bx + c \) denote an arbitrary quadratic. Determine the second derivative of \( y \) with respect to \( x \). \( y'' = \quad \).
Laboratory Four: Further Explorations of Derivatives

How does the second derivative compare with the coefficient of the squared term?

Relate the above statements for parabolas to conditions on the second derivative by completing the following statements.

If the second derivative of $y$ with respect to $x$ is \textbf{positive}, the parabola opens ________.

If the second derivative of $y$ with respect to $x$ is \textbf{______________}, the parabola opens ________.

3. Obtain a graph of the function $y = x \cos(x) - x$ with a viewing range of $[-6, 6]$ for $x$ and $[-7, 7]$ for $y$. It will be helpful to resize (enlarge) the graph using the square at the bottom right hand corner of the graph to about 5 in by 7 in. It may help to print this graph.

Determine the intervals (approximately) over which the function is concave upward.

Determine the intervals (approximately) over which the function is concave downward.

In the following questions, you will investigate how the concavity relates to attributes of the first and second derivative. Keep in mind the terms: positive, negative, zero, increasing, decreasing, and constant.

Over an interval on which the function is concave upward, what can be said about the slope of the tangent lines? What does this imply about the derivative of a function when the function is concave upward?

Over an interval on which the function is concave downward, what can be said about the slope of the tangent lines?

The first derivative of the function is \textbf{__________} when the function is concave upward.

The first derivative of the function is \textbf{__________} when the function is concave downward.

Calculate the first and second derivatives for the function $y = x \cos(x) - x$.

$$y' = \text{______________________________}$$

$$y'' = \text{______________________________}$$

Print the plot of the derivatives along with the function on the same viewing area. How can you use this graph to verify your answers on the previous page?
4. Label the various curves with the appropriate function name.

(a) For each of the rows in the table, identify and label across the x-axis of your graph, the points or intervals for which the function $y$ satisfies the conditions stated. Some cases may not exist.

(b) Check your answers on part (a) with those of the instructor. Then, fill in the last column with concave upward, concave downward, increasing and/or decreasing. Use all the terms that apply.

<table>
<thead>
<tr>
<th>Label</th>
<th>$y'$</th>
<th>$y''$</th>
<th>$y$ is:</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>&lt; 0</td>
<td>&lt; 0</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>&lt; 0</td>
<td>= 0</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>&lt; 0</td>
<td>&gt; 0</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>= 0</td>
<td>&lt; 0</td>
<td></td>
</tr>
<tr>
<td>E</td>
<td>= 0</td>
<td>= 0</td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>= 0</td>
<td>&gt; 0</td>
<td></td>
</tr>
<tr>
<td>G</td>
<td>&gt; 0</td>
<td>&lt; 0</td>
<td></td>
</tr>
<tr>
<td>H</td>
<td>&gt; 0</td>
<td>= 0</td>
<td></td>
</tr>
<tr>
<td>I</td>
<td>&gt; 0</td>
<td>&gt; 0</td>
<td></td>
</tr>
</tbody>
</table>

Complete the following sentences.

When the function is concave upward the ________ derivative is increasing.

When the first derivative is increasing the second derivative must be ____________.

When the second derivative is positive the function is _________________.

When the function is concave downward the ________ derivative is decreasing.

When the second derivative is negative the function is _________________.

When the function reaches a (relative) maximum, the first derivative is ________________ and the second derivative is ________________.

When the function reaches a (relative) minimum, the first derivative is ________________ and the second derivative is ________________.
Laboratory Four: Further Explorations of Derivatives

5. Points where the function changes from being concave upward to concave downward are called points of inflection. Determine the points of inflection for the function \( y = x \cos(x) - x \) within 0.01 on the interval \([-6, 6]\).

What is true about the sign of the second derivative near a point of inflection?

It is possible to have a second derivative which is zero at a given point and yet the point not be a point of inflection. By investigating functions of the form \( y = x^n \), find a function whose second derivative is zero at \( x = 0 \) and which is concave upward everywhere.

6. Sketch a graph of a function \( f \) that is decreasing everywhere and is concave upward everywhere. You do not have to find an equation for the function.

Sketch a graph of a function \( f \) that is increasing everywhere and is concave downward everywhere.

Sketch a graph of a function \( f \) that is decreasing and for which the slopes of the tangent line are decreasing.
Laboratory Four: Further Explorations of Derivatives

7. In a short paragraph, explain how the fact that the second derivative at a point is positive implies that the function is concave upward at that point. Begin by describing what you know about the first derivative if the second derivative is positive. Relate the behavior of the first derivative to the characteristic of being concave upward. Finish your paragraph with a summary statement.

8. We have seen that when a differentiable function has a maximum or minimum at a point, the first derivative must be zero. But the first derivative being zero at a point does not guarantee a maximum or minimum. Find a function whose first derivative is zero at a point but does not have a (relative) maximum or minimum at that point. Justify your answer.

9. Why do you think we study the first and second derivatives of functions? Can you think of any applications of the information we have gained so far?
Laboratory Five: More Uses of the Derivative

Laboratory Five

Name: _____________________

More Uses of the Derivative

So far your use of the derivative has been to describe and determine information about the behavior of a function. In this lab, you will apply and extend your work on derivatives to solve several problems.

1. Your investigation into derivatives began with looking at linear approximations to a function and tangent lines. As a review, find the equation of the tangent line to the curve 
   \( y = x^2 \sin(x) - 3x + 2 \) at \( x = 1 \). Attach a graph of the function and the tangent line together.

   The equation of the tangent line to the curve at \( x = 1 \) is ______________________.

2. Observe the graph of this function intersects the x-axis near \( x = 1 \). Explain how you could use the tangent line above to approximate where the zero of this function occurs.

   Using your method, approximate the zero of this function. Calculate the true value of the function at this point. Do you think this value is close enough to zero? Explain.

   Repeat this process by finding the equation of the tangent line at your new approximation, and obtaining a second approximation. Calculate the true value of the function at this point.

3. You can also use the tangent line to approximate values of the function. This is a technique used by all types of scientists, psychologists, and engineers. For example, you may have selected observations of population size at various times, and you wish to determine a reasonable population size at a time when no observation was made.

   Assume you wish to find an approximation to a function \( P \) near \( t = 36 \), and we know that \( P(39) = 7 \) and \( P'(39) = 0.65 \). Draw a diagram or picture which represents the given information. By finding the equation of the tangent line at \( t = 39 \), determine an approximation to \( P \) at \( t = 36 \). Illustrate on your picture your approximation.

   Based upon the magnitude of \( P \) and \( P' \), do you feel this will be a reasonable approximation? What additional information might be relevant in your decision?
Laboratory Five: More Uses of the Derivative

4. Complete the following table. You may check your function $f(x)$ by using Theorist to substitute $u(x)$ into $v(x)$. You should use Theorist to find any derivative about which you are uncertain.

<table>
<thead>
<tr>
<th>$u(x)$</th>
<th>$u'(x)$</th>
<th>$v(x)$</th>
<th>$v'(x)$</th>
<th>$f(x) = v(u(x))$</th>
<th>$f'(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^2 + 3x - 4$</td>
<td></td>
<td>$\sin(x)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x^7$</td>
<td></td>
<td>$\cos(x)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\cos(x)$</td>
<td></td>
<td>$x^7$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$3x$</td>
<td></td>
<td>$x^7$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\cos(x)$</td>
<td></td>
<td>$\sin(x)$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Use this table to develop a general rule for the derivative $f'$ where $f(x) = v(u(x))$ in terms of the functions $u$, $u'$, $v$, and $v'$. You will need to compose and multiply these functions appropriately.

Describe your rule in sentence form using phrases such as 'derivative of the outside', 'derivative of the inside', 'evaluated at', and 'multiplied by'.

Using your rule, determine the derivative of the function $h(x) = (f(x))^5$ in terms of $f$ and $f'$.

5. A rule for finding the derivative of a quotient has not yet been discussed. In this problem, you will explore one method of finding the derivative of a quotient by using the product rule and your rule from item 4. Rewrite the function $f(x) = \frac{\sin(x)}{x^2 + 1}$ as a product of two functions $h(x)$ and $g(x)$. This will require the use of a negative exponent.

$f(x) = h(x) \cdot g(x)$ where $h(x) =$ ___________ and $g(x) =$ ___________.

Use the rules we have developed, including item 4, to find the derivative of $f$.

$f'(x) =$ ___________________________________.

Use Theorist to compute the derivative. Do the answers agree? If the two answers are not of the same form, explain how you could determine if the answers are equivalent. Are they equivalent?
Laboratory Five: More Uses of the Derivative

6. The generalized rule for finding the derivative of a quotient function \( f(x) = \frac{u(x)}{v(x)} \) is

\[
f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{(v(x))^2}.
\]

Use this formula to verify your answer to question 5.

Many students find this formula difficult to remember correctly. Below are the two common difficulties. From your work for item 5, give a rationale (not a proof) for each.

Why is the numerator a difference (-) rather than a sum (+)?

Why is the denominator squared?

7. Find the equation in variables \( x \) and \( y \) of a circle with radius 7 and center at (2, 3).

To find the equation of the tangent line at a point, you could solve for \( y \) (choosing either the top half or the bottom half appropriately), and proceed using the differentiation rules you have thus far. However, this is not necessary, and with some equations, this is not possible. You can implicitly differentiate the equation and then solve for the derivative of \( y \) with respect to \( x \). Enter the equation for the circle. Highlight both sides of the equation but not the equal sign itself. Click on the derivative sign from the function palette. Choose EXPAND. Using Theorist, isolate the terms involving the derivative. Refer back to previous labs if you have forgotten how to move terms to the other side of the equation. Use COLLECT from the MANIPULATE menu. Note the important difference between COLLECT and FACTOR.

\[ \frac{dy}{dx} = \quad \text{__________}. \]

You can do this for any equation. Implicitly differentiate the equation \( x^2 + xy^2 + y\cos(x) = y \) to find \( \frac{dy}{dx} \). You do not have to use Theorist, but you may.
8. In my many travels to Fort Wayne, Indiana, I have found that the 120 mile trip almost always takes 2 hours - no matter how fast or slow I drive. What is my average velocity?

The speed limit all the way to Fort Wayne is 65 mph. Must I at some point along the trip have been going the speed you found above? Why or why not?

On toll roads, the total time you spend on the road and the distance covered is automatically recorded. Could this information be used to check if you were speeding? Imagine coming off the toll road, paying the toll, and then being handed a ticket!

9. Let \( f(x) = x^4 + ax + 1 \). Which of the following four statements are true?

I \( f \) always has a relative minimum on \( -\infty < x < \infty \).

II \( f \) never has a relative minimum on \( -\infty < x < \infty \).

III \( f \) always has a relative maximum on \( -\infty < x < \infty \).

IV \( f \) never has a relative maximum on \( -\infty < x < \infty \).

(a) I only (b) IV only (c) I and III (d) I and IV (e) II and IV

10. A conical tank has height 10 ft and the diameter of the top is 10 ft. The tank is being filled with water at the rate of 10 ft\(^3\)/min. How fast is the depth of the water changing when it is 5 ft deep?

(a) \( \frac{1}{5\pi} \) (b) \( \frac{5}{8\pi} \) (c) \( \pi \) (d) \( \frac{2}{5\pi} \)

(e) \( \frac{8}{5\pi} \) (f) \( 5\pi \)
Laboratory Six: Areas, Volumes, and the Definite Integral

Names: 1. ____________________  
2. ____________________  
3. ____________________

We now turn to another aspect of calculus, namely finding areas and volumes. There are basic formulas for finding the area and volume of various standard shapes. In this lab you will investigate how these measures can be calculated for a much wider variety of shapes.

1. A few years ago Exxon had a major oil spill. Both Exxon and ecologists were trying to assess the damage. Their models included components for currents, tides and coast line. In this lab you will determine the area of a spill while it is still at sea. For the diagram below, approximate the area of the spill. Each square represents one square mile.

Approximation: ___________

To aid in judging the precision of your approximation, determine an upper bound (over approximation) and a lower bound (under approximation) by counting complete squares only.

Upper bound: ___________
Lower bound: ___________

How do you know your upper bound and lower bound are over and under approximations?

How could we improve your upper and lower bounds?

2. Unfortunately, when a spill occurs, it is usually some time before the hole is sealed and oil continues to flow out of a tanker. Thus, investigators try to determine the volume of oil that is flowing out of the tanker.

A similar problem occurs across the nation around the beginning of summer as people prepare their pools for the dog days of summer. Suppose you are filling a swimming pool at a rate of three gallons per minute for the first 10 minutes. How much water has entered the pool?

After 10 minutes, you're getting anxious and so try to turn up the rate at which the water is flowing. The water is now flowing at a rate of 5 gallons per minute. After 10 more minutes, how much water is in the pool?
You become impatient and borrow the hose from your neighbor which adds water to the pool at a rate of 4 gallons per minute. What is the rate at which the water is flowing into the pool and how much water is in the pool 10 minutes after you added the second hose (30 minutes since you began)?

Sketch a graph of the rate at which water is flowing into the pool over the thirty minutes. Write a piece-wise defined function expressing the flow rate, $f$, of the water with respect to time $t$.

Complete the following statement.

The amount of water in the pool over a time interval is the product of:

Explain how the amount of water in the pool can be considered as the area under the curve in your sketch.

Write an expression for the amount of water in the tank after 30 minutes in terms of $f(10), f(20),$ and $f(30)$.

Of course, if you are filling a small wading pool, using a garden hose is okay. But large in-ground pools are filled by water trucks. Suppose the graph gives the flow rate of water from a truck. The flow rate will depend upon the size of the pipe and amount of water left in the tank.

Give an upper bound of the amount of water dispensed after 30 minutes by approximating the amount of water pumped at 15 minute intervals.

Approximate the amount of water in the pool after 30 minutes using 10 minute intervals.
Laboratory Six: Areas, Volumes, and the Definite Integral

Approximate the amount of water in the pool after 30 minutes using 5 minute intervals.

As the length of the time intervals decrease, how does the approximation compare to the area under the curve from \( t = 0 \) to \( t = 30 \)?

Assuming the flow rate at time \( t \) is given by a function \( f(t) \), write your approximation for the amount of water in the pool using 5 minute intervals in terms of \( f(5), f(10), \ldots, f(30) \).

3. The same process can be used to determine the area under a curve for any function. For this problem consider the function \( f(x) = 16 - x^2 \) for \( x \) between 1 and 4. The Maple commands `leftbox` and `rightbox` are useful for our investigation. These commands are part of a group of procedures written for calculus. Type the following commands and print one or both of the plots produced.

\[
\begin{align*}
\text{• with (student);} & \quad [\text{This tells Maple to retrieve the stored procedures}] \\
\text{• } f := \text{proc}(x) 16 - x^2 \text{ end;} & \quad [\text{Defining the function as a procedure}] \\
\text{• } \text{leftbox}(f(x), x=1..4, 5); & \\
\text{• } \text{rightbox}(f(x), x=1..4, 5); & \\
\end{align*}
\]

How does the procedure `leftbox` use the range \( x=1..4 \)? What is the purpose of the third item or parameter, in this case a '5'?

For this function \( f(x) \), how does the area of the rectangles drawn by the `leftbox` command compare to the area under the curve from \( x = 1 \) to \( x = 4 \)? How do the `leftbox` and `rightbox` commands differ?

If you wish to use 5 intervals in the approximation, what would be the width of each interval?

Write an equation (of sums and products) which expresses the approximate area under the curve from \( x = 1 \) to \( x = 4 \) using these 5 intervals. What is the approximate area under the curve?
Laboratory Six: Areas, Volumes, and the Definite Integral

If you wish to use 10 intervals, what would be the width of each interval?

Write an equation approximating the area using 10 intervals.

Looking at the endpoints of the intervals above, what pattern emerges?

Writing the expression for 100 intervals would be tedious and unproductive. Invent a short hand notation for writing your equation approximating the area using 10 intervals. Use your notation to write an approximation for 100 intervals.

If you were interested in determining the area from $x = a$ to $x = b$ using $n$ intervals, what would be the width of each interval?

What are the end points of the first interval? of the second interval? of the third interval? of the $i$-th interval?

For the function $f(x) = 16 - x^2$, which of the two commands leftbox or rightbox produces rectangles which can be used to over approximate the area of interest? Which of the two commands produces rectangles which would lead to an under approximation?

Give an example of a function for which the roles of being an over or under approximation would change for the leftbox and rightbox commands.
PART TWO: THE DEFINITE INTEGRAL

4. The sigma notation is useful for writing sums of terms with a pattern. For example:

\[
\sum_{i=1}^{3} 6 = 6 + 6 + 6 \quad \sum_{i=1}^{3} i = 1 + 2 + 3 \quad \sum_{i=1}^{3} 6i^2 = 6 \cdot 1^2 + 6 \cdot 2^2 + 6 \cdot 3^2
\]

\[
\sum_{i=0}^{2} \cos(i) = \cos(0) + \cos(1) + \cos(2) \quad \sum_{i=0}^{2} f(a + i) = f(a + 0) + f(a + 1) + f(a + 2)
\]

*Maple* allows you to enter sums using the command *sum*. For example the sum \(\sum_{i=1}^{3} i = 1 + 2 + 3\) is equivalent to the command "sum(i, i=1..3)" and produces a result if possible. The command *Sum* can be used to enter and see the appearance of a sum in *Maple* which can then be evaluated using the command *value*. For each of the sums below, write the sum using sigma notation and then use *Maple* to evaluate the sum. Next to each sum, suggest a way of judging the results for correctness.

\[1 + 2 + 3 + \ldots + 500 = \sum_{i=1}^{500} i = \ldots\]

\[\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{100} = \sum_{i=2}^{100} \frac{1}{i} = \ldots\]

\[\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{4096} = \sum_{i=1}^{\infty} \frac{1}{2^i} = \ldots\]

Use the sigma notation to write the approximation you found in question 3 using 10 intervals.

Enter and evaluate this expression in *Maple*.

Similar to the *leftbox* and *rightbox* commands, the *leftsum* and *rightsum* commands can be used to write in sigma notation an approximation for the area under a curve using rectangles. Issue the command "leftsum(f(x), x=1..4, 10);" or "rightsum(f(x), x=1..4, 10);". How does the result compare to your expression?

Use the command *value* instead of *evalf* to evaluate the results from a *leftsum* or *rightsum* command.
Complete the following table (refer back to problem 3) using Maple.

<table>
<thead>
<tr>
<th>Number of Intervals</th>
<th>Approximate area under the curve of the function $f(x) = 16 - x^2$ from $x = 1$ to $x = 4$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td></td>
</tr>
<tr>
<td>1000</td>
<td></td>
</tr>
</tbody>
</table>

What number do these values appear to be approaching?

5. As the number of intervals increases, what happens to the length of the intervals?

The exact area is the limit of the approximations as the number of intervals $n$ gets large or equivalently as the size of the intervals gets smaller. As $n$ approaches infinity, the length of the intervals approaches zero. It is customary to investigate the limit as a function of the number of intervals rather than the length of the intervals. When the limit exists, it is denoted by the definite integral $\int_a^b f(x) \, dx$ where $a$ and $b$ are the end points of the region.

If you think of $dx$ as the width of the interval (think 'Δx') and $f(x)$ as the height of the rectangle at a given value of $x$, then the product $f(x)dx$ represents the small (rectangular) change in area. The integral sign $\int$ denotes summing the areas of the rectangles and taking the limit of the sum as the number of rectangles increases (note how it resembles the 'limit of a capital letter S'). One possible way of denoting this process is by the equation below.

$$\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(a + i \cdot \frac{b-a}{n}) \cdot \frac{b-a}{n}.$$  

Approximating areas by summing up the area of lots of rectangles is something well suited for a computer. Maple can calculate definite integrals using this concept - although the algorithms used are much more sophisticated than just using rectangles. The corresponding Maple command is:

- `int(f(x), x = a..b);`  
  [The definite integral of $f$ from $x = a$ to $x = b$.]

Use the Maple `int` to find the area of problem 3 using the definite integral $\int_1^4 (16 - x^2) \, dx$. How does the value obtained compare with your results at the end of item 4?
6. Compute the area of a semicircle of radius 3 centered at the origin by using the definite integral. You will need to determine the equation for the top half of the semicircle. Include in your answer the definite integral you used.

Compute the area of the semicircle above using the known rule for the area of a circle. Do the values agree?

7. For the following questions refer to the sketch. By thinking of the definite integral \( \int_a^b f(x) \, dx \) as the area under the curve from \( x = a \) to \( x = b \), write the following as a single definite integral.

\[
\int_a^c f(x) \, dx + \int_a^b f(x) \, dx =
\]

\[
\int_a^c f(x) \, dx - \int_a^b f(x) \, dx =
\]

How should we define \( \int_a^b f(x) \, dx \)? Give a justification of your answer.

How do you think the value of \( \int_a^b f(x) \, dx \) and \( \int_a^c f(x) \, dx \) should compare?

Test your conjecture by using Maple to compute the last two definite integrals for the function \( f(x) = 16 - x^2 \) from \( x = 1 \) to \( x = 4 \). If your conjecture is incorrect, reformulate it based on the computer evidence. Give a plausible explanation of this relationship.
8. Complete the following table using *Maple* to calculate the definite integrals.

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>$\int_0^1 f(x),dx$</th>
<th>$g(x)$</th>
<th>$\int_0^1 g(x),dx$</th>
<th>$h(x)=f(x)+g(x)$</th>
<th>$\int_0^1 h(x),dx$</th>
<th>$p(x)=f(x)-g(x)$</th>
<th>$\int_0^1 p(x),dx$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>$x$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$7x^2$</td>
<td></td>
<td>$3x^2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sin(x)$</td>
<td></td>
<td>$x$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$5\sin(x)$</td>
<td></td>
<td>$\cos(x)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Make a conjecture on how the sum $\int_a^b f(x)\,dx + \int_a^b g(x)\,dx$ can be expressed as a single definite integral involving $f$ and $g$.

Make a conjecture on how the difference $\int_a^b f(x)\,dx - \int_a^b g(x)\,dx$ can be expressed as a single definite integral involving $f$ and $g$.

Thinking of the definite integral as the area under a curve on an interval, give a graphical explanation for your above conjecture.

Compute $\int_0^1 (-x)\,dx$. Graph the function $y = -x$ and shade the region of interest.

Use your conjecture above to explain why the definite integral of a region lying below the $x$-axis should be "negative". Hint: Consider also the function $f(x) = 0$.

By using the geometric interpretation of the definite integral, determine the definite integral $\int_2^3 10\,dx$. Make a conjecture about the definite integral $\int_a^c dx$ for any constant $c$. Give an explanation for your conjecture.
Laboratory Six: Areas, Volumes, and the Definite Integral

Using the patterns in your table and the fact that the definite integral \( \int_a^b f(x) \, dx \) of a certain function \( f \) has a value of 10, determine the value of the definite integral \( \int_a^b (5f(x) - 6) \, dx \).

9. Use Maple to determine the area bounded by the two curves \( y = \cos(x) \) and \( y = \frac{x}{3} \). To do so you will need to determine the intersection points graphically. Check your answer by making a rough estimate of the area from the graph.

PART THREE: VOLUMES

Recall that the definite integral \( \int_a^b f(x) \, dx \) can be thought of as the limit of adding up little changes of area \( \Delta A \) given by \( f(a + i \Delta x) \Delta x \) as the width of the intervals, \( \Delta x \), gets smaller and smaller. This idea can be used to determine certain volumes.

10. Maple's 3-dimensional graphics are useful for visualizing a curve rotating about an axis. If you are already in Maple, restart the program so that you get a new worksheet. Issue the command:

\[ \text{read 'rotations.m';} \]

The graph of the function \( f(x) = \text{abs}(x - 1) + 1 \) on the interval \([0, 2]\) is given at the right. Sketch by-hand what you think will be the appearance of the shape generated by rotating the curve about the \( x \)-axis and the shape generated by rotating the curve about the \( y \)-axis.

Enter the function \( f \) as a procedure. Issue the following two commands to see how Maple can present the resulting surfaces.

\[ \text{rotate_about_x_axis (f, 0, 2);} \quad \text{and} \quad \text{rotate_about_y_axis (f, 0, 2);} \]

How do these compare with your sketches?
Laboratory Six: Areas, Volumes, and the Definite Integral

When Maple finishes the plot, click and hold the mouse in the middle of the plot. Move the mouse down a bit and release the button. You may also wish to vary some of the 3-D plot options from the format menu. What changes aid in visualizing the surfaces?

For the remainder of these questions, instead of considering the surface generated by rotating a curve about an axis, you will consider the volume generated by rotating the area bounded by the curve and the axis of rotation. Keep in mind Maple can only show us the surface generated. You must mentally fill in the surface to create a solid.

For the given function and the rotation about the x-axis, if you slice the object vertically (with a plane perpendicular to the x-axis), what do the cross sections resemble? Hint: it is a very familiar two dimensional shape.

What determines the dimensions of the cross section? What is the area of the cross section if you slice the solid at \( x = 1 \)?

What is the area of cross section if you slice the solid at \( x = a \)? Note: use \( f(a) \) in your answer.

Suppose the slice has thickness \( \Delta x \). Find an expression for \( \Delta V \), the volume of this slice.

\[
\Delta V = \frac{}{\Delta x}.
\]

Therefore the volume of the object generated by rotating the curve \( f(x) \) between \( x = 0 \) to \( x = 2 \), could be calculated by the definite integral

\[
V = \int_0^2 \frac{}{dx}.
\]

Use this to determine the volume of this object.

11. The surface of a sphere can be generated by rotating a semicircle about a diameter. Modify the function in your notebook from item 10 so that it is the semicircle of question 6. When the curve between -3 and 3 is rotated about the x-axis, does it look like a sphere? Use Maple to compute the volume of a sphere using a definite integral similar to the one above. Check your answer by computing the volume of a sphere of radius 3 using the standard formula.
Laboratory Six: Areas, Volumes, and the Definite Integral

12. For each of the following functions, view the curve on the given interval about both axes. Sketch (by hand) the appearance of both rotations. You may use Maple to plot the function and/or to plot the surfaces. The goal here is to develop a little skill in drawing 3-dimensional surfaces.

a. \( y = |x - 1| + 1 \) on \([0, 2]\)

b. \( y = 2 + \cos(x) \) on \([0, \pi]\)

c. \( y = x \) on \([2, 3]\)

d. \( y = \sqrt{x} \) on \([0, 1]\)

13. For each function in item 12, write a definite integral which will compute the volume of the resulting solid when the function is rotated about the \(x\)-axis.

a. 

b. 

c. 

d. 

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PART ONE: DISTANCE FROM VELOCITY

Imagine the following trip from Kalamazoo to Fort Wayne. Frank Burns starts off from Kalamazoo driving 65 mph. After one hour, he hits construction at rush hour and for the next hour travels at 30 mph. Finally turning off, he travels for one more hour down a rural road at 55 mph. Sketch a reasonable graph of Frank's velocity \( v(t) \) as a function of time. Will this graph be discontinuous?

Approximate the total distance traveled by Frank. How does this approximation relate to the graph of the velocity function above?

How far had Frank traveled after 30 minutes? after 90 minutes? after 2 hours?

Is it possible to determine the distance \( s \) that Frank has traveled for any time \( t = b \) during the first three hours? Explain.

How does the distance \( s(t) \) traveled by Frank from Kalamazoo relate to the area under the curve? State this relationship in words and mathematically using a definite integral and the functions \( s \) and \( v \). Explain the limits of integration that you choose.

What was the relationship found in your work with derivatives between the distance function \( s(t) \) and the velocity function \( v(t) \)? State this relationship in words and mathematically using proper notation.
Laboratory Seven: The Indefinite Integral

Combine the two statements on the previous page to conjecture a relationship between the derivative of a definite integral and the velocity function.

If Frank originally came from Chicago, he would have an initial distance of 120 miles traveled. How would this affect the velocity function above measured from Kalamazoo? How would the distance function change? Include in your explanation related properties of the derivative.

Is it possible to determine the initial displacement of an object given its velocity as a function of time? If so, explain how, and if not, explain why not.

2. For any function $f(t)$, we can associate a new function $A(x)$ which is the definite integral of $f$ from $t = 0$ to $t = x$. Complete the following equation.

$$A(x) = \int_{0}^{x} _______ \, dt$$

Keep in mind that we can associate the definite integral with the area under the curve - with some caution when the curve lies below the x-axis.

Without doing any calculations, what should be the value of $A(0)$?

Using properties of the definite integral, write a definite integral which is equivalent to the expression $A(7) - A(4)$.

For the remaining questions use the function $f(t) = t^2 + 1$. Enter both functions $f$ and $A$ in Theorist. Don't forget to use bold letters where appropriate and to define both $f$ and $A$ as functions. When entering the definite integral to define $A$, $t$ should not be bold since it is not the variable of the function.
Laboratory Seven: The Indefinite Integral

Obtain a graph of \( y = A(x) \) on the range \([-2, 2]\). What happens to the graph when the lower limit of the definite integral defining \( A \) is changed from 0 to 1? to 5? to -5? What plausible explanation can you give for this? Here it will be beneficial to examine the four plots simultaneously.

Graph \( y' = \frac{\partial}{\partial x} A(x) \) and \( z = f(t) \). How do the graphs of these two functions compare? State the relationship in words and then mathematically without using \( A, A', y', \) or \( z \).

PART TWO: ANTIDERIVATIVES

3. You know several rules for calculating derivatives. These rules can be "used in reverse" to find functions having a specific derivative. For each function \( f \) in the table, find a function \( F \) whose derivative \( F' \) is the function \( f \), i.e. \( F'(x) = f(x) \) using your knowledge of derivatives.

<table>
<thead>
<tr>
<th>( f(x) )</th>
<th>( F(x) )</th>
<th>( f(x) )</th>
<th>( F(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2x</td>
<td>5</td>
<td>( 3x^2 - 4x + 3 )</td>
<td>( \cos(x) )</td>
</tr>
<tr>
<td>( 2x\sin(x) + x^2\cos(x) )</td>
<td>( \sin(x) )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Hint: The first entry in the last row was the result of differentiating a product. Check your answers by finding the derivative of \( F(x) \) using Theorist. Are they equal to \( f(x) \)?

We call a function \( F \) an antiderivative of the function \( f \) if \( F'' = f \).

4. Let \( f \) be the function defined by \( f(t) = 3t^2 - 4t \). Find an antiderivative \( F_1 \) of the function \( f \).

Find a second function, \( F_2 \), which is also an antiderivative of \( f \).

Check both antiderivatives by differentiating the function either by hand or using Theorist.
Laboratory Seven: The Indefinite Integral

Define the function $A$ by $A(x) = \int_{t}^{x} f(t) \, dt$. Complete the following table by evaluating each function at the given value of $x$. If all three functions are entered as functions in Theorist, then the first row can be obtained by using $A(1), F_1(1),$ and $F_2(1)$, and the other rows similarly.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$A(x)$</th>
<th>$F_1(x)$</th>
<th>$F_2(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

5. Using the table of question 4, conjecture a relationship between the functions $A$ and $F_1$. Does knowing $F_1(1)$ help in the conjecture?

Using the table of question 4, conjecture a relationship between $A$ and $F_2$. Does knowing $F_2(1)$ help in the conjecture?

If $F$ were yet another antiderivative of $f$, what relationship should exist between the functions $A$ and $F$?

Write the relationship above mathematically using only the definite integral and the antiderivative $F$. 

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6. Use this relationship to calculate the area under the curve $y = 3x^2 - 2$ from $x = 1$ to $x = 3$. Specifically, find an antiderivative and use it to evaluate the definite integral for the given area.

Use Theorist to compute the definite integral to check your work.

In questions 1 and 2, you should have found certain relationships between the definite integral and the derivative. We saw how the distance function $s$ could be determined using the definite integral of the velocity function $v$ - up to the initial displacement. But we also know that the velocity function is the derivative of the distance function. This can be summarized by the First Fundamental Theorem of Calculus.

Given a continuous function $f$ and if we define the function $A(x)$ by $A(x) = \int_0^x f(t)dt$, then

$$A'(x) = \frac{d}{dx} \int_0^x f(t)dt = f(x)$$

In question 5, we saw that an antiderivative can be useful for evaluating definite integrals. We also saw that there are many antiderivatives for a given function and any two differ by a constant. This can be summarized in the Second Fundamental Theorem of Calculus.

If $F$ is any antiderivative of a continuous function $f$, that is $F'(x) = f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Since any two different antiderivatives differ only by a constant, we refer to the family of all antiderivatives by the term indefinite integral. We denote the indefinite integral or family of all antiderivatives of a function $f$ by $\int f(x)dx$. If we can find one antiderivative $F$ of a function $f$, then any other antiderivative differs from $F$ by just a constant. Thus we also denote the family of all antiderivatives of $f$ by $F(x) + C$. That is,

$$\int f(x)dx = F(x) + C.$$

For example the indefinite integral $\int 2x\,dx$ is the family of quadratics $x^2 + C$. However, finding an antiderivative of a function is not always easy. In fact much of Calculus II, second semester calculus, is devoted to finding indefinite integrals.
Laboratory Seven: The Indefinite Integral

7. The graph of a function \( y = f(t) \) is shown on the right. Let \( F \) be the function defined by the equation \( F(x) = \int_a^x f(t) \, dt \).

Is \( F(c) \) greater than or less than \( F(b) \)? Explain.

Is \( F(d) \) greater than or less than \( F(c) \)? Explain.

Is \( F(e) \) greater than or less than \( F(d) \)? Explain.

Determine all relative extrema for the function \( F \). Explain your choice(s).

Determine all points of inflection for the function \( F \). Explain your choice(s).

Draw a rough sketch of the graph of \( y = F(x) \).
Laboratory Seven: The Indefinite Integral

8. Finding antiderivatives or indefinite integrals is not always easy. For each function below, try to find an antiderivative. Use Theorist to check your answer by taking the derivative and checking it against the original function.

a. \[ \int (7x - 4) \, dx \]

b. \[ \int (2\sin(x) + 4x^2) \, dx \]

c. \[ \int x^2(7x^3 - 4)^3 \, dx \]

d. \[ \int \cos(x)\sin(x) \, dx \]

e. \[ \int \sqrt{9 - x^2} \, dx \]

Give an ordering of these from most difficult to least difficult. ______________________

What made the most difficult indefinite integral challenging?

What made the next difficult indefinite integral challenging?
Laboratory Seven: The Indefinite Integral

9. In the Calculus I folder, you will find a notebook called "Approx Int" which contains the formulas for the left, right, midpoint, trapezoid and Simpson's approximation for a definite integral. Use this notebook to complete the following table of approximations for the definite integral
\[ \int_{1}^{3} (x^3 - 3x + 4) \, dx. \]

<table>
<thead>
<tr>
<th>n</th>
<th>Left</th>
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<th>Midpoint</th>
<th>Trapezoid</th>
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Compute the actual area using the Second Fundamental Theorem of Calculus.

Complete the following table of approximations for the definite integral
\[ \int_{0}^{2} 4\sqrt{x} \, dx. \]

<table>
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<tr>
<th>n</th>
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Note: This integral can not be computed using the Second Fundamental Theorem.

Given the table, approximate the value of the definite integral \( \int_{0}^{2} 4\sqrt{x} \, dx \) and specify the accuracy of your answer.

Considering both tables, how do the various approximations compare?
This Theorist notebook was used in conjunction with Lab 2. This notebook will animate secant lines to a function through one fixed point.

**INSTRUCTIONS:** Highlight just the left hand side of the equation \( b = 3 \). Choose ANIMATE from the GRAPH menu. Set the Speed to 1 frames per second (fps). Set the value of \( b \) to vary FROM 3 UP TO 1.

Observe what is happening to the secant line as the value of \( b \) approaches the value of \( a \). The values of \( b \) are shown in the upper right hand of the graph during animation.

If you 'trash' this notebook beyond repair and wish to start over. Close the notebook and reopen it. I have turned off the Icons (Show Icons option from the Pref menu) to make things more readable.

Instead of making lots of substitutions, we can use Theorist's function definition capabilities. To modify the function or to enter your own function you must use the bold shadow variables from the variable palette menu. You may have to adjust the graphing window below by zooming in/out. You may also wish to try different values for \( a \) and \( b \).

\[ f(x) = x^2 - 1 \]

The next two lines specify the constant values for \( a \) and \( b \).

\[ a = 1 \]

\[ b = 3 \]

However to graph a function define as above, we still must assign it to another variable.

\[ y = f(x) \]

Normally this line would contain a comment describing the next line. This is your job as part of the lab.

\[ m = \frac{f(b) - f(a)}{b - a} \]

Normally this line would contain a comment describing the next line. This is your job as part of the lab.

\[ s = m(x - a) + f(a) \]
The following Theorist notebook was used in exploring surfaces and volumes of revolution. It is reproduced here at 75% of normal size.

This notebook shows the rotation of a function about the \( x \)-axis and about the \( y \)-axis on a specified interval. The portion of the curve to be rotated is given by the initial and final values below.

\[
\begin{align*}
\alpha &= 0 \\
\beta &= 2
\end{align*}
\]

When rotating about the \( x \)-axis, the \( y \)-ranges is set from \(-y_2\) to \(y_2\). When rotating about the \( y \)-axis, the \( y \)-range is set from \(y_1\) to \(y_2\).

\[
\begin{align*}
y_1 &= 0 \\
y_2 &= 2
\end{align*}
\]

TRY NOT TO MODIFY THE GRAPHS - INSTEAD CHANGE THE RANGES ABOVE. However in the graph details you may wish to change the following options depending upon the function.

- True Proportions <-> Stretch to Fit
- Translucent <-> Illuminated and Mesh <-> No Mesh

TO SEE THE CURVE ROTATED - HIGHLIGHT the variable Angle and the graph containing the rotation you desire. Choose ANIMATE from the graph menu.

And of course, we need to specify the function.

\[ y = |x - 1| + 1 \]

Rotated about the \( x \)-axis.

Rotated about the \( y \)-axis

\[
\begin{align*}
\text{Angle} &= 0
\end{align*}
\]
The following Theorist Notebook was provided as examples of how to code various methods of approximating definite integrals and how to create a function which is defined using a definite integral. This notebook is distributed after the methods are discussed.

\[
\boxed{\int_a^b f(x) \, dx}
\]

- **Left** \((a, b, n)\): 
  \[
  \sum_{i=0}^{n-1} f(a + i \frac{b-a}{n}) \frac{b-a}{n}
  \]

- **Right** \((a, b, n)\): 
  \[
  \sum_{i=1}^{n} f(a + i \frac{b-a}{n}) \frac{b-a}{n}
  \]

- **Mid** \((a, b, n)\): 
  \[
  \sum_{i=0}^{n-1} f(a + i \frac{b-a}{n}) \frac{b-a}{n}
  \]

- **Trap** \((a, b, n)\): 
  \[
  \frac{1}{2} \left( f(a) + \sum_{i=1}^{n-1} f(a + i \frac{b-a}{n}) \frac{b-a}{n} \right) + f(b) \frac{b-a}{n}
  \]

- **Simp** \((a, b, n)\): 
  \[
  f(a) + \sum_{i=1}^{n-1} \left[ 2^{\text{mod}(i, 2)} + 1 \right] f(a + i \frac{b-a}{2n}) + f(b) \frac{b-a}{n}
  \]

- \(F(x) = \int_a^b f(x) \, dx\)
The following image presents the parallel information given to the Maple to that of the Theorist Notebook for Laboratory 2. It is reproduced here at 80%.

Worksheet "secants"

```maple
secants := proc(f, a, b)
    local s, c, m;
    s := (f(x));
    for c from b by -(b - a)/5 to a + (b - a)/5 do
        m := (f(c) - f(a))/(c - a);
        s := s union {m * (x - a) + f(a)}
    od;
    plot(s, x = a-(b-a)..a+(b-a));
end;
```

```maple
plot(x=a-(b-a)..a+(b-a));
```

```maple
f := proc(x) x^2+1 end;
```

```maple
secants(f, 1, 3);
```

The following images illustrate the parallel worksheets for surfaces and volumes of revolution for Maple. The images are 90% of their original size.
with(plots):

rotate_about_y_axis := proc (f, a, b)
cylinderplot([x, t, f(x)], t=0..2*Pi, x=a..b, style=PATCH,
orientation=[0,60], axes=NORMAL)
end:

rotate_about_x_axis := proc (f, a, b)
cylinderplot([f(x), t, x], t=0..2*Pi, x=a..b, style=PATCH,
orientation=[0,60], axes=NORMAL)
end:

f := proc(x) abs(x-1)+1 end;
f := proc(x) abs(x-1)+1 end

plot(f, x=0..2);

rotate_about_x_axis(f, 0, 2);

rotate_about_y_axis(f, 0, 2);
Appendix B

Pretest and Posttest Instruments
Directions for Instructors Participating in the Calculus Research

Dates:

Pretest: January 4, 1993 [First day of class]
Posttest: April 6, 1993

It is very important that all students take the exam on the specified date or at the first possible opportunity. Direct any late arriving students to see me in 4406 Everett Tower. I will be available to proctor individual exams for students who miss the scheduled exam period. Until all students have taken the exam, continue to ask at the beginning of the hour if there are any students who have not taken the exam. I will be outside the classroom at the end of the hour to schedule a time with these students. It is important, especially for the pretest, that students take the exam before sitting through as little classroom discussion as possible.

For the first day, identify yourself, hand out a syllabus, and give any initial assignment. Please start the pretest administration at 10 minutes into the hour. On average, most students take 40 minutes for the pretest. For the posttest, start the exam immediately at the beginning of the hour. Most students will use the entire hour for the posttest. Please include in your syllabus a statement similar to the one below:

There will be [four] exams ... A [fifth] exam will be given on April 6, 1992. Performance on this exam will be factored in when determining borderline grades. A uniform comprehensive final will ...

These questions are likely to be asked by students.

- How will the pretest affect my grade?
  The pretest provides a measure of your readiness for calculus. It will be used primarily as a diagnostic tool.

- How will the posttest affect my grade?
  The posttest will give an indication of the progress of the student since the beginning of the semester. It will provide another performance indicator that will be factored in when determining borderline grades.

- I did very poorly on the pretest and am worried about how well I will do in this course. - or - I do not know (remember) anything about trig. Should I drop 122 and take 118?
  These questions occur without the pretest, but occur more frequently because of the pretest, and should be handled on an individual basis. Reassure these students that we will take into account some background deficiencies as we teach the course.
Procedure for Administering the Pretest and Posttest

• State the following purpose to the students.
  Pretest:  
The purpose of this exam is to determine students' readiness for calculus. This information will be used to guide instruction throughout the semester and to identify specific topics which students need to review.
  Posttest:  
The purpose of this exam is to determine students' understanding of calculus. The information will indicate students' progress since the beginning of the semester, and will provide information to aid students in preparing for the final. It will also provide another performance indicator that will be factored in when determining borderline grades.

• Instruct all students to clear their desk - only a pencil or pen is necessary. Calculators or 3x5 note cards are NOT permitted.

• Instruct all students to read the directions but not to begin the exam. Allow 2-4 minutes.

• Ask if there are any questions over the directions. Clarify any questions regarding the directions. In past administrations of this exam, there have been no questions. Record any questions asked.

• Stress: It is important to attempt every problem and not to spend too much time on any one problem.

• Indicate that the student has until the end of class time to finish.

• State that you will be willing to answer questions during the exam.

• After beginning the exam, wander around the room for at least the first 5 minutes, and frequently thereafter to answer questions. Record any questions asked and the clarification given as soon as possible during the exam. You may answer questions regarding the nature of responses: more than one response may be correct, fill in the empty boxes, partial credit will be given. Do not answer questions which clarify mathematical content or symbols. If you are uncertain, a good respond is "I'm sorry, I can't answer that."

• Collect any remaining exams at five minutes till the hour (five additional minutes after the period ends).
MATH 122: Calculus I Pretest

DIRECTIONS

This test is intended to provide information on your understanding of precalculus topics. You are to record your answers directly on the test pages. Do not spend too much time on any one question. DO NOT SIMPLIFY YOUR WORK. This test contains three types of items as illustrated below.

One group consists of multiple choice items. For these items, there may be more than one response that is correct. Circle the letter of ALL correct responses.

Sample:

Which of the following values of $x$ satisfy $x + 2 > 0$?

- $0$
- $-1$
- $2$
- $-2$
- $1000000$
- $-4$

A second group consists of completion or short answer items. Record your work in the space given.

Samples:

Given the relationship $y = x + 2$, complete the following statement.

When $x$ is 0, the value of $y$ is ______.

What is the sum of the first three positive integers? ______

because $1 + 2 + 3 = 6$

In a third group of items, you are to supply missing steps and missing information. For missing steps, record what you think a natural intermediate step would be.

Sample:

Fill in the missing steps and missing information to make the equations true.

$3 \left( \frac{12}{6} + x \right) = 3 \left( 2 + x \right) = 4 + 3x$

DO NOT BEGIN UNTIL INSTRUCTED TO DO SO.
MATH 122: Calculus I Pretest

1. Which of the following are equivalent to $|x| + 7 < 9$?

(a) $|x| + 7 < 9$  
(b) $|x| - 7 > -9$

(c) $x + 7 < 9$ and $x + 7 > 9$.  
(d) $x + 7 < 9$ or $x + 7 > 9$.

(e) $x + 7 < 9$ and $x + 7 > -9$.  
(f) $x + 7 < 9$ or $x + 7 > -9$.

2. Assuming $f(x)$ is the polynomial $x^7 + 6x^5 - 4x^3 + 8x^2 + 124$ and $n$ is the number of real solutions to $f(x) = 0$, which of the following could be possible values of $n$?

(a) 9  
(b) 8

(c) 7  
(d) 5

(e) 3  
(f) 0

3. Circle the letter of the equation(s) whose graph would look like that below.

(a) $y = (x - 3)(x + 1)$  
(b) $y = (x + 3)(x - 1)$

(c) $y = 6(x - 3)(x + 1)$  
(d) $y = 6(x + 3)(x - 1)$

(e) $y = 2(x - 3)(x + 1)$  
(f) $y = 2(x + 3)(x - 1)$

4. Circle the letter of the equation(s) whose graph would look like:

(a) $y = 2\cos(2x)$  
(b) $y = 2\cos(x)$

(c) $y = \cos(x) + 2$  
(d) $y = 2\cos(x) + 2$

(e) $y = \cos(2x)$  
(f) $y = \cos(x)$
MATH 122: Calculus I Pretest

5. Which of the following expressions are equivalent to $\frac{\cos(-x)}{-x}$?
   (a) $\cos(1)$  
   (c) $\cos$  
   (e) $0$
   (b) $-\frac{\cos(-x)}{x}$  
   (d) $-\frac{\cos(x)}{-x}$  
   (f) $\frac{\cos(x)}{x}$

6. Circle the letter of the equation(s) whose graph would look like that below.

   ![Graph](image)

   (a) $y = \frac{1}{(x - 1)(x + 1)}$
   (c) $y = \frac{5}{(x - 1)(x + 1)}$
   (e) $y = \frac{1}{(x - 5)(x + 5)}$
   (b) $y = \frac{-1}{(x - 1)(x + 1)}$
   (d) $y = \frac{-5}{(x - 1)(x + 1)}$
   (f) $y = \frac{-1}{(x - 5)(x + 5)}$

7. Which of the following expressions are equivalent to $\sqrt{a^2x^2 + a^2}$?
   (a) $ax + a$
   (c) $|ax| + |a|$
   (e) $a|ax| + a$
   (b) $|a|\sqrt{x^2 + 1}$
   (d) $a\sqrt{x^2 + 1}$
   (f) $a\sqrt{x^2} + a$

8. Which of the following functions will have a maximum value?
   (a) $f(x) = x^2 + 9x + 18$
   (c) $h(x) = \tan(x)$
   (e) $p(x) = 100 - x^4$
   (b) $g(x) = x^3 + 9x^2 + 18$
   (d) $s(x) = -x^2 - 15x - 56$
   (f) $t(x) = \sin(x)$

9. Given that the function $h(x)$ has four zeroes, how many zeroes can $h(x + c)$ have?
   (a) 4  
   (c) At least 4  
   (e) Cannot be determined without knowing $c$.  
   (b) 5  
   (d) 0  
   (f) Cannot be determined without knowing $h$.  

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10. Given that a function $f$ has the form $f(x) = \frac{x^3 + ax^2 + bx + c}{x^2 + 1}$, which of the following could be a graph of $f$?

(a)  
(b)  
(c)  
(d)  
(e)  

11. Assume $F$ stands for the number of females and $M$ stands for the number of males. Given that for participants in a recent survey at Western Michigan University, $F = \frac{5}{4} M$, complete the following statement.

There are ___________ times as many males as females.

12. Let $f$ be the function defined by $f(x) = 2\sqrt{x+7} + x^2$. Determine each of the following:

(a) $f(x + h) =$ ____________________.

(b) $f(-x) =$ ____________________.

13. If $f(x)g(x) + f(x)h(x) = 0$ and if $g(2) + h(2) = 5$, what is $f(2)$?

$f(2) =$ ______.
14. If \((x_1, y_1)\) and \((x_2, y_2)\) are two points, name the following expressions.

(a) \(\frac{y_2 - y_1}{x_2 - x_1}\) is the ______________________ between the two points.

(b) \(\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\) is the ______________________ between the two points.

15. Given the functions \(f(x) = 2x - 1\) and \(g(x) = x + 2\), what is \(f(g(f(x)))\)?

\(f(g(f(x))) = \) ______________________.

16. Let \(x_1 = 0\), \(x_2 = 1\), and for \(n \geq 3\), \(x_n = 2x_{n-2} - x_{n-1}\). Determine \(x_4\).

\(x_4 = \) ________.

17. Given the sequence of functions:

\[ f_1(x) = x + 1 \]
\[ f_2(x) = \frac{x^2}{2!} + x + 1 \]
\[ f_3(x) = \frac{x^3}{3!} + \frac{x^2}{2!} + x + 1 \]

Presuming the pattern continues, find an expression for the function \(f_n(x)\).

\(f_n(x) = \) ______________________.

18. Write an expression equivalent to \(\sum_{k=1}^{4} (k^2 - 1)\). Do not simplify your expression.

\(\sum_{k=1}^{4} (k^2 - 1) = \) ______________________.

19. During the fall semester at WMU, there were approximately six times as many students, \(S\), as parking spaces, \(P\). Write an equation involving \(S\) and \(P\) which expresses this relationship.

21. Without actually expanding, what is the likely degree of \((x - 1)^5 - (x + 2)^5\)?

22. Let \(f\) be the function given by \(f(x) = \begin{cases} x + 1, & x \leq 0 \\ 2x, & x > 0 \end{cases}\).
   (a) Name a point on the graph of \(f\).
   (b) On the axes at the right sketch a graph of \(f\).

23. What is the smallest value the expression \(\sqrt{(x - a)^2 + 3}\) can take on?

24. For what values of \(x\) will the function \(y = (x - 1)(x + 2)\) have positive value?

25. Give an equation involving \(x\) and \(y\) whose graph is that given below.
MATH 122: Calculus I Pretest

In items 26-30, supply the missing steps and missing information to make the equations true.

26.
\[ x^2 - 8x + \square = (x - 3)(x - \square) - 6 \]

27.
\[ \frac{x^3 + 7x^4 + x^3 + 8x^2 - 2x - 12}{x^2 + 2} = x^3 + \square - x + \square \]

28.
\[ \sin^2 x + 2 \cos^2 x = 1.5 \]
\[ 1 + \cos^2 x = 1.5 \]

29.
\[ (3^{a+2})^a \cdot 3 = 3 \square \cdot 3 = 3 \square = 3^{(a+1)^2} \]

30.
\[ \frac{1 + \sec \theta}{\sin \theta + \tan \theta} = \frac{\cos \theta + 1}{\cos \theta} \frac{\cos \theta}{\cos \theta} = \frac{\cos \theta + 1}{\sin \theta \cos \theta + \sin \theta} = \frac{1}{\sin \theta} \]

IF YOU FINISH BEFORE TIME IS CALLED, TURN YOUR TEST OVER.

DO NOT CHANGE ANY OF YOUR ANSWERS.
MATH 122: Calculus I Posttest

DIRECTIONS

This test is intended to provide information on your understanding of calculus topics. You are to record your answers directly on the test pages. Do not spend too much time on any one question. DO NOT SIMPLIFY YOUR WORK. This test contains three types of items as illustrated below.

One group consists of multiple choice items. For these items, there may be more than one response that is correct. Circle the letter of ALL correct responses.

Sample:
Which of the following values of \( x \) satisfy \( x + 2 > 0? \)

- (a) 0
- (b) -1
- (c) 2
- (d) -2
- (e) 1000000

A second group of items requires short answers. Record your work in the space given.

Sample:
What is the sum of the first three positive integers? \( 6 \)

because \( 1 + 2 + 3 = 6 \)

In a third group of items, you are to supply missing steps and missing information. For missing steps, record what you think a natural intermediate step would be.

Sample:
Fill in the missing steps and missing information to make the equations true.

\[
3 \left( \frac{12}{6} + x \right) = \boxed{3(x + x)} = \boxed{6} + 3x
\]

DO NOT BEGIN UNTIL INSTRUCTED TO DO SO.
1. Assuming \( f(x) \) is the polynomial \( x^6 + 7x^5 - 1024x^3 + 20414x^2 + 124000 \) and \( n \) is the number of solutions to \( f'(x) = 0 \), which of the following could be possible values of \( n \)?

(a) 7  
(b) 6  
(c) 5  
(d) 3  
(e) 1  
(f) 0

2. Which of the following functions are increasing functions over their domains?

(a) \( f(x) = x^3 - 4 \)  
(b) \( g(x) = x^2 + 4 \)  
(c) \( h(x) = \sqrt{x} \)  
(d) \( p(x) = x^5 \)  
(e) \( s(x) = \sin(x) \)  
(f) \( t(x) = \tan(x) \)

3. Circle the letter(s) of the equation(s) whose graph would look like that below.

4. Which of the following expressions are equivalent to \( \sqrt{a^2 - a^2 \cos^2(x)} \)?

(a) \( a - a\cos(x) \)  
(b) \( a| - a\cos(x)| \)  
(c) \( |a| + a\cos(x) \)  
(d) \( |a|\sqrt{1 - \cos^2(x)} \)  
(e) \( |a|\sqrt{1 - \cos^2(x)} \)  
(f) \( a\sqrt{1 - \cos^2(x)} \)

5. Given that a function \( h(x) \) has four zeroes, how many zeroes can \( h(x + c) \) have?

(a) 4  
(b) 5  
(c) At least 4  
(d) 0  
(e) Cannot be determined without knowing \( c \).  
(f) Cannot be determined without knowing \( h \).
6. Given the graph of the function $f$ on the right, which of the following statements are true?

(a) $f'(b) > 0$
(b) $f'(b) < f'(a)$
(c) $\frac{f(c) - f(a)}{c - a} < 0$
(d) $f'(a) > 0$
(e) $f'(b) > f'(a)$
(f) $\frac{f(c) - f(a)}{c - a} > 0$

7. Which of the following expressions are equivalent to $\frac{d}{dx} \left[ \int_{1}^{2} f(x) \, dx \right]$?

(a) $2 - 1$
(b) $0$
(c) $f(2) - f(1)$
(d) $f(2)$
(e) $f'(2) - f'(1)$
(f) $f'(2)$

8. Which of the following expressions are equivalent to $\frac{\sin^2(-x^2)}{-x^2}$?

(a) $\frac{-\sin^2(-x^2)}{x^2}$
(b) $\frac{-\sin^2(x^2)}{-x^2}$
(c) $\frac{\sin^2(1)}{1}$
(d) $\frac{\sin^2(1)}{-1}$
(e) $\left(-\frac{\sin(-x)^2}{x}\right)$
(f) $\left(-\frac{\sin(-x^2)^2}{x}\right)$

9. Let $f$ be the function defined by $f(x) = x^2$ and let $g$ denote a second unknown function. Which of the following statements are always true?

(a) $f(x + 2) = f(x) + f(2)$
(b) $f(-x) = -f(x)$
(c) $f(x + 2) = f(x) + 2$
(d) $f(g(x)) = g(f(x))$
(e) $xf(x) = f(x^2)$
(f) $(f - g)(x) = f(x) - g(x)$

10. Each of the following expressions is the result of taking the derivative of various functions. Which expressions are the direct result of taking the derivative of a product of two functions?

(a) $x^2 + 2x \sin(x)$
(b) $x^2 \cos(x) + 2x \sin(x)$
(c) $7x^6 + 3x^2$
(d) $6(x^2 + 4x)^3(2x + 4)$
(e) $7x^6(x^2 + 2) + x^7(2x)$
(f) $\sin(x) \cos(x) + \cos(x) \sin(x)$
MATH 122: Calculus I Posttest

11. Circle the letter(s) of the equation(s) whose graph would look like that on the right.

(a) \( y = \frac{1}{(x - 1)(x + 1)} \)  
(b) \( y = \frac{-1}{(x - 1)(x + 1)} \)  
(c) \( y = \frac{5}{(x - 1)(x + 1)} \)  
(d) \( y = \frac{-5}{(x - 1)(x + 1)} \)  
(e) \( y = \frac{1}{(x - 5)(x + 5)} \)  
(f) \( y = \frac{-1}{(x - 5)(x + 5)} \)

12. Assume \( A \) and \( B \) stand for the amounts of water in two basins and that these amounts vary with respect to time. The rate at which water is flowing into the basins is five times as fast for basin \( A \) as for basin \( B \). Write an equation which expresses this relationship.

13. Let \( f \) be the function defined by \( f(x) = \sqrt{x^2 + 7} + x \). Determine the difference quotient \( \frac{f(x + h) - f(x)}{h} \), but do not simplify.

\[
\frac{f(x + h) - f(x)}{h} =
\]

14. Assume you are walking in downtown Kalamazoo. For the first hour you walk at a pace of 5 miles per hour, after which time you walk at a pace of 4 miles per hour. Write a function which expresses the distance, \( s \), traveled with respect to time, \( t \).

15. Give an example of a function for which the tangent line at \( x = 1 \) has slope 3.
16. Let $f$ be the function given by 
$$f(x) = \begin{cases} 
  x + 2, & x < -2 \\
  -x, & -2 \leq x \leq 2 \\
  x - 2, & x > 2
\end{cases}$$

On the axes at the right, sketch a graph of $f$.

17. Without actually expanding, what is the likely degree of $(x - 1)^5 - (x + 2)^5$?

18. If $f(x)g(x) + f(x)h(x) = 0$ and if $g(2) + h(2) = 6$, what is $f(2)$?

19. What is the largest value the expression $2 + \sqrt{2\sin^2(x) + 3}$ can take on?

20. Using sigma (Σ) notation, write an expression equivalent to $\frac{2}{3} + \frac{3}{5} + \frac{4}{27} + \frac{5}{81} + \ldots$
21. Give the equation for the semicircle shown on the right.

22. Given the rule for differentiation: \( \frac{d}{dx} g(x) = \frac{1}{x} \), find the indefinite integral \( \int \frac{x + 1}{x} dx \).

23. Using sigma notation, write an expression for the derivative with respect to \( x \) of \( \sum_{j=0}^{9} x^j \).

24. The interval from \( x = 2 \) to \( x = 4 \) is to be subdivided into 10 intervals. Write an expression for either one of the endpoints of the \( i \)-th interval.

25. Determine the following multiple indefinite integral \( \int \int 3x^2 dx \).
MATH 122: Calculus I Posttest

In items 26-30, supply the missing steps and missing information to make the equations true.

26. 
\[ x^2 + 12x + \square = (x + 3)(x + \square) - 7 \]

27. 
\[ \frac{x + 7x^4 + x^3 + 8x^2 - 2x - 12}{x^2 + 2} = x^3 + \square - x + \square \]

28. 
\[ \frac{d}{dx} \left( \cos^2(x) \right) = \cos^2(x) \left( \cos(x) \right) - \sin(x) \left( 2 \cos(x) \left( -\sin(x) \right) \right) \]
\[ = \cos^2(x) + 2 \sin^2(x) \]
\[ = \cos^2(x) \cos^2(x) \]

29. 
\[ \frac{\cos^2(x) + 2 \sin^2(x)}{\cos^2(x)} = \frac{1 + \sin^2(x)}{\cos^2(x)} \]

30. 
\[ \int 2x (7 - 3x^2) \, dx = \frac{(7 - 3x^2) \square}{c} + c \]

IF YOU FINISH BEFORE TIME IS CALLED, TURN YOUR TEST OVER.
DO NOT CHANGE ANY OF YOUR ANSWERS.
Appendix C

Pretest and Posttest Scoring Rubric
Rubric for Pretest

All items are based on a 4 point scale. All other responses not noted were scored as 0.

1. e - 4 pts, f - 2 pts e,f - 1 pt.

This question examines students ability to judge equivalent expressions. Students ought to eliminate options a, b, c and d, and may recognize options e and f as results of 'standard' procedures.

2. c,d,e - 4 pts c,d,e,f - 3 pts c+ anything but a,b - 2 pts. c - 1 pt.

This question may evoke either a graphical or symbolic reasoning. Either way, students should expect a polynomial of degree \( n \) to have from 0 to \( n \) roots. Many students do not have a sense beyond this stage, and so the scoring is meant to distinguish different levels.

3. f - 4 pts b or d - 2 pts b, d, f - 2 pts e or e, f - 2 pts 2 of b,d,f - 1 pt.

Features of this question are the identification and use of roots and determining the effect of leading coefficient. Students should be able to use the zeros to eliminate a, c and e. Selecting just e and/or f suggests an examination of the y-intercept.

4. a - 4 pts a, e or a, b - 2 pts a, b, d - 2 pts b - 2 pts e or ae - 2 pts.

This question requires the determination of period, amplitude and shifts, and their corresponding roles in symbolic expressions. Incorrect responses may have been the result of many confounding factors. For example, interviews suggest that some students while thinking about amplitude, picked the expressions dealing with period. However, options c and d should not be present in general. Students selecting a, b, and d, may have quickly determined amplitude without going further.

5. b - 4 pts b, d or b, f or b, d, f - 2 pts Other subsets of b, d, f - 1 pt.

Clearly options a, c, and e should not be present (unfortunately frequently they are). Most students do not select among a, b, and f based upon whether the cosine function is even or odd, but instead think of functions in general.

6. d - 4 pts d - 2 pts c, d - 1 pt.

With identification of symmetric asymptotes, the question deals primarily with the y-intercept, and the observation of the differences in the symbolic expressions.

7. b - 4 pts d - 2 pts b, d - 2 pts

This question examines students abilities to predict the form of the result. All options but b and d should be ignored. Interviews with students suggested selection of b and d stating verbally conditions on the constant a. Others viewed b and d as equivalent.

8. d, f, e - 4 pts d, e - 3 pts Other subsets of d, f, e - 1 pt.
b, d, e - 2 pts  d, f or c, d, e, f - 2 pts

Students should approach this question through identification of the basic graphical behavior of each of the functions. Some students may select b thinking of relative maximum versus absolute maximum. Options c and e are likely to be the functions whose graphs are not readily recalled.

9. a - 4 pts.

This question may be approached either graphically or symbolically in determining what a shift in the parameter variable does to the roots of a function.

10. c, d - 4 pts  c or d or c, d, e - 2 pts

11. 4/5 or .8 (or clear attempts at .8) - 4 pts.

Typically there are two responses to this question: 4/5 or 5/4.

12. Each part 2 pts. If the addition in part b becomes a subtraction 1 pt for item b.

While this question is predominantly skills based, it examines students' ability to deal with the generality (h is not specified) and with function composition. The goal is to measure the extent to which the students deals with functional notation.

13. 0 - 4 pts  Evidence of factoring - 2 pts.

This question investigates students ability to see linearity in the expression and to deal with abstract functions.

14. Each part 2 pts and must be different - i.e. slope and slope is scored 0.

Recognition of forms of symbolic expressions is the primary interest. Specifying the same word for both does suggest more symbol sense than no response, but also suggests inappropriate symbol sense.

15. 4x+1 - 4 pts  2x + 3 or 2x + 1 - 2 pts
   Evidence of f(f(g(x))) or g(f(f(x)) error - 1 pt.

Students have experience with composition of two functions. Dealing with the composition of three functions involve a regulation process. The students work on this problem is critical. If the student has written an intermediate step which suggests that a distribution error occurred the response is scored as 2 pts.

16. 3 - 4 pts.  \( x_4 = 2x_2 - x_3 \) - 1 pt.

Again, this question seems more skills based, but it aids in determining students ability to deal with abstraction and subscripts.

17. Correct either using ... or sigma - 4 pts
No edding terms - 2 pts
Next in sequence, \( f_4 \) - 1 pt.
Sigma notation with variable upper limit - 3 pts  infinite upper limit - 2 pts

Ability to perceive and express patterns is the object of this question. Students may use either sigma or ... notation. Of primary interest was the second and third terms in which students must decrement the index variable in an abstract fashion.

18. Correct finite expression involving numbers - 4 pts.
Split the sum into two sigmas - 4 pts.
Factor the summand - 4 pts.

Given the nature of the question, the few responses which were equivalent, but unexpected were considered correct.

19. \( S = 6P \) - 4 pts.

There are typically two responses: \( S = 6P \) or \( P = 6S \). A student with well-developed symbol sense should be able to detect this error.

20. \( y = 6x + ? \) - 4 pts.

Students may use any variables (including those of question 19, although no one did). This questions examines students' ability to transfer verbally stated conditions into symbolic form.

21. 4 - 4 pts  5 - 2 pts  0 - 0 pts

These are the three most popular responses. The awarding of 2 pts is primarily to distinguish between those students and the students who responded that the expression would end up as a constant.

22. Correct graph not plotting more than two points per line and correct point - 4 pts.
Correct graph not plotting points, but no specified point on the graph - 3 pts.
Near correct graph - missing open/close circles or no values near 0 - 2 pts.
Correct point on the graph -1 pt.

This question test students ability to reason through piece-wise defined functions. But also the perception of linearity in each piece. Students plotting more than 2 or 3 points per line segment recieved 1 less point.

23. \( \sqrt{3} \) -4pts  0 - 2 pts.

This question examines students ability to estimate the results of expressions. A response of 0 may suggest that the student recognized that square roots are nonnegative.

24. \( (x < -2 \text{ or } x > 1) \) - 4 pts (without the 'or' - 3 pts).
\[ x < -2 \text{ - 2 pts} \quad x > 1 \text{ - 2 pts.} \]
Evidence of integerization of $x$ was scored as 0 pts.

25. Correct equation of circle - 4 pts. Wrong radius or signs on center - 2 pts.

Ability to predict the form of the resulting expression was of interest. So even responses which incorrectly identified the value or role of the radius or center points were scored as 2 pts.

26. 9, 5 - 4 pts. 15, 5 or 21, 5 - 3 pts Nothing, 5 - 2 pts.

The 3 pt options come from an addition error of the constant terms.

27. 5, 7$x^2$, -6 - 4 pts. 5, 7$x^2$, 6 - 3 pts. 2 of 3 - 2 pts 1 of 3 - 1 pt.

Students actually performing long division were given one less point.

28. $\sin^2(x) + \cos^2(x) + \cos^2(x) = 1.5$ or similar expressions - 4 pts. Noting the trigonometric identity - 1 pt.

Students who wrote down the trigonometric identity for sines and cosines may have had a sense of what was involved.

29. $a^2 + 2a, a^2 + 2a + 1$ - 4 pts. First box only - 2 pts Last box only - 2 pt.

30. All 3 boxes - 4 pts 2 of 3 boxes - 3 pts 1 box - 1 pt.

There are a couple of alternative intermediate steps which can be used to fill in the box. The expression should however move or lead to the subsequent expression.
### Rubric for Posttest

1. c, d, e - 4 pts  
   c, d, e, f - 3 pts  
   c - 2 pts  
   b+ subset of c, d, e, f - 1 pt  
   b - 0 pts.

See question 2 of the pretest.

2. a, c, d - 4 pts  
   a, c, d, f - 3 pts  
   a, ? or a, d, ? or c, d - 2 pts  
   a, b, c, d - 2 pts  
   a, ? or c, ? - 1 pt.

Some students think only of positive values or associate increasing with moving up as  
moving away from the origin. Similar to question 8 of pretest.

3. c - 4 pts  
   a or e or d - 2 pts  
   a, c, e - 2 pts  
   c, d - 2 pts  
   a, c or e - 1 pt.

See question 3 of pretest.

4. e, f - 4 pts  
   e or f or d, f, e - 3 pts  
   d or d, f or d, e - 2 pts.

See question 7 of pretest.

5. a - 4 pts.

See question 9 of pretest.

6. b, d, f - 4 pts  
   b, d or d, f - 2 pts  
   b or d or f - 1 pt.  
   f or d, e, f - 1 pt.  
   2 of b, d, f + ? - 1 pt.

The pairs of opposition options: {b, e} and {c, f} should not occur together.

7. b - 4 pts  
   c - 1 pt.

Selection of c is usually based upon the perceived similarity to the fundament  
theorem of calculus.

8. a, f - 4 pts  
   a or f or a, f, b - 2 pts  
   a but no c or d - 1 pt  
   b, f - 1 pt.

See question 5 of pretest.

9. f - 4 pts.  
   d, f - 2 pts  
   f, ? - 1 pt.

10. b, e, f - 4 pts  
    b, e - 3 pts  
    b or e, f - 2 pts  
    b or e or f or a, b, e, f - 1 pt

11. c - 4 pts  
    d - 2 pts  
    c, d - 1 pt.

See question 10 of pretest.
12. \( A' = 5 B' \) (any appropriate notation) - 4 pts. \( A = 5 B \) - 2 pts

13. Correct - 4 pts  
Without parentheses - 3 pts.

14. Correct piece-wise - 4 pts, close piece-wise - 3 pts, rough piece-wise 2 pts
A very wrong piece-wise - 1 pt. \[ s(t) = 5 + 4(t - 1) \] - 2 pts

15. Any appropriate function - 4 pts. If the work suggests the student was finding the actual tangent line (which also works) - 2 pts.

16. Correct graph without many plotted points - 4 pts.
Essentially correct graph without many plotted points - 3 pts.
Correct jumps and at least 1/2 the graph correct - 2 pts.
Point approach to graphing - 1 less from above scoring.

See question 22 from pretest.

17. 4 - 4 pts 5 - 2 pts 0 - 0 pts.

See question 21 from pretest.

18. 0 - 4 pts Factored - 2 pts.

See question 13 from pretest.

19. \( 2 + \sqrt{5} \) - 4 pts \( 2 + \sqrt{3} \) - 2 pts.

See question 23 from pretest.

20. \( \sum_{i=1}^{\infty} \frac{i+1}{3^i} \) - 4 pts \( \sum_{i=1}^{\infty} \frac{i+1}{3^i} \) - 3 pts \( \sum_{i=1}^{\infty} \frac{i}{3^i} \) or \( \sum_{i=1}^{4} \frac{i+1}{3^i} \) - 2 pts
Mismatched index variables - 1 pt.

This question investigates students ability to express numeric patterns in symbolic form and the role of index variables. See question 18 of the pretest.

21. \( y = \sqrt{4 - x^2} \) - 4 pts \( x^2 + y^2 = 4 \) - 3 pts. \( y = \sqrt{2 - x^2} \) or \( \sqrt{4 - x} \) - 2 pts
Attempts on 3 pt form to restrict it to a semi-circle such as absolute value - 1 pt.

See question 25 on pretest.

22. \( x + g(x) + \text{constant?} \) - 4 pts \( 1 + g(x) + \text{constant?} \) - 2 pts
Separation into two indefinite integrals - 1 pt.
23. Correct without expanding and then differentiating - 4 pts.
Rewriting with derivative notation on the inside - 2 pts.
Expanding and then differentiating and then rewriting using sigmas - 3 pts.
Expanding and then differentiating - 2 pts.

24. $2 + (i - 1) \times 0.2$ or $2 + i \times 0.2$ - 4 pts
Correct area expression - 1 pt.

25. $\frac{x^4}{4} + cx + \text{constant?}$ - 4 pts
$\frac{x^4}{2} + cx + \text{constant?}$ - 3 pts.
$\frac{x^4}{4} + \text{constant?}$ - 2 pts
No + constant - (± 1) pt.

26. 20, 9 - 4 pts 34, 9 or 20, 9 - 3 pts Nothing, 9 - 2 pts
See pretest question 26.

27. 5, 7$x^2$, -6 - 4 pts. 5, 7$x^2$, 6 - 3 pts. 2 of 3 - 2 pts 1 of 3 - 1 pt.
Students actually performing long division were given one less point. See pretest question 27.

28. Each box worth 2. Second box can be any appropriate simplification.

29. Separation of numerator using $2\sin^2(x) = \sin^2(x) + \sin^2(x)$ - 4 pts.

30. 5, -15 - 4 pts 5, 5 or -3 - 3 pts 5, Nothing - 2 pts.
Appendix D

Comprehensive Final Exams and Laboratory Practicals
1. Find the following limit, if it exists: \( \lim_{x \to 3} \frac{x^2 - 2x - 3}{x^2 - 8x + 15} \).
   
   (a) -2  \hspace{1cm} (b) 0  \hspace{1cm} (c) -\frac{1}{5}  \hspace{1cm} (d) 1  \hspace{1cm} (e) \text{Does not exist}

2. Find the following limit: \( \lim_{x \to -1^+} \frac{3}{(x + 1)^3} \).
   
   (a) -\infty  \hspace{1cm} (b) -3  \hspace{1cm} (c) 0  \hspace{1cm} (d) 3  \hspace{1cm} (e) \infty

3. The range of the function \( f(x) = -x + \sqrt{x^2 - 1} \) is
   
   (a) \([-1, 0) \cup [1, \infty)\)  \hspace{1cm} (b) \(( -\infty, -1] \cup [1, \infty)\)  \hspace{1cm} (c) \([-1, \infty)\)

   (d) \([-1, 0] \cup [1, \infty)\)  \hspace{1cm} (e) All real numbers
4. Which one of the following functions has a graph symmetric with respect to the origin?

(a) \( f(x) = \sin x + x \)  
(b) \( f(x) = \cos x - x^2 \)  
(c) \( f(x) = x^2 + x - 1 \)  
(d) \( f(x) = x^2 |x| \)  
(e) \( f(x) = 1 - \frac{1}{x} \)

5. If \( g \) is differentiable and \( f(x) = g(x^2) \), find \( f'(4) \) in terms of \( g' \).

(a) \( 8g'(4) \)  
(b) \( g'(8) \)  
(c) \( 8g'(8) \)  
(d) \( g'(16) \)  
(e) \( 8g'(16) \)

6. Find the following limit: \( \lim_{x \to 0} \frac{\sin^2 3x}{x^2} \).

(a) \( \frac{1}{9} \)  
(b) \( \frac{1}{3} \)  
(c) 1  
(d) 3  
(e) 9

7. Find an expression for \( \frac{dy}{dx} \) if \( y^2 = \frac{1 + x}{1 - x} \).

(a) \( \frac{-x}{y(1 - x)^2} \)  
(b) \( \frac{1}{y(1 - x)^2} \)  
(c) \( \frac{-x}{(1 - x)^2} \)  
(d) \( \frac{1}{1 - x^2} \)  
(e) \( \frac{-x}{1 - x^2} \)

8. Find an equation for the line tangent to the graph of \( y = \sqrt{x^3 + 1} \) at \( (2, 3) \).

(a) \( 2x - y = 4 \)  
(b) \( x - 6y = -16 \)  
(c) \( 2x - y = 1 \)  
(d) \( x - 6y = 20 \)  
(e) \( \sqrt{12}x - y = 2\sqrt{12} - 3 \)

9. The derivative of \( f(x) = x^3 \sin 5x \) is

(a) \( 3x^2 \cos 5x \)  
(b) \( 5x^3 \cos 5x + 3x^2 \sin 5x \)  
(c) \( 3x^2 \sin 5x - x^3 \cos 5x \)  
(d) \( x^3 \cos 5x + 3x^2 \sin 5x \)  
(e) \( -5x^3 \cos 5x + 3x^2 \sin 5x \)
10. A farmer wants to fence in 800 ft² of land in a rectangular plot \( x \) ft wide and \( y = \frac{800}{x} \) ft long and then subdivide it into three equal plots with fences parallel to one of the sides as shown in the picture. What are the \((x, y)\) outside dimensions of the rectangular plot which require the least amount of fence?

(a) \((20, 40)\)  
(b) \((16, 50)\)  
(c) \(\left(\frac{20}{3}, \frac{20}{\sqrt{6}}\right)\)  
(d) \((25, 32)\)  
(e) \((10, 80)\)

11. The function \( f(x) = \frac{12x - 12}{x^2} \) has

(a) a relative minimum point at \((2, 3)\)  
(b) a relative maximum point at \((2, 3)\)  
(c) a relative minimum point at \((-2, -9)\)  
(d) a relative maximum point at \((-2, -9)\)  
(e) no relative extreme points

12. Which of the following answers is the best approximation for the positive root of the equation \(x^4 - 4x^3 + 8x - 16 = 0\)? [Hint: Use Newton's method with \(x_1\) about 3 or 4.]

(a) 3.7335208  
(b) 3.7350087  
(c) 3.7351328  
(d) 3.7777778  
(e) 5.375

13. A spherical balloon is being inflated at the constant rate of \(\frac{dV}{dt} = 4000 \text{ cm}^3/\text{min}\). What is the approximate rate \(\frac{dr}{dt}\) that the radius \(r\) is increasing when \(r = 100\text{ cm}\)? [Hint: \(V = \frac{4}{3}\pi r^3\).]

(a) 0.32 cm/min  
(b) 0.032 cm/min  
(c) 0.081 cm/min  
(d) 0.16 cm/min  
(e) 0.24 cm/min

14. For the function \(f\) whose graph is shown at the right, which point could be used to verify the Mean Value Theorem for Derivatives on the interval shown?

(a) Point A  
(b) Point B  
(c) Point C  
(d) Point D  
(e) Point E
15. The number of relative extrema for the function \( f(x) = x \sin \frac{1}{x} \) on the open interval \((0.05, 0.1)\) is:
   (a) 0  (b) 1  (c) 2  (d) 3  (e) More than 3

16. The value of \( x \) at which \( f(x) = x^3 - x^2 - 40x + 8 \) has a local maximum is:
   (a) -2  (b) -10/3  (c) 1/3  (d) 1  (e) 4

17. The points of inflection of \( f(x) = 3x^5 - 5x^4 - 7x + 4 \) occur at
   (a) \( x = 0 \) and \( x = 1 \)  (b) \( x = 0 \) only  (c) \( x = 1 \) only
   (d) no points of inflection  (e) \( x = 4/3 \) only

18. Find \( \int (x^3 + 1)^5x^2 \ dx \).
   (a) \( \frac{(x^3 + 1)^6}{3} + C \)  (b) \( \frac{(x^3 + 1)^6}{6} + C \)  (c) \( \frac{(x^3 + 1)^6}{18} + C \)
   (d) \( \frac{(x^3 + 1)^6x^3}{18} + C \)  (e) \( \frac{(x^3 + 1)^6x^2}{6} + \frac{(x^3 + 1)^5x^3}{3} + C \)

19. The limit of the Riemann sum \( \lim_{n \to \infty} \sum_{k=1}^{n} \left(-1 + \frac{3k}{n}\right)^2 \left(\frac{3}{n}\right) \) is equal to:
   (a) \( \int_{0}^{3} x^2 \ dx \)  (b) \( \int_{-1}^{2} 3x^2 \ dx \)  (c) \( \int_{-1}^{2} x^2 \ dx \)
   (d) \( \int_{0}^{3} (-1 + x^2) \ dx \)  (e) \( \int_{0}^{3} (-1 + 3x) \ dx \)

20. Evaluate the following indefinite integral: \( \int (-3 \cos x - 4 \sec^2 x) \ dx \)
   (a) \( 3 \sin x + 4 \tan x + C \)  (b) \( 3 \sin x - 4 \cot x + C \)  (c) \( -3 \sin x - 4 \tan x + C \)
   (d) \( -3 \sin x + 4 \cot x + C \)  (e) \( 3 \sin x + 4 \cot x + C \)
21. A region in the first quadrant is bounded by \( y = x^2 \), \( y = \frac{1}{x^2} \), and \( y = 9 \). Find an integral representing the area of this region.

(a) \( \int_1^9 \left( x^2 - \frac{1}{x^2} \right) \, dx \)
(b) \( \int_1^3 \left( x^2 - \frac{1}{x^2} \right) \, dx \)
(c) \( \int_1^3 (9 - x^2) \, dx \)
(d) \( \int_1^9 \left( 9 - \sqrt{y} \right) \, dy \)
(e) \( \int_1^9 \left( \sqrt{y} - \frac{1}{\sqrt{y}} \right) \, dy \)

22. Find \( \frac{d}{dx} \int_x^1 \left( 3 - \sqrt{1 + t^2} \right) \, dt \).

(a) \( 3 - \sqrt{1 + x^2} \)  
(b) \( 3 - \sqrt{1 + x} \)  
(c) \( \sqrt{1 + x^2} \)  
(d) \( \sqrt{1 + x^2} - 3 \)  
(e) \( -\frac{t}{\sqrt{1 + t^2}} \)

23. Which of the following is the best approximation to \( \int_{-1}^1 \sqrt{1 - \frac{x^2}{4}} \, dx \)?

(a) 1.913203710  
(b) 1.913207215  
(c) 1.913218144  
(d) 1.913222955  
(e) 1.91323577

24. If \( a \) and \( b \) are arbitrary constants, then \( \int_1^2 \frac{a \cdot x^2 + b}{x^2} \, dx \) is:

(a) \( a + \frac{7b}{24} \)  
(b) \( a + \frac{b}{2} \)  
(c) \( 3a - \frac{3b}{8} \)  
(d) \( a - \frac{3b}{8} \)  
(e) \( a - \frac{3b}{2} \)

25. The equation of the tangent line to the graph of a differentiable function \( y = f(x) \) at the point \( (2, f(2)) \) is \( y + 2 = 3(x - 2) \). Suppose that \( \lim_{x \to 2} f(x) = a \) and \( \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = b \). Which of the following statements is true?

(a) \( a = 2, \, b = 3 \)  
(b) \( a = -2, \, b = 3 \)  
(c) \( a = 2, \, b = 2 \)  
(d) \( a = -2, \, b = 0 \)  
(e) \( a = 0, \, b = 2 \)
26. One student found that \( \frac{1}{3}(x + 1)^3 \) is an antiderivative for \( f(x) = (x + 1)^2 \) and another student found by a different method that \( \frac{1}{3}x^3 + x^2 + x \) is an antiderivative for \( f(x) \).

Then:
(a) Only the first student is correct.
(b) Only the second student is correct.
(c) Neither student is correct
(d) The two answers are exactly the same.
(e) Both students are correct, but their answers differ by a constant.

27. A swimmer is drowning 50 yards out from a straight shore line. A lifeguard on the beach is 100 yards away from the point on the shore line closest to the swimmer. The lifeguard runs \( x \) yards down the shore line, jumps into the ocean, and swims directly to the distressed swimmer. (See figure.) The lifeguard's speed is 8 yards per second on land and 3 yards per second in the water. Let \( t \) be the number of seconds it takes the lifeguard to reach the swimmer from the instant she starts running. Express \( t \) as a function of \( x \).

\[
\begin{align*}
\text{(a)} & \quad t = \frac{x + \sqrt{(100 - x)^2 + 50^2}}{8} \\
\text{(b)} & \quad t = 3\sqrt{(100 - x)^2 + 50^2} + 8x \\
\text{(c)} & \quad t = 8x + 3(100 - x + 50) \\
\text{(d)} & \quad t = \frac{x + \sqrt{(100 - x)^2 + 50^2}}{0.5(8 + 3)} \\
\text{(e)} & \quad t = \frac{\sqrt{x^2 + 50^2}}{3} + \frac{100 - x}{8}
\end{align*}
\]

28. The length of the curve \( y = \sin(3x) \) from \( x = 0 \) to \( x = \pi \) is given by:

\[
\begin{align*}
\text{(a)} & \quad \int_0^\pi \sin(3x) \, dx \\
\text{(b)} & \quad \int_0^\pi \sqrt{1 + \cos^2(3x)} \, dx \\
\text{(c)} & \quad \int_0^\pi \sqrt{1 + 9\cos^2(3x)} \, dx \\
\text{(d)} & \quad \int_0^\pi \sqrt{1 + \sin^2(3x)} \, dx \\
\text{(e)} & \quad \int_0^\pi (1 + \cos(3x)) \, dx
\end{align*}
\]
29. Use Simpson's rule to estimate the area of the piece of lakeside property in the figure. [Assume that the measurements were made at equally spaced intervals along the baseline of the property.]

(a) 8,433 ft²
(b) 9,300 ft²
(c) 9,400 ft²
(d) 10,467 ft²
(e) 10,900 ft²

30. The region bounded by \( y = 3x - x^2 \) and the \( x \)-axis is rotated about the \( y \)-axis. Find the volume of the resulting solid.

(a) \( \int_{0}^{3} \pi (3x - x^2) \, dx \)
(b) \( \int_{0}^{2} \pi y^2 \, dy \)
(c) \( \int_{0}^{3} 2\pi x (3x - x^2) \, dx \)
(d) \( \int_{0}^{2} 2\pi xy \, dy \)
(e) \( \int_{0}^{3} \pi (3x - x^2)^2 \, dx \)
MATH 122 FINAL EXAM

1. Use the graph to estimate the range (not the domain) of the function \( \frac{\sqrt{x} + 4}{x^2 + 1} \).
   (a) \([-4, 2]\)  
   (b) \([0, 2]\)  
   (c) \([-\infty, 0]\)  
   (d) \([0, \infty]\)  
   (e) \((-\infty, \infty)\)

2. Find the derivative of \( f(x) = x^3 \tan x \).
   (a) \(3x^2 \sec^2 x\)  
   (b) \(3x^2 \cot x\)  
   (c) \(3x^2 \tan x - x^3 \cot x\)  
   (d) \(3x^2 \tan x + x^3 \sec^2 x\)  
   (e) \(3x^2 \cot x + x^3 \tan x\)

3. The formula to express the radius of a cylinder as a function of its volume and height is
   (a) \(r = \frac{V^2}{\pi h^2}\)  
   (b) \(r = \pi V h\)  
   (c) \(r = \frac{V}{\frac{4}{\pi} h}\)  
   (d) \(r = \sqrt{\frac{V}{\pi h}}\)  
   (e) \(r = \sqrt{\frac{V}{\pi h}}\)
4. The functions \( f(x) \) and \( g(x) \) shown here are related by what equation?

(a) \( g(x) = f(x + 3) + 1 \)
(b) \( g(x) = f(x + 1) + 3 \)
(c) \( g(x) = f(x - 1) + 3 \)
(d) \( g(x) = f(x - 1) - 3 \)
(e) \( g(x) = f(x - 3) + 1 \)

5. Suppose that \( f(x) \) is defined by
\[
f(x) = \begin{cases} 
  x + 3, & x \neq -3 \\
  m, & x = -3 
\end{cases}
\]
Find the value of \( m \) so that the given function is continuous.

(a) \( f \) cannot be continuous  (b) 3  (c) -3  (d) 6  (e) -6

6. Find the following limit, if it exists:
\[
\lim_{x \to 1} \frac{x^2 + x - 2}{x - 1}
\]

(a) 1  (b) 3  (c) 0  (d) \( \infty \)  (e) Does not exist

7. Find the following limit, if it exists:
\[
\lim_{x \to -\infty} \frac{x^3 - 2x + 1}{3 - 4x^3}
\]

(a) Does not exist  (b) 1/3  (c) 1/4  (d) -1/4  (e) -1/2
8. Consider the graph of the function \( f(x) \) shown here. Determine which of the following statements is true.

(a) \( \lim_{x \to a} f(x) \) exists

(b) \( \lim_{x \to a^-} f(x) \) exists and \( \lim_{x \to a^+} f(x) \) does not exist

(c) \( \lim_{x \to a^-} f(x) \) does not exist and \( \lim_{x \to a^+} f(x) \) exists

(d) both \( \lim_{x \to a^-} f(x) \) and \( \lim_{x \to a^+} f(x) \) exist

(e) neither \( \lim_{x \to a^-} f(x) \) nor \( \lim_{x \to a^+} f(x) \) exists

9. The derivative of \( f(x) = \sin (\cos (3x)) \) is

(a) \( \cos (\cos (3x)) + \sin (-\sin (3x)) \)

(b) \( \cos (\cos (3x)) + 3 \sin (-\sin (3x)) \)

(c) \( \cos (-\sin (3x)) \)

(d) \( 3 \cos (\cos (3x)) (-\sin (3x)) \)

(e) \( 3 \cos (-\sin (3x)) \)

10. Find an equation for the line tangent to the graph of \( y = \frac{3}{\sqrt{5}} x - 8 \) at \((7,3)\).

(a) \( 5y + x = 22 \)

(b) \( 5y - x = 8 \)

(c) \( 9y - 5x = -8 \)

(d) \( 8y + 9x = 27 \)

(e) \( 27y - 5x = 46 \)

11. Which of the following answers is the best approximation for the positive root of the equation \( x^2 - \sin x = 0 ? \) \[\text{Hint: Use Newton's method.}\]

(a) 0.8759273

(b) 0.8761349

(c) 0.8765542

(d) 0.8767262

(e) 0.8776151

12. If a balloonist drops a sand bag (called ballast) from a balloon 500 feet above the ground, the bag's distance above the ground after \( t \) seconds is given by \( 500 - 16 t^2 \). With what velocity does the bag strike the ground?

(a) \(-500 \text{ ft/sec}\)

(b) \(-201.25 \text{ ft/sec}\)

(c) \(-178.89 \text{ ft/sec}\)

(d) \(-32 \text{ ft/sec}\)

(e) \(-31.25 \text{ ft/sec}\)
13. If \( f(x) = \frac{3x^2 - x + 8}{2 - 9x} \), find \( f'(0) \).

\[ \begin{array}{cc}
(a) & 16 \\
(b) & 2 \\
(c) & 37 \\
(d) & 18 \\
(e) & \text{none of these}
\end{array} \]

14. If \( x^2 + y^5 + \cos x = 0 \), find the slope of the tangent line at \((0, -1)\).

\[ \begin{array}{cc}
(a) & -1/3 \\
(b) & 1/3 \\
(c) & 0 \\
(d) & -1/5 \\
(e) & 1/5
\end{array} \]

15. Water is draining from a conical reservoir of height 10 m and radius 10 m at the constant rate of 100 m³/min. What is the approximate rate \( \frac{dh}{dt} \) when \( h = 5 \) m? \([\text{Hint: } V = \frac{1}{3} \pi r^2 h]\)

\[ \begin{array}{cc}
(a) & 0 \text{ m/min} \\
(b) & -1.27 \text{ m/min} \\
(c) & -2.55 \text{ m/min} \\
(d) & -3.82 \text{ m/min} \\
(e) & -5.09 \text{ m/min}
\end{array} \]

16. The value of \( x \) at which \( f(x) = x^3 - x^2 - 5x + 2 \) has a local maximum is:

\[ \begin{array}{cc}
(a) & 5/3 \\
(b) & 1 \\
(c) & -1 \\
(d) & -5/2 \\
(e) & -2
\end{array} \]

17. The function \( f(x) \) whose graph is given does not satisfy the hypothesis of the Mean Value Theorem (for derivatives) on \([a, b]\) because:

\[ \begin{array}{cc}
(a) & f'(x) \text{ is a constant on } (a, b) \\
(b) & f'(x) \text{ can never be zero} \\
(c) & f(x) \text{ is not continuous} \\
(d) & f(a) \neq f(b) \\
(e) & f(x) \text{ is not differentiable on } (a, b)
\end{array} \]
18. Use the graph and the derivatives to approximately determine the largest interval on which the graph of the function \( f(x) = 0.2x^5 - x^3 + 0.5x^2 + 1 \) is concave upward.

(a) \(( -\infty, -1.3)\)  (b) \((-1.3, 0.2)\)  (c) \((0.2, 1.3)\)  (d) \((-2.5, \infty)\)  (e) \((1.3, \infty)\)

19. The function \( f(x) = ax^5 + bx + c \) with \( a > 0 \) and \( b > 0 \) has how many critical points?

(a) 4  (b) 3  (c) 2  (d) 1  (e) None

20. The integral \( \int_{-1}^{3} x^2 \, dx \) is equivalent to the limit of which Riemann sum below?

(a) \( \lim_{n \to \infty} \sum_{k=1}^{n} \left( -1 + \frac{4k}{n} \right)^2 \left( \frac{4}{n} \right) \)  
(b) \( \lim_{n \to \infty} \sum_{k=1}^{n} \left[ -1 + \frac{4k}{n} \right] \left( \frac{4}{n} \right) \)  

(c) \( \lim_{n \to \infty} \sum_{k=1}^{n} \left( -1 + \frac{x_k}{n} \right)^2 \left( \frac{3}{n} \right) \)  
(d) \( \lim_{n \to \infty} \sum_{k=1}^{n} \left( \frac{3}{n} \right)^2 \left( \frac{3}{n} \right) \)  

(e) \( \lim_{n \to \infty} \sum_{k=1}^{n} \left( \frac{4}{n} \right)^2 \left( \frac{4}{n} \right) \)

21. If \( \int_{0}^{2} f(x) \, dx = 7 \) and \( \int_{0}^{5} f(x) \, dx = 9 \), find the value of \( \int_{2}^{5} (3f(x) + 2) \, dx \).

(a) 4  (b) 6  (c) 8  (d) 10  (e) 12

22. Evaluate \( \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos(2x) \, dx \).

(a) \(-2\)  (b) \(-1\)  (c) 0  (d) 1  (e) 2
23. A region is bounded by $y = x$ and $x = 2 - y^2$. Find an integral representing the area of this region.

\[ \begin{align*}
(a) & \int_{-2}^{1} \left[ y - (2 - y^2) \right] \, dy \\
(b) & \int_{-2}^{1} \left[ 2 - y^2 - y \right] \, dy \\
(c) & \int_{-2}^{1} \left[ x - \sqrt{2x} \right] \, dx \\
(d) & \int_{-2}^{2} \left[ x - \sqrt{2 - x} \right] \, dx \\
(e) & \int_{-2}^{2} \left[ \sqrt{2 - x} - x \right] \, dy
\end{align*} \]

24. The region to the right of the line $x = 1$ and bounded by $x = 1$, $y = 0$, and $y = \sqrt{2 - x}$ is rotated about the $x$-axis. Find the volume of the resulting solid of revolution.

\[ \begin{align*}
(a) & \pi \\
(b) & \frac{\pi}{2} \\
(c) & \frac{3\pi}{2} \\
(d) & \frac{11\pi}{12} \\
(e) & \frac{28\pi}{15}
\end{align*} \]

25. Find the area of the region bounded by $y = \sqrt{x}$ and $y = x^2$.

\[ \begin{align*}
(a) & \frac{1}{3} \\
(b) & \frac{1}{2} \\
(c) & 1 \\
(d) & 2 \\
(e) & 3
\end{align*} \]

26. The value of \( \int_{1}^{4} \frac{x + 1}{\sqrt{x}} \, dx \) is:

\[ \begin{align*}
(a) & 6 \\
(b) & \frac{19}{3} \\
(c) & \frac{20}{3} \\
(d) & 7 \\
(e) & \frac{22}{3}
\end{align*} \]

27. An antiderivative of $f(x) = (3 - x)^{49}$ is:

\[ \begin{align*}
(a) & F(x) = -48(3 - x)^{48} \\
(b) & F(x) = 48(3 - x)^{48} \\
(c) & F(x) = -\frac{(3 - x)^{50}}{50} \\
(d) & F(x) = \frac{(3 - x)^{50}}{50} \\
(e) & F(x) = \frac{(3 + x)^{50}}{50}
\end{align*} \]
28. Find the derivative of the function $F(x) = \int_3^{x^2} (1 + t)^3 \, dt$.

(a) $F'(x) = (1 + x)^3$  
(b) $F'(x) = 3(1 + x^2)^2$  
(c) $F'(x) = (1 + x^2)^3$  
(d) $F'(x) = (1 + x^2)^3 \cdot 2x$  
(e) $F'(x) = 3(1 + x^2)^3 \cdot 2x$

29. Find the best numerical approximation for the arc length of the part of the graph of $y = (1/2) x^2$ from $x = 0$ to $x = 2$.

(a) 2.57551  
(b) 2.953195  
(c) 2.957886  
(d) 2.957901  
(e) 2.959913

30. If in the integral $\int_0^3 x \sqrt{1 + x^2} \, dx$ one makes the substitution $u = 1 + x^2$,

then the integral becomes:

(a) $\int_0^3 u \, du$  
(b) $\int_1^{10} u^{1/5} \, du$  
(c) $\int_0^3 u^{1/5} \, du$  
(d) $\frac{1}{2} \int_1^{10} u^{1/5} \, du$  
(e) $\frac{1}{2} \int_0^3 u^{1/5} \, du$
Name: ____________________________

Theorist LAB PRACTICAL

Directions: Solve each of the following problems using Theorist. Record your answers here as well as a print out of all your Theorist work. You are not permitted to use your calculator. All work must be done in Theorist.

Warning: Avoid assignments of values to variables such as $x = 1$!

1. Find the derivative of $y = \frac{\sin(x^2 + 1)}{x^3 + 1}$.

   $y' =$ ________________________________

2. For the function of question 1, find the equation of the tangent line at $x = 1$.

   The equation of the tangent line is: ________________________________

3. Obtain a graph of the function and the tangent line together so that the behavior of the function and the tangent line near $1$ is clear.

4. For the function of question 2, find the second derivative $y''$. List the first few terms of the derivative below.

   $y'' =$ ________________________________

5. Implicitly differentiate the equation $x^3y + y^2x = y^4$.

   $\frac{dy}{dx} =$ ________________________________
6. Define the function \( f(x) = \begin{cases} \frac{x^2}{x-2}, & x < -2 \\ \sqrt{4 - x^2}, & -2 \leq x < x < 2 \\ x \sin(x), & x \geq 2 \end{cases} \).

Using your function, determine \( f(f(-3)) \). \( f(f(-3)) = \) ________________

Obtain an appropriate graph for this function on the range \(-5 \leq x \leq 5\).

7. Enter the sum \( \sum_{j=1}^{10} \frac{j}{3^j} \). Determine the terms of this sum and the value of the sum.

\[ \sum_{j=1}^{10} \frac{j}{3^j} = \] ________________

8. Determine the value of the definite integral \( \int_{-2}^{2} x \cos(x+1) \, dx \).

\[ \int_{-2}^{2} x \cos(x+1) \, dx = \] ________________

9. Let \( f(t) = \cos(t) \) and define \( A(x) = \int_{0}^{x} f(t) \, dt \). Enter \( A(x) \) as a function and determine \( A(2\pi) \).

\( A(2\pi) = \) ________________
Directions: Solve each of the following problems using Maple. Record your answers here as well as a print out of all your Maple work. You are not permitted to use your calculator. All work must be done in Maple.

Warning: Don't forget to unassign variables between unrelated problems!

1. Find the derivative of \( y = \frac{\sin(x^2 + 1)}{x^3 + 1} \).

\[ y' = \quad \]

2. For the function of question 1, find the equation of the tangent line at \( x = 1 \).

The equation of the tangent line is:

3. Obtain a graph of the function and the tangent line together so that the behavior of the function and the tangent line near 1 is clear.

4. For the function of question 2, find the second derivative \( y'' \). List the first few terms of the derivative below.

\[ y'' = \quad \]

5. Implicitly differentiate the equation \( x^3 y + y^2 x = y^4 \) using Maple's 'D' command.

\[ \frac{dy}{dx} = \quad \]
6. Define the function \( f(x) = \begin{cases} \frac{x^2}{x-2}, & x < -2 \\ \sqrt{4-x^2}, & -2 \leq x \leq 2 \\ x \sin(x), & x \geq 2 \end{cases} \) as a procedure and using an if-then-else-fi command.

Using your function, determine \( f(f(-3)) \). \( f(f(-3)) = \) ________________

Obtain an appropriate graph for this function on the range \(-5 \leq x \leq 5\).

7. Enter the sum \( \sum_{j=1}^{10} \frac{j}{3^j} \). Determine the terms of this sum and the value of the sum.

\[ \sum_{j=1}^{10} \frac{j}{3^j} = \] ________________

8. Determine the value of the definite integral \( \int_{-2}^{2} x \cos(x+1) \, dx \).

\[ \int_{-2}^{2} x \cos(x+1) \, dx = \]

9. Let \( f(t) = \cos(t) \) and define \( A(x) = \int_{0}^{x} f(t) \, dt \). Enter \( A(x) \) as a procedure and determine \( A(2\pi) \).

\( A(2\pi) = \) ________________
Appendix E

Additional Interview Tasks
Calculus I Discussion Questions

1. Without actually finding each of the following limits, which of the limits are likely to exist? Explain your reasoning in each case.

   (a) \( \lim_{x \to 4} \frac{x^2 - x - 12}{x - 4} \)

   (b) \( \lim_{x \to -1} \frac{x^2 + ax - 3}{(x + 1)^3} \)

   (c) \( \lim_{x \to 0} \frac{3x}{\sin(x)} \)

   (d) \( \lim_{x \to 0} \frac{1 - \cos^2(x)}{\sin^2(x)} \)

   (e) \( \lim_{x \to \infty} \frac{x^4 - 3x}{x^6 - x \cos(x)} \)

2. Give an equation of a function which is discontinuous at \( x = 2 \). Explain how you know your function is discontinuous at \( x = 2 \).

3. Which of the following limits can be used to define the derivative of \( f(x) \) at the point \( (a, f(a)) \)? Explain your reasoning in each case.

   (a) \( \lim_{x \to 0} \frac{f(x) - f(a)}{x - a} \)

   (b) \( \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \)

   (c) \( \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \)

   (d) \( \lim_{a \to b} \frac{f(b) - f(a)}{b - a} \)

   (e) \( \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x} \)

   (f) \( \lim_{h \to 0} \frac{f(a - h) - f(a)}{-h} \)

4. Let \( f(x) \) and \( g(x) \) be two functions. Recall that, in general, the derivative of the product \( f(x)g(x) \) is:

\[
\frac{d}{dx} [f(x)g(x)] = f(x)g'(x) + f'(x)g(x).
\]

Suppose for a certain function \( g(x) \), the derivative of the product \( f(x)g(x) \) is:

\[
\frac{d}{dx} [f(x)g(x)] = f'(x)g(x).
\]

What type of a function must \( g(x) \) be in this case?
Calculus I Discussion Questions II

1. Find the derivative of $f(x) = \sin(\cos(3x))$.

2. Let $G$ be a differentiable function. What is $\frac{d}{dx}[G(-x^2)]^2$?

3. Given that $n$ is a positive integer, find a formula for the derivative $\frac{d^n}{dx^n} x^n$.

4. Find the absolute extrema for the function $y = (x - 1)(x + 1)$.

5. Determine where the function $g(x) = (x - 2)^4$ is increasing.

6. Determine where the function $f(x) = |1 - x^2|$ is concave up.

7. Determine a function whose first derivative is each of the following.
   a. $x^2$
   b. $2x \sin(x) + x^2 \cos(x)$
   c. $72x^2(2x^3 + 5)^5$

8. Given the equation $x^2 + x^3 y^3 + y^4 = 7$, find $\frac{dx}{dy}$.
Appendix F

Results of Tests for Homogeneity of Regression
## TESTS FOR SEMESTER EFFECTS

### Table 15
Results of Test for Semester Effects on Symbol Sense Posttest

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### Table 16
Results of Test for Semester Effects on Comprehensive Final Exam

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### Table 17
Results of Test for Semester Effects on Laboratory Practical

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### TESTS FOR POSTTEST

#### Table 18

Results of Test for Homogeneity of Regression of Posttest Score on Pretest Score — Fall

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#### Table 19

Results of Test for Homogeneity of Regression of Posttest Score on Pretest Score — Winter

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### TESTS FOR COMPREHENSIVE FINAL EXAM

#### Table 20
Results of Test for Homogeneity of Regression of Comprehensive Final Exam Score on Pretest Score — Fall

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<th>F</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>Treatment*Pretest</td>
<td>2</td>
<td>14.78</td>
<td>0.383</td>
<td>0.682</td>
</tr>
<tr>
<td>Error</td>
<td>97</td>
<td>1873.71</td>
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</tr>
</tbody>
</table>

#### Table 21
Results of Test for Homogeneity of Regression of Comprehensive Final Exam Score on Pretest Score — Winter

<table>
<thead>
<tr>
<th>Source</th>
<th>df</th>
<th>Seq. SSE</th>
<th>F</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>Treatment*Pretest</td>
<td>2</td>
<td>7.56</td>
<td>0.210</td>
<td>0.811</td>
</tr>
<tr>
<td>Error</td>
<td>86</td>
<td>1549.55</td>
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</tbody>
</table>

### TEST FOR LABORATORY PRACTICAL

#### Table 22
Results of Test for Homogeneity of Regression of Laboratory Practical Score on Pretest Score

<table>
<thead>
<tr>
<th>Source</th>
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<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>Treatment*Pretest</td>
<td>1</td>
<td>11.45</td>
<td>0.116</td>
<td>0.734</td>
</tr>
<tr>
<td>Error</td>
<td>121</td>
<td>11971.11</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Appendix G

Human Subjects Institutional Review Board
Research Protocol Clearance
Date: July 15, 1992
To: Brian A. Keller
From: Mary Anne Bunda, Chair
Re: HSIRB Project Number: 92-06-17

This letter will serve as confirmation that your research protocol, "Differences in Students' Understanding of Symbols across two Instructional Methods for Teaching First-semester Calculus" has been approved under the exempt category of review by the HSIRB. The conditions and duration of this approval are specified in the Policies of Western Michigan University. You may now begin to implement the research as described in the approval application.

You must seek reapproval for any changes in this design. You must also seek reapproval if the project extends beyond the termination date.

The Board wishes you success in the pursuit of your research goals.

xc: Hirsch, Math & Statistics

Approval Termination: July 15, 1993
BIBLIOGRAPHY


