Common Moment Sets of Complementary Graphs

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COMMON MOMENT SETS OF COMPLEMENTARY GRAPHS

by

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COMMON MOMENT SETS OF COMPLEMENTARY GRAPHS

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Two sequences of nonnegative integers have the k^{th} common moment if they have equal sums of k^{th} powers. We intend to study common moment sets of the degree sequences of complementary graphs, and similarly, of the score sequences of complementary tournaments.

In Chapter I, we first study common moment sets of arbitrary sequences of nonnegative integers. Some basic concepts are introduced. The relations between characteristic functions and initial common moments are discovered. We extend Hua's discussion of the Tarry-Escott problem. We conclude that any finite subset of nonnegative integers can be a common moment set of some sequences, and conversely, any common moment set must be finite. Complementary sequences are also discussed.

We study common moments problems of the degree sequences of complementary graphs in Chapter II. Some interesting results on testing degree sequences are given. We explicitly construct complementary graphs which have the first $2p+1$ moments in common for any p. Furthermore, we seek the smallest such graphs. This can be achieved with $cp^2 \ln p$ vertices. We introduce Pascal submatrices and study their singularity. Finally we characterize the sets which can be the common moment sets of complementary graphs.

In Chapter III, we study common moment sets of complementary tournaments. We give some new observations for testing score sequences. We present Schwenk's construction of certain complementary tournaments. This provides us with a means of finding complementary tournaments of small order with desired initial common moments. We construct some examples of tournaments which

possess a given common moment set. Finally, we characterize common moment sets using the concept of initial density.

Several open problems are presented in Chapter IV for further study.

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For

Mom and Dad

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TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER I

INTRODUCTION TO MOMENTS OF SEQUENCES

1.1 Introduction

We intend to study properties of the degree sequences of complementary graphs, and similarly, the score sequences of complementary tournaments. We begin by defining several concepts relating to arbitrary sequences of real numbers.

The k^{th} *moment* of a number sequence $R = \{r_i\}_1^n$ is defined to be the sum of the k^{th} powers of the elements r_i, that is, $m_k(R) = \sum_{i=1}^{n} r_i^k$. It is convenient to assume that $r^0 = 1$ for any number r. Two sequences R and Q of nonnegative integers are said to share the k^{th} moment if $m_k(R) = m_k(Q)$. The *common moment set* of R and Q is $P = \{k \mid m_k(R) = m_k(Q)\}$. The *initial interval* of the common moment set is defined to be $P_0 = \{0, 1, \ldots, M(R,Q)\}$, where $M(R,Q) = M = \max\{p \mid m_k(R) = m_k(Q),\ 0 \leq k \leq p\}$. Thus $P = P_0 \cup A$ where $A \subset \{M+1, M+2, \ldots\}$ with $M+1 \notin A$. We notice that if R and Q are two sequences of length n then $m_0(R) = m_0(Q) = n$. If R and Q are the same multiset, we interpret $M(R,Q) = \infty$. In this dissertation, we only consider two distinct sequences which implies $M(R,Q) \leq n-1$. For general graph theoretic terminology the reader is referred to Chartrand and Lesniak [2].

We shall discuss common moments problems when R and Q are the degree sequences of a graph G and its complement $\overline{G}$. Although it is possible for nonisomorphic graphs to have identical degree sequences, we will not be concerned with that possibility since we only consider the existence of some graph possessing the degree sequence in question. Thus we shall let G denote both a degree sequence and any of the possible graphs possessing that sequence, so that $G = \{d_i\}_1^n$. We

shall assume G and $\overline{G}$ have distinct degree sequences. Thus the common moment set P is a finite subset of $\mathbb{N}\cup\{0\}$. In fact, we shall show that $P = \{0, 1, \ldots, 2p\} \cup A$ where $p \geq 0$ and $A \subset \{2p+3, 2p+4, \ldots\}$. For every integer p, we explicitly construct a graph which shares the first $2p+1$ common moments with its complement, and furthermore, we seek the smallest such graph. This can be achieved with $cp^2 \ln p$ vertices. We characterize the sets P which can be the common moment set of G and $\overline{G}$.

We also study common moments problems for score sequences of a tournament T and its complement T^c. We shall not consider nonisomorphic tournaments that share a common score sequence. Thus we let T denote both a tournament score sequence and any of the possible tournaments possessing that sequence, so that $T = \{t_i\}_1^n$. We shall also assume that T and T^c have distinct score sequences. This leads to the interesting question whether $M(T, T^c)$ can be large without having $T = T^c$. On the 1966 Putnam Examination Problem B2 stated, in effect, that $M(T, T^c) \geq 2$ for every tournament of order n. This is straightforward because

$$\begin{aligned} m_1(T^c) &= m_1(T) = \binom{n}{2}, \\ m_2(T^c) &= \sum_{i=1}^{n}(t_i^c)^2 = \sum_{i=1}^{n}(n-1-t_i)^2 \\ &= n(n-1)^2 - 2(n-1)\sum_{i=1}^{n} t_i + \sum_{i=1}^{n}(t_i)^2 = \sum_{i=1}^{n}(t_i)^2 \\ &= m_2(T). \end{aligned}$$

E. Wang and Z. Shan [11] showed that for any p, there exists a tournament T with $M(T, T^c) \geq p$, but they only gave an existence proof that does not give useful bounds. We shall show that $M(T, T^c)$ is always even. For every even integer $2p$, we explicitly construct a tournament T with $M(T, T^c) = 2p$. We shall present Schwenk's construction of certain complementary tournaments. This provides us a means of constructing a tournament of small order with $M(T, T^c)$ large.

Erdös asked whether any tournament and its complement yield a nonempty set A. For a long time we could not find any example with A nonempty. We shall show that the common moment set of T and T^c is $P = \{0, 1, \ldots, 2p\} \cup A$ where $p \geq 1$ and $A \subset \{2p+1, 2p+2, \ldots\}$ with $2p+1, 2p+2 \notin A$. We shall present some examples of tournaments which possess nonempty A. Finally, we characterize the sets P which can be the common moment set of T and T^c.

We shall use multiset notation for these sequences. Let $\{n_1 \cdot r_1, \ldots, n_p \cdot r_p\}$ denote a sequence consisting of elements r_i with the repetition number n_i, for $1 \leq i \leq p$. For $R = \{r_i\}_1^n$ and $Q = \{q_i\}_1^m$, we use $\cup$ to denote the multiset union, so that $R \cup Q = \{r_i\}_1^n \cup \{q_i\}_1^m = \{r_1, \ldots, r_n, q_1, \ldots, q_m\}$. We write $k \cdot R$ to mean the multiset union of k copies of R. We also denote $\delta + R = R + \delta = \{\delta + r_i\}_1^n$ as the translation of sequence R by the number δ. For a sequence $R = \{r_i\}_1^n$, we define Ru to be the result of multiplying each score in R by u, that is, $Ru = \{r_i u\}_1^n$. Be careful to distinguish Ru from $u \cdot R$ which replicates each score in R exactly u times. We also notice that $m_k(Ru) = u^k m_k(R)$.

1.2 Characteristic Functions and the Initial Interval

For two sequences of length n both arranged in nonincreasing order, we say that $R = \{r_i\}_1^n$ *dominates* $Q = \{q_i\}_1^n$ if there is an index k such that $r_i = q_i$ for $i < k$ and $r_k > q_k$. For distinct sequences, one must always dominate the other. We shall assume, without loss of generality, that R dominates Q. The *characteristic function* of R and Q is

$$f(x; R, Q) = \sum_{i=1}^{n} (x^{r_i} - x^{q_i}).$$

Lemma 1.1 *Let R and Q be two sequences of length n, and let R dominate Q. Then $f(x; R, Q) = (x-1)^{1+M(R,Q)} g(x)$ with $g(1) \neq 0$.*

Proof. By definition $f(1; R, Q) = 0$, so select $p \geq 0$ so that

$$f(x; R, Q) = (x-1)^{p+1} g(x),$$

with $g(1) \neq 0$. We define $F_k(x)$ recursively by

$$\begin{aligned} F_0(x) &= f(x; R, Q), \\ F_k(x) &= xF'_{k-1}(x), \quad (k \geq 1). \end{aligned} \tag{1.1}$$

Evaluating $F_k(x)$ at $x = 1$ yields

$$F_k(1) = \sum_{i=1}^{n}(r_i^k - q_i^k) = m_k(R) - m_k(Q), \text{ for } k \geq 1.$$

Thus $x - 1$ is a factor of $F_k(x)$ for $k \leq p$ and so

$$F_k(x) = (x-1)g_k(x), \text{ for } 0 \leq k \leq p.$$

But we find $F_{p+1}(x)$ by repeatedly applying (1.1) to $f(x; R, Q) = (x-1)^{p+1}g(x)$. After collecting all terms having at least one factor of $x - 1$ we get

$$F_{p+1}(x) = (x-1)g_{p+1}(x) + (p+1)!x^{p+1}g(x).$$

Therefore, $F_k(1) = 0$ for $0 \leq k \leq p$, and $F_{p+1}(1) = (p+1)!g(1) \neq 0$. Hence, $p = \max\{\, j \mid m_k(R) = m_k(Q),\ 0 \leq k \leq j \,\} = M(R, Q)$. ∎

Corollary 1.2 *If $f(x; R, Q) = (x-1)^{p+1}g(x)$ and $g(x)$ is a polynomial of all positive terms, then $m_k(R) > m_k(Q)$ for $k > p$.*

Proof. In the proof of Lemma 1.1, we observe that $F_k(x) = (x-1)g_k(x) + h_k(x)$. If $k > p$, then $h_k(x)$ is a polynomial of all positive terms. Therefore, we have $m_k(R) - m_k(Q) = F_k(1) > 0$, that is, $m_k(R) > m_k(Q)$. ∎

Lemma 1.3 *Let R, Q be two sequences of length n, and let δ be any number. Then $M(R, Q) = M(\delta + R, \delta + Q)$.*

Proof. By Lemma 1.1, $f(x; R, Q) = (x-1)^{p+1}g(x)$ where $p = M(R, Q)$ and $g(1) \neq 0$. Therefore, $f(x; \delta + R, \delta + Q) = (x-1)^{p+1}g(x)x^{\delta}$. So by Lemma 1.1, $p = M(\delta + R, \delta + Q)$. ∎

The next lemma constructs new sequences which increase $M(R, Q)$ by two.

Lemma 1.4 *Let $R = \{r_i\}_1^n$, $Q = \{q_i\}_1^n$ be two sequences with $M(R,Q) = p$. For any pair of integers $\Delta > \delta \geq 0$, let*

$$\begin{aligned} R' &= (R-\Delta)\cup(Q-\delta)\cup(Q+\delta)\cup(R+\Delta), \\ Q' &= (Q-\Delta)\cup(R-\delta)\cup(R+\delta)\cup(Q+\Delta). \end{aligned} \tag{1.2}$$

Then $M(R',Q') = p+2$.

Proof. By Lemma 1.1, we have $f(x;R,Q) = (x-1)^{p+1}g(x) = f(x)$ where $g(1) \neq 0$. The characteristic function of R' and Q' is

$$\begin{aligned} f(x;R',Q') &= x^{-\Delta}f(x) - x^{-\delta}f(x) - x^{\delta}f(x) + x^{\Delta}f(x) \\ &= (x-1)^{p+1}g(x)x^{-\Delta}(x^{\Delta+\delta}-1)(x^{\Delta-\delta}-1) \\ &= (x-1)^{(p+2)+1}g(x)x^{-\Delta}(\sum_{i=0}^{\Delta+\delta-1} x^i)(\sum_{i=0}^{\Delta-\delta-1} x^i) \end{aligned}$$

By Lemma 1.1, $M(R',Q') = p+2$, since $g(1)1^{-\Delta}(\sum_{i=0}^{\Delta+\delta-1} 1^i)(\sum_{i=0}^{\Delta-\delta-1} 1^i) \neq 0$ ∎

1.3 The Tarry-Escott Problem and Common Moments

There is considerable research on common moment problems in the number theory literature. Dickson [4] reports that Tarry stated that the first $2^p(2c+1)$ integers can be separated into two sets each of $2^{p-1}(2c+1)$ integers having k^{th} common moments for $k = 1, 2, \ldots, p$. In addition, Escott [4] showed how to find all sequences $R = \{r_i\}_1^n$ and $Q = \{q_i\}_1^n$ having first and second common moments. Hua discussed the Equal Power Sums Problem in [8]. Here, we state some results from Hua's discussion without proof. All results have been transcribed into our notation for the convenience of the reader.

Theorem 1.5 *Let R and Q be two distinct integer sequences of length n. Then $M(R,Q) \leq n-1$.*

p	n	sequences R and Q
0	1	$R=\{1\}, \quad Q=\{0\}$
1	2	$R=\{0,3\}, \quad Q=\{1,2\}$
2	3	$R=\{1,2,6\}, \quad Q=\{0,4,5\}$
3	4	$R=\{0,4,7,11\}, \quad Q=\{1,2,9,10\}$
4	5	$R=\{1,2,10,14,18\}, \quad Q=\{0,4,8,16,17\}$
5	6	$R=\{0,4,9,17,22,26\}, \quad Q=\{1,2,12,14,24,25\}$
6	7	$R=\{0,18,27,58,64,89,101\}, \quad Q=\{1,13,38,44,75,84,102\}$
7	8	$R=\{0,4,9,23,27,41,46,50\}, \quad Q=\{1,2,11,20,30,39,48,49\}$
8	9	$R=\{0,24,30,83,86,133,157,181,197\}$, $Q=\{1,17,41,65,112,115,168,174,198\}$
9	10	$R=\{0,3083,3301,11893,23314,24186,35607,44199,44417,47500\}$, $Q=\{12,2865,3519,11869,23738,23762,35631,43981,44635,47488\}$

Figure 1.1 Sequences of Smallest Order n with $M(R,Q)=p\leq 9$.

From Theorem 1.5 and Figure 1.1, we have the next theorem.

Theorem 1.6 *For any integer $0\leq p\leq 9$, the shortest pair of sequences R and Q with $M(R,Q)=p$ have order $n=p+1$.*

Theorem 1.7 *For any integer $p\geq 0$, there exists two distinct sequences R and Q of length $n\leq \frac{1}{2}p(p+1)+1$ such that $M(R,Q)\geq p$.*

Theorem 1.8 *For any integer $p\geq 0$, there exists two sequences R and Q of length n such that $M(R,Q)=p$ and $n\leq 2^p$.*

Hua sketches an existential inductive proof. But we will construct explicit pairs that achieve $M(R,Q)=p$ with length $n=2^p$. These pairs will be frequently used in Chapters II and III.

Theorem 1.9 *For any $p \geq 0$, there exist two sequences R_p and Q_p of length 2^p so that the common moment set $P = \{0, 1, \ldots, p\}$.*

Proof. If p is even, then let

$$R_p = \{\binom{p+1}{2i+1} \cdot (p+1-2i) \mid i = 0, 1, \ldots, \frac{p}{2}\},$$

$$Q_p = \{\binom{p+1}{2i} \cdot (p-2i) \mid i = 0, 1, \ldots, \frac{p}{2}\}.$$

If p odd, then let

$$R_p = \{\binom{p+1}{2i} \cdot (p+1-2i) \mid i = 0, 1, \ldots, \frac{p+1}{2}\},$$

$$Q_p = \{\binom{p+1}{2i+1} \cdot (p-2i) \mid i = 0, 1, \ldots, \frac{p-1}{2}\}.$$

In both cases, the characteristic function of R_p and Q_p is $f(x; R_p, Q_p) = (x-1)^{p+1}$. By Lemma 1.1 and Corollary 1.2, $m_k(R_p) = m_k(Q_p)$ for $k \leq p$ and $m_k(R_p) \neq m_k(Q_p)$ for $k > p$. That is, the common moment set $P = \{0, 1, \ldots, p\}$. ∎

Observe that these results all deal with the initial interval P_0. The next lemma constructs a pair with common moment set $P = P_0 \cup A$ with $A \neq \emptyset$.

Lemma 1.10 *For any integers p_1 and p_2, $1 \leq p_1 < p_2$, there exist two sequences R and Q such that $m_k(R) = m_k(Q)$ for $1 \leq k \leq p_1$, and $k = p_2 + 1$, but $m_k(R) \neq m_k(Q)$ for $p_1 < k \leq p_2$.*

Proof. Let R_{p_i}, Q_{p_i} be the sequences in Lemma 1.9. Then $m_k(R_{p_i}) - m_k(Q_{p_i}) = 0$ if and only if $1 \leq k \leq p_i$ for $i = 1, 2$. Let n_1, n_2 be the smallest positive integers such that $n_1[m_{p_2+1}(R_{p_1}) - m_{p_2+1}(Q_{p_1})] = n_2[m_{p_2+1}(R_{p_2}) - m_{p_2+1}(Q_{p_2})]$. We define

$$R = n_1 \cdot R_{p_1} \cup n_2 \cdot Q_{p_2}, \quad Q = n_1 \cdot Q_{p_1} \cup n_2 \cdot R_{p_2}.$$

Then $m_k(R)-m_k(Q) = n_1[m_k(R_{p_1})-m_k(Q_{p_1})]-n_2[m_k(R_{p_2})-m_k(Q_{p_2})]$. Therefore $m_k(R) - m_k(Q) = 0$ for $1 \leq k \leq p_1$ and $k = p_2 + 1$. But $m_k(R) - m_k(Q) \neq 0$ for $p_1 < k \leq p_2$. ∎

Theorem 1.11 *No infinite set $P \subset \mathrm{N} \cup \{0\}$ can be a common moment set. On the other hand, for any finite set $P \subset \mathrm{N} \cup \{0\}$ there exists a pair of nonnegative integer sequences R and Q whose common moment set is P.*

Proof. Suppose $R = \{r_i\}_1^n$ dominates $Q = \{q_i\}_1^n$. For convenience of notation, let us assume that any common values have been removed from both R and Q, so that $r_n > q_n$ where both sets have order n. Now for every p we have

$$m_p(R) - m_p(Q) = \sum_{i=1}^{n} r_i^p - \sum_{i=1}^{n} q_i^p.$$

Since each $r_i \geq 0$ and each $q_i \leq r_n - 1$, we see that $m_p(R)-m_p(Q) \geq r_n^p - n(r_n-1)^p$. This will be strictly positive for all p greater than

$$\frac{\ln n}{\ln r_n - \ln(r_n - 1)}.$$

Thus, the common moment set must be finite.

We observe that $0 \in P$ if and only if R and Q have the same length. Inserting a 0 to a sequence increases its length by one but does not change the k^{th} moment for any $k > 0$. Therefore, we may assume that P is a finite subset of N. To attain each P, we imitate the proof of Lemma 1.10 to add elements to P one by one. Let p_k be the largest element of P and $P' = P - \{p_k\}$. We shall proceed by induction on the number of elements in P. For any singleton $P = \{p\}$ we may set $R = \{2^p \cdot 1\}$ and $Q = \{(2^p - 1) \cdot 0, 1 \cdot 2\}$. Clearly the only matching moment is $m_p(R) = m_p(Q) = 2^p$. Now assume by induction that we have already constructed R' and Q' that attains P'. We wish to add p_k to the common moment set. Select n_1 and n_2 to be the smallest pair of positive integers with $n_1[m_{p_k}(R') - m_{p_k}(Q')] = n_2[m_{p_k}(R_{p_k-1}) - m_{p_k}(Q_{p_k-1})]$. We define the new sequences to be

$$R = n_1 \cdot Q' \cup n_2 \cdot R_{p_k - 1}, \qquad Q = n_1 \cdot R' \cup n_2 \cdot Q_{p_k - 1}.$$

Now $m_j(R) = m_j(Q)$ for all $j \in P$. It is conceivable (although highly unlikely) that we may accidentally have some additional matching moments larger than p_k, let us say $p_k < i_1 < i_2 < \ldots < i_j$. Since R dominates Q (without loss of generality) we may presume that for some $p_0 > i_j$ we have $m_i(R) - m_i(Q) > 0$ for $i \geq p_0$. Now augment R and Q with $R_{i_1 - 1}$ and $Q_{i_1 - 1}$ to remove the accidental matching at i_1. Do this so that R remains dominant over Q. Since $m_i(R_{i_1-1}) - m_i(Q_{i_1-1}) > 0$ for all $i \geq i_1$, we shall still have $m_i(R) - m_i(Q) > 0$ for all $i \geq p_0$. We have altered the list of accidental matches, but now any new matches must lie between i_1 and p_0. Since $i_1 > p_k$, we have reduced the length of the interval containing accidental matches. We repeat this argument until all accidental matches have been removed. Thus, we have constructed R and Q which attain precisely P as their common moment set. ∎

1.4 Complementary Sequences

We consider special sequences whose elements are integers between 0 and $n-1$. Two sequences of nonnegative integers $R = \{r_i\}_1^n$ and $Q = \{q_i\}_1^n$ are said to be *complementary* if $q_i = n - 1 - r_i$ for $i = 1, 2, \ldots, n$. Thus, $r_i = n - 1 - q_i$. The *midrange* of a sequence is the middle of the range of potential elements, that is, for a sequence of order n, it has the value $a = \frac{n-1}{2}$. For complementary sequences R and Q, there exists a sequence $S = \{s_i\}_1^n$ so that $r_i = a + s_i$, $q_i = a - s_i$ for $i = 1, 2, \ldots, n$. The sequence S is called the (midrange) translated sequence. Hence the k^{th} power of r_i and q_i can be writen in the following forms.

$$\begin{aligned} r_i^k &= (a + s_i)^k = \sum_{j=0}^{k} \binom{k}{j} a^{k-j} s_i^j, \\ q_i^k &= (a - s_i)^k = \sum_{j=0}^{k} \binom{k}{j} a^{k-j} (-1)^j s_i^j. \end{aligned}$$

Therefore, the k^{th} moments of R and Q are

$$\begin{aligned} m_k(R) &= \sum_{j=0}^{k} \binom{k}{j} a^{k-j} m_j(S), \\ m_k(Q) &= \sum_{j=0}^{k} \binom{k}{j} a^{k-j} (-1)^j m_j(S)\,. \end{aligned} \tag{1.3}$$

Lemma 1.12 *Let R and Q be complementary sequences of nonnegative integers. Then $M(R,Q) \geq p$ if and only if $m_{2j+1}(S) = 0$ for $j = 0, 1, \ldots, \lfloor \frac{p-1}{2} \rfloor$.*

Proof. From (1.3), we observe that $m_k(R) = m_k(Q)$ if and only if

$$\sum_{j=0}^{k} \binom{k}{j} a^{k-j} m_j(S) = \sum_{j=0}^{k} \binom{k}{j} a^{k-j} (-1)^j m_j(S).$$

This is equivalent to

$$2 \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k}{2j+1} a^{k-2j-1} m_{2j+1}(S) = 0. \tag{1.4}$$

Now, $M(R,Q) \geq p$ if and only if (1.4) holds for $k = 1, 2, \ldots, p$ if and only if $m_{2j+1}(S) = 0$ for $j = 0, 1, \ldots, \lfloor \frac{p-1}{2} \rfloor$. ∎

Theorem 1.13 *Let two complementary sequences of nonnegative integers R and Q be of order $n \geq 2$. Then $M(R,Q)$ is always an even integer $2p \leq n-2$.*

Proof. We have $M(R,Q) \geq 0$ since R and Q have the same length. Assume that $M(R,Q) \geq 2k-1$. By Lemma 1.12, $M(R,Q) \geq 2k$. That is, $M(R,Q)$ is always an even integer $2p$.

By Lemma 1.1, the characteristic function $f(x; R, Q) = (x-1)^{2p+1} g(x)$ where $2p = M(R,Q)$, $g(1) \neq 0$. Therefore,

$$2p + 1 \leq deg(f(x; R, Q)) \leq \max\{r \mid r \in R \cup Q\} \leq n - 1.$$

That is, $2p \leq n-2$. ∎

Theorem 1.14 *Let R and Q be complementary sequences of nonnegative integers with $M(R,Q) = 2p$. Then $m_j(R) \neq m_j(Q)$ for $j = 2p+1, 2p+2$.*

Proof. Automatically $m_{2p+1}(R) \neq m_{2p+1}(Q)$ by definition of $M(R,Q) = 2p$. Let S be the degree sequence translated by the midrange a. By Lemma 1.12, we conclude sequentially that $m_{2j+1}(S) = 0$ for $0 \leq j \leq p-1$, but $m_{2p+1}(S) \neq 0$. Now in (1.3) with $k = 2p+2$ we observe that all terms cancel except for $j = 2p+1$ leaving

$$m_k(R) - m_k(Q) = 2\begin{pmatrix} 2p+2 \\ 2p+1 \end{pmatrix} a m_{2p+1}(S).$$

Thus, $m_{2p+2}(R) \neq m_{2p+2}(Q)$. ∎

CHAPTER II

COMMON MOMENTS OF COMPLEMENTARY GRAPHS

2.1 Graphical Sequences

Let $G = \{d_i\}$ and $\overline{G} = \{\overline{d_i}\}$ denote the graph G and its complement $\overline{G}$. To construct a complementary pair of graphs with given common moments we shall construct a potential sequence of numbers. Then we need to check if the sequence so constructed is a legal degree sequence for some graph. The following fundamental theorem of testing degree sequences is due to Erdös and Gallai [5].

Theorem 2.1 *Let $\{d_i\}_1^n$ be a nonincreasing sequence of nonnegative integers. Then $\{d_i\}_1^n$ is graphical if and only if $\sum_{i=1}^{n} d_i$ is even, and for any $1 \leq t \leq n-1$,*

$$\sum_{i=1}^{t} d_i \leq t(t-1) + \sum_{i=t+1}^{n} min\{t, d_i\}.$$

From the Erdös-Gallai theorem, we find an interesting theorem which guarantees the existence of a graph with a given degree sequence provided the range of degrees is not too large.

Theorem 2.2 *Let $\{d_i\}_1^n$ be a nonincreasing sequence of nonnegative integers, and let $\sum_{i=1}^{n} d_i$ be even. If $x \leq d_i \leq 2\sqrt{nx} - x - 1$ for some $0 < x < \frac{n}{2}$, then $\{d_i\}_1^n$ is graphical.*

Proof. Since $x \leq d_i \leq 2\sqrt{nx} - x - 1$, we have the following inequalities.

$$\sum_{i=1}^{t} d_i \leq 2t\sqrt{nx} - tx - t, \quad \text{for } 1 \leq t \leq n-1.$$

And,

$$t(t-1)+\sum_{i=t+1}^{n} min\{t,d_i\} \geq \begin{cases} t(t-1)+(n-t)t, & \text{if } t \leq d_n; \\ t(t-1)+(n-t)x, & \text{otherwise.} \end{cases}$$

Either bound leads to the verification of the Erdös-Gallai condition. In the first case, for any $1 \leq t \leq n-1$, we have $t(t-1)+(n-t)t = t(n-1) \geq \sum_{i=1}^{t} d_i$. For the second case, we need to observe that $(t-\sqrt{nx})^2 \geq 0$ implies that $t^2+nx \geq 2t\sqrt{nx}$. Consequently, $t(t-1)+(n-t)x \geq 2t\sqrt{nx} - tx - t$. Therefore both cases yield $\sum_{i=1}^{t} d_i \leq t(t-1) + \sum_{i=t+1}^{n} min\{t,d_i\}$. By Theorem 2.1, $\{d_i\}_1^n$ is graphical. ∎

When the lower bound of the sequence $x \geq \frac{n}{2}$, we shall apply this theorem to its complementary sequence $\{\overline{d_i}\}_1^n$ where $\overline{d_i} = n-1-d_i$.

Corollary 2.3 *Let $\{d_i\}_1^n$ be a nonincreasing sequence of nonnegative integers and let $\sum_{i=1}^{n} d_i$ be even. If $\lceil \frac{n}{4} \rceil \leq d_i \leq \lfloor \frac{3n}{4} \rfloor - 1$, then $\{d_i\}_1^n$ is graphical.*

Proof. In Theorem 2.2, we let $x = \frac{n}{4}$. Then,

$$2\sqrt{nx} - x - 1 = 2\sqrt{n\tfrac{n}{4}} - \tfrac{n}{4} - 1 = \tfrac{3n}{4} - 1$$

Hence, $x \leq \lceil \frac{n}{4} \rceil \leq d_i \leq \lfloor \frac{3n}{4} \rfloor - 1 \leq 2\sqrt{nx} - x - 1$. By Theorem 2.2 the sequence $\{d_i\}_1^n$ is graphical. ∎

In Theorem 2.2, setting $x = 1$ yields a theorem by Erdös, Jacobson, and Lehel [6] that we now state as a corollary.

Corollary 2.4 *If $k \geq d_1 \geq d_2 \geq \cdots \geq d_n \geq 1$, $n \geq (\frac{k+2}{2})^2$ and $\sum_{i=1}^{n} d_i$ is even, then $\{d_i\}_1^n$ is graphical.*

2.2 Initial Common Moments

The *midrange degree* of a graph is the middle of the range of potential degrees, that is, for a graph of order n, it has the value $a = \frac{n-1}{2}$. Let $G = \{d_i\}_1^n$ and $\overline{G} = \{\overline{d_i}\}_1^n$ be the degree sequences of a graph and its complement, then there

exists a sequence $S = \{s_i\}_1^n$ so that $d_i = a + s_i$, $\overline{d_i} = a - s_i$ for $1 \le i \le n$. We refer to S as the (midrange degree) translated degree sequence. Hence,

$$\begin{aligned} m_k(G) &= \sum_{j=0}^{k} \binom{k}{j} a^{k-j} m_j(S), \\ m_k(\overline{G}) &= \sum_{j=0}^{k} \binom{k}{j} a^{k-j} (-1)^j m_j(S) \,. \end{aligned} \tag{2.1}$$

Therefore, G and $\overline{G}$ have the k^{th} common moment if and only if

$$\sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k}{2j+1} a^{k-2j-1} m_{2j+1}(S) = 0. \tag{2.2}$$

From Theorem 1.13 and Lemma 1.12, we state the next two lemmas.

Lemma 2.5 *Let G be a graph of order $n \ge 2$ and its degree sequence $G \ne \overline{G}$. Then $M(G, \overline{G})$ is always an even integer $2p \le n - 2$.*

Lemma 2.6 *Let G be a graph. Then $M(G, \overline{G}) \ge 2p$ if and only if $m_{2j+1}(S) = 0$ for $0 \le j \le p - 1$.*

For $p = 0$, we observe that the complete graph K_2 and its complement $\overline{K_2}$ have $M(K_2, \overline{K_2}) = 2p = 0$.

Theorem 2.7 *For any integer $p > 0$, there exists a graph G_{2p} of order 4^p such that $M(G_{2p}, \overline{G_{2p}}) = 2p$.*

Proof. Let the sequence

$$G_{2p} = \{ \binom{2p+1}{2i} \cdot (2p + 1 - 2i + \Delta) \mid 0 \le i \le p \},$$

and its complementary sequence

$$\overline{G_{2p}} = \{ \binom{2p+1}{2i+1} \cdot (2p - 2i + \Delta) \mid 0 \le i \le p \},$$

where $\Delta = 2(4^{p-1}) - p - 1$. Then the order of G_{2p} is $n = 2^{2p} = 4^p$. Now, we have

$$\begin{aligned}\min\{d \in G_{2p}\} &= 1 + \Delta = 2(4^{p-1}) - p > \frac{n}{4},\\ \max\{d \in G_{2p}\} &= 2p + 1 + \Delta = 2(4^{p-1}) + p < \frac{3n}{4} - 1.\end{aligned}$$

By Corollary 2.3, there exist complementary graphs with G_{2p} and $\overline{G_{2p}}$ as their degree sequences. The characteristic function for the complementary graphs is $f(x; G_{2p}, \overline{G_{2p}}) = (x-1)^{2p+1}x^{\Delta}$ which can be verified by expanding $(x-1)^{2p+1}$. Lemma 1.1, shows that $M(G_{2p}, \overline{G_{2p}}) = 2p$. ∎

While this construction is simple, it suggests that the order of G_{2p} grows exponentially in p. This need not be the case as the next theorem shows. This result closely parallels a result for tournaments in [3] and is proved by probabilistic methods.

Theorem 2.8 *There exists a constant c such that for every positive integer p there exists a graph G of order at most $cp^2 \ln p$ and with $M(G, \overline{G}) \geq 2p$.*

Proof. We shall seek a graph G of order $n = 4k + 1$ where k remains to be determined as a function of p. We shall work with the (midrange degree) translated sequence S. Let $S = Q \cup \{0\} \cup (-R)$, where $Q = \{q_i\}_1^{2k}$ with $q_i \geq 0$, and $R = \{r_i\}_1^{2k}$ with $-r_i \leq 0$. By Lemma 2.6, $M(G, \overline{G}) \geq 2p$ if and only if $m_{2j-1}(S) = 0$ for all $j \leq p$. This in turn forces $m_{2j-1}(R) = m_{2j-1}(Q)$.

If we require the absolute value of each translated degree to be at most $k-1$, then by Theorem 2.2 the resulting graph degree sequence must be legal. We shall count the number of such sequences and also count the number of possible p-tuples

$$(m_1(Q), m_3(Q), \ldots, m_{2p-1}(Q)).$$

If we can show there are more sequences than p-tuples, then the pigeonhole principle guarantees we can find two sequences R and Q with precisely the same odd

moments. Then the translated sequence S gives us a graph $G = a + S$ with $M(G, \overline{G}) \geq 2p$.

The number of sequences $0 \leq q_1 \leq q_2 \leq \cdots \leq q_{2k} \leq k-1$ can be viewed as a lattice walk in the plane, starting at $(0,0)$ and ending at $(2k, k-1)$. Each integer q_i gives us a horizontal step from $(i-1, q_i)$ to (i, q_i). All other steps are upward steps. The sequences are in one-to-one correspondence with the walks and each walk has $2k$ horizontal steps and $k-1$ vertical steps. Thus, there are $\binom{3k-1}{k-1}$ possible sequences for Q.

Each moment $m_{2j-1}(Q)$ is at most $2k(k-1)^{2j-1}$. Since we cannot get a suitable example by choosing Q to be identically 0, we may presume $m_i(Q) \geq 1$. Thus, the j^{th} position in the p-tuple can be filled in at most $2k(k-1)^{2j-1}$ ways. Consequently, the number of p-tuples is $(2k)^p(k-1)^{p^2}$.

Now the pigeonhole principle forces a repeated p-tuple if the number of nontrivial sequences exceeds the number of p-tuples, that is,

$$\binom{3k-1}{k-1} - 1 \geq (2k)^p(k-1)^{p^2}.$$

We shall ignore the -1 and use Stirling's formula to estimate the binomial coefficient. We find that k must satisfy

$$\frac{3^{3k}}{\sqrt{12\pi k}\; 2^{2k}} > (2k)^p(k-1)^{p^2}.$$

Taking logarithms we require

$$k > \frac{(p+0.5)\ln k + p^2\ln(k-1) + p\ln 2 + 0.5\ln(12\pi)}{3\ln 3 - 2\ln 2}.$$

Since k can be chosen to be at most p^3+1, $\ln(k-1)$ is less than $3\ln p$. Thus, roughly speaking, k must be chosen to be a constant times $p^2 \ln p$. Since $n = 4k+1$, we have verified the order of growth claimed in the theorem. ∎

2.3 Pascal Submatrices

In this section, we shall consider a special matrix. Let $r_1 < r_2 < \cdots < r_n$ and $c_1 < c_2 < \cdots < c_n$ be two sequences of nonnegative integers. A *Pascal submatrix* H is an $n \times n$ matrix whose entry at the ij position is the binomial coefficient $\binom{r_i}{c_j}$. That is,

$$H = \begin{pmatrix} \binom{r_1}{c_1} & \binom{r_1}{c_2} & \cdots & \binom{r_1}{c_n} \\ \binom{r_2}{c_1} & \binom{r_2}{c_2} & \cdots & \binom{r_2}{c_n} \\ \vdots & \vdots & & \vdots \\ \binom{r_n}{c_1} & \binom{r_n}{c_2} & \cdots & \binom{r_n}{c_n} \end{pmatrix}.$$

Thus, H can be viewed as a submatrix of the lower triangular matrix formed by Pascal's Triangle.

Lemma 2.9 *Let H be any $n \times n$ Pascal submatrix. Then $det(H) \geq 0$.*

Proof. Let $c_i = c + \delta_i$ for $i \geq 1$ with $\delta_1 = 0$. We prove this lemma by induction on the order of the matrix. If $n = 1$, then

$$det(H) = \binom{r}{c} \geq 0.$$

Thus the lemma is true for $n = 1$. We now assume it is true for n and prove it is true when n is replaced by $n+1$. If $r_1 < c_1 = c$, then $det(H) = 0$ since each entry in row 1 of the matrix is 0. Now we assume that $r_1 \geq c_1$. By using the identity (see Brualdi [1], p.69, Exercise 16) that

$$\binom{x}{b+y} = \binom{x}{b}\binom{x-b}{y}\binom{b+y}{y}^{-1},$$

we can rewrite each entry in H. Then we take common factors $\binom{r_i}{c}$ from the i^{th} row and $\binom{c_j}{\delta_j}^{-1}$ from the j^{th} column of $det(H)$. Thus

$$det(H) = \left[\prod_{i=1}^{n+1}\binom{r_i}{c}\right]\prod_{j=2}^{n+1}\binom{c_j}{\delta_j}^{-1} det(H'),$$

where

$$H' = \begin{pmatrix} 1 & \binom{r_1 - c}{\delta_2} & \cdots & \binom{r_1 - c}{\delta_{n+1}} \\ 1 & \binom{r_2 - c}{\delta_2} & \cdots & \binom{r_2 - c}{\delta_{n+1}} \\ \vdots & \vdots & & \vdots \\ 1 & \binom{r_{n+1} - c}{\delta_2} & \cdots & \binom{r_{n+1} - c}{\delta_{n+1}} \end{pmatrix}.$$

From the formula $\binom{r}{b} = \sum_{i=0}^{r-1}\binom{i}{b-1}$ (see Brualdi [1], p.60, formula (4.3.9)), we have, for $r_2 > r_1$,

$$\binom{r_2}{b} - \binom{r_1}{b} = \sum_{i=r_1}^{r_2-1}\binom{i}{b-1}. \tag{2.3}$$

By subtracting consecutive rows and applying (2.3), we obtain

$$det(H') = \begin{vmatrix} 1 & \binom{r_1 - c}{\delta_2} & \cdots & \binom{r_1 - c}{\delta_{n+1}} \\ 0 & \sum_{i=r_1}^{r_2-1}\binom{i-c}{\delta_2 - 1} & \cdots & \sum_{i=r_1}^{r_2-1}\binom{i-c}{\delta_{n+1} - 1} \\ \vdots & \vdots & & \vdots \\ 0 & \sum_{i=r_n}^{r_{n+1}-1}\binom{i-c}{\delta_2 - 1} & \cdots & \sum_{i=r_n}^{r_{n+1}-1}\binom{i-c}{\delta_{n+1} - 1} \end{vmatrix}.$$

Now the form of the first column allows us to delete row 1 and column 1 without altering $det(H')$. To complete the proof by induction we need to replace this single matrix by a sum of many matrices each in the form of a Pascal submatrix. But we may think of each row as a vector sum and apply linearity of the determinant one row at a time to transport the summations within row k to the exterior of the det operator. Thus $\det(H') = \sum_j det(H_j)$, where the summation is over multiple indices, one for each row. But each individual H_j is again a Pascal submatrix of order n, by induction, $det(H_j) \geq 0$, so $det(H') = \sum_j det(H_j) \geq 0$. Therefore,

$$det(H) = \left[\prod_{i=1}^{n+1} \binom{r_i}{c}\right] \prod_{j=2}^{n+1} \binom{c_j}{\delta_j}^{-1} det(H') \geq 0. \qquad \blacksquare$$

Lemma 2.10 *Let H be an $n \times n$ Pascal submatrix. Then $det(H) > 0$ if and only if $r_i \geq c_i$ for $i = 1, 2, \ldots, n$.*

Proof. If $r_i \geq c_i$ for every i, we prove $det(H) > 0$ by applying Lemma 2.9 and by modifying its proof. Notice that if $n = 1$ then

$$det(H) = \binom{r}{c} > 0$$

since $r \geq c$. The other place needing modification is $det(H') > 0$ instead of $det(H') \geq 0$. In fact, among the H_j's, there exists an $n \times n$ Pascal submatrix

$$H_{j_0} = \begin{pmatrix} \binom{r_2 - 1 - c}{\delta_2 - 1} & \cdots & \binom{r_2 - 1 - c}{\delta_{n+1} - 1} \\ \vdots & & \vdots \\ \binom{r_{n+1} - 1 - c}{\delta_2 - 1} & \cdots & \binom{r_{n+1} - 1 - c}{\delta_{n+1} - 1} \end{pmatrix}.$$

We note that in this matrix every $r_i - 1 - c \geq \delta_i - 1$. By induction, $det(H_{j_0}) > 0$. Since each other $det(H_j) \geq 0$ by Lemma 2.9, $det(H) > 0$.

Conversely, if there exists some i_0 such that $r_{i_0} < c_{j_0}$, then $\binom{r_i}{c_i} = 0$ for $1 \le i \le i_0$ and $i_0 \le j \le n$ since $r_1 < r_2 < \cdots < r_n$ and $c_1 < c_2 < \cdots < c_n$. Therefore, the $n \times n$ Pascal submatrix is

$$H = \begin{pmatrix} * & O_{i_0 \times (n-i_0-1)} \\ * & * \end{pmatrix},$$

where $O_{i_0 \times (n-i_0-1)}$ is an $i_0 \times (n - i_0 - 1)$ block whose every entry is 0. By applying Laplace's Theorem in the first i_0 rows to expand the determinant, we have $det(H) = 0$. This completes the proof. ‖

2.4 Common Moment Sets of Complementary Graphs

We shall analyze the actual set P of common moments. Recall that we write $P = \{0, 1, \ldots, 2p\} \cup A$. In Theorem 2.7, we constructed graphs with $M(G, \overline{G}) \ge 2p$ for each positive p. Applying Corollary 1.2 to the characteristic functions, we find the common moment set $P = \{0, 1, \ldots, 2p\}$, that is, $A = \emptyset$. Our goal is to show that nonempty A's can occur. Furthermore, we characterize sets P which can be the common moment set of complementary graphs.

As a direct corollary of Lemma 1.14, the next lemma shows that the initial omission must delete at least two values, that is, $2p + 1,\ 2p + 2 \notin P$.

Lemma 2.11 *Let G be a graph with $M(G, \overline{G}) = 2p$. Then $m_j(G) \ne m_j(\overline{G})$ for $j = 2p + 1, 2p + 2$.*

The next lemma will be used to translate and scale sequences while maintaining common moment sets. Thus we shall construct sequences which possess given common moment set P. When the sequence fails to be graphical, we apply this lemma to find a pair of graphs which has the same common moment set.

Lemma 2.12 *Let $S' = \{s'_i\}$ be a sequence of integers, and let $a' \ge 1 + 2\max\{|s'_i|\}$ be an integer. Then there exists a graph $G = a + S$ and an integer $u > 0$ such that $m_k(G) - m_k(\overline{G}) = u^k[m_k(a' + S') - m_k(a' - S')]$.*

Proof. We add 0's to the sequence $S'u$ to form $S = \{s_i\}_1^n$ so that $a = a'u = \frac{n-1}{2}$ for some even integer $u \geq 6$. Let $G = a + S$. Then $\sum_{d \in G} d$ is even. Since $|s_i'| \leq \frac{a'-1}{2}$ for every i, we have $|s_i| \leq \frac{a'-1}{2}u$. Thus,

$$\begin{aligned} \min\{d \mid d \in G\} &\geq a - \frac{a'-1}{2}u = \frac{n}{4} - \frac{1}{4} + \frac{u}{2} \geq \lceil \frac{n}{4} \rceil, \\ \max\{d \mid d \in G\} &\leq a + \frac{a'-1}{2}u = \frac{3n}{4} - \frac{3}{4} - \frac{u}{2} \leq \lfloor \frac{3n}{4} \rfloor - 1. \end{aligned}$$

By Lemma 2.3, $G = a + S$ is graphical. From (2.1) subtraction leaves only the odd indexed terms, so that

$$\begin{aligned} m_k(G) - m_k(\overline{G}) &= 2 \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k}{2i+1} a^{k-2i-1} m_{2i+1}(S) \\ &= 2 \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k}{2i+1} (a')^{k-2i-1} u^{k-2i-1} m_{2i+1}(S') u^{2i+1} \\ &= u^k [m_k(a' + S') - m_k(a' - S')]. \qquad \blacksquare \end{aligned}$$

The following propositions give two examples of graphs which have the common moment set $P = \{0, 1, \ldots, 2p\} \cup A$ with $A \neq \emptyset$. They also provide a method of constructing graphs which possess a given P with nonempty A.

Proposition 2.13 *There exists a graph of order* 21 *with the common moment set* $P = \{0, 3\}$.

Proof. Let $n = 2a + 1$ be the order of the graph G. Then $G = a + S$ where S is the translated degree sequence. We want the common moment set $P = \{0, 3\}$. Examining the third moment, by (1.4), we require $m_1(S) \neq 0$, and

$$\binom{3}{1} a^2 m_1(S) + \binom{3}{3} m_3(S) = 0 \tag{2.4}$$

We begin with two sequences $R = \{1\}$, and $Q = \{1 \cdot 2, 2 \cdot (-1)\}$ to build S. To increase flexibility, we let $S = (n_1 \cdot Ru) \cup (Qv)$. Then

$$m_1(n_1 \cdot Ru) = n_1 u, \quad m_3(n_1 \cdot Ru) = n_1 u^3; \qquad m_1(Qv) = 0, \quad m_3(Qv) = 6v^3.$$

A convenient selection is $n_1 = 3$ and $u = 2$ to give $m_1(S) \neq 0$ and $-v^3 = 3a^2 + 4$ from the equation (2.4). Generally, to find a solution which is suitable to both this equation and a graph is difficult. To reduce the difficulty we replace Qv by the union of several Qv_i's so that we obtain $-\sum v_i^3 = 3a^2 + 4$. Since the v_i's may be negative or positive, this equation is nothing more than an instance of Waring's problem [10] which asserts that for each power p there exists a number of terms $h(p)$ such that every positive integer can be written as the sum of at most $h(p)$ terms with each term a p^{th} power. Thus, if we allow ourselves $h(3)$ terms we can certainly solve the equation. But this doesn't quite complete our task. Observe that

$$S = \left(\bigcup_{i=1}^{h(3)} Qv_i\right) \cup (3 \cdot R2)$$

has $4h(3) + 3$ individual translated degrees. To be a valid solution we require $n = 2a+1 > 4h(3)+3$. Since $h(3)$ is a presumably known constant, we can select n (and hence a) large enough to guarantee a solution. But this is extravagantly large. With a little searching, we can find values for n and a that fortuitously give a solution with fewer terms than $h(3)$. In particular, we find $n = 21$, $a = 10$ has the solution

$$3a^2 + 4 = 304 = 5^3 + 5^3 + 3^3 + 3^3.$$

The resulting sequence is $S = \{4 \cdot 5, 4 \cdot 3, 3 \cdot 2, 2 \cdot (-6), 2 \cdot (-10)\}$. Now $|S| = 15$ whereas $n = 21$. We correct this discrepancy by adding $6 \cdot 0$ to S. Upon translating by 10 we get the graph degree sequence

$$G = \{4 \cdot 15, 4 \cdot 13, 3 \cdot 12, 6 \cdot 10, 2 \cdot 4, 2 \cdot 0\}.$$

Now its complementary graph $\overline{G}$ is $a - S$. The common moment set is $P = \{0, 3\}$ since $0 \in P$ for any graph. ∎

Proposition 2.14 *There exists a graph G of order 601 with the common moment set $P = \{0, 3, 5\}$.*

Proof. We begin with the translated degree sequence

$$R = \{4 \cdot 5, 4 \cdot 3, 3 \cdot 2, 2 \cdot (-6), 2 \cdot (-10)\}$$

from the graph that has the common moment set $P = \{0, 3\}$ with its midrange degree 10. Therefore, we have

$$m_1(n_1 \cdot Ru) = 6n_1u, \quad m_3(n_1 \cdot Ru) = -1800n_1u^3, \quad m_5(n_1 \cdot Ru) = -201984n_1u^5.$$

We wish to combine R with a sequence whose first two odd moments are 0, specifically

$$Q = \{1 \cdot 5, 3 \cdot 1, 2 \cdot (-4)\}.$$

Upon scaling Q by a factor v, we find that the first three odd moments are

$$m_1(Qv) = 0, \quad m_3(Qv) = 0, \quad m_5(Qv) = 1080v^5.$$

Let us consider the translated degree sequence $S = (n_1 \cdot Ru) \cup (Qv)$. Then $m_1(S) \neq 0$ since we select positive n_1 and u. By (1.4), we must have

$$\binom{3}{1} a^2 m_1(S) + \binom{3}{3} m_3(S) = 0$$

to ensure the third common moment. This is valid if and only if $a = 10u$, so we shall take this as a condition defining the midrange degree a. To match the fifth moment, it forces

$$\binom{5}{1} a^4 m_1(S) + \binom{5}{3} a^2 m_3(S) + \binom{5}{5} m_5(S) = 0.$$

Setting $n_1 = 1$, $u = 30$, and $a = 10u = 300$ leads to $v^5 = 38294640000$. Let $v^5 = \sum v_i^5$. We find a collection of 76 terms for v_i which solve the equation

$$\sum v_i^5 = 38294640000.$$

Each v_i introduces 6 degrees into the sequence. After deleting complementary terms, $S = (R30) \cup (\cup Qv_i)$ has 463 terms. But $n = 2a + 1 = 601$. So we need to add $138 \cdot 0$ to the S to form the graph $G = a + S$, that is,

$$\begin{aligned} G &= \{53 \cdot 595, 1 \cdot 560, 4 \cdot 450, 2 \cdot 430, 4 \cdot 390, 2 \cdot 360, 159 \cdot 359, \\ &\quad 3 \cdot 352, 2 \cdot 348, 2 \cdot 336, 6 \cdot 326, 4 \cdot 320, 1 \cdot 312, 28 \cdot 304, \\ &\quad 138 \cdot 300, 42 \cdot 299, 6 \cdot 297, 20 \cdot 295, 3 \cdot 291, 2 \cdot 285, 2 \cdot 275, \\ &\quad 1 \cdot 255, 4 \cdot 196, 2 \cdot 120, 2 \cdot 92, 106 \cdot 64, 2 \cdot 0\}. \end{aligned}$$

Thus, $\overline{G}$ is just $a - S$. The common moment set is $P = \{0, 3, 5\}$. ∎

We can now present the main theorem in this chapter which characterizes precisely which sets can occur as A.

Theorem 2.15 *Let $A = \{r_1, r_2, \ldots, r_k\}$ be a finite subset of $\mathbf{N}$ with $2p+2 < r_1 < r_2 < \cdots < r_k$, where $p \geq 0$. Then there exists a graph G with common moment set $P = \{0, 1, \ldots, 2p\} \cup A$ if and only if $r_i > 2p + 2i$ for $1 \leq i \leq k$.*

Proof. Let G be a graph of order n, and let S be the degree sequence translated by the midrange degree a. Then $G = a + S$ and its complement $\overline{G} = a - S$. If G and $\overline{G}$ have common moment set $P = \{0, 1, \ldots, 2p\} \cup A$, then from Lemma 2.6 and (2.2), we observe that $m_{2i+1}(S) = 0$ for $0 \leq i \leq p - 1$ and

$$\sum_{j=p}^{\lfloor \frac{r_i - 1}{2} \rfloor} \binom{r_i}{2j+1} a^{r_i - 2j - 1} m_{2j+1}(S) = 0, \quad j \in A. \tag{2.5}$$

First, if there exists some $r_i \in A$ such that $r_i \leq 2p + 2i$, then we can find the smallest such $r_{i_0+1} \in A$ since $2p + 2 < r_1$. Let $A' = \{r_1, r_2, \ldots, r_{i_0}\}$. Then $r_i > 2p + 2i$ for every $r_i \in A'$. Observe that the system

$$\sum_{j=p}^{\lfloor \frac{r_i - 1}{2} \rfloor} \binom{r_i}{2j+1} a^{r_i - 2j - 1} m_{2j+1}(S) = 0, \quad r_i \in A' \tag{2.6}$$

consists of the first i_0 equations of the system (2.5). We notice that $2p+2i_0 < r_{i_0} < r_{i_0+1} \leq 2p+2(i_0+1)$. Thus, $r_{i_0} = 2p+2i_0+1$ and $r_{i_0+1} = 2p+2(i_0+1)$. So, the coefficient matrix of the system (2.6) is a $i_0 \times i_0$ Pascal submatrix. By Lemma 2.10, this Pascal submatrix is nonsingular. Therefore system (2.6) has a unique solution which is $m_{2j+1}(S) = 0$ for $2j+1 \leq r_{i_0}$. This implies that $m_{2p+1}(S) = 0$. By Lemma 2.6, $2p+1 \in P$, which contradicts our assumption. Therefore, no graph exists if any $r_i \in A$ satisfies $r_i \leq 2p+2i$.

Conversely, let us assume $r_i > 2p+2i$ for $1 \leq i \leq k$. In Theorem 2.7, we constructed graphs which have common moment set P with $A = \emptyset$. Now, we may assume $|A| = k > 0$. We shall construct a translated degree sequence S with the midrange degree a such that the graph $G = a+S$ and its complement $\overline{G} = a - S$ have the common moment set $P = \{0, 1, ..., 2p\} \cup A$. Since $r_i > 2p+2i-1$, the equations in (2.5) are independent by Lemma 2.10. Therefore, if the the sequence S satisfies $m_{2i+1}(S) = 0$ for $0 \leq i \leq p-1$ and the system (2.5), then G and $\overline{G}$ meet the common moment requirements up to r_k. Now, we reduce (2.5) to the following equivalent system

$$\sum_{i=p}^{\frac{i_j-1}{2}} c_{i_j(2i+1)} a^{i_j-2i-1} m_{2i+1}(S) = 0, \quad j = 1, 2, \ldots, k \tag{2.7}$$

with $c_{i_j i_j} \neq 0$, where $2p+1 < i_1 < i_2 < ... < i_k$ and every i_j is odd.

We begin with a list of sequences Q_j's. We wish to combine these sequences one by one in a certain way to satisfy the equations in (2.7) one by one. The generating function $(x-1)^j$ provides the sequence

$$Q_j = \{\begin{pmatrix} j \\ i \end{pmatrix} \cdot (-1)^i (j-i) \mid i = 0, 1, \ldots, j\}.$$

We observe that $m_{2i+1}(Q_j) = 0$ for $2i+1 < j-1$, $m_{2i+1}(Q_j) = j!$ for $2i+1 = j$, and $m_i(Q_j) > 0$ for $i > j$. Let $a_0 = 2i_k + 1$. We form $S_{i_1} = Q_{2p+1}(i_1! c_{i_1 i_1}) \cup v_1 \cdot Q_{i_1}$ with v_1 to be chosen below. Now, we take $a_1 = a_0(i_1! c_{i_1 i_1})$ to satisfy $m_{2i+1}(S_{i_1}) = 0$ for $i = 0, 1, \ldots, p-1$ and the first equation in (2.7), that is,

$$\sum_{i=p}^{\frac{i_1-1}{2}} c_{i_1(2i+1)} a_1^{i_1-2i-1} m_{2i+1}(S_{i_1}) = 0.$$

In fact, we select the integer v_1 to satisfy

$$\sum_{i=p}^{\frac{i_1-1}{2}} c_{i_1(2i+1)} a_1^{i_1-2i-1} (i_1! c_{i_1 i_1})^{2i+1} m_{2i+1}(Q_{2p+1}) = -c_{i_1 i_1} v_1 i_1! \ .$$

Therefore, the sequence S_{i_1} meets the common moment requirements up to i_1. Then we consider the next equation. In general, for $1 < j < k$, we construct $S_{i_j} = S_{i_{j-1}}(i_j! c_{i_j i_j}) \cup v_j \cdot Q_{i_j}$ with $a_j = a_{j-1}(i_j! c_{i_j i_j})$ then select an integer v_j to satisfy the j^{th} equation in (2.7). Finally, $S_{i_k} = S_{i_{k-1}}(i_k! c_{i_k i_k} L) \cup v_k \cdot Q_{i_k}$ with $a_k = a_{k-1}(i_k! c_{i_k i_k} L)$ where L is a suitable integer chosen to avoid the accidental matching at the $(i_k+1)^{th}$ moment since $(i_k+1) \notin A$. Now, the sequence S_{i_k} meets the common moment requirements up to $i_k + 1$. Let $S' = S_{i_k} = \{s_i'\}$ and $a' = a_k$. Then $a' > 1 + 2\max\{|s_i'|\}$. By Lemma 2.12, we find the midrange degree a and the S to form the graph $G = a + S$.

Since we assume that G dominates $\overline{G}$, there exists an integer n_0 such that $m_i(G) - m_i(\overline{G}) > 0$ for $i > n_0$. If there are some accidental matching moments between $i_k + 2$ and n_0, then we modify the graph degree sequence to remove these matching moments. Assume l_0 is the smallest accidental matching moment between $i_k + 2$ and n_0. Then, depending on whether l_0 is odd or even, we modify the translated degree sequence as $Su \cup S_{l_0}$ or $Su \cup S_{l_0-1}$ with the midrange degree au where u is selected large to ensure G remains a legal graph degree sequence. Now, the accidental matching moments can only occur between $l_0 + 1$ and n_0. And, $m_i(G) - m_i(\overline{G}) > 0$ still holds for $i > n_0$. We proceed in this way to remove successively all accidental matching moments. Therefore, G has its common moment set $P = \{0, 1, \ldots, 2p\} \cup A$. ∎

CHAPTER III

COMMON MOMENTS OF COMPLEMENTARY TOURNAMENTS

We shall study the common moment problems for the score sequences of a tournament T and its complement T^c. To construct a T of small order with large $M(T,T^c)$, we shall present Schwenk's construction of tournaments.

3.1 Score Sequences

Let $T = \{t_i\}_1^n$ and $T^c = \{t_i^c\}_1^n$ be complementary tournaments, where $t_i^c = n-1-t_i$. Then there exists a sequence $S = \{s_i\}_1^n$ so that $T = a+S$, and $T^c = a-S$ where $a = \frac{n-1}{2}$ is the *average score* of T. We refer to S as the (average score) translated score sequence. Hence,

$$\begin{aligned} m_k(T) &= \sum_{j=0}^{k} \binom{k}{j} a^{k-j} m_j(S), \\ m_k(T^c) &= \sum_{j=0}^{k} \binom{k}{j} a^{k-j} (-1)^j m_j(S) . \end{aligned} \tag{3.1}$$

Thus, T and T^c share the k^{th} moment if and only if

$$\sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k}{2j+1} a^{k-2j-1} m_{2j+1}(S) = 0. \tag{3.2}$$

As special cases of Theorem 1.13 and Lemma 1.12, we state the following two results for tournaments.

Lemma 3.1 *Let T be a tournament of order $n \geq 2$ and its score sequence $T \neq T^c$. Then $M(T,T^c)$ is always an even integer $2p \leq n-2$.*

Lemma 3.2 *Let T be a tournament of order n. Then $M(T,T^c) \geq 2p$ if and only if $m_{2j+1}(S) = 0$ for $0 \leq j \leq p-1$.*

We often construct tournaments with given properties. It is necessary to check if a sequence constructed is a tournament score sequence. The fundamental theorem for testing score sequences is due to Landau [9] and there is a corollary due to L.Moser (see Harary and Moser [7]).

Theorem 3.3 *The nondecreasing sequence $\{t_i\}_1^n$ is the score sequence of a tournament if and only if*

$$(1)\ \sum_{i=1}^{k} t_i \geq \binom{k}{2} \quad \text{for } 1 \leq k < n, \quad \text{and}$$

$$(2)\ \sum_{i=1}^{n} t_i = \binom{n}{2}.$$

Corollary 3.4 *The nondecreasing sequence $\{t_i\}_1^n$ is the score sequence of a strong tournament if and only if*

$$(1)\ \sum_{i=1}^{k} t_i > \binom{k}{2} \quad \text{for } 1 \leq k < n, \quad \text{and}$$

$$(2)\ \sum_{i=1}^{n} t_i = \binom{n}{2}.$$

It is useful to realize that if Landau's condition fails for the k smallest scores of a prospective tournament T, then it also fails for the $n-k$ smallest scores of T^c. Thus, it suffices to check the first half of T and the first half of T^c rather than all of T. This apparently new observation forms the next lemma.

Lemma 3.5 *Let $T = \{t_i\}_1^n$ be a sequence with $\sum_{i=1}^{n} t_i = \binom{n}{2}$. Then Landau's inequality fails for the k smallest scores of T if and only if it fails for the $n-k$ smallest scores of T^c.*

Proof. Assume $\sum_{i=1}^{k} t_i < \binom{k}{2}$. Then

$$\begin{aligned}
\sum_{i=k+1}^{n} t_i^c &= \sum_{i=k+1}^{n} (n-1-t_i) \\
&= (n-1)(n-k) - \sum_{i=k+1}^{n} t_i \\
&= (n-1)(n-k) - \binom{n}{2} + \sum_{i=1}^{k} t_i \\
&< (n-1)(n-k) - \binom{n}{2} + \binom{k}{2} \\
&= \binom{n-k}{2}.
\end{aligned}$$

To obtain the reverse implication repeat the same argument using T^c for T and $n-k$ for k. ∎

This allows us to state a theorem guaranteeing the existence of a tournament with a given score sequence provided the range of scores is not too large.

Theorem 3.6 *If the nondecreasing integer sequence $\{t_i\}_1^n$ satisfies $\sum_{i=1}^{n} t_i = \binom{n}{2}$ and $t_n - t_1 \leq \frac{n+1}{2}$, then a tournament exists with this score sequence. Moreover, if $t_n - t_1 \leq \frac{n-1}{2}$, then the tournament is strong.*

Proof. If no tournament exists, Landau's Theorem says for some k that

$$\sum_{i=1}^{k} t_i < \binom{k}{2}.$$

Now $t_1 \leq \frac{1}{k}\sum_{i=1}^{k} t_i < \frac{k-1}{2}$. Thus, we conclude $t_1 \leq \frac{k-2}{2}$. But Lemma 3.5 also requires

$$\sum_{i=k+1}^{n} t_i^c < \binom{n-k}{2},$$

so that

$$\sum_{i=k+1}^{n}(n-1-t_i)=(n-1)(n-k)-\sum_{i=k+1}^{n}t_i<\binom{n-k}{2}.$$

Consequently, $t_n \geq \frac{1}{n-k}\sum_{i=k+1}^{n} t_i > n-1-\frac{n-k-1}{2}=\frac{n+k-1}{2}$. That is, $t_n \geq \frac{n+k}{2}$. But now we have $t_n - t_1 \geq \frac{n+k}{2}-\frac{k-2}{2}=\frac{n+2}{2}$ contrary to the hypothesis.

Similarly, Moser's Corollary assures that if no strong tournament exists then $t_n - t_1 \geq \frac{n}{2}$, again violating the hypothesis. ∎

Corollary 3.7 *Let $T=\{t_i\}_1^n$ be a nondecreasing integer sequence with even $\sum_{i=1}^{n} t_i$, and let $M(T,T^c)>0$. If $x \leq t_i \leq 2\sqrt{nx}-x-1$ for some $0<x<\frac{n}{2}$, then T is both the degree sequence for a graph and the score sequence for a tournament.*

Proof. Let $T=a+S$. By Theorem 2.2, T is graphical. On the other hand, we observe that $t_n-t_1 \leq t_n - x \leq 2\sqrt{nx}-2x-1 \leq \frac{n-2}{2}$. Now, by Lemma 1.12, $M(T,T^c)>0$ implies $m_1(S)=0$ so that

$$\sum_{i=1}^{n} t_i = m_1(T)=m_1(a+S)=na+m_1(S)=na=\binom{n}{2}.$$

By Theorem 3.6, T is a score sequence for some tournament. ∎

The next corollary is directly from Corollary 3.7 by setting $x=\frac{n}{4}$.

Corollary 3.8 *Let $T=\{t_i\}_1^n$ be a nondecreasing integer sequence with even $\sum_{i=1}^{n} t_i$, and let $M(T,T^c)>0$. If $\lceil\frac{n}{4}\rceil \leq t_i \leq \lfloor\frac{3n}{4}\rfloor - 1$, then T is both the degree sequence for a graph and the score sequence for a tournament.*

3.2 Schwenk's Construction of Tournaments

For any integer $p>0$, by Corollary 3.8, every sequence constructed in the proof of Theorem 2.7 is a score sequence for some tournament T with $M(T,T^c)=2p$. But the order of T is 4^p. In order to find such a tournament of smaller order, we now present Schwenk's construction in the next theorem.

Theorem 3.9 *For any integer $p > 0$, there exists a tournament T_{2p} of order 4^p with $M(T_{2p}, T_{2p}^c) = 2p$.*

Proof. The proof is by induction on p. For $p = 1$, let $T_2 = \{3 \cdot 1, 1 \cdot 3\}$ of order 4, then $T_2^c = \{1 \cdot 0, 3 \cdot 2\}$. Now $M(T_2, T_2^c) = 2$, since $m_3(T_2) - m_3(T_2^c) \neq 0$. We now proceed by induction. Assume the theorem is true for a given $p \geq 1$, that is, assume there exists a tournament T_{2p} of order 4^p such that $M(T_{2p}, T_{2p}^c) = 2p$. For $p+1$, we construct a tournament T_{2p+2} of order 4^{p+1} with $M(T_{2p+2}, T_{2p+2}^c) = 2p+2$. Use two copies of T_{2p} and two copies of T_{2p}^c joined as shown in Figure 3.1, where $n = \frac{1}{2}4^p$. Each oval contains 4^p vertices. The line joining a pair of ovals represents a $K_{4^p,4^p}$ which must be suitably oriented so that the pair of numbers given for this line becomes the outdegrees within that $K_{4^p,4^p}$ subgraph. Observe that each pair is required to sum to 4^p. This process does not specify a unique tournament, but it does specify a unique score sequence. Select a, b, c so that $0 \leq a,\ b,\ c \leq \frac{1}{2}4^p$, and let $\Delta = a + b + c \geq 1$.

T_{2p+2}: T_{2p+2}^c:

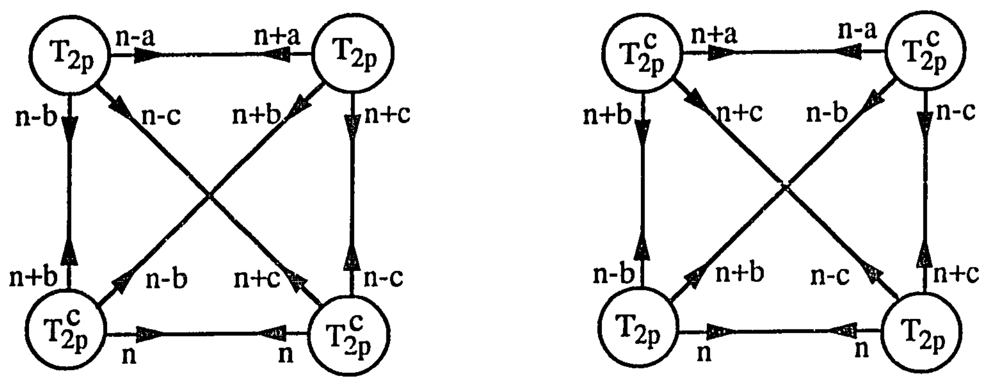

Figure 3.1 Schwenk's Construction of Tournaments T_{2p+2} and T_{2p+2}^c.

We have the following score sequences

$$T_{2p+2} = (3(\frac{4^p}{2}) - \Delta + T_{2p}) \cup (3(\frac{4^p}{2}) + T_{2p}^c) \cup (3(\frac{4^p}{2}) + T_{2p}^c) \cup (3(\frac{4^p}{2}) + \Delta + T_{2p}),$$

$$T_{2p+2}^c = (3(\frac{4^p}{2}) - \Delta + T_{2p}^c) \cup (3(\frac{4^p}{2}) + T_{2p}) \cup (3(\frac{4^p}{2}) + T_{2p}) \cup (3(\frac{4^p}{2}) + \Delta + T_{2p}^c).$$

Let $R = 3(\frac{4^p}{2}) + T_{2p}$ and $Q = 3(\frac{4^p}{2}) + T_{2p}^c$. Then

$$\begin{aligned} T_{2p+2} &= (R-\Delta)\cup Q\cup Q\cup(R+\Delta), \\ T_{2p+2}^c &= (Q-\Delta)\cup R\cup R\cup(Q+\Delta). \end{aligned}$$

By Lemma 1.3 and the inductive assumption, $M(R,Q) = M(T_{2p}, T_{2p}^c) = 2p$. Now, in Lemma 1.4, setting $\delta = 0$ we have

$$M(T_{2p+2}, T_{2p+2}^c) = M(R,Q) + 2 = 2p+2.$$

Thus the theorem is true by mathematical induction. ||

This theorem shows that we can construct tournaments of order 4^p with $M(T_{2p}, T_{2p}^c) = 2p$ for arbitarily large p. But this order 4^p is a good deal larger than necessary in order to obtain $2p$ matching moments. Often in Schwenk's construction of T_{2p} and T_{2p}^c we find common scores being duplicated in both T_{2p} and T_{2p}^c. When this happens it is usually (though not always) possible to delete duplicated pairs and reduce the remaining scores by a suitable translation to obtain a smaller pair T_{2p}, T_{2p}^c.

Lemma 3.10 *Let T be a tournament of order n. Then the score $d \in T \cap T^c$ if and only if the score $n-1-d \in T \cap T^c$.*

Proof. This follows from $d \in T$ if and only if $n-1-d \in T^c$. ||

Lemma 3.11 *Let T be a tournament of order n, and let $d \in T \cap T^c$. Let R, Q be obtained by removing a pair of scores d and $n-1-d$ from both T, T^c and decreasing all other scores by 1. Then $M(R,Q) = M(T,T^c)$.*

Proof. Since we delete a pair of common scores d and $n-1-d$ from both T and T^c, we do not alter $m_k(T) - m_k(T^c)$ for any k. By Lemma 1.3, we have $M(R,Q) = M(T,T^c)$. ||

Roughly speaking, this amounts to deleting two vertices from the tournament. But when we do so, the average score must drop by 1 whereas our average is unchanged since $d + (n - 1 - d) = n - 1 = 2a$. Thus we translate both sequences by -1 to produce the anticipated average for the smaller tournament. Unfortunately, we cannot guarantee that a tournament actually exists with this new reduced score sequence. For example, consider the tournament T of order 8 in Figure 3.2. Here all arcs not shown point downward and the numbers shown are the scores.

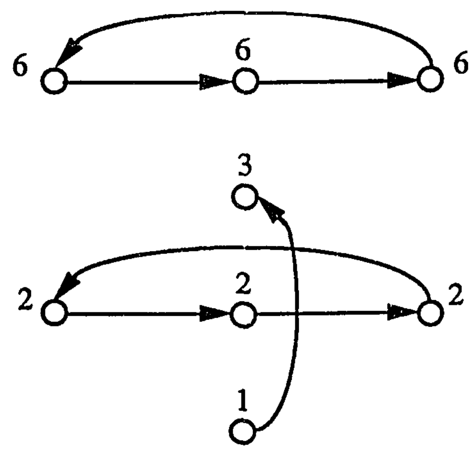

Figure 3.2 A Tournament of Order 8.

$T = \{1 \cdot 1, 3 \cdot 2, 1 \cdot 3, 3 \cdot 6\}$ and $T^c = \{3 \cdot 1, 1 \cdot 4, 3 \cdot 5, 1 \cdot 6\}$. Since scores 6 and 1 are duplicated, we might delete and translate to get $R = \{3 \cdot 1, 1 \cdot 2, 2 \cdot 5\}$ and $Q = \{2 \cdot 0, 1 \cdot 3, 3 \cdot 4\}$. But Q cannot be a tournament score sequence because it has two 0's.

We have attempted to construct the smallest possible examples of tournament with $M(T, T^c) = 2p$ for $p \leq 8$. Theorem 3.9 guarantees solutions of order 4^p, but we have found much smaller examples by repeated applications of Lemma 1.4 and Lemma 3.11. We shall just list the best examples discovered without any detail of how they were constructed (see Figure 3.3).

$2p$	4^p	smallest order found	score sequence T
2	4	4	$\{3\cdot 1, 1\cdot 3\}$
4	16	9	$\{3\cdot 1, 2\cdot 4, 1\cdot 5, 2\cdot 6, 1\cdot 8\}$
6	64	17	$\{3\cdot 2, 3\cdot 5, 3\cdot 7, 1\cdot 8, 1\cdot 10, 4\cdot 12, 1\cdot 13, 1\cdot 15\}$
8	256	27	$\{4\cdot 4, 7\cdot 7, 1\cdot 10, 1\cdot 13, 3\cdot 14, 3\cdot 18, 3\cdot 20, 4\cdot 21, 1\cdot 23\}$
10	1024	43	$\{2\cdot 1, 2\cdot 7, 1\cdot 8, 2\cdot 9, 2\cdot 10, 1\cdot 12, 5\cdot 18, 2\cdot 19, 6\cdot 20,$ $5\cdot 21, 1\cdot 26, 2\cdot 27, 1\cdot 28, 4\cdot 29, 2\cdot 31, 1\cdot 36, 2\cdot 37,$ $1\cdot 38, 1\cdot 42\}$
12	4096	60	$\{3\cdot 8, 1\cdot 11, 2\cdot 13, 4\cdot 14, 1\cdot 17, 3\cdot 19, 4\cdot 22, 2\cdot 24,$ $4\cdot 25, 10\cdot 30, 3\cdot 31, 1\cdot 32, 4\cdot 33, 2\cdot 36, 1\cdot 39, 5\cdot 41,$ $2\cdot 43, 2\cdot 44, 1\cdot 47, 3\cdot 49, 1\cdot 50, 1\cdot 52\}$
14	16384	96	$\{3\cdot 15, 1\cdot 18, 2\cdot 20, 4\cdot 21, 1\cdot 24, 1\cdot 27, 6\cdot 28, 2\cdot 29,$ $3\cdot 34, 7\cdot 37, 4\cdot 38, 2\cdot 44, 8\cdot 46, 12\cdot 47, 2\cdot 52, 4\cdot 53,$ $1\cdot 54, 3\cdot 55, 2\cdot 56, 1\cdot 59, 5\cdot 60, 4\cdot 63, 2\cdot 64, 3\cdot 69,$ $3\cdot 70, 2\cdot 72, 2\cdot 73, 1\cdot 76, 3\cdot 78, 1\cdot 79, 1\cdot 81\}$
16	65536	168	$\{3\cdot 42, 1\cdot 45, 2\cdot 47, 4\cdot 48, 3\cdot 53, 8\cdot 55, 2\cdot 58, 1\cdot 60,$ $8\cdot 61, 2\cdot 65, 2\cdot 66, 5\cdot 68, 9\cdot 71, 3\cdot 72, 6\cdot 74, 6\cdot 78,$ $9\cdot 79, 18\cdot 84, 6\cdot 85, 5\cdot 87, 2\cdot 90, 4\cdot 91, 8\cdot 92, 11\cdot 97,$ $3\cdot 98, 1\cdot 100, 6\cdot 103, 2\cdot 104, 2\cdot 105, 1\cdot 108, 8\cdot 110,$ $2\cdot 111, 1\cdot 113, 1\cdot 115, 5\cdot 116, 2\cdot 118, 1\cdot 121, 3\cdot 123,$ $1\cdot 124, 1\cdot 126\}$

Figure 3.3 Smallest Examples Found for $M(T, T^c) = 2p \leq 16$.

Obviously, these orders are growing much more slowly than 4^p, but it is not clear whether or not they grow exponentially. By Corollary 3.8, the graph

degree sequence with which we worked in the proof of Theorem 2.8 is also the score sequence for a tournament. Thus, the same proof leads to the next theorem.

Theorem 3.12 *There exists a constant c such that for every positive integer p there exists a tournament T of order at most* $cp^2 \ln p$ *and with* $M(T, T^c) \geq 2p$.

3.3 Tournaments With Given Common Moment Set

Now, we shall analyze the actual set $P = \{0, 1, \ldots, 2p\} \cup A$. Several examples of tournament possessing a given P will be presented. As a special case of Lemma 1.14, the next lemma shows that, for a tournament, $2p+1,\ 2p+2 \notin P$.

Lemma 3.13 *Let T be a tournament with* $M(T, T^c) = 2p$. *Then* $m_j(T) \neq m_j(T^c)$ *for* $j = 2p+1, 2p+2$.

The following lemma will be used to translate and scale sequences while maintaining common moment sets.

Lemma 3.14 *Let* $S' = \{s'_i\}$ *be a sequence of integers with* $m_1(S') = 0$, *and let* $a' \geq 2\max\{|s'_i|\}$ *be an integer. Then there exists a tournament* $T = a + S$ *and an integer* $u > 0$ *such that* $m_k(T) - m_k(T^c) = u^k[m_k(a' + S') - m_k(a' - S')]$.

Proof. We add 0's to the sequence $S'u$ to form $S = \{s_i\}_1^n$ so that $a = a'u = \frac{n-1}{2}$ for some integer $u > 0$. Therefore,

$$\sum_{i=1}^{n}(a + s_i) = na + m_1(S') = \binom{n}{2},$$

and, $\max_i\{s_i | s_i \in S\} - \min_i\{s_i | s_i \in S\} \leq a$. By Theorem 3.6, $T = a + S$ and $T^c = a - S$ are complementary tournaments. From (3.2), we find

$$\begin{aligned} m_k(T) - m_k(T^c) &= 2\sum_{i=0}^{\lfloor\frac{k-1}{2}\rfloor}\binom{k}{2i+1} a^{k-2i-1} m_{2i+1}(S) \\ &= 2\sum_{i=0}^{\lfloor\frac{k-1}{2}\rfloor}\binom{k}{2i+1} (a')^{k-2i-1} u^{k-2i-1} m_{2i+1}(S') u^{2i+1} \\ &= u^k[m_k(a' + S') - m_k(a' - S')] \qquad \blacksquare \end{aligned}$$

Theorem 3.15 *Let $A = \{r_1, r_2, \ldots, r_k\}$ be a finite subset of* N *with $2p + 2 < r_1 < r_2 < \cdots < r_k$, where $p \geq 0$. Then there exists a tournament T with common moment set $P = \{0, 1, \ldots, 2p\} \cup A$ if and only if $r_i > 2p + 2i$ for $1 \leq i \leq k$.*

We notice that the proof of this theorem is almost the same as the proof of Theorem 2.15 which presents a parallel result for graphs. Of course we are constructing a tournament score sequence instead of a graph degree sequence. Following the proof, we find the translated sequence S. If $a + S$ fails to be the score sequence of a tournament, then we apply Lemma 3.14 to find a pair of complementary tournaments which maintain the common moment set possessed by $a + S$.

The above theorem ensures that we can construct a tournament T such that its common moment set $P = \{0, 1, \ldots, 2p\} \cup A$ if $r_i > 2p + 2i$. But such a systematic construction gives very large order tournaments. We shall illustrate more flexible methods for a few small sets P in order to obtain solutions dramatically smaller than the general proof would provide. We list these constructions as corollaries.

Corollary 3.16 *There exists a tournament T of order 39 with $P = \{0, 1, 2, 5\}$.*

Proof. We assume the order $n = 2a + 1$ is odd and we use the translated score sequence S. We want $P = \{0, 1, 2, 5\}$. Then, $M(T, T^c) = 2$. From equation (3.2) and Lemma 3.2, we require $m_1(S) = 0$, $m_3(S) \neq 0$, and

$$\binom{5}{3} a^2 m_3(S) + \binom{5}{5} m_5(S) = 0. \tag{3.3}$$

We shall build a set S with several parameters that allow us to adjust $m_5(S)$ until it attains the value forced upon it by equation (3.3).

First, consider $R = \{2, -1, -1\}$. To increase flexibility, we shall consider $n_1 \cdot Ru$ for which we find

$$m_1(n_1 \cdot Ru) = 0, \quad m_3(n_1 \cdot Ru) = 6n_1u^3, \quad m_5(n_1 \cdot Ru) = 30n_1u^5.$$

Similarly, we may select $Q = \{5, 2 \cdot (-4), 3 \cdot 1\}$ and we consider Qv. We find

$$m_1(Qv) = 0, \quad m_3(Qv) = 0, \quad m_5(Qv) = 1080v^5.$$

If we attempt to form S as the union $S = Qv \cup (n_1 \cdot Ru)$ we immediately have $m_1(S) = 0$ and $m_3(S) = 0$ as required. It merely remains to satisfy (3.3). Thus $0 = 10a^2(6n_1u^3) + 30n_1u^5 + 1080v^5$ or equivalently $-36v^5 = 2a^2n_1u^3 + n_1u^5$. A convenient selection is $u = 2$ and $n_1 = 3$ to give $-v^5 = \frac{4a^2+8}{3}$. It would seem we have converted our question into a problem in number theory which could be quite difficult. But wait. If we replace Qv by the union of several Qv_i's we obtain $-\sum v_i^5 = \frac{4a^2+8}{3}$. Since the v_i's may be negative or positive, this equation is nothing more than an instance of Waring's problem [10] which asserts that for each power p there exists a number of terms $h(p)$ such that every positive integer can be written as the sum of at most $h(p)$ terms with each term a p^{th} power. Thus, if we allow ourselves $h(5)$ terms we can certainly solve the equation. But this doesn't quite complete our task. Observe that $S = \bigcup_{i=1}^{h(5)} Qv_i \cup 3 \cdot R2$ has $6h(5) + 9$ individual scores. To be a valid solution we require $n = 2a + 1 > 6h(5) + 9$. Since $h(5)$ is a presumably known constant, we can select n (and hence a) large enough to guarantee a solution. But this is extravagantly large. With a little searching, we can find values for n and a that fortuitously give a solution with fewer terms than $h(5)$. In particular, we find $n = 39$, $a = 19$ has the solution $\frac{4a^2+8}{3} = 484 = 3^5 + 3^5 + (-1)^5 + (-1)^5$. The resulting S is $S = \{2 \cdot 15, 4 \cdot (-12), 2 \cdot (-5), 1 \cdot 4, 6 \cdot 3, 6 \cdot 2, 6 \cdot (-1)\}$. Now $|S| = 27$ whereas $n = 39$. We correct this discrepancy by adding $12 \cdot 0$ to S. Upon translating by 19 we get the tournament score sequence

$$T = \{4 \cdot 7, 2 \cdot 14, 6 \cdot 18, 12 \cdot 19, 6 \cdot 21, 6 \cdot 22, 1 \cdot 23, 2 \cdot 34\}.$$

Of course T^c is just $19 - S$. ‖

To further illustrate the method, here are two more examples.

Corollary 3.17 *There exists a tournament of order* 45 *with* $P = \{0, 1, 2, 6\}$.

Proof. The analogous steps lead to the equation $-1080 \sum v_i^5 = 20n_1u^3a^2 + 30n_1u^5$. This time we select $u = 3$ and $n_1 = 4$ to obtain $-\sum v_i^5 = 2a^2 + 27$. For $a = 22$ we find $2a^2 + 27 = 995 = 4^5 - 2^5 + 3 \cdot 1^5$. This gives

$$S = \{1 \cdot 20, 2 \cdot (-16), 1 \cdot (-10), 2 \cdot 8, 4 \cdot (-6), 3 \cdot 5, 3 \cdot (-4), 8 \cdot 3, 3 \cdot (-2), 9 \cdot 1\}.$$

Upon inserting nine 0's and translating by 22, we get the score sequence

$$T = \{2 \cdot 6, 1 \cdot 12, 4 \cdot 16, 3 \cdot 18, 3 \cdot 20, 9 \cdot 22, 9 \cdot 23, 8 \cdot 25, 3 \cdot 27, 2 \cdot 30, 1 \cdot 42\}.$$

Its complement is $T^c = 22 - S$. ∎

Corollary 3.18 *There exists a tournament of order* 2281 *with* $P = \{0, 1, 2, 5, 7\}$.

Proof. We begin with the translated scores

$$R = \{2 \cdot (-15), 4 \cdot 12, 2 \cdot 5, 1 \cdot (-4), 6 \cdot (-3), 6 \cdot (-2), 6 \cdot 1\}$$

from the tournament that achieves $P = \{0, 1, 2, 5\}$. We wish to combine R with a sequence whose first three odd moments are 0, specifically

$$Q = \{1 \cdot 10, 2 \cdot (-9), 3 \cdot 6, 2 \cdot (-5), 1 \cdot 4, 4 \cdot (-1)\}.$$

To build flexibility into the model, let us consider $\bigcup_{i=1}^{h(7)} Qv_i \cup (n_1 \cdot Ru)$. Of course the first two moments are always 0. The Q sequence will not affect $m_5(T)$, so we need only examine the effect of $n_1 \cdot Ru$. We must have

$$\binom{5}{3} 144n_1u^3a^2 - \binom{5}{5} 519840n_1u^5 = 0.$$

This is valid if and only if $a = 19u$, so we shall take this as a condition defining a (and hence $n = 2a + 1$).

Next, examining the seventh moment forces

$$\begin{aligned} 1134000\sum v_i^7 &= -\binom{7}{3}144n_1u^3a^4 + \binom{7}{5}519840n_1u^5a^2 \\ &+\binom{7}{7}198265536n_1u^7. \end{aligned}$$

Substituting $n_1 = 1$, $u = 30$, $a = 19u = 570$, and $n = 2a+1 = 1141$ leads to $\sum v_i^7 = 67159698480000$. We find a collection of 72 terms for v_i which solve this equation. But each v_i introduces 13 scores into the sequence under construction along with the 27 from R giving 963 scores. It would seem that this should lead to a legitimate example, but our set of v_i's include 56 repeated 38 times. Since Q includes 10, this leads the translated score to have 560 appearing 38 times, or an actual score of $a - 560 = 10$ appearing 38 times. But Landau's theorem allows 10 to appear at most 21 times. Thus, the constructed sequence cannot be the score sequence of any tournament.

But all is not lost. By Lemma 3.14, we can indeed provide a legitimate tournament of order 2281 with the common moment set $P = \{0, 1, 2, 5, 7\}$. Specifically, this sequence is $T = \{2\cdot 114, 76\cdot 132, 1\cdot 220, 2\cdot 240, 2\cdot 570, 76\cdot 580, 3\cdot 588, 2\cdot 654, 1\cdot 772, 2\cdot 870, 1\cdot 880, 1\cdot 900, 7\cdot 960, 2\cdot 978, 3\cdot 984, 7\cdot 1020, 4\cdot 1026, 152\cdot 1028, 1\cdot 1036, 2\cdot 1050, 6\cdot 1068, 4\cdot 1086, 22\cdot 1104, 34\cdot 1120, 44\cdot 1128, 24\cdot 1132, 36\cdot 1136, 1352\cdot 1140, 64\cdot 1142, 32\cdot 1150, 9\cdot 1156, 28\cdot 1158, 27\cdot 1164, 4\cdot 1166, 2\cdot 1172, 5\cdot 1180, 5\cdot 1188, 8\cdot 1200, 2\cdot 1220, 4\cdot 1232, 5\cdot 1248, 2\cdot 1270, 1\cdot 1284, 4\cdot 1356, 2\cdot 1374, 2\cdot 1440, 1\cdot 1464, 1\cdot 1500, 38\cdot 1588, 1\cdot 1596, 2\cdot 1600, 1\cdot 1680, 114\cdot 1812, 3\cdot 1824, 2\cdot 1860, 2\cdot 1968, 38\cdot 2260, 1\cdot 2280\}$. ∎

For $P = \{0, 1, 2, 7, 8\}$, we have also constructed an example of tournament of order 2241 with P as the common moment set. These last two examples illustrate the importance of our methods of construction, because any attempt to verify directly the matching powers for $P = \{0, 1, 2, 5, 7\}$ and $n = 2281$ requires us to sum 2281 scores each to the seventh power. The total has 26 digits which should

perfectly match the total for T^c. But this computation exceeds the accuracy of our system. On the other hand, our method of construction verifies the equality without reaching such large numbers. We have attempted to double check our results for $P = \{0, 1, 2, 7, 8\}$ by computing $m_7(T) = m_7(T^c)$ directly, and then directly modulo one billion. Both computations matched as they were predicted to do based on the average translated scores used in the construction. For our last example of $P = \{0, 1, 2, 7, 8\}$ we trusted the translated score equations and did not bother to raise the actual scores to the eighth power.

3.4 Initial Density

In our characterization of common moment sets, $P = P_0 \cup A$, the conditions constraining the subset A can be viewed intuitively as follows. After closing the initial interval P_0, we are not allowed to include too many small moments in A. This can made more precise by introducing a new concept.

Let $B = \{b+1, b+2, \ldots, b+n, \ldots\} \subset \mathrm{N}$, and let $A \subset B$. We define the *initial density* $d(A|B)$ of A relative to B as

$$d(A|B) = \sup_{j \geq 1} \frac{1}{j} |A \cap \{b+1, b+2, \ldots, b+j\}|.$$

Notice that $0 \leq d(A|B) \leq 1$, and if A is finite then $d(A|B) = 1$ if and only if $b+1 \in A$. We first discuss two properties of the initial density.

Lemma 3.19 *Let $A \subset B \subset \mathrm{N}$. For any integer δ, let $B' = \{i + \delta \mid i \in B\}$, and let $A' = \{i + \delta \mid i \in A\}$. Then $d(A'|B') = d(A|B)$.*

Proof. For any $j \geq 1$,

$$|A' \cap \{b+\delta+1, b+\delta+2, \ldots, b+\delta+j\}| = |A \cap \{b+1, b+2, \ldots, b+j\}|.$$

Therefore, by definition, $d(A'|B') = d(A|B)$. ∎

Lemma 3.20 *Let $A = \{r_1, r_2, \ldots, r_k\}$ and $A' = \{r_2 - 2, \ldots, r_k - 2\}$ be subsets of B, where $r_1 < r_2 < \cdots < r_k$. If $d(A|B) \leq \frac{1}{2}$, then $d(A'|B) \leq d(A|B)$.*

Proof. For any $j \geq 1$, if $r_n - 2 < b + j < r_{n+1} - 2$, then since $j > r_n - 2 - b$ we find

$$\frac{1}{j}|A' \cap \{b+1, b+2, \ldots, b+j\}| < \frac{1}{r_n - 2 - b}|A' \cap \{b+1, b+2, \ldots, r_n - 2\}|.$$

Thus $d(A'|B) = \max_{2 \leq n \leq k} \frac{1}{r_n - 2 - b}|A' \cap \{b+1, b+2, \ldots, r_n - 2\}|$. So, there exists an integer n_0 such that

$$d(A'|B) = \frac{1}{r_{n_0} - 2 - b}|A' \cap \{b+1, b+2, \ldots, r_{n_0} - 2\}|,$$

that is, $d(A'|B) = \frac{n_0 - 1}{r_{n_0} - 2 - b}$. On the other hand,

$$\frac{1}{r_{n_0} - b}|A \cap \{b+1, b+2, \ldots, r_{n_0}\}| \leq d(A|B) \leq \frac{1}{2},$$

that is, $\frac{n_0}{r_{n_0} - b} \leq \frac{1}{2}$. Thus, we have

$$\frac{n_0 - 1}{r_{n_0} - 2 - b} \leq \frac{n_0}{r_{n_0} - b}.$$

Therefore, $d(A'|B) \leq d(A|B)$. ∎

Lemma 3.21 *Let $B = \{2p+1, 2p+2, \ldots\}$, and let $A = \{r_1, r_2, \ldots, r_k\}$ be a subset of B. Then $d(A|B) < \frac{1}{2}$ if and only if $r_i > 2p + 2i$ for $1 \leq i \leq k$.*

Proof. If there exists some $r_i \in A$ such that $r_i \leq 2p + 2i$, then, for $1 \leq i \leq k$, $\frac{1}{i}|A \cap \{2p+1, \ldots, 2p+2i\}| \geq \frac{1}{2}$. Thus, $d(A|B) \geq \frac{1}{2}$.

Conversely, let us assume $r_i > 2p + 2i$ for $1 \leq i \leq k$. Then, for $1 \leq i \leq k$, $\frac{1}{i}|A \cap \{2p+1, \ldots, 2p+i\}| < \frac{1}{2}$. Since A is a finite set, by definition of initial density, we have $d(A|B) < \frac{1}{2}$. ∎

In our problem, we study the potential $P = \{0, 1, \ldots, 2p\} \cup A$. We consider the initial density $d(A|B)$ of A relative to $B = \{2p+1, 2p+2, \ldots\}$. We find that initial density of $\frac{1}{2}$ is the critical value in the following sense. Applying Lemma 3.21, we can restate Theorem 2.15 and Theorem 3.15 as following two theorems.

Theorem 3.22 *Let* $B = \{2p+1, 2p+2, \ldots\}$, $p \geq 0$, *and let* $A = \{r_1, r_2, \ldots, r_k\}$ *be a finite subset of* B. *Then there exists a graph* G *with common moment set* $P = \{0, 1, \ldots, 2p\} \cup A$ *if and only if the initial density* $d(A|B) < \frac{1}{2}$.

Theorem 3.23 *Let* $B = \{2p+1, 2p+2, \ldots\}$, $p \geq 1$, *and let* $A = \{r_1, r_2, \ldots, r_k\}$ *be a finite subset of* B. *Then there exists a tournament* T *with common moment set* $P = \{0, 1, \ldots, 2p\} \cup A$ *if and only if the initial density* $d(A|B) < \frac{1}{2}$.

CHAPTER IV

OPEN PROBLEMS

In this final chapter, we would like to suggest the following problems for future study.

1. In Section 1.2, we discussed the relations between the characteristic function $f(x; R, Q)$ and the initial interval P_0 of common moments. Is it possible to find the rest of P (that is A) directly from $f(x; R, Q)$?

2. We tried to construct small complementary graphs and tournaments which possess the first $2p + 1$ moments in common. According to Theorem 2.8 and Theorem 3.12, the order of such a graph or a tournament grows at the rate of $p^2 \ln p$. Is this the smallest order that can be achieved?

3. More specifically, for any specified common moment set P what is a complementary pair of smallest order realizing P? However, we expect all such questions to be extremely difficult.

4. The value of the determinant of a Pascal submatrix is always a non-negative integer. Herbert Wilf has suggested that perhaps this indicates that the determinant in fact "counts" some combinatorial object. What indeed does it count, if anything?

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