Surgical Techniques for Constructing Minimal Orientable Imbeddings of Joins and Compositions of Graphs

David L. Craft
Western Michigan University

Follow this and additional works at: https://scholarworks.wmich.edu/dissertations
Part of the Applied Mathematics Commons

Recommended Citation
https://scholarworks.wmich.edu/dissertations/2024

This Dissertation-Open Access is brought to you for free and open access by the Graduate College at ScholarWorks at WMU. It has been accepted for inclusion in Dissertations by an authorized administrator of ScholarWorks at WMU. For more information, please contact maira.bundza@wmich.edu.
SURGICAL TECHNIQUES FOR CONSTRUCTING MINIMAL ORIENTABLE IMBEDDINGS OF JOINS AND COMPOSITIONS OF GRAPHS

by

David L. Craft

A Dissertation
Submitted to the
Faculty of The Graduate College
in partial fulfillment of the
requirements for the
Degree of Doctor of Philosophy
Department of Mathematics and Statistics

Western Michigan University
Kalamazoo, Michigan
June 1991
INFORMATION TO USERS

This manuscript has been reproduced from the microfilm master. UMI films the text directly from the original or copy submitted. Thus, some thesis and dissertation copies are in typewriter face, while others may be from any type of computer printer.

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleedthrough, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps. Each original is also photographed in one exposure and is included in reduced form at the back of the book.

Photographs included in the original manuscript have been reproduced xerographically in this copy. Higher quality 6" x 9" black and white photographic prints are available for any photographs or illustrations appearing in this copy for an additional charge. Contact UMI directly to order.
Surgical techniques for constructing minimal orientable imbeddings of joins and compositions of graphs

Craft, David L., Ph.D.
Western Michigan University, 1991
For Stacy,
the center of my
sphere
ACKNOWLEDGEMENTS

First and foremost, I would like to thank Professor Arthur T. White, my advisor and committee chairman, for his counsel, his encouragement, his trust, and for the countless hours he spent in seeing me through all facets of writing this dissertation. It has truly been an honor to work with him and to know him as a friend.

I offer my deep appreciation to Professor Allen Schwenk for serving as second reader on my committee and for allowing me the opportunity to begin my publication record as his coauthor.

I owe Professor S. F. Kapoor my profuse thanks for offering his vast expertise as well as allowing me the use of his computer with all of his marvelous software.

I would like to thank Professors Erik Schreiner, Ajay Gupta, and Joseph McCanna for serving on my committee and for offering many helpful suggestions.

It has been my good fortune to have known many fine educators and mathematicians during my formal training. Each of the following has had a profound impact on my academic and professional life for which I am extremely grateful: Karl Gartner, Edgar Franz, Ts'ing-Hi Tong, Sheldon Davis, and Zevi Miller.

Thank you to my wonderful wife Stacy for helping with the overwhelming chore of typing this manuscript.

Finally, I would like to thank my family: my parents Donald and Jean, my sisters Andrea and Trudy, my brother Steve, and especially my wife Stacy. The debt I owe for their love and support is beyond measure.

David L. Craft
# TABLE OF CONTENTS

ACKNOWLEDGEMENTS ........................................................................................................ ii

LIST OF TABLES .................................................................................................................. v

CHAPTER

I. INTRODUCTION .............................................................................................................. 1
   1.1 A Short History of the Genus Parameter for Graphs ................................................. 1
   1.2 Elementary Concepts ............................................................................................... 4
   1.3 A Survey of Techniques ......................................................................................... 8
   1.4 A Preview .............................................................................................................. 13

II. GRAPHICAL SURFACES .......................................................................................... 16
   2.1 The Surface $\bar{s}(G)$ ......................................................................................... 16
   2.2 A Descriptive Geometry for $\bar{s}(G)$ ................................................................ 18
   2.3 Imbedding Graphs in $\bar{s}(G)$ ........................................................................... 22
   2.4 An Alternate Construction of $\bar{s}(G)$ and an Alternate Proof of the Euler-Poincaré Formula ................................................................. 26

III. ARCCHAINS ............................................................................................................... 29
   3.1 Orderings and Chains ......................................................................................... 29
   3.2 Orientations and Arcchains ............................................................................... 30
   3.3 Collections of Arcchains ..................................................................................... 33
   3.4 Collections of Bipartite Arcchains ......................................................................... 43

IV. CROWNING ............................................................................................................... 52
   4.1 The Basic Crown ................................................................................................. 52
   4.2 Additional Adjacencies ......................................................................................... 58
   4.3 Variations of the Crown ...................................................................................... 58
Table of Contents — Continued

4.4 Multiple Clones, Double Crowns, and Crown Clusters .......... 62

V. THE GENUS OF THE GRAPHICAL JOIN ................................................. 70

5.1 The Complete Bipartite Graphs ................................................. 70

5.2 The Genus of $\overline{K_a} + H$ ..................................................... 73

5.3 The Complete Tripartite Graphs ................................................. 80

5.4 The Genus of $G + H$ ................................................................. 86

VI. THE GENUS OF THE GRAPHICAL COMPOSITION .......................... 94

6.1 The Genus of $G[\overline{K_n}]$ ......................................................... 94

6.2 The Genus of $G[H]$ ................................................................. 96

6.3 The Generalized Composition of Graphs ................................. 101

VII. OPEN PROBLEMS ................................................................. 104

REFERENCES ........................................................................................................... 107
LIST OF TABLES

1. Common Composite Graphs ................................................................. 6
2. Second-Order Diagonal of Quadrilateral Formed by Adjacent Tubes........ 26
3. Couples Formed by Adjacent Oriented Links ......................................... 32
4. Regions of a Crowned Surface ............................................................. 57
5. Increase in Genus Due to Crowning ...................................................... 66
CHAPTER I

INTRODUCTION

1.1 A Short History of the Genus Parameter for Graphs

As with so much of modern day Graph Theory, the history of the problem of determining the genus of a graph can be traced to the early attempts to prove the famous Four-Color Conjecture. For this reason, we begin with the conjecture itself. (See [9] for a more complete history of the conjecture.)

At some point during or shortly before 1852, Francis Gutherie communicated an observation and a request to his brother Frederick who was then a student of Augustus De Morgan at University College in London. Francis had observed that in coloring a map of connected countries, under the restriction that countries sharing a common boundary line be colored differently, no more than four colors are needed. He asked his brother to discuss this with De Morgan and to obtain a proof of this "fact." On October 23, 1852, De Morgan wrote to Sir William Hamilton about the problem and apparently continued to discuss the problem with students and colleagues. (See [9].)

During a session of the London Mathematical Society in 1878, Cayley presented the problem and discussed the difficulties inherent in it. Less than a year later Kemp published his "proof" of the Four-Color Conjecture which was enthusiastically received and widely accepted (See [23] or [24]). The Kemp proof stood for over a decade, until 1890, when Heawood published an article [20] exposing a fatal flaw.

This same article by Heawood contains a discussion of coloring maps on the orientable surfaces $S^k$ of genus $k > 0$ and proffers his Heawood Map-Coloring.
Conjecture. This apparently more ambitious conjecture states that the maximum number of colors needed to color a map on $S_k$ is given by

$$\chi(S_k) = f(k) = \left\lceil \frac{7 + \sqrt{1 + 48k}}{2} \right\rceil$$

for $k > 0$.

[Note that the case $k = 0$ gives the Four Color Conjecture.] Heawood showed $\chi(S_k) \leq f(k)$, leaving the reverse inequality to be established (although, he claimed it was obvious). By considering the dual graph (or multigraph) of a map on $S_k$ it can be shown that $\chi(S_k) \geq f(k)$ for $k > 0$, provided the genera of the complete graphs $K_n$ are given by

$$\gamma(K_n) = g(n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$$

for $n \geq 7$.

(See Section 8.4 of White [43].) Heawood showed that $\gamma(K_7) = 1$ and the following year, Heffter [21] established $\gamma(K_n) = g(n)$ for $8 \leq n \leq 12$. Thus began the study of the genus of a graph.

No further progress was made toward establishing the genera of the complete graphs (and hence proving the Heawood Map-Coloring Theorem) until 1952 when Ringel [28] showed that $\gamma(K_{13}) = g(13) = 8$. Two years later, Ringel [27] proved that $\gamma(K_n) = g(n)$ for $n \equiv 5 \pmod{12}$. This breakthrough set the tone for the remainder of the work on the complete graphs. Residue classes modulo 12 fell steadily throughout the 1960s, much of the work due to Ringel, Youngs and Gustin. The formula for $\gamma(K_n)$ given above was finally established for all $n \geq 3$ in 1969. A complete discussion of this monumental work can be found in Ringel's Map Color Theorem [31]. (For a very readable description of the proof, see [42].)

The tremendous level of activity involved in the proof of the Heawood Map Coloring Theorem induced interest in finding the genera of arbitrary graphs. Battle, Harary, Kodama, and Youngs showed, in 1962 [5], that the genus of a graph is the
sum of the genera of its blocks; this narrows the problem to 2-connected graphs. In 1965, Ringel found the genera of the complete bipartite graphs [30]. In that same year, Ringel [29] and, independently, Beineke and Harary [6] established the genera of the n-cubes $Q_n$. In his doctoral dissertation [38] (and in subsequent papers [39] and [40]) White extended the technique used by Beineke and Harary to establish several genus formulae for cartesian products of graphs. (Pisanski has extended this work even further, cf. [26].) In the same work, White obtained genus formulae for two infinite classes of complete tripartite graphs; one of these included the regular case $K_{3(n)}$ obtained independently by Ringel and Youngs [32]. Stahl and White published two new genus formulae for complete tripartite graphs in 1976 [35]. Jungerman found the genus of $K_{4(n)}$ in 1975 [22], except for the case $n = 3$, which was settled by White [43, page 169]. Several infinite cases of $K_{m(n)}$ fell during the following two years, primarily at the hands of Bouchet (cf. [10],[7]). During the last decade many of the results giving genus formulae for specific classes of graphs have been in the category of amalgamations of graphs. (See [2], [3], [4], [13] and [34].)

The foregoing list is not intended to be complete and certainly omits substantial contributions. The excellent survey by Stahl [33] and White's *Graphs, Groups and Surfaces* [43] both contain lists of known genus formulae through 1978 and 1984 respectively. References to more recent work can be found in the bibliography of Gross and Tucker's *Topological Graph Theory* [18].

It should be understood that, although our current work is restricted to minimal or nearly minimal imbeddings of graphs, there are many problems closely associated with the genus parameter. These include finding the maximum genus of graphs, imbedding graphs in nonorientable surfaces or in pseudosurfaces, and determination of the genera of groups. The reader is again referred to White [43] for discussion of these and other interesting topics.
1.2 Elementary Concepts

A graph $G$ consists of a finite nonempty set of vertices $V(G)$ together with a set of edges $E(G)$ which are unordered pairs of distinct elements of $V(G)$. The order of a graph $G$, denoted $p$ or $p(G)$, is the cardinality of $V(G)$. The size of a graph $G$, denoted $q$ or $q(G)$, is the cardinality of $E(G)$. A $(p, q)$-graph is a graph having order $p$ and size $q$. If $E(G)$ is permitted to be a multiset, $G$ is called a multigraph. If the elements of $E(G)$ are ordered pairs of vertices, called arcs or directed edges, then $G$ is called a digraph.

Vertices $u$ and $v$ are said to be adjacent (to each other) provided $uv \in E(G)$, in which case $u$ and $v$ are said to be incident with the edge $uv$. The degree of a vertex $u$, denoted $\deg u$ or $\deg_G u$, is the number of edges with which $u$ is incident. The neighborhood of a vertex $u$ is given by $N(u) = \{v \in V(G) \mid uv \in E(G)\}$. In a graph, $\deg u = |N(v)|$. A graph $G$ is said to be $n$-regular if $\deg u = n$ for every vertex $u$ in $V(G)$. The complement $\overline{G}$ of a graph $G$ is given by $V(\overline{G}) = V(G)$, $E(\overline{G}) = \{uv \mid u \neq v; u, v \in V(G)$ and $uv \notin E(G)\}$.

Given two graphs $G$ and $H$, $G$ is said to be isomorphic to $H$, denoted $G \cong H$, if there exists a bijection $\phi : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $\phi(u)\phi(v) \in E(H)$. The bijection $\phi$ is called an isomorphism.

The path of order $n$, denoted $P_n$, is defined by $V(P_n) = \{v_1, v_2, ..., v_n\}$, $E(P_n) = \{v_iv_{i+1} \mid i = 1, 2, ..., n - 1\}$. A graph $G$ is a path provided there exists a positive integer $n$ such that $G \cong P_n$. The cycle of order $n$, denoted $C_n$, is defined for $n \geq 3$ by $V(C_n) = V(P_n)$, $E(C_n) = E(P_n) \cup \{v_nv_1\}$; that is, $C_n = P_n \cup \{v_nv_1\}$. A graph $G$ is a cycle provided there exists a positive integer $n$ such that $G \cong C_n$.

The complete graph of order $n$, denoted $K_n$, is defined by $V(K_n) = \{v_1, v_2, ..., v_n\}$, $E(K_n) = \{uv \mid u \neq v \text{ and } u, v \in V(K_n)\}$. Notice, $K_n$ could
have been defined as the unique graph of order \( n \) for which \( E(K_n) \) is empty. We call \( K_n \) the empty graph of order \( n \).

The complete multipartite graph \( K_{p_1, p_2, \ldots, p_n} \) is constructed as follows. Let \( p = \sum_{i=1}^{n} (p_i) \) and partition \( V(K_p) \) into sets \( A_i \) for which \( |A_i| = p_i \) for \( i = 1, 2, \ldots, n \). Now delete all edges \( uv \) from \( E(K_p) \) for which \( u,v \in A_i \) for some \( i \). The resultant graph is denoted \( K_{p_1, p_2, \ldots, p_n} \). If \( p_i = m \) for all \( i = 1, 2, \ldots, n \) then we denote the graph by \( K_n(m) \).

Although the present work is primarily concerned with graphs, certain digraphs will play a role. The directed path \( DP_n \) and the directed cycle \( DC_n \) are defined exactly as were \( P_n \) and \( C_n \), except that the former edges \( uv \) are now arcs, i.e., \( uv \) is now an ordered pair. The complete symmetric digraph \( K_n^* \) is defined exactly as was \( K_n \) except that now, for each pair of distinct vertices \( u \) and \( v \), we have \( uv, vu \in E(K_n^*) \).

Given two graphs \( G \) and \( H \), where \( V(G) \) and \( V(H) \) are disjoint, there are several ways to construct new graphs \( G*H \). Table 1 gives the four most common ones.

The concept of union can be generalized to a finite collection of graphs. If all of the graphs are the same, we have the notation \( nG = \bigcup_{i=1}^{n}(G) \). Thus, \( K_n = nK_1 \).

The complete multipartite graph can be represented as the complement of such a union, i.e., \( K_{p_1, p_2, \ldots, p_n} = \bigcup_{i=1}^{n}(K_{p_i}) \). A similar concept is that of (recursively defined) repeated cartesian products. For example, the n-cube \( Q_n \) is defined as follows: \( Q_1 = K_2 \), and for \( n \geq 2 \), \( Q_n = K_2 \times Q_{n-1} \).

If \( G \) and \( H \) are graphs for which \( V(H) \subseteq V(G) \) and \( E(H) \subseteq E(G) \) then we say \( H \) is a subgraph of \( G \), or that \( G \) is a supergraph of \( H \), and denote this fact by \( H \subseteq G \). Furthermore, if \( V(H) = V(G) \) we say \( H \) is a spanning subgraph of \( G \).
Table 1

Common Composite Graphs

<table>
<thead>
<tr>
<th>Name</th>
<th>Notation</th>
<th>Vertex Set</th>
<th>Edge Set</th>
</tr>
</thead>
<tbody>
<tr>
<td>union</td>
<td>$G \cup H$</td>
<td>$V(G) \cup V(H)$</td>
<td>$E(G) \cup E(H)$</td>
</tr>
<tr>
<td>join</td>
<td>$G + H$</td>
<td>$V(G) \cup V(H)$</td>
<td>$E(G) \cup E(H) \cup {uv \mid u \in V(G), v \in V(H)}$</td>
</tr>
<tr>
<td>cartesian product</td>
<td>$G \times H$</td>
<td>$V(G) \times V(H)$</td>
<td>${(u_1, v_1)(u_2, v_2) \mid (u_1 = u_2 \text{ and } v_1v_2 \in E(H)) \text{ or } (v_1 = v_2 \text{ and } u_1u_2 \in E(G))}$</td>
</tr>
<tr>
<td>composition (or lexicographic product)</td>
<td>$G[H]$</td>
<td>$V(G) \times V(H)$</td>
<td>${(u_1, v_1)(u_2, v_2) \mid (u_1 = u_2 \text{ and } v_1v_2 \in E(H)) \text{ or } u_1u_2 \in E(G)}$</td>
</tr>
</tbody>
</table>

A path (cycle) in $G$ is a subgraph of $G$ which is isomorphic to $P_n$ (respectively, $C_n$) for some positive integer $n$. A spanning path (cycle) is called a hamiltonian path (respectively, hamiltonian cycle.) A graph containing a hamiltonian cycle is called a hamiltonian graph.

A path $P$ in a graph is called a $u$-$v$ path if $\deg_G u = \deg_G v = 1$. A graph $G$ is said to be connected provided $G$ contains a $u$-$v$ path for all $u, v \in V(G)$ with $u \neq v$. Maximal connected subgraphs are called connected components or simply components. Furthermore, if $G$ is connected, and $G - v$ is connected and nontrivial for each $v \in V(G)$, we say $G$ is a block or that $G$ is 2-connected.

A tree is a connected graph which contains no cycles. It is easily shown that each connected graph has a spanning tree. Hence, if $G$ is a $(p, q)$-graph with $k$ connected components we can form a spanning forest $F$ (a graph without cycles) in $G$ by choosing a spanning tree for each component. The Betti number of $G$ is the
The cardinality of \( E(G) - E(F) \) and is given by \( \beta(G) = q - p + k \). Notice, that if \( G \) is connected then \( \beta(G) = q - p + 1 \).

This completes our review of general graph theoretic vocabulary and notation. For terminology and notation not explicitly mentioned above, we follow Chartrand and Lesniak [12]. Terminology specific to Topological Graph Theory which is not explicitly defined below can be found in White [43].

A **surface** is a closed, orientable 2-manifold. Adding a **handle** to a surface \( S \) refers to the procedure of removing two disjoint open disks \( D_1 \) and \( D_2 \) from \( S \), removing two disjoint open disks \( D_1' \) and \( D_2' \) from the sphere \( S_0 \) (disjoint from \( S \)) and "gluing" \( S - \{D_1, D_2\} \) and \( S_0 - \{D_1', D_2'\} \) together by identifying \( \partial D_i \) with \( \partial D_i' \) for \( i = 1, 2 \). (Recall that an open disk is a subspace which is homeomorphic to \( \{(x,y) \mid x^2 + y^2 < 1\} \) and that \( \partial \) is the boundary operator.) The **handle** \( S_0 - \{D_1', D_2'\} \) is sometimes referred to as a **tube** or **topological cylinder** and could alternately be described as being homeomorphic to \( S^0 \times [0,1] \) where \( S^0 \) is a circle.

We recursively define the surfaces \( S_k \) for all nonnegative integers \( k \) as follows. Let \( S_0 \) be the sphere (as indicated above.) Now, for \( k > 0 \), \( S_k \) is the surface which results from adding a handle to \( S_{k-1} \). The Classification Theorem from surface topology (cf. [15]) implies that if \( S \) is a surface, then \( S \) is homeomorphic to \( S_k \) for some \( k \). In this case, we say that \( S \) has **genus** \( k \).

An **imbedding** of a graph \( G \) in a surface \( S \) is a mapping of \( V(G) \) into distinct points of \( S \) and an extension of this map to \( E(G) \) by mapping each edge \( uv \) into a curve (i.e., a homeomorphic image of \( [0,1] \)) whose endpoints are the images of \( u \) and \( v \), so that the corresponding open curves are pairwise nonintersecting. By a standard abuse of terminology, we identify \( G \) with its image under this mapping; that is, we shall speak of \( G \) as being a subspace of \( S \). The connected components of...
$S - G$ are called the regions of the imbedding of $G$ in $S$. A 2-cell imbedding is one for which each region is homeomorphic to the open disk.

It is well known that every connected graph has a 2-cell imbedding in some surface $S_k$. The genus of a graph $G$, denoted $\gamma(G)$, is the smallest $k$ for which $G$ has an imbedding in $S_k$. If $G$ is connected, this must be a 2-cell imbedding. For this reason and in light of the theorem by Battle, Harary, Kodama, and Youngs we will restrict our attention in this dissertation to the problem of imbedding connected graphs.

An imbedding of $G$ in $S_{\gamma(G)}$ is called a minimal (or genus) imbedding.

We complete this section with the very useful Euler-Poincaré formula. If a $(p, q)$-graph has a 2-cell imbedding in $S_k$ with $r$ regions, then $p - q + r = 2 - 2k$.

1.3 A Survey of Techniques

Virtually all genus formulae are established in the same manner: A lower bound is found for $\gamma(G)$, usually by using the Euler-Poincaré formula; then an imbedding of $G$ is constructed which realizes this lower bound. It is this later construction for which various techniques have been developed. We will briefly discuss the derivation of a lower bound and then we will outline the techniques which account for the vast majority of results.

When we consider the Euler-Poincaré formula $p - q + r = 2 - 2k$, we see that, since $p$ and $q$ are fixed for a particular graph, $k$ is minimized by maximizing the number $r$ of regions. This, in turn, is accomplished by reducing the lengths of the individual region boundaries as much as possible. The best possible is a triangular imbedding (i.e., all regions are bounded by three edges) which yields a genus of $\frac{q}{6} - \frac{p}{2} + 1$. Hence, for all $(p, q)$-graphs $G$, $\gamma(G) \geq \left\lceil \frac{q}{6} - \frac{p}{2} + 1 \right\rceil$. If $G$ is triangle-free, the best possible is a quadrilateral imbedding which yields a genus of $\frac{q}{4} - \frac{p}{2} + 1$. Thus, for all triangle-free $(p, q)$-graphs $G$, $\gamma(G) \geq \left\lceil \frac{q}{4} - \frac{p}{2} + 1 \right\rceil$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Perhaps a more obvious lower bound for the genus of a graph is the genus of a subgraph. Consider the following.

**Theorem 1.1:** If $H$ and $G$ are graphs with $H \subseteq G$ then $\gamma(H) \leq \gamma(G)$.

A very natural consequence of this is the following pinching theorem for graphs.

**Corollary 1.1:** If $K$, $H$, and $G$ are graphs with $K \subseteq H \subseteq G$ and $\gamma(K) = \gamma(G) = k$, then $\gamma(H) = k$.

Once the lower bound $b$ has been established for $\gamma(G)$, an imbedding of $G$ in $S_b$ must be found, to conclude that $\gamma(G) = b$. The techniques commonly used for these constructions are (i) rotation schemes, (ii) current or voltage graphs, or (iii) surgery.

The first of these techniques was introduced by Heffter [21], rediscovered by Edmonds [14], explained carefully by Youngs [44], and first used extensively by Ringel [31]. Observe that if a graph $G$ is imbedded in a surface $S$ and $v$ is any vertex of $G$, then the vertices adjacent to $v$ are cyclically arranged around $v$ in $S$. In fact, the imbedding induces a **rotation scheme** on $G$ which consists of a cyclic permutation of $N(v)$ for each $v \in V(G)$. It has been shown that there is a one-to-one correspondence between such rotation schemes on connected graphs $G$ and 2-cell imbeddings of $G$ in (orientable) surfaces. Hence, a 2-cell imbedding of $G$ can be given via a rotation scheme.

The genus corresponding to a rotation scheme is calculated by combinatorially tracing out all region boundaries to find the number $r$ of regions, which is then placed into the Euler-Poincaré formula. Finding rotation schemes which give minimal imbeddings is usually extremely tedious work, but this technique has been utilized quite successfully by Ringel for the complete graphs as well as other classes of graphs.
Example 1.1: If \( V(K_{4,4}) = \{0, 1, 2, \ldots, 7\} \) with partite sets \( \{0, 2, 4, 6\} \) and \( \{1, 3, 5, 7\} \), then the following rotation scheme yields eight quadrilateral regions, thus corresponding to a quadrilateral (and hence minimal) imbedding of \( K_{4,4} \) in the torus \( S_1 \).

\[
\begin{align*}
P_0 &= (1,3,5,7) & P_1 &= (0,2,4,6) \\
P_2 &= (7,5,3,1) & P_3 &= (6,4,2,0) \\
P_4 &= (1,3,5,7) & P_5 &= (0,2,4,6) \\
P_6 &= (7,5,3,1) & P_7 &= (6,4,2,0)
\end{align*}
\]

Strictly speaking, current graph and voltage graph constructions are different, though very closely related methods. The former was developed by Gustin in 1963 [19] and used extensively by Gustin, Ringel and Youngs in their contributions to the proof of the Heawood Map Coloring Theorem. In oversimplified terms, the idea underlying current graphs is to describe an imbedding of a Cayley graph by "modding out" similarities via some subgroup (see [43] for definition), thus taking advantage of the group structure. This method works well, as mentioned, but is a bit cumbersome.

In 1974, Gross [17] developed the theory of voltage graphs which is a natural dual of the theory of current graphs. A voltage graph consists of a group \( \Gamma \), a graph \( G \) and a labeling function \( \phi \), which assigns an element of \( \Gamma \) to every arc \( e \) of \( G \) (regarded as a symmetric digraph), so that \( \phi(e^{-1}) = [\phi(e)]^{-1} \). The derived graph \( G \times_{\phi} \Gamma \) has vertex set \( V(G) \times \Gamma \) and two vertices \( (g_1, \alpha_1), (g_2, \alpha_2) \) are adjacent if and only if \( g_1g_2 \in E(G) \) such that \( \phi(g_1g_2) = \alpha_1^{-1}\alpha_2 \). For example Figure 1.1 shows a one vertex, three-loop voltage graph on the torus \( S_1 \). If \( \Gamma = Z_7 \) this "lifts" to a toroidal imbedding of \( K_7 \) (which is a 7-fold covering of the voltage graph shown and where the subgroup "modding out" the similarities is \( K_7 \) itself.) Two advantages
of voltage graphs over current graphs are that (1) the covering structure is immediately the desired graph imbedding and (2) the construction is not as tightly tied to the imbedding of the base graph, i.e., various imbeddings of a voltage graph can be investigated, each giving an imbedding of the derived graph in a surface whose genus is easily calculated.

Both current graphs and voltage graphs have the advantage of powerful theory coupled with elegant arguments. The one drawback is the requirement that the graph being imbedded be a Cayley graph or at least have a highly symmetric structure which is describable in terms of a group.

Surgery is a scissors-and-paste approach to imbedding graphs. In some cases, a major portion of the graph \( G \) being imbedded is contained in a surface \( S \). Then handles can be added to \( S \) (in the same way that \( S_k \) was defined) in order to imbed the missing elements of \( G \). In other cases, disjoint surfaces contain disjoint subgraphs of \( G \). Here, tubes are added joining these surfaces into a single surface and
accommodating missing elements of $G$ (usually edges.) The two examples below illustrate these situations.

**Example 1.2:** To imbed $K_5$ in $S_1$, first imbed $K_5$ (an edge) in $S_0$ (see Figure 1.2a) and then add a handle to accommodate the missing edge (see Figure 1.2b).

![Figure 1.2](image)

**Example 1.3:** $Q_4$ can be imbedded in $S_1$ by beginning with two copies of $Q_3$ in $S_0$ (see Figure 1.3a) and then adding tubes joining two corresponding pairs of opposite regions (see Figure 1.3b).

The construction shown in Example 1.3 was used recursively by Beineke and Harary [6] to obtain minimal imbeddings for $Q_n$. White generalized this approach to establish many genus formulae for cartesian, lexicographic, and strong tensor product graphs. (See [38], [39], and [40].) Also, Alpert [2] used surgery to obtain his results for amalgamations. In addition to purely surgical arguments, surgery can be used to augment imbeddings derived from other methods, as in the additional adjacency constructions of Ringel and Youngs [31]. Recently, Abay Asmerom [1] has combined voltage graph theory and surgery to construct a variety of imbeddings.
A fourth method, called generative m-valuations, was developed by Bouchet to obtain genus results in the class $K_{n(m)}$. Due to its restricted applicability we will not describe it here, but refer the reader to the Stahl survey [33] or to Bouchet [10].

1.4 A Preview

The purpose of this dissertation is to introduce two new surgical techniques for imbedding graphs in orientable surfaces. It will be seen that these techniques apply very well to joins and compositions of graphs and that they yield imbeddings which are frequently minimal or, at least, nearly minimal.

Chapter II describes our first technique, that of graphical surfaces. The idea, here, is to pattern an orientable surface after a graph. We show that the genus of such a surface is easily calculated from the order and size of the underlying graph. We also specify how a graph is imbedded in such a surface.
The technique of graphical surfaces differs dramatically from other methods in its global perspective; the surface is viewed as a whole and the imbedded graph is viewed in its entirety. Compare this to rotation schemes which are pictures of individual imbedded vertices and their immediate neighborhoods. In current graphs, one does not view the imbedded graph at all, but rather views the dual of a generating graph. Voltage graphs are slightly more suggestive in that the view is of an imbedded graph (directly). However, each vertex in this base graph represents several vertices in the covering graph—the one in which our interest lies and one we do not see. Surgery views the imbedded graph as it is, but even the process of adding a tube restricts view to the vicinity of attachment and to the elements imbedded in the tube itself. This rarely involves the entire graph.

The description of a graphical surface with its imbedded graph is not without subtlety. The task of assembling several spheres and tubes involves combinatorial problems of order and orientation. Chapter III abstracts these notions from the geometric and graph theoretic environment, in order to develop the language and theory to complete the description of graphical surfaces and to give them power.

We postpone application of this new technique in order to describe the second technique—crowning. Chapter IV constructs the basic crown as well as variations of the basic crown, the double crown, and clusters of crowns. The purpose served by crowning is that of adding clones of vertices (and their incident edges) to imbedded graphs. This is particularly useful in imbedding complete multipartite graphs. That is, given a complete $n$-partite graph imbedded in a surface, the addition of a basic crown gives an imbedding of a second complete $n$-partite graph (the first graph with one partite set increased by one). The crown nicely augments our first technique in light of the fact that graphs imbedded in graphical surfaces are required to have even order.
Chapter V applies both techniques to joins of graphs. Section 5.1 offers two new proofs of the genus formula for complete bipartite graphs. Section 5.2 contains results for joins in which one of the graphs is empty. Section 5.3 vastly expands knowledge of the genus of complete tripartite graphs. Finally, Section 5.4 presents a few results for the joins of nonempty graphs, including a complete solution of the genus question for the joins of cycles and paths.

Chapter VI applies our techniques to compositions of graphs, where surprisingly strong results are obtained. We also define the generalized composition of graphs and generalize the genus results for compositions.

We conclude with a very short chapter of open problems.
CHAPTER II

GRAPHICAL SURFACES

In this chapter, we define a surface \( S(G) \) whose construction is based on a graph \( G \). It will then be shown that the genus of \( S(G) \) is easily calculated. A careful study of \( S(G) \) will allow the introduction of several geometric terms which will then be used to describe how a second graph \( H \) is imbedded in this new surface.

2.1 The Surface \( S(G) \)

Let \( G \) be a connected graph. We define the graphical surface \( S(G) \) as follows. Replace each vertex \( v \) of \( G \) by a sphere \( S_{v} \) and for each each edge \( e = uv \) of \( G \), add a tube \( T_{e} \) joining the spheres \( S_{u} \) and \( S_{v} \). Moreover, do this so that these individual components are exterior to one another and pairwise nonintersecting, except for pairs of the form \( T_{e} \) and \( S_{u} \), where \( e \) is incident with \( u \). (See Figure 2.1.)

\[ 
\begin{align*}
\text{Figure 2.1}
\end{align*}
\]
We note that in joining tubes to spheres, disks are removed from the spheres rendering them objects which are no longer spheres in a strict geometric sense. For this reason, when one or more disks have been removed from $\text{Sph}_u$ for the purpose of attaching handles or tubes, we denote the new object by $S_u$; but for ease of exposition we continue to call $S_u$ a sphere.

**Lemma 2.1:** If $T$ is a tree then $\gamma(S(T)) = 0$.

**Proof:** We proceed by induction on the order $p$ of the tree $T$.

If $p = 1$ then $S(T)$ is a sphere, whose genus is zero.

Assume the lemma is true for trees of order $p = k$.

Now, suppose $T$ is a tree of order $p = k + 1$. Let $u$ be an endvertex of $T$ and $v$ the unique neighbor of $u$ in $T$. Then $S_u$ is the sphere $\text{Sph}_u$ minus the disk which was removed to attach $T_{uv}$. Hence $S_u$, itself, is homeomorphic to an open disk. Also, $T_{uv}$ is a sphere minus the two disjoint open disks which were removed to attach it to $S_u$ and $S_v$. Thus the combination of $S_u$ and $T_{uv}$ is homeomorphic to an open disk. Therefore, $S(T)$ is homeomorphic to $S(T - u)$, which implies $\gamma(S(T)) = \gamma(S(T - u))$. By the induction hypothesis, $\gamma(S(T - u)) = 0$. Therefore, $\gamma(S(T)) = 0$. □

**Theorem 2.1:** If $G$ is a connected $(p, q)$-graph, then $\gamma(S(G)) = \beta(G)$.

**Proof:** Let $T$ be a spanning tree of the connected $(p, q)$-graph $G$. We note that $S(G)$ can be constructed from $S(T)$ by adding a handle for each edge in $E(G) - E(T)$. By Lemma 2.1, $\gamma(S(T)) = 0$ and since $|E(G) - E(T)| = q - (p - 1) = q - p + 1 = \beta(G)$, we have $S(G)$ represented as a sphere with $\beta(G)$ handles. Hence $S(G) \cong S\beta(G)$. That is, $\gamma(S(G)) = \beta(G)$. □
Example 2.2: The graph $K_{a,b}$ has order $a + b$ and size $ab$. This gives $\beta(K_{a,b}) = ab - (a + b) + 1 = (a - 1)(b - 1)$. So, by Theorem 2.1, $\gamma(S(K_{a,b})) = (a - 1)(b - 1)$.

2.2 A Descriptive Geometry for $\mathcal{S}(G)$

The purpose of this section is twofold. We will demonstrate that $\mathcal{S}(G)$ can be described as a topological subspace of Euclidean 3-space $\mathbb{R}^3$ in a very natural way and, as a consequence of this construction, we will develop the geometric vocabulary required in the following section.

We begin with a 2-cell imbedding of $G$ in a surface $S$, where $S$ is a smooth (in the sense of differential geometry) topological subspace of $\mathbb{R}^3$. In fact, $S$ can be flattened locally and retain its smooth character. So, suppose $S$ has zero curvature within a distance $D$ of each vertex of $G$, where $D$ is a positive real number which is "small" with respect to the distances (in $\mathbb{R}^3$) between vertices of $G$, say $D < \frac{1}{3}$ (minimum Euclidean distance between pairs of vertices of $G$). Furthermore, assume that each edge is imbedded in $S$ as a smooth arc and that for each vertex $v$ in $G$, the edges incident with $v$ are equally spaced radial rays around $v$ to a distance of at least $D$ from $v$ (i.e., the angle between consecutive edges at $v$ is $\frac{2\pi}{\deg v}$) and that no edges other than those incident with $v$ pass within a distance $D$ of $v$. (Reduce $D$ if necessary.)

Once $G$ has been imbedded in $S$ as described, the surface $\mathcal{S}(G)$ will consist of "spheres" of radius $r$ centered at the vertices of $G$, and tubes of cross sectional radius $t$ centered on the edges of $G$. In determining $r$ and $t$, we require that (a) the spheres be pairwise disjoint, (b) the tubes be disjoint from the spheres (except at circles of attachment) and from each other, and (c) no tube be self-intersecting.

First, we determine $r$. For each vertex $v$ of $G$, find a positive real number $r_v < \frac{1}{2}D$ such that $B(v, r_v) \cap G$ is topologically connected (where $B(v, r_v)$ denotes...
the open ball of radius $r_v$ centered at $v$). Let $r = \frac{1}{2} \min\{r_v \mid v \in V(G)\}$. By our choice of $D$, $r_v \leq \frac{1}{2} D$ for all $v \in V(G)$, which gives us objective (a). Also, since $S$ is flat within a distance $D$ of each vertex $v$, we have guaranteed that no portion of an edge, outside of this flat area, is closer than a distance $r$ to the sphere $S_{ph_v}$. If we require that $t < r$, this accomplishes the first part of objective (b).

Secondly, we determine an upper bound for $t$ so that distinct tubes do not intersect. Define the \textbf{r-interior} of an edge $e$ by $I_r(e) = \{x \in e \mid d_E(x, u) > r \text{ for } u \text{ incident with } e\}$ (where $d_E$ denotes euclidean distance). Let $m = \{d_E(I_r(e), I_r(f)) \mid e, f \in E(G)\}$. This must be positive since $\{I_r(e) \mid e \in E(G)\}$ is a pairwise disjoint collection of closed subsets of $R^3$. Let $t < \frac{m}{3}$. Observe that this, together with the description of the edges incident with a vertex $v$, guarantees that the tubes attached to $S_v$ will be equally spaced around $S_v$, with consecutive tubes separated by at least $\frac{m}{3}$.

Thirdly, we determine an upper bound for $t$ so that no tube $T_e$ intersects itself. For $x \in I_r(e)$, let $P_x$ be the plane in $R^3$ which is perpendicular to the edge $e$ at $x$. If $I_r(e) \cap P_x = \{x\}$, set $m_x = \frac{r}{2}$; else, let $m_x = d_E(x, (I_r(e) \cap P_x) - x)$. Note that $(I_r(e) \cap P_x) - x$ is a closed subset of $R^3$ which does not contain $x$, so $m_x > 0$ for each $x \in I_r(e)$. Now let $t_0 = \min \{\{m_x \mid x \in I_r(e)\} \cup \{\frac{r}{2}\}\}$. We claim that $t_0 > 0$. If not, there is a sequence $x_1, x_2, \ldots$ in the closed set $I_r(e)$ which converges to a point $x_{\infty}$ in $I_r(e)$ for which $m_{x_{\infty}} = 0$. This contradicts the fact that $m_x > 0$ for each $x \in I_r(e)$. Thus $t_0 > 0$. Hence $\{t_0 \mid e \in E(G)\}$ is a finite collection of positive real numbers. Let $t \leq \min \{t_e \mid e \in E(G)\}$. So, objectives (b) and (c) are met by finding a positive real number $t$ for which $t \leq \min \{r \cup \{\frac{1}{2} m\} \cup \{t_0 \mid e \in E(G)\}\}$.

We are now in a position to redefine $\bar{S}(G)$. For each $v \in V(G)$, define the (holed) sphere $S_v$ by $S_v = \{x \in R^3 \mid d_E(v, x) = r \text{ and } d_E(e, x) \geq t \text{ for all edges incident with } v \text{ in } G\}$. Also, for each $e = uv \in E(G)$, define the tube $T_e$ by $T_e = T_{uv} = \{x \in R^3 \mid d_E(e, x) = t, d_E(u, x) \geq r \text{ and } d_E(v, x) \geq r\}$. Finally,
\[ S(G) = \left( \bigcup_{v \in V(G)} (S_v) \right) \cup \left( \bigcup_{e \in E(G)} (T_e) \right). \]

The great circle \( S_v \cap S \) is called the **equator of** \( S_v \). Observe that \( \mathbb{R}^3 - S \) consists of two components, an unbounded one and a bounded one, which we will call the **north side of** \( S \) and the **south side of** \( S \), respectively. Consequently, the components of \( S_v - S \) lying on the north and south sides of \( S \) are respectively called the **northern and southern hemispheres of** \( S_v \). Let \( L_v \) be the line normal to \( S \) at \( v \). Then \( S_v \cap L_v \) consists of two points in different hemispheres of \( S_v \) which we will call the **north and south poles of** \( S_v \) accordingly. The north pole of \( S_v \) is denoted \( v^N \) and the south pole is denoted \( v^S \). Continuing to exploit the geographical analogy, we describe each of the two semicircular portions of a great circle whose endpoints are \( v^N \) and \( v^S \) as **meridians of** \( S_v \). A circle in \( S_v \) which is parallel to the equator is called a **line of latitude** or a **parallel**. The direction of travel along such a parallel is referred to as **west** or **east**, respectively, depending on whether it is clockwise or counterclockwise around \( S_v \) from the vantage point of \( v^N \).

Now consider the tubes \( T_e \). The top of \( T_e \) is the curve in \( T_e \) given by \( \{ x \in T_e \mid d_E(S, x) = t \text{ and } x \text{ is on the north side of } S \} \). Similarly, the bottom of \( T_e \) is the curve in \( T_e \) given by \( \{ x \in T_e \mid d_E(S, x) = t \text{ and } x \text{ is on the south side of } S \} \). Suppose \( e = uv \) and \( M \) is the meridian of \( S_v \) which intersects \( e \). Then the **left side of** \( T_e \) **relative to** \( u \) (i.e., looking away from \( u \)) is the component of \( T_e - (\text{the top and bottom of } T_e) \) which is attached to \( S_u \) east of \( M \). Naturally then, the **right side of** \( T_e \) **relative to** \( u \) is the component of \( T_e - (\text{the top and bottom of } T_e) \) which is attached to \( S_u \) west of \( M \). These names are, of course, interchanged relative to \( v \). (See Figure 2.2.)

Movement in \( T_e = T_{uv} \) can also be described relative to \( u \) or \( v \). For instance, we may speak of moving toward or away from \( u \) or of moving clockwise or counterclockwise around \( T_e \) relative to \( u \). Now, let \( C \) be a curve in \( T_e \) which has
as endpoints $P_u$ and $P_v$ where $P_u \in S_u$ and $P_v \in S_v$. If movement along $C$ (starting from $P_u$) is away from $u$ and clockwise around $T_c$ relative to $u$, then $C$ is called a positive spiral in $T_c$. Similarly, if movement along $C$ (starting from $P_u$) is away from $u$ and counterclockwise around $T_c$ relative to $u$, then $C$ is called a negative spiral in $T_c$. It is easily seen that a positive or negative spiral would have the same description from either end, rendering unnecessary the qualifying phrase "relative to $u$" or "relative to $v". (See Figure 2.3.)

We may now present the general description of a graphical surface imbedding.
2.3 Imbedding Graphs in $\mathbb{S}(G)$

Let $G$ be a connected graph. What properties must a graph $H$ have in order to be imbedded in $\mathbb{S}(G)$ and how can such a graph be imbedded?

Clearly, $V(H)$ could be imbedded as any set of $p(H)$ distinct points in $\mathbb{S}(G)$. However, to facilitate a description of the imbedding of the elements of $E(H)$, we restrict the imbedding of $V(H)$ to the poles of the spheres of $\mathbb{S}(G)$ and, in fact, each such pole will correspond to a vertex of $H$. So, if $\text{Pole}(\mathbb{S}(G)) = \bigcup_{v \in V(G)} \{v^N, v^S\}$ then an imbedding of $V(H)$ in $\mathbb{S}(G)$ begins with a bijection $\phi: V(H) \rightarrow \text{Pole}(\mathbb{S}(G))$; or rather, with the image of such a bijection. This requires that $p(H) = 2p(G)$. In general, only graphs of even order may have imbeddings of the type being described here. (See Chapter 4 for a discussion of imbedding odd order graphs.)

We further restrict the imbedding of $V(H)$ as follows. Suppose $x$ and $y$ are adjacent vertices of $H$, imbedded in $\mathbb{S}(G)$ as poles of $S_u$ and $S_v$, respectively. In this case, we say that $xy$ is an edge of order $n$, where $n = d_G(u,v)$. In order to limit the "length" of the imbedded edge $xy$, we require that $d_E(u,v) \leq 2$, i.e., that all edges have order $n \leq 2$. (Note, if all first-order edges are present, then edges of order $> 2$ cannot exist.) Note that the order of an edge of $H$ is a measure of the number of tubes it traverses in $\mathbb{S}(G)$ from end to end. It will also be shown that the order of an edge is an indicator of the difficulty of imbedding the edge. (See Figure 2.4.)

![Figure 2.4](image-url)
An edge of order 0 joins \( v^N \) and \( v^S \) for some \( v \in V(G) \). Such an edge is easily imbedded in \( S_v \) as a meridian which intersects the equator of \( S_v \) between successive tubes.

An edge \( e \) of order 1 joins a pole of \( S_u \) to a pole of \( S_v \), where \( uv \in E(G) \). In all such cases, \( e \) is imbedded in \( S(G) \) so that \( e \cap S_u \) is a connected portion of a meridian of \( S_u \) which does not intersect the equator of \( S_u \). Similarly for \( e \cap S_v \). If \( e = u^Nv^N \) then \( e \cap T_{uv} \) is the top of \( T_{uv} \). If \( e = u^Sv^S \) then \( e \cap T_{uv} \) is the bottom of \( T_{uv} \). The remaining edges \( e_1 = u^Nv^S \) and \( e_2 = u^Sv^N \) are imbedded in one of two ways. Either \( C_1 = e_1 \cap T_{uv} \) and \( C_2 = e_2 \cap T_{uv} \) are both positive spirals which are, respectively, contained in the right and left sides of \( T_{uv} \) relative to \( S_u \); or \( C_1 \) and \( C_2 \) are negative spirals, respectively contained in the left and right sides of \( T_{uv} \) relative to \( S_u \). If \( C_1 \) and \( C_2 \) are positive spirals, we say that \( T_{uv} \) is positively oriented or has positive orientation. If \( C_1 \) and \( C_2 \) are negative spirals, we say that \( T_{uv} \) is negatively oriented or has negative orientation. (See Figure 2.5, where (a) shows two positively oriented tubes and (b) shows two negatively oriented tubes.)

It should be observed that no two edges of order \( \leq 1 \) can possibly intersect one another if imbedded as described. Furthermore, if each tube carries a full complement of four first-order edges, then these edges alone separate \( S(G) \) into \( 2q(G) \) quadrilateral regions. Each such region is bounded by a 4-cycle in \( H \) of the form \( u^Nv^Sx^S \) where \( x \) is a pole of \( S_v \) and \( y \) is a pole of \( S_w \), such that \( u, v \) and \( w \) are distinct vertices of \( G \). [Note, if \( \deg_G u = 1 \) then, and only then, may \( v = w \).] If \( u^Nv^S \in E(H) \), this zeroeth-order edge is imbedded as a diagonal in one of the \( \deg_G u \) quadrilateral regions having \( u^N \) and \( u^S \) as opposite vertices.

An edge \( e = xy \) of order 2 joins a pole of \( S_v \) to a pole of \( S_w \), where \( d_G(v, w) = 2 \). The edge \( e \) is imbedded in the portion of \( S(G) \) given by \( S_v \cup T_{uv} \cup S_u \cup T_{uw} \cup S_w \), where \( u \in V(G) \) with \( vu, uw \in E(G) \) and such that \( v \)
and \( w \) are consecutive neighbors of \( u \) in the rotation scheme giving the imbedding of \( G \) in \( S \). In particular, \( e \cap S_v \) is a connected portion of a meridian of \( S_v \) which does not intersect the equator of \( S_v \), similarly with \( e \cap S_w \); \( e \cap T_{vu} \) and \( e \cap T_{uw} \) are spirals; and \( e \cap S_u \) is a connected portion of a parallel of \( S_w \).

Just as with zeroeth-order edges, second-order edges \( xy \) are imbedded as diagonals of the quadrilateral regions formed by the first-order edges. Whether or not a quadrilateral region is formed having \( x \) and \( y \) as opposite vertices depends not only on \( T_{vu} \) and \( T_{uw} \) being attached consecutively around the equator of an intermediate sphere \( S_u \), but also on the orientations of \( T_{vu} \) and \( T_{uw} \) induced by the first-order edges.

For instance, if \( xy \in E(H) \) is to be imbedded as a second-order edge, \( x = v^N \), \( y = w^S \) and \( u \in N_G(v) \cap N_G(w) \), then the quadrilateral \( xu^N vy^S \) is formed in one of two ways. Either (i) \( T_{vu} \) is attached to \( S_u \) immediately to the west of \( T_{uw} \) and both tubes are positively oriented (as is shown in Figure 2.5 a) or (ii) \( T_{vu} \) is attached to \( S_u \) immediately to the east of \( T_{uw} \) and both tubes are negatively oriented (which is similar to the situation in Figure 2.5 b except that \( S_v \) and \( S_w \) are interchanged). If negative orientation of a tube is denoted by a tilda "\( \sim \)" over the "\( T \)", with the default orientation being positive, we denote the two cases just given by \( T_{vu}T_{uw} \) and \( \sim T_{uw}\sim T_{vu} \). Figure 2.5 shows the effect of tube attachment orders (i.e., the rotation scheme giving the imbedding of \( G \) in \( S \)) and the tube orientations on the imbedding of second-order edges. Table 2 summarizes this geometric situation.

It should be noted here, that the imbedding of \( G \) in \( S \), as well as the orientations of the tubes, are incidental to the process of imbedding edges of order 0 or 1, and are only pertinent to the imbedding of edges of order 2. For this reason,
these two aspects of the construction may remain unspecified (in the absence of second-order edges) or may be specified only in part (as needed to imbed these longest edges). To accomplish the substantial task of imbedding a large number of second-order edges, we study the interplay between orientation and order in the form of a combinatorial object called an arcchain, which is the topic of the next chapter.
In this section, we construct a surface \( \bar{S}(G) \) which will be shown to be homeomorphic to \( S(G) \). The consequent equality of the genera of these surfaces yields the famous Euler-Poincaré formula.

As in the construction of \( S(G) \) given in Section 2.2, begin with the connected \((p, q)\)-graph \( G \), 2-cell imbedded in a surface \( S_k \), with \( r \) regions \( p_1, p_2, \ldots, p_r \). Thicken \( S_k \) to a solid whose surface is two copies of \( S_k \); call these \( S_k(1) \) and \( S_k(2) \). (If \( S_k \) is a smooth subspace of \( R^3 \), this solid is \( \{ x \in R^3 | d_E(x, S_k) = d \} \) for some, appropriately small, positive real number \( d \). Then \( S_k(1) \) and \( S_k(2) \) are the two components of \( \{ x \in R^3 | d_E(x, S_k) = d \} \) each of which is homeomorphic to \( S_k \).)

Suppose both surfaces of the solid contain a copy of \( G \), say \( G_i \) in \( S_k(i) \) for \( i = 1, 2 \); in the "same position" as \( G \) in \( S_k \). (Say, \( G_i \) is the set of points in \( S_k(i) \) which are a distance \( d \) from \( G \) in \( R^3 \).) Denote the regions of \( G_i \) by \( \rho_1(i), \rho_2(i), \ldots, \rho_r(i) \) for \( i = 1, 2 \). Now for each \( j = 1, 2, \ldots, r \), drill a hole through this solid which intersects \( S_k(i) \) in the interior of \( \rho_j(i) \). Call the surface of the resultant solid \( \bar{S}(G) \).
Lemma 2.3: (a) \(\chi(\mathcal{S}'(G)) = 2k + r - 1\)

(b) \(\mathcal{S}'(G) \cong \mathcal{S}(G)\)

Proof: (a) \(\mathcal{S}'(G)\) is essentially two surfaces of genus \(k\) joined by \(r\) tubes. The first tube joins the two surfaces, resulting in a single surface of genus \(2k\). The remaining \(r - 1\) tubes act as handles, each adding one to the genus.

(b) Observe that, by construction, \(\mathcal{S}'(G)\) contains the graph \(2G (= G \cup G)\).

The surface \(\mathcal{S}(G)\) also contains \(2G\); with the first copy, \(G_1\), of \(G\) having all north poles of \(\mathcal{S}(G)\) as vertices and the other, \(G_2\), having all south poles as vertices. The edge sets \(E(G_1)\) and \(E(G_2)\) consist of all first order edges joining north and south poles, respectively.

Now the regions of \(2G\) in \(\mathcal{S}'(G)\) and the regions of \(2G\) in \(\mathcal{S}(G)\) have the same description. That is, each is tubular (homeomorphic to \(S^0 \times [0,1]\)) and they are in one-to-one correspondence with the regions \(\rho_j\), \(j = 1, 2, \ldots r\) of the imbedding of \(G\) in \(S_k\). Furthermore, the boundary of the tubular region corresponding to \(\rho_j\) is \(\partial(\rho_j(1)) \cup \partial(\rho_j(1))\).

Thus, \(\mathcal{S}'(G) - 2G \equiv \mathcal{S}(G) - 2G \equiv (\text{the disjoint union of } r \text{ copies of } S^0 \times [0,1])\), and since the positions of those regions are the same relative to \(2G\) in both \(\mathcal{S}'(G)\) and \(\mathcal{S}(G)\), therefore \(\mathcal{S}'(G) \equiv \mathcal{S}(G)\).

Theorem 2.3 [Euler-Poincaré]: If \(G\) is a connected \((p, q)\)-graph with a 2-cell imbedding in \(S_k\) having \(r\) regions, then \(p - q + r = 2 - 2k\).

Proof: Lemma 2.3.b implies that \(\gamma(\mathcal{S}'(G)) = \gamma(\mathcal{S}(G))\), so by Lemma 2.3.a and Theorem 2.1 we have \(2k + r - 1 = q - p + 1\). Thus, \(p - q + r = 2 - 2k\).
Figure 2.6 shows (a) a graph $G$ imbedded in a surface, (b) the surface $\mathcal{S}(G)$ and (c) the surface $\mathcal{S}'(G)$.
CHAPTER III

ARCCHAINS

A chain is an object commonly studied in combinatorics and elsewhere in mathematics. In this chapter we introduce a generalization of this concept. The vocabulary and results introduced here will play a key role in describing how surfaces are constructed in Chapters V and VI.

3.1 Orderings and Chains

Let \( L \) be a set of cardinality \( |L| = n < \infty \). A linear ordering of \( L \) is a bijection \( \tau : L \rightarrow \{0, 1, ..., n - 1\} \). The pair \((L, \tau)\) is called an \( L \)-chain or linear \( L \)-chain and is normally given as a list \( \tau^{-1}(0) \tau^{-1}(1) \ldots \tau^{-1}(n - 1) \). Observe that an \( L \)-chain is nothing more than a permutation of the elements of \( L \). As such, it is well known that there are \( n! \) distinct \( L \)-chains.

Now consider the bijections \( \mu : L \rightarrow Z_n \). We define an equivalence relation on such bijections as follows: \( \mu_1 \equiv \mu_2 \) if and only if there is an \( a \in Z_n \) such that \( \mu_2(\ell) = \mu_1(\ell) + a \) for all \( \ell \in L \). A cyclic ordering of \( L \) is an equivalence class \([\mu]\) of bijections \( \mu : L \rightarrow Z_n \). The pair \((L, [\mu])\) is called a cyclic \( L \)-chain and is normally given as a list \( \mu^{-1}(0) \mu^{-1}(1) \ldots \mu^{-1}(n - 1) \mu^{-1}(0) \), where \( \mu \in [\mu] \). Observe that \( \mu^{-1}(0) \) is repeated at the end of the list to distinguish the list from that of a linear \( L \)-chain. Also, since a cyclic ordering \([\mu]\) consists of \( n \) bijections \( \mu \), there are \( n \) different ways to represent a cyclic \( L \)-chain as a list (as opposed to the unique list representing a linear \( L \)-chain.)
Example 3.1: If $L = \{a, b, c, d\}$ then each of the following represents the same cyclic $L$-chain: $abced$, $bcda$, $cdabc$ and $dabcd$.

We see that cyclic $L$-chains are cyclic permutations of $L$ and recall that they number $\frac{n!}{n} = (n - 1)!$.

Whether the context is linear $L$-chains or cyclic $L$-chains we call the elements of $L$ links and $L$ the link set.

3.2 Orientations and Arcchains

The foregoing description of chains views links as zero dimensional objects, aptly modeled as points. Suppose the links in a chain were modeled as directed edges or arcs, i.e., as objects with two distinct ends. In constructing a chain of arcs, in addition to linearly or cyclicly ordering the arcs, it would be necessary to specify an orientation for each arc. These ideas motivate the concept of an arcchain.

An orientation of a set $L$ is a function $\sigma : L \rightarrow \{+1, -1\}$. The elements of an oriented set are denoted $\sigma(\ell) \cdot \ell$ for $\ell \in L$ (i.e., $+\ell$ if $\sigma(\ell) = +1$ and $-\ell$ if $\sigma(\ell) = -1$.)

A linear $L$-arcchain is a triple $(L; \sigma, \tau)$ where $L$ is a finite set, $\sigma$ is an orientation of $L$ and $\tau$ is a linear ordering of $L$. So, a linear $L$-arcchain is a linear arrangement of oriented links. Since context will generally make clear that a list represents an arcchain as opposed to a chain, $+\ell$ will be denoted $\ell$ while $-\ell$ will be denoted $\tilde{\ell}$ (e.g., $\tilde{\ell}_4 \tilde{\ell}_2 \tilde{\ell}_1 \tilde{\ell}_0 \tilde{\ell}_3$.) By replacing the linear ordering $\tau$ with a cyclic ordering $[\mu]$, we obtain both definition and notation $(L; \tau, [\mu])$ for a cyclic $L$-arcchain.

An arrangement of oriented links, whether linear or cyclic, allows relationships between adjacent links other than their order. For the purpose of exploring these
relationships, we describe an unoriented link $\ell$ as having two ends or poles; a north pole $\ell^N$ and a south pole $\ell^S$. To orient a link $\ell$ means to order its poles. In this way we associate $+\ell$ with the ordered pair $(\ell^S, \ell^N)$ and $-\ell$ or $\bar{\ell}$ with $(\ell^N, \ell^S)$. The set $\bigcup_{\ell \in L} \{\ell^S, \ell^N\}$ is called the pole set of $L$ and is denoted Pole $(L)$. Furthermore, we assume that for $\ell_i, \ell_j \in L$ and $i \neq j$ we have $|\{\ell_i^S, \ell_i^N, \ell_j^S, \ell_j^N\}| = 4$. Thus $|\text{Pole}(L)| = 2|L|$.

Observe that a linear L-chain (respectively, cyclic L-chain) induces a linear Pole $(L)$-chain (respectively, cyclic Pole $(L)$-chain) which is obtained by replacing each oriented link, $\ell$ or $\bar{\ell}$ with the ordered pair of poles to which it corresponds, $\ell^S \ell^N$ or $\ell^N \ell^S$. A couple is defined as an (unordered) pair of poles, belonging to different links, which appear as adjacent elements in this induced chain.

**Example 3.2:** The cyclic L-chain $\ell_1 \ell_0 \ell_2 \ell_1$ induces the cyclic Pole $(L)$-chain $\ell_1^S \ell_1^N \ell_0^S \ell_0^N \ell_2^N \ell_2^S \ell_1^S$. Hence, we say this cyclic L-arcchain forms the three couples $\{\ell_1^N, \ell_0^N\}$, $\{\ell_0^S, \ell_2^S\}$, and $\{\ell_2^N, \ell_1^S\}$.

Clearly, each pair of adjacent links forms a couple. Thus any linear L-arcchain forms $|L| - 1$ couples and any cyclic L-arcchain forms $|L|$ couples. Also, each pair of links $\ell_i$ and $\ell_j$ represents four possible couples. These are formed by an arcchain only if $\ell_i$ and $\ell_j$ are adjacent in the ordering; the particular couple formed is determined by the order of $\ell_i$ and $\ell_j$ as well as their orientations. The various possibilities are given below. The pattern observed in Table 3 motivates several definitions.

If $\sigma$ is an orientation of $L$ then $-\sigma$ is called the opposite of $\sigma$, that is $-\sigma(\ell) = (-1)^{\sigma(\ell)}$ for all $\ell \in L$.

If $\tau$ is a linear ordering of $L$ with $|L| = n < \infty$, then the reverse of $\tau$, denoted $\bar{\tau}$, is defined for all $\ell \in L$ by $\bar{\tau}(\ell) = n - 1 - \tau(\ell)$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
If $\mu : L \rightarrow \mathbb{Z}_n$ is bijective, then define $\overline{\mu}$ for all $\ell \in L$ by $\overline{\mu}(\ell) = n - 1 - \mu(\ell)$. Thus if $[\mu]$ is a cyclic ordering of $L$ then $[\overline{\mu}]$ is called the reverse of $[\mu]$.

Next, if $C = (L ; \sigma , \tau)$ is a linear $L$-arcchain, then the conjugate of $C$ is the linear $L$-arcchain $\overline{C} = (L ; -\sigma , \overline{\tau})$. Similarly, the conjugate of cyclic $L$-arcchain $C = (L ; \sigma , [\mu])$ is the cyclic $L$-arcchain $\overline{C} = (L ; -\sigma , [\overline{\mu}])$.

Returning to Table 3, we see that the eight pairs of oriented links constitute all possible linear $\{\ell_1, \ell_2\}$-arcchains. Furthermore, couples formed by conjugate pairs are identical. This conclusion easily generalizes.

**Theorem 3.1:** A couple is formed by an arcchain $C$ (linear or cyclic) if and only if it is formed by $\overline{C}$.

We finish this section with a few more definitions.

A couple consisting of a north pole and a south pole is called attractive. Any arcchain which forms only attractive couples is called an attractive arcchain. Observe that $(L ; \sigma , \tau)$ or $(L ; \sigma , [\mu])$ is attractive if and only if $\sigma$ is constant.
A couple which is not attractive is called **repellent**. That is, a couple is repellent if both elements are the same type (north pole or south pole.) Any arcchain which forms only repellent couples is called a **repellent arcchain**. Observe that \((L; \sigma, \tau)\) or \((L; \sigma, [\mu])\) is repellent if and only if \(\sigma\) alternates relative to the ordering of \(L\). In the cyclic case, this requires \(|L|\) to be even. Note that there are arcchains which are neither attractive nor repellent.

**Example 3.3:** We see that \(\overline{t}_0 \overline{t}_1 \overline{t}_2 \overline{t}_3\) is an attractive arcchain, forming the attractive couples \(\{t_0^S, t_1^N\}, \{t_1^S, t_2^N\}\) and \(\{t_2^S, t_3^N\}\).

Also \(\overline{t}_0 \overline{t}_1 \overline{t}_2 \overline{t}_3\) is a repellent arcchain, forming the repellent couples \(\{t_0^N, t_1^N\}, \{t_1^S, t_2^S\}\) and \(\{t_2^N, t_3^N\}\).

Thus, by defining the **constant orientations** \(\sigma^+\) and \(\sigma^-\) as \(\sigma^+(\ell) = +1\) for all \(\ell \in L\) and \(\sigma^-(\ell) = -1\) for all \(\ell \in L\), we have \((L; \sigma, \tau)\) attractive if and only if \(\sigma = \sigma^+\) or \(\sigma^-\). Similarly, we define the **alternating orientation of \(L\)** with respect to a linear ordering \(\tau\) by \(\sigma^\tau(\ell) = (-1)^\tau(\ell)\) for all \(\ell \in L\). Hence, \((L; \sigma, \tau)\) is repellent if and only if \(\sigma = \sigma^\tau\) or \(-\sigma^\tau\). Also if \(\mu\) represents a cyclic ordering \([\mu]\) of \(L\), \(\sigma^\mu(\ell) = (-1)^\mu(\ell)\) for all \(\ell \in L\). Note that \(\sigma^\mu\) is not the same for all representatives of \([\mu]\) and that if \(n\) is odd, the adjacent links \(\mu^{-1}(1)\) and \(\mu^{-1}(n)\) will both be negatively oriented.

### 3.3 Collections of Arcchains

Our primary interest in this section is the construction of collections of arcchains of the same type—linear or cyclic, defined on the same link set, which together form all couples of a given description. Such a collection will be called a **C^4**, meaning a Couple Complete Collection of arcChains. For the remainder of this chapter,
"arcchain" will refer to an $L$-arcchain (linear or cyclic), where $|L| = n < \infty$ and $L = \{e_0, e_1, ..., e_{n-1}\}$.

Eight adjectives will be used to modify $C^4$ in this section. These are: minimal, even, odd, linear, cyclic, total, attractive, and repellent. A "minimal" $C^4$ of a given description is such a collection containing the smallest number of arcchains possible. "Even" and "odd" refer to $n (= |L|)$. "Linear" and "cyclic" refer to the arcchains in the collection, in that they must all be of the type specified. The remaining three refer to the couples being formed. Specifically, a "total" $C^4$ forms all possible couples. An "attractive" $C^4$ forms all possible attractive couples. Note that an attractive $C^4$ may contain arcchains which are not attractive (i.e., an attractive $C^4$ may form some repellent couples), but if all arcchains in an attractive $C^4$ are attractive then it must be minimal. Similarly, a "repellent" $C^4$ forms all possible repellent couples, but may contain arcchains which are not repellent.

To streamline notation somewhat, we introduce the notation $(t_1, t_2)-C^4(n)$. The value of $t_1$ indicates whether the $C^4$ is linear ($t_1 = 1$) or cyclic ($t_1 = 2$). The value of $t_2$ indicates whether the $C^4$ is total ($t_2 = 1$), attractive ($t_2 = 2$) or repellent ($t_2 = 3$). If $n$ is not to be specified, the "(n)" is omitted. We also introduce the notation $c_{t_1 t_2}(n)$ as the cardinality of a minimal $(t_1, t_2)-C^4(n)$.

Our strategy in attacking the problem of constructing minimal $C^4$'s will be to construct an appropriate collection of chains and then to orient the links to achieve our goal. We begin by associating $L$ with $K_n$; that is, thinking of the links as the vertices in $K_n$. A linear $L$-chain is then a directed hamiltonian path in $K_n$ and a cyclic $L$-chain is a directed hamiltonian cycle in $K_n$. With this in mind, we present two well known results of Lucas [25].
Theorem 3.2: For every positive integer \( m \), the graph \( K_{2m+1} \) can be decomposed into \( m \) hamiltonian cycles.

Proof [see Chartrand and Lesniak (12) p. 237]: Since the result is clear for \( m=1 \), we may assume that \( m \geq 2 \). Let \( V(K_{2m+1}) = \{v_0, v_1, \ldots, v_{2m}\} \). Arrange the vertices \( v_0, v_1, \ldots, v_{2m-1} \) in a regular \( 2m \)-gon and place \( v_{2m} \) in some convenient position. Join every two vertices by a straight line segment, thereby producing \( K_{2m+1} \). We define the edge set of \( F_1 \) to consist of \( v_{2m}v_0, v_{2m}v_1, \) all edges parallel to \( v_0v_1 \), and all edges parallel to \( v_{2m-1}v_1 \). In general, for \( i = 1, 2, \ldots, m \), we define the edge set of the factor \( F_i \) to consist of \( v_{2m}v_{i-1}, v_{2m}v_{m+i-1}, \) all edges parallel to \( v_{i-1}v_i \), and all edges parallel to \( v_{i-2}v_i \), where the subscripts are expressed modulo \( 2m \). Then \( K_{2m+1} = F_1 \oplus F_2 \oplus \ldots \oplus F_m \), where \( \oplus \) is the direct product and \( F_i \) is the hamiltonian cycle,

\[ v_{2m}, v_{i-1}, v_i, v_{i-2}, v_{i+1}, v_{i-3}, \ldots, v_{m+i-2}, v_{m+i}, v_{m+i-1}, v_{2m}. \]

Corollary 3.2: For every positive integer \( m \), the graph \( K_{2m} \) can be decomposed into \( m \) hamiltonian paths.

Proof: Decompose \( K_{2m+1} \) into hamiltonian cycles. Now delete \( v_{2m} \) from \( K_{2m+1} \) to obtain \( K_{2m} \) and from each cycle to obtain a hamiltonian path.

We will also find the following results of Tillson (36) helpful. Recall that \( K^*_m \) is the complete symmetric digraph of order \( m \).

Theorem 3.3: For \( 2m \geq 8 \), \( K^*_m \) can be decomposed into \( 2m - 1 \) directed hamiltonian cycles.

Corollary 3.3: For \( 2m \geq 8 \), \( K^*_{2m-1} \) can be decomposed into \( 2m - 1 \) directed hamiltonian paths.
We are now in a position to state and prove the solution to the main problem.

**Theorem 3.4:** For all positive integers $m$ and $n$,

(i) $c_{11}(n) = 2n$ ;

(ii) $c_{12}(n) = n$ except that $c_{12}(3) = 4$ and $c_{12}(5) = 6$ ;

(iii) $c_{13}(n) = n$ ;

(iv) $c_{21}(n) = 2(n-1)$ ;

(v) $c_{22}(n) = n - 1$ except that $c_{22}(4) = 4$ and $c_{22}(6) = 6$ ;

(vi) $c_{23}(2m) = 2m - 1$ ;

(vii) $c_{23}(2m - 1) = 2m - 1$.

**Proof:** The total number of couples from the $n$ links $L$ is $4 \left(\frac{n}{2}\right) = 2n(n-1)$. Of these, half are attractive and half are repellent.

Each linear arcchain forms $n - 1$ couples. Thus, $c_{11}(n) \geq \frac{2n(n-1)}{n-1}$, $c_{12}(n) \geq n$, and $c_{13}(n) \geq n$. Also, each cyclic arcchain forms $n$ couples. Thus, $c_{21}(n) \geq 2(n-1)$, $c_{22}(n) \geq n - 1$ and $c_{23}(n) \geq n - 1$.

With the exception of (vii) and the four cases noted in (ii) and (v), we will prove equality by constructing $C^4$'s which attain these lower bounds. In the exceptional cases, it will be necessary to show the impossibility of attaining the aforementioned lower bound and then to construct $C^4$'s with one additional arcchain.

We divide the proof into four cases.

**Case 1 (even linear):** Let $n = 2m$ and let $P_1, P_2, \ldots, P_m$ be a decomposition of $K_{2m}$ into hamiltonian paths, via Corollary 3.2. Then by way of any bijection between $L$ and $V(K_{2m})$, each path $P_k$, $k = 1, 2, \ldots, m$, corresponds to two (reverse) linear orderings of $L$; call them $\tau_k$ and $\overline{\tau}_k$. Now each pair of links $\ell_i$ and $\ell_j$ appear as an adjacent pair in exactly one of the chains $(L; \tau_k)$ for $k = 1, 2, \ldots, m$. Furthermore,
if \( \tau_k(\tilde{t}_j) = \tau_k(\tilde{t}_j) + 1 \) (i.e., \( \cdot \cdot \cdot \tilde{t}_j \cdot \cdot \cdot \) then all four of the \( \tilde{t}_i - \tilde{t}_j \) couples are formed by the arcchains \( (L; \sigma^+\tau_k) \), \( (L; \sigma^-\tau_k) \), \( (L; \sigma^k\tau_k) \), and \( (L; -\sigma^k\tau_k) \). Specifically, the attractive couples \( \tilde{t}_i^N \tilde{t}_j^S \) and \( \tilde{t}_i^S \tilde{t}_j^N \) are formed by \( (L; \sigma^+\tau_k) \) and \( (L; \sigma^-\tau_k) \) respectively; and the repellent couples \( \tilde{t}_i^N \tilde{t}_j^S \) and \( \tilde{t}_i^S \tilde{t}_j^N \) are formed by \( (L; \sigma^k\tau_k) \) and \( (L; -\sigma^k\tau_k) \), (respectively, if \( \tau_k(\tilde{t}_i) \) is even and inversely if \( \tau_k(\tilde{t}_i) \) is odd.)

Consequently, \( \bigcup_{k=1}^{m} \{(L; \sigma^+\tau_k), (L; \sigma^-\tau_k), (L; \sigma^k\tau_k), (L; -\sigma^k\tau_k)\} \) is a \( (1,1)-C^{4}(2m) \) of cardinality \( 4m \). Also, \( \bigcup_{k=1}^{m} \{(L; \sigma^+\tau_k), (L; \sigma^-\tau_k)\} \) is a \( (1,2)-C^{4}(2m) \) of cardinality \( 2m \), and \( \bigcup_{k=1}^{m} \{(L; \sigma^k\tau_k), (L; -\sigma^k\tau_k)\} \) is a \( (1,3)-C^{4}(2m) \) of cardinality \( 2m \). Since these attain the lower bounds established above, \( c_{11}(n) = 2n \) and \( c_{12}(n) = c_{13}(n) = n \) if \( n \) is even.

**Case 2 (even cyclic):** Again, let \( n = 2m \).

We consider \( (2,2)-C^{4}(2m) \) first. By Theorem 3.3, \( K_{2m}^* \) has a decomposition \( C_1, C_2, \ldots, C_{2m-1} \) into directed hamiltonian cycles when \( 2m \geq 8 \). Obviously \( K_2^* \) has such a decomposition as well. Bermond and Faber [8] have shown that \( K_4^* \) and \( K_6^* \) do not have such a decomposition.

For \( 2m = 2 \) or \( 2m \geq 8 \), each hamiltonian cycle \( C_k \) for \( k = 1, 2, \ldots, 2m-1 \) gives a cyclic ordering \( [\mu_k] \) of \( L \). Moreover, each pair of links occur adjacent to one another in exactly two of these orderings, but in different orders with respect to each other. Thus \( \bigcup_{k=1}^{2m-1} \{(L; \sigma^+\tau_k, [\mu_k])\} \) is a \( (2,2)-C^{4}(2m) \) which attains the lower bound of \( 2m-1 \) on \( c_{22}(2m) \). Therefore, if \( n \) is even and \( n \neq 4 \) or \( 6 \), then \( c_{22}(n) = n - 1 \).

We see that the same technique cannot be applied to the subcases \( 2m = 4 \) and \( 6 \). However, the negative result of Bermond and Faber [8] can be used to conclude that \( c_{22}(4) \neq 3 \) and \( c_{22}(6) \neq 5 \) as we will show. Suppose a \( (2,2)-C^{4}(2m) \) exists which has cardinality \( 2m-1 \). Then it must be composed entirely of attractive arcchains \( (L; \sigma_k, [\mu_k]) \), \( k = 1, 2, \ldots, 2m-1 \), where \( \sigma_k = \sigma^+ \) or \( \sigma^- \) for all \( k \).
Without loss of generality, we can assume that $\sigma_k = \sigma^+$ for all $k$ (for, if $(L; \sigma^-, [\mu_k])$ is in the collection, it can be replaced by its conjugate $(L; \sigma^+, [\bar{\mu}])$). Each cyclic ordering $[\mu_k]$ corresponds to a directed hamiltonian cycle $C_k$ in $K_{2m}^*$ and, in fact, the collection $C_1, C_2, \ldots, C_{2m-1}$ so formed decomposes $K_{2m}^*$. But no such decomposition exists for $2m = 4$ or $6$. Thus $c_{22}(4) > 3$ and $c_{22}(6) > 5$. But $c_{12}(4) = 4$ and $c_{12}(6) = 6$ and each linear arcchain can be transformed into a cyclic arcchain by making its endlinks adjacent. Therefore $c_{22}(4) = 4$ and $c_{22}(6) = 6$.

We now turn our attention to $(2,1)$- and $(2,3)$-s. Proceed in a manner similar to that of the proof of Theorem 3.2. Let $V(K_{2m}) = \{b_0, b_1, \ldots, b_{2m-1}\}$. (Here we view the subscripts of $V(K_{2m}) - b_{2m-1}$ as the elements of $Z_{2m-1}$.) Arrange the vertices $b_0, b_1, \ldots, b_{2m-2}$ in a regular $(2m-1)$-gon and place $b_{2m-1}$ in some convenient position. Join every two vertices by a straight line segment, thereby producing $K_{2m}$. In general, for $i = 1, 2, \ldots, 2m-1$, we define the edge set of $F_i$ to consist of $b_{2m-1}b_{1}, b_{2m-1}b_{m+i-1},$ all edges parallel to $b_{i-1}b_{i}$, and all edges parallel to $b_{i}b_{i-2}$. This edge set induces the hamiltonian cycle

$$C_i: b_{2m-1}, b_{1-i}, b_{-1}, b_{2}, b_{i+1}, b_{i-3}, \ldots, b_{m+i-3}, b_{m+i}, b_{m+i-2}, b_{m+i-1}, b_{2m-1}.$$  

This collection of cycles is not a decomposition of $K_{2m}$, but rather a double cover of $E(K_{2m})$. To gain a better understanding of this collection of cycles, form a $(2m-1) \times (2m+1)$ array where the entries in the $i^{th}$ row are the subscripts of the links in $C_i$, in order. Figure 3.1 shows the array for $2m = 8$.

We will use this array to show that these cycles double cover $E(K_{2m})$. First observe that all columns, except for the first and last, are generated from the top downward by adding 1 modulo $2m-1$ (i.e., for $i = 1, 2, \ldots, 2m-2$ we have $a_{i+1,j} \equiv a_{i,j+1} \pmod{2m-1}$). So each such column contains each of the elements of $Z_{2m-1}$ exactly once.
One implication of this is that each link pair \( \ell_2^{-m-1} \ell_j \) occurs between columns 1 and 2, and each link pair \( \ell_j \ell_2^{-m-1} \) occurs between columns \( 2m \) and \( 2m+1 \).

Also, each row is generated from left to right, between columns 2 and \( 2m \) as follows: 
\[
ai,3 = ai,2 + 1, \quad ai,4 = ai,3 - 2, \quad ai,5 = ai,4 + 3, \quad \ldots.
\]
In general, 
\[
ai,j = ai,j-1 + (-1)^{j-1}(j - 2)
\]
for \( j = 3, 4, \ldots, 2n \). But since addition is modulo \( 2m-1 \), for \( 2 \leq j \leq m-1 \), we have
\[
ai,2m+3-j = ai,2m+2-j + (-1)^{2m+3-j-1}(2m + 3 - j - 2)
\]
\[
= ai,2m+2-j + (-1)^j(2 - j)
\]
\[
= ai,2m+2-j + (-1)^{j-1}(j - 2)
\]

So, in order to locate a pair \( \{r, s\} \) where \( r, s \in \mathbb{Z}_{2m+1} \) in the array, find an integer \( j, \ 3 \leq j \leq m + 1 \), such that either \( r - s = (-1)^{j-1}(j - 2) \) in which case \( s \) appears first, or \( s - r = (-1)^{j-1}(j - 2) \) in which case \( r \) appears first. Find this first number \( r \) or \( s \) in the \((j - 1)\)th column, say entry \( ai,j-1 \); then the second number will occur as entry \( ai,j \). This pair can also be found, in the same order, straddling columns \( 2m+2-j \) and \( 2m+3-j \) in the same way.

Returning to the hamiltonian cycles \( C_1, C_2, \ldots, C_{2m-1} \) of \( K_{2m} \), we see that each corresponds to a cyclic ordering \([\mu_k]\) of \( L \). For each of these, let \( \mu_k \) be the representative of \([\mu_k]\) for which \( \mu_k(\ell_2^{-m-1}) = 1 \). Then \( \mu_k \) corresponds to the \( k \)th
row of the array. We can now orient the links in these cyclic chains \((L; [\mu_k])\) to construct a \((2,3)\)-\(C^4(2m)\) and a \((2,1)\)-\(C^4(2m)\).

Claim 1: \(\{(L; \sigma^{Hk}, [\mu_k]) \mid k = 1, 2, \ldots, 2m-1\}\) is a \((2,3)\)-\(C^4(2m)\), thus establishing \(c_{23}(n) = n - 1\) for \(n\) even.

Proof of Claim 1: We have shown that if neither \(i\) nor \(j\) is \(2m-1\), then the links \(\ell_i\) and \(\ell_j\) appear adjacent to each other and in the same order, say \(\ell_i \ell_j\), in two of the chains \((L; [\mu_k])\), say \((L; [\mu_{k1}])\) and \((L; [\mu_{k2}])\). But we have also shown that \(\mu_{k1}(\ell_i)\) and \(\mu_{k2}(\ell_i)\) must have different parity, which implies that \((L; \sigma^{H_{k1}}, [\mu_{k1}])\) and \((L; \sigma^{H_{k2}}, [\mu_{k2}])\) form the two repellent \(\ell_i - \ell_j\) couples.

Also, all link pairs of the form \(\{\ell_i, \ell_{2m-1}\}\) occur in reverse orders in, say \((L; [\mu_{k1}])\) and \((L; [\mu_{k2}])\) with \(\sigma^{H_{k1}}(\ell_{2m-1}) = \sigma^{H_{k2}}(\ell_{2m-1}) = -1\). Thus, again, both repellent \(\ell_i - \ell_{2m-1}\) couples are formed by \((L; \sigma^{H_{k1}}, [\mu_{k1}])\) and \((L; \sigma^{H_{k2}}, [\mu_{k2}])\).

Therefore, \(\{(L; \sigma^{Hk}, [\mu_k]) \mid k = 1, 2, \ldots, 2m-1\}\) is a \((2,3)\)-\(C^4(2m)\) and since it attains the lower bound on \(c_{23}(n)\), it must be minimal. [End of Claim 1]

Before construction of our last \(\mathcal{C}^4\) of Case 3, we define two additional orientations, \(\sigma^{Hk}_1\) and \(\sigma^{Hk}_2\). (Recall, \(\mu_k\) is as defined above.)

\[
\sigma^{Hk}_1(\ell_i) = \begin{cases} 
-1 & \text{if } \mu_k(\ell_i) = m+2, m+4, m+6, \ldots < 2m+1 \\
+1 & \text{else}
\end{cases}
\]

\[
\sigma^{Hk}_2(\ell_i) = \sigma^{Hk}_1(\ell_i) \text{ except that } \sigma^{Hk}_2(\ell_{2m-1}) = +1
\]

Claim 2: \(\bigcup_{k=1}^{2m-1} \{(L; \sigma^{Hk}_1, [\mu_k]), (L; \sigma^{Hk}_2, [\mu_k])\}\) is a \((2,1)\)-\(C^4(2m)\), thus establishing \(c_{21}(n) = 2(n-1)\) for \(n\) even.

Proof of Claim 2: We begin by showing that for \(i \neq 2m-1\), all \(\ell_i - \ell_{2m-1}\) couples are formed. If \(m\) is even, then \(\ell_{2m-1} \ell_i\) and \(\ell_i \ell_{2m-1}\) appear in \(\{(L; \sigma^{Hk}_1, [\mu_k]) \mid k = 1, 2, \ldots, 2m-1\}\); \(\ell_{2m} \ell_i\) and \(\ell_i \ell_{2m-1}\) appear in
\[(L; \sigma^k_2, [\mu_k]) \mid k = 1, 2, \ldots, 2m - 1\). If \(m\) is odd, then \(\ell_{2m-1} \ell_i\) and \(\ell_i \ell_{2m-1}\) appear in \[((L; \sigma^k_1, [\mu_k]) \mid k = 1, 2, \ldots, 2m - 1)\), \(\ell_{2m-1} \ell_i\) and \(\ell_i \ell_{2m-1}\) appear in \[((L; \sigma^k_2, [\mu_k]) \mid k = 1, 2, \ldots, 2m - 1)\).

If neither \(i\) nor \(j\) is \(2m - 1\), then, for \(k = 1, 2, \ldots, 2m - 1\), both occurrences of the pair \(\ell_i, \ell_j\) in \[((L; [\mu_k]) \mid k = 1, 2, \ldots, 2m - 1)\) appear in the linear subarcchains \(\mu^{-1}_k(2) \mu^{-1}_k(3) \ldots \mu^{-1}_k(2m)\). In fact, we have \(A = \{ \mu^{-1}_k(2) \mu^{-1}_k(3) \ldots \mu^{-1}_k(m + 1) \mid k = 1, 2, \ldots, 2m - 1\}\) in one and \(B = \{ \mu^{-1}_k(m + 1) \mu^{-1}_k(m + 2) \ldots \mu^{-1}_k(2m) \mid k = 1, 2, \ldots, 2m - 1\}\) in the other. But the orientations \(\sigma^k_1\) and \(\sigma^k_2\) restricted to \(A\) are opposite constant orientations forming all attractive couples, and restricted to \(B\) are opposite alternating orientations forming all repellent couples. [End of Claim 2.]

So in Case 2 we have shown that for \(n\) even, \(c_{22}(n) = n - 1\) provided \(n \neq 4, 6\); \(c_{22}(4) = 4, c_{22}(6) = 6, c_{23}(n) = n - 1\) and \(c_{21}(n) = 2(n - 1)\).

**Case 3** (odd linear): Let \(n = 2m - 1\). Observe that a \((1,t_2)\)-C\(^4\)(2m - 1) can be obtained from a \((2,t_2)\)-C\(^4\)(2m) by removing \(\ell_{2m} (= \ell_0)\) from all arcchains in the collection. Hence \(c_{1t_2}(2m - 1) \leq c_{2t_2}(2m)\) for \(t_2 = 1, 2\), or 3. But these upper bounds on \(c_{1t_2}(2m - 1)\) given by Case 2 are precisely the lower bounds given at the beginning of the proof. Thus \(c_{11}(2m - 1) = 2(2m - 1), c_{13}(2m - 1) = 2m - 1\), and for \(2m - 1 \neq 3\) or 5, \(c_{12}(2m - 1) = 2m - 1\). To show that \(c_{12}(3) = 4\) and \(c_{12}(5) = 6\) we need only mention that in showing that \(K_4^*\) and \(K_6^*\) have no decomposition into directed hamiltonian cycles, Bermond and Faber also showed that \(K_3^*\) and \(K_5^*\) have no decomposition into directed hamiltonian paths. Thus Case 3 is merely a corollary of Case 2. Therefore, for \(n\) odd, \(c_{11}(n) = 2n, c_{12}(n) = n\) except that \(c_{12}(3) = 4\) and \(c_{12}(5) = 6\), and \(c_{13}(n) = n\).
Case 4 (odd cyclic): Again, let $n = 2m - 1$.

We begin by considering repellent $C^d$'s. It has been shown that $c_{23}(2m - 1) \geq 2m - 2$ with equality if and only if all arcchains in the collection are repellent. But it is impossible for a cyclic arcchain on an odd number of links to be repellent, so $c_{23}(2m - 1) \geq 2m - 1$. We establish equality here by observing that $c_{23}(2m - 1) \leq c_{13}(2m - 1)$ (since a $(1,3)$-$C^d(2m - 1)$ can be used to form a $(2,3)$-$C^d(2m - 1)$ by wrapping each linear arcchain around to form a cyclic arcchain) and by recalling, from Case 3, that $c_{13}(2m - 1) = 2m - 1$. Thus $c_{23}(2m - 1) = 2m - 1$.

Now decompose $K_{2m-1}$ into $m - 1$ hamiltonian cycles $C_1, C_2, \ldots, C_{m-1}$ via Theorem 2.2. Each of these cycles $C_k$ corresponds to two reverse cyclic orderings of $L$, which we denote $[\mu_k]$ and $[\tilde{\mu}_k]$. For each $k$, $k = 1, 2, \ldots, m - 1$, let $\mu_k$ be the representative of $[\mu_k]$ for which $\mu_k(0) = 0$.

Clearly, \[ m \] \[ \begin{cases} (L ; \sigma^+, [\mu_k]) , (L ; \sigma^+, [\tilde{\mu}_k]) \end{cases} \] is a minimal $(2,2)$-$C^d(2m - 1)$, thus establishing $c_{22}(2m - 1) = 2(m - 1)$.

Before constructing a minimal $(2,1)$-$C^d(2m - 1)$, we define two orientations $\sigma'$ and $\sigma^{hk}_3$ as follows:

\[ \sigma'(i) = \begin{cases} +1 & \text{if } i = 0 \\ -1 & \text{if } i \neq 0 \end{cases} \quad \text{and} \quad \sigma^{hk}_3(i) = -\sigma^{hk}(i) \quad \text{except that } \sigma^{hk}(0) = -1. \]

Claim: \[ m \] \[ \begin{cases} (L ; \sigma^+, [\mu_k]) , (L ; \sigma', [\mu_k]), (L ; \sigma^{hk}, [\mu_k]), (L ; \sigma^{hk}_3, [\mu_k]) \end{cases} \] is a $(2,1)$-$C^d(2m - 1)$, thus establishing $c_{21}(2m - 1) = 4(m - 1)$.

Proof of Claim: Fix $k$ and suppose $\ell_i = \mu_k^{-1}(1)$ and $\ell_j = \mu_k^{-1}(2m - 2)$.

Then
\[ (L ; \sigma^+, [\mu_k]) \text{ forms } \bar{\ell}_0 \bar{\ell}_i \text{ and } \ell_j \ell_0, \]
\[ (L ; \sigma', [\mu_k]) \text{ forms } \bar{\ell}_0 \bar{\ell}_i \text{ and } \ell_j \ell_0, \]
\[ (L ; \sigma^{hk}, [\mu_k]) \text{ forms } \bar{\ell}_0 \bar{\ell}_i \text{ and } \ell_j \ell_0, \text{ and} \]

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Now for pairs \( \ell_i \) and \( \ell_j \), where \( 1 \leq \mu_k(\ell_i) < \mu_k(\ell_j) \leq 2m - 2 \), we observe that \( \sigma^+ \) and \( \sigma^- \), restricted to \( L - \ell_0 \), are opposite constant orientations forming all attractive \( \ell_i - \ell_j \) couples; and that \( \sigma^{H_k} \) and \( \sigma^{H_k} \) restricted to \( L - \ell_0 \) are opposite alternating orientations forming all repellent \( \ell_i - \ell_j \) couples. [End of Claim]

Therefore, in Case 4 we have established that for \( n \) odd, \( c_{23}(n) = n \), \( c_{22}(n) = n - 1 \), and \( c_{21}(n) = 2(n - 1) \). □

3.4 Collections of Bipartite Arcchains

As before, we begin with a link set \( L = \{ \ell_0, \ell_1, \ldots, \ell_{n-1} \} \). Suppose Pole (L) is partitioned as \( A \cup B \) where \( |A| = n_1 \geq n_2 = |B| \geq 1 \). We are interested in forming couples (by arcchains) which consist of one element from each of \( A \) and \( B \). Such a couple will be called a bicouple. (Note, couples other than bicouples may be formed.)

A linear \( \mathcal{C}^4(n_1, n_2) \) is a collection of linear arcchains which, together, form all possible bicouples. A cyclic \( \mathcal{C}^4(n_1, n_2) \) is similarly defined. Observe that this notation does not specify \( A \) and \( B \), but only their cardinalities. By convention, the partition \( L = A \cup B \) is common to all arcchains in a given \( \mathcal{C}^4 \), but different \( \mathcal{C}^4(n_1, n_2) \)'s may relate to different partitions. Thus, in constructing these bipartite \( \mathcal{C}^4 \)'s, it is first necessary to fix a partition of \( L \) to best serve the purpose of forming all bicouples.

It will be convenient to develop notation for links to indicate the partition. Let \( a_0, a_1, \ldots, a_{k_1} \) be the links \( \ell \) for which \( \ell^A, \ell^N \in A \). Similarly, \( b_0, b_1, \ldots, b_{k_2} \) will denote the links \( \ell \) for which \( \ell^S, \ell^N \in B \). Also let \( x_0, x_1, \ldots, x_{k_3} \) denote the links \( \ell \) for which \( \ell^N \in A \) and \( \ell^S \in B \). If \( k_i = 0 \) for \( i = 1, 2, \) or 3 then the subscript 0 may be omitted.
We show that it is unnecessary to have links $\ell$ such that $\ell \in A$ and $\ell \in B$ when our focus is on the cardinality of the $C^4$ constructed. Let $\mathcal{C}$ be a $C^4(n_1, n_2)$ relative to $L = A \cup B$, with $\ell_0, \ell_1, \ldots, \ell_k$ having north poles in $B$ and south poles in $A$ (and $\ell_{k+1}, \ldots, \ell_n$ of the types defined above.) Define a second partition $A' \cup B'$ of $L$ so that $A'$ and $B'$ differ from $A$ and $B$, respectively, only in that $\ell_i \in A'$ and $\ell_i \in B'$ for $i = 0, 1, \ldots, k$. Now form a $C^4(n_1, n_2)$ relative to $L = A' \cup B'$, call it $\mathcal{C}'$, from $\mathcal{C}$ by replacing all occurrences of $\ell_i$ by $\bar{\ell}_i$ and vice versa for $i = 0, 1, \ldots, k$. Observe that $\mathcal{C}'$ forms distinct bicouples with precisely the same pairs of links as does $\mathcal{C}$ and that $|\mathcal{C}| = |\mathcal{C}'|.$

We make just a few comments before proving the main results of this section. If there are no type-x links, then the total number of bicouples to be formed by a $C^4(n_1, n_2)$ is $n_1n_2$. At the other extreme, if there are no type-b links, then the total number of bicouples is reduced to $(n_1 - 1)n_2$, where the missing $n_2$ bicouples correspond to the $n_2$ links of type-x. (Recall that the elements of a single link do not constitute a couple.) Also, an arcchain in a $C^4(n_1, n_2)$ can form at most $n_2$ bicouples ($n_2 - 1$ if $n_1 = n_2$ and the arcchains are linear), one for each element of $B$. Thus, it is reasonable to hope that a minimal $C^4(n_1, n_2)$ has cardinality $n_1 - 1$ or $n_1$. Theorems 3.5 and 3.6 below show this to be true in all but four particular instances.

**Theorem 3.5:** For all positive integers $n$, a minimal linear $C^4(n, n)$ has cardinality $c_{12}(n)$ and a minimal cyclic $C^4(n, n)$ has cardinality $c_{22}(n)$.

**Proof:** Begin by observing that an attractive $C^4(n)$ is a $C^4(n, n)$ where all links are type-x (i.e., $A$ is the set of north poles and $B$ is the set of south poles). Furthermore, except for $(1,2)-C^4(3)$, $(1,2)-C^4(5)$, $(2,2)-C^4(4)$, and $(2,2)-C^4(6)$, minimal $C^4(n)$'s are obviously minimal $C^4(n, n)$'s, since each pair of adjacent links in each

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
arcchain forms a bicouple and none of the collections form duplicate bicouples. It therefore remains to verify that the four exceptional cases cannot be improved by considering alternative partitions of $L$. 

**Case 1:** Each linear arcchain on 3 links forms at most 2 bicouples. The partition $L = \{a, b, x\}$ requires 8 bicouples to be formed, so at least 4 arcchains are needed. Thus, it is not possible to construct a linear $C_{4}(3,3)$ of fewer than 4 arcchains.

**Case 2:** Each linear arcchain on 5 links forms at most 4 bicouples. The partition $L = \{a, b, x_0, x_1, x_2\}$ requires 22 bicouples to be formed, so at least 6 arcchains are needed. The partition $L = \{a_0, a_1, b_0, b_1, x\}$ requires 24 bicouples to be formed, so at least 6 arcchains are needed. Thus, it is not possible to construct a linear $C_{4}(5,5)$ of fewer than 6 arcchains.

**Case 3:** Each cyclic arcchain on 4 links forms at most 4 bicouples. The partition $L = \{a, b, x_0, x_1\}$ requires 14 bicouples to be formed, so at least 4 arcchains are needed. The partition $L = \{a_0, a_1, b_0, b_1\}$ requires 16 bicouples to be formed, so at least 4 arcchains are needed. Thus, it is not possible to construct a cyclic $C_{4}(4,4)$ of fewer than 4 arcchains.

**Case 4:** Each cyclic arcchain on 6 links forms at most 6 bicouples. The partition $L = \{a, b, x_0, x_1, x_2, x_3\}$ requires 32 bicouples, the partition $L = \{a_0, a_1, b_0, b_1, x_0, x_1\}$ requires 34 bicouples, and the partition $L = \{a_0, a_1, a_2, b_0, b_1, b_2\}$ requires 36 bicouples; so at least 6 arcchains are needed. Thus, it is not possible to construct a cyclic $C_{4}(6,6)$ of fewer than 6 arcchains. □

The following theorem considers construction of $C_{4}(n_1, n_2)$'s for which $n_1 > n_2$. When $n_1 \neq n_2$, the distinction between cyclic and linear arcchains is
unnecessary for the following reason. The number of bicouples formed by an arcchain in a cyclic $C^4(n_1, n_2)$ is at most $n_2$, which is strictly less than the $n = \frac{1}{2}(n_1 + n_2)$ pairs of adjacent links. Hence, there is at least one pair of adjacent links which does not form a bicouple. By "cutting" the arcchain between these links, we obtain a linear arcchain which forms the same bicouples as did the original. You will notice that the statement of Theorem 3.6 below refers to $C^4(n_1, n_2)$'s without qualifying them as linear or cyclic. The proof uses the notation for cyclic arcchains, but it is now clear that these could be made linear by the cutting process just described.

In addition, observe that Theorem 3.6 differs from Theorems 3.4 and 3.5 in that no claim of minimality is being made. This reflects the difficulty inherent in this more general setting, but it also reflects a need for easily describable collections which will find future application. It should not, however, be thought that this result is far from optimal. Indeed, in light of the calculations made prior to Theorem 3.5, the $C^4$'s constructed below are either minimal or contain only one arcchain more than would a minimal $C^4$.

**Theorem 3.6:** For all positive integers $n_1$ and $n_2$ with $n_1 > n_2$ and $n_1 + n_2$ even, there exists a $C^4(n_1, n_2)$ consisting of $n_1$ arcchains. (Note, $n_1 + n_2 = 2n$.)

**Proof:** In each of the three cases that follow, the collection will be given as an $(n_1 \times (n + 1))$-array, where the entries are oriented links and each row represents a cyclic arcchain.

**Case 1** ($n_1$ and $n_2$ even): Let $n_1 = 2m_1$, and $n_2 = 2m_2$. We will use the partition $L = \{a_0, a_1, \ldots, a_{m_1-1}, b_0, b_1, \ldots, b_{m_2-1}\}$. For $i = 1, 2, \ldots, m_1$, construct row $2i - 1$ as follows. Begin with the type-a links cyclicly ordered by subscript, i.e., $a_0, a_1, \ldots, a_{m_1-1}, a_0$. Now for each $j$, $j = 0, 1, 2, \ldots, m_2 - 1$, insert $b_j$ between
\(a_{j+i-1}\) and \(a_{j+i}\) (where the addition in the subscripts is modulo \(m_1\).) Note that all links in the odd numbered rows just described are positively oriented. Now, for \(i = 1, 2, \ldots, m_1\), let row \(2i\) be identical to row \(2i - 1\) except that all type-b links are negatively oriented.

It remains to show that all bicouples have been formed. In particular, for any \(k\), \(k = 0, 1, \ldots, m_1 - 1\), and for any \(j\), \(j = 0, 1, \ldots, m_2 - 1\), we need to show that all \(a_k - b_j\) bicouples are formed. Let \(i = k - j \pmod{m_1}\) where \(0 \leq i \leq m_1 - 1\). Then \(a_k\) and \(b_j\) appear as adjacent links as follows:

\[
\begin{align*}
\text{row } 2i - 1 : & \quad \ldots b_j a_k \ldots \\
\text{row } 2i : & \quad \ldots b_j a_k \ldots \\
\text{row } 2i + 1 : & \quad \ldots a_k b_j \ldots \\
\text{row } 2i + 2 : & \quad \ldots a_k b_j \ldots .
\end{align*}
\]

(It should be noted that this construction does not prohibit \(n_1\) from equaling \(n_2\). This cannot be said of the remaining cases.)

**Case 2** \((n_1 \equiv 3 \pmod{4} \text{ and } n_1 > n_2)\): Let \(n_1 = 2m_1 + 1\) (where \(m_1\) is odd) and \(n_2 = 2m_2 + 1\). We will use the partition \(L = \{a_0, a_1, \ldots, a_{m_1-1}, b_0, b_1, \ldots, b_{m_2-1}, x\}\). Begin with the \(m_1\) rows as constructed for \(n_1 = 2m_1\) and \(n_2 = 2m_2\) in Case 1 above.

For each \(i\), \(i = 1, 2, \ldots, m_1\) insert \(x\) in rows \(2i - 1\) and \(2i\) as follows. If there is no type-b link between \(a_{m_1-i}\) and \(a_{m_1-i+1}\), place \(x\) there in row \(2i - 1\) and place \(\bar{x}\) there in row \(2i\). If there is a type-b link between \(a_{m_1-i}\) and \(a_{m_1-i+1}\) it will be \(b_{m_1-2i+1}\). In this case, insert \(x\) between \(a_{m_1-i}\) and \(b_{m_1-2i+1}\) in row \(2i - 1\) and alter row \(2i\) by first changing the orientation on \(b_{m_1-2i+1}\) to positive and then inserting \(\bar{x}\) between \(b_{m_1-2i+1}\) and \(a_{m_1-i+1}\). Finally, construct row \(2m_1 + 1\). Begin
with the type-a links in order \((a_0, a_1, \ldots, a_{m_1-1}, a_0)\). Insert \(x\) between \(a_{m_1-1}\) and \(a_0\) and for \(i = 1, 2, \ldots, m_1\) insert \(\beta_{m_1-2i+1}\) between \(a_{m_1-i}\) and \(a_{m_1-i+1}\).

It remains to show that all bicouples have been formed. Consider the same portion of the following four rows:

- **row 2i - 1**: \(\ldots am_{1-i} x bm_{1-2i+1} am_{1-i+1} \ldots\)
- **row 2i**: \(\ldots am_{1-i} bm_{1-2i+1} x am_{1-i+1} \ldots\)
- **row 2i + 2**: \(\ldots x am_{1-i} bm_{1-2i+1} am_{1-i+1} \ldots\)
- **row 2m_1 + 1**: \(\ldots am_{1-i} \tilde{\beta}_{m_1-2i+1} am_{1-i+1} \ldots\)

We first verify the formation of all \(a - b\) bicouples. Since we started with arcchains containing all \(a - b\) bicouples, the only such bicouples which could possibly be missing from this new collection are those which have been blocked by the insertion of the \(x\) link. The \(x\) in row 2i - 1 blocks the pair \(am_{1-i} bm_{1-2i+1}\). Row 2i contains this pair, but neither \(am_{1-i} \tilde{\beta}_{m_1-2i+1}\) nor \(\tilde{\beta}_{m_1-2i+1} am_{1-i+1}\) which it had previously contained. These two are contained in row 2m_1 + 1. So, all \(a - b\) bicouples have been formed.

Note that rows 2i - 1 and 2i contain both \(x - bm_{1-2i+1}\) bicouples. Since the subscripts on the \(b\)'s are elements of \(Z_{m_2}\) and \(m_2\) is odd, thus \(-2\) generates \(Z_{m_2}\), i.e., each element of \(Z_{m_2}\) can be expressed as \(m_2 - 2(i - 1)\) for some \(i\), \(i = 1, 2, \ldots, m_1\). Therefore, all \(x - b\) bicouples have been formed.

Similarly, rows 2i - 1 and 2i + 2 contain both \(x - am_{1-i}\) bicouples. Since the subscripts of the \(a\)'s are elements of \(Z_{m_1}\) and \(-1\) generates \(Z_{m_1}\); therefore, all \(x - a\) bicouples have been formed.

**Case 3** \((n_1 \equiv 1 \pmod{4} \text{ and } n_1 > n_2)\): Before presenting the general construction, we consider a small exceptional subcase. We have the following \(C^4(5,3)\) of cardinality 4: \(a x_0 x_1 x_2, \tilde{x}_0 \tilde{x}_1 \tilde{x}_2 a, \tilde{x}_1 a x_2 x_0, \tilde{x}_2 \tilde{x}_0 a x_1\).

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
It will become clear in the general development below, that \( n_1 \) is assumed to be at least 3, so we need to consider the case \( n_1 = 1 \) separately. In this setting, we partition the links as \( a_0, a_1, \ldots, a_{m_1-1}, x \) where \( n_1 = 2m_1 + 1 \). This situation is very similar to Case 2, with the simplifying fact that there are no b-type links. Consequently, we accomplish our goal by constructing row \( 2i - 1 \) as \( a_0 a_1 \ldots a_{i-1} x a_i \ldots a_{m_1-1} \) and row \( 2i \) as \( a_0 a_1 \ldots a_{i-1} \tilde{x} a_i \ldots a_{m_1-1} \).

Now let \( n_1 = 2m_1 + 3 \) (where \( m_1 \) is odd) and \( n_2 = 2m_2 + 3 \) for which it is not true that \( n_1 = 5 \) and \( n_2 = 3 \). We will use the partition \( L = \{ a_0, a_1, \ldots, a_{m_1-1}, b_0, b_1, \ldots, b_{m_2-1}, x_0, x_1, x_2 \} \). Begin with the \( 2m_1 + 1 \) rows as constructed for \( n_1 = 2m_1 + 1 \) and \( n_2 = 2m_2 + 1 \) in Case 2 above, recalling that we must now use \( x_0 \) instead of \( x \).

For each \( i = 1, 2, \ldots, m_1 \) insert \( x_1 \) and \( x_2 \) in rows \( 2i - 1 \) and \( 2i \) as follows. If there is no type-b link between \( a_{m_1-i-1} \) and \( a_{m_1-i} \), place \( x_1 \tilde{x}_2 \) there in row \( 2i - 1 \) and place \( x_2 \tilde{x}_1 \) there in row \( 2i \). The only exception to this is in rows \( m_1 \) and \( m_1 + 1 \), in which \( x_1 x_2 \) and \( \tilde{x}_1 \tilde{x}_2 \) are placed between \( a_{(m_1-3)/2} \) and \( a_{(m_1-1)/2} \) respectively. If there is a type-b link between \( a_{m_1-i-1} \) and \( a_{m_1-i} \) it will be \( b_{m_1-2i} \). In this case, this portion of row \( 2i - 1 \) will be constructed as \( \ldots a_{m_1-i-1} x_1 b_{m_1-2i} \tilde{x}_2 a_{m_1-i} \ldots \) and the same portion of row \( 2i \) will appear as \( \ldots a_{m_1-i-1} x_2 b_{m_1-2i} \tilde{x}_1 a_{m_1-i} \ldots \). We insert \( x_1 x_2 \) into row \( 2m_1 + 1 \) between \( x_0 \) and \( a_0 \). Next, construct rows \( 2m_1 + 2 \) and \( 2m_1 + 3 \) as follows. Begin with the type-a links in order. For row \( 2m_1 + 2 \), insert \( b_{m_1-2i} \) between \( a_{m_1-i-1} \) and \( a_{m_1-i} \) for \( i = 1, 2, \ldots, m_1 \) and insert \( x_2 x_0 x_1 \) between \( a_{(m_1-3)/2} \) and \( a_{(m_1-1)/2} \). Row \( 2m_1 + 3 \) duplicates row \( 2m_1 + 2 \) except that the type-b and type-x links are all negatively oriented.

As was the situation in Case 2, the insertion of \( x_1 \) and \( x_2 \) blocked some \( a-b \) bicouples. Rows \( 2m_1 + 2 \) and \( 2m_1 + 3 \) contain these. Also, as in Case 2, rows...
1 through $m_1$ contain all $x - b$ bicouples and all $x - a$ bicouples except $a_{(m_1-1)/2}^N$, $x_2^S$ and $x_1^S a_{(m_1-1)/2}^N$, but these appear in rows $2m_1 + 2$ and $2m_1 + 3$ respectively. Finally, all $x - x$ bicouples are formed in rows $m_1$, $m_1 + 1$, $2m_1 + 2$, and $2m_1 + 3$.

We conclude with examples of the three constructions used in this theorem. Note that the construction of (ii) is based on (i) just as the $C^4$ constructed in case 2 was constructed from the $C^4$ of case 1. Similarly, (iii) follows from (ii) just as case 3 followed from case 2.

**Example 3.4:**

(i) A $C^4(6,4)$ of cardinality 6.

\[
\begin{array}{cccccc}
  a_0 & b_0 & a_1 & b_1 & a_2 & a_0 \\
  a_0 & \tilde{b}_0 & a_1 & \tilde{b}_1 & a_2 & a_0 \\
  a_0 & a_1 & b_0 & a_2 & b_1 & a_0 \\
  a_0 & a_1 & \tilde{b}_0 & a_2 & \tilde{b}_1 & a_0 \\
  a_0 & b_1 & a_1 & a_2 & b_0 & a_0 \\
  a_0 & \tilde{b}_1 & a_1 & a_2 & \tilde{b}_0 & a_0 \\
\end{array}
\]

(ii) A $C^4(7,5)$ of cardinality 7.

\[
\begin{array}{cccccc}
  a_0 & b_0 & a_1 & b_1 & a_2 & x & a_0 \\
  a_0 & \tilde{b}_0 & a_1 & \tilde{b}_1 & a_2 & \tilde{x} & a_0 \\
  a_0 & a_1 & x & b_0 & a_2 & b_1 & a_0 \\
  a_0 & a_1 & b_0 & \tilde{x} & a_2 & \tilde{b}_1 & a_0 \\
  a_0 & x & b_1 & a_1 & a_2 & b_0 & a_0 \\
  a_0 & b_1 & \tilde{x} & a_1 & a_2 & \tilde{b}_0 & a_0 \\
  a_0 & \tilde{b}_1 & a_1 & \tilde{b}_0 & a_2 & x & a_0 \\
\end{array}
\]
(iii) A $C^d(9,7)$ of cardinality 9.

$$
\begin{array}{cccccccc}
  a_0 & b_0 & a_1 & x_1 & b_1 & \tilde{x}_2 & a_2 & x_0 & a_0 \\
  a_0 & \tilde{b}_0 & a_1 & x_2 & b_1 & \tilde{x}_1 & a_2 & \tilde{x}_0 & a_0 \\
  a_0 & x_1 & x_2 & a_1 & x_0 & b_0 & a_2 & b_1 & a_0 \\
  a_0 & \tilde{x}_1 & \tilde{x}_2 & a_1 & b_0 & \tilde{x}_0 & a_2 & \tilde{b}_1 & a_0 \\
  a_0 & x_0 & b_1 & a_1 & a_2 & x_1 & b_0 & \tilde{x}_2 & a_0 \\
  a_0 & b_1 & \tilde{x}_0 & a_1 & a_2 & x_2 & b_0 & \tilde{x}_1 & a_0 \\
  a_0 & \tilde{b}_1 & a_1 & \tilde{b}_0 & a_2 & x_0 & x_1 & x_2 & a_0 \\
  a_0 & x_2 & x_0 & x_1 & a_1 & b_1 & a_2 & b_0 & a_0 \\
  a_0 & \tilde{x}_2 & \tilde{x}_0 & \tilde{x}_1 & a_1 & \tilde{b}_1 & a_2 & \tilde{b}_0 & a_0 \\
\end{array}
$$

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
CHAPTER IV

CROWNING

By the convention established in Chapter II, a graph imbedded in the surface \( \mathcal{S}(G) \) must be of even order. In this chapter we introduce a surgical construction, called a crown, which can be used to add a single vertex to an imbedded graph in an "efficient" manner. Hence, imbeddings of even order graphs can be augmented to produce imbeddings of odd order graphs.

4.1 The Basic Crown

If \( G \) is a connected graph and \( x \in V(G) \), then define a second graph \( G_{2x} \) as follows: \( V(G_{2x}) = V(G) \cup \{x' \} \) where \( x' \notin V(G) \) and \( E(G_{2x}) = E(G) \cup \{x'v \mid v \in V(G) \text{ and } xv \in E(G)\} \). The procedure of forming \( G_{2x} \) from \( G \) will be called cloning the vertex \( x \) (or simply cloning \( x \)) and \( x' \) will be called a clone of \( x \). For instance, we see that \( K_{a+1,b} \) can be formed from \( K_{a,b} \) by cloning one of the vertices in the first partite set.

For the remainder of this section, we will assume that the graph \( G \) is imbedded in the orientable surface \( S_k \) where \( k \geq \gamma(G) \). Our purpose here is to describe a surgical procedure, which we call crowning \( x \), whereby \( G_{2x} \) is imbedded in \( S_{k+h} \), where \( h = \lceil \frac{1}{4}(\deg x - 2) \rceil \). The name is intended to conjure up the image of \( x \) on the surface and a torus suspended "above" \( x \), supported by tubes running from the torus to the immediate vicinity of \( x \), with \( x' \) residing on the torus. (See Figure 4.1.) The torus and its tubes form a "crown" for the vertex \( x \). All edges incident with \( x' \) in \( G_{2x} \) will run down these supporting tubes. These tubes will also
carry some edges incident with $x$, which must be detoured due to the attachment of the crown-tubes to the original surface.

Suppose the neighbors of $x$, arranged in a clockwise fashion around $x$ (prior to crowning), are $y_1, y_2, \ldots, y_{\deg x}$. Figures 4.2 and 4.3 show the crowning procedure for a vertex $x$ of degree $4h + 2$. (For $\deg x = 4h + 1$, $4h$, or $4h - 1$, these diagrams are simply modified by deleting all edges labeled $a_i$ and $b_i$ for $i > \deg x$.) The edges $xy_i$ and $x'y_i$ are denoted by $a_i$ and $b_i$ and are represented by solid lines and dotted lines respectively for $1 \leq i \leq 4h + 2$. Figure 4.2 shows the surface near $x$, in which the numbered circles represent the attachment locations (in the original surface) for the $h$ crown-tubes. Figure 4.3 shows the crown-body, which is a torus represented in the usual way (with the pairs of opposite sides of the bounding rectangle identified).
Figure 4.2

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Figure 4.3
Notice that as you approach $x$ along either $a_{4i-1}$ or $a_{4i}$ ($1 \leq i \leq h-1$) the attachment of the crown-tubes to $S_k$ requires a detour which runs up to the crown along tube $i$, across the crown to tube $i+1$, then down along this tube to $S_k$, and finally to $x$. For $i = h$ the edges $a_{4h-1}$, $a_{4h}$, $a_{4h+1}$, and $a_{4h+2}$ run up tube $h$, across the crown, and down tube 1.

We now determine the regions of $G_{2x}$ in $S_{k+h}$. Let $R_i$, $1 \leq i \leq \deg x$, denote the regions of $G$ in $S_k$ incident with $x$; $R_i$ has edges $a_i$ and $a_{i+1}$ adjacent in its boundary. (See Figure 4.4 a.) (The boundary of $R_{\deg x}$ has adjacent edges $a_{4h+2}$ and $a_1$.) Also, let $R'_i$ denote a region of $G_{2x}$ in $S_{k+h}$ whose boundary is the same as that of $R_i$ except that $a_i$ and $a_{i+1}$ have been replaced by $b_i$ and $b_{i+1}$ respectively, i.e., if $\partial R_i : \ldots y_i x y_{i+1} \ldots$ then $\partial R'_i : \ldots y_i x' y_{i+1} \ldots$. (Addition in the subscripts is modulo the degree of $x$.)

![Figure 4.4](image)

Since regions whose boundaries do not contain $x$ are unaffected by the crowning of $x$, it suffices to describe the regions incident with $x$ and $x'$ in $S_{k+h}$. Table 4 lists these regions in clockwise order around $x$ and $x'$ according to $\deg x$.

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Table 4

Regions of a Crowned Surface

<table>
<thead>
<tr>
<th>deg x</th>
<th>regions (clockwise) around x</th>
<th>regions (clockwise) around x'</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>description of Tr (if any): Tr has the same boundary as R_{4h+2-r} except for the change indicated.</td>
<td></td>
</tr>
<tr>
<td>4h + 2</td>
<td>R_1 Q R_{4h} + 1 Q R_{4h-1} Q R_5 Q R_3 Q R_9 ... R_4i - 3 Q R_{4i-5} Q R_{4i+1} ... R_{4h-7} Q R_{4h-9} Q R_{4h-3} Q R_{4h-5} Q R_1</td>
<td>R'_2 Q R'_6 Q R'_10 ... R'_4i - 2 ... R'_4h - 6 Q R'_4h - 2 Q R'_4h + 2 Q R'_4h - 4 ... R'_4i ... R'_12 Q R'_8 Q R'_4 Q R'_4h Q R'_2</td>
</tr>
<tr>
<td>4h + 1</td>
<td>R_1 Q T_1 R_{4h-1} Q R_5 ... (complete as above)</td>
<td>R'_2 Q R'_6 Q ... R'_4h-2 T_1 Q R'_4h-4 ... (complete as above)</td>
</tr>
<tr>
<td></td>
<td>... y_1 x y_{4h+1} ... is replaced by ... y_1 x' y_{4h-1} x y_{4h+1} ...</td>
<td>...</td>
</tr>
<tr>
<td>4h</td>
<td>R_1 T_2 R_{4h-1} Q R_5 ... (complete as above)</td>
<td>R'_2 Q R'_6 Q ... R'_4h-2 T_2 Q R'_4h-4 ... R'_8 Q R'_4 Q T_2 R'_2</td>
</tr>
<tr>
<td></td>
<td>... y_1 x y_{4h} ... is replaced by ... y_1 x' y_{4h-1} x y_2 x' y_{4h} ...</td>
<td>...</td>
</tr>
<tr>
<td>4h - 1</td>
<td>R_1 T_3 T_3 R_5 ... (complete as above)</td>
<td>R'_2 Q R'_6 Q ... R'_4h-2 T_3 Q R'_4h-4 ... R'_8 Q R'_4 Q T_3 R'_2</td>
</tr>
<tr>
<td></td>
<td>... y_1 x y_{4h-1} ... is replaced by ... y_1 x' y_{4h-1} x y_2 x' y_5 x y_{4h-1} ...</td>
<td>...</td>
</tr>
</tbody>
</table>
The symbol $Q$ in these lists refers to a quadrilateral region having $x$ and $x'$ as opposite vertices in the boundary. Figure 4.4 b shows the quadrilateral referred to in $\ldots R_i Q R_j \ldots$ and $\ldots R_{j-1}' Q R_{i+1}' \ldots$. The symbol $T_r$ denotes a region which is none of the aforementioned region types. (See Table 4 for descriptions of these.)

### 4.2 Additional Adjacencies

Crowning a vertex $x$ forms several quadrilaterals of the form $xy_ix'y_j$ where $y_i, y_j \in N(x)$ and such that $|i - j| \geq 2$. (Specifically, for $\deg x = 4h + r$, with $r \in \{-1, 0, 1, 2\}$, the number of quadrilaterals formed is $2h + r - 1$.) Once $x$ has been crowned, the edges $y_i y_j$ can be imbedded as diagonals of these quadrilateral regions. In this way, a crown can be used for the second purpose (in addition to cloning a vertex) of adding edges between certain pairs of neighbors of the cloned vertex.

**Example 4.1:** Begin with $K_2 + C_6$ imbedded in $S_0$ as shown in Figure 4.5 a, where $V(K_2) = \{x_1, x_2\}$ and $V(C_6) = \{y_1, y_2, y_3, y_4, y_5, y_6\}$. By crowning $x_1$ we obtain an imbedding of $\overline{K_2} + C_6$ in $S_1$. In this new imbedding, the regions incident with $x_1$ appear as in Figure 4.5 b. We can now add the edges $y_1 y_4$, $y_2 y_5$, and $y_3 y_6$ as diagonals of the regions labeled $Q$ to obtain a toroidal imbedding of $K_{3(3)}$ (with partite sets $\{x_1, x'_1, x_2\}$, $\{y_1, y_3, y_5\}$, and $\{y_2, y_4, y_6\}$).

### 4.3 Variations of the Crown

The basic crown constructed in Section 4.1 accomplishes the task of adding a clone of a vertex (to an imbedded graph) in an efficient manner. In Section 4.2 we saw that there is a side benefit of the crown which allows additional edges to be imbedded in the new surface. In this section we will discuss three ways in which the basic
construction could be altered. The first two of these point to arbitrary choices that were made in the description of the basic construction. These bear little on the primary purpose, but they do affect which additional edges may be imbedded. The third variation generalizes the construction to apply both to the original goal of cloning vertices and beyond.

The first variation concerns the crown-tubes. These are attached to the original surface around the vertex being cloned, at one end, and around the crown-body, at the other end. Both of these sets of attachments represent cyclic orderings of the crown-tubes and, in the basic constructions, these orderings are the same (or the reverse of one another, depending on viewpoint). It is easily observed that permuting the attachment order, say at the crown-body end, will affect the new quadrilaterals \( Q \) formed, without affecting the genus of the new surface formed. For instance, we could define a reverse crown as identical to a basic crown, except that the order of attachments of crown-tubes to the crown-body is the reverse of that shown in Figure 4.3, say keeping the tube labeled "h" fixed.
The second variation concerns the modification of the crown diagrams—Figures 4.2 and 4.3 (for crowning a vertex $x$ such that $\deg x \equiv 2 \pmod{4}$). As was noted in Section 4.1, when $\deg x \not\equiv 2 \pmod{4}$ we describe the crown by deleting the edges $a_j$ and $b_j$ for $j > \deg x$. This is clearly an unnecessary restriction. If $\deg x = 4h + 2 - r$, where $h$ is a positive integer and $r \in \{1, 2, 3\}$, we can delete any $r$ pairs of edges $\{a_j, b_j\}$ from Figures 4.2 and 4.3. Then, simply relabel the remaining edges with subscripts $1, 2, \ldots, \deg x$ (cyclicly arranged around $x$). This will greatly impact the quadrilateral regions formed as well as alter the description of the enlarged region $T_r$ described in Table 4. In fact, if the deleted edges are not consecutively labeled and $r > 1$, there will be more than one enlarged region.

Before presenting the third variation, the generalized crown, we require two new definitions. For the first, suppose $G$ is an imbedded graph with vertices $x$ and $y$. We say $y$ is accessible from $x$ (equivalently, $x$ is accessible from $y$) provided there is a region $R$ of $G$ for which $x, y \in \partial R$. In particular, we say $y$ is accessible from $x$ via $R$. For the second definition, a region $R_i$ (of the imbedded $G$, before the crown is attached) incident with $x$ is said to be split by the crown provided the edges of $G_{2x}$ incident with $x'$ are partially imbedded in the interior of $R_i$. The term refers to the fact that the set of edges incident with $x$ are "split" (or partitioned) by the edges incident with the clone. (See Figure 4.2.) Note that in the basic construction with $\deg x \equiv 2 \pmod{4}$, all even numbered regions are split because $N(x)$ is accessible from $x$ via these regions. (We could just as easily have chosen to split all odd numbered regions.) Also note, in all cases (of the degree of $x$ modulo 4) each crown-tube $T_i$, for $i < h$, splits two regions, whereas tube $T_h$ splits three regions when $\deg x \equiv 2$ or $1 \pmod{4}$ and splits two regions in the other two cases. Thus, a crown increases the genus of the surface by $\left\lfloor \frac{1}{2} \right\rfloor$ (the number of regions split by the crown).
Now suppose $G$ is an imbedded graph and $x$ is a vertex of $G$. Suppose further that we desire to add a vertex $z$ to $G$ and join it to vertices of $G$ which are all accessible from $x$. Choose a minimal set of regions incident with $x$, of even cardinality, such that the intended neighborhood of $z$ is accessible from $x$ via this set of regions. Denote these regions $S_1, S_2, \ldots, S_k$ (labeled clockwise around $x$). Now attach a generalized crown, containing $z$, which splits only the regions $S_i$, for $i = 1, 2, \ldots, k$, as shown (for odd $k$) in Figure 4.6 below.

![Figure 4.6](image_url)

In this generalized crown, $z$ plays a role analogous to $x'$ in the basic crown in that it resides in the crown-body and all edges incident with it run down the crown-tubes between pairs of detoured edges (incident with $x$) and out through the interiors of the regions being split. Observe that the basic crown is a particular form of the generalized crown.
4.4 Multiple Clones, Double Crowns, and Crown Clusters

We begin this section by defining the graph $G_{nx}$. Suppose $G$ is a graph and $x$ is a vertex of $G$. Then, for an integer $n \geq 2$, the graph $G_{nx}$ is constructed from $G$ by adding $n - 1$ clones of $x$ to $G$. That is, $V(G_{nx}) = V(G) \cup \{x^{(i)} | i = 1, 2, \ldots, n-1\}$ and $E(G_{nx}) = E(G) \cup \{x^{(i)}y | i = 1, 2, \ldots, n-1, \text{ and } xy \in E(G)\}$. (Note that we encountered $G_{2x}$ in Section 4.1 where $x' = x^{(1)}$.) If we further suppose that $G$ is imbedded in a surface, the problem of augmenting this imbedding to obtain an efficient imbedding of $G_{nx}$ naturally arises. Before attempting to solve this problem, we must carefully consider the concept of the "efficiency" of an imbedding.

Just as $G_{2x}$ served as a model for $G_{nx}$, we use the basic crown as a model of efficiency for imbedding $G_{nx}$. Once again we point to the fact that the basic crown forms quadrilaterals (with at most one exception) and in the case $\deg x = 2 \pmod{4}$, all new regions formed by the crown are quadrilaterals. Thus, an imbedding of $G_{nx}$ will be deemed efficient if it is as nearly quadrilateral as possible.

Suppose, for the sake of simplicity of exposition, that $G$ is quadrilaterally imbedded in $S_k$. If $G$ is a $(p, q)$-graph and the number of regions is $r$, we recall the Euler-Poincaré formula $p - q + r = 2 - 2k$. Also, the fact that each region is quadrilateral implies that $4r = 2q$. Replacing $r$ with $\frac{1}{2}q$ and solving the Euler-Poincaré formula for $k$ yields $k = \frac{1}{4}(q - 2p + 4)$. Thus, augmenting the imbedding of $G$ in $S_k$ to an efficient imbedding of $G_{nx}$ in $S_{k'}$ requires that

$$k' = \left\lceil \frac{1}{4}(q(G_{nx}) - 2p(G_{nx}) + 4) \right\rceil$$
$$= \left\lceil \frac{1}{4}((q + (n - 1)\deg x) - 2(p + (n - 1)) + 4) \right\rceil$$
$$= k + \left\lceil \frac{1}{4}(n - 1)\deg x - 2 \right\rceil.$$

This brings us to the statement of the main result of the section.
Theorem 4.1: If $G$ has an imbedding in $S_k$, $x$ is a vertex of $G$, and $n \geq 2$ is an integer, then $G_{nx}$ has an imbedding in $S_{k'}$ where $k' = k + \left\lceil \frac{1}{4} (n - 1) (\deg x - 2) \right\rceil$.

Notice that we do not require $G$ to be quadrilaterally imbedded in $S_k$. We postpone proving Theorem 4.1 until the end of the section.

All constructions described thus far in Chapter 4 add a single vertex to an imbedded graph and could consequently be called single crowns. It stands to reason then that a double crown would add two vertices to an imbedded graph $G$. The double crown constructed below adds two clones of the same vertex $x$, i.e., it augments an imbedding of $G$ to an imbedding of $G_{3x}$.

Suppose $G$ is imbedded in a surface $S_k$. Denote the neighbors of $x$ by $y_1, y_2, \ldots, y_{\deg x}$ in a clockwise fashion around $x$. Also, denote the regions incident with $x$ by $R_1, R_2, \ldots, R_{\deg x}$ so that $\partial R_i : y_i, x, y_{i+1}, \ldots$ (traced counterclockwise) as was done in Section 4.1; see Figure 4.7.

The (basic) double crown consists of a sphere, which contains $x^{(1)}$ and $x^{(2)}$ at its north and south poles (respectively) and $\left\lfloor \frac{\deg x}{2} \right\rfloor$ tubes, which join the sphere to the surface $S_k$. These tubes are attached to the sphere along its equator (as in a graphical surface) and are attached to $S_k$ in the interiors of the odd numbered regions $R_1, R_3, \ldots$. Number these tubes $T_1, T_3, \ldots$ so that tube $T_i$ joins the sphere to region $R_i$. Each tube $T_i$ carries four edges of $G_{3x}$, namely $x^{(1)}y_i, x^{(1)}y_{i+1}, x^{(2)}y_i, \text{ and } x^{(2)}y_{i+1}$. (As always, addition in the subscripts is modulo the degree of $x$.) The single exception to this is that $T_{\deg x}$ carries only the edges $x^{(1)}y_{\deg x}$ and $x^{(2)}y_{\deg x}$ when $\deg x$ is odd.

The double crown described thus far has a great deal in common with the basic crown. Both consist of a surface (or crown-body) attached to $S_k$ in the vicinity of the vertex $x$ being cloned. The clone(s) reside in the crown-body, with incident edges

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
running down the crown-tubes and out through the interiors of the regions $R_i$ to the neighbors of $x_i$. Moreover, in both cases, the primary purpose of the crown is served, independent of the order of attachment of the crown-tubes to the crown-body. However, this attachment order does affect which quadrilaterals are formed in the process.

There are also several differences between the two constructions. Some superficial ones are: the number of clones being added, the number of crown-tubes, and the "shape" of the crown-body (torus or sphere). The double crown seems to be simpler than the basic (single) crown in two important respects: none of the edges of $G$ are required to detour over any portion of the crown and there are only two variations dependent on $\deg x \equiv 0 \text{ or } 1 \pmod 2$ as opposed to the multiple variations of the basic crown based on $\deg x \pmod 4$ discussed in the previous section.

Another point of comparison between the two types of crowns is the way in which the new edges $E(G_{nx}) - E(G)$ are imbedded in the new surface. In both cases each tube (with the exception of at most one) carries four new edges. For the basic crown, the imbedding of these edges, all of which are incident with the one clone, is fixed by the order of tube attachment and the subsequent reimbedding of those edges of $G$ which are detoured over the crown. For the double crown, the edges running along tube $T_i$, except for $i = \deg x$ odd, are $x^{(1)}y_i, x^{(2)}y_i, x^{(2)}y_{i+1}, x^{(1)}y_{i+1}$ in this order and independent of the order of tube attachment. However, whether the indicated order is clockwise or counterclockwise around $T_i$ (say, with respect to the crown-body) is an arbitrary choice which affects the regions formed. For $i = \deg x$ odd, there is no distinction between clockwise and counterclockwise order for the two edges $x^{(1)}y_{\deg x}$ and $x^{(2)}y_{\deg x}$ around $T_{\deg x}$; however, these edges can be imbedded to have either order in the rotation scheme at $y_{\deg x}$.
Hence, to completely specify a double crown it is necessary to (i) order the crown tubes around the equator of the crown, (ii) specify the order of the new edges in each tube, and (iii) if \( \text{deg } x \) is odd, specify the order of the edges \( x^{(1)}y_{\text{deg } x} \) and \( x^{(2)}y_{\text{deg } x} \) in the rotation scheme at \( y_{\text{deg } x} \). The following is offered as a default choice. Order the tubes \( T_1, T_3, \ldots \) from east to west around the crown-body. For \( i = 1, 3, \ldots \), order the edges \( x^{(1)}y_i, x^{(2)}y_i, x^{(2)}y_{i+1}, x^{(1)}y_{i+1}, x^{(1)}y_i \) in counterclockwise fashion around \( T_i \) with respect to (i.e., viewed from inside) the crown-body. Finally, if \( \text{deg } x \) is odd, imbed the edges \( x^{(1)}y_{\text{deg } x} \) and \( x^{(2)}y_{\text{deg } x} \) so that \( x^{(2)}y_{\text{deg } x} \) appears immediately prior to \( x^{(1)}y_{\text{deg } x} \) in the clockwise ordering of edges around \( y_{\text{deg } x} \). Note that the imbedding here described gives identical rotation schemes at \( x^{(1)} \) and \( x \). (See Figures 4.7 and 4.8.)

Figure 4.7

To summarize, if \( G \) is imbedded in \( S_k \), then double-crowning a vertex \( x \) of \( G \) results in \( G_{3x} \) imbedded in a surface of genus \( k + \left\lceil \frac{\text{deg } x}{2} \right\rceil - 1 \). (Recall, if two
surfaces $S_{k_1}$ and $S_{k_2}$ are joined by $t$ tubes, the resulting surface has genus $k_1 + k_2 + t - 1$. In this case, $k_1 = k$, $k_2 = 0$, and $t = \lceil \frac{\deg x}{2} \rceil$. Note that $\lceil \frac{\deg x}{2} \rceil - 1 = \lceil \frac{1}{4} (3 - 1) (\deg x - 2) \rceil$.

**Proof of Theorem 4.1:** We begin with the graph $G$ imbedded in $S_k$. Our goal is to augment this imbedding so that $(n - 1)$ clones of $x$ can be added to $G$ while increasing the genus by $\lceil \frac{1}{4} (n - 1) (\deg x - 2) \rceil$. Clearly, it suffices to show this for $n - 1 = 1, 2, 3, 4$ and for $\deg x = 4h + 2, 4h + 1, 4h, 4h - 1$. The entries of Table 5 are the various values of $\lceil \frac{1}{4} (n - 1) (\deg x - 2) \rceil$. (The reader may wish to refer to this table while reading this proof.)

**Table 5**

Increase in Genus Due to Crowning

<table>
<thead>
<tr>
<th>$n - 1$</th>
<th>$\deg x$</th>
<th>$4h + 2$</th>
<th>$4h + 1$</th>
<th>$4h$</th>
<th>$4h - 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>h</td>
<td>h</td>
<td>h</td>
<td>h</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$2h$</td>
<td>$2h$</td>
<td>$2h - 1$</td>
<td>$2h - 1$</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$3h$</td>
<td>$3h$</td>
<td>$3h - 1$</td>
<td>$3h - 2$</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$4h$</td>
<td>$4h - 1$</td>
<td>$4h - 2$</td>
<td>$4h - 3$</td>
<td></td>
</tr>
</tbody>
</table>

If the number of clones $n - 1$ is 1, then a basic (single) crown adds $\lceil \frac{1}{4} (\deg x - 2) \rceil = \lceil \frac{1}{4} (1) (\deg x - 2) \rceil$ to the genus, as was shown in Section 4.1. Also, we have just shown in this section that a double crown adds two clones of $x$ while adding $\lceil \frac{1}{2} \deg x \rceil - 1 = \lceil \frac{1}{4} (2) (\deg x - 2) \rceil$ to the genus.

Now suppose $\deg x$ is even. Then $\lceil \frac{1}{4} (3) (\deg x - 2) \rceil = (\frac{1}{2} \deg x - 1) + \lceil \frac{1}{4} (\deg x - 2) \rceil$, so we can accomplish the addition of three clones of $x$ by first adding a double crown and then a basic (single) crown. Also,
\[ \left\lfloor \frac{1}{4} (4) (\deg x - 2) \right\rfloor = 2 \left( \frac{1}{2} \deg x - 1 \right) \], so we can add four clones of \( x \) by first adding a double crown and then a second double crown.

Unfortunately, things do not proceed as smoothly when \( \deg x \) is odd. If \( \deg x \equiv 1 \pmod{4} \), the third clone can be added after a double crown with a basic (single) crown. However, if \( \deg x \equiv -1 \pmod{4} \) we have \( \left\lfloor \frac{1}{4} (3) (\deg x - 2) \right\rfloor = \left( \left\lfloor \frac{1}{2} \deg x \right\rfloor - 1 \right) + \left\lfloor \frac{1}{4} (\deg x - 2) \right\rfloor - 1 \). That is, a double crown followed by a basic (single) crown will add one more to the genus than is necessary. Also, if \( \deg x \) is any odd integer, \( \left\lfloor \frac{1}{4} (4) (\deg x - 2) \right\rfloor = 2 \left( \left\lfloor \frac{1}{2} \deg x \right\rfloor - 1 \right) - 1 \), which means that if four clones of \( x \) are added to \( G \) via two double crowns, the genus will be increased by one more than is necessary. In these problem cases, the first double crown is added in such a way as to require the second crown (single or double) to have one fewer tube. Figure 4.8 shows the enlarged region (shaded) formed by the first double crown, where all default orders are in effect and \( d = \deg x \).

\[ a \]

\[ b \]

Figure 4.8

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
We can trace the boundary of this region in a counterclockwise direction, to obtain the following sequence of vertices: \( \ldots \, y_d \, x^{(2)} \, y_{d-1} \, x^{(1)} \, y_d \, x \, y_1 \, \ldots \). Call this region \( R_0 \). Observe that after the first double crown is added, \( N(x) \) is accessible from \( x \) via regions \( R_0, R_2, R_4, \ldots, R_{d-3} \). Now by adding a generalized (single) crown to the surface containing \( G_{2x} \), in the case \( \deg x \equiv -1 \pmod{4} \), we add a third clone \( x^{(3)} \) of \( x \) and simultaneously add \( \frac{1}{4} (\deg x - 3) = \left[ \frac{1}{4} (\deg x - 2) \right] - 1 \) to the genus.

Furthermore, we claim that by adding a variant of the double crown, after the first double crown, we can add clones \( x^{(3)} \) and \( x^{(4)} \) while increasing the genus by \( \left\lceil \frac{\deg x}{2} \right\rceil - 2 \). Construct this second double crown exactly like the basic double crown described in this section with the following exceptions. Place \( x^{(3)} \) at the north pole and \( x^{(4)} \) at the south pole. Join the crown-body to the surface by \( \frac{\deg x - 1}{2} \) tubes attached in the interiors of the regions \( R_0, R_2, R_4, \ldots, R_{d-3} \). For \( i = 2, 4, \ldots, d - 3 \), \( T_i \) will carry the edges \( x^{(3)}y_i \), \( x^{(3)}y_{i+1} \), \( x^{(4)}y_i \), and \( x^{(4)}y_{i+1} \). The tube \( T_0 \) will carry the six edges \( x^{(3)}y_{d-1} \), \( x^{(3)}y_d \), \( x^{(3)}y_1 \), \( x^{(4)}y_{d-1} \), \( x^{(4)}y_d \), and \( x^{(4)}y_1 \). Figure 4.9 shows how these edges are imbedded.

So, in general, in order to add \( n - 1 \) clones of \( x \) to \( G \) in \( S_k \) efficiently, we add a cluster of \( \left\lfloor \frac{n-1}{2} \right\rfloor \) crowns. If \( \deg x \) is even, all of these are double crowns. If \( \deg x \) is odd, the last is a single crown with all others being double. □

This marks the end of the formal development of the crown constructions and, in fact, of all surgical constructions which will be presented in this dissertation. We are now ready to apply these techniques to imbed specific graphs and classes of graphs.
Figure 4.9
CHAPTER V

THE GENUS OF THE GRAPHICAL JOIN

We begin this chapter with two new proofs of the genus formula for the simplest join—the complete bipartite graph. We use the second of these as a starting point to show that several graphical joins imbed in the same surface and hence have the same genus.

5.1 The Complete Bipartite Graphs

Recall that the join $G + H$ of two graphs $G$ and $H$ is the disjoint union $G \cup H$ together with all edges of the form $gh$ where $g \in V(G)$ and $h \in V(H)$. These edges which join $G$ and $H$ are called the join edges of $G + H$ and induce a spanning subgraph isomorphic to $K_{a,b}$, where $a = p(G)$ and $b = p(H)$. In fact, if $G$ and $H$ are both empty, $G + H = K_{a,b}$. We begin here.

As noted in Chapter 1, Ringel [30] found the genus of the complete bipartite graphs in 1965. His method of proof was to produce a suitable rotation scheme for each of several cases. In 1974, Gross [17] published a second proof using voltage graphs and, in 1978, Bouchet [11] published a third proof using an ad hoc method which involved surgery to a small degree. Below, we present two surgical proofs which bear little resemblance to prior proofs.

It should be observed that the "hard part" of the proof is establishing the upper bound by describing an imbedding. The lower bound follows from the Euler-Poincaré formula.

**Lemma 5.1:** For integers $a$ and $b$, each greater than 1, $\gamma(K_{a,b}) \geq \left\lceil \frac{1}{4} (a - 2)(b - 2) \right\rceil$.
Proof: As observed in Chapter 1, the genus of a triangle-free, \((p, q)\)-graph is bounded below by \(\left\lceil \frac{a - b}{2} + 1 \right\rceil\). Since the genus of a graph is an integer, this bound is actually \(\left\lceil \frac{a - b}{2} + 1 \right\rceil\). In the case of \(K_{a, b}\), \(p = a + b\) and \(q = ab\) so

\[
\gamma(K_{a, b}) \geq \left\lceil \frac{ab}{4} - \left\lceil \frac{a + b}{2} \right\rceil + 1 \right\rceil = \left\lceil \frac{1}{4} (ab - 2 (a + b) + 4) \right\rceil = \left\lceil \frac{1}{4} (a - 2)(b - 2) \right\rceil.
\]

**Theorem 5.1:** \(\gamma(K_{a, b}) = \left\lceil \frac{1}{4} (a - 2)(b - 2) \right\rceil\) for \(a \geq 2\) and \(b \geq 2\).

**Proof 1:** Begin with a quadrilateral imbedding of \(K_{2, b}\) in \(S_0\); see Figure 5.1.

![Figure 5.1](image)

Let \(u_1\) be a vertex in the first partite set of \(K_{2, b}\). We obtain \(K_{a, b}\) by adding \(a - 2\) clones of \(u_1\) to \(K_{2, b}\). By Theorem 4.1, \(K_{a, b}\) has an imbedding in the surface of genus \(0 + \left\lceil \frac{1}{4} (a - 2) (\deg u_1 - 2) \right\rceil = \left\lceil \frac{1}{4} (a - 2)(b - 2) \right\rceil\).

Thus, \(\gamma(K_{a, b}) \geq \left\lceil \frac{1}{4} (a - 2)(b - 2) \right\rceil\), which together with Lemma 5.1 gives \(\gamma(K_{a, b}) = \left\lceil \frac{1}{4} (a - 2)(b - 2) \right\rceil\). □

**Proof 2:** As in Proof 1, in light of Lemma 5.1, it remains to imbed \(K_{a, b}\) in the surface of genus \(\left\lceil \frac{1}{4} (a - 2)(b - 2) \right\rceil\).
Case 1 (a and b even): Say $a = 2m$ and $b = 2n$. Clearly, $K_{a,b}$ imbeds in $\mathcal{S}(K_{m,n})$ with all edges of first-order. Note that $\gamma(\mathcal{S}(K_{m,n})) = mn - (m + n) + 1 = (m - 1)(n - 1) = \frac{1}{4} (a - 2)(b - 2)$.

Case 2 (a + b odd): Say $a = 2m + 1$ and $b = 2n$. Begin with the imbedding of $K_{a-1,b}$ in $\mathcal{S}(K_{m,n})$ as in Case 1. Now add a crown to clone an element of the first partite set. The result is $K_{a,b}$ imbedded in the surface of genus

$$(m - 1)(n - 1) + \left[ \frac{1}{4} (b - 2) \right] = \left[ \frac{1}{4} [(2m - 2)(2n - 2) + (b - 2)] \right] = \left[ \frac{1}{4} (a - 2)(b - 2) \right].$$

Case 3 (a and b odd): Say $a = 2m + 1$ and $b = 2n + 1$. Begin with the imbedding of $K_{a-1,b-1}$ in $\mathcal{S}(K_{m,n})$ as in Case 1. We will add two crowns to clone an element of each partite set. However, as we shall see, the order of crowning is crucial.

Before crowning, we have a surface of genus $(m - 1)(n - 1)$. If we clone an element of the first partite set first, this adds $\left[ \frac{1}{4} (b - 1) - 2 \right]$ to the genus. If we now clone an element of the second partite set, we add another $\left[ \frac{1}{4} (a - 2) \right]$. (The numerator is one more than $(a - 1) - 2$ because the first clone added one to the degree of the second vertex being cloned.) The final surface has genus

$$(m - 1)(n - 1) + \left[ \frac{b-3}{4} \right] + \left[ \frac{a-2}{4} \right] = (m - 1)(n - 1) + \left[ \frac{2n-2}{4} \right] + \left[ \frac{2m-1}{4} \right].$$

Routine calculation shows that $\left[ \frac{1}{4} (a - 2)(b - 2) \right] = (m - 1)(n - 1) + \left[ \frac{2m + 2n - 3}{4} \right]$, so the question becomes, does $\left[ \frac{2n-2}{4} \right] + \left[ \frac{2m-1}{4} \right]$ equal $\left[ \frac{2m + 2n - 3}{4} \right]$? If $n$ is odd, the answer is yes because $\frac{2n-2}{4}$ is an integer. If both $m$ and $n$ are even, the answer is again yes since then both expressions equal $\frac{m+n}{2}$. In the remaining case, $n$ even and $m$ odd, the answer is no. Thus, we accomplish our task in all cases, taking care when $a \equiv b (\text{mod } 4)$ to first crown a vertex whose degree is congruent to 0 modulo 4.
For the sake of completeness, it should be noted that if either \( a = 1 \) or \( b = 1 \),
\( \gamma(K_{a,b}) = 0 \). Specifically, \( K_{1,1} = K_2 \) so \( \gamma(K_{1,1}) = \gamma(K_2) = 0 \). Also, \( K_{n,1} \subseteq K_{n,2} \)
for any positive integer \( n \geq 2 \) and \( \gamma(K_{n,2}) = 0 \) by Theorem 5.1, so \( \gamma(K_{n,1}) = 0 \) by
Theorem 1.1. This together with Theorem 5.1 gives the following formula
\[
\gamma(K_{a,b}) = \begin{cases} 
\left\lfloor \frac{1}{4} (a-2)(b-2) \right\rfloor & \text{if } a \geq 2 \text{ and } b \geq 2 \\
0 & \text{else.}
\end{cases}
\]
Let \( g(a, b) \) denote the function given by the right-hand side of this equation.

Then, by an observation that introduced this section, we have the following.

**Theorem 5.2:** If \( G \) is a graph of order \( a \) and \( H \) is a graph of order \( b \), then
\( \gamma(G + H) \geq g(a, b) \).

**Proof:** Since \( K_{a,b} \subseteq G + H \) we have \( \gamma(K_{a,b}) \leq \gamma(G + H) \) by Theorem 1.1. \( \Box \)

The sole purpose of this chapter is to demonstrate that for several pairs of
graphs \( G \) and \( H \), with respective orders \( a \) and \( b \), \( \gamma(G + H) = g(a, b) \). That this is
true when \( G \) and \( H \) are both empty has just been established. Sections 5.2 and 5.3
deal with the case of \( G \) empty and \( H \) nonempty and Section 5.4 gives a few results
for joins \( G + H \) where neither \( G \) nor \( H \) is empty.

### 5.2 The Genus of \( \overline{K}_a + H \)

For the purpose of establishing the results of this section we assume that \( H \) is
of even order; say \( p(H) = 2n \). All of our results here will be of the form
"\( \gamma(\overline{K}_a + H) = g(a, 2n) \) provided \( a \) is 'large enough' relative to \( n \)". Theorem 5.3
below is used to establish a general lower bound for \( a \) which depends only on the
order of \( H \). The remaining results improve (lower) this bound in certain cases,
depending on the structure of \( H \).
Lemma 5.3: For positive integers $r$ and $n$, if $H$ is a graph of order $2n$ and $\gamma(K_{2r} + H) = g(2r, 2n)$ then $\gamma(K_a + H) = g(a, 2n)$ for any integer $a \geq 2r$.

**Proof:** Begin with $K_{2r} + H$ imbedded in a surface of genus $g(2r, 2n)$. Choose $x \in V(K_{2r})$ and add $a - 2r$ clones of $x$ by crowns. By Theorem 4.1, the genus of the resulting surface is $g(2r, 2n) + \left[ \frac{1}{4} (a - 2r)(2n - 2) \right] = \left[ \frac{1}{4} (2r - 2)(2n - 2) + \frac{1}{4} (a - 2r)(2n - 2) \right] = \left[ \frac{1}{4} (a - 2)(2n - 2) \right] = g(a, 2n)$. Thus $g(a, 2n) \geq \gamma(K_a + H)$. By Theorem 5.2, $g(a, 2n) \leq \gamma(K_a + H)$. So we have $\gamma(K_a + H) = g(a, 2n)$.

Theorem 5.3: For positive integers $a$ and $n$, $\gamma(K_a + K_{2n}) = g(a, 2n)$, provided $a \geq 4(n - 1)$.

**Proof:** As the result is obvious for $n = 1$, assume $n \geq 2$. In light of Lemma 5.3 and Theorem 5.2, it suffices to show that $K_{4(n-1)} + K_{2n}$ imbeds in a surface of genus $g(4(n - 1), 2n)$, namely $\mathcal{S}(K_{2(n-1)},n)$. Denote the partite sets of the underlying graph $K_{2(n-1),n}$ by $U = \{u_1, u_2, \ldots, u_{2n-2}\}$ and $V = \{v_1, v_2, \ldots, v_n\}$. Imbed $V(K_{4(n-1)})$ as $\{u_i^S, u_i^N\}_{i=1}^{2n-2}$ and imbed $V(K_{2n})$ as $\{v_i^S, v_i^N\}_{i=1}^n$. Drawing all possible first-order edges results in a quadrilateral imbedding of $K_{4(n-1)} + K_{2n}$. We will imbed the edges of $K_{2n}$ as second-order edges which form diagonals of these quadrilaterals, as outlined in Section 2.4, using the results of Chapter 3.

For each $u \in U$, the attachment order of tubes from $S_{v_1}, S_{v_2}, \ldots, S_{v_n}$ around the equator of $S_u$, is essentially a cyclic ordering of $\{S_{v_i}\}_{i=1}^n$. By also specifying orientations for the tubes $\{T_{uv_i}\}_{i=1}^n$, we see that each $u \in U$ corresponds to a cyclic arcchain on the link set $\{(v_i^S, v_i^N)\}_{i=1}^n$. If we let the $2(n - 1)$ elements of $U$ correspond to the $2(n - 1)$ arcchains in a $(2,1)$-$C^d(n)$, we can imbed all edges of $K_{2n}$ as second-order edges in the resulting quadrilaterals. Thus, we have $K_{4(n-1)} + K_{2n}$ imbedded in $\mathcal{S}(K_{2(n-1)},n)$, which was our goal.
**Corollary 5.3:** For positive integers \( n \) and \( a \), if \( H \) is any graph of order \( 2n \) and \( a \geq 4(n - 1) \) then \( \gamma(\overline{K_a} + H) = g(a, 2n) \).

**Proof:** If \( K_{a,2n} \subseteq \overline{K_a} + H \subseteq \overline{K_a} + K_{2n} \) and \( \gamma(K_{a,2n}) = \gamma(\overline{K_a} + K_{2n}) = g(a, 2n) \), then, by Corollary 1.1, \( \gamma(\overline{K_a} + H) = g(a, 2n) \). \( \square \)

**Theorem 5.4:** For positive integers \( n \) and \( a \), \( \gamma(\overline{K_a} + K_{n,n}) = g(a, 2n) \), provided \( a \geq 2(n - t) \), where \( t = 0 \) if \( n = 4 \) or \( 6 \) and \( t = 1 \) otherwise.

**Proof:** The argument simply mimics the proof of Theorem 5.3. The variation in the bound on \( a \) results from requiring the elements of \( U \) to correspond to the arcchains of a \((2,2)\)-C\((n)\); hence, \( |U| \) need only be \( c_{22}(n) \). (See Theorem 3.4.) \( \square \)

**Corollary 5.4:** For positive integers \( n \) and \( a \), if \( H \) is any spanning subgraph of \( K_{n,n} \) and \( a \geq c_{22}(n) \) then \( \gamma(\overline{K_a} + H) = g(a, 2n) \).

**Proof:** Similar to that of Corollary 5.3. \( \square \)

It should be noted that \( \overline{K_a} + K_{n,n} = K_{a,n,n} \). In Section 5.3, we extend this result to several larger classes of complete tripartite graphs.

**Theorem 5.5:** For positive integers \( n \) and \( a \), \( \gamma(\overline{K_a} + 2K_n) = g(a, 2n) \), provided \( a \geq 2(n - t) \), where \( t = 1 \) if \( n \) is odd and \( t = 0 \) if \( n \) is even.

**Proof:** Again, the proof mimics that of Theorem 5.3. In this case, the elements of \( U \) correspond to the arcchains of a \((2,3)\)-C\((n)\); so \( |U| \) need only be \( c_{23}(n) \). (See Theorem 3.4.) \( \square \)

**Corollary 5.5:** For positive integers \( n \) and \( a \), if \( H_1 \) and \( H_2 \) are any graphs of order \( n \) and \( a \geq c_{23}(n) \) then \( \gamma(\overline{K_a} + (H_1 \cup H_2)) = g(a, 2n) \).
Theorem 5.5 and Corollary 5.5 represent only a sample of a large family of formulae of the form \( \gamma(\overline{K_n + H}) = g(a, p(H)) \) for \( H \) disconnected which can be generated using the techniques developed thus far. We present a few more in this family before proceeding to another type of result.

**Theorem 5.6:** If \( a, n_1 \geq n_2 \geq \ldots \geq n_k \) are positive integers and \( n = \sum_{j=1}^{k} n_j \), then

(i) \( \gamma(\overline{K_n + \bigcup_{j=1}^{k} K_{2n_j}}) = g(a, 2n) \), provided \( a \geq 4 n_1 \);

(ii) \( \gamma(\overline{K_n + \bigcup_{j=1}^{k} 2K_{n_j}}) = g(a, 2n) \), provided \( a \geq 2 n_1 \).

**Proof:** Part (i) As in the proof of Theorem 5.3, it suffices to show that \( \overline{K_{4n_1} + \bigcup_{j=1}^{k} K_{2n_j}} \) imbeds in \( \overline{\Delta(K_{2n_1}, n)} \). Let \( U = \{u_1, u_2, \ldots, u_{2n_1}\} \) and \( V = \{v_1, v_2, \ldots, v_n\} \) denote the partite sets of \( K_{2n_1, n} \). Also, for \( j = 1, 2, \ldots, k \) let \( m_j = \sum_{i=1}^{j} n_i \).

Imbed \( V(\overline{K_{4n_1}}) \) as the poles of \( \{S_{uj}\}_{j=1}^{2n_1} \). Imbed \( V(K_{2n_1}) \) as the poles of \( \{S_{uj}\}_{j=1}^{n_1} \). Imbed \( V(K_{2n_2}) \) as the poles of \( \{S_{v_j}\}_{j=n_1+1}^{n_1+n_2} \). In general, imbed \( V(K_{2n_j}) \) as the poles of \( \{S_{v_j}\}_{j=m_{j-1}+1}^{m_j} \). Clearly, all join edges imbed as all possible first-order edges. It remains to describe the attachment of all tubes around the equators of \( S_u \) for all \( u \in U \), and to orient all tubes in order to imbed all edges within \( K_{2n_j} \) for \( j = 1, 2, \ldots, k \).

For each \( u \in U \), divide the equator of \( S_u \) into \( k \) sections and, for each \( j = 1, 2, \ldots, k \), attach the tubes \( \{T_{uv_j}\}_{j=m_{j-1}+1}^{m_j} \) (not necessarily in this order) within the \( j \)th section. By specifying an order (within the \( j \)th section) and an orientation for these tubes, we see that each \( u \in U \) corresponds to a linear arcchain on the link set \( \{(v_i^S, v_i^N)\}_{j=m_{j-1}+1}^{m_j} \). Moreover, each \( u \in U \) corresponds to \( k \) arcchains; that is, an arcchain for \( j = 1, 2, \ldots, k \), and these arcchains are independent of one another. Thus, for each \( j = 1, 2, \ldots, k \), we order and orient the tubes from \( \{S_{v_j}\}_{j=m_{j-1}+1}^{m_j} \) to \( \{S_{uj}\}_{j=1}^{c_1(n_j)} \) so that each of these later spheres corresponds to an arcchain in a
(1,1)–$C^4(n_j)$. For $i = c_{11}(n_j) + 1, \ldots, 2n_1$ order and orient the tubes $\{T_{u_i}v_k\}_{k=m_{j-1}+1}^{m_j}$ arbitrarily. The resulting quadrilateral regions will now accommodate the required additional edges as diagonals.

**Part (ii)** Mimic the proof of part (i) by imbedding $\overline{K_{2n_1}} + \bigcup_{j=1}^{k} 2K_{n_j}$ in $\mathcal{S}(K_{n_1}, n)$ in roughly the same manner. The argument is altered as follows. Instead of imbedding $V(K_{2n_j})$ as the poles of $\{S_{v_j}\}_{j=m_{j-1}+1}^{m_j}$, we imbed one copy of $V(K_{n_j})$ as the north poles of these spheres and imbed the other copy of $V(K_{n_j})$ as the south poles. Section the equators of $S_u$ for $u \in U$ and attach the $k$ groups of tubes as above. Here, however, we require that for each $j = 1, 2, \ldots, k$, the spheres $\{S_{u_j}\}_{j=1}^{c_{11}(n_j)}$ correspond to the arcchains in a $(1,3)$–$C^4(n_j)$.

**Corollary 5.6 a:** If $a, m,$ and $n$ are positive integers with $mn$ even, then $\gamma(\overline{K_a + mK_n}) = \gamma(\overline{K_a + K_{m(n)}}) = g(a, mn)$, provided $a \geq 2n$.

**Proof:** If $n$ is even, this follows immediately from part (i) of Theorem 5.6. If $m$ is even, it follows from part (ii). □

**Corollary 5.6 b:** If $a, m,$ and $n$ are positive integers with $mn$ even and $H$ is any graph of order $n$, then $\gamma(\overline{K_a + mH}) = g(a, mn)$, provided $a \geq 2n$.

**Proof:** This is an immediate consequence of Corollary 5.6 a and Corollary 1.1. □

**Corollary 5.6 c:** If $H$ is a graph of order $n$ such that all components of $H$ have even order, the largest having order $m$, then $\gamma(\overline{K_a + H}) = g(a, n)$, provided $a \geq 2m$.

**Proof:** This follows immediately from Theorem 5.6 (i) and Corollary 1.1. □
The final results of this section differ from those presented thus far in that they
do not rely on Theorem 3.4.

**Lemma 5.7:** Suppose \( H \) is a graph of order \( 2n \). If \( E(H) \) can be partitioned as
\( E_0 \cup E_1 \cup \ldots \cup E_k \) so that

(i) \( E_j \) is an independent (i.e., pairwise nonadjacent) set of edges for \( j = 1, \ldots, k \),

(ii) \( |E_0| = n \) (i.e., \( \langle E_0 \rangle \) is a 1-factor of \( H \)), and

(iii) \( \langle E_0 \cup E_j \rangle \) is acyclic for \( j = 1, 2, \ldots, k \),

then \( \gamma(\overline{K_a} + H) = g(a, 2n) \), provided \( a \geq 2k \).

**Proof:** We begin by repeating a familiar pattern. In light of Lemma 5.3 and Theorem
5.2, it suffices to imbed \( \gamma(\overline{K_{2k}} + H) \) in \( \mathcal{S}(K_{k,n}) \). Let \( U = \{ u_1, u_2, \ldots, u_k \} \) and
\( V = \{ v_1, v_2, \ldots, v_n \} \) denote the partite sets of \( K_{k,n} \). Let \( V(H) = \{ x_1, x_2, \ldots, x_{2n} \} \) so that
\( E_0 = \{ x_1x_2, x_3x_4, \ldots, x_{2n-1}x_{2n} \} \).

Imbed \( V(\overline{K_{2k}}) \) as the poles of \( \{ S_{u_j} \}_{j=1}^k \). Imbed \( V(H) \) as the poles of
\( \{ S_{v_i} \}_{i=1}^n \) so that \( v_1^S = x_2i-1 \) and \( v_i^N = x_2i \). Imbed the join edges as all possible
first-order edges. Imbed the edges of \( E_0 \) as zeroth-order edges. It remains to imbed
the edges of \( E_i \) for \( i = 1, 2, \ldots, k \) as diagonals of quadrilateral regions formed (i.e., as
second-order edges) by appropriately ordering (around \( S_{u_j} \)) and orienting the
tubes \( \{ T_{u_jv_k} \}_{k=1}^n \).

Observe that, for \( j = 1, 2, \ldots, k \), \( \langle E_0 \cup E_j \rangle \) is a linear forest. Thus \( \langle E_0 \cup E_j \rangle \)
could be imbedded in \( \mathbb{R}^1 \) (the real line) by arbitrarily selecting a linear arrangement of
the components of \( \langle E_0 \cup E_j \rangle \) and, also, selecting one of two orientations for each
component. But this gives a linear ordering and an orientation of \( E_0 \). Since there is a
natural one-to-one correspondence between \( E_0 \) and the tubes \( \{ T_{u_jv_k} \}_{k=1}^n \) (i.e.,
\( x_{2i-1}x_{2i} \) corresponds to \( T_{u_jv_{i-1}} \)) we order these tubes around the equator of \( S_{u_j} \) and
orient them as the edges of \( E_0 \) are ordered and oriented in the imbedding of.
(E₀ ∪ Eᵢ) in R¹ (i.e., replace ..., x₂ⱼ₋₁x₂ⱼ ..., with Tᵢⱼⱼ and replace x₂ⱼx₂ⱼ₋₁ with Tᵢⱼⱼ). Thus, the edges of Ej can be imbedded as diagonals in the quadrilateral regions formed by the tubes joining {Sᵢⱼⱼ}|₁≤ᵢ≤ᵣ to Sₙ₁

**Theorem 5.7:** For positive integers a and n, if H is a graph of order 2n, then \( \gamma( \overline{K_a} + H) = g(a, 2n) \) provided \( a > 4 (\chi_1(H) - 1) \) (where \( \chi_1(H) \) is the edge-chromatic number of H).

**Proof:** Let \( F_0, F_1, ..., F_{\chi_1(H)-1} \) be color classes of a minimal edge coloring of H such that \( F_0 \) is as large as possible. If \( \langle F_0 \rangle \) is a 1-factor of H let \( E_0 = F_0 \) and \( H' = H \); otherwise, let \( E_0 = F_0 ∪ \{x₁x₂, x₃x₄, ..., x₂m₋₁x₂m\} \) and \( H' = H + x₁x₂ + x₃x₄ + ... + x₂m₋₁x₂m \) where \( V(H) - V(\langle F_0 \rangle) = \{x₁, x₂, ..., x₂m\} \).

For \( j = 1, 2, ..., \chi_1(H) - 1 \), if \( \langle E₀ ∪ Eᵢ \rangle \) is acyclic, let \( E₂j₋₁ = F_j \) and \( E₂j = \emptyset \); otherwise let \( E₂j \) consist of one element of \( F_j \) chosen from each cycle of \( \langle E₀ ∪ Eᵢ \rangle \) and let \( E₂₋₁ = F_j - E₂j \). Then \( E₀ ∪ E₁ ∪ ... ∪ E₂(\chi₁(H)-1) \) is a partition of E(H) which satisfies conditions (i)–(iii) of Lemma 5.7, so the result follows. □

It has been shown by Vizing [37] that for any nonempty graph G, \( \chi_1(G) \) is either \( \Delta(G) \) or \( \Delta(G) + 1 \). If \( \chi_1(G) = \Delta(G) \) then G is said to be a class 1 graph; if \( \chi_1(G) = \Delta(G) + 1 \), G is called a class 2 graph. This leads to the following.

**Corollary 5.7:** For positive integers a and n, if H is a nonempty graph of order 2n, then \( \gamma( \overline{K_a} + H) = g(a, 2n) \) provided \( a ≥ 4 (\Delta(H) - t) \), where \( t = 1 \) if H is of class 1 and \( t = 0 \) if H is of class 2.

It should be noted here that the bound on a given by Corollary 5.7 is not necessarily better than that given by Corollary 5.3. For example, Corollary 5.7 gives \( \gamma( \overline{K_a} + K₁₀₀) = g(a, 100) \) provided \( a ≥ 392 \) (since \( K₁₀₀ \) is of class 1, as it is
1-factorable) whereas Corollary 5.3 gives the same equation provided $a \geq 196$.

However, Corollary 5.7 may be quite a bit better than Corollary 5.3. Consider a graph $H$ of order 100 which is a cubic, planar block and hence of class 1. (See Section 10.3 of [12].) Corollary 5.3 gives $\gamma(\overline{K_{a}} + H) = g(a, 100)$ provided $a \geq 196$, whereas Corollary 5.7 gives the same equation provided $a \geq 8$.

Before moving on to joins of two nonempty graphs, which will be the topic of Section 5.4, we devote Section 5.3 to a study of complete tripartite graphs.

### 5.3 The Complete Tripartite Graphs

In this section, we adopt the convention that $a$, $b$, and $c$ are positive integers with $a \geq b \geq c$. Observe that $K_{a,b,c} = K_a + K_{b,c} \cong K_{a,b+c}$, so we have $\gamma(K_{a,b,c}) \geq g(a, b+c)$. In 1969, White (in Chapter 8 of his thesis [38]) conjectured that $\gamma(K_{a,b,c}) = g(a, b+c)$ for all choices of $a$, $b$, and $c$. He proceeded to prove this conjecture true for $K_{mn,n,n}$ and for $K_{a,b,c}$ when $b+c \leq 6$. His method used rotation schemes. The following year, Ringel and Youngs [32] published their rotation scheme proof for $K_{3(n)}$ (which actually predates, but is subsumed by, White's work). In 1976, Stahl and White [35] published voltage graph proofs for $K_{3(n)}$, $K_{2n,2n,n}$ and, when $n$ is even, for $K_{n,n,n-2}$. In his 1978 survey, Stahl [33] announced an unpublished voltage graph proof for $K_{2n,2n,n-m}$ provided $m+1$ generates $Z_{2n}$. All of these results support White's conjecture.

Our purpose here is to add substantially to the list of complete tripartite graphs for which the White conjecture is true. Before presenting our main results, we note that Theorem 4.1 can be used to extend some of the known classes listed above.

**Theorem 5.8:** For $a \geq b$, $\gamma(K_{a,b,b}) = g(a, 2b)$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Proof: Begin with $K_{b,b,b}$ imbedded in a surface of genus $g(b, 2b)$. Choose a vertex $x$ in the first partite set and add $a - b$ clones of $x$ by crowning. The resultant surface has genus $g(b, 2b) + \left[\frac{1}{4}(a - b)(2b - 2)\right] + \left[\frac{1}{4}(b - 2)(2b - 2)\right] + \frac{1}{4}(a - b)(2b - 2) = \left[\frac{1}{4}(a - 2)(2b - 2)\right] = g(a, 2b)$. (It should be noted that this argument "works" because $\frac{1}{4}(b - 2)(2b - 2)$ is an integer.) □

Observe that Theorem 5.8 subsumes Theorem 5.4.

The proofs of the following two theorems are very similar to that of Theorem 5.8 (following immediately from the results of Stahl and White) and will be omitted.

Theorem 5.9: For $a \geq b$ and $b$ even, $\gamma(K_{a,b,b-2}) = g(a, 2b - 2)$.

Theorem 5.10: For $a \geq b$ and $b$ even, $\gamma(K_{a,b,b \frac{3}{2}}) = g(a, \frac{3}{2}b)$.

We now present the main results of this section.

Theorem 5.11: For positive integers $a$, $b$, and $c$ with $a \geq b \geq c$ and $b + c$ even, $\gamma(K_{a,b,c}) = g(a, b + c)$ provided $a \geq 2b$.

Proof: In light of Lemma 5.3 (with $H = K_{b,c}$ and $r = b$), it suffices to imbed $K_{2b,b,c}$ in $S(K_{b,n})$ where $b + c = 2n$. Let $U = \{u_i\}_{i=1}^{b}$ and $V = \{v_i\}_{i=1}^{n}$ be the partite sets of $K_{b,n}$. Imbed the elements of the first partite set of $K_{2b,b,c}$ as the poles of $S_{u_1}, S_{u_2}, \ldots, S_{u_b}$.

By Theorem 3.6, there exists a $C^4(b, c)$ consisting of $b$ arcchains. Imbed the elements of the second and third partite sets of $K_{2b,b,c}$ as the poles of $S_{v_1}, \ldots, S_{v_n}$, so that each $S_{v_i}$ corresponds to a link in this $C^4(b, c)$. Then by appropriately ordering and orienting the tubes $T_{uv_1}, T_{uv_2}, \ldots, T_{uv_n}$ for each $u \in U$, we find that $S_u$ corresponds to a bipartite arcchain in this $C^4(b, c)$. The edges joining the second and third partite sets of $K_{2b,b,c}$ can now be imbedded as second-order edges. □
Theorems 5.11 (above) and 5.12 (below) are a matched set; both give the formula \( \gamma(K_{a,b,c}) = g(a, b + c) \), where \( b + c \) is even in Theorem 5.11 and \( b + c \) is odd in Theorem 5.12. It is tempting to jump to the conclusion that Theorem 5.12 is simply a Corollary of Theorem 5.11, reasoning as follows. To imbed \( K_{a,b,c} \), where \( b + c \) is odd, first imbed \( K_{a,b-1,c} \) and then crown a vertex of the second partite set. This approach does indeed result in an imbedding of \( K_{a,b,c} \) and it could be used to establish an upper bound for \( \gamma(K_{a,b,c}) \). (See Theorem 5.13 for an example of this type of result.) Unfortunately, it is not minimal. In general, the genus given by the described procedure exceeds the minimal genus by approximately \( \frac{c}{4} \).

**Example 5.1:** We will show that \( \gamma(K_{34,11,8}) = g(34, 11 + 8) = 136 \). However, using the method described above, we begin with \( K_{34,10,8} \) imbedded in \( S_{128} \) and crown a vertex \( x \) of the second partite set. This crown increases the genus of the surface by \( \lceil \frac{1}{4} (\deg x - 2) \rceil = \lceil \frac{1}{4} (42 - 2) \rceil = 10 \). So we have \( K_{34,11,8} \) imbedded in \( S_{138} \).

One solution to this problem is to duplicate all of the regions incident with the vertex being crowned and through which vertices of the third partite set are accessible, in such a way that one such collection of regions is split by the crown and the other is not. Then diagonals can be imbedded in the first such collection which join the clone to the third partite set, and diagonals of the second collection of regions can be imbedded which join the vertex clones to the third partite set. The proof below accomplishes this for each vertex in the second partite set, making it possible to clone each such vertex. Consequently, we need only begin with a second partite set which is approximately half the size of the desired second partite set.
Theorem 5.12: For positive integers $a, b,$ and $c$ with $a \geq b \geq c$ and $b + c$ odd, if $b'$ is the smallest integer such that $b' \geq \frac{b}{2}$ and $b' + c \equiv 2 \pmod{4}$ then 
\[ \gamma(K_{a,b,c}) = g(a, b + c) \], provided $a \geq 4 \max\{b', c\} + 2$.

Proof: Let $a, b, c,$ and $b'$ be positive integers, as specified, and let $a' = \max\{b', c\}$.

Imbed $K_{4a',b'+c}$ in the surface $\mathcal{S}(K_{2a',\frac{1}{2}(b'+c)})$. Let $G$ denote the underlying graph $K_{2a',\frac{1}{2}(b'+c)}$ and say its partite sets are $U = \{u_i\}_{i=1}^{2a'}$ and $V = \{v_i\}_{i=1}^{\frac{1}{2}(b'+c)}$. Let $H$ denote the imbedded graph $K_{4a',b'+c}$ whose first partite set is $X = \{x_i\}_{i=1}^{4a'}$. Also, partition the second partite set of $H$ into $Y = \{y_i\}_{i=1}^{b'}$ and $Z = \{z_i\}_{i=1}^{c}$.

Imbed $X$ and $Y \cup Z$ as the poles of $\{S_{v_i}\}_{i=1}^{2a'}$ and $\{S_{v_i}\}_{i=1}^{\frac{1}{2}(b'+c)}$, respectively, pairing these (as north and south poles of the individual spheres) in a way consistent with a cyclic $C^4(b', c)$. Order and orient the tubes incident with $\{S_{u_i}\}_{i=1}^{a'}$ to correspond to the bipartite arcchains of a minimal, cyclic $C^4(b', c)$. Also, order and orient the tubes incident with $S_{v_{i+a'}}$ to correspond to the same arcchain as $S_{v_i}$, for $i = 1, 2, ..., a'$. Lastly, for each $i = 1, 2, ..., \frac{1}{2}(b'+c)$, order the tubes incident with $S_{v_i}$ as $T_{v_iu_1}, T_{v_iu_2}, ..., T_{v_iu_{2a'}}$ from east to west around the equator of $S_{v_i}$.

Now, consider an element $y$ of $Y$. For each $j = 1, 2, ..., 2a'$, let $R_j(y)$ denote the region, incident with $y$, whose boundary includes the edges $yu_{j}^{N}$ and $yu_{j}^{S}$. (Recall that $u_{j}^{N}, u_{j}^{S} \in X$.) Note that this labels only half of the regions incident with $y$. In fact, these regions form an alternating class of regions around $y$. All regions of the remaining alternating class around $y$ have the form $y x_m \omega(y) x_n$, where $x_m, x_n \in X$ and $\omega(y)$ is the element of $Y \cup Z$ which shares the same sphere with $y$ (i.e., $y$ and $\omega(y)$ are the poles of $S_{v_i}$ for some $i$). Also, note that the regions $\{R_i(y)\}_{i=1}^{2a'}$ are arranged around $y$ in the order indicated by the subscripts (clockwise, if $y$ is a north pole, counterclockwise if $y$ is a south pole). By this
construction, all elements of Z—except ω(y) if ω(y) ∈ Z—are accessible from y via the regions \( \{R_i(y)\}_{i=1}^{a'} \) and also via \( \{R_i(y)\}_{i=a'+1}^{2a'} \).

Next, clone \( u_1^N \) and \( u_{a'+1}^N \) by crowning so that neither \( R_1(y) \) nor \( R_{a'+1}(y) \) is split for any \( y \in Y \). For each \( y \in Y \), the two new edges \( y(u_1^N)' \) and \( y(u_{a'+1}^N)' \) increase the total number of regions incident with \( y \) by two in such a way that the collections of regions \( \{R_i(y)\}_{i=1}^{a'} \) and \( \{R_i(y)\}_{i=a'+1}^{2a'} \) are in separate alternating classes around \( y \). Also, \( ω(y) \) is accessible from \( y \) via each of the remaining regions incident with \( y \) (i.e., other than \( \{R_i(y)\}_{i=1}^{a'} \)) except the region with edges \( yu_1^N \) and \( y(u_1^N)' \) and the region with edges \( yu_{a'+1}^N \) and \( y(u_{a'+1}^N)' \). Thus, all elements of \( Z \) are accessible from \( y \) via either alternating class of regions around \( y \).

At this point, clone \( b - b' \) elements of \( Y \) and observe that all elements of \( Z \) remain accessible from each \( y \in Y \) as well as from each of the new \( b - b' \) clones. This is due to the fact that when a vertex \( y \) (with deg \( y \equiv 2 \pmod{4} \)) is cloned by crowning, the regions of one alternating class around \( y \) remain incident with \( y \), while the regions of the other alternating class are "lifted" to become incident with the clone \( y' \). We complete our imbedding of \( K_{a,b,c} \) by imbedding all edges joining elements of the second partite set with those of the third partite set and by choosing \( x \in X \) and adding \( a - (4a' + 2) \) clones of \( x \) by means of a cluster of crowns (in the fashion of Theorem 4.1).

To verify that the imbedding here described is minimal, we summarize the steps taken.

(i) Begin with \( K_{4a',b'+c} \) quadrilaterally imbedded in \( s(K_{2a',2(b'+c)}) \), which has genus \( g(4a', b' + c) \).

(ii) Clone two elements of the first partite set by crowning. Since each of the vertices cloned has degree \( = b' + c \equiv 2 \pmod{4} \), the resultant imbedding of \( K_{4a'+2,b'+c} \) is quadrilateral and the crowns add \( 2\left\lfloor \frac{1}{4} ((b' + c) - 2) \right\rfloor = \frac{1}{2} (b' + c - 2) \) to the genus.
(iii) Clone \( b - b' \) elements of the second partite set \((Y)\) by crowning. Here, each cloned vertex has degree \( 4a' + 2 \equiv 2 \) (mod 4), so the resultant imbedding of \( K_{4a'+2,b+c} \) is quadrilateral. These crowns add \((b - b')\left\lceil \frac{1}{4} \left((4a' + 2) - 2\right)\right\rceil = a' (b - b')\) to the genus. At this point the surface has genus \( g(4a', b' + c) + \frac{1}{2} (b' + c - 2) + a' (b - b') = (a' - \frac{1}{2}) (b' + c - 2) + \frac{1}{2} (b' + c - 2) + a' (b - b') = a' (b + c - 2) = \frac{1}{4} ((4a' + 2) - 2)((b + c) - 2) = g(4a' + 2, b + c)\).

Since imbedding the edges joining the second and third partite sets does not increase the genus of the surface, and since \( g(4a' + 2, b + c) \) is a lower bound for \( \gamma(K_{4a'+2,b+c}) \), we have \( \gamma(K_{4a'+2,b+c}) = g(4a' + 2, b + c) \). By Theorem 4.1 this gives us \( \gamma(K_{a,b,c}) = g(a, b + c) \) when \( a \geq 4a' + 2 \).

Although the previous theorems show that \( \gamma(K_{a,b,c}) = g(a, b + c) \) for a large number of complete tripartite graphs, our work here does not prove the White conjecture completely. We round out this section by establishing an upper bound for the remaining graphs in this class, using an approach similar to the one outlined prior to Theorem 5.12.

**Theorem 5.13:** For positive integers \( a, b, \) and \( c \) with \( a \geq b \geq c \),

\[
\gamma(K_{a,b,c}) \leq g(a, 2c) + (b - c)\left\lceil \frac{1}{4} (a + c - 2)\right\rceil.
\]

**Proof:** Begin with \( K_{a,c,c} \) imbedded in a surface of genus \( g(a, 2c) \). (See Theorem 5.8.) Then choose an element of the second partite set and add \( b - c \) clones of this chosen vertex by crowning. By Theorem 4.1, the result is \( K_{a,b,c} \) imbedded in a surface of genus \( g(a, 2c) + (b - c)\left\lceil \frac{1}{4} (a + c - 2)\right\rceil \).

A small amount of calculation shows that this upper bound exceeds \( g(a, b + c) \) by approximately \( \frac{c}{4} (b - c) \).
5.4 The Genus of $G + H$

As advertised, in this last section of Chapter 5, we consider the join of two nonempty graphs. Our goal here is to produce the formula $\gamma(G + H) = g(p(G), p(H))$ for certain pairs of graphs $G$ and $H$. This is clearly not always the case. For instance, $\gamma(K_a + K_b) = \gamma(K_{a+b})$ which, for most positive integers $a$ and $b$, is strictly larger than $g(a, b) = \gamma(K_{a,b})$.

Suppose $G$ and $H$ are nonempty graphs of even order, say $p(G) = 2m$ and $p(H) = 2n$. We attempt to imbed $G + H$ in $\mathcal{S}(K_{m,n})$. Let $U = \{u_i\}_{i=1}^m$ and $V = \{v_i\}_{i=1}^n$ be the partite sets of the underlying graph $K_{m,n}$. Imbed $V(G)$ as the poles of $\{S_{u_i}\}_{i=1}^m$ and imbed $V(H)$ as the poles of $\{S_{v_i}\}_{i=1}^n$. The join edges of $G + H$ are imbedded as all first-order edges, giving a quadrilateral imbedding of $K_{2m,2n}$ in $\mathcal{S}(K_{m,n})$. It remains to imbed $E(G)$ and $E(H)$ as diagonals of these quadrilateral regions.

Two obstructions to obtaining such an imbedding are the number and types of quadrilateral regions. There are a total of $2mn$ quadrilateral regions, so success hinges on $q(G) + q(H) \leq 2mn$. (This bound is derivable directly from the Euler-Poincaré formula.) Our construction is even more restrictive. Each of these regions is of the form $x_i y_j x_k y_l$, where $x_i, x_k \in V(G)$ and $y_j, y_l \in V(H)$. Moreover, either $x_i$ and $x_k$ are opposite poles of the same sphere--call these type-$g$ regions--or $y_j$ and $y_l$ are opposite poles of the same spheres--call these type-$h$ regions. So, for a type-$g$ region, $x_i x_k$ could be imbedded as a zeroth-order edge of $G$ or $y_j y_l$ could be imbedded as a second-order edge of $H$ (but not both). A similar argument applies to type-$h$ regions (interchanging "zeroth-order" and "second-order"). It is easily shown that there are $mn$ regions of each type. Since $G$ has at most $m$ zeroth-order edges,
we have \( q(G) \leq \lvert \{\text{type-h regions}\} \rvert + m = m(n + 1) \). Also, \( H \) has at most \( n \) zeroeth-order edges, so \( q(H) \leq n(m + 1) \).

Aside from the question of the sizes of \( G \) and \( H \), we have a third obstruction which could be called \"conflicting tube orientations.\" It would be convenient to proceed as follows. To imbed \( E(G) \), orient and order the tubes of \( S(K_m,n) \) around the equators of \( \{S_i\}_{i=1}^{n} \). At the same time, to imbed \( E(H) \), orient and order the tubes around the equators of \( \{S_i\}_{i=1}^{m} \). However, this may not be possible. Although the ordering of tubes around the equator of one sphere is independent of the ordering of tubes around another, orienting the tubes is not.

The general problem of designing a consistent pattern of tube orientations is unsolved and appears to be difficult. Two ways of avoiding this problem are (i) consider graphs \( G \) and \( H \) whose edges can be imbedded using tubes of the same orientation and (ii) consider graphs \( G \) and \( H \) which have \"several\" isolated vertices so that disjoint collections of tubes can be used to imbed \( E(G) \) and \( E(H) \). Below, we present examples of both approaches.

The first of these results was proved by Jungerman in 1975 [22] using current graphs and later, independently, by Garman [16] (for \( n \) even), also using current graphs. Jungerman also proved that \( \gamma(K_{4(3)}) \geq 5 \). White [41] used a voltage graph to show \( \gamma(K_{4(3)}) \leq 5 \), thus establishing \( \gamma(K_{4(3)}) = 5 \). We offer the first surgical proof of the general case.

**Theorem 5.14:** If \( n \) is a positive integer with \( n \neq 3 \) or \( 5 \), then

\[
\gamma(K_{4(n)}) = g(2n, 2n) = (n - 1)^2.
\]

**Proof:** First observe that \( K_{4(n)} = K_{n,n} + K_{n,n} \). We imbed this graph in \( S(K_{n,n}) \) as described above. That is, let \( U = \{u_i\}_{i=1}^{n} \) and \( V = \{v_i\}_{i=1}^{n} \) be the partite sets of the underlying graph \( K_{n,n} \). Imbed the partite sets of \( K_{4(n)} \) as \( \{u_i^N\}_{i=1}^{n} \), \( \{u_i^S\}_{i=1}^{n} \),...
As always, the join edges are imbedded as all first-order edges, so it remains to imbed the edges of the two copies of $K_{n,n}$. It suffices to show that in all cases of $n$ (except 3 and 5), there is an attractive, linear $C^4$ of cardinality $n$, all of whose links are positively oriented. (We require linear arcchains because each sphere contains a zeroth-order edge.) In this way, the tubes can be ordered around $S_{ui}$ for $i = 1, 2, ..., n$, to correspond to the arcchains of this $C^4$, and also ordered around the equators of $S_{vi}$ for $i = 1, 2, ..., n$ in similar fashion. Since all tubes are positively oriented, no conflict occurs.

We turn to the proof of Theorem 3.4 to verify that such $C^4$'s exist. For $n$ even, we found an attractive linear $C^4$ all of whose arcchains were of the form $(L; \sigma^+, \tau_k)$ or $(L; \sigma^-, \tau_k)$. By replacing each $(L; \sigma^-, \tau_k)$ with $(L; \sigma^+, \tau_k)$ (which forms the same couples) we obtain the desired $C^4$. For $n$ odd, with $n \neq 3$ or 5, the $C^4$ constructed in the proof of Theorem 3.4 consisted entirely of positively oriented links. □

The next two results use the second approach discussed above.

**Theorem 5.15:** For positive integers $a$ and $b$ and graphs $G$ and $H$,
\[
\gamma((G \cup \overline{K}_a) + (H \cup \overline{K}_b)) = g(p(G) + a, p(H) + b),
\]
provided $a \geq 4 \left\lceil \frac{p(H)}{2} \right\rceil + 2$ and $b \geq 4 \left\lceil \frac{p(G)}{2} \right\rceil + 2$.

**Proof:** Let $m = \left\lfloor \frac{p(G) + a}{2} \right\rfloor$ and $n = \left\lfloor \frac{p(H) + b}{2} \right\rfloor$. Also, if $p(G)$ is even let $G' = G$, otherwise let $G' = G \cup K_1$. Similarly, if $p(H)$ is even let $H' = H$; otherwise, let $H' = H \cup K_1$. Finally, let $a' = 2m - p(G')$ and $b' = 2n - p(H')$. Note that $a'$ and $b'$ are even with $a' \geq 2p(H')$ and $b' = 2p(G')$.

We imbed $(G' \cup \overline{K}_a') + (H' \cup \overline{K}_b')$ in $S(K_{m,n})$ as follows. Let $U = \{u_i\}_{i=1}^m$ and $V = \{v_i\}_{i=1}^n$ be the partite sets of the underlying graph $K_{m,n}$. Imbed $\overline{K}_a'$ as the
poles of \( \{S_u_i\}_{i=1}^{a'} \) and imbed \( V(G') \) as the poles of \( \{S_u_i\}_{i=1}^{m} \). Also, imbed \( \overline{K_b'} \) as the poles of \( \{S_v_j\}_{j=1}^{b'} \) and imbed \( V(H') \) as the poles of \( \{S_v_j\}_{j=1}^{n} \). Imbed the join edges as all first-order edges.

For each \( i = 1, 2, \ldots, \frac{1}{2} a' \), divide the equator of \( G' \) into two sections. In one section, attach the tubes \( \{T_{u_i u_j}\}_{j=1}^{a'} \). In the other section, attach (and orient) the tubes \( \{T_{u_i u_j}\}_{j=1}^{b'} \) to form the \( i \)th arcchain in a minimal \( (1,1)-C^4(\frac{p(H)}{2}) \). Similarly, for each \( i = 1, 2, \ldots, \frac{1}{2} b' \), attach and orient the tubes \( \{T_{v_i u_j}\}_{j=1}^{a'} \) to correspond to the \( i \)th arcchain in a minimal \( (1,1)-C^4(\frac{p(G)}{2}) \), attaching the tubes \( \{T_{v_i u_j}\}_{j=1}^{b'} \) to \( S_{v_i} \) in a separate section of its equator. By this construction, we can imbed \( E(G') \) and \( E(H') \) as second-order edges.

Observe that \( G \cup \overline{K_a} = G' \cup \overline{K_{a'}} \) or \( G' \cup \overline{K_{a'}} \cup \overline{K_1} \) depending on whether \( p(G) + a \) is even or odd, respectively. Likewise, \( H \cup \overline{K_b} = H' \cup \overline{K_{b'}} \) or \( H' \cup \overline{K_{b'}} \cup \overline{K_1} \) depending on the parity of \( p(H) + b \). If \( p(G) + a \) and \( p(H) + b \) are both even we are done. If only one of these, say \( p(G) + a \), is odd, we clone \( u_i^N \) with a single crown. If both \( p(G) + a \) and \( p(H) + b \) are odd, we clone both \( u_i^N \) and \( v_i^N \), keeping in mind that if \( \deg u_i^N \neq \deg v_i^N \pmod{4} \), we must first clone the vertex whose degree is congruent to 0 modulo 4. (See Proof 2 of Theorem 5.1.) □

**Theorem 5.16:** For positive integers \( a \) and \( b \) and graphs \( G \) and \( H \),
\[
\gamma((G \cup \overline{K_a}) + (H \cup \overline{K_b})) = g(p(G) + a, p(H) + b),
\]
provided \( a \geq 4 \chi_1(G) - 2 \) and \( b \geq 4 \chi_1(H) - 2 \).

**Proof:** Simply repeat the construction of the previous proof, except that instead of ordering and orienting tubes to form minimal \( (1,1)-C^4 \)'s, we use the construction found in the proof of Theorem 5.7. □
We complete our study of joins by determining the genera of the joins of cycles and paths.

**Theorem 5.17:** For positive integers $a \geq 3$ and $b \geq 3$, $\gamma(C_a + C_b) = g(a, b)$.

**Proof:** Without loss of generality, assume $a \leq b$. The proof is in two cases.

**Case 1 ($a = 3$):** First observe that $g(3, b) \leq \gamma(C_3 + C_b) \leq \gamma(K_{3+b})$. For $b = 3$ or $4$, $g(3, b) = \gamma(K_{3+b}) = 1$ so $\gamma(C_3 + C_3) = \gamma(C_3 + C_4) = 1$.

Next, consider values of $b$ for which $b = 4h + 2$ with $h = 1, 2, \ldots$. Let $V(C_3) = \{u_i\}_{i=1}^3$ and $V(C_b) = \{v_i\}_{i=1}^b$. (It should be mentioned here that we will add edges among the vertices $\{v_i\}_{i=1}^b$ to form a cycle, but that this cycle will not correspond to the natural one indicated by subscripts.) We begin with the plane imbedding of $K_{2,b}$ shown in Figure 5.1.

Add $u_3$ as a clone of $u_2$. Since the neighbors of $u_2$ are arranged in Figure 5.1 as the neighbors of $x$ are arranged in the description of the basic crown construction (prior to crowning) we use all of the notation of Section 4.1, except that $u_2$ replaces $x$ and $v_i$ replaces $y_i$.

Observe that $u_1$ is accessible from $u_2$ via each odd numbered region $(R_{2i+1})$, $u_1$ is accessible from $u_3$ via each even numbered region $(R'_{2i})$, and $u_3$ is accessible from $u_2$ via each of the quadrilateral regions $(Q)$ formed by crowning. Our goal is to imbed edges among the vertices $\{v_i\}_{i=1}^b$ to form a cycle in such a way that at least one of each type of region is unaffected by these cycle edges. With this goal accomplished, the edges $u_1u_2$, $u_1u_3$, and $u_2u_3$ can be imbedded as diagonals of these "unblocked" regions. This is done as follows:

1. If $h = 1$ the cycle $C_6$ is $v_1v_2v_3v_6v_5v_4v_1$.
2. If $h = 2$ the cycle $C_{10}$ is $v_1v_2v_3v_6v_7v_{10}v_9v_8v_5v_4v_1$.

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
(3) If \( h \geq 3 \) the cycle \( C_b \) is \( v_1 v_2 v_3 v_6 v_5 v_4 v_9 v_8 v_7 v_{10} v_{11} v_{12} \ldots \ v_b v_1 \),

(where "..." means the subscripts continue to increase by increments of one).

For \( h = 1 \), the cycle edges block the regions \( R_1, R'_2, Q(u_2 v_6 u_3 v_3), R_5, R'_4, \) and \( Q(u_2 v_4 u_3 v_1) \). This leaves \( R_3, R'_6, Q(u_2 v_2 u_3 v_5) \) unblocked. For \( h = 2 \), one set of unblocked regions is \( \{R_3, R'_10, Q(u_2 v_2 u_3 v_9)\} \). For \( h \geq 3 \), one set of unblocked regions is \( \{R_3, R'_6, Q(u_2 v_2 u_3 v_b)\} \).

For \( b \geq 5 \) and \( b \neq 2(\text{mod } 4) \), the cycle is obtained, from the appropriate sequence above, by deleting the highest numbered one, two, or three entries. In each case, accessibility of each \( u_i \) from another \( u_j \) is easily established, although access may be via the expanded region \( T_r \) for one or two pairs of vertices of \( C_3 \).

**Case 2 (\( a \geq 4 \)):** First, suppose that \( a \) and \( b \) are even, say \( a = 2m \) and \( b = 2n \).

Imbed \( K_{a,b} \) in \( S(K_{m,n}) \) so that all tubes are positively oriented. Let \( U = \{u_i\}_{i=1}^m \) and \( V = \{v_j\}_{j=1}^n \) be the partite sets of the underlying graph \( K_{m,n} \). For each \( i = 1, \ldots, m \), order the tubes incident with \( S_{u_i} \) as \( T_{u_i v_1}, T_{u_i v_2}, \ldots, T_{u_i v_n} \) from west to east around \( S_{u_i} \). Similarly, for each \( j = 1, 2, \ldots, n \), order the tubes incident with \( S_{v_j} \) as \( T_{v_j u_1}, T_{v_j u_2}, \ldots, T_{v_j u_m} \) from west to east around \( S_{v_j} \).

The cycle \( C_a \) will be \( u_1^S u_1^N u_2^S u_2^N \ldots u_m^S u_m^N u_1^S \). Imbed \( u_1^S u_1^N \) as a zeroeth-order diagonal of the region \( u_1^N v_n^N u_1^S v_1^S \) and, for \( i=1,2,\ldots,m \), imbed \( u_i^S u_i^N \) as a zeroeth-order diagonal of \( u_i^N v_1^N u_i^S v_2^S \). Also, imbed \( u_m^N u_1^S \) as a second-order diagonal of \( u_m^N v_2^S u_1^S v_2^N \) and, for \( i = 1, 2, \ldots, m-1 \), imbed \( u_i^N u_{i+1}^S \) as a second-order diagonal of \( u_i^N v_1^S u_{i+1}^S v_1^N \). If we repeat this by replacing \( u \) by \( v \) and interchanging \( m \) and \( n \) we have a description of \( C_b \) and its imbedding.

Next, suppose \( a + b \) is odd, say \( a = 2m + 1 \) and \( b = 2n \). Begin by imbedding \( K_{a-1,b} \) in \( S(K_{m,n}) \) as described above. Now clone \( u_1^N \) with a basic
single crown which splits the alternating class of regions (around $u_1^N$) which contains $u_1^N v_n^N u_1^S v_1^S$. The edges of $C_b$ are imbedded now, precisely as above. The edges of $C_a$: $u_1^S (u_1^N) v_1^N u_1^S u_2^N u_2^S ... u_m^S u_m^N u_1^S$ are also imbedded as above except that $u_1^S u_1^N$ is no longer imbedded, $u_1^S (u_1^N) v_1^N$ is imbedded as a diagonal of the region $(u_1^N) v_n^N u_1^S v_1^S$, and $(u_1^N) u_1^N$ is imbedded as a diagonal of one of the quadrilaterals $(Q)$ formed by the crown.

Finally, if $a$ and $b$ are both odd, say $a = 2m + 1$ and $b = 2n + 1$, then we begin with $K_{a-1,b-1}^{a-1,b-1}$ in $S(K_m,n)$ and crown both $u_1^N$ and $v_1^N$ using the construction just described. Of course, care must be taken to crown in the proper order as dictated at the end of Proof 2 of Theorem 5.1. □

**Corollary 5.17 a:** Let $a$ and $b$ be positive integers.

(i) If $a \geq 3$ and $b \neq 2$, then $\gamma(C_a + P_b) = g(a, b)$.

(ii) If $a \geq 3$, then $\gamma(C_a + P_2) = 1$.

(iii) $\gamma(P_a + P_b) = g(a, b)$.

**Proof:** If $a \geq 3$ and $b \geq 3$, these results follow from Theorems 5.17 and 1.1, since $P_n \subseteq C_n$ for all integers $n \geq 3$.

If $a + b \leq 4$, then $\gamma(C_a + P_b) = \gamma(P_a + P_b) = 0$, since both $C_a + P_b$ and $P_a + P_b$ are subgraphs of $K_4$.

If $b = 1$, we have $C_a + P_1 = C_a + K_1$, which is a wheel and hence planar, i.e., $\gamma(C_a + P_1) = 0$. Since $P_a + P_1 \subseteq C_a + P_1$ for $a \geq 3$, we have $\gamma(P_a + P_1) = 0$.

It remains to show that for $a \geq 3$, $\gamma(P_a + P_2) = 0$ and $\gamma(C_a + P_2) = 1$. Observe that $C_a + \overline{K}_2$ is the double wheel, which is a maximal planar graph. Since $C_a + P_2$ is obtained from $C_a + \overline{K}_2$ by adding a single edge, we have $\gamma(C_a + P_2) = 1$.

However, if $C_a + \overline{K}_2$ is imbedded in the plane, we can obtain a plane imbedding of
\( P_a + P_2 \) by deleting an edge of \( C_a \) and imbedding the "other diagonal" of the resultant quadrilateral region. □

**Corollary 5.17 b:** If \( G \) and \( H \) are linear forests, then \( \gamma(G + H) = g(p(G), p(H)) \).

**Proof:** This follows immediately from part (iii) of Corollary 5.17 a. □
CHAPTER VI

THE GENUS OF THE GRAPHICAL COMPOSITION

In this chapter we apply the ideas of graphical surfaces and crowns to derive genus formulae for several classes of graph compositions. We also define a generalized composition of graphs and apply our techniques to these graphs.

6.1 The Genus of \( G[ K_n] \)

The composition \( G[H] \) of graphs \( G \) and \( H \) is defined in Table 1 (on page 6). If \( V(G) = \{u_i\}_{i=1}^p \) and \( V(H) = \{v_j\}_{j=1}^n \), then, for ease of notation, we denote the vertices of \( G[H] \) by \( w_{i,j} \) for \( i = 1, 2, \ldots, p \) and \( j = 1, 2, \ldots, n \). Also, for \( i = 1, 2, \ldots, p \), let \( H_i = \langle \{w_{i,j}\}_{j=1}^n \rangle \) and note that \( H_i \cong H \). Thus, we see that \( G[H] \) can be formed from the disjoint union \( \bigcup_{i=1}^p H_i \) of \( p \) copies of \( H \) by adding all edges between \( H_i \) and \( H_j \) whenever \( u_iu_k \in E(G) \). In particular, each edge \( u_iu_k \) of \( G \) corresponds to an induced subgraph of \( G[H] \) which is isomorphic to \( H + H \), namely \( \langle \bigcup_{i=1}^p \{w_{i,j}, w_{k,j}\} \rangle \). We will exploit this connection between the concepts of join and composition by using our knowledge of joins to explore the genus of compositions. Just as we began our study of joins with empty graphs, we begin our study of compositions by considering graphs \( G[H] \) for which \( H \) is empty. The following lemma lays the foundation for most of our work here.

Lemma 6.1: For any nontrivial, connected graph \( G \) and any positive integer \( n \), \( G[ \overline{K_{2n}}] \) has a quadrilateral imbedding in \( \mathbb{S}(G[ \overline{K_n}]) \).
Proof: This result is really more in the nature of an observation. The underlying graph \( G[\overline{K_n}] \) consists of \( p(G) \) copies of \( \overline{K_n} \) joined together according to the edges of \( G \). Each such copy of \( \overline{K_n} \) corresponds to a collection of \( n \) spheres in \( S(G[\overline{K_n}]) \) which are joined (to other such collections of spheres) according to the edges of \( G \). The poles of each such collection of spheres form a copy of \( \overline{K_{2n}} \) imbedded in \( S(G[\overline{K_n}]) \). Finally, these copies of \( \overline{K_{2n}} \) can be joined together, according to the edges of \( G \), by imbedding all first-order edges. The resulting imbedding of \( G[\overline{K_{2n}}] \) in \( S(G[\overline{K_n}]) \) is quadrilateral. □

**Theorem 6.1:** For any nontrivial, connected graph \( G \) and any positive integer \( n \),

\[
\gamma(G[\overline{K_{2n}}]) \leq \beta(G[\overline{K_n}]).
\]

Moreover, we have equality if \( G \) is triangle-free.

Proof: The inequality follows immediately from Lemma 6.1. The remainder follows from the observation that if \( G \) is triangle-free, then \( G[\overline{K_{2n}}] \) is also triangle-free. □

The following is a corollary of a result of White [40].

**Corollary 6.1:** For any nontrivial, connected, triangle-free, \((p, q)\)-graph \( G \) and any positive integer \( n \),

\[
\gamma(G[\overline{K_{2n}}]) = n^2q - np + 1.
\]

In light of these results, our primary objective will be to determine the genus of \( G[H] \) under the restrictions that \( G \) be connected and triangle-free and that \( H \) have even order. It should be noted that genus results for compositions have been obtained beyond these restrictions. For instance, \( \gamma(K_p[K_n]) = \gamma(K_{pn}) \) is completely determined, as mentioned in Chapter 1. Also, much has been done with the class
K_m(n) = K_m[ \overline{K_n} ]. We have already discussed K_3(n). The reader is also directed to Bénard and Bouchet [7], Bouchet [10], [11], and Garman [16].

Before completely abandoning the compositions G[H] for which H has odd order, we offer the following.

**Theorem 6.2:** Let \( n \) be a positive integer. If \( G \) has a quadrilateral imbedding in some surface and all vertices of \( G \) have even degree, then \( G[ \overline{K_{2n+1}} ] \) has a quadrilateral imbedding. Hence, if \( G \) is triangle-free, then

\[
\gamma(G[ \overline{K_{2n+1}} ]) = \frac{q}{4} (2n + 1)^2 - \frac{p}{2} (2n + 1) + 1.
\]

**Proof:** Begin with a quadrilateral imbedding of \( G \). Order \( V(G) \) and, one-by-one, add 2n clones of each vertex by adding n double crowns. Referring to the description of the double crown given in Section 4.4, we see that when a double crown is applied to a vertex \( v \) of even degree, all new regions are quadrilateral and all previously existing regions are either unaltered or altered (by replacing \( v \) by \( v^{(1)} \) or \( v^{(2)} \)) so as to have the same boundary length. Since a double crown adds two to the degree of each neighbor of \( v \), all intermediate graphs throughout this procedure are quadrilaterally imbedded and have no vertices of odd degree.

The final equation follows immediately from the Euler-Poincaré formula, assuming a quadrilateral imbedding in an orientable surface. \( \square \)

### 6.2 The Genus of \( G[H] \)

In this section, we deal with the question: For which pairs of graphs \( G \) and \( H \), with \( G \) connected and triangle-free and \( H \) nonempty and of even order 2n, does \( \gamma(G[H]) = \gamma(G[ \overline{K_{2n}} ]) \)? Our approach is similar to that of Chapter V. Begin with the quadrilateral imbedding of \( G[ \overline{K_{2n}} ] \) described in the previous section; then orient and order tubes so as to imbed the edges of the various copies of \( H \) as diagonals of
quadrilateral regions. Since each edge of $G$ corresponds to the join of nonempty graphs $(H + H)$ we must avoid the problem of conflicting tube orientations.

The edges of the underlying graph $G[\overline{K_n}]$ can be partitioned into collections corresponding to the edges of $G$. That is, each edge of $G[\overline{K_n}]$ joins vertices in different copies of $\overline{K_n}$, say $(\overline{K_n})_i$ and $(\overline{K_n})_k$ where $u_iu_k \in E(G)$. This edge belongs to the collection of edges joining the vertices of $(\overline{K_n})_i$ to the vertices of $(\overline{K_n})_k$. The tubes of $S(G[\overline{K_n}])$ corresponding to such a collection of edges of $G[\overline{K_n}]$ will be called a bundle. Let $B_{i,k}$ denote the bundle of tubes of $S(G[\overline{K_n}])$ corresponding to the edge $u_iu_k$ of $G$. Also, let $S(i)$ denote the spheres of $S(G[\overline{K_n}])$ corresponding to $(\overline{K_n})_i$.

In order to maximize the number of second-order diagonals among vertices of a given copy of $\overline{K_{2n}}$ (in the imbedded graph $G[\overline{K_{2n}}]$) tubes are attached to spheres so that the tubes within a given bundle are attached consecutively around the equator of the sphere. Thus, the equator of each sphere $S_{w_i,j}$ (where $u_i \in E(G)$ and $v_j \in \overline{K_n}$) is divided into $\deg_G u_i$ sections by the attachment of the subbundles of tubes corresponding to the neighbors of $u_i$ in $G$. In this way, each sphere of $S(G[\overline{K_n}])$ can be seen to correspond to $\deg_G u_i$ linear arcchains (cyclic of $\deg_G u_i = 1$), one for each $(\overline{K_{2n}})_k$ for which $u_iu_k \in E(G)$. We say that $B_{i,k}$ is incident with $(\overline{K_{2n}})_i$ and $(\overline{K_{2n}})_k$.

We may now begin to describe the imbedding of $G[H]$ in $S(G[\overline{K_n}])$. Imbed $V(H_i)$ as the poles of $S(i)$, pairing these vertices as links to form linear arcchains. Each of the $\deg_G u_i$ bundles of tubes incident with $H_i$ corresponds to a set of $n$ linear arcchains on the links of $V(H_i)$.

As in Section 5.4, we avoid conflicting tube orientations within a tube bundle $B_{i,k}$ in one of two ways: (1) by positively orienting all tubes in $B_{i,k}$ and allowing $H_i$ and $H_k$ to "share" the bundle or (2) by assigning each bundle $B_{i,k}$ to $H_i$ or...
H_k, i.e., using B_{i,k} to accomplish adjacencies within either H_i or H_k, but not both. Both approaches are seen to be useful in proving the following results.

**Theorem 6.3:** If G is a nontrivial, connected, (p, q)-graph and H is a spanning subgraph of K_{n,n}, where n ≠ 3 or 5, then
\[ \gamma(G[H]) \leq n^2q - np + 1. \]
Moreover, if G is triangle-free, we have equality.

**Proof:** In light of Theorem 6.1 and Corollaries 6.1 and 1.1, it suffices to show that G[K_{n,n}] imbeds in S(G[Kn,n]). As outlined above, for each i = 1, 2, ..., p imbed V((K_{n,n})_i) as the poles of S(i) so that one partite set imbeds as the north poles and the other as the south poles. Choose any u_i ∈ N_G(u_i) and use B_{i,k} to accomplish all adjacencies within K_{n,n}. That is, positively orient the tubes of B_{i,k} and order these along the equators of the spheres S(k) so that each such sphere corresponds to an arccoin in a minimal (1,2)-C^4(n). There can be no conflicting tube orientations since all tubes are positively oriented. □

Observe that K_{4(n)} = K_2[Kn,n], so Theorem 5.14 is simply a corollary of Theorem 6.3. The following is a sampling of other immediate consequences of this Theorem.

**Corollary 6.3 a:** For positive integers m and n,
\[ \gamma(Q_m(Q_n)) = m \cdot 2^{m+2n-3} - 2^{m+n-1} + 1. \]

**Corollary 6.3 b:** For positive integers m and n with n ≠ 3 or 5,
\[ \gamma(K_{m,m}[K_{n,n}]) = (mn - 1)^2. \]

**Corollary 6.3 c:** For positive integers m and k ≠ 3 or 5,
(i) \[ \gamma(C_m[C_{2k}]) = \gamma(C_m[P_{2k}]) = k^2m - km + 1, \text{ provided } m \geq 4. \]
(ii) \( \gamma(P_m[C_{2k}]) = \gamma(P_m[P_{2k}]) = k^2(m - 1) - km + 1 \). In fact, if \( T \) is any tree of order \( m \), then \( \gamma(T[C_{2k}]) = \gamma(T[P_{2k}]) = k^2(m - 1) - km + 1 \).

We will now utilize the second approach for avoiding conflicting tube orientations—assigning each bundle \( B_{i,k} \) to \( H_i \) or \( H_k \). This is equivalent to directing each edge \( u_iu_k \) of \( G \) toward the endpoint to which it is being assigned. Thus, the number of bundles of \( S(G[\overrightarrow{K_n}]) \) which can be used to accomplish adjacencies within \( H_i \) equals the indegree of \( u_i \) in this orientation of \( G \). The best orientation of \( G \), to suit our purpose, would be one which assigns as many edges to each vertex as possible. That is, we want to maximize the minimum indegree. With this in mind, we make the following definition.

For a digraph \( D \), the minimum indegree is denoted \( \delta^-(D) \). For a graph \( G \) we define \( \delta^*(G) \) by \( \delta^*(G) = \max \delta^-(D) \), where the maximum is taken over all orientations \( D \) of \( G \). We are now in a position to present our next result.

**Theorem 6.4:** If \( G \) is a nontrivial, connected \((p, q)\)-graph with \( \delta^*(G) \geq 2 \) and \( H \) is any graph of order \( 2n \), then

\[ \gamma(G[H]) < qn^2 - 2pn + 1. \]

Moreover, if \( G \) is triangle-free, we have equality.

**Proof:** In light of Theorem 6.1 and Corollaries 6.1 and 1.1, it suffices to show that \( G[K_{2n}] \) imbeds in \( \overrightarrow{S}(G[\overrightarrow{K_n}]) \). Again, as outlined above, for each \( i = 1, 2, \ldots, p \) imbed \( V((K_{2n})_i) \) as the poles of \( S(i) \). Assign each bundle \( B_{i,k} \) to either \((K_{2n})_i\) or \((K_{2n})_k\) so that each copy of \( K_n \) is assigned at least two bundles. Each of the bundles assigned to \((K_{2n})_i\) represents \( n \) linear arcchains on the links of \((K_{2n})_i\). Recalling, from Theorem 3.4, that \( c_{11}(n) = 2n \), we can order and orient the tubes of two of these bundles to accomplish all adjacencies within \((K_{2n})_i\) for each \( i = 1, 2, \ldots, p \). □
Corollary 6.4: For positive integers \( m, n, a, \) and \( b \) with \( m, n \geq 4, \) and \( a + b = 2k \),
\[
\gamma(K_{m,n}[K_{a,b}]) = mnk^2 - 2(m + n)k + 1.
\]

Proof: We need only show that \( \delta^*(K_{m,n}) \geq 2 \). Number the first partite set from one to \( m \) and the second from one to \( n \). Direct edges joining two vertices of different parity toward the first partite set and all others toward the second partite set. Then the minimum indegree of this digraph is \( \min\{\left\lfloor \frac{m}{2} \right\rfloor, \left\lfloor \frac{n}{2} \right\rfloor\} \) which, in view of the lower bounds on \( m \) and \( n \), is at least \( 2 \). Thus, \( \delta^*(K_{m,n}) \geq 2 \). □

(Observe that Corollary 6.4 subsumes Corollary 6.3 b.)

The proof of the following offers a variation on this type of argument.

Theorem 6.5: If \( G \) is a nontrivial, \((p, q)\)-graph with \( \delta^*(G) \geq 1 \) and \( H \) is a graph of order \( 2n \) with \( n \geq 2 (\chi_1(H) - 1) \), then
\[
\gamma(G[H]) \leq qn^2 - 2pn + 1,
\]
with equality when \( G \) is triangle-free.

Proof: Mimic the proof of Theorem 6.4 except that the tubes of a single bundle assigned to \((H)_i\) are ordered and oriented according to the construction given in Theorem 5.7. □

Corollary 6.5: For positive integers \( a \geq 4, b \geq 9, \) and \( c \geq 8 \) with \( bc = 2n \),
\[
\gamma(C_a[C_b[C_c]]) = an(n - 2) + 1.
\]

Proof: Clearly, \( C_a \) is triangle-free with \( \delta^*(C_a) = 1 \). It remains to show that \( n \geq 2 (\chi_1(C_b[C_c]) - 1) \). By Vizing's Theorem, \( \chi_1(H) - 1 \leq \Delta(H) \). Note that \( C_b[C_c] \) is a \( 2(c + 1) \)-regular graph. Thus, \( 2(\chi_1(C_b[C_c]) - 1) \leq 4(c + 1) \leq \frac{bc}{2} = n \). □
It should be clear that many more theorems and corollaries of a similar type could be derived at this juncture. However, at this point we choose to apply our techniques to a generalization of the concept of the compositions of graphs.

6.3 The Generalized Composition of Graphs

In forming \( G[H] \), we replace each vertex \( u_i \) of \( G \) with a graph \( H_i \) isomorphic to \( H \). We also replace each edge \( u_iu_j \) of \( G \) with all join edges between \( H_i \) and \( H_j \). For a generalized composition, we eliminate the requirement that \( H_i \) be isomorphic to the same graph \( H \) for \( i = 1, 2, \ldots, p(G) \). That is, if \( V(G) = \{u_i\}_{i=1}^p \) and \( H_1, H_2, \ldots, H_p \) is a collection of \( p = p(G) \) graphs, then the generalized composition graph \( G[H_1, H_2, \ldots, H_p] \) is formed from the disjoint union \( \biguplus_{i=1}^p H_i \) by adding all join edges between \( H_i \) and \( H_j \) if and only if \( u_iu_k \in E(G) \).

The following results are stated without proof, as being entirely analogous with the corresponding results of Section 6.1.

**Lemma 6.6:** If \( n_1, n_2, \ldots, n_p \) are positive integers and \( G \) is a nontrivial, connected graph of order \( p \), then \( G[\overline{K_{2n_1}}, \overline{K_{2n_2}}, \ldots, \overline{K_{2n_p}}] \) has a quadrilateral imbedding in \( \Xi(G[\overline{K_{n_1}}, \overline{K_{n_2}}, \ldots, \overline{K_{n_p}}]) \).

**Theorem 6.6:** If \( n_1, n_2, \ldots, n_p \) are positive integers and \( G \) is a nontrivial, connected graph of order \( p \), then

\[
\gamma(G[\overline{K_{2n_1}}, \overline{K_{2n_2}}, \ldots, \overline{K_{2n_p}}]) \leq \beta(G[\overline{K_{n_1}}, \overline{K_{n_2}}, \ldots, \overline{K_{n_p}}]).
\]

Moreover, we have equality if \( G \) is triangle-free.

To express this Betti number, we define the weighted degree of a vertex \( u_i \in V(G) \) by \( dw(u_i) = \sum \{n_j \mid u_iu_j \in E(G)\} \). Then \( G[\overline{K_{n_1}}, \overline{K_{n_2}}, \ldots, \overline{K_{n_p}}] \) has...
order $\sum_{i=1}^{p} n_i$ and size $\frac{1}{2} \sum_{i=1}^{p} (n_i \cdot \text{dw}(u_i))$. So we have the following analogue of Corollary 6.1.

**Corollary 6.6:** If $n_1, n_2, \ldots, n_p$ are positive integers and $G$ is a nontrivial, connected, triangle-free graph of order $p$, then

$$\gamma(G[ \overline{K_{2n_1}}, \overline{K_{2n_2}}, \ldots, \overline{K_{2n_p}}]) = 1 + \frac{1}{2} \sum_{i=1}^{p} n_i (\text{dw}(u_i) - 2).$$

**Theorem 6.7:** Let $G$ be a graph of order $p$ and let $n_1, n_2, \ldots, n_p$ be positive integers. If $G$ has a quadrilateral imbedding in some surface and all vertices of $G$ have even degree, then $G[ \overline{K_{2n_1+1}}, \overline{K_{2n_2+1}}, \ldots, \overline{K_{2n_p+1}}]$ also has a quadrilateral imbedding. This imbedding is minimal if $G$ is triangle-free.

If we work under the restriction that $p(H_i) = 2n$ for $i = 1, 2, \ldots, p$ we have the following analogues of Theorems 6.3, 6.4, and 6.5.

**Theorem 6.8:** If $G$ is a nontrivial, connected $(p, q)$-graph and, for $i = 1, 2, \ldots, p$,

$H_i$ is a spanning subgraph of $K_{n,n}$, where $n \neq 3$ or 5, then

$$\gamma(G[H_1, H_2, \ldots, H_p]) \leq n^2q - np + 1.$$

Moreover, if $G$ is triangle-free, we have equality.

**Theorem 6.9:** If $G$ is a nontrivial, connected $(p, q)$-graph with $\delta^*(G) \geq 2$ and, for $i = 1, 2, \ldots, p$,$H_i$ is any graph of order $2n$, then

$$\gamma(G[H_1, H_2, \ldots, H_p]) \leq n^2q - np + 1.$$

Moreover, if $G$ is triangle-free, we have equality.

**Theorem 6.10:** If $G$ is a nontrivial, $(p, q)$-graph with $\delta^*(G) \geq 1$ and, for $i = 1, 2, \ldots, p$,$H_i$ is a graph of order $2n$ with $n \geq 2 (\chi_1(H_i) - 1)$, then

$$\gamma(G[H_1, H_2, \ldots, H_p]) \leq n^2q - np + 1.$$

Moreover, if $G$ is triangle-free, we have equality.
We conclude this section with an analogue of Theorem 6.4 which does not require the graphs $H_1, H_2, \ldots, H_p$ to have equal order (although, they are still required to have even order). Our goal is to embed $G[H_1, H_2, \ldots, H_p]$ in $\delta(G[\overline{K_{n_1}}, \overline{K_{n_2}}, \ldots, \overline{K_{n_p}}])$ under a suitable substitute for the restriction "\(\delta^*(G) \geq 1.\)" Toward this end, we require a few definitions.

Suppose $D$ is a digraph of order $p$ with $V(G) = \{u_i\}_{i=1}^p$. Label each vertex $u_i$ with a positive integer $n_i$. Then, the **weighted indegree** $\text{id}_w(u_i)$ of a vertex $u_i \in V(D)$ is the sum of the weights of the vertices from which $u_i$ is adjacent, i.e., $\text{id}_w(u_i) = \sum\{n_j | u_ju_i \in E(D)\}$. Define $\delta_w^-(D)$ by $\delta_w^-(D) = \min_{1 \leq i \leq p} \frac{\text{id}_w(u_i)}{n_i}$.

Finally, for a graph whose vertices have been labeled with positive integers, define $\delta_w^*(G)$ by $\delta_w^*(G) = \max \delta_w^-(D)$ where the maximum is taken over all orientations $D$ of $G$.

**Theorem 6.11:** If $G$ is a nontrivial, connected $(p, q)$-graph, $H_i$ is any graph of order $2n_i$ for $i = 1, 2, \ldots, p$, and $\delta_w^*(G) \geq 2$ under the labeling which assigns the value $n_i$ to $u_i$, then

$$\gamma(G[H_1, H_2, \ldots, H_p]) \leq n^2q - np + 1,$$

with equality if $G$ is triangle-free.

**Proof:** The argument is almost identical to the proof of Theorem 6.4. The restriction $\delta_w^*(G) \geq 2$ means that bundles can be assigned to $\overline{K_{2n_1}}, \overline{K_{2n_2}}, \ldots, \overline{K_{2n_p}}$ in such a way that, for $i = 1, 2, \ldots, p$, the bundles assigned to $K_{2n_i}$ together represent at least $2n_i$ arcchains on the links of $K_{2n_i}$. This is sufficient to embed all edges of $K_{2n_i}$ as second-order diagonals. \qed
CHAPTER VII

OPEN PROBLEMS

As with any work in mathematics, each answer raises more questions. The following are offered as possible directions of future study.

1. Many restrictions were placed on the surgical constructions introduced in this dissertation. One of these was the requirement that all poles hold vertices of the imbedded graph, consequently requiring that the imbedded graph have even order. Is it possible to leave one or more poles vacant, but retain the minimality of the imbedding?

2. The restriction that all imbedded edges be of order $\leq 2$ was due to the presence of all first-order edges for the graphs considered here. Are there other imbeddings of these graphs, or of other graphs, in graphical surfaces which contain higher order edges? If so, how much more complicated would be the description of such imbeddings?

Closely related to these questions is: Does every even order graph have a minimal imbedding as a proper imbedding in a graphical surface? (I believe the answer here is clearly in the affirmative.) If so, is such an imbedding possible with all edges of order $\leq 2$? If not, we could define the imbedding order of a graph to be the minimum, over all such imbeddings, of the maximum edge order required. It seems that determining this parameter for graphs in general would be quite difficult.

3. The description of $\mathcal{S}(G)$ given in Section 2.2 begins with a 2-cell imbedding of $G$. Yet, throughout this dissertation, this imbedding was never given explicitly and was only given implicitly by specifying attachment orders of tubes around equators (which corresponds to giving a rotation scheme for $G$). Is it ever
desirable to begin a construction with a complete description of the imbedding of the underlying graph?

4. The alternate description of \( \mathcal{S}(G) \) given in Section 2.4 seems closely tied to the imbedding of \( G \). Is this construction ever preferable to the standard one? Is it ever useful to alter this second construction by not drilling all holes?

As a partial answer to this second question consider the following: If \( G \) is triangularly imbedded with bichromatic dual and if all tubes are positively oriented, then each second-order edge duplicates a first-order edge. By drilling holes only through regions of one of the color classes, we eliminate the duplication of edges. (That is, a triangular imbedding of \( G = K_3(n) \) yields a triangular imbedding of \( K_3(2n) \) in this fashion.) How useful are such variations?

5. If a tube is sewed to a sphere "backwards" the resultant surface is nonorientable. If one pair of opposite sides of the bounding rectangle in Figure 4.3 are identified "in reverse" the crown body becomes a Klein bottle. Could these alterations be used to obtain minimal, nonorientable imbeddings?

6. Can the graphical surface technique be used to obtain maximum genus results? Can it be used to produce symmetric imbeddings?

7. The proof of Theorem 3.6 contains several collections of bipartite arcchains; however, no claim of minimality was made or proven. Are any (or all) of these collections minimal? If not, can minimal collections be found?

8. Are there (properly imbedded) joins, other than \( K_4(n) \), which triangulate a graphical surface? One obstruction to producing triangular imbeddings seems to be the fact that, with all first-order diagonals present, the same zeroth-order diagonal appears in several quadrilateral regions. Specifically, the poles of \( S_v \) appear as opposite vertices in \( \text{deg}_G v \) quadrilaterals (where \( G \) is the underlying graph). Is there a way to modify the construction to eliminate this duplication?
9. We have constructed a single crown and a double crown. Is there a general construction for an n-crown, i.e., one crown which adds n clones of a given vertex of an imbedded graph? Is there a construction to add the clone of a subgraph of an imbedded graph?

10. Our approach to the problem of conflicting tube orientations in Section 5.4 and in Chapter VI was really one of avoidance. It seems that bundles of tubes could be shared without requiring uniform orientations. Which pairs of collections of arcchains are compatible in this sense? In particular, which collections are self-compatible?

11. Our work in Section 5.3 seems to put the problem of determining the genus of all complete tripartite graphs within reach. Can the solution be completed using the techniques discussed in this dissertation or using some modification of these? Can these techniques be used to determine genus formulae for other classes of complete n-partite graphs?

12. We have focused our attention on joins and compositions of graphs. Which other classes of graphs might be susceptible to our techniques?
REFERENCES


SURGICAL TECHNIQUES FOR CONSTRUCTING MINIMAL ORIENTABLE IMBEDDINGS OF JOINS AND COMPOSITIONS OF GRAPHS

David L. Craft, Ph.D.

Western Michigan University, 1991

The various cases within the proof of the Heawood Map-Coloring Theorem, which established the genus of the complete graphs, utilized various techniques—some for the first time. This activity spurred interest in determining the genus of various other classes of graphs. However, very few generally applicable techniques have been developed, beyond those used in the proof of this famous theorem. Finding genera of arbitrary graphs remains a very difficult problem.

In this dissertation, we describe two surgical techniques for imbedding graphs. The first construction, called a graphical surface, views an orientable surface as a fattened graph, i.e., vertices become spheres and edges become tubes. We describe how a graph is imbedded in such a surface using a combinatorial structure which we call an arcchain (an ordered set of oriented objects). The second construction, called a crown, is primarily used to add clones of individual vertices of an imbedded graph. These constructions are developed independently but can be used in conjunction with one another.

Our results on joins include new (surgical) proofs of the well known genus formulae for complete bipartite graphs and regular complete 4-partite graphs. Among the new results are several genus formulae for graphs of the form $\overline{K_a} + G$ including large classes of complete tripartite graphs. We also establish genus formulae for all joins of cycles and paths.
A brief study of compositions completes our work. For compositions $G[H]$ we require $H$ to have even order. It is shown that under modest restrictions on $G$, the genus of $G[H]$ is independent of the size and structure of $H$. Other results restrict $H$ but require only that $G$ be connected and triangle-free. We also define the concept of a generalized composition graph and determine genera for several classes of these.