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Graph Products and Covering Graph Imbeddings

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GRAPH PRODUCTS AND COVERING GRAPH IMBEDDINGS

by

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In this dissertation, several graph products of the form $H * G$, with vertex set $V(H * G) = V(H) \times V(G)$, are investigated. Throughout the dissertation the second factor, G, is assumed to be a Cayley graph $G_\Delta(\Gamma)$, for a finite group Γ and a generating set Δ. Each such graph product is regarded as an $|\Gamma|$-fold covering graph of a voltage graph H^*, obtained by modifying the first factor H according to the product $*$ and the graph G.

Chapter I gives a brief history of the graph products that exist in the literature. It also gives an overview of the graph-theoretic terms used in this dissertation with special emphasis on terms of topological graph theory.

In Chapter II new graph products are defined, and the modifications needed to see $H * G$ as an $|\Gamma|$-fold covering graph of a voltage graph H^* are justified. Together with this, examples are given showing the different graph products and their corresponding H^*. A new imbedding technique called the surgical-voltage method is introduced and for comparison an example for each of the three familiar methods is presented.

Chapter III is devoted mostly to establishing necessary and sufficient conditions for the connectedness and bipartiteness of the newly defined graph products. In addition a table containing the sizes of the graph products in terms of the sizes and orders of the factors is given. The last section of this chapter deals with the study of small order graphs relative to these graph products.

In Chapter IV several genus formulae are established using the new imbedding technique developed in Chapter II. As a corollary of one of the results a short proof for the genus of the famous n-cube (whose genus was established by Ringel in 1955, and later independently by Beineke and Harary) is given.

The first section of Chapter V gives isogonal imbeddings of some graph products. In the second section examples are given showing how the new method of graph imbedding can be used in getting an upper bound for the genus of a product graph. In the last chapter some open problems are mentioned.

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Graph products and covering graph imbeddings

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For my parents Abay and Letegerges,
who gave up every thing for their children's education,

my sister Hiwet,
whose sacrifice changed my educational life,

and

In memory and honor of all Eritrean students,
who, because of war and displacement, did not get the opportunity to continue their education.

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CHAPTER I

INTRODUCTION

1.1 History

As one begins to study graphs, an important question to ask is how to produce a new graph from those that are well known, and how to "break up" a given graph into familiar ones. Decomposition and factoring are some of the methods associated with the latter. There are several binary operations used for the former.

In this research we examine and study how to produce new graphs from given graphs. We first consider some of the known graph products for two graphs **H** and **G**. We then introduce some interesting new ones. These are two matters that we consider. In particular, special attention will be given to an imbedding technique for graph products.

Throughout our discussion, the vertex set of a graph product **H*G** of two graphs **H** and **G**, will be **V(H)** × **V(G)**, the cartesian product of the two vertex sets **V(H)** and **V(G)**. There are several graph products that have been studied over the years. Among these are: the **cartesian product** (Sabidussi [23]); the **lexicographic product,** some times known as composition, (Harary [11] and Sabidussi [22]); the **tensor product** (Weichsel [31]); the **disjunction, symmetric difference** and **rejection** (Harary and Wilcox [12]); the **strong cartesian product** defined earlier by Sabidussi and studied further by White ([23], [29], [30]); the **strong tensor product** defined and studied by Garman, Ringeisen and White [8]; and the **augmented tensor product** (White [28]).

In most of these, except for [8] and [28], the focus of the study was on defining and studying various properties other than those connected with imbeddings. As research in topological graph theory increased, researchers showed interest in answering questions related to genus imbeddings and related topological questions about graph products. Some of these were: Ringel in 1955 ([19]) worked on the genus of Q_n, the n-cube, which can be defined as a repeated cartesian product of K_2, the complete graph on two vertices. He found the genus of Q_n to be $1 + 2^{n-3}(n-4)$ for $n \geq 2$. The same result was established by Beineke and Harary in 1965 ([3]) using a different method. Ringel's method was the use of rotation schemes, while that of Beineke and Harary was surgery. White, in 1970, using surgery similar to that of Beineke and Harary, worked on the genus of repeated product of bipartite graphs ([25]). He also worked on the genus of the general product ([29]) and the lexicographic product ([26]). Garman, Ringeisen and White in 1976 studied the genus of the Strong tensor product ([8]). Additional work has also been done on the cartesian product by Pisanski ([17]), Mohar, Pisanski, and White ([16]), and on the tensor product by Železnik ([32]). In 1985 White, while working on the genus of the repeated cartesian product of three triangles ([15]), introduced a new imbedding technique and reported several results on previously defined graph products and a "new" product, the augmented tensor product ([28]). In this technique, the second factor of the two graphs in a graph product is assumed to be a Cayley graph. This technique will be the one that we use in this research for the purpose of finding graph imbeddings and, when possible, a minimal one for a given graph product.

1.2 General Graph Theoretic Definitions and Results

We present a few definitions and working notations that will be needed throughout our discussion. This list of definitions and results is not exhaustive. Several other important definitions and results will be mentioned in later sections and chapters when the need for them arises. For any other definitions and results that are not mentioned here the reader is referred to Chartrand and Lesniak [5] and White [24].

A **graph G** consists of a finite nonempty set **V(G)** of **vertices** together with a possibly empty set **E(G)** of unordered pairs of distinct vertices called **edges.** An edge joining vertex u and v will be denoted by uv. The notations |V(G)|, **p**(G), and **p** when there is no confusion, will be used to denote the cardinality of the vertex set V(G) of G. This number will be called the **order** of G. The **size** of G is the cardinality of the edge set E(G) and will be denoted by |**E(G)**|, **q(G)** and, when it is clear which graph we are talking about, simply as **q**. A **loop** is an edge of the form uu, where u is a vertex, and a **multiple edge** is an edge that appears more than once in E(G). If the edge set E(G) admits loops and/or multiple edges then G is called a **pseudo-graph**. A **walk** of a graph G is an alternating sequence of vertices and edges $v_0, v_0v_1, v_1, \ldots, v_{n-1}, v_{n-1}v_n, v_n$. A **path** is a walk of distinct vertices, and a closed walk with $n \geq 3$ distinct vertices is a **cycle.** A cycle is said to be **even** if it has an even number of vertices. We say a graph G is **connected** if for any two vertices u and v of V(G) there exists a path joining u and v. A **bipartite** graph G is a graph whose vertex set V(G) can be partitioned into two nonempty subsets V_1 and V_2 so that every edge of G has one vertex in V_1 and the other in V_2.

The following notations will be used in this paper: $\mathbf{P_n}$ for the path of n vertices; C_n for the cycle of n vertices, ($n \geq 3$); $\mathbf{K_n}$ for the complete graph of n vertices; $K_{p_1,p_2,\ldots,p_n}$ for the complete n-partite graph with n partite sets having orders p_1, $p_2, \ldots, p_n$. If $p_i = m$, $1 \leq i \leq n$, we write $\mathbf{K}_{p_1,p_2,\ldots,p_n}$ as $K_n(m)$.

Theorem 1.2.1 (The first theorem of graph theory, see [5], p.7). For any graph

$$G, \quad 2q = \sum_{u \in V(G)} \deg(u), \quad \text{where } \deg(u) = |\{v \in V(G) \mid uv \in E(G)\}|.$$

Theorem 1.2.2 (see [5], p.34) A nontrivial graph G is bipartite if and only if all its cycles are even.

1.3 Topological Graph Theory: Preliminary Definitions and Results

In this section we give some of the frequently used definitions and results of topological graph theory, deferring others for later mention as we need them. Others that are not mentioned here can be found in White [24].

By a **surface** we mean a compact orientable 2-manifold. It is known that every surface M is homeomorphic to a sphere with γ (≥ 0) handles. In this case, we write $M = S_\gamma$ and say that M has **genus** γ. A graph G is said to be **imbedded** in a surface M, if it is "drawn" in M so that edges intersect only at their common vertices. The components of $M - G$ are called regions of the imbedding. A region of an imbedding of a graph G in a surface M is said to be **2-cell** if it is homeomorphic to the open unit disk. If every region for an imbedding is a 2-cell, the imbedding is said to be a **2-cell imbedding**. If G is 2-cell imbedded in S_k we will denote it by $\mathbf{G \lhd S_k}$. The number of regions of an imbedding will be denoted by $\mathbf{r}$, and r_i will stand for the number of regions having i sides. The **genus,** $\gamma(G)$, of a graph G is the minimum

genus among all surfaces in which G can be imbedded. An imbedding of a graph G in a surface S_k is said to be a **minimal imbedding** if $\gamma(G) = k$. A 2-cell imbedding is said to be **triangular, quadrilateral** or **n-gonal** if $r = r_3$, $r = r_4$ or $r = r_n$ $(n \geq 3)$ respectively.

The following three theorems can be found in White [24] (p. 48 & 62):

Theorem 1.3.1 Let G be a connected graph 2-cell imbedded in S_k, with p vertices, q edges, and r regions, then

$$p - q + r = 2 - 2k.$$

Theorem 1.3.2 Suppose a graph G having a minimum degree of at least three is 2-cell imbedded in a surface S_k. Let V_i stand for the number of vertices of degree i and let r_i be as defined above, then

$$p = \sum_{i \geq 3} V_i \; ; \qquad r = \sum_{i \geq 3} r_i \; ; \qquad \sum_{i \geq 3} iV_i = 2q = \sum_{i \geq 3} ir_i$$

Theorem 1.3.3 Let G be a connected graph, with $p \geq 3$, then

$$\gamma(G) \geq \frac{q}{6} - \frac{p}{2} + 1.$$

Furthermore, equality holds if and only if G admits a triangular imbedding.

The **girth** g(G) of a graph G with at least one cycle is the length of the shortest cycle in G. The following result is a generalization of the above theorem.

Theorem 1.3.4 Let G be a connected graph with girth $g(G) \geq n$, $n \geq 3$; then

$$\gamma(G) \geq \left(\frac{1}{2} - \frac{1}{n}\right)q - \frac{p}{2} + 1,$$

with equality holding if and only if we have a minimal n-gonal imbedding for G.

Corollary 1.3.5 Let G be a connected bipartite graph, then $\gamma(G) \geq \frac{q}{4} - \frac{p}{2} + 1$, with equality holding if and only if G has a quadrilateral imbedding.

Theorem 1.3.6 (Battle, Harary, Kodama and Youngs; [4]) Let G be a graph with $G_1, G_2, \ldots, G_n$ as its components; then

$$\gamma(G) = \sum_{i=1}^{n} \gamma(G_i)$$

The following corollary follows from Corollary 1.3.5 and Theorem 1.3.6.

Corollary 1.3.7 Let G be a bipartite graph with a quadrilateral imbedding and with n(c) number of components, then

$$\gamma(G) = \frac{q}{4} - \frac{p}{2} + n(c).$$

Definition 1.3.8 A function $\rho: \tilde{X} \to X$ from one path connected topological space to another is called a **covering projection** if every point $x \in X$ has a neighborhood U_x which is evenly covered. This means that ρ maps each component of $\rho^{-1}(U_x)$ homeomorphically onto U_x. If $Y \subseteq X$ and $\tilde{Y} \subseteq \tilde{X}$ is such that ρ maps $\tilde{Y}$ homeomorphically onto Y, we say that Y **lifts** to $\tilde{Y}$. We call $\tilde{X}$ a **covering space** for X. (See White [24] p. 157).

It is known that $|\rho^{-1}(x)|$ is independent of $x \in X$. If $|\rho^{-1}(x)| = n$, then ρ is said to be an **n-fold** covering projection.

Definition 1.3.9 A function $\rho: \tilde{X} \to X$ from one path connected topological space to another is called a **branched** covering projection (and $\tilde{X}$ is a **branched covering space** of X) if there exists a finite set $B \subseteq X$ such that the restricted

function $\rho: \tilde{X} - \rho^{-1}(B) \to X - B$ is a covering projection. The points of B are called the branch points. (See White [24] p. 159).

In Chapter II, we define several graph products; some have been used before, while others are new. We will give examples illustrating some of the known graph imbedding techniques and graphs showing the different graph products. In Chapter III, we will give necessary and sufficient conditions for the connectedness and the bipartiteness of the graph products discussed in Chapter II. In Chapter IV, several genus imbeddings for graph products will be given. All these are achieved by using an imbedding technique called surgical-voltage graph theory. The first section of Chapter V will be devoted to establishing isogonal imbeddings, while the second section will show how the new imbedding technique can be used in establishing good genus upper bounds. The last chapter gives some open questions.

CHAPTER II

GRAPH PRODUCTS DEFINED

2.1 Graph Products: an Overview

There are numerous ways to define a graph product on two given graphs H and G. In all of the graph products that we will be discussing, the vertex set of H*G, the graph product of H and G, will be the cartesian product $V(H) \times V(G)$ of the vertex sets of H and G respectively. Depending on how the edge set is defined, there are exactly 256 distinct graph products; of course, some of these are not as interesting as others and some might be more commonly encountered than others. It is not our intention here to consider all products. We will focus on the graph products that we think are interesting from a topological point of view. Some are very familiar, some will be new.

As was mentioned above, we will consider only a fraction of the possible 256 graph products. We organize our graph products, for the sake of clarity, into two groups: those that can be defined by modifying edges and vertices of H and those that require modifying complementary edges of H. We will focus on the former in Section 2, while the latter will be discussed in section 3.

2.2 Graph Products Using Factor Edges

If H and G are two given graphs, the graph products involving H and G that we are going to discuss will have a vertex set $V(H * G) = V(H) \times V(G)$. We define five types of possible edges $\{(u_1,v_1)\ (u_2,v_2)\}$ for these products:

(i) $u_1 = u_2$ and $v_1v_2 \in E(G)$;

((ii)) $v_1 = v_2$ and $u_1u_2 \in E(H)$;

(iii) $u_1u_2 \in E(H)$;

(iv) $u_1u_2 \in E(H)$ and $v_1v_2 \in E(G)$;

(v) $v_1 \neq v_2$ and $u_1u_2 \in E(H)$.

We note that we can have added $v_1v_2 \in E(G)$ for type (iii)' and $u_1 \neq u_2$ and $v_1v_2 \in E(G)$ for type (v)', but the listed five will prove to be sufficient for the products that we are going to consider in this section.

By taking one or more of the above five types we define ten graph products as given in Table 2.1.

We should mention here that the cartesian, tensor, strong tensor, augmented tensor, strong cartesian, and lexicographic products appear in the literature ([23], [31], [8], [28], [23], and [11]), respectively. Replication is usually mentioned as the union of $|V(H)|$ disjoint copies of G; it is mentioned here for the sake of comparison in what is going to follow. We will not have much to say about it. It is clear that edge type (iii) is abstractly redundant, for it can be replaced by (ii) and (v); we maintain it for ease of expression.

Remark 2.2.1 $H \otimes' G \cong G \underline{\otimes} H$.

Thus we see the augmented tensor product is abstractly redundant, but because we are going to insist on making the second factor a Cayley graph we keep both products under consideration. This will also prove to be helpful in Chapter IV when we try to find genus imbeddings for the two graph products.

Table 2.1

Symbols and Edge Types of the First Ten Graph Products

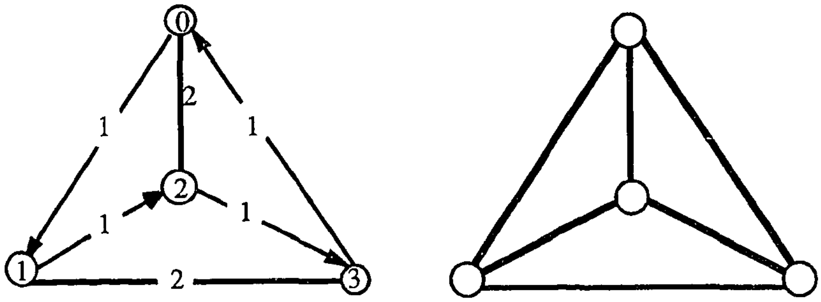
	Name	Symbol	Edge Type
1.	*Replication*	$\lvert V(H)\rvert G$	(i)
2.	*Cartesian*	$H \times G$	(i), (ii)
3.	*Tensor*	$H \otimes G$	(iv)
4.	*Strong Tensor*	$H \underline{\otimes} G$	(i), (iv)
5.	*Augmented Tensor*	$H \otimes' G$	(ii), (iv)
6.	*Strong Cartesian*	$H \underline{\times} G$	(i), (ii), (iv)
7.	*Lexical*	$H \otimes_L G$	(v)
8.	*Strong Lexical*	$H \underline{\otimes}_L G$	(i), (v)
9.	*Sub Lexicographic*	$H \otimes_{SL} G$	(iii) or (ii), (v)
10.	*Lexicographic*	$H[G]$	(i), (iii) or (i), (ii), (v)

We end this section by defining a Cayley graph. Given a finite group Γ with a set Δ of generators of Γ, the **Cayley color graph** $C_\Delta(\Gamma)$ has vertex set Γ, with (g, g') a directed edge (our pair of vertices are ordered) labeled with generator δ_i if and only if $g' = g\delta_i$. We assume that e, the group identity, is not in Δ, and if $\delta_i \in \Delta$ then $\delta_i^{-1} \notin \Delta$ unless δ_i has order 2. In this latter case, the two directed edges $(g,g\delta_i)$ and $(g\delta_i,g)$ are represented by a single undirected edge $[g,g\delta_i]$,

labeled with δ_i. The graph obtained by deleting all labels (colors) and arrows (directions) from the arcs of $C_\Delta(\Gamma)$ is called a **Cayley graph,** $G_\Delta(\Gamma)$. Then $V(G_\Delta(\Gamma)) = \Gamma$ and $E(G) = \{\{g, g\delta\} \mid g \in \Gamma,\ \delta \in \Delta^*\}$; $\Delta^* = \Delta \cup \Delta^{-1}$; $\Delta^{-1} = \{\delta^{-1} \mid \delta \in \Delta\}$.

Example 2.2.1

Let $\Gamma = Z_4$, the cyclic group of order 4, and $\Delta = \{1, 2\}$; then Figure 2.1 (a) is the Cayley color graph $C_\Delta(Z_4)$, while Figure 2.1 (b) is the Cayley graph $G_\Delta(Z_4)$.



$C_\Delta(Z_4)$ (a) $G_\Delta(Z_4)$ (b)

Figure 2.1

2.3 Graph Products Using Complementary Edge Modification

We now define some graph products that involve the complements $\overline{H}$ and $\overline{G}$ of the graphs G and H. A graph $\overline{H}$ is said to be the **complement** of H if $V(\overline{H}) = V(H)$ and the edge set $E(\overline{H}) = \{uv \mid u \neq v, uv \notin E(H)\}$. Some of these products can be defined without the use of the complementary edges; for example, the edge set for product (11) below can be defined as consisting of edges of types (iii), (iii)'. The main reason for such a classification here is based on what needs to be done when the time to modify H comes.

We now define the following edge types in addition to the five types we had above:

(vi) $u_1 = u_2$ and $v_1v_2 \in E(\overline{G})$;

(vii) $u_1u_2 \in E(H)$ and $v_1v_2 \in E(\overline{G})$;

(viii) $v_1 = v_2$ and $u_1u_2 \in E(\overline{H})$;

(ix) $u_1u_2 \in E(\overline{H})$ and $v_1v_2 \in E(G)$;

(x) $u_1u_2 \in E(\overline{H})$ and $v_1v_2 \in E(\overline{G})$.

Using these edge types together with edge types (i)–(v), we define six additional graph products, shown in Table 2.2.

Table 2.2

Symbols and Edge Types of the Next Six Graph Products

	Name	Symbol	Edge Type
11	*Disjunction*	$H \vee G$	(i), (iii), (ix)
12	*Symmetric Difference*	$H \nabla G$	(i), (ii), (vii), (ix)
13	*Rejection*	H / G	(x)
14	*Strong Rejection*	$H \angle G$	(vi), (viii), (x)
15	*Exclusion*	$H \Theta G$	(ii), (vii)
16	*Total Exclusion*	$H \underline{\Theta} G$	(vii)

Since type (iii) is composed of types ((ii), (iv), and (vii)) we can define (11) by (i), (ii), (iv), (vii) and (ix) to display its relationship to (12). However we prefer the brief definition given. There is a difference between $u_1u_2 \notin E(H)$ and $u_1u_2 \in E(\overline{H})$.

Since type (iii) is composed of types ((ii), (iv), and (vii)) we can define (11) by (i), (ii), (iv), (vii) and (ix) to display its relationship to (12). However we prefer the brief definition given. There is a difference between $u_1u_2 \notin E(H)$ and $u_1u_2 \in E(\overline{H})$. In the first case u_1 could be equal to u_2, while in the second case this is not possible. The following remarks follow easily from the definitions.

Remark 2.3.1 $H / G \cong \overline{H} \otimes \overline{G}$;

Remark 2.3.2 $H \angle G \cong \overline{H} \underline{\times} \overline{G}$;

Remark 2.3.3 $H \Theta G \cong H \otimes' \overline{G}$;

Remark 2.3.4 $H \underline{\Theta} G \cong H \otimes \overline{G}$.

Thus products 13-16 are redundant; we retain them, however, as we will wish to maintain $G = G_\Delta(\Gamma)$ in the second factor.

2.4 Imbedding Techniques for Graphs

There are several methods of constructing and describing a graph imbedding. These include **rotation scheme, surgery, voltage graphs** (and their duals **current graphs**). A rotation scheme describes the imbedding of a graph algebraically; voltage graphs and current graphs use covering spaces; surgery typically involves cutting and attaching handles on one or more surfaces of an original imbedding to come up with a new surface and a new imbedding on that surface. Voltage graphs are especially helpful in imbedding Cayley graphs of a group. If a graph is factorable by a product operation, surgery seems to be a good candidate. One usually turns to the

rotation scheme approach only after trying the others without success. The imbedding technique that we are going to use for this research was introduced by White [28], combining the ideas of surgery and voltage graphs. It seems to work especially well for graphs that are factorable by a product operation with the second factor being a Cayley graph. This technique will be the subject of our next section, but before we study it we give examples for the three basic methods so as to appreciate our new method.

Example 2.4.1 Rotation Scheme

Consider the imbedding of $K_{4,4}$ in S_1 depicted in Figure 2.2 below. Let $V(K_{4,4}) = \{0, 1, 2, 3, 4, 5, 6, 7\}$; for each $i \in V(K_{4,4})$ let $N(i) = \{j \mid ij \in E(K_{4,4})\}$. Let π_i: $N(i) \to N(i)$, $0 \leq i \leq 7$, be a cyclic permutation on N(i) of length $n_i = |N(i)|$. We see from this imbedding that $N(0) = N(2) = N(4) = N(6) = \{1, 3, 5, 7\}$ and $N(1) = N(3) = N(5) = N(7) = \{0, 2, 4, 6\}$.

We get also the following cyclic permutations:

$\pi_0 = (1753)$ $\qquad$ $\pi_1 = (0462)$

$\pi_2 = (1357)$ $\qquad$ $\pi_3 = (0264)$

$\pi_4 = (1357)$ $\qquad$ $\pi_5 = (0462)$

$\pi_6 = (1753)$ $\qquad$ $\pi_7 = (0264)$.

Then $P^* = \{\pi_i\}_{i=0}^{7}$ completely describes the imbedding of Figure 2.2. Here $\pi_i(j) = k$ where ik immediately succeeds ij (in the counterclockwise direction) in the imbedding. The regions of $K_{4,4}$ imbedded in S_1 can be determined from these π_i's, using the following method:

Let $D^*=\{(i,j) \mid ij \in E(K_{4,4})\}$, and define $\pi^*: D^* \to D^*$ by $\pi^*(ij) = (i, \pi_j(i))$. Then the orbits of π^* will give the regions of $K_{4,4} \lhd S_1$ (whose boundary edges are traced in clockwise manner). This will give us the following orbits:

Region (1): (0,1) (1,4) (4,3) (3,0) <u>(0,1)</u> abbreviated as 0 – 1 – 4 – 3;

(2): 0 – 7 – 2 – 1; (3): 0 – 5 – 4 – 7; (4): 0 – 3 – 2 – 5;

(5): 3 – 4 – 5 – 6; (6): 1 – 6 – 7 – 4; (7): 1 – 2 – 3 – 6;

(8): 2 – 7 – 6 – 5.

For additional details on rotation schemes see [6] and White [24] Chapter 5.

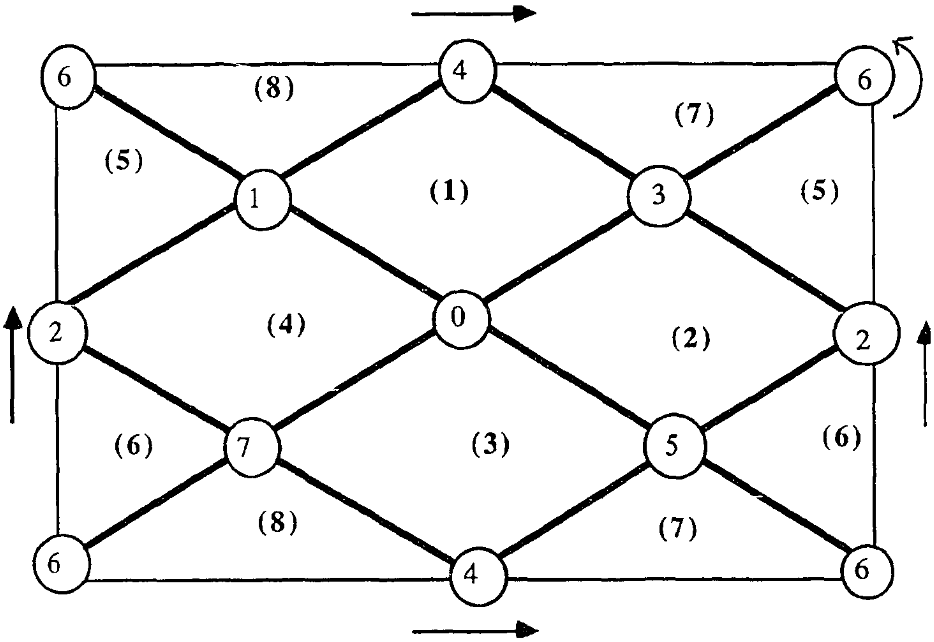


Figure 2.2

To make it easy for us to follow this example, we started with a graph imbedding and proceeded to describe it with a rotation scheme. But in applying this imbedding technique, normally one selects (by skill or by luck) from among the

$\prod_{i=1}^{n}(n_i - 1)!$ possible permutations P^*, one which gives the maximum number r, of orbits, and hence determines the genus k of the graph by using the Euler formula $p - q + r = 2 - 2k$. Thus it is not hard to see why this is a method of last resort.

Now we will imbed $K_{4,4}$ using the voltage graph imbedding technique, but before doing that we define some terms that are going to be helpful in the process of this example and for the work that follows.

Definition 2.4.2 A **voltage graph** is a triple (K, Γ, ϕ) where K is a pseudograph, Γ is a finite group and $\phi: K^* \rightarrow \Gamma$ satisfies $\phi(e^{-1}) = (\phi(e))^{-1}$ for all $e \in K^*$, where $K^* = \{(u,v) \mid uv \in E(K)\}$ and $e = (u,v)$, while $e^{-1} = (v,u)$. The **covering graph** $K \times_\phi \Gamma$ for (K, Γ, ϕ) has vertex set $V(K) \times \Gamma$ and $E(K \times_\phi \Gamma) = \{(u,g)(v,g\phi(e)) \mid e = uv \in E(K) \text{ and } g \in \Gamma\}$. Thus every vertex v of K is covered by $|\Gamma|$ vertices (v,g), $g \in \Gamma$; and each edge $e = uv$ of K is covered by $|\Gamma|$ edges $(u,g)(v,g\phi(e))$, $g \in \Gamma$. Then $K \times_\phi \Gamma$ is an $|\Gamma|$-fold covering graph of K if we regard K, as a topological space.

For additional information on voltage graphs, Chapter 10 of White [24], Chapter 2 of Gross and Tucker [10], and Gross [7] will prove to be helpful.

A 2-cell imbedding of K on a surface S with rotation scheme Π, determines a permutation scheme Π' for $K \times_\phi \Gamma$ S as follows:

If $\pi_v(u) = w$, then $\tilde{\pi}_{(v,g)}((u,g\phi(v,u)) = (w, g\phi(v,w))$. The scheme $\tilde{\pi}$ is said to be the **lift** of the scheme π. If R is a region of $K \triangleleft S$, then $|R|_\phi$ is the order of $\phi(w)$, where w is the boundary of the region R and $\phi(w)$ is the product of the voltages on that boundary (in clockwise order); i.e., $\phi(w) = \prod_{i=1}^{m}\phi(e_i)$, where $w = e_1, e_2, \ldots, e_m = e_1$. If $|R|_\phi = 1$ then **Kirchoff's Voltage Law** (KVL) is said to

The following theorem, due to Alpert and Gross (see [1] and Thm. 10.8 of White [24]), is the central result of voltage graph theory. Thus it should not be a surprise if we appeal to it time and again in our future work.

Theorem 2.4.3 Let (K, Γ, ϕ) be a voltage graph with rotation scheme π and $\tilde{\pi}$ the lift of π to $K \times_\phi \Gamma$. Let π and $\tilde{\pi}$ determine 2-cell imbeddings of K and $K \times_\phi \Gamma$ on the orientable surfaces S and $\tilde{S}$ respectively. Then there exists a (possibly branched) covering projection $\rho\colon \tilde{S} \to S$ such that:

(a) $\rho^{-1}(K) = K \times_\phi \Gamma$;

(b) if b is a branch point of multiplicity n, then b is in the interior of a region R such that $|R|_\phi = n$;

(c) if R is a region of the imbedding of K which is a k-gon, then $\rho^{-1}(R)$ has $\frac{|\Gamma|}{|R|_\phi}$ components, each of which is a $k|R|_\phi$-gon region of the covering imbedding of $K \times_\phi \Gamma$.

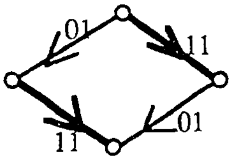
We define the **local group** at v by $\Gamma_v = \{\phi(w) \mid w \text{ is a closed walk at } v\}$. It is known that Γ_v is a subgroup of Γ and $[\Gamma: \Gamma_v]$ is independent of v, if K is connected.

Theorem 2.4.4 (Alpert and Gross [2], see also Thm. 10-7 in White [24]): For a connected voltage graph (K, Γ, ϕ), the number of components of the covering graph $K \times_\phi \Gamma$ is given by $[\Gamma : \Gamma_v]$, for any $v \in V(K)$.

Corollary 2.4.5 For a connected voltage graph (K, Γ, ϕ), the covering graph $K \times_\phi \Gamma$ is connected if and only if $[\Gamma : \Gamma_v] = 1$, for any $v \in V(K)$.

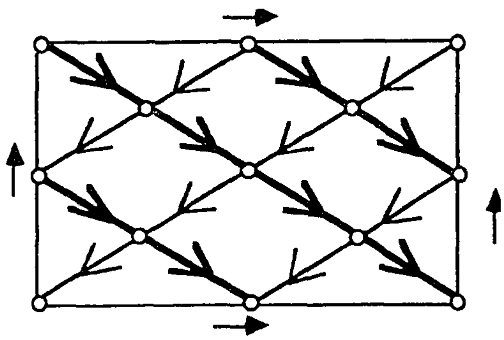
Example 2.4.6 Voltage graphs

Going back to our ongoing example of imbedding $K_{4,4}$, let us take $\Gamma = Z_2 \times Z_4$; $\Delta = \{01,11\}$; we will see below that the voltage graph representation of Figure 2.3 (a) gives all the necessary information about the imbedding of $K_{4,4}$ in Figure 2.3 (b).



(a)

$(K, \Gamma, \phi) \lhd S_1$



(b)

$K_{4,4} = K \times_\phi \Gamma \lhd S_1$

Figure 2.3

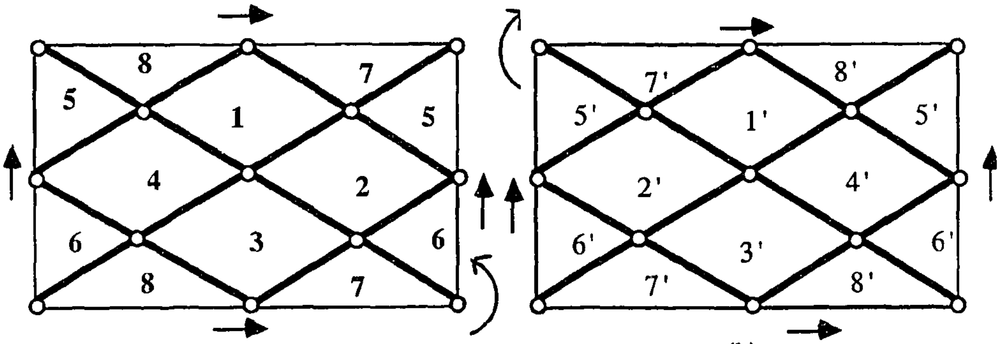
We should note here that K has 1 vertex and $K \times_\phi \Gamma$ has $8 = |\Gamma|$ vertices, K has 2 edges and $K \times_\phi \Gamma$ has $16 = 2|\Gamma|$ edges. There is only one 4-gon, R, in K and it satisfies the KVL; i.e., $|R|_\phi = 1$ and we see that there are $\frac{|\Gamma|}{|R|_\phi} = \frac{8}{1} = 8$ $4|R|_\phi$-gons, i.e., 8 4-gons in the lift $K \times_\phi \Gamma$ of K. Let us also observe that $\tilde{\pi}_g = (g+\delta_1, g+\delta_2, g-\delta_1, g-\delta_2)$, where $\delta_1 = 01$ and $\delta_2 = 11$, for every $g \in \Gamma = Z_2 \times Z_4$.

Thus with the knowledge of p, q, r of $K \times_\phi \Gamma$, we can find the genus of $\tilde{S}$ where $K \times_\phi \Gamma$ is imbedded. The beauty of this method (voltage graph theory) is that a simpler imbedding of K below gives much information about the more complicated

imbedding of $K \times_{\phi} \Gamma$ above. This is a tremendously powerful device for topological graph theory.

Example 2.4.7 Surgery

To demonstrate how surgery in graph imbeddings works let us try to find $\gamma(K_2 \times K_{4,4})$. We start by an imbedding of $K_{4,4}$ on S_1 and a mirror image imbedding of the same graph on another S_1 as shown on Figure 2.4.



(a) (b)

Figure 2.4

We observe that we have a set {1,6} of 2 vertex-disjoint quadrilateral regions in the imbedding of Figure 2.4(a). This set contains all 8 vertices of $K_{4,4}$. We could have chosen {2,8} or {3,5} or {4,7} just as well. Similarly we get the set {1',6'} in the imbedding of Figure 2.4 (b); again here our choice could as well be {2', 8'} or {3',5'} or {4',7'} respectively. We first cut one disk in each of the regions 1, 6, 1' and 6'; then we attach two tubes. The first tube will run from the disk cut in region 1 to the one in region 1', while the second will do the same with regions 6 and 6'. Next we join the four vertices of region 1 to the corresponding ones on region 1' and those of region 6 to the corresponding ones of region 6' by running edges

through the respective tubes, four edges per tube. The mirror image construction allows this to be done without the edges intersecting one another. The resulting graph is $K_2 \times K_{4,4}$ and it is easy to see that all the regions of this imbedding of $K_2 \times K_{4,4}$ are quadrilateral. Since $K_2 \times K_{4,4}$ is bipartite, our imbedding is minimal. The genus of the new surface for which this minimal imbedding was achieved can be calculated by using the relation $\gamma = 2k + n - 1$, where k is the genus of the original surface S_k and n is the number of tubes added to create the new surface. Thus our surface is S_3 and our graph has genus 3. Figure 2.5 illustrates our last calculation. We can see that in addition to the two holes numbered (1) and (2) from the original two surfaces, the two handles (tubes) that we added gave another hole that is numbered here as (3). Thus the resulting surface is a sphere with three handles.

We could have reached the same conclusion as to the genus of the graph, once we knew that we had a minimal quadrilateral imbedding, by using Corollary 1.3.5. This is because $p(K_2 \times K_{4,4}) = 16$, $q(K_2 \times K_{4,4}) = 40$, and using $\gamma = \frac{q}{4} - \frac{p}{2} + 1$, we get 3.

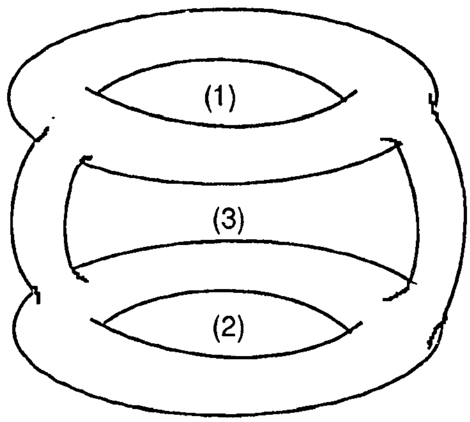


Figure 2.5

Examples 2.4.1, 2.4.6, and 2.4.7 showed the different methods of constructing imbeddings. In the next section we are going to discuss a new method due to White [28]. This method is a marriage of surgery and voltage graph theory. We will use its power in our pursuit of genus imbeddings for different graph products. To help us utilize the full power of this method we shall always assume and insist (unless stated otherwise) that the second factor, G, of H*G be a Cayley graph $G_\Delta(\Gamma)$ for some finite group Γ and an appropriate generating set Δ.

2.5 Surgical-Voltage Imbedding Technique

Let Γ be a finite group with a set of generators Δ; $\Delta^{-1} = \{\delta^{-1} \mid \delta \in \Delta\}$ and $\Delta^* = \Delta \cup \Delta^{-1}$. Defining $G = G_\Delta(\Gamma)$, we regard H*G as an $|\Gamma|$-fold covering graph of the voltage graph (H^*, Γ, ϕ) obtained by modifying H. Once H is modified appropriately it will be lifted to $H^* \times_\phi \Gamma$, which will be the graph product H*G. This modification and its justification will be the main thrust of this section. This new technique is called a surgical-voltage method because it combines the concepts of both methods. We start with an appropriate imbedding of H on some surface S; this is similar to what we do in surgery. Even though we look for a "suitable" type of imbedding in S we do not require the imbedding of H in S to be minimal, as for the usual surgical application. After getting the "suitable" imbedding of H in S, we will modify it using the scheme discussed below. This modification is dependent on the graph product, the group Γ, and the set of generators Δ. The last two choices are of course determined by what type of graph G is. After this the properties of H*G, especially those that pertain to graph imbeddings, are studied using the theory of voltage graphs.

We proceed to modify H according to the product involved. In this process, as in our defining sections, we present first the ten graph products, 1–10, that require the modification of vertices and edges of H. We then consider the six products, 11–16, that require additional modification on edges that are in the complement of H, or vertices of H using loops for group elements other than those in Δ.

Table 2.3 gives the modifications that are necessary to each vertex **u** of **V(H)** and edge **uv** of **E(H)** in order to see **H * G** as an $|\Gamma|$-fold covering of the resulting voltage graph H^*, for the ten products of section 2.2. These modifications will be justified in detail after giving examples for each product.

Table 2.3

Modifications of the Vertices and Edges of H for the First Ten Graph Products

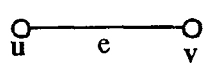
	Product	$u \in V(H)$		$uv \in E(H)$
1.	Replication		$\forall\delta \in \Delta$	
2.	Cartesian		$\forall\delta \in \Delta$	
3.	Tensor			 $\forall\delta \in \Delta^*$

Table 2.3— Continued

	Product	u ∈ V(H)		uv ∈ E(H)
4.	Strong Tensor	$\underset{u}{\overset{\delta}{\circlearrowleft}}$	$\forall\delta \in \Delta$	$u \xrightarrow{\delta} v$ $\forall\delta \in \Delta^*$
5.	Augmented Tensor	$\underset{u}{\circ}$		$u \xrightarrow{\delta} v$ $\forall\delta \in \Delta^* \cup \{e\}$
6.	Strong Cartesian	$\underset{u}{\overset{\delta}{\circlearrowleft}}$	$\forall\delta \in \Delta$	$u \xrightarrow{\delta} v$ $\forall\delta \in \Delta^* \cup \{e\}$
7.	Lexical	$\underset{u}{\circ}$		$u \xrightarrow{g} v$ $\forall g \in \Gamma - \{e\}$
8.	Strong Lexical	$\underset{u}{\overset{\delta}{\circlearrowleft}}$	$\forall\delta \in \Delta$	$u \xrightarrow{g} v$ $\forall g \in \Gamma - \{e\}$
9.	Sub Lexicographic	$\underset{u}{\circ}$		$u \xrightarrow{g} v$ $\forall g \in \Gamma$
10.	Lexicographic	$\underset{u}{\overset{\delta}{\circlearrowleft}}$	$\forall\delta \in \Delta$	$u \xrightarrow{g} v$ $\forall g \in \Gamma$

Example 2.5.1

Let us take **H** to be $\mathbf{P_3}$, a path with three vertices, and **G** to be $\mathbf{C_4}$, a cycle of four vertices. Here we should note that $\mathbf{G} = \mathbf{C_4} = \mathbf{G_\Delta(Z_4)}$, where $\Delta = \{1\}$. With these two given factor graphs, we present in Figure 2.7 ten graphs corresponding to the ten products that are under discussion. Below each graph there is a corresponding voltage graph $\mathbf{H^*}$ obtained by modifying **H** as was mentioned earlier. An edge without direction represents a group element of order two, i.e., $\delta = \delta^{-1}$. In all the other cases each group element or generator is represented by a directed arc.

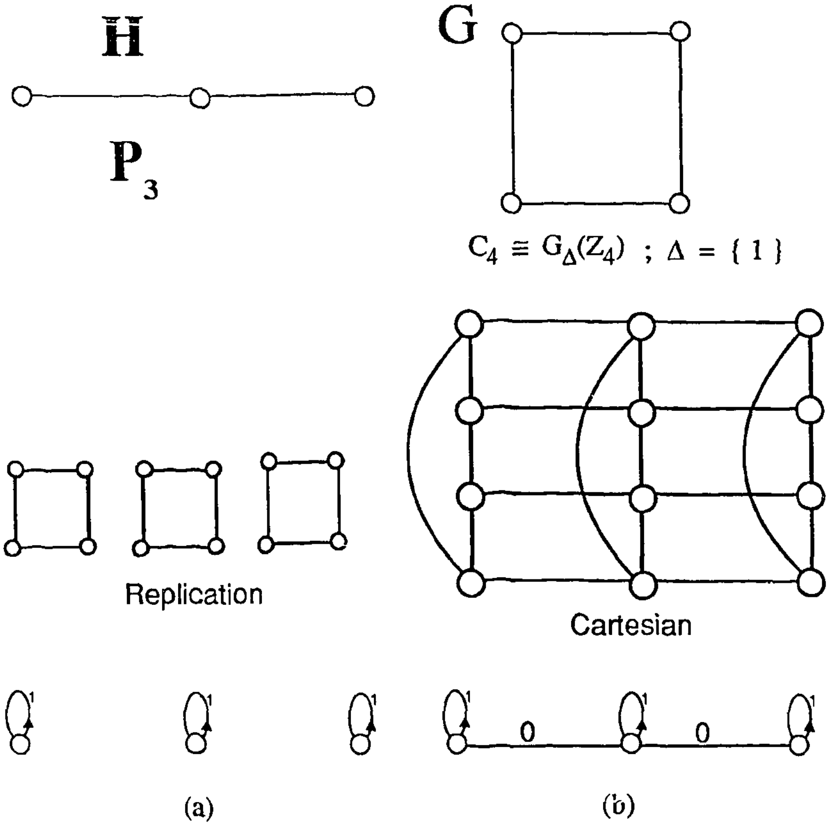


Figure 2.7

Figure 2.7 — Continued

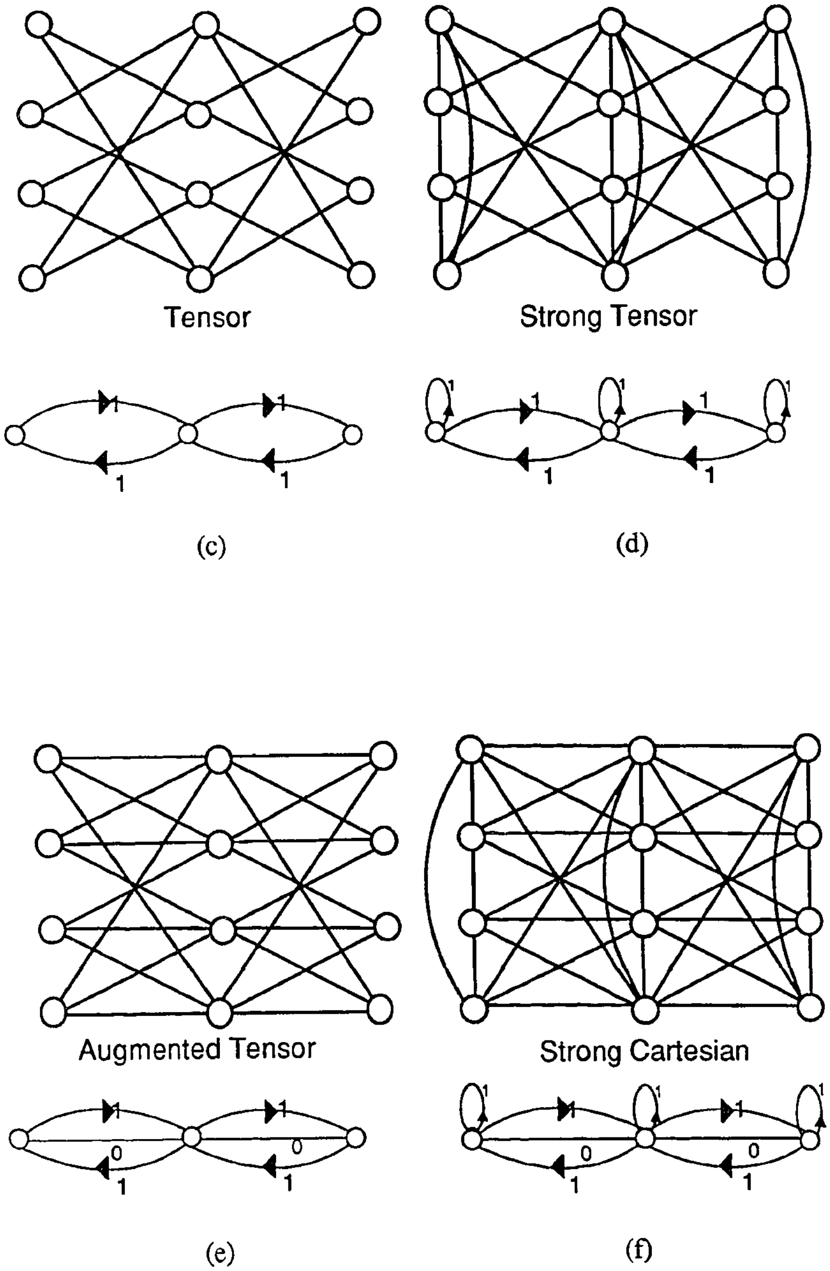
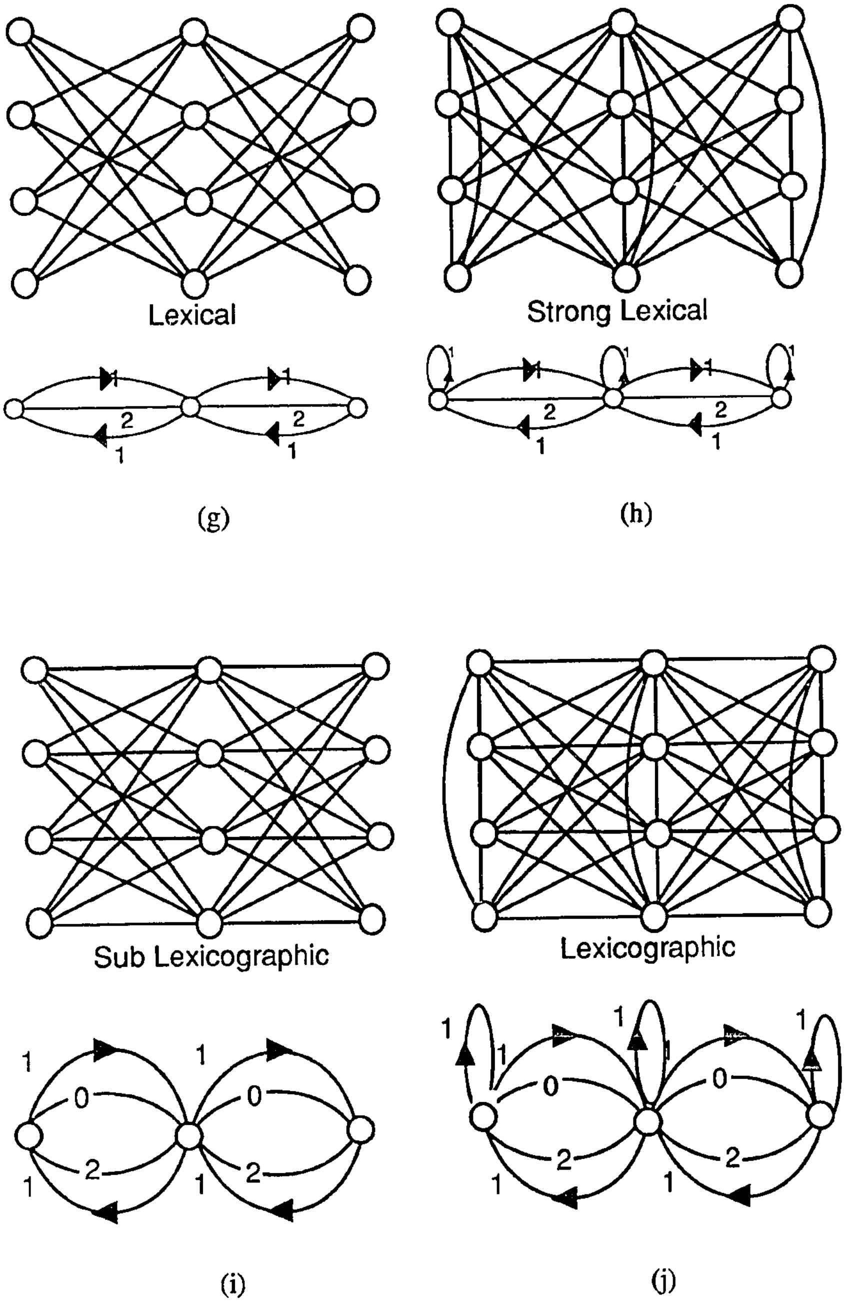


Figure 2.7 — Continued



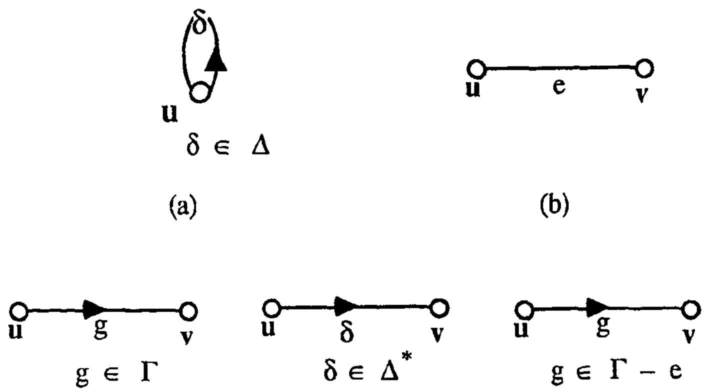
From the above examples we see that each product was represented by an H^* of order 3; thus H*G, which is the covering graph $H^* \times_\phi \Gamma$, has $3|\Gamma| = 3(4) = 12$ vertices for all ten products. This is consistent with $|V(H{*}G)| = |V(H) \times V(G)| = 12$. From the theory of voltage graphs, we find that $|E(H{*}G)| = |E(H^* \times_\phi \Gamma)| = |\Gamma| q(H^*)$; $|\Gamma| = 4$ for all. These are:

Replication product, $2|\Gamma| = 8$ edges; cartesian product, $5|\Gamma| = 20$; tensor product, $4|\Gamma| = 16$; strong tensor product, $7|\Gamma| = 28$; augmented tensor product, $6|\Gamma| = 24$; strong cartesian product, $9|\Gamma| = 36$; lexical product, $6|\Gamma| = 24$; strong lexical product, $9|\Gamma| = 36$; sublexicographic product, $8|\Gamma| = 32$; and the lexicographic product, $11|\Gamma| = 44$.

By considering the sizes of these graphs we observe that most are distinct except for those of the tensor product and the lexical product, and the strong cartesian product and the strong lexical product, which have the same number of edges pair-wise. Further examination of the latter graphs shows that they are indeed isomorphic (in the preceding pairs). This is coincidental, because in the general case $|\Delta^* \cup \{e\}| \neq |\Gamma - \{e\}|$ and the number of involutions (elements of order two) in $\Gamma - \{e\}$, is not necessarily equal to those in $\Delta \cup \{e\}$, as is the case with these two pairs of isomorphic graph products in our example. Thus, in general these four graph products are distinct as well.

The next lemma and the theorem that follows it will justify the modifications of H given by Table 2.3 to obtain H^* and why we view H*G as being the covering graph $H^* \times_\phi \Gamma$ of H^*.

Let Figure 2.8 give the modifications we will employ, to obtain H^* from H.



(a) (b)

(c) (d) (e)

Figure 2.8

Recall that we had the following five edge types in Section 2.2.

(i) $u_1 = u_2$ and $v_1v_2 \in E(G)$;

(ii) $v_1 = v_2$ and $u_1u_2 \in E(H)$;

(iii) $u_1u_2 \in E(H)$;

(iv) $u_1u_2 \in E(H)$ and $v_1v_2 \in E(G)$;

(v) $v_1 \neq v_2$ and $u_1u_2 \in E(H)$.

Lemma 2.5.2 Given two graphs H and G, where $G = G_\Delta(\Gamma)$, then the five edge types: (i), (ii), (iii), (iv), and (v), defined above are, respectively, the covers of the five modifications a, b, c, d, and e of Figure 2.8 in $H^* \times_\phi \Gamma$.

Proof:

Case (a): Let us say that u, a vertex of H, is modified as in Figure 2.8 (a); then we show that this modification stands for the type (i) edges. From the theory of voltage graphs a loop on u, for each $\delta \in \Delta$, is covered by the edge set $\{(u,g)(u,g\delta) \mid g \in \Gamma\}$ in $H^* \times_\phi \Gamma$. This means $\{u_1 = u_2$ and $\{g, g\delta\} \in E(G)\}$. But because G is a

Cayley graph, $G_\Delta(\Gamma)$, we get $\{u_1=u_2$ and $v_1v_2 \in E(G)\}$. These are exactly type (i) edges.

Case (b): If an edge u_1u_2 of H is modified as shown in Figure 2.8 (b), then its cover in $H^* \times_\phi \Gamma$ consists of $|\Gamma|$ edges of the form: $(u_1, g)\,(u_2, ge)$, $g \in \Gamma$. This means $v_1=v_2$ and $u_1u_2 \in E(H)$, and these are exactly the type (ii) edges.

Case (c): If $u_1u_2 \in E(H)$ is modified as seen on (c) of Figure 2.8, then its lift in $H^* \times_\phi \Gamma$ will be the edges of the type: $(u_1, h)\,(u_2, hg)$, where $h, g \in \Gamma$. This is the same as saying $u_1u_2 \in E(H)$ and we have type (iii) edges.

Case (d): If we modify $u_1u_2 \in E(H)$ as on Figure 2.8 (d), then the cover in $H^* \times_\phi \Gamma$ consists of the edges $(u_1,g)\,(u_2,g\delta)$, $g \in \Gamma$ and $\delta \in \Delta^*$. This means $u_1u_2 \in E(H)$ and $\{g, g\delta\} \in E(G)$, and because G is a Cayley graph $\{g, g\delta\} \in E(G)$ means $v_1v_2 \in E(G)$. Hence the cover of this modification in $H^* \times_\phi \Gamma$ corresponds to those edges defined as type (iv) edges, or the tensor edges.

Case (e): If each edge u_1u_2 of H is replaced by an arc labeled g for each $g \in \Gamma - \{e\}$ as in Figure 2.8 (e) to get H^*, then its lift in $H^* \times_\phi \Gamma$ is the set of edges: $(u_1, h)(u_2, hg)$, $h \in \Gamma$ and $g \in \Gamma - \{e\}$. This means $u_1u_2 \in E(H)$ and $v_1 \neq v_2$, which are type (v) edges. ❑

Theorem 2.5.3 If H is modified as shown in Table 2.3 to give H^* for each graph product H*G, then $H^* \times_\phi \Gamma = H*G$ for all ten products 1-10, respectively.

Proof: We consider each case:

1. The Replication: If our product is the replication product, we have modification (a) of Figure 2.8; however by Lemma 2.5.2 this modification is

covered by type (i) edges. This was exactly how we defined the replication product in Table 2.1.

2. **The Cartesian Product:** If we are working with the cartesian product, according to table 2.3 our modification will involve two things: modification type (a) for each vertex of H and modification (b) for each edge of H. These, from Lemma 2.5.2, give rise to edge types (i) and (ii), respectively; these are exactly the type of edges used in defining the cartesian product.

3. **The Tensor Product:** The tensor product involves modification (d), which in turn is covered by edge type (iv); this is in harmony with the tensor edge definition by our lemma.

4. **The Strong Tensor Product:** If H uses modifications (a) and (d), by Lemma 2.5.2 the cover in $H^* \times_\phi \Gamma$ will be type (i) and type (iv) edges, respectively, and these are the edge types for the strong tensor product.

5. **The Augmented Tensor Product:** Modifications (b) and (d) were the ones mentioned for this product; these are lifted to edge types (ii) and (iv), respectively, in $H^* \times_\phi \Gamma$, which are the edge types used in defining the augmented tensor product.

6. **The Strong Cartesian Product:** If H^* is H modified using (a), (b) and (d), then by Lemma 2.5.2 they will be lifted to type edges (i), (ii), and (iv), respectively; these are the ones used to define the strong cartesian product.

7. **The Lexical Product:** If we modify H as specified in Table 2.3, our modification will be that of (e) in Figure 2.8. By our lemma this will be lifted to

edge type (v) in $H^* \times_\phi \Gamma$, but this means we have the lexical product for our product graph.

8. The Strong Lexical Product: According to Table 2.3, H has to be modified using (a) and (e) of Figure 2.8; these from Lemma 2.5.2 give type edges (i) and (v) in $H^* \times_\phi \Gamma$; these are exactly the edge types used in defining the strong lexical product.

9. The Sublexicographic Product: Let us modify H as in Figure 2.8 (c); then the covering edges in $H^* \times_\phi \Gamma$ will be those of type (iii). This means our product graph is the sublexicographic product.

10. The Lexicographic Product: Finally, if we modify H using modifications (a) and (c) of Figure 2.8, then our cover graph consists of edge types (i) and (iii), which are exactly those edges used in defining the lexicographic product. ❑

We are now going to show the necessary modifications of H for the graphs defined in Section 2.3. In this case, in order to regard H*G as an $|\Gamma|$-fold covering of H^*, a modification of vertices and edges of H is not going to be sufficient; we have to also modify some edges that are in the complement of H. All the product graphs defined in Section 2.3 require either the modification of edges in the complement of H or the vertices of H using loops that are not in Δ^*. This was part of the reason why we classified them under products using complementary edges.

To simplify our notation we are going to define $\Omega^* = \Gamma - (\Delta^* \cup \{e\})$ and Ω as follows: $\Omega \subseteq \Omega^*$ and if $\omega \in \Omega$ then $\omega^{-1} \notin \Omega$ unless $\omega = \omega^{-1}$.

Let us recall the following edge types we had in Section 2.3:

(vi) $u_1 = u_2$ and $v_1v_2 \in E(\overline{G})$;

(vii) $u_1u_2 \in E(H)$ and $v_1v_2 \in E(\overline{G})$;

(viii) $v_1 = v_2$ and $u_1u_2 \in E(\overline{H})$;

(ix) $u_1u_2 \in E(\overline{H})$ and $v_1v_2 \in E(G)$;

(x) $u_1u_2 \in E(\overline{H})$ and $v_1v_2 \in E(\overline{G})$.

Let Figure 2.9 give the modifications we employ to obtain H^* from H. In (g) $uv \in E(H)$, while in (h), (i), and (j), $uv \in E(\overline{H})$. In the listing we are avoiding the use of modification labeled (i) to prevent confusion with edge type (i).

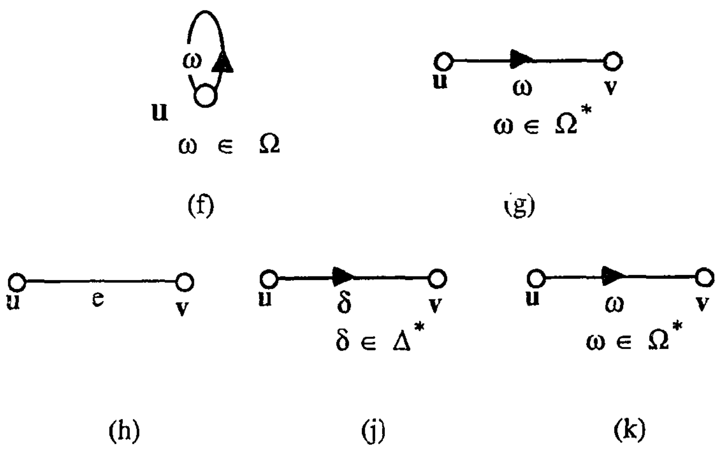


Figure 2.9

Lemma 2.5.4 Given two graphs H and G, where $G = G_\Delta(\Gamma)$, if (g) above is a modification of an edge of H while those of (h), (j), and (k) are the edges in the complement of H, then the five edge types: (vi), (vii), (viii), (ix), and (x), defined above are, respectively, the covers of the five modifications (f), (g), (h), (j), and (k) of Figure 2.9 in $H^* \times_\phi \Gamma$.

Proof:

Case (f): A loop labeled $\omega \in \Omega$ at u of H as in (f) of Figure 2.9 implies that its cover will be (u, g) (u, gω), $\omega \in \Omega$, which means $u_1 = u_2$ and $v_1v_2 \in E(\overline{G})$. These are exactly edge type (vi).

Case (g): If u_1u_2, an edge of H, is modified as shown in Figure 2.9 (g), then the edges in its lift are of the following type: $(u_1, g)\,(u_2, g\omega)$, $\omega \in \Omega^*$. This means $u_1u_2 \in E(H)$ and $v_1v_2 \in E(\overline{G})$, and these are type (vii) edges.

Case (h): If edge u_1u_2 of $\overline{H}$ is modified as indicated in Figure 2.9 (h), then the cover edges in $H^* \times_\phi \Gamma$ are: $(u_1, g)\,(u_2, ge)$. These are the same as those of the type $v_1 = v_2$ and $u_1u_2 \in E(\overline{H})$, and these were the edges we referred to as type (viii).

Case (j): If edge u_1u_2 of $\overline{H}$ is modified as indicated in Figure 2.9 (i), then the cover edges in $H^* \times_\phi \Gamma$ are: $(u_1, g)\,(u_2, g\delta)$. These are of type $u_1u_2 \in E(\overline{H})$ and $v_1v_2 \in E(G)$, and these we referred to as type (ix) edges.

Case (k): If edge u_1u_2 of $\overline{H}$ is modified as indicated in Figure 2.9 (j), then the cover edges in $H^* \times_\phi \Gamma$ are: $(u_1, g)\,(u_2, g\omega)$. This means $u_1u_2 \in E(\overline{H})$ and $v_1v_2 \in E(\overline{G})$, hence we have edges of type (x). ❑

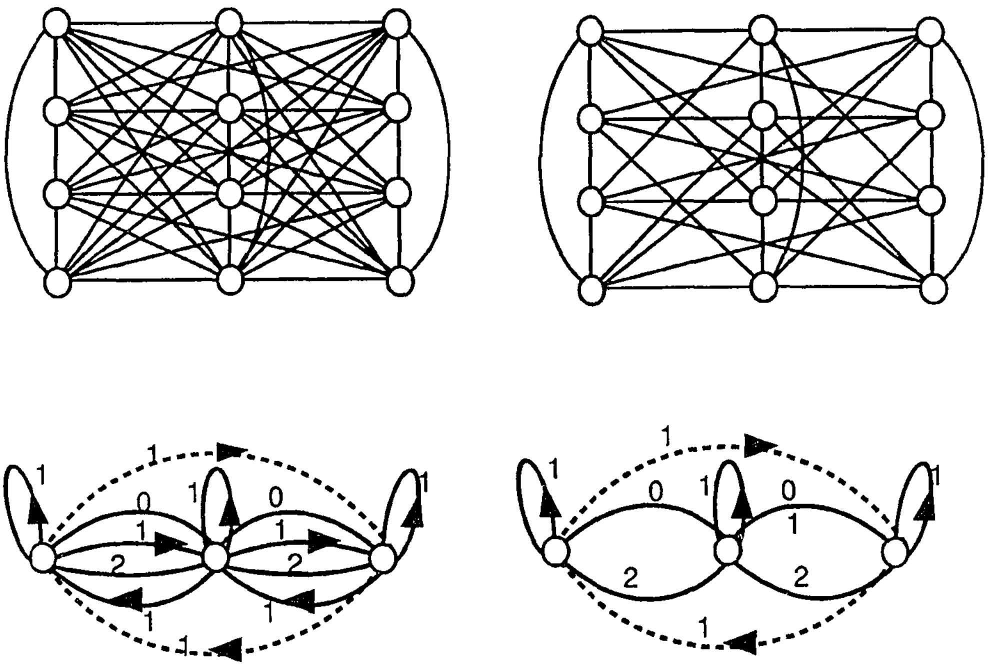
Table 2.4 gives the necessary modifications needed to get the voltage graph H^* from H. This makes H*G to be viewed as $H^* \times_\phi \Gamma$ for each of the last six products 11- 16. These modifications come from the edge type definitions and their corresponding modifications in Figure 2.8 and Figure 2.9.

Table 2.4

Modifications of the Vertices and Edges of H for the Last Six Graph Products

	Product	$u \in V(H)$	$uv \in E(H)$	$uv \in E(\overline{H})$
11.	Disjunction	$\overset{\delta}{\circlearrowleft} u$ $\quad \forall \delta \in \Delta$	$u \xrightarrow{g} v$ $\forall g \in \Gamma$	$u \xrightarrow{\delta} v$ $\forall \delta \in \Delta^*$
12.	Symmetric Difference	$\overset{\delta}{\circlearrowleft} u$ $\quad \forall \delta \in \Delta$	$u \xrightarrow{\omega} v$ $\omega \in \Omega^* \cup \{e\}$	$u \xrightarrow{\delta} v$ $\forall \delta \in \Delta^*$
13.	Rejection	u		$u \xrightarrow{\omega} v$ $\omega \in \Omega^*$
14.	Strong Rejection	$\overset{\omega}{\circlearrowleft} u$ $\quad \forall \omega \in \Omega$		$u \xrightarrow{\omega} v$ $\omega \in \Omega^* \cup \{e\}$
15.	Exclusion	u	$u \xrightarrow{\omega} v$ $\omega \in \Omega^* \cup \{e\}$	
16.	Total Exclusion	u	$u \xrightarrow{\omega} v$ $\omega \in \Omega^*$	

Example 2.5.5: To illustrate our six products, 11-16, let us take $H = P_3$ and $G = G_\Delta(Z_4)$, $\Delta = \{1\}$ as in Example 2.5.1. Then the six graphs and their corresponding modifications are shown in Figure 2.10. Here the dashed arcs (edges, when δ or ω has order two) represent modifications on edges that belong to the complement of H. The solid arcs or edges stand for the modifications done to edges of H.



Disjunction

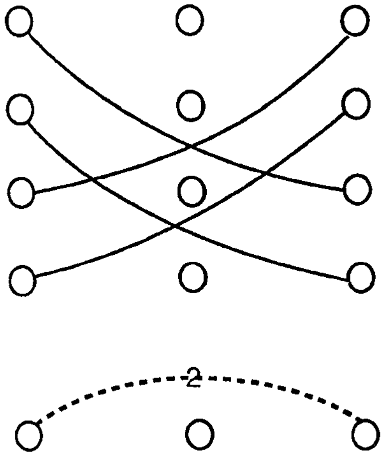
(a)

Symmetric Difference

(b)

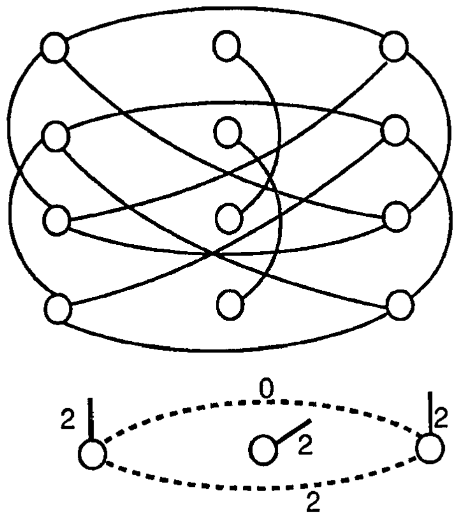
Figure 2.10

Figure 2.10 — Continued



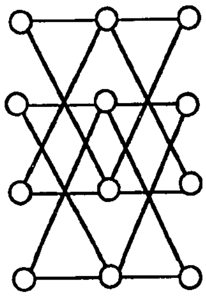
Rejection

(c)

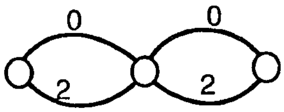


Strong Rejection

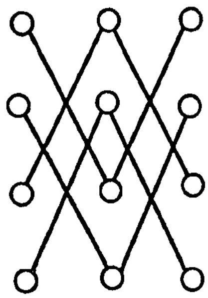
(d)



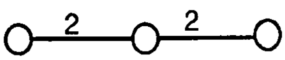
Exclusion



(e)



Total Exclusion



(f)

We observe that in each case H^* contains important information about $H*G$, when the latter is seen as $H^* \times_\phi \Gamma$. When our product is disjunction, H^* has 13 edges and a simple count of the edges of $H \vee G$ shows that we have 52 edges; this is exactly $13\,|\Gamma|$. In the case of symmetric difference H^* has 9 edges; this agrees with 36 which is $9\,|\Gamma|$ for the graph product. If our graph product is the rejection, then H^* has only one edge, but we can easily see that H/G has 4 edges which is the same as $|\Gamma|$. For the strong rejection in H^* we have $\frac{7}{2}$ edges (here each half edge carries an involution); and multiplication by $|\Gamma|$ gives us the desired 14 edges for the strong rejection graph. Similarly, the exclusion and the total exclusion have four and two edges for H^* respectively, giving 16 and 8 edges for the respective graph products.

By proving the following theorem we will justify Table 2.4 for each of the six products 11-16.

Theorem 2.5.6 If H is modified to give H^* for each graph product $H*G$ as shown in Table 2.4, then $H^* \times_\phi \Gamma = H*G$ for all six products 11–16 respectively.

Proof:

11. Disjunction: From Table 2.4 we see that the modifications needed on H consist of (a) and (c) of Figure 2.8 and (j) of Figure 2.9; but from Lemmas 2.5.2 and 2.5.4 these are covered by type edges (i), (iii), and (ix), respectively. These were the type edges we used in defining the disjunction of two graphs, in Table 2.2.

12. Symmetric difference: We have modifications (a) and (b) of Figure 2.8 and modifications (g) and (i) of Figure 2.9 for this case; but Lemma 2.5.2 tells us

that in $H^* \times_\phi \Gamma$ we have the edge types (i) and (ii) covering (a) and (b), respectively, while Lemma 2.5.4 tells us that type (vii) covers (g) and type (ix) covers (j). These were the edge types used in defining the symmetric difference of two graphs.

13. Rejection: This, according to Table 2.4, involves the use of modification (k) of Figure 2.9, which from Lemma 2.5.4 is covered by type (x) in $H^* \times_\phi \Gamma$. This was exactly how rejection was defined.

14. Strong rejection: The modifications employed in this case were those of (f), (h), and (k) of Figure 2.9, which by Lemma 2.5.4 are covered in $H^* \times_\phi \Gamma$ by edge types (vi), (viii), and (x), respectively. These were the edge types used in defining the strong rejection of two graphs.

15. Exclusion: According to Table 2.4 we need modification (b) of Figure 2.8 and (g) of Figure 2.9; but then these will be covered by edge types (ii) and (vii) by applying Lemma 2.5.2 and 2.5.4, respectively. This gives us the definition needed for the exclusion of two graphs.

16. Total exclusion: Finally, if we modify H as in Table 2.4 (16) we see that we have used only (g) of Figure 2.9, and this agrees with our definition (type (vii) only) for the total exclusion of two graphs. ❑

It would be nice, but too much to ask, if the graphs in Examples 2.5.1 and 2.5.5 could be imbedded in some surface, perhaps the surface S_0, so that all these graphs can be drawn on the plane without crossing. Instead, for a given graph we will find, whenever possible, a minimum k so that the graph can be imbedded in S_k. In short, we will find the genus of these and other related graph products. In most

cases our tool will be surgical-voltage graph theory; in addition, it will prove useful to know when our graph product is connected, and if it is bipartite. These and other properties will be discussed in the next chapter.

CHAPTER III

BASIC PROPERTIES OF GRAPH PRODUCTS

In this chapter we discuss some of the main properties of graph products that we will need in Chapter IV and Chapter V. Knowing whether a graph is connected, and if disconnected, how many components it has, and if these components are isomorphic, is crucial in graph imbeddings. Another important property in graph imbeddings is bipartiteness, for then the improved lower bound of Corollary 1.3.5 applies. In addition to studying these two properties we will calculate the degree of each vertex in a graph product and the size of the product itself. Degrees and sizes will be studied in Section 1, connectedness in Section 2, bipartiteness in Section 3, and Section 4 will be devoted to studying general properties and observations about our different product graphs.

3.1 Degrees and Sizes

Some results that will be discussed here appear in Harary and Wilcox [12]. Because of their fundamental nature they appear elsewhere in the literature as well. These are included here for completeness, and because the method of proof used in establishing some of them is new. The results for the cartesian, tensor, and lexicographic products together with those of disjunction, symmetric difference and rejection can be found in Harary and Wilcox [12]. The strong tensor is considered in Garman, Ringeisen and White [8]. The results for the remaining products appearing here, we believe, are new.

We continue to regard a vertex (u, v) of $V(H^*G)$ to mean that $u \in V(H)$ and $v \in V(G)$. By q_H or q_G we mean the size of H or G, respectively, while $\deg_H u$ will stand for the degree of vertex u in H and deg(u,v) will mean the degree of the vertex (u, v) in H^*G. Since we are viewing H^*G as a covering graph of H^* we will use H^* in our proof to establish the degree of a vertex and from there use the first theorem of graph theory in computing the size of the product graph. The proofs of most of these results are similar; thus we only provide a sample proof for some of the graph products and the reader can provide the rest.

Theorem 3.1.1 The degree of the vertex (u,v) in $H \otimes' G$ is given by:

$$\deg(u,v) = \deg_H u\,[\deg_G v + 1].$$

Proof: We recall that every edge uu_1 of H was replaced by $|\Delta^*| + 1$ edges; thus every vertex (u,v) of $H \otimes' G$ will have $\deg_H u\,[\,|\Delta^*| + 1\,]$ neighbors. But because G is a Cayley graph, $|\Delta^*| = \deg_G v$. Thus the degree of (u,v) in $H \otimes' G$ is $\deg_H u\,[\,\deg_G v + 1\,]$. ❑

Let us note here that the above result remains the same in general even if G is not a Cayley graph. This is true because at every vertex (u,v) of H^*G we have the following edges to consider: $v_1 = v_2$ and $u_1u_2 \in E(H)$, and $u_1u_2 \in E(H)$ and $v_1v_2 \in E(G)$. We have $\deg_H u$ edges of the first type and $\deg_H u \bullet \deg_G v$ edges of the second type. All together we have $\deg_H u + \deg_H u \bullet \deg_G v = \deg_H u\,[\,\deg_G v + 1\,]$ edges at (u, v).

Corollary 3.1.2 $q(H \otimes' G) = q_H\,[\,2q_G + p_G\,]$.

Proof: By the first theorem of graph theory,

$$2q(H \otimes' G) = \sum_{u \in V(H)} \sum_{v \in V(G)} \deg_H u \, [\, \deg_G v + 1 \,]$$

$$= \sum_{u \in V(H)} \deg_H u \, [\, 2q_G + p_G \,]$$

$$= 2q_H [\, 2q_G + p_G \,]$$

thus $$q(H \otimes' G) = q_H [\, 2q_G + p_G \,]. \square$$

Similar proofs work for each of the first ten products. Since the last six products are of a slightly different nature, the proofs are also slightly different. Here is a sample.

Theorem 3.1.3 The degree of the vertex (u, v) in $H \nabla G$, the symmetric difference, is given by:

$$\deg(u,v) = p_H \deg_G v + p_G \deg_H u - 2 \deg_H u \deg_G v.$$

Proof: As we recall H was modified by putting a loop for each $\delta \in \Delta$ at every vertex u, and replacing an edge uu_1 of H by an edge for each $\omega \in \Omega^* \cup \{e\}$, and an edge uu_2 in the complement of H by an edge for each $\delta \in \Delta^*$. Thus if (u,v) is a vertex of $H \nabla G$, then the total number of edges incident with it is:

$$|\Delta^*| + |\Omega^* \cup \{e\}| \deg_H u + |\Delta^*| (P_H - 1 - \deg_H u)$$

$$= |\Delta^*| + (P_G - |\Delta^*|) \deg_H u + |\Delta^*| (P_H - 1 - \deg_H u)$$

$$= P_G \deg_H u + |\Delta^*| P_H - 2 |\Delta^*| \deg_H u$$

$$= P_G \deg_H u + P_H \deg_G v - 2 \deg_H u \bullet \deg_G v. \square$$

Corollary 3.1.4 $q(H \nabla G) = q_G p_H^2 + q_H p_G^2 - 4 q_H q_G$.

Proof:

$$2q(H \nabla G) = \sum_{u \in V(H)} \sum_{v \in V(G)} (p_G \deg_H u + p_H \deg_G v - 2 \deg_H u \bullet \deg_G v)$$

$$= \sum_{u \in V(H)} \left(p_G^2 \deg_H u + 2 p_H q_G - 4 q_G \deg_H u \right)$$

$$= 2 q_H p_G^2 + 2 q_G p_H^2 - 8 q_H q_G.$$

Hence $q(H \nabla G) = q_G p_H^2 + q_H p_G^2 - 4 q_H q_G$. ❑

By using proofs similar to the above two lemmas and their corollaries we get the results listed in Table 3.1 on the next page. It should be noted that the results we have for rejection are different from those mentioned in Harary and Wilcox [12]. Their definition of rejection is what we are calling strong rejection; however, the degree they give for their rejection agrees with that of our rejection. (In this table $\mathbf{deg_G v}$ is denoted by $\mathbf{d_G v}$).

3.2 Connectedness of Graph Products

In this section we give the necessary and sufficient conditions for a graph product H*G to be connected. Most of these results are already in the literature, thus we will simply state the conditions and cite references. For those that are new, we will provide proofs.

Theorem 3.2.1 Replication is connected if and only if $| V(H) | = 1$ and G is connected.

Table 3.1

The Degrees of Vertices and Sizes in Graph Products

Product	deg(u,v)	q(H*G)
Replication	$d_G u$	$p_H q_G$
Cartesian	$d_H u + d_G v$	$p_H q_G + p_G q_H$
Tensor	$d_H u \, d_G v$	$2 q_H q_G$
Strong tensor	$d_G v [d_H u + 1]$	$2 q_H q_G + p_H q_G$
Augmented tensor	$d_H u [d_G v + 1]$	$2 q_H q_G + p_G q_H$
Strong Cartesian	$(d_H u + 1)(d_G v + 1) - 1$	$p_H q_G + 2 q_H q_G + p_G q_H$
Lexical	$(p_G - 1) d_H u$	$q_H [p_G^2 - p_G]$
Strong lexical	$(p_G - 1) d_H u + d_G v$	$q_H [p_G^2 - p_G] + p_H q_G$
Sublexicographic	$p_G \, d_H u$	$q_H p_G^2$
Lexicographic	$p_G \, d_H u + d_G v$	$p_H q_G + q_H p_G^2$
Disjunction	$p_H \, d_G v + p_G \, d_H u - d_H u d_G v$	$q_G p_H^2 + q_H p_G^2 - 2 q_H q_G$
Symmetric Difference	$p_H d_G v + p_G d_H u - 2 \, d_H u \, d_G v$	$q_G p_H^2 + q_H p_G^2 - 4 q_H q_G$
Rejection	$(p_H - 1 - d_H u)(p_G - 1 - d_G v)$	$\frac{1}{2}[p_H^2 - p_H - 2 q_H][p_G^2 - p_G - 2 q_G]$
Strong Rejection	$(p_H - d_H u)(p_G - d_G v) - 1$	$\frac{1}{2}[(p_H^2 - 2 q_H)(p_G^2 - 2 q_G) - p_H p_G]$
Exclusion	$d_H u [p_G - d_G v]$	$q_H [p_G^2 - 2 q_G]$
Total exclusion	$d_H u [p_G - 1 - d_G v]$	$q_H [p_G^2 - p_G - 2 q_G]$

Proof: This product consists of $|V(H)|$ disjoint copies of G; thus it that is connected if and only if $|V(H)| = 1$ and G is connected. That is, H is K_1 and G is connected.

Theorem 3.2.2 (Harary and Wilcox [12]): The cartesian product $H \times G$ is connected if and only if H and G are both connected.

Theorem 3.2.3 (Weichsel [31]): The tensor product $H \otimes G$ is connected if and only if H and G are both connected and either H or G contains an odd cycle.

Theorem 3.2.4 (Garman, Ringeisen and White [8]): The strong tensor product $H \underline{\otimes} G$ is connected if and only if H and G are both connected and $G \neq K_1$.

Corollary 3.2.5 The augmented tensor product $H \otimes' G$ is connected if and only if H and G are both connected and $H \neq K_1$.

Proof: This follows from Remark 2.2.1 and Theorem 3.2.4. ❑

Theorem 3.2.6 The strong cartesian product $H \underline{\times} G$ is connected if and only if H and G are both connected.

Proof: Let us assume that $H \underline{\times} G$ is connected, and let one of H or G, say (without loss of generality) H be disconnected. Let H_1 be one component of H with $H = H_1 \cup H_2$. Consider the vertex sets $V(H_1) \times V(G)$ and $V(H_2) \times V(G)$. First, we see that $V(H \underline{\times} G) = (V(H_1) \times V(G)) \cup (V(H_2) \times V(G))$. Second, we claim that there are no edges that join the first set of vertices with the second. This is true because if $(u_1, v_1) \in V(H_1) \times V(G)$ and $(u_2, v_2) \in V(H_2) \times V(G)$, every path between them must use only edges of type (ii) or type (iv). But this is impossible because

there are no edges between vertices of H_1 and vertices of H_2 in H. This says that $H \underline{\times} G$ is not connected, contrary to assumption. Therefore, H must be connected. A similar argument shows that G also must be connected.

Conversely, if H and G are both connected, then $H \times G$ is connected. Since $H \times G$ is a spanning subgraph of $H \underline{\times} G$, $H \underline{\times} G$ is connected. (By a spanning subgraph G_1 of G_2 we mean that G_1 is a subgraph of G_2 and $V(G_1) = V(G_2)$). ❑

Lemma 3.2.7a If $p_G = 2$, then $H \otimes_L G \cong H \otimes K_2$.

Proof: The vertex set in both cases has $2\,|V(H)|$ vertices. The edges in $H \otimes_L G$ are of type (v): $v_1 \neq v_2$ and $u_1u_2 \in E(H)$, and those of $H \otimes K_2$ are: $u_1u_2 \in E(H)$ and $v_1v_2 \in E(K_2)$; but $v_1v_2 \in E(K_2)$ if and only if $v_1 \neq v_2$. Thus the two edge sets are the same and hence the conclusion. ❑

Theorem 3.2.7b Let H and G both be nontrivial;

(a) If $p_G = 2$, then the lexical product $H \otimes_L G$ is connected if and only if H is connected and has an odd cycle.

(b) If $p_G > 2$, then the lexical product $H \otimes_L G$ is connected if and only if H is connected.

Proof: (a) If $p_G = 2$, then by Lemma 3.2.7a $H \otimes_L G \cong H \otimes K_2$; but from Theorem 3.2.3, $H \otimes K_2$ is connected if and only if H is connected and has an odd cycle.

(b) If $p_G > 2$, let us first assume that $H \otimes_L G$ is connected and H is not connected. If H_1 is a component of H such that $H = H_1 \cup H_2$, and u_1 and u_2 are two vertices that belong to two different components, with v_1 and v_2 two distinct

vertices of G, then there is no path of $H \otimes_L G$ that joins (u_1, v_1) to (u_2, v_2). This is true because there are no type (v) edges between them since there is no u_1–u_2 path in H. This means $H \otimes_L G$ is not connected, which contradicts our assumption; hence H must be connected.

Conversely, assume $p_G > 2$ and H is connected. If (u_1, v_1) and (u_i, v_j) are any two vertices of $H \otimes_L G$, then we are going to show that there exists a path joining (u_1, v_1) to (u_i, v_j). Since H is connected, we know that a u_1–u_i path exists in H.

Case 1: Let u_1–u_i be an even path (a path with an even number of vertices).

If $v_1 \neq v_j$, take vertices on the u_1–u_i path and alternate v_1 and v_j; thus if $u_1, u_2, u_3, \ldots, u_{i=2k}$ is our path in H, then $(u_1, v_1)\,(u_2, v_j)\,(u_3, v_1)\ldots(u_i, v_j)$ is the desired path.

On the other hand if $v_1 = v_j$, take $v_2 \neq v_1$ and $v_3 \neq v_1, v_2$; then if k=1 take $(u_1, v_1)\,(u_2, v_2)\,(u_1, v_3)\,(u_2, v_1)$ and this will be the path that we want. Otherwise, take $(u_1, v_1)\,(u_2, v_2)\,(u_3, v_3)\,(u_4, v_1)\ldots$ (continue by alternating v_1 and v_2) ... $(u_{i=2k}, v_1)$ and we will have the path we need.

Case 2: If u_1–u_i is an odd path, $i = 2k+1$, and $v_1 \neq v_j$ then let us take $v_2 \neq v_1, v_j$. If k=1 then $(u_1, v_1)\,(u_2, v_2)\,(u_3, v_j)$ will suffice. If $k \neq 1$, then $(u_1, v_1)\,(u_2, v_2)\,(u_3, v_j)\,(u_4, v_1)\,(u_5, v_j)\ldots, (u_i, v_j)$ will be the desired path.

On the other hand if $v_1 = v_j$, then by taking $v_2 \neq v_1$ we see that $(u_1, v_1)\,(u_2, v_2)\,(u_3, v_1)$ if $k = 1$, and $(u_1, v_1)\,(u_2, v_2)\,(u_3, v_1)\,(u_4, v_2)\ \ldots\ (u_{i=2k+1}, v_1)$ if $k \neq 1$, will suffice. ❑

Lemma 3.2.8a: If $G = \overline{K}_2$, then $H \underline{\otimes}_L G \cong H \otimes_L G \cong H \otimes K_2$.

Proof: If $G = \overline{K}_2$, then $H \underline{\otimes}_L G$ has only type (v) edges just like $H \otimes_L G$, hence the first part of the lemma; the second one follows from Lemma 3.2.7a. ❑

In view of the above lemma we would like to make the following remarks: If $H = K_1$, then $H \underline{\otimes}_L G \cong G$ and consequently $H \underline{\otimes}_L G$ is connected if and only if G is connected; and if $G = K_1$, then $H \underline{\otimes}_L G$ is totally disconnected.

Theorem 3.2.8b If H and G are both nontrivial, then:

(a) If $G = \overline{K}_2$, then the strong lexical product $H \underline{\otimes}_L G$ is connected if and only if H is connected and has an odd cycle;

(b) If $G \neq \overline{K}_2$, then $H \underline{\otimes}_L G$ is connected if and only if H is connected.

Proof:

(a) If $G = \overline{K}_2$, then from Lemma 3.2.8a $H \underline{\otimes}_L G \cong H \otimes K_2$. Thus the result follows from Theorem 3.2.3.

(b) Suppose $H \underline{\otimes}_L G$ is connected and $G \neq \overline{K}_2$, then we show that H is connected. Let us say that u_1 and u_2 are any two distinct vertices of H; we show that they are joined by a path. Take (u_1, v_1) and (u_2, v_2), two vertices of $H \underline{\otimes}_L G$; then we know that a (u_1, v_1)—(u_2, v_2) path exists in $H \underline{\otimes}_L G$, and the type of edges that make up this path are either of type (i): $u_j = u_i$ and $v_jv_i \in E(G)$ or of type (v): $v_i \neq v_j$ and $u_iu_j \in E(H)$. Since $u_1 \neq u_2$ we know that we have at least one edge of type (v). By selecting the first coordinates from the path in $H \underline{\otimes}_L G$, gives us a u_1—u_2 walk, and thus H is connected.

Conversely, if H is connected and $p(G) > 2$, then since $H \otimes_L G$ is a spanning subgraph of $H \underline{\otimes}_L G$ and the former is connected so is the latter. If $p(G) = 2$, then the only case to consider is $G = K_2$ because the other case was

handled in part (a) above. We take (u_1, v_1) and (u_2, v_2) as any two distinct vertices of $H \otimes_L G$. If $u_1 = u_2$, then $v_1 \neq v_2$ and $v_1v_2 \in E(K_2)$, so (u_1, v_1) (u_2, v_2) is a desired path. If on the other hand $u_1 \neq u_2$ and $v_1 = v_2$, using the u_1—u_2 path in H and the vertex v distinct from v_1 in K_2, together with type (v) edges, we get the paths: (u_1, v_1)—(u_2, v) or (u_1, v_1)—(u_2, v_1) depending upon whether the path $u_1 - u_2$ is even or odd, respectively. In the first case we use a type (i) edge to add (u_2, v) (u_2, v_2); while in the second case if $u_1 \neq u_2$, and $v_1 = v_2$ we get a (u_1, v_1)—(u_2, v_1) path in $H \otimes_L G$. For the remaining case, where $u_1 \neq u_2$, and $v_1 \neq v_2$, combine the above two cases. Thus $H \otimes_L G$ is connected. ❑

Theorem 3.2.9 The sublexicographic product $H \otimes_{SL} G$ is connected if and only if H is nontrivially connected.

Proof: Let us assume that $H \otimes_{SL} G$ is connected. If $H \cong K_1$, then $H \otimes_{SL} G$ is totally disconnected; but this is impossible, thus $H \ncong K_1$. Hence let us assume that $p(H) \geq 2$, and let u_1 and u_2 be arbitrary vertices of H. Take a (u_1, v_1)—(u_2, v_2) path in $H \otimes_{SL} G$. Such a path exists because $H \otimes_{SL} G$ is connected. Since the edges that make up this path are of the type $u_iu_j \in E(H)$, we have a u_1—u_2 path in H. Therefore, H is connected.

Conversely, let us assume that H is nontrivially connected and let (u_1, v_1) and (u_2, v_2) be any two vertices of $H \otimes_{SL} G$. Because H is connected, a nontrivial u_1— u_2 path exists in H, provided $u_1 \neq u_2$. Using these path edges we get type (iii) edges joining (u_1, v_1) and (u_2, v_2) in $H \otimes_{SL} G$. Thus, $H \otimes_{SL} G$ is connected. If $u_1 = u_2$, let u_3 be any other vertex of H (such vertex exists because H is nontrivial). Then a u_1—u_3 path exists and consequently a (u_1, v_1)—(u_3, v_2) path exists. Similarly a (u_3, v_2)—(u_2, v_2) path also exists. This means a (u_1, v_1)—(u_2, v_2)

path exists in $H \otimes_{SL} G$. Therefore, if H is nontrivially connected then $H \otimes_{SL} G$ is connected. ❑

Theorem 3.2.10 (Harary and Wilcox [12]): The lexicographic product H[G] is connected if and only if H is nontrivially connected or $H \cong K_1$ and G is connected.

Theorem 3.2.11 (Harary and Wilcox [12]): Let H and G be nontrivial graphs. If neither H nor G is totally disconnected, then the disjunction $H \vee G$ is connected if and only if H and G do not both contain isolated vertices. If exactly one of H or G is totally disconnected, then $H \vee G$ is connected if and only if the other is connected.

Theorem 3.2.12 (Harary and Wilcox [12]): The symmetric difference $H \nabla G$ is connected if and only if $H \vee G$ is connected.

Theorem 3.2.13 The rejection H/G is connected if and only if both $\overline{H}$ and $\overline{G}$ are connected and either $\overline{H}$ or $\overline{G}$ contains an odd cycle.

Proof: This follows from Remark 2.3.1 and Theorem 3.2.3. ❑

Theorem 3.2.14 The strong rejection $H \angle G$ is connected if and only if both $\overline{H}$ and $\overline{G}$ are connected.

Proof: This follows from Remark 2.3.2 and Theorem 3.2.6. ❑

Theorem 3.2.15 The exclusion $H \ominus G$ is connected if and only if H and $\overline{G}$ are both connected and $H \neq K_1$.

Proof: This follows immediately from Remark 2.3.3 and Corollary 3.2.5. ❑

Theorem 3.2.16 The total exclusion $H \ominus G$ is connected if and only if H and $\bar{G}$ are both connected and either H or $\bar{G}$ contains an odd cycle.

Proof: This follows easily from Remark 2.3.4 and Theorem 3.2.3. ❑

Before leaving this section, we remark that for each of the products that we discussed above, if we maintain G to be a Cayley graph, then connectedness of $H*G$ is easily verified by the use of Theorem 2.4.4. Thus all that we have to check is $[\Gamma : \Gamma_v]$, for any vertex v of H^*; from this we know that $H*G$ will be connected if and only if $[\Gamma : \Gamma_v] = 1$.

3.3 Bipartiteness of Graph Products

In this section we give the necessary and sufficient conditions for a graph product to be bipartite. If we find out that a graph product is bipartite, then this knowledge will first and foremost give us an improved lower bound for the genus of this particular graph and if we happen to find a quadrilateral imbedding for our graph, we will be guaranteed that we have a genus imbedding for our graph. This is basically what we are going to do in our quest for genus imbeddings in Chapter IV; hence the characterizations of this section are vital.

Theorem 3.3.1 Replication is bipartite if and only if G is bipartite.

Proof: This follows from the simple observation that the replication product is $|V(H)|$ disjoint copies of G. ❑

Theorem 3.3.2 The Cartesian product $H \times G$ is bipartite if and only if H and G are both bipartite.

Proof: Let us assume that $H \times G$ is bipartite. Because $H \times G$ contains, isomorphically, all edges of H and of G, both H and G are bipartite.

Conversely, let us assume that H and G are both bipartite, with partite sets H_1, H_2 and G_1, G_2 respectively. Then the vertex sets $(H_1 \times G_1) \cup (H_2 \times G_2)$ and $(H_1 \times G_2) \cup (H_2 \times G_1)$ form partite sets of $H \times G$; for neither type (i) nor type (ii) edges are present between vertices that belong to the same set. Hence $H \times G$ is bipartite. ❑

Theorem 3.3.3 The tensor product $H \otimes G$ is bipartite if and only if at least one of H or G is bipartite.

Proof: Suppose $H \otimes G$ is bipartite and H and G are both not bipartite. If we take one odd cycle from H of length n and another odd cycle from G of length m, then we have a cycle corresponding to these two cycles in $H \otimes G$. Its length is [n,m], the least common multiple of n and m. But [n,m] is odd, for the least common multiple of two odd numbers is odd, contrary to our assumption of $H \otimes G$ being bipartite; thus at least one of H or G has to be bipartite.

Conversely, if at least one of H or G is bipartite, without loss of generality say H, let H_1 and H_2 be the partite sets of H. Because there are no edges between H_1 and H_2 in H, the vertex sets $H_1 \times V(G)$ and $H_2 \times V(G)$ form partite sets for $H \otimes G$. Thus $H \otimes G$ is bipartite. ❑

Theorem 3.3.4 The strong tensor product $H \underline{\otimes} G$ is bipartite if and only if G is bipartite.

Proof: If $H \underline{\otimes} G$ is bipartite, then since it contains isomorphically all the edges of G it follows that G is bipartite.

Conversely, if G is bipartite with partite sets G_1 and G_2, then $V(H) \times G_1$ and $V(H) \times G_2$ form partite sets for $H \underline{\otimes} G$; because if (u_1, v_1) and (u_2, v_2) are both in $H \times G_1$ neither type (i) nor type (iv) edges exist between them, and similarly if both were in $H \times G_2$. Thus $H \underline{\otimes} G$ is bipartite. ❑

Corollary 3.3.5 The augmented tensor product $H \otimes' G$ is bipartite if and only if H is bipartite.

Proof: This follows immediately from Remark 2.2.1 and Theorem 3.3.4. ❑

Theorem 3.3.6 The strong cartesian product $H \underline{\times} G$ is bipartite if and only if H is empty and G is bipartite or H is bipartite and G is empty.

Proof: Let $H \underline{\times} G$ be bipartite. If we have any tensor edge $(u_1, v_1)(u_2, v_2)$ in $H \underline{\times} G$, then $H \underline{\times} G$ is not bipartite, for edges $(u_2, v_2)(u_2, v_1)$ and $(u_2, v_1)(u_1, v_1)$ will also be present in $H \underline{\times} G$, giving a 3-cycle; but this is impossible because $H \underline{\times} G$ is bipartite. Therefore, we can not have tensor edges; and this is only possible if one factor is empty. If H is empty then $H \underline{\times} G \cong p_H G$, and G must be bipartite; whereas if G is empty, then $H \underline{\times} G = p_G H$, and H must be bipartite.

Conversely, if H is empty and G is bipartite, then $H \underline{\times} G \cong p_H G$; thus $H \underline{\times} G$ is bipartite; and similarly $H \underline{\times} G$ is bipartite if G is empty and H bipartite. ❑

Theorem 3.3.7 The lexical product $H \otimes_L G$ is bipartite if and only if $p_G \leq 2$ or H is bipartite.

Proof: Suppose $H \otimes_L G$ is bipartite and H is not bipartite. Then if $p_G \geq 3$, let us take three distinct vertices v_1, v_2, and v_3 of G, and an odd cycle $u_1, u_2, \ldots, u_n, u_1$ of H. By using $(u_1, v_1)\,(u_2, v_2)\,(u_3, v_1) \ldots (u_n, v_3)\,(u_1, v_1)$ (here alternate v_1 and v_2 until we get to u_n as far as the cycle in H is concerned, there use v_3 instead of v_1), we get an odd cycle in $H \otimes_L G$, contradicting our assumption of $H \otimes_L G$ being bipartite; thus $p_G \leq 2$.

Conversely, let $p_G \leq 2$. If G is trivial then $H \otimes_L G$ is totally disconnected and there is nothing to prove. If $p_G = 2$, then let v_1 and v_2 be vertices of G. By taking the vertex sets $V(H) \times \{v_1\}$ and $V(H) \times \{v_2\}$ we form partite sets for $H \otimes_L G$. This is because no type (v) edges exist between vertices of the same vertex set $V(H) \times \{v_i\}$, $i = 1, 2$; hence $H \otimes_L G$ is bipartite.

On the other hand, if H is bipartite, with H_1 and H_2 as its partite sets, then $H_1 \times V(G)$ and $H_2 \times V(G)$ form partite sets for $H \otimes_L G$. ❑

Theorem 3.3.8 The strong lexical product $H \underline{\otimes}_L G$ is bipartite if and only if G is empty and $H \otimes_L G$ bipartite; or H is empty and G bipartite; or $p_G \leq 2$.

Proof: If $H \underline{\otimes}_L G$ is bipartite and G empty, then $H \underline{\otimes}_L G \cong H \otimes_L G$, because the only edges for the product are those of type (v).

If G is not empty, but H is empty, then $H \underline{\otimes}_L G \cong p_H G$; thus G is bipartite. On the other hand, if both G and H are nonempty, let u_1u_2 and v_1v_2 be edges in H and G, respectively. If $p_G \geq 3$, taking v_3 a vertex distinct from v_1 and v_2 we get $(u_1, v_3)\,(u_2, v_2)$, $(u_2, v_2)\,(u_2, v_1)$ and $(u_2, v_1)\,(u_1, v_3)$, three edges that form a 3-cycle; thus $H \underline{\otimes}_L G$ is not bipartite, a contradiction. Thus $p_G \leq 2$.

Conversely, if G is empty and $H \otimes_L G$ bipartite, since $H \otimes_L G \cong H \underline{\otimes}_L G$, the latter is bipartite. If H is empty and G is bipartite then as we remarked above $H \underline{\otimes}_L G \cong p_H G$ and we obtain $H \underline{\otimes}_L G$ as bipartite. Finally, if $p_G \leq 2$ and say $p_G = 1$, then $H \underline{\otimes}_L G$ is totally disconnected and thus it is bipartite. However, if $p_G = 2$, then by taking the vertex sets $V(H) \times \{v_1\}$ and $V(H) \times \{v_2\}$ we can see that $H \underline{\otimes}_L G$ is bipartite. ❑

Theorem 3.3.9 The sublexicographic product $H \otimes_{SL} G$ is bipartite if and only if H is bipartite.

Proof: Let us recall that all edges of $H \otimes_{SL} G$ are of type (iii): $u_1u_2 \in E(H)$. Thus if $H \otimes_{SL} G$ is bipartite and H is not, then using an odd cycle $u_1, u_2, \ldots, u_n, u_1$ of H we can get $(u_1, v_1), (u_2, v_1), \ldots, (u_n, v_1), (u_1, v_1)$ an odd cycle of $H \otimes_{SL} G$; but this can not happen, thus H is bipartite. We could have arrived at the same conclusion by observing that $H \otimes_{SL} G$ contains edges of H isomorphically; thus $H \otimes_{SL} G$ bipartite forces H to be bipartite.

Conversely, if H is bipartite with partite sets H_1 and H_2, then the vertex sets $H_1 \times V(G)$ and $H_2 \times V(G)$ form partite sets of $H \otimes_{SL} G$. Thus $H \otimes_{SL} G$ is bipartite. ❑

Theorem 3.3.10 The lexicographic product $H[G]$ is bipartite if and only if one factor is bipartite and the other empty.

Proof: Let us assume $H[G]$ is bipartite; then at least one factor has to be empty, otherwise, if $u_1u_2 \in E(H)$ and $v_1v_2 \in E(G)$ then $(u_1, v_1)(u_2, v_1)$, $(u_2, v_1)(u_1, v_2)$ and $(u_1, v_2)(u_1, v_1)$ are edges of $E(H[G])$ giving a triangle. This is impossible because $H[G]$ is bipartite. Furthermore if we assume that G is empty and H is not

bipartite, then we can find an odd cycle $u_1, u_2, \ldots, u_{n-1}, u_n = u_1$ in H. Using these edges and v_1 of V(G) we get (u_1, v_1) (u_2, v_1) $(u_3, v_1) \ldots (u_{n-1}, v_1)$ (u_1, v_1) an odd cycle in H[G], which is again contrary to our assumption of H[G] being bipartite. Therefore H must be bipartite. Similarly, if H is empty, G must be bipartite.

Conversely, if H is empty and G bipartite, then $H[G] \cong p_H G$, which is bipartite. On the other hand, if G is empty and H bipartite, but H[G] is not bipartite, then any odd cycle in H[G] would correspond to an odd cycle in H, since G is empty. But this can not happen, because H is assumed to be bipartite. Thus H[G] is bipartite. ❑

Theorem 3.3.11 The disjunction $H \vee G$ is bipartite if and only if H is empty and G is bipartite or if G is empty and H is bipartite.

Proof: Let us assume that $H \vee G$ is bipartite and H is empty; since $H \vee G$ contains all edges of G isomorphically, G must be bipartite. If on the other hand H is not empty, let $u_1u_2 \in E(H)$; if G is not empty, then there exists $v_1v_2 \in E(G)$. Using these two edges, one in H and the other in G, we get (u_1, v_1) (u_1, v_2), (u_1, v_2) (u_2, v_2) and (u_2, v_2) (u_1, v_1) that form a triangle in $H \vee G$, a contradiction. This forces G to be empty, so that $H \vee G$ contains all edges of H isomorphically and H is bipartite. This is the other case of the theorem.

Conversely, if H is empty and G bipartite, let G_1 and G_2 be the partite sets of G; then $V(H) \times G_1$ and $V(H) \times G_2$ form partite sets for $H \vee G$. A similar analysis applies to the case when G is empty and H is bipartite. ❑

Theorem 3.3.12 The symmetric difference $H \nabla G$ is bipartite if and only if one of H or G is empty and the other is bipartite; or both H and G are complete bipartite graphs.

Proof: Suppose $H \nabla G$ is bipartite and that one of H and G (by symmetry, without loss of generality let it be G) is empty; then since $H \nabla G$ contains all edges of H isomorphically, H is bipartite.

On the other hand, if both H and G are nonempty, as we saw above, $H \nabla G$ bipartite implies H and G are both bipartite. If we assume one of them, without loss of generality say G, is not a complete bipartite graph, then since the first case of a nonempty bipartite graph that is not complete bipartite is of order at least three, in G there exists a vertex w with the property that there is at least one pair of adjacent vertices of G that w is not adjacent to. Let these vertices be v_1 and v_2. Since H is nonempty, for an edge u_1u_2 of H we get the three edges $(u_1, w)\,(u_2, v_1)$, $(u_2, v_1)\,(u_2, v_2)$, and $(u_2, v_2)\,(u_1, w)$ of $H \nabla G$ giving a triangle, a contradiction to the fact that $H \nabla G$ is bipartite. Thus G is a complete bipartite graph. Similarly we can see that H is a complete bipartite graph.

Conversely if H is bipartite with partite sets H_1 and H_2 and G is empty, then $H_1 \times V(G)$ and $H_2 \times V(G)$ form partite sets of $H \nabla G$; this is because no edges exist between vertices of $H_1 \times V(G)$ since neither edges due to H nor to G exist. Similarly, for the vertices of $H_2 \times V(G)$; hence $H \nabla G$ is bipartite. By symmetry, if G is bipartite and H empty, then $H \nabla G$ is bipartite.

Finally, let us assume that both H and G are complete bipartite graphs, and let their partite sets be H_1, H_2 and G_1, G_2 respectively. Then the edges of $H \nabla G$ are

only of the types shown in Figure 3.1. But from Figure 3.1 we can observe that every cycle of $H \nabla G$ is an even cycle. Hence $H \nabla G$ is bipartite. ❑

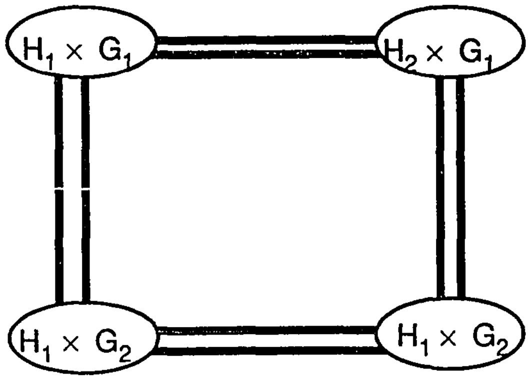


Figure 3.1

Theorem 3.3.13 The rejection H / G is bipartite if and only if at least one of $\overline{H}$ or $\overline{G}$ is bipartite.

Proof: This follows from Remark 2.3.1 and Theorem 3.3.3. ❑

Theorem 3.3.14 The strong rejection $H \angle G$ is bipartite if and only if $\overline{H}$ is empty and $\overline{G}$ is bipartite or $\overline{G}$ is empty and $\overline{H}$ is bipartite.

Proof: This follows from Remark 2.3.2 and Theorem 3.3.6. ❑

Theorem 3.3.15 The exclusion $H \ominus G$ is bipartite if and only if H is bipartite.

Proof: This follows from Remark 2.3.3 and Corollary 3.3.5. ❑

Theorem 3.3.16 The total exclusion $H \underline{\ominus} G$ is bipartite if and only if at least one of H or $\overline{G}$ is bipartite.

Proof: This follows from Remark 2.3.4 and Theorem 3.3.3. ❑

3.4 Some Observations on Small Order Factors

If our factors are of small order, then we can observe that several of the graph products under discussion overlap, even more than already indicated. This section will mention some of these overlapping products. Since the n-cube can be defined as a repeated cartesian product of K_2, a natural question would be: what graph do we obtain if we take the repeated product, *, of K_2 for each of the products that we are studying? This, together with some other properties, is discussed below.

Theorem 3.4.1 If H is bipartite, then $H \otimes K_2 = 2\,H$.

Proof: Let H_1 and H_2 be partite sets of H, if v_1 and v_2 are the vertices of K_2, then $H_1 \times \{v_1\}$ and $H_2 \times \{v_2\}$ have an edge between them if and only if a corresponding edge exists in H; that is, the edges between $H_1 \times \{v_1\}$ and $H_2 \times \{v_2\}$ are in one-to-one correspondence with those of H. Hence we have one copy of H. Similarly the same can be said about the edges between $H_1 \times \{v_2\}$ and $H_2 \times \{v_1\}$ and these give a second disjoint copy of H. As this exhausts all the edges of $H \otimes K_2$, we have the result. ❑

Corollary 3.4.2 Let $G_1 = K_2$ and $G_n = G_{n-1} \otimes K_2$, for $n \geq 2$; then

$$G_n = 2^{n-1} K_2.$$

It can be seen from the definition that, in general, the tensor product is both commutative and associative.

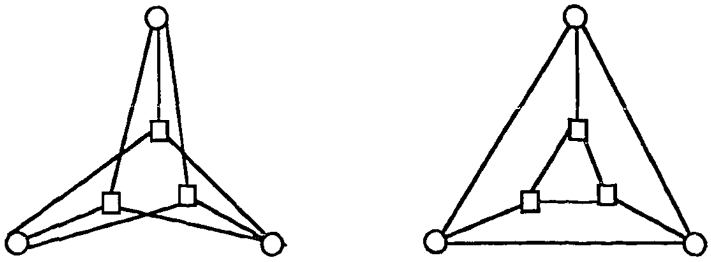
Theorem 3.4.3 $C_n \otimes K_2 = C_{2n}$, for $n \geq 3$ and odd.

Proof: Modify C_n to get C_n^* as specified in Table 2.3 for the tensor product and apply voltage graph theory; that is, use $\Gamma = Z_2$, and $\phi(e) = 1$ for $e \in E(C_n)$. From this we find that $| C_n^* |_\phi = 2$; thus by Theorem 2.4.3 the lift of C_n^* to $C_n^* \times_\phi \Gamma$ is C_{2n} and we have only one such C_{2n}. ❑

Theorem 3.4.4 (Garman, Ringeisen and White [8]): If H is bipartite,then

$$H \underline{\otimes} K_2 = H \times K_2.$$

The condition that H be bipartite is important because if we say H is not bipartite, for simplicity let H be C_3, then we see that $C_3 \underline{\otimes} K_2 = K_{3,3} \neq C_3 \times K_2$. See Figure 3.2. By using Theorems 3.3.4 and 3.3.2 we can show that if $H \underline{\otimes} K_2 = H \times K_2$ then H is bipartite.



$C_3 \underline{\otimes} K_2$ $C_3 \times K_2$.

Figure 3.2

Corollary 3.4.5: (Garman, Ringeisen and White [8]) If $H_1 = K_2$, and for $n \geq 2$,

$$H_n = H_{n-1} \underline{\otimes} K_2, \text{ then } H_n = Q_n.$$

The following theorem and two corollaries are also due to Garman, Ringeisen and White [8]:

Theorem 3.4.6 $K_m \underline{\otimes} K_{p_1,p_2,p_3,\dots,p_n} = K_{mp_1,mp_2,mp_3,\dots,mp_n}$.

Corollary 3.4.7 (a) $K_n \underline{\otimes} K_2 = K_{n,n}$;

(b) $K_2 \underline{\otimes} K_n = K_{n(2)}$.

Corollary 3.4.8 Let $G_1 = K_2$, and $G_n = K_2 \underline{\otimes} G_{n-1}$, for $n \geq 2$; then

$$G_n = K_{2^{n-1},2^{n-1}}.$$

Since $(K_2 \underline{\otimes} K_2) \underline{\otimes} K_2 = Q_3 \neq K_{4,4} = K_2 \underline{\otimes} (K_2 \underline{\otimes} K_2)$, we can say that, in general, the strong tensor product is neither associative nor commutative.

Remark 3.4.9 Since, by Remark 2.2.1, $H \otimes' G = G \underline{\otimes} H$, all the above mentioned results and comments on the strong tensor product apply to the augmented tensor product, as well.

Lemma 3.4.10 If H is an r-regular graph of order p, then $H \underline{\times} K_2$ is a $(2r + 1)$-regular graph of order 2p.

Proof: Since both H and G are regular, so is $H \underline{\times} K_2$, and from Table 3.1 the degree of each vertex of $H \underline{\times} K_2$ is $2r + 1$. ❑

Theorem 3.4.11 Let Let $H_1 = K_2$, and $H_n = H_{n-1} \underline{\times} K_2$, for $n \geq 2$; then

$$H_n = K_{2^n}.$$

Proof: Apply Lemma 3.4.10 and use induction. ❑

We remark here that the strong cartesian product is both commutative and associative and these properties are apparent from its definition. Note also that $H \underline{\times} \overline{K_n} = nH$.

Let us recall from Lemma 3.2.7a that $H \otimes_L K_2 = H \otimes K_2$. Using Corollary 3.4.2, 3.4.3 and Lemma 3.2.7a we get the following results:

Corollary 3.4.12 Let $G_1 = K_2$, and $G_n = G_{n-1} \otimes_L K_2$, for $n \geq 2$; then

$$G_n = 2^{n-1} K_2.$$

Corollary 3.4.13 $C_n \otimes_L K_2 = C_{2n}$, for $n \geq 3$ and odd.

According to Lemma 3.2.7a the above two corollaries will hold even if K_2 is replaced by $\overline{K_2}$. From Corollary 3.4.12 $(K_2 \otimes_L K_2) \otimes_L K_2 = 4K_2$, but Figure 3.3 shows that $K_2 \otimes_L (K_2 \otimes_L K_2) = K_2 \otimes_L 2K_2 = Q_3 \neq 4K_2$. Thus we conclude that, in general, the lexical product is neither associative nor commutative.

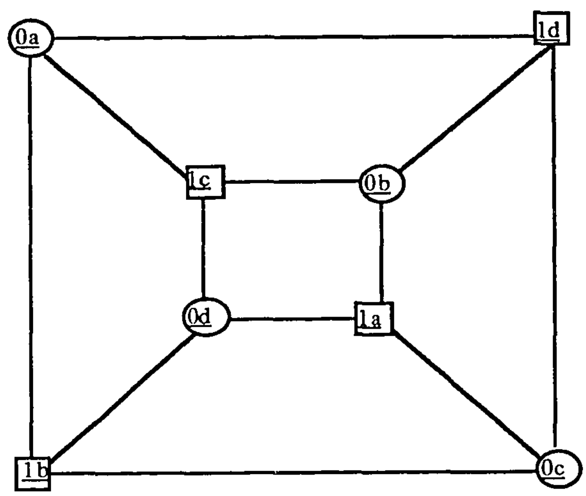


Figure 3.3

The above remark prompts the question: if we let $H_1 = K_2$, and $H_n = K_2 \otimes_L H_{n-1}$, for $n \geq 2$, then how does H_n compare with G_n? The answer is provided below.

Lemma 3.4.14 If G is any graph, then $K_2 \otimes_L G$ is a $(p_G - 1)$-regular bipartite graph.

Proof: Since both factors are regular, then $K_2 \otimes_L G$ is regular. The fact that $K_2 \otimes_L G$ is bipartite is the result of Theorem 3.3.7, and the degree of regularity follows from Table 3.1. ❑

Theorem 3.4.15 Let $H_1 = K_2$, and $H_n = K_2 \otimes_L H_{n-1}$, for $n \geq 2$; then

H_n, $n \geq 2$, is a $(2^{n-1} - 1)$-regular bipartite graph.

Proof: This follows from Lemma 3.4.14 and induction. ❑

Theorem 3.4.16 $H \underline{\otimes}_L K_2 = H \underline{\otimes} K_2$.

Proof: The edges of $H \underline{\otimes}_L K_2$ are all of type (i): $u_1 = u_2$ and $v_1v_2 \in E(K_2)$ and type (v): $v_1 \neq v_2$ and $u_1u_2 \in E(H)$; but $v_1 \neq v_2$ means $v_1v_2 \in E(K_2)$. Thus under this condition type (v) edges are the same as type (iv) (tensor edges) and hence the conclusion. ❑

As a result of Theorem 3.4.16, Theorem 3.4.4, and Corollary 3.4.5, we get the following:

Corollary 3.4.17 If H is bipartite, then $H \underline{\otimes}_L K_2 = H \underline{\otimes} K_2 = H \times K_2$.

Corollary 3.4.18 If $H_1 = K_2$, and $H_n = H_{n-1} \underline{\otimes}_L K_2$, for $n \geq 2$; then $H_n = Q_n$.

Theorem 3.4.19 If G is of order n, then $K_2 \otimes_{SL} G = K_{n,n}$.

Proof: From the definition of the sublexicographic product, its edge set consists only of type (iii) edges; this means $u_1u_2 \in E(K_2)$, i.e., if we replace the two

vertices of K_2 by a copy of G each, we get no edges between vertices that belong to this copy (vertices with the same first coordinate); on the other hand every vertex in this copy is adjacent to all the other vertices in the second copy. But this is what we mean by $K_{n,n}$. ❑

Theorem 3.4.20 If $p_G = n$, then

$$K_{p_1, p_2, p_3, \ldots, p_m} \otimes_{SL} G = K_{np_1, np_2, np_3, \ldots, np_m}.$$

Proof: Let P_i, $1 \le i \le m$, be the m partite sets of the given m-partite graph. Then $P_i \times V(G)$, $1 \le i \le m$, partition the vertex set of $K_{p_1, p_2, p_3, \ldots, p_m} \otimes_{SL} G$. Since there are no edges between vertices that belong to the same P_i, $1 \le i \le m$, we do not have edges between vertices that belong to the same set $P_i \times V(G)$, $1 \le i \le m$. On the other hand, because of the definition of type (iii) edges, every vertex of $P_i \times V(G)$ is joined to all vertices of $P_j \times V(G)$ for $j \ne i$. This gives $K_{np_1, np_2, np_3, \ldots, np_m}$. ❑

Corollary 3.4.20 (a) If $G_1 = K_2$, and $G_n = K_2 \otimes_{SL} G_{n-1}$, for $n \ge 2$, then

$$G_n = K_{2^n, 2^n};$$

(b) If $H_1 = K_2$, and $H_n = H_{n-1} \otimes_{SL} K_2$, for $n \ge 2$, then

$$H_n = K_{2^n, 2^n}.$$

Corollary 3.4.21 (a) $K_2 \otimes_{SL} K_n = K_{n,n}$;

(b) $K_n \otimes_{SL} K_2 = K_{n(2)}$.

From Table 3.1 we find that if G is an r-regular graph of order p, then $K_2 \vee G$ is a $(p + r)$-regular graph.

Theorem 3.4.22 If $G_1 = K_2$, and $G_n = K_2 \vee G_{n-1}$, f or $n \ge 2$, then $G_n = K_{2^{n-1}}$.

Proof: This follows from the above remark and induction. ❑

From the definition of the edge types for the symmetric difference and the lexicographic product of two graphs we have the following:

Theorem 3.4.23 $H \nabla \overline{K_n} = H[\overline{K_n}]$.

Corollary 3.4.24 $K_2 \nabla \overline{K_n} = K_{n,n}$.

Corollary 3.4.25 Let $H_1 = K_2$, and $H_n = H_{n-1} \nabla \overline{K_2}$, for $n \geq 2$; then

$H_n = K_{2^{n-1},2^{n-1}}$.

Finally, we conclude this section by making the following remark.

Remark 3.4.26

(a) If H is bipartite, then $H \times K_2 = H \underline{\otimes} K_2 = K_2 \otimes' H = H \underline{\otimes}_L K_2$;

(b) $H[K_2] = H \underline{\times} K_2$;

(c) $H \otimes_{SL} K_2 = H \otimes' K_2 = H[\overline{K_2}]$;

(d) $H \otimes K_2 = H \otimes_L K_2$.

Proof: The equalities of (a) follow from Theorem 3.4.4, Remark 2.2.1, and Corollary 3.4.17. The results in the rest follow from definition.

This chapter was devoted to studying the main properties that will be needed in the next chapter. In Chapter IV we establish genus imbeddings for several of the graph products discussed so far. To check whether we have a minimal imbedding, we need to know if our graph is connected and/or bipartite. So the results that we established in Chapter III are going to prove useful.

CHAPTER IV

GENUS IMBEDDINGS OF PRODUCT GRAPHS

In this chapter we will provide several genus imbeddings for the different graph products that have been discussed. The method of graph imbedding used will be the surgical-voltage technique. Some of the necessary preparation for this method was discussed in Section 2.5. This method assumes that in $H * G$, G is a Cayley graph, $G_\Delta(\Gamma)$, for some finite group Γ and generating set Δ. Minimal imbeddings are more readily obtained, by this method, for graphs which do not use edge type (i), as such edges often introduce 3-cycles into the product graph. We are going to use the properties discussed in Chapter III repeatedly. Corollary 1.3.5 will often be invoked to calculate the genus of a surface after we establish that we have a minimal quadrilateral imbedding for our graphs.

Throughout this chapter and the next one we encounter abelian groups $Z_m \times Z_m \times \ldots \times Z_m$, n factors in all, which we write as Z_m^n. By a standard (canonical) generating set Δ for Z_m^n, we mean the set of n m-tuples, that have 1 as one coordinate and 0 for all the others; for example, the standard generating set Δ for Z_2^3 is $\Delta = \{100, 010, 001\}$. For simplicity of notation an element of a standard generating set with its non zero entry at position i will be denoted by ε_i or sometimes simply as i. The identity element of a group, whenever used, will be denoted by 0 instead of its usual representation as a string of zeros. In this chapter, unless otherwise stated, we assume H to be connected.

4.1 The Tensor Product

Lemma 4.1.1 The graph $P_3 \otimes Q_n$ has two isomorphic components.

Proof: Take $\Gamma = Z_2^n$ and let Δ be the set of standard generators of Γ; then $G_\Delta(\Gamma) = Q_n$. We modify P_3 as given in Table 2.3(3) for the tensor product to get P_3^*; see Figure 4.1.

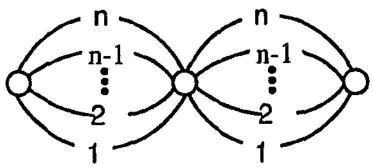


Figure 4.1

We note that the local group at v is given by:

$$\Gamma_v = \{(a_1, a_2, \ldots, a_n) \in Z_2^n \mid \sum_{i=1}^{n} a_i \equiv 0 \bmod 2\ \}.$$

This is true because to obtain a closed walk in Figure 4.1 we always need an even number of edges. Each edge contributes a 1 to the sum, thus an even number of 1's contributes 0 mod 2. From this we conclude that $[\,\Gamma: \Gamma_v\,] = 2$. From Theorem 2.4.4 we have the number of isomorphic components of our voltage graph equal to $[\,\Gamma : \Gamma_v\,]$ and thus the lemma. ❑

We remark that $P_3 \otimes Q_n$ is disconnected; this follows also from Theorem 3.2.3; however, Theorem 2.4.4 also tells us exactly how many components there are.

Theorem 4.1.2 $\gamma(P_3 \otimes Q_n) = 2^{n-1}(n-3) + 2$.

Before proving this theorem we illustrate it using the special case of $n = 2$. We take $\Gamma = Z_2 \times Z_2$, $\Delta = \{10, 01\}$ and $G_\Delta(\Gamma) = Q_2 = C_4$.

The modification of P_3 is shown in Figure 4.2.

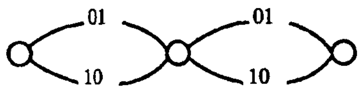


Figure 4.2

By Lemma 4.1.1, $P_3 \otimes C_4$ has two components that are isomorphic. Moreover, $P_3 \otimes C_4$ has 12 vertices and 16 edges. A close examination of Figure 4.2 shows that we have two types of regions in the imbedded P_3^*, the two 2-sided regions and a 4-gon. Then $|R|_\phi$ for the 2-sided regions is 2, and by Theorem 2.4.3 their lifts will be two 4-gons each, and the only 4-gon of the figure satisfies the KVL or $|R|_\phi = 1$; thus by Theorem 2.4.3 it will be lifted to four 4-gons. This means then we have a 4-gon imbedding (all the regions in the imbedding are 4-gons) for $P_3 \otimes C_4$. If we can establish that this imbedding is minimal, then the genus of our graph will be found after a routine calculation. This is true because, as a result of Theorem 3.3.3, $P_3 \otimes C_4$ is bipartite. By Corollary 1.3.7 $\gamma(P_3 \otimes C_4) = \frac{q}{4} - \frac{p}{2} + 2 = \frac{16}{4} - \frac{12}{2} + 2 = 0$. This says that our product graph is planar; this also agrees with $\gamma(P_3 \otimes G) = 2^{n-1}(n-3) + 2$ for $n = 2$ in the theorem. (This graph, $P_3 \otimes C_4$, was the graph we saw in Figure 2.7 (c); however, here $\Gamma = Z_2 \times Z_2$ instead of Z_4.)

Now let us proceed to prove the general case:

Proof of Theorem 4.1.2: Let $\Gamma = Z_2^n$ and Δ be the standard set of generators; then $G_\Delta(\Gamma) = Q_n$. Let us modify P_3 to get P_3^* as shown in Figure 4.1. Since every element of Γ except for the identity element has order two, $|R|_\phi = 2$ for all the 2-sided regions of Figure 4.1. Each of these will be lifted to 2^{n-1} 4-gons in

$P_3^* \times_\phi \Gamma$. The only 4-gon region in P_3^* satisfies the KVL and thus it will be lifted to 2^n 4-gons by Theorem 2.4.3. Lemma 4.1.1 tells us that we have two isomorphic components for this product. The product $P_3 \otimes G$ has $3(2^n)$ vertices and $2n(2^n)$ edges; the application of Corollary 1.3.7 for the number of components $n(c) = 2$ gives $\gamma(P_3 \otimes G) = \frac{2n2^n}{4} - \frac{3(2^n)}{2} + 2 = 2^{n-1}(n-3) + 2$. ❑

If Γ is a group and Δ is its generating set, we say Δ contains **odd redundancy** if we can get an odd cycle in $G_\Delta(\Gamma)$; for example when $\Gamma = Z_2^3$ and $\Delta = \{100, 010, 001, 101\}$ then Δ has an odd redundancy, since we obtain a 3-cycle from the generators 100, 001, and 101.

Theorem 4.1.3 If $\Gamma = Z_2^n$, Δ is a generating set with odd redundancy, $|\Delta| = k > n$, and $G = G_\Delta(\Gamma)$, then $\gamma(P_3 \otimes G) = 2^{n-1}[k-3] + 1$.

Proof: The proof of this theorem is similar to the proof of Theorem 4.1.3; the difference is that in Theorem 4.1.3 we had n generators, but here we have k generators, where $k > n$. Also, since Δ has an odd redundancy, G contains an odd cycle. This means that by Theorem 3.2.3 $P_3 \otimes G$ is connected. As for the proof of Theorem 4.1.3 we can see that all the regions of $P_3^* \times_\phi \Gamma$ are 4-gons; thus we have a 4-gon imbedding for $P_3 \otimes G$. Furthermore by Theorem 3.3.3, $P_3 \otimes G$ is bipartite and this makes our imbedding minimal. We also note that P_3^* has three vertices and 2k edges; this means that $P_3 \otimes G$ has $3(2^n)$ vertices and $2k(2^n)$ edges. Then using Corollary 1.3.5, we find

$$\gamma(P_3 \otimes G) = \frac{2k(2^n)}{4} - \frac{3(2^n)}{2} + 1 = 2^{n-1}(k-3) + 1. \square$$

Let $\Gamma = Z_2^n \times Z_4$, $n \geq 0$. We define Δ_n by: $\Delta_0 = \{1\}$ and Δ_{k+1} is obtained from Δ_k as follows: for each of the 2^k generators (each a $k + 1$ tuple) of Δ_k, form

two $k+2$ tuples: one by adding a " 0 " at the beginning and another by adding a "1" at the beginning. These 2^{k+1} (k+2)-tuples form Δ_{k+1}. For example, $\Delta_0 = \{1\}$, $\Delta_1 = \{01, 11\}$, $\Delta_2 = \{001, 011, 101, 111\}$, etc.

Theorem 4.1.4 If $\Gamma = Z_2^n \times Z_4$, $n \geq 0$, and Δ_n is as defined above, then for $G = G_{\Delta_n}(\Gamma)$, $\gamma(P_3 \otimes G) = 2^{n+1}(2^{n+1} - 3) + 2$.

Proof: Let us modify P_3 consistent with Table 2.3(3) to get P_3^* as seen in Figure 4.3. In this figure let δ_i stand for the i-th generator, $1 \leq i \leq m = 2^n$.

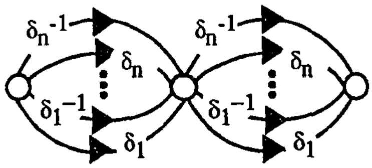


Figure 4.3

For Γ_v, the local group at v, we have $\Gamma_v = \{a \in \Gamma \mid a^2 = \text{identity}$ (i.e, last coordinate is even) $\}$. Thus $[\Gamma : \Gamma_v] = 2$ and hence $P_3 \otimes G$ has two components. From the figure we can see that P_3^* has 3 vertices and $2(2m) = 2(2^{n+1})$ edges; this in turn says that $P_3 \otimes G$ has $3|\Gamma| = 3(2^{n+2})$ vertices and $2(2^{n+1})|\Gamma| = 2^{2n+4}$ edges. From the arrangements of the generators every 2-sided region satisfies $|R|_\phi = 2$; thus they will be lifted to 4-gons. The only 4-gon of the figure satisfies the KVL and hence it also lifts to a 4-gon. This means we have a 4-gon imbedding for $P_3 \otimes G$. Since P_3 is bipartite, Theorem 3.3.3 tells us that $P_3 \otimes G$ is also bipartite; thus our imbedding is minimal. The calculation of the genus using Corollary 1.3.7 for $n(c) = 2$ is as follows:

$$\gamma(P_3 \otimes G) = \frac{2^{2n+4}}{4} - \frac{3(2^{n+2})}{2} + 2 = 2^{n+1}(2^{n+1} - 3) + 2. \quad \square$$

Theorem 4.1.5 Let H be a bipartite (p, q) graph with an orientable 4-gon imbedding; then:

(a) $\gamma(H \otimes Q_n) = 2^{n-2}(nq - 2p) + 2$;

(b) If $\Gamma = Z_2^n \times Z_4$ and Δ_n is as defined in Theorem 4.1.4, then $\gamma(H \otimes G) = 2^{n+1}\big(q(2^n) - p\big) + 2$.

Before proving the theorem, we illustrate (a) with the case when $n = 2$. We take $\Gamma = Z_2 \times Z_2$ and $\Delta = \{10,01\}$. Then $G_\Delta(\Gamma) = C_4$. Let us also take $H = C_4$. Modify each edge of H shown on the left to look like the one shown on the right of Figure 4.4.

Figure 4.4

The modification of Figure 4.4 can be done consistently with all edges because H is bipartite. Let C_4 be modified according to Figure 4.4 as shown in Figure 4.5 to give C_4^*.

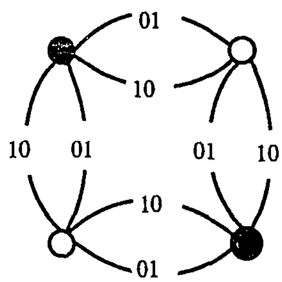


Figure 4.5

In Figure 4.5 we have four 2-sided regions; all will lift to 4-gons, by Theorem 2.4.3, in $C_4^* \times_\phi \Gamma$. We also have two 4-gons, that satisfy the KVL and hence will lift to 4-gons as well. Thus $C_4 \otimes C_4$ has a 4-gon imbedding. Furthermore we can see (using Theorem 2.4.4) that $C_4 \otimes C_4$ has two isomorphic components. Since this product has 32 edges and 16 vertices, using Corollary 1.3.7 for two components we get: $\gamma(C_4 \otimes C_4) = \frac{32}{4} - \frac{16}{2} + 2 = 2$. This is in harmony with the given formula for $p = q = 4$ and $n = 2$.

Proof of Theorem 4.1.5: For case (a) we take Γ to be Z_2^n and Δ the standard set of generators, giving $G_\Delta(\Gamma) = Q_n$. Let us modify H by following the modification indicated for each edge in Figure 4.6. This modification is consistent since H is bipartite. Each 4-gon of H will be modified as seen in Figure 4.5 except this time we have n edges instead of two for case (a) and 2^{n+2} edges when it comes to case (b). Since by Theorem 2.4.3 all the regions lift to 4-gons, we get a 4-gon imbedding for $H^* \times_\phi \Gamma = H \otimes G$. From Theorem 3.3.3 and Theorem 2.4.4 we find $H \otimes G$ to be bipartite with two isomorphic components. It is not hard to see also that $H \otimes G$ has $p(2^n)$ vertices and $q(n)2^n$ edges. Thus from Corollary 1.3.7 we have:

$$\gamma(H \otimes G) = \frac{q(n)2^n}{4} - \frac{p(2^n)}{2} + 2 = 2^{n-2}(nq - 2p) + 2.$$

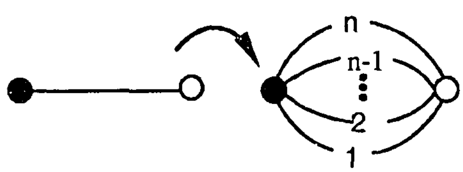


Figure 4.6

For case (b), we first note that $|\Delta^*| = 2^{n+1}$ instead of n as in case (a), and the figure shown in Figure 4.6 will have directed edges this time; we will order them $\delta_1, \delta_2, \ldots, \delta_{2^n}, (\delta_1)^{-1}, \ldots, (\delta_{2^n})^{-1}$. This means that H^* has $q2^{n+1}$ edges and p vertices, giving $H \otimes G$ $q2^{2n+3}$ edges and $p2^{n+2}$ vertices. Again $H \otimes G$ is not connected but it has two isomorphic components which are bipartite. An examination of the regions of H^* also shows that all its regions will lift to 4-gons. Thus we have a minimal 4-gon imbedding for $H \otimes G$. From Corollary 1.3.7 we get that $\gamma(H \otimes G) = \frac{q2^{2n+3}}{4} - \frac{p(2^{n+2})}{2} + 2 = 2^{n+1}(q2^n - p) + 2$. ❑

Corollary 4.1.6 (White [28]) If H is a bipartite (p,q) graph with an orientable 4-gon imbedding, then $\gamma(H \otimes K_2) = \frac{q}{2} - p + 2$.

Proof: Take $n = 1$ in the above theorem; thus $\gamma(H \otimes K_2) = 2^{n-2}(nq - 2p) + 2 = \frac{q}{2} - p + 2$.❑

Theorem 4.1.7 Let H be a bipartite (p,q) graph with an orientable 4-gon imbedding. If $\Gamma = Z_2^n$, Δ has odd redundancy with $|\Delta| = k > n$, and $G = G_\Delta(\Gamma)$, then $\gamma(H \otimes G) = 2^{n-2}(kq - 2p) + 1$.

Proof: The proof of this theorem is essentially the same as that of Theorem 4.1.5, except this time the existence of odd redundancy in Δ makes G have an odd cycle. This, by Theorem 3.2.3, guarantees that $H \otimes G$ is connected. Also, the size of $H \otimes G$ is $qk(2^n)$. Once more the use of Corollary 1.3.5 gives us:

$$\gamma(H \otimes G) = 2^{n-2}(kq - 2p) + 1. \quad ❑$$

Theorem 4.1.8 (White [28]): Let H be a (p,q) graph with an odd cycle and an orientable 4-gon imbedding; then $\gamma(H \otimes K_2) = \frac{q}{2} - p + 1$.

Proof: In modifying H the only thing we do is label each edge of H by the group element 1 (we are using $\Gamma = Z_2$). This means every 4-gon of H^* satisfies the KVL and hence all will be lifted to 4-gons. Since H is not bipartite, $H \otimes K_2$ is connected by Theorem 3.2.3 and because K_2 is bipartite, $H \otimes K_2$ is also bipartite from Theorem 3.3.3. It is easy to see that $H \otimes K_2$ has 2p vertices and 2q edges; thus the use of Corollary 1.3.5 gives: $\gamma(H \otimes K_2) = \frac{q}{2} - p + 1$.□

A graph G imbedded in some surface S_k is said to have a **bichromatic dual** for that particular imbedding if the geometric dual of G imbedded in S_k has chromatic number equal to two.

Theorem 4.1.9 Let H be a non-bipartite (p,q) graph with an orientable 4-gon imbedding and bichromatic dual. Let $G = G_\Delta(\Gamma)$ then:

(a) If $\Gamma = Z_4$ and $\Delta = \{1\}$, then $\gamma(H \otimes G) = 2(q-p) + 1$;

(b) If $\Gamma = Z_2^n$ and Δ is the standard set of generators, then

$$\gamma(H \otimes G) = 2^{n-2}(nq - 2p) + 1;$$

(c) If $\Gamma = Z_2^n \times Z_4$ and Δ_n is as in Theorem 4.1.4, then

$$\gamma(H \otimes G) = 2^{n+1}\left(q(2^n) - p\right) + 1.$$

Proof: Since H has a bichromatic dual imbedding, let us assume the regions of this imbedding are colored black and white. For cases (a) and (c) we modify each edge of H indicated in Figure 4.7(a) by the ones shown in Figure 4.7(b); here $k = |\Delta^*|$ and the ordering of the generators looks like the arrangement of Figure 4.3. For case (b) we have to remove direction from the edges, since all generators in Δ have order two. The arrangement here is from $\delta_1, \delta_2, \ldots, \delta_n$ for $k = n$. Because of the coloring of the regions, this modification can be done consistently in all three cases. In addition

to this, we observe the condition $(\delta_i\ \delta_{i+1}^{-1})^2$ = identity, for δ_i and δ_{i+1} consecutive arcs in the modification of H. This will guarantee that every 2-sided region of H^* will have $|R|_\phi = 2$, giving 4-gons in the lift. The coloring of the regions divides the 4-gons of H^* into two types, half of them having all four sides labeled δ_1 and the rest of them having all four sides labeled δ_k. This means all the 4-gons satisfy the KVL and they too will be lifted to 4-gons; thus we have a 4-gon imbedding for $H \otimes G$. Since in all cases G is bipartite, by Theorem 3.3.3 $H \otimes G$ is also bipartite; as a result of these our imbedding is minimal. Because H is not bipartite, from Theorem 3.2.3 $H \otimes G$ is connected in all cases.

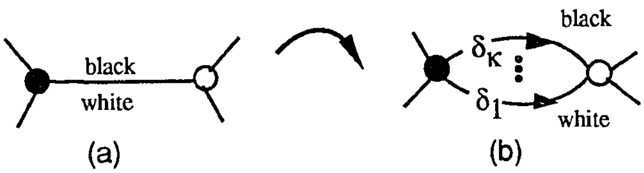


Figure 4.7

Let us now proceed to get specific formulae for the corresponding Γ and Δ.

(a) When $\Gamma = Z_4$, $H \otimes G$ has 4p vertices and 8q edges and the result follows from Corollary 1.3.5; that is $\gamma(H \otimes G) = \frac{8q}{4} - \frac{4p}{2} + 1 = 2(q-p) + 1$.

(b) When $\Gamma = Z_2^n$, then $|\Delta^*| = n$; thus H^* has nq edges and p vertices implying $H \otimes G$ has $nq(2^n)$ edges and $p(2^n)$ vertices. The result then follows from Corollary 1.3.5.

(c) Here as we saw in Theorem 4.1.4 $|\Delta| = 2^n$. Since none of the elements of Δ is of order 2, $|\Delta^*| = 2^{n+1}$. This gives $q(2^{n+1})$ edges for H^* and the size of

$H \otimes G$ is $q(2^{n+1})|\Gamma| = q(2^{2n+3})$; of course its order is $p(2^{n+2})$. If we apply Corollary 1.3.5 we get:

$$\gamma(H \otimes G) = \frac{q(2^{2n+3})}{4} - \frac{p(2^{n+2})}{2} + 1 = 2^{n+1}\left(q(2^n) - p\right) + 1. \square$$

The following theorem will overlap Theorem 4.1.5 when both r and s are even.

Theorem 4.1.10: Let H be the cartesian product $C_r \times C_s$ of two cycles C_r and C_s; $r, s \geq 4$. Then :

(a) $$\gamma(H \otimes Q_n) = \begin{cases} 2^{n-1}(n-1)rs + 2; & s \text{ and } r \text{ even} \\ 2^{n-1}(n-1)rs + 1; & \text{otherwise} \end{cases}$$

(b) If $\Gamma = Z_2^n \times Z_4$ and Δ_n is as defined in Theorem 4.1.4, with $G = G_{\Delta_n}(\Gamma)$, then

$$\gamma(H \otimes G) = \begin{cases} 2^{n+1}\left(2^{n+1} - 1\right)rs + 2; & s \text{ and } r \text{ even} \\ 2^{n+1}\left(2^{n+1} - 1\right)rs + 1; & \text{otherwise} \end{cases}$$

Proof: First of all, we note that $C_r \times C_s$ has an orientable 4-gon imbedding in S_1 as seen in Figure 4.8. For case (a) we take $\Gamma = Z_2^n$ and Δ the standard set of generators, so that $G_\Delta(\Gamma) = Q_n$.

When r and s are both even, the results are special cases of Theorem 4.1.5. For the remaining cases (at least one of r or s odd) we proceed as follows. In both (a) and (b) G is bipartite: in (a) G is the n-cube, and in (b) we may take V_1 to be those vertices that end in either 0 or 2 and V_2 to be those vertices that end in 1 or 3. Thus $H \otimes G$ is bipartite. In addition to this, $H \otimes G$ is connected because H is

not bipartite. Figure 4.9 shows how a typical 4-gon of H will be modified in H^*. (Here we observe that in the first case the figure will be as it is, while for the second case we have to add directions to the edges and their arrangement is like that of Figure 4.3.) Here let i stand for δ_i, where $1 \leq i \leq k = |\Delta^*|$. In case (b) note that $(\delta_i \, \delta_{i+1}^{-1})^2 = e$.

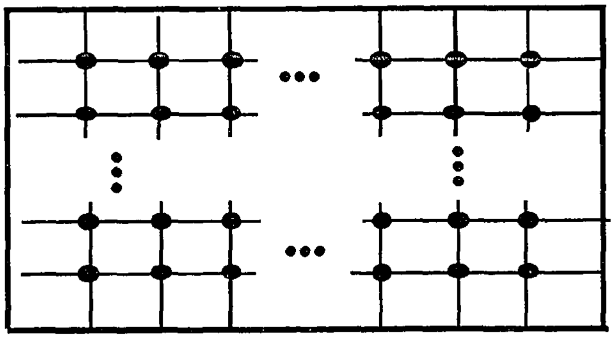


Figure 4.8

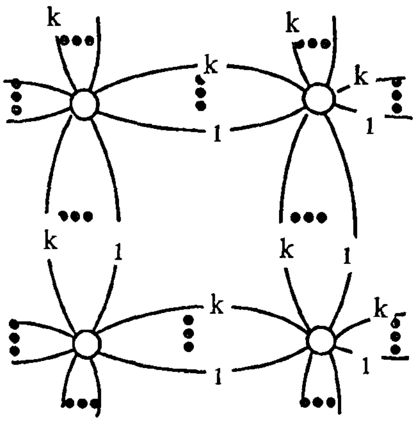


Figure 4.9

It will be helpful to remind ourselves that H has rs vertices and 2rs edges; as a result of this H* will have 2rsk edges. This means $H \otimes G$ has $2rsk\,|\Gamma|$ edges and $rs\,|\Gamma|$ vertices. The modification also shows that the 2-sided regions have $|R|_\phi = 2$ and the 4-gons satisfy the KVL; hence all the regions lift to 4-gons, giving us a 4-gon imbedding of $H \otimes G$. But $H \otimes G$ is bipartite, because G is bipartite, and connected because H is not bipartite, which means our imbedding is minimal. All that remains is to count the number of edges and apply Corollary 1.3.5 as usual. In (a) $k = n$, $|\Gamma| = 2^n$, and an application of Corollary 1.3.5 gives

$$\gamma(H \otimes G) = \frac{2rsn(2^n)}{4} - \frac{rs2^n}{2} + 1 = 2^{n-1}(n-1)rs + 1.$$

On the other hand in (b) $k = 2^{n+1}$ and $|\Gamma| = 2^{n+2}$. Thus using the above cited Corollary we have

$$\gamma(H \otimes G) = \frac{2rs2^{n+1}(2^{n+2})}{4} - \frac{rs2^{n+2}}{2} + 1 = 2^{n+1}(2^{n+1}-1)rs + 1. \quad \square$$

4.2 The Augmented Tensor Product

We begin this section by giving genus results of some products involving K_2, C_n, K_n, and P_n. The methods that have been used are the common imbedding techniques. Some of these results are already in the literature and these will be indicated as such. The remaining part of the section will be devoted to new results obtained by using the surgical–voltage method.

Theorem 4.2.1 For $n \geq 3$, $\gamma(K_2 \otimes' C_n) = \begin{cases} 0; & \text{if } n \text{ is even} \\ 1; & \text{otherwise.} \end{cases}$

Proof: If n is even, then from Remark 2.2.1 and Theorem 3.4.4 $K_2 \otimes' C_n = C_n \underline{\otimes} K_2 = C_n \times K_2$ and the last one is planar. If on the other hand n is odd,

then $K_{3,3}$ is a subgraph (homeomorphically) of $K_2 \otimes' C_n$ and by the famous theorem of Kuratowski (see [14] also Theorem 4.9 in [5]) $K_2 \otimes' C_n$ is non–planar. Hence $\gamma(K_2 \otimes' C_n) \geq 1$. Figure 4.10 shows that $K_2 \otimes' C_n$ can be imbedded in S_1, so that $\gamma(K_2 \otimes' C_n) \leq 1$. Combining these two we obtain $\gamma(K_2 \otimes' C_n) = 1$.

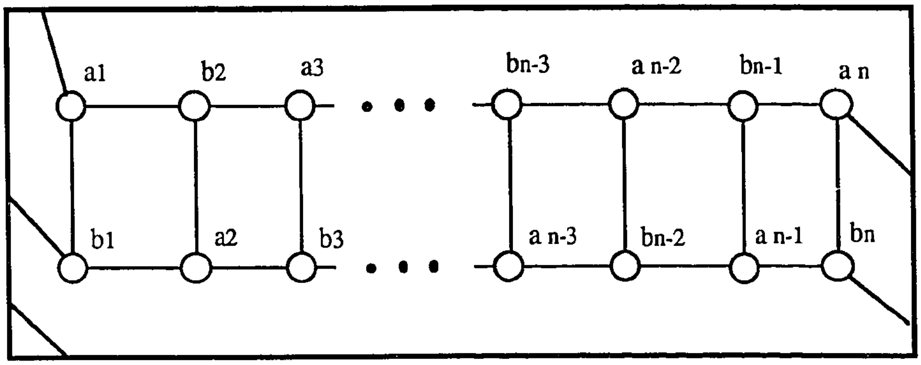


Figure 4.10

Theorem 4.2.2 For $n \geq 4$, $\gamma(C_n \otimes' K_2) = 1$.

Before proving the theorem we note that we are considering $n \geq 4$ because for $n = 3$, $C_3 \otimes' K_2 = K_2 \underline{\otimes} C_3 = K_2 \underline{\otimes} K_3 = K_{3(2)}$. The first equality is the result of Remark 2.2.1 and the last one is from Corollary 3.4.7 (b); the fact that $K_{3(2)}$ is planar then makes $C_3 \otimes' K_2$ planar too. Thus $\gamma(C_3 \otimes' K_2) = 0$.

Proof: First we claim that for $n \geq 4$, the girth of $C_n \otimes' K_2$ is four. This is true because if we were going to have a triangle in $C_n \otimes' K_2$, then at least two of the second coordinates are the same because our second factor is K_2. Also we can not have all three of them the same; because, for $n \geq 4$, C_n does not contain a triangle. Thus without loss of generality we can assume (u_1, v_1), (u_2, v_1), (u_3, v_2) to give our triangle. This forces both (u_2, v_1) (u_3, v_2) and (u_3, v_2) (u_1, v_1) to be tensor

edges, giving a triangle in C_n, a contradiction. Thus the girth of $C_n \otimes' K_2$ is at least four; but if u_1, u_2, and u_3 are consecutive vertices of C_n, then $(u_1, v_1), (u_2, v_1), (u_3, v_2), (u_2, v_2), (u_1, v_1)$ is a 4-cycle in $C_n \otimes' K_2$, giving the girth to be exactly four. From Corollary 1.3.5 we see that $\gamma(C_n \otimes' K_2) \geq \frac{4n}{4} - \frac{2n}{2} + 1 = 1$, because the size and order of $C_n \otimes' K_2$ are 4n and 2n respectively (see Table 3.1). Figure 4.11 shows that $\gamma(C_n \otimes' K_2) \leq 1$; thus for $n \geq 4$, $\gamma(C_n \otimes' K_2) = 1$.

From Remark 2.2.1 and Corollary 3.4.7(a) we find that $K_2 \otimes' K_n = K_{n,n}$ and thus from Ringel's result ([20]) we see that $\gamma(K_2 \otimes' K_n) = \left\lceil \frac{(n-2)^2}{4} \right\rceil$; on the other hand using the above cited remark and Corollary 3.4.7(b) we see that $K_n \otimes' K_2 = K_{n(2)}$ and from a result due to Jungerman and Ringel ([13]) we find that $\gamma(K_n \otimes' K_2) = \frac{(n-3)(n-1)}{3}$, for $n \not\equiv 2(\text{mod}3)$. We can also show that $\gamma(K_2 \otimes' P_n) = 0$ and $\gamma(P_n \otimes' K_2) = 1$, for $n \geq 4$.

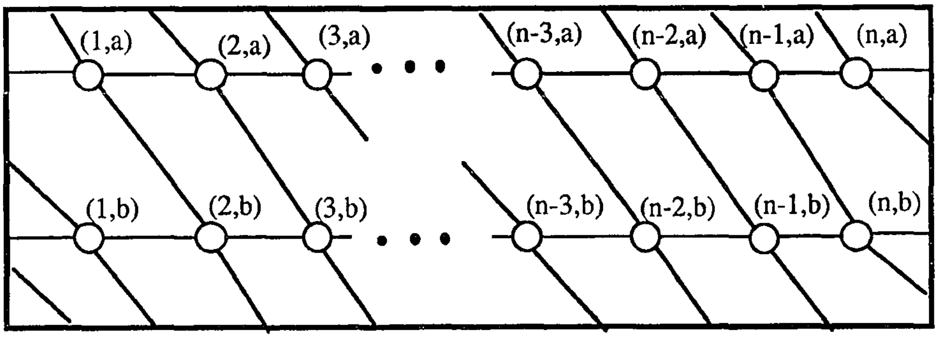


Figure 4.11

Theorem 4.2.3 (White [28]): If H is a (p,q) bipartite graph with a quadrilateral imbedding, then $\gamma(H \otimes' K_2) = \beta(H)$ (where $\beta(H) = q - p + 1$ is called the Betti number of H).

Proof: We modify each edge of H as seen in Figure 4.12(a) and thus every quadrilateral of H will be modified to look like the one shown in Figure 4.12(b). We are using $\Gamma = Z_2$, and Table 2.3(5). A look at Figure 4.12(b) shows that all the 2-sided regions satisfy $|R|_\phi = 2$ and the 4-gons $|R|_\phi = 1$; this means from Theorem 2.4.3 that the lift of each type of region will be a 4-gon, giving a 4-gon imbedding for $H \otimes' K_2$. From Corollary 3.2.5 $H \otimes' K_2$ is connected; also since H is bipartite, Corollary 3.3.5 tells us that $H \otimes' K_2$ is also bipartite. The product graph $H \otimes' K_2$ has 4q edges and 2p vertices. This means from Corollary 1.3.5

$$\gamma(H \otimes' K_2) = \frac{4q}{4} - \frac{2p}{2} + 1 \quad = q - p + 1 = \beta(H). \ \square$$

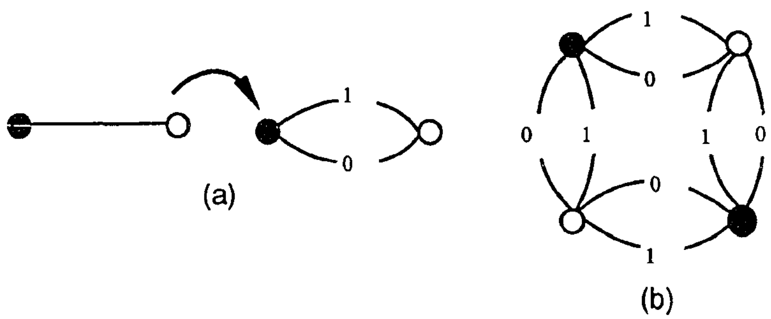


Figure 4.12

Corollary 4.2.4 Let $G_1 = K_2$ and $G_n = G_{n-1} \otimes' K_2$ for $n \geq 2$; then

$$\gamma(G_n) = 2^{2(n-2)} - 2^{n-1} + 1, \text{ for } n \geq 2.$$

Proof: From Remark 2.2.1 and Corollary 3.4.8 $G_{n-1} = K_{2^{n-2},2^{n-2}}$, meaning G_{n-1} has $(2^{n-2})^2$ edges and 2^{n-1} vertices, and we can show inductively that G_{n-1} has a quadrilateral imbedding. Thus from Theorem 4.2.3 $\gamma(G_n) = 2^{2(n-2)} - 2^{n-1} + 1$. $\square$

The last expression can be viewed as $(2^{n-2} - 1)^2$, and this is in harmony with Ringel's result on $\gamma(K_{m,m})$ where Corollary 4.2.4 is a special case with $m = 2^{n-1}$ for some positive integer n.

Theorem 4.2.5 Let H be a bipartite (p,q) graph with an orientable 4-gon imbedding. If $\Gamma = Z_2^n$, Δ is any generating set with k elements, and $G = G_\Delta(\Gamma)$, then $\gamma(H \otimes' G) = 2^{n-2}[(k+1)q - 2p] + 1$.

Proof: Modify each edge of H as in Figure 4.12(a). This time, however, instead of two edges use k + 1 edges labeled 0, 1, 2,..., k. Figure 4.13 shows how a 4-gon of H is modified in H^* according to the specifications of Table 2.3(5).

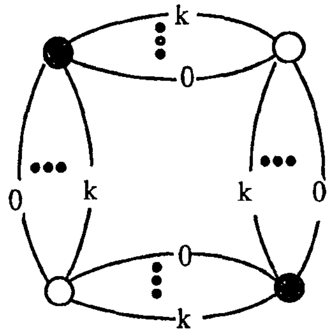


Figure 4.13

Every 2-sided region of H^* has $|R|_\phi = 2$, and thus it will lift to a 4-gon; every 4-gon of H^* satisfies the KVL and hence will also lift to a 4-gon. This means $H \otimes' G$ has a 4-gon imbedding; but from Corollaries 3.2.5 and 3.3.5 $H \otimes' G$ is connected and bipartite respectively. Thus our imbedding is minimal. We can see that H^* has $(k+1)q$ edges and p vertices, implying that $H \otimes' G$ has $2^n(k+1)q$ edges and $2^n p$

vertices. Then Corollary 1.3.5 gives us $\gamma(H \otimes' G) = \frac{2^n(k+1)q}{4} - \frac{2^n p}{2} + 1 = 2^{n-2}[(k+1)q - 2p] + 1$. ❑

Corollary 4.2.6 $\gamma(H \otimes' Q_n) = 2^{n-2}[(n+1)q - 2p] + 1$.

Proof: Take Δ as the set of standard generators of Γ in the above theorem. ❑

Corollary 4.2.7 $\gamma(C_4 \otimes' Q_n) = 2^n[n-1] + 1$.

Proof: Let $H = C_4$ in the above corollary. ❑

Theorem 4.2.8 Let $\Gamma = Z_2^n$, $|\Delta| = k \geq n$, and $G = G_\Delta(\Gamma)$; then $\gamma(K_2 \otimes' G) = 2^{n-2}[k-3] + 1$.

Proof: We modify K_2 as in Figure 4.14. Every region in K_2^* is 2-sided and satisfies $|R|_\phi = 2$; this means they all will be lifted to 4-gons. Thus we have a 4–gon imbedding for $K_2 \otimes' G$; but from Corollary 3.3.5 $K_2 \otimes' G$ is bipartite and from Corollary 3.2.5 $K_2 \otimes' G$ is connected. Hence our imbedding is minimal. The number of edges and vertices in $K_2 \otimes' G$ are $(k+1)2^n$ and 2^{n+1} respectively. The theorem then follows as a result of Corollary 1.3.5. ❑

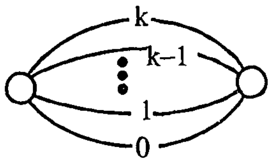


Figure 4.14

Now we give a new proof of an old result:

Corollary 4.2.9 (Ringel [19], Beineke and Harary [3]): $\gamma(Q_n) = 2^{n-3}[n-4] + 1$.

Proof: First of all from Remark 2.2.1 and Corollary 3.4.5 we see that $Q_n = K_2 \otimes' Q_{n-1}$. Then in the above theorem we replace n with n – 1 and we let $|\Delta| = n - 1$; thus $\gamma(Q_n) = \gamma(K_2 \otimes' Q_{n-1}) = 2^{n-3}[n-4] + 1$. ❑

Corollary 4.2.10 $\gamma(K_2 \otimes' K_m) = \frac{m}{4}[m-4] + 1$, when $m = 2^n$, $n \geq 1$.

Proof: Take $k = 2^n - 1$, $\Gamma = Z_2^n$ in the above theorem. ❑

Theorem 4.2.11 $\gamma(P_3 \otimes' Q_n) = 2^{n-1}[n-2] + 1$.

Proof: Let $\Gamma = Z_2^n$, Δ the standard set of generators, and $G = G_\Delta(\Gamma)$. Let P_3 be modified as shown on Figure 4.15. Then we see that we have two types of regions in P_3^*; the 2-sided regions that will be covered by 4-gons, and the outer 4-gon that will lift to 4-gons. This means we have a 4-gon imbedding for $P_3 \otimes' G$; but because this product is bipartite and connected our imbedding is minimal. Since $P_3 \otimes' G$ has $(n+1)2^{n+1}$ edges and $3(2^n)$ vertices, we see from Corollary 1.3.5 that $\gamma(P_3 \otimes' G) = 2^{n-1}[n-2] + 1$. ❑

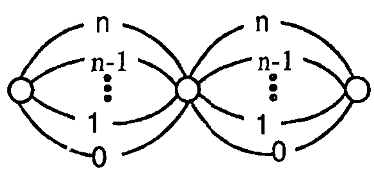


Figure 4.15

Let us note here that if $G = C_4$ (i.e., n = 2) in the above theorem, then $\gamma(P_3 \otimes' C_4) = 1$; this was the graph we had in Figure 2.7(e) (although here $C_4 = G_\Delta(Z_2 \times Z_2)$ instead of $G_{\Delta'}(Z_4)$).

Corollary 4.2.12 Let $|\Delta| = k > n$ with redundant generators; then

$$\gamma(P_3 \otimes' G) = 2^{n-1}[k-2] + 1.$$

Proof: The proof is like that of the above theorem, except now $P_3 \otimes' G$ has $(k+1)2^{n+1}$ edges and $3(2^n)$ vertices. This gives $P_3 \otimes' G$ a genus equal to $\frac{(k+1)2^{n+1}}{4} - \frac{3(2^n)}{2} + 1 = 2^{n-1}[k-2] + 1$. ❑

Corollary 4.2.13 $\gamma(P_3 \otimes' K_m) = \frac{m}{2}[m-3] + 1$, where $m = 2^n$ for $n \geq 1$.

Proof: In the above corollary take $k = 2^n - 1$ and $\Gamma = Z_2^n$. ❑

The following Theorem overlaps Theorem 4.2.5 when both r and s are even.

Theorem 4.2.14 Let $H = C_s \times C_r$, $s, r \geq 4$; then $\gamma(H \otimes' Q_n) = nrs(2^{n-1}) + 1$.

Proof: First of all, as seen in Figure 4.8, $C_s \times C_r$ has an orientable 4-gon imbedding on S_1, and a sample modification looks like the one on Figure 4.8 except now we have $n+1$ edges labeled 0 to n. Every reasoning of the theorem resembles the preceding ones, except we should note that $H \otimes' G$ is connected, by Theorem 3.2.3. By an argument similar to that in the proof of Theorem 4.2.2, $H \otimes' Q_n$ does not have a triangle and if u_2 is a vertex adjacent to both u_1 and u_3 in H, and v_1 and v_2 are adjacent in Q_n, then $(u_1, v_1), (u_2, v_1), (u_3, v_2), (u_2, v_2), (u_1, v_1)$ is a 4-cycle in $H \otimes' Q_n$ and so it has girth four. Thus any 4-gon imbedding for $H \otimes' Q_n$ would be minimal. Since our construction gives us a 4-gon imbedding, it is minimal. We know that $H \otimes' G$ has $(n+1)rs2^{n+1}$ edges and $2^n rs$ vertices; hence from Corollary 1.3.5 $\gamma(H \otimes' G) = \frac{2^{n+1}(n+1)rs}{4} - \frac{2^n rs}{2} + 1 = 2^{n-1}rsn + 1$. ❑

Corollary 4.2.15 If $|\Delta| = k > n$ in the above theorem, then

$$\gamma(H \otimes' G) = 2^{n-1}krs + 1.$$

Corollary 4.2.16 Let H be as in the above theorem and $G = K_m$, where m is a power of two, say $m = 2^n$; then $\gamma(H \otimes' K_m) = \frac{m}{2} rs(m-1) + 1$.

Proof: Take $k = 2^n - 1$ in the above corollary. ❑

4.3 The Lexical and Sublexicographic Products

Theorem 4.3.1 Let H be a bipartite (p, q)-graph with an orientable quadrilateral imbedding, and G be any graph of order $m = 2^n$ for $n \geq 2$; then

$$\gamma(H \otimes_L G) = \frac{m}{4}[(m-1)q - 2p] + 1.$$

We will illustrate this theorem with the case of $H = C_4$ and G of order four; thus we will take Γ to be Z_2^2. Let us modify C_4 according to Table 2.3(7) as seen in Figure 4.16 below.

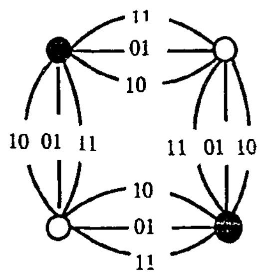


Figure 4.16

Here, as we can see from the Figure 4.16, each of the 2-sided regions has $|R|_\phi = 2$ and the two 4-gons satisfy the KVL; hence by Theorem 2.4.3 all the regions lift to 4-gons. As we can see in Figure 4.16 again, C_4^* has 12 edges and four vertices; thus $C_4 \otimes_L G$ will have 48 edges and 16 vertices. From Theorems 3.3.7 and 3.2.7(b) we get that $C_4 \otimes_L G$ is bipartite and connected; thus our imbedding is minimal. This means $\gamma(C_4 \otimes_L G) = \frac{48}{4} - \frac{16}{2} + 1 = 5$.

One interesting thing about this formula is that it is independent of the size of G; it only depends on the order of G, as long as H is fixed.

Proof of Theorem 4.3.1: Let us take $\Gamma = Z_2^n$ and modify every 4-gon of H as seen in Figure 4.16 using the bipartite nature of H, except now we will have $2^n - 1$ edges, one for each nonzero element of Γ, instead of three. Since every edge label in this modification is of order two, the 2-sided regions will lift to 4-gons; so do the 4-gons, as they all satisfy the KVL. Our modification guarantees us a 4-gon imbedding for $H \otimes_L G$. Since H is bipartite, by Theorem 3.3.7 $H \otimes_L G$ is bipartite; and because H is connected and the order of G is greater than two by Theorem 3.2.7b, $H \otimes_L G$ is connected. Thus using Corollary 1.3.5 we can get the genus of $H \otimes_L G$ provided we know the order and size of $H \otimes_L G$. The order is $p2^n$ and the size is $(2^n - 1)q2^n$.

$$\text{Thus } \gamma(H \otimes_L G) = \frac{(2^n - 1)q2^n}{4} - \frac{p2^n}{2} + 1$$

$$= 2^{n-2}[(2^n - 1)q - 2p] + 1$$

$$= \frac{m}{4}[(m - 1)q - 2p] + 1, \text{ because } m = 2^n. \ \square$$

Theorem 4.3.2 Let G be of order m, a power of two; that is $m = 2^n$, for $n \geq 2$. Then $\gamma(P_3 \otimes_L G) = \frac{m}{2}[m-4] + 1$.

Proof: We let Γ be Z_2^n . The resulting modification of P_3 looks like that in Figure 4.15, except this time we will have $2^n - 1$ edges for each edge of P_3. This gives $2(2^n - 1)\,|\Gamma|$ edges in $P_3 \otimes_L G$. We can easily show that $P_3 \otimes_L G$ is both connected and bipartite. Also, since every 2-sided region satisfies $|R|_\phi = 2$ all such regions lift to 4-gons, and the only 4-gon of P_3^* satisfies the KVL and thus it also lifts to 4-gons. We thus have as a result of Corollary 1.3.5 $\gamma(P_3 \otimes_L G) = \frac{2^{n+1}(2^n-1)}{4} - \frac{3(2^n)}{2} + 1 = 2^{n-1}[(2^n - 4] + 1$; but because $2^n = m$ we get $\gamma(P_3 \otimes_L G) = \frac{m}{2}[m-4] + 1$. ❑

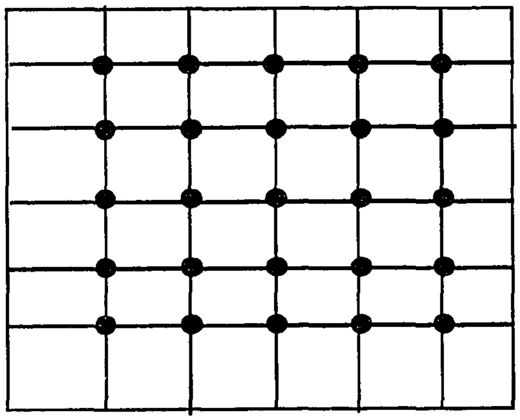
Corollary 4.3.3 $\gamma(P_3 \otimes_L C_4) = 1$.

Proof: Take Z_2^2 for Γ, giving four for the order of G. Now $m = 4$ in the above theorem gives us the desired result. (We would like to note that this graph is what we had in Example 2.5.1 as Figure 2.7(g)). Theorem 4.3.4 below overlaps Theorem 4.3.1 when both r and s are even.

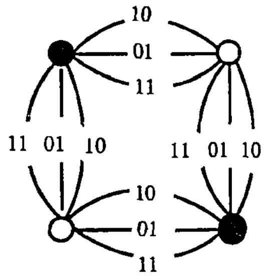
Theorem 4.3.4 Let H be the cartesian product $C_r \times C_s$ of two cycles C_r and C_s; $r, s \geq 4$. If G is of order $m = 2^n, n \geq 1$, then $\gamma(H \otimes_L G) = \frac{rsm}{2}(m-2) + 1$.

Before proving this theorem, we illustrate the case $H = C_5 \times C_5$ and $G = C_4$. Let us take Γ to be Z_2^2. Figure 4.17(a) shows a 4-gon imbedding of H on S_1. We can easily show that $C_5 \times C_5$ has girth four. And consequently $H \otimes_L G$ has girth four. If we modify every 4-gon of Figure 4.17(a) as seen in figure 4.17(b), we see that our modification is consistent and $H^* \times_\phi \Gamma$ will have a 4-gon imbedding. Also from Theorem 3.2.7b, $H^* \times_\phi \Gamma = H \otimes_L C_4$ is connected, because H is connected.

Hence given the fact that the girth of $H \otimes_L C_4$ is four we see that our imbedding is minimal. Since $H \otimes_L C_4$ has 24rs edges and 4rs vertices, Corollary 1.3.5 gives the genus to be $4rs + 1$. ❑



(a)



(b)

Figure 4.17

Proof of Theorem 4.3.4: The proof of the Theorem resembles that of our example except now we have $H \otimes_L G$ with $2^{n+1}rs(2^n - 1)$ edges and $2^n rs$ vertices. Thus, after noting that $H \otimes_L G$ is a connected graph with girth four, we conclude by Corollary 1.3.5 that $\gamma(H \otimes_L G) = \frac{2^{n+1}rs(2^n - 1)}{4} - \frac{2^n r s}{2} + 1$ which by substituting m for 2^n gives the result $\frac{rsm}{2}[m - 2] + 1$. ❑

Theorem 4.3.5 Let H be a bipartite (p,q)-graph with an orientable 4-gon imbedding. If G is any graph of order $m = 2^n$, for $n \geq 1$, then $\gamma(H \otimes_{SL} G) = \frac{m}{4}[mq - 2p] + 1$.

Proof: Let us take $\Gamma = Z_2^n$ and $G = G_\Delta(\Gamma)$ with Δ any generating set for Γ. The proof of this theorem is analogous to the proof of Theorem 4.3.1, except for the fact that now we add an edge labeled with the identity element in the modifications. We put this new edge next to the edge labeled **10.** We observe that all the regions of H^* lift to 4-gons, giving a 4-gon imbedding of $H \otimes_{SL} G$. Since H is bipartite, by Theorem 3.3.9 $H \otimes_{SL} G$ is bipartite and because H is connected and nontrivial, by Theorem 3.2.9 $H \otimes_{SL} G$ is connected. Thus the 4-gon imbedding we have is minimal. We can also see from Table 3.1 that $H \otimes_{SL} G$ has m^2q edges and mp vertices; this gives, using Corollary 1.3.5, $\gamma(H \otimes_{SL} G) = \frac{m}{4}[mq - 2p] + 1$. ❑

Theorem 4.3.6 If G has order $m = 2^n$, $n \geq 1$, then

$$\gamma(K_2 \otimes_{SL} G) = \frac{(m-2)^2}{4}.$$

Proof: Let us take Γ to be Z_2^n and modify K_2 as shown in Figure 4.18.

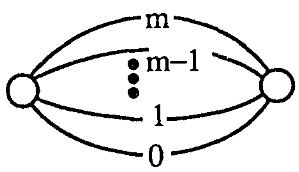


Figure 4.18

From the above figure we see that the lift of K_2^* is 4-gonal. We can also observe that $K_2 \otimes_{SL} G$ is both connected and bipartite as a result of Theorem 3.2.9 and 3.3.9 in that order. This graph is of order 2m and size m^2; thus Corollary 1.3.5 applies and we get: $\gamma(K_2 \otimes_{SL} G) = \frac{m^2}{4} - m + 1 = \frac{(m-2)^2}{4}$. ❑

We would like to mention here that, from Corollary 3.4.21(i), $K_2 \otimes_{SL} K_m = K_{m,m}$ and our result agrees with Ringel's result for $K_{m,m}$, $(m = 2^n)$.

Theorem 4.3.7 If G is of order $m = 2^n$, $n \geq 1$, then $\gamma(P_3 \otimes_{SL} G) = \frac{m}{2}(m-3)+1$.

Proof: Let us take $\Gamma = Z_2^n$ and modify P_3 as shown in Figure 4.19 below. This shows that all the regions of $P_3^* \times_\phi \Gamma$ are 4-gons. As a result of Theorem 3.2.9 and 3.3.9 $P_3 \otimes_{SL} G$ is bipartite and connected; thus by Corollary 1.3.5 we get

$$\gamma(P_3 \otimes_{SL} G) = \frac{2m^2}{4} - \frac{3m}{2} + 1 = \frac{m}{2}(m-3) + 1. \;\square$$

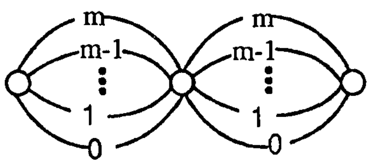


Figure 4.19

Corollary 4.3.8 $\gamma(P_3 \otimes_{SL} C_4) = 3$.

Proof: In the above theorem take $m = 4$. $\square$

We note that this was the graph we encountered in Figure 2.7(i) in Example 2.5.1. The following theorem overlaps Theorem 4.3.5 when both r and s are even.

Theorem 4.3.9 Let H be the cartesian product $C_r \times C_s$ of two cycles C_r and C_s, $r, s \geq 4$. If G is of order $m = 2^n$, $n \geq 1$, then $\gamma(H \otimes_{SL} G) = \frac{rsm}{2}(m-1) + 1$.

Proof: Let us take $\Gamma = Z_2^n$ and a 4-gon imbedding of H on S_1, as in Theorem 4.3.4. We modify H by replacing every edge with edges labeled 0, 1, 2, ..., 2^n, one for each element of Γ. In the modification of H we discussed in the proof of Theorem 4.3.4 insert an edge labeled 0, and we can see that still we have all the 2-sided regions satisfying $|R|_\phi = 2$, and the 4-gons satisfy the KVL. This gives us a 4-gon imbedding for $H \otimes_{SL} G$. From Theorem 3.2.10 $H \otimes_{SL} G$ is connected; since the girth of H is four, the girth of $H \otimes_{SL} G$ is also four. This makes our imbedding minimal, and an application of Corollary 1.3.5 with size $2rsm^2$ and order rsm gives $\gamma(H \otimes_{SL} G) = \frac{rsm}{2}[\, m - 1] + 1$. ❑

4.4 Rejection, Exclusion, and Total Exclusion

If G_1 and G_2 are two disjoint graphs, then the graph $G_1 \cup G_2$ is the graph where $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$.

Theorem 4.4.1 Let $H = K_2 \cup K_1$; then for $n \geq 3$, $\gamma(H/Q_n) = 2^{n-1}[\, 2^n - n - 4\,] + 1$.

Before going on with the proof of the theorem, we present an example for the case n = 3. We take $\Gamma = Z_2^3$ and $\Delta = \{100, 010, 001\}$ so that $G_\Delta(\Gamma) = Q_3$; then $\Omega^* = \{111, 011, 101, 110\}$. Modify the edges of $\overline{H}$ as shown in Figure 4.20.

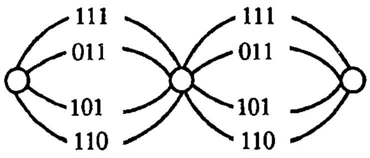


Figure 4.20

This figure shows that all the digons have $|R|_\phi = 2$, which means $H^* \times_\phi \Gamma$ has all its regions 4-gons, giving a 4-gon imbedding for H/G. Since it is not hard to see that both $\overline{H}$ and $\overline{G}$ are connected and that $\overline{G}$ contains a 3-cycle, by Theorem 3.2.13 we obtain that H/G is connected. Also, by Theorem 3.3.13, because $\overline{H}$ is bipartite, H/G is bipartite; these two results allow us to use Corollary 1.3.5 to find that $\gamma(H/G) = 5$. ❑

Proof of Theorem 4.4.1: For the general case we take $\Gamma = Z_2^n$, $n \geq 3$, and Δ the standard generating set; then $G_\Delta(\Gamma) = Q_n$. The proof is similar to the above example. First we see that $\overline{H} = \overline{K_2 \cup K_1} = P_3$, and when $n \geq 3$ $|\Omega^*| = 2^n - n > 2^{n-1}$. The latter says that the degree of each vertex in $\overline{G}$ is greater than 2^{n-1}, implying that $\overline{G}$ is connected. Also it is not hard to see that $\overline{G}$ contains a 3-cycle; thus by Theorem 3.2.13 H/G is connected. Because $\overline{H} = P_3$ is bipartite, from Theorem 3.3.13 H/G is bipartite. In the modification of H we have $|\Omega^*|$ edges for each edge in $\overline{H}$; thus the number of edges in H^* is $2(2^n - n - 1)\, 2^n$ and of course the number of vertices is $3(2^n)$. Thus it follows that $\gamma(H/G) = 2^{n-1}[\,2^n - n - 4] + 1$. ❑

Theorem 4.4.2 Let H be a graph such that $\overline{H}$ is a connected bipartite graph of order p and size q with an orientable 4-gon imbedding; then

$$\gamma(H/Q_n) = 2^{n-2}[(2^n - n - 1)q - 2p] + 1.$$

Proof: We take $\Gamma = Z_2^n$, $n \geq 3$, and Δ the standard generating set; then $G_\Delta(\Gamma) = Q_n$. From the hypotheses we see that $\overline{H}$ is both connected and bipartite, and $\overline{Q_n}$ is connected with an odd cycle from the proof of Theorem 4.4.1. Thus from Theorems 3.2.13 and 3.3.13 we see that H/G is connected and bipartite. For the remaining part of the proof we refer the reader to the proof of Theorem 4.1.7. The

only thing we should realize is that H/G has $|\Omega^*| q2^n$ edges and $p2^n$ vertices, but $|\Omega^*| = (2^n - n - 1)$; thus $\gamma(H/G) = 2^{n-2}[(2^n - n - 1)q - 2p] + 1$. ❑

Theorem 4.4.3 For $n \geq 3$, $\gamma(K_2 \Theta Q_n) = 2^{n-2}[2^n - n - 4] + 1$.

Proof: If we take $\Gamma = Z_2^n$, $n \geq 3$, and Δ the standard generating set, then $G_\Delta(\Gamma) = Q_n$. Replace the only edge of K_2 by $2^n - n$ edges to get K_2^*; this will give a modification with only 2-sided regions and for each region $|R|_\phi = 2$. This means that in $K_2^* \times_\phi \Gamma$ all the regions are 4-gons. From Theorem 3.2.15 and Theorem 3.3.15, together with the facts that K_2 is connected and bipartite and that $\overline{Q_n}$ is connected, we conclude that $K_2 \Theta Q_n$ is connected and bipartite. In addition to these we can easily see that $K_2 \Theta Q_n$ has order and size of 2^{n+1} and $2^n(2^n - n)$ respectively. Then Corollary 1.3.5 gives $\gamma(K_2 \Theta Q_n) = 2^{n-2}[2^n - n - 4] + 1$. ❑

Theorem 4.4.4 The graph $P_3 \Theta C_4$ is planar; that is $\gamma(P_3 \Theta C_4) = 0$.

Proof: We regard C_4 as $G_\Delta(\Gamma)$ for $\Gamma = Z_2^2$ and $\Delta = \{10, 01\}$; then $\Omega^* \cup \{e\} = \{11, 00\}$. We modify P_3 as shown in Figure 4.21.

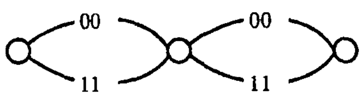


Figure 4.21

We can see that the two digons each satisfy $|R|_\phi = 2$ and the only 4-gon of the figure satisfies the KVL; thus $P_3^* \times_\phi \Gamma$ has a 4-gon imbedding. A further examination of the figure and the local group Γ_v at a vertex v shows that $[\Gamma : \Gamma_v] = 2$, which from Theorem 2.4.3 tells us that $P_3 \Theta C_4$ has two isomorphic components. By the bipartiteness of P_3, Theorem 3.3.15 guarantees the

bipartiteness of $P_3 \Theta C_4$. Then the use of Corollary 1.3.7 shows that $\gamma(P_3 \Theta C_4) = \frac{16}{4} - \frac{12}{2} + 2 = 0$. ❑ (This graph was the graph we saw in Figure 2.10(e).)

Theorem 4.4.5 For $n \geq 3, \gamma(P_3 \Theta Q_n) = 2^{n-1}[2^n - n - 3] + 1$.

Proof: Let us take $\Gamma = Z_2^n$, $n \geq 3$, and Δ the standard generating set; then $G_\Delta(\Gamma) = Q_n$. Modify P_3 as shown in Figure 4.21, except that we have $|\Omega^* \cup \{e\}|$ edges for each edge of H this time. Thus $P_3 \Theta Q_n$ will have $2(2^n - n)\ 2^n$ edges and $3(2^n)$ vertices. And since P_3 and $\overline{Q_n}$, for $n \geq 3$, are both connected and P_3 is bipartite, by Theorem 3.2.15 and Theorem 3.3.15 $P_3 \Theta Q_n$ is both connected and bipartite. Similar to the way we reasoned for Theorem 4.4.3, we get a 4-gon imbedding for $P_3 \Theta Q_n$. Thus our imbedding is minimal and by Corollary 1.3.5 $\gamma(P_3 \Theta Q_n) = 2^{n-1}[2^n - n - 3] + 1$. ❑

Theorem 4.4.6 Let $n \geq 3$; then $\gamma(K_2 \Theta Q_n) = 2^{n-2}[2^n - n - 5] + 1$.

Proof: We take Γ to be Z_2^n for $n \geq 3$ and for Δ we take the standard generating set, so that $G_\Delta(\Gamma) = Q_n$. We modify K_2 by replacing the edge of K_2 by $2^n - n - 1$ edges, one for each element in Ω^*. This means that $K_2 \Theta Q_n$ has $2^n(2^n - n - 1)$ edges. Since K_2 and $\overline{Q_n}$ are both connected, the former is bipartite and the latter contains an odd cycle, Theorem 3.2.16 tells us that $K_2 \Theta Q_n$ is connected, while Theorem 3.3.16 guarantees that it is bipartite. Since the order of our graph is 2^{n+1}, an application of Corollary 1.3.5 gives the desired result:

$$\gamma(K_2 \Theta Q_n) = 2^{n-2}[2^n - n - 5] + 1. \ ❑$$

Corollary 4.4.7 Let $\Gamma = Z_2^n$, $G = G_\Delta(\Gamma)$ for some generating set Δ, where $|\Delta^*| = m$ and $\overline{G}$ is connected with an odd cycle. Then

$$\gamma(K_2 \Theta G) = 2^{n-2}[2^n - m - 5] + 1.$$

Proof: The same reasoning as in the proof of the theorem works here except that now $|\Omega^*| = 2^n - m - 1$. ❑

Theorem 4.4.8 Let $n \geq 3$; then $\gamma(P_3 \underline{\Theta} Q_n) = 2^{n-1}[2^n - n - 4] + 1$.

Proof: Let $\Gamma = Z_2^n$, $n \geq 3$, with Δ the standard generating set so that $G_\Delta(\Gamma) = Q_n$. The proof is similar to that of Theorem 4.4.5 with the only exception that we have $|\Omega^*|$ edges replacing each edge of H this time. This means that the size of $P_3 \underline{\Theta} Q_n$ is $2^{n+1}(2^n - n - 1)$ and of course its order is $3(2^n)$. We can also verify that $P_3 \underline{\Theta} Q_n$ is connected and bipartite from Theorems 3.2.16 and 3.3.16 and the properties of P_3 and $\overline{Q_n}$ that we discused in the previous theorems. Thus from Corollary 1.3.5 we have $\gamma(P_3 \underline{\Theta} Q_n) = \frac{2^{n+1}[2^n - n - 1]}{4} - \frac{3(2^n)}{2} + 1 = 2^{n-1}[2^n - n - 4] + 1$. ❑

We observe that when $n = 2$ in Z_2^n, P_3^* shows that $[\Gamma : \Gamma_v] = 4$, so that $P_3 \underline{\Theta} C_4 = 4P_3$ and thus $\gamma(P_3 \underline{\Theta} Q_n) = 0$; this was the graph we saw in Figure 2.10(f).

4.5 The Strong Tensor Product

As we have seen in Remark 2.2.1, $H \otimes' G \cong G \underline{\otimes} H$; thus all the genus results we got in Section 2 of this chapter will apply for this product as well. Since the second factor is always regarded, in this dissertation, as a Cayley graph, the results that we are going to establish in this section are going to be different from those we already obtained in Section 4.2.

Theorem 4.5.1 $\gamma(K_4 \underline{\otimes} C_4) = 9$.

Proof : Take $C_4 = G_\Delta(\Gamma)$ where $\Gamma = Z_4$ and $\Delta = \{1\}$; thus we modify K_4 as seen in Figure 4.22. From this figure we see that we have three types of regions: the 4-

gons, the digons, and the loops. The 4-gons satisfy the KVL, the digons all have $|R|_\phi = 2$, and for the loops $|R|_\phi = 4$; thus all three will lift to 4-gons. This means we have a 4-gon imbedding for $K_4^* \times_\phi Z_4 = K_4 \underline{\otimes} C_4$. Since both factors are connected and the second factor is bipartite, $K_4 \underline{\otimes} C_4$ is connected and bipartite by Theorems 3.3.4 and 3.2.4 respectively. Thus the 4-gon imbedding we have is minimal. We can see that the size and order of $K_4 \underline{\otimes} C_4$ are 64 and 16 in that order; hence from Corollary 1.3.5 $\gamma(K_4 \underline{\otimes} C_4) = 9$. ❑

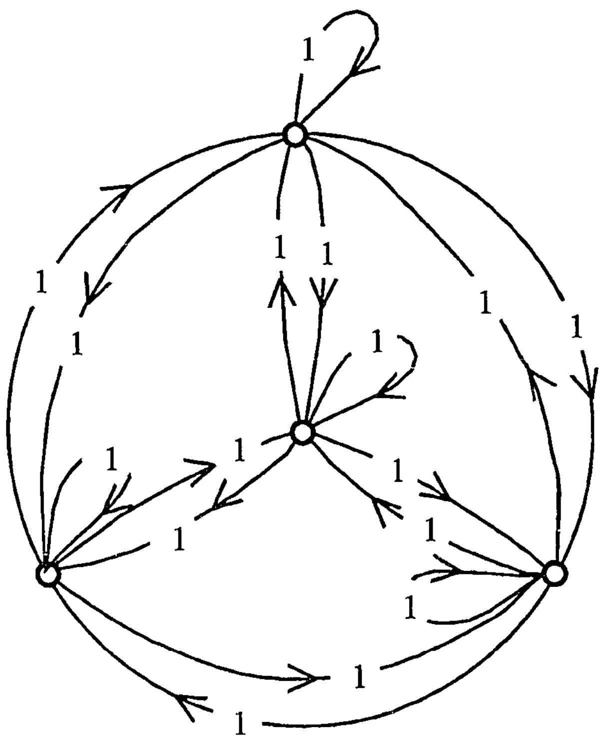


Figure 4.22

If we view C_4 as $K_{2,2}$ and use Theorem 3.4.6, we observe that $K_4 \underline{\otimes} C_4 = K_{8,8}$ and thus Theorem 4.5.1 is a special case of Ringel's result for $K_{m,n}$. It is included here because of the new proof.

By taking the modification of K_2 in Figure 4.23, using $\Gamma = Z_3$ and $\Delta = \{1\}$, we get $\gamma(K_2 \underline{\otimes} C_3) = 0$. This result follows from the fact that, in the modification, we have a triangular imbedding for $K_2 \underline{\otimes} C_3$. Then an application of Theorem 1.3.4 for $p = 6$ and $q = 12$ gives genus 0. We could also have used Corollary 3.4.7 to see this graph as $K_{3(2)}$ and this is well known to be planar. Our method usually gives a minimal 4-gon imbedding; but here is a case of a triangular imbedding, even though it is a small order case.

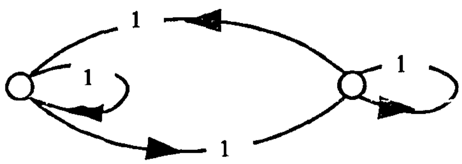


Figure 4.23

If in Figure 4.23 the loop on the vertex on the right is put inside by reflection on a vertical axis, the resulting modification of K_2 gives a genus imbedding for $K_2 \underline{\otimes} C_4$ on the torus, where Γ is now taken to be Z_4; of course this can be independently verified from Theorem 3.4.6, because $K_2 \underline{\otimes} C_4 = K_2 \underline{\otimes} K_{2,2} = K_{4,4}$ and the last one is known to be of genus one.

Theorem 4.5.2 (White [28]): If $H = G_{\Delta'}(\Gamma_1)$, with $\Delta' = \{a, b, a + b\}$ for Γ_1 abelian, then $\gamma(H \underline{\otimes} K_{4,4}) = 1 + 24\,|\Gamma_1|$.

Proof: We first observe that H can be viewed as $G_{\Delta'}(\Gamma_1)$ imbedded in S_1, where $\Delta' = \{a, b, a+b\}$, as an $|\Gamma_1|$-fold covering graph of the voltage graph imbedded in the torus as shown in Figure 4.24.

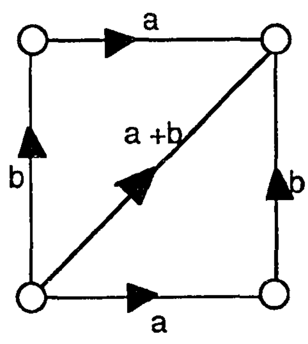


Figure 4.24

For more discussion on this see White [24, page 167]. As for the $K_{4,4}$, let us regard it as $G_\Delta(\Gamma)$ where $\Gamma = Z_2 \times Z_4$ and $\Delta = \{01, 11\}$; see Example 2.4.2. Because $K_{4,4}$ is bipartite, Theorem 3.3.4 tells us that $H \underline{\otimes} K_{4,4}$ is bipartite. Since H and $K_{4,4}$ are both connected, by Theorem 3.2.4 we conclude that $H \underline{\otimes} K_{4,4}$ is also connected. Now let us modify H as seen in Figure 4.25 below. (In this figure the group elements 01, 11, 03, and 13 are represented for an ease of writing by 1, 2, 3, and 4 in that order.) This Figure shows that H^* has three types of regions: 4-gons, digons, and loops. The 4-gons satisfy the KVL, the digons have $|R|_\phi = 2$, and the loops have $|R|_\phi = 4$; thus all of them will lift to 4-gons. This gives a 4-gon imbedding for $H \underline{\otimes} K_{4,4}$. Since this graph is bipartite, our imbedding is minimal. If we can find out what the size and order of $H \underline{\otimes} K_{4,4}$ are, then since $H \underline{\otimes} K_{4,4}$ is connected, Corollary 1.3.5 as usual will give us $\gamma(H \underline{\otimes} K_{4,4})$.

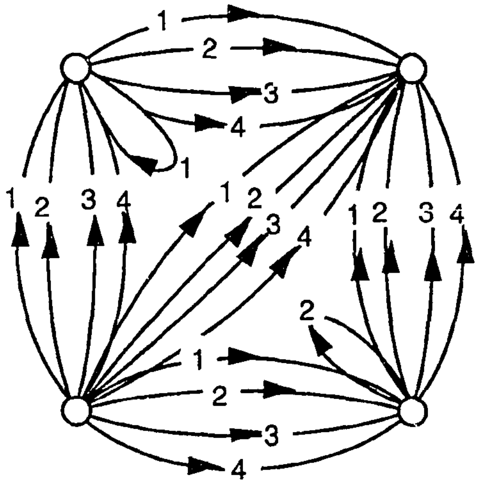


Figure 4.25

Since H is seen as an $|\Gamma_1|$-fold covering of the one-vertex, three-loop pseudograph in Figure 4.24, we see that H has $|\Gamma_1|$ vertices and $3|\Gamma_1|$ edges. But from Figure 4.25 we see that H^* has $|\Gamma_1|$ vertices and $14|\Gamma_1|$ edges. This means that $H^* \times_\phi \Gamma = H \underline{\otimes} K_{4,4}$ has $8|\Gamma_1|$ vertices and $112|\Gamma_1|$ edges. Therefore :

$$\gamma(H \underline{\otimes} K_{4,4}) = \frac{112|\Gamma_1|}{4} - \frac{8|\Gamma_1|}{2} + 1 = 24|\Gamma_1| + 1. \square$$

Corollary 4.5.2a $\gamma(K_7 \underline{\otimes} K_{4,4}) = 169$.

Proof: Take $a = 1$ and $b = 2$, $\Gamma_1 = Z_7$ in the theorem. $\square$

Corollary 4.5.2b $\gamma(K_{4(2)} \underline{\otimes} K_{4,4}) = 193$.

Proof: Take $a = 1$ and $b = 2$, $\Gamma_1 = Z_8$ in the above theorem. $\square$

Corollary 4.5.2c $\gamma(K_{3(3)} \underline{\otimes} K_{4,4}) = 217$.

Proof: Take a= 10 and b = 01, $\Gamma_1 = Z_3 \times Z_3$ in Theorem 4.5.2. $\square$

Theorem 4.5.3 Let $H = G_{\Delta'}(\Gamma_1)$, with $\Delta' = \{a, b, c, d, b-a, c-b, d-c, a-d\}$ for Γ_1 abelian, with no generator an involution; then $\gamma(H \underline{\otimes} K_{8,8}) = 264\,|\Gamma_1| + 1$.

Proof: We first note that H can be viewed as an $|\Gamma_1|$-fold covering of the following one-vertex, eight-loop pseudo graph in Figure 4.26 imbedded in S_2. From this figure we see that H has $|\Gamma_1|$ vertices and $8\,|\Gamma_1|$ edges. (The latter is true due to the fact that the eight edges on the boundary of the octagon will be identified in pairs to give only four edges, since we have the standard representation for S_2 given by those edges as $aba^{-1}b^{-1}cdc^{-1}d^{-1}$.)

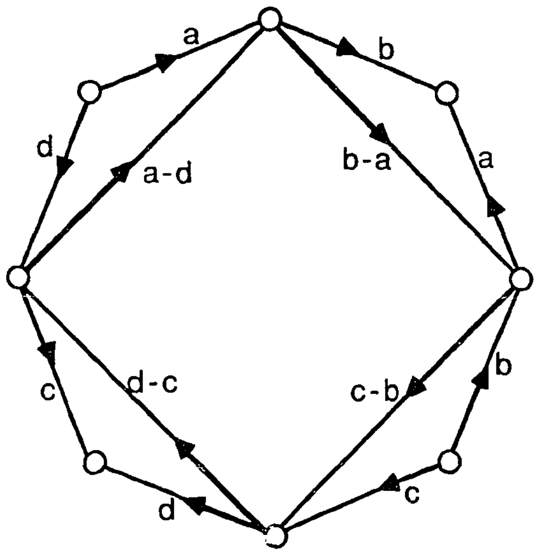
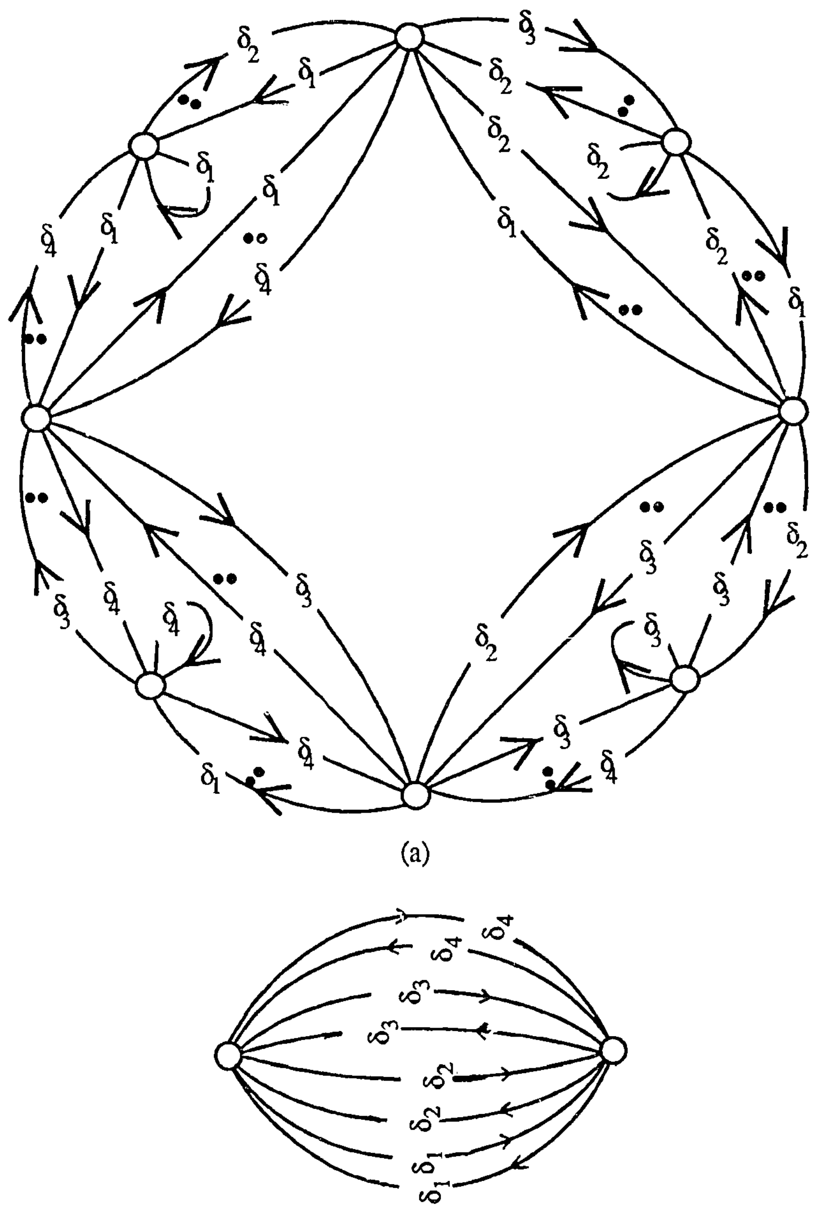


Figure 4.26

We next proceed to modify H. (It should be noted here that we are not actually modifying H but the pseudograph imbedded in S_2 that will lift to the modification of H). Note that $K_{8,8}$ can be regarded as $G_\Delta(\Gamma)$ where $\Gamma = Z_2 \times Z_2 \times Z_4$ and

$\Delta = \{001, 101, 011, 111\}$. We arrange the arcs of Δ^* as seen in Figure 4.27(b) and Figure 4.27(a) shows the modification.



(a)

(b)

Figure 4.27

We use $\delta_1, \delta_2, \delta_3, \delta_4$ to stand for 001, 101, 011, 111 in that order. Figure 4.27 (a) shows that H^* has $68\,|\Gamma_1|$ edges and $|\Gamma_1|$ vertices. This means that $H^* \times_\phi \Gamma = H \underline{\otimes} K_{8,8}$ will have $16\,|\Gamma_1|$ vertices and $1088\,|\Gamma_1|$ edges. Since $K_{8,8}$ is bipartite, by Theorem 3.3.4 $H \underline{\otimes} K_{8,8}$ is also bipartite. It is not hard to see, using Theorem 3.2.4, that it is connected. We also see that we have three types of regions in H^* : 4-gons, digons, and loops. The 4-gons satisfy the KVL, the digons have $|R|_\phi = 2$, and the loops have $|R|_\phi = 4$; thus all of them will lift to 4-gons. This gives a 4-gon imbedding for $H \underline{\otimes} K_{8,8}$. Since $H \underline{\otimes} K_{8,8}$ is bipartite our imbedding is minimal. The genus of $H \underline{\otimes} K_{8,8}$ then follows from Corollary 1.3.5 as follows:

$$\gamma(H \underline{\otimes} K_{8,8}) = \frac{1088\,|\Gamma_1|}{4} - \frac{16\,|\Gamma_1|}{2} + 1 = 264\,|\Gamma_1| + 1. \ \square$$

Theorem 4.5.4 If $H = G_{\Delta'}(\Gamma_1)$, with $\Delta' = \{a, b, b-a\}$ for Γ_1 abelian and no element of Δ' an involution, then $\gamma(H \underline{\otimes} Q_2) = 5\,|\Gamma_1| + 1$.

Proof: We first observe that H can be viewed as $G_{\Delta'}(\Gamma_1) \triangleleft S_1$, where $\Delta' = \{a, b, b-a\}$, as an $|\Gamma_1|$–fold covering graph of a voltage graph imbedded in the torus similar to the one shown in Figure 4.24 except this time the arrows on the edge labeled a in that figure are in the opposite direction, and the label on the diagonal arc is $b-a$ instead of $a+b$. As for Q_2, let us regard it as $G_\Delta(\Gamma)$ where $\Gamma = Z_2 \times Z_2$ and $\Delta = \{10, 01\}$. Because $Q_2 = C_4$ is bipartite, Theorem 3.3.4 tells us that $H \underline{\otimes} Q_2$ is bipartite. Since H and Q_2 are both connected, by Theorem 3.2.4 we conclude that $H \underline{\otimes} Q_2$ is also connected. Now let us modify H as seen in Figure 4.28. In this figure we have two types of regions, 4–gons and digons. The 4–gons satisfy the KVL and the digons have $|R|_\phi = 2$. As a result of Theorem 2.4.3 all of these regions will lift to 4–gons, giving a 4–gon imbedding for a bipartite graph $H \underline{\otimes} Q_2$; thus we conclude that our imbedding is minimal. Since H has $|\Gamma_1|$ vertices and $3|\Gamma_1|$

edges, H^* has $|\Gamma_1|$ vertices and $7|\Gamma_1|$ edges. Thus $H \underline{\otimes} Q_2$ has order and size $4|\Gamma_1|$ and $28|\Gamma_1|$, respectively. Using Corollary 1.3.5 we get :

$$\gamma(H \underline{\otimes} Q_2) = \frac{28|\Gamma_1|}{4} - \frac{4|\Gamma_1|}{2} + 1 = 5|\Gamma_1| + 1. \square$$

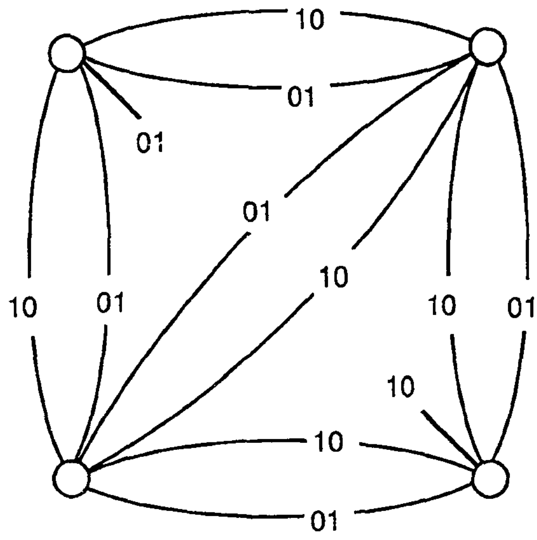


Figure 4.28

Theorem 4.5.5 If $H = G_{\Delta'}(\Gamma_1)$, with $\Delta' = \{a, b, c, d, b-a, c-b, d-c, a-d, b-d\}$ for Γ_1 abelian and no element of Δ' an involution, then

$$\gamma(H \underline{\otimes} Q_6) = 880\,|\Gamma_1| + 1.$$

Proof: Let us regard H as an $|\Gamma_1|$-fold covering of the one-vertex, nine-loop pseudograph of Figure 4.29 imbedded in S_2. For Q_6 we take $\Gamma = Z_2^6$, and Δ as the set of standard generators. Use the modification of H as shown in Figure 4.30. From this modification we see that we have two types of regions in H^*: digons that have $|R|_\phi = 2$ and 4-gons that have $|R|_\phi = 1$; thus all of these regions will be lifted to 4-gons, giving a 4-gon imbedding for $H \underline{\otimes} Q_6$. Since Q_6 is bipartite Theorem 3.3.4

tells us that $H \underline{\otimes} Q_6$ is also bipartite; by Theorem 3.2.4 $H \underline{\otimes} Q_6$ is also connected; hence our imbedding is minimal. We then observe that H has $|\Gamma_1|$ vertices and $9|\Gamma_1|$ edges. This makes H^* to have $2^6[\,6(9|\Gamma_1|) + 3|\Gamma_1|\,] = 2^6(57|\Gamma_1|)$ edges and $2^6|\Gamma_1|$ vertices; then from Corollary 1.3.3 we get:

$$\gamma(H \underline{\otimes} Q_6) = \frac{2^6(57|\Gamma_1|)}{4} - \frac{2^6|\Gamma_1|}{2} + 1 = 880|\Gamma_1| + 1. \ \square$$

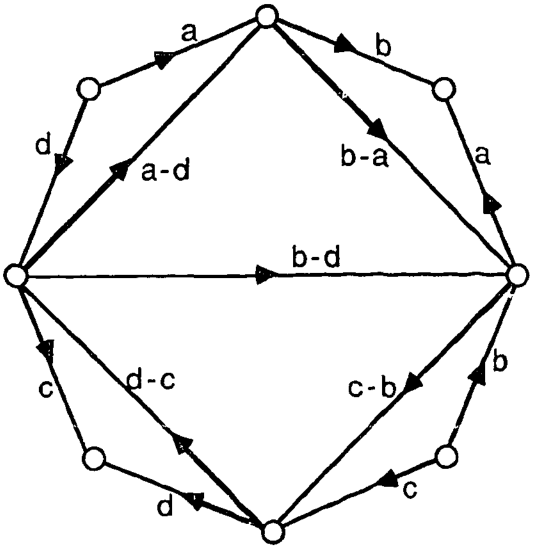


Figure 4.29

If in Theorems 4.5.4 and 4.5.5 Δ' is defined to be $\{a_1, a_2, a_2 - a_1\}$ and $\{a_1, a_2, a_3, a_4, a_2 - a_1, a_3 - a_2, a_4 - a_3, a_1 - a_4, a_4 - a_2\}$ respectively we get the following theorem as the next case.

Theorem 4.5.6 Let $H = G_{\Delta'}(\Gamma_1)$, with $\Delta' = \{a_1, a_2, a_3, a_4, a_5, a_6, a_2 - a_1, a_3 - a_2, a_4 - a_3, a_5 - a_4, a_6 - a_5, a_1 - a_6, a_2 - a_6, a_4 - a_2, a_6 - a_4\}$ for Γ_1 abelian and Δ' with no involutions; then $\gamma(H \underline{\otimes} Q_{10}) = 39168\,|\Gamma_1| + 1$.

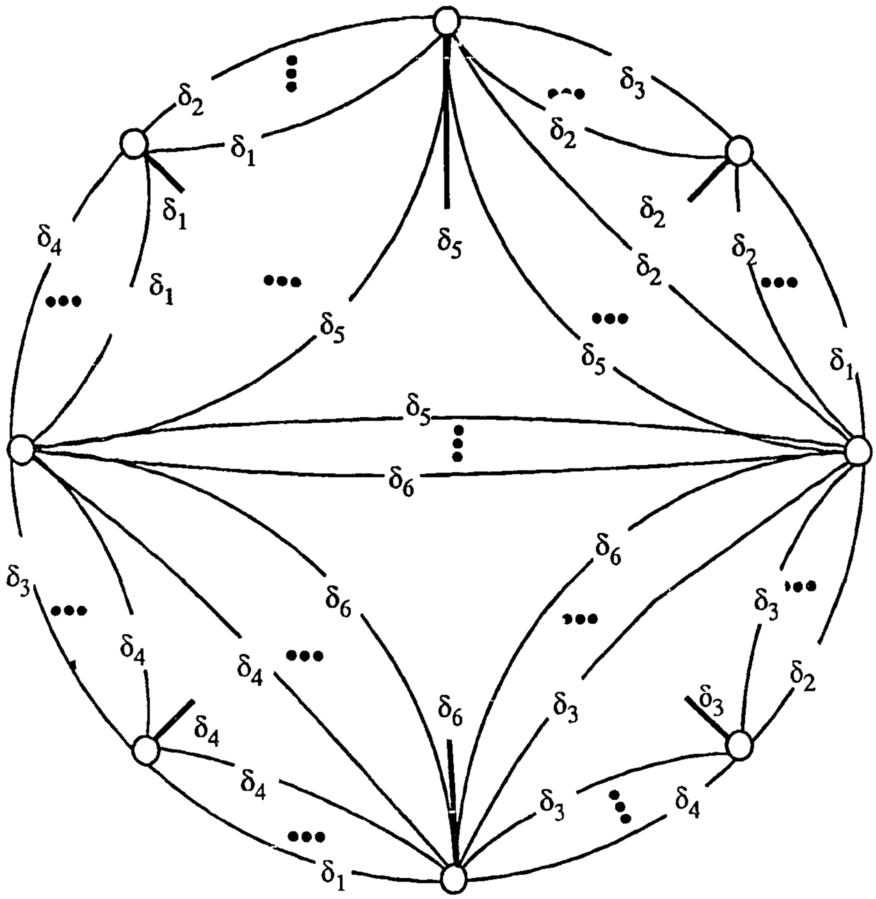


Figure 4.30

Proof: Let us view H as a covering of the one-vertex, 15-loop, voltage graph imbedded in S_3. The technique will almost be similar to the one we encountered in the proof of Theorem 4.5.4, but this time we are using the standard form of S_3. See Figure 4.31. Then H will have $15|\Gamma_1|$ edges and $|\Gamma_1|$ vertices. We take $\Gamma = Z_2^{10}$, with Δ the standard set of generators, to get Q_{10}. Thus H^* will have $155|\Gamma_1|$

edges and still $|\Gamma_1|$ vertices. Since Q_{10} is bipartite, $H \underline{\otimes} Q_{10}$ is also bipartite by Theorem 3.3.4, and by Theorem 3.2.4 it is connected.

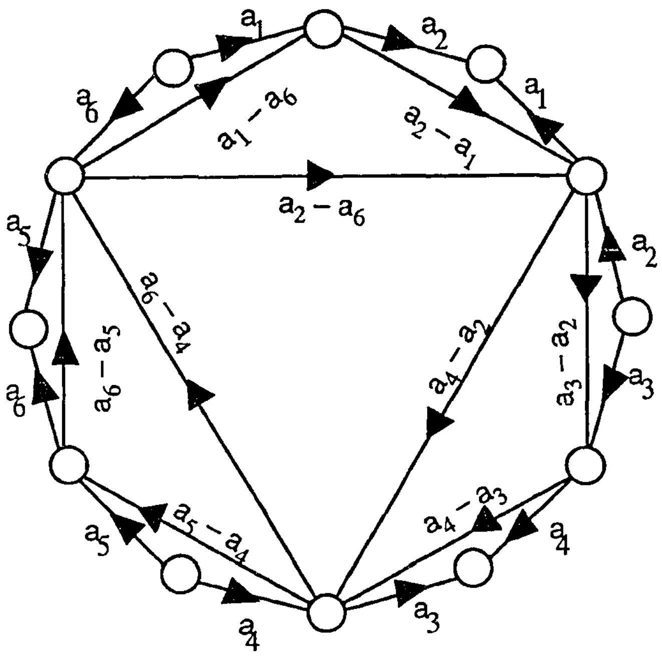


Figure 4.31

Using arguments similar to those we had in the proof of Theorem 4.5.4 and Figure 4.32, we find out that we have a 4-gon imbedding for $H \underline{\otimes} Q_{10}$; but due to the fact that $H \underline{\otimes} Q_{10}$ is bipartite, our imbedding becomes minimal. Now $H \underline{\otimes} Q_{10}$ has $2^{10}(155|\Gamma_1|)$ edges and $2^{10}|\Gamma_1|$ vertices; hence from Corollary 1.3.5 $\gamma(H \underline{\otimes} Q_{10})$ $= \dfrac{2^{10}(155|\Gamma_1|)}{4} - \dfrac{2^{10}|\Gamma_1|}{2} + 1 = 39168|\Gamma_1| + 1.$ ❑

We extend the construction of Theorems 4.5.4, 4.5.5, and 4.5.6 for $k = 1, 2,$ and 3 by extending Δ' for Γ_1 abelian in a natural way.

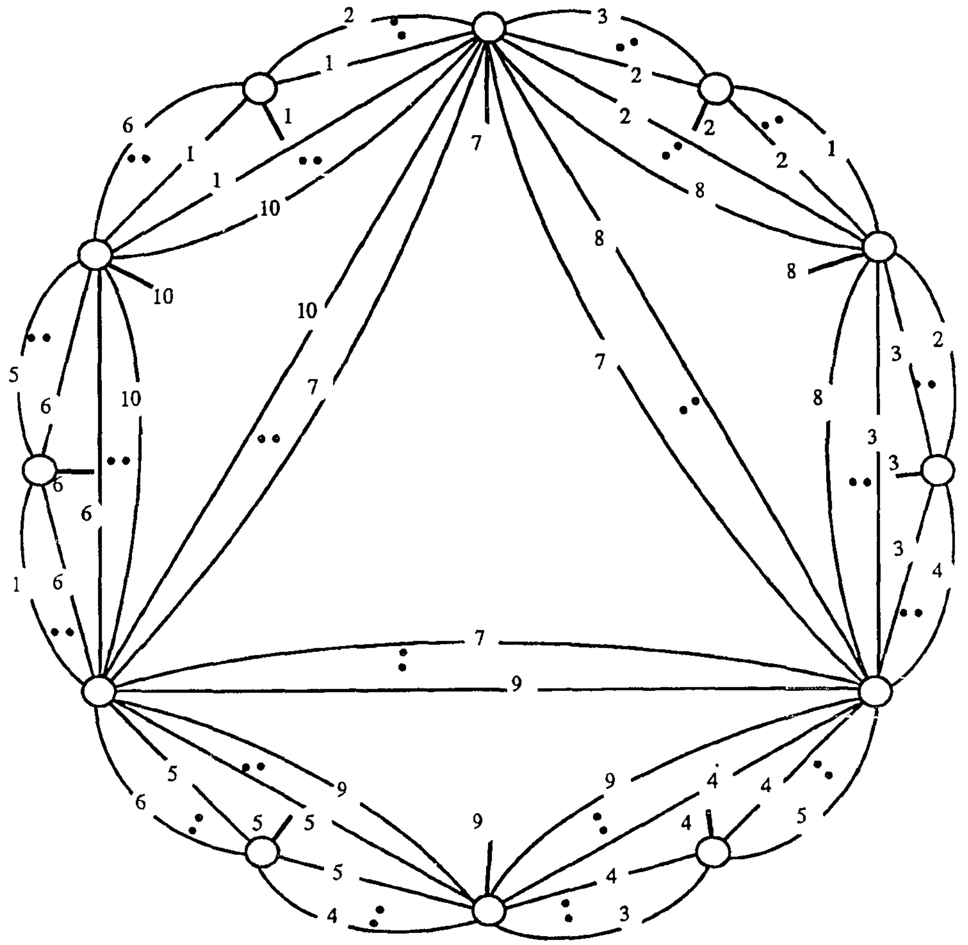


Figure 4.32

Theorem 4.5.7 Let H be $G_{\Delta'}(\Gamma_1)$ for Δ' extended as mentioned above for Γ_1 abelian, so as H could be regarded as an $|\Gamma_1|$-fold covering of a one-vertex, $6k-3$ loop, pseudo-graph imbedded in S_k. Then for $n = 4k-2$, $k \geq 1$,

$$\gamma(H \underline{\otimes} Q_n) = 2^{4k-4}|\Gamma_1|\left[(2k-1)(12k-5)-2\right] + 1.$$

Proof: The modification of H extends those of the preceding theorems, and the proof is an extension as well. We have $|\Gamma_1|(6k-3)n + |\Gamma_1|\frac{n}{2}$ edges for H^*. This is because each of the $|\Gamma_1|(6k-3)$ edges is replaced by n edges (one for each generator of Z_2^n in the standard generating set Δ of Z_2^n). Also we have n half edges below, giving $\frac{n}{2}|\Gamma_1|$ edges above. It is also clear that, as in the earlier proofs, $H \underline{\otimes} Q_n$ is bipartite and all the regions of H^* lift to 4-gons giving us a 4-gon imbedding that is minimal. Thus from Corollary 1.3.5 we have

$$\gamma(H \underline{\otimes} Q_n) = \frac{2^n [\,|\Gamma_1|(6k-3)\,n + |\Gamma_1|\frac{n}{2}\,]}{4} - \frac{2^n|\Gamma_1|}{2} + 1$$

$$= 2^{n-2}\left(|\Gamma_1|\, n\,(6k-\tfrac{5}{2})\right) - 2^{n-1}|\Gamma_1| + 1$$

$$= 2^{n-3}(|\Gamma_1|\, n\,(12k-5) - 4|\Gamma_1|) + 1$$

$$= 2^{4k-4}|\Gamma_1|\big[(2k-1)(12k-5) - 2\big] + 1. \quad \square$$

Theorem 4.5.8: If G is empty, then $H[G] = H \otimes_{SL} G$.

Proof: Because G is empty, the only edges that appear in H[G] are of type (iii); this means $H[G] = H \otimes_{SL} G$. $\square$

As a result of this theorem, all the results listed in Theorems 4.3.5 – 4.3.7 and 4.3.9, referring to the sublexicographic product work for the lexicographic product if the second factor G is empty.

Theorem 4.5.9 If G is empty, then $H \nabla G = H \vee G = H \otimes_{SL} G = H[G]$.

Proof: When G is empty the edge types (ii) and (vii) are, in combination, equivalent to edge type (iii); this is the only type present in $H \nabla G$ or $H \vee G$ when G is

empty. This gives the first equality. Edge type (iii) is also the only edge type in the definition of $H \otimes_{SL} G$, giving the second equality. The last equality is that of Theorem 4.5.8. ❑

As a consequence of Theorem 4.5.9, all that we proved for the sublexicographic product in Theorems 4.3.5 —4.3.7 and 4.3.9 remain true for both the disjunction and symmetric difference when the second factor G is taken to be empty.

In this chapter we were dealing mostly with graph products that did not use edge type (i). In most cases the use of this edge type makes our imbedding technique inefficient. One of the reasons for this is that the introduction of loops at every vertex of H gives regions with order of δ sides and this does not lead, in most cases, to an efficient imbedding. In some products also, the inclusion of this edge type together with others, introduced triangles, which meant that a 4-gon imbedding would not be minimal.

CHAPTER V

ISOGONAL IMBEDDINGS AND GENUS UPPER BOUNDS

Among the many 2- cell imbeddings of a connected graph, most of the interest of topological graph theorists has been focused on four types. The first two are the minimum genus imbedding, or imbedding G on $S_{\gamma(G)}$, and the **maximum genus** imbedding on $S_{\gamma_M(G)}$, where $\gamma_M(G)$ is the maximum among all the genera of all surfaces on which G can be 2-cell imbedded. The other two are special imbeddings for some particular graphs. There is what is called a **self-dual** imbedding, where $G \cong G^*$, G^* being the dual of G in a particular imbedding. Before presenting our fourth type of imbedding, we will define some related terms.

Let $\rho = (\rho_1, \rho_2, \ldots, \rho_n)$ be a rotation scheme for a connected graph G of order n; ρ_i is the rotation at vertex i, $1 \leq i \leq n$. It is known that ρ determines a 2-cell imbedding of G in a closed orientable 2-manifold S_k, where k can be uniquely determined by the number of orbits of the permutation ρ^* ($\rho^* : D^* \to D^*$ defined by $\rho^*(a,b) = (b, \rho_b(a))$, where $D^* = \{(a,b) | ab \in E(G)\}$. For further details see White [24, Section 6.6]. An **automorphism** of an imbedding (G, ρ) is a graph automorphism $a \in \mathcal{a}(G)$ such that $a(\rho) = \rho$. (That is, ρ is equivalent with itself, under the action of a.) This imbedding automorphism group, sometimes called a map automorphism group, we denote by $\mathcal{a}(G, \rho)$. If $|\mathcal{a}(G, \rho)| = 2|E(G)|$, we say (G, ρ) is a **symmetrical imbedding.** Thus a symmetrical imbedding has the maximum number of orientation-preserving symmetries. See White [24, Chapter 14]. In addition to these four types of imbeddings there is another type of imbedding that is

worth studying. This is an imbedding of a graph where $r = r_n$; we call this imbedding an **isogonal imbedding** or more specifically an **n-gon imbedding.** Thus duals of imbeddings of regular graphs are iosogonal, which implies that self-dual imbeddings of regular graphs are isogonal. In particular, self-dual imbeddings of Cayley graphs are also isogonal. Also, symmetrical imbeddings are necessarily isogonal, but not conversely, as our several constructions in the next section will indicate. The first section of this chapter will discuss isogonal imbeddings of some product graphs where the first factors include trees, cycles and the one-vertex, $6k - 3$ loop pseudograph imbedded in S_k.

Finally, in Section 5.2 we present examples showing how the surgical-voltage graph method can be used in giving an upper bound for product graphs for which the genus is unknown.

5.1 Isogonal Imbeddings

In this section by an n-gon imbedding of G we mean a 2-cell imbedding of G where each region of the imbedding is bounded by n edges. The repeated cartesian product of the cycle C_k, $k \geq 3$, n factors in all, will be denoted by C_k^n. We let γ_i denote the genus of the surface on which an isogonal imbedding occurs.

Theorem 5.1.1 The graph $C_{2k} \otimes C_k^n$, $n \geq 1$ and $k \geq 3$ and odd, has a 2k-gon imbedding on S_{γ_i}, for $\gamma_i = k^n[(2n-1)k - 2n] + 1$.

Before proving this result, we give an example for the case $k = 3$ and $n = 2$. We will take $C_3^2 = C_3 \times C_3$ to be $G_\Delta(Z_3^2)$ with $\Delta = \{10, 01\}$. We want to show that $C_6 \otimes (C_3 \times C_3)$ has a 6-gon (hexagonal) imbedding in S_{46}. We will modify C_6 according to Table 2.3(3), as shown in Figure 5.1(a). Each edge of C_6 is

modified as seen on Figure 5.1(b). In this modification we have two types of regions: the digons and the 6-gons. The digons satisfy $|R|_\phi = 3$ and sixgons satisfy the KVL. This means that the lift of C_6^*, $C_6^* \times_\phi Z_3^2 = C_6 \otimes (C_3 \times C_3)$ is a 6-gon; hence we have a 6-gon imbedding for our graph. Now let us find the genus of the surface where we obtained our imbedding. Before going to a routine calculation using euler's equation (Theorem 1.3.1), we first see that because both factors are connected and $(C_3 \times C_3)$ contains an odd cycle, from Theorem 3.2.3 $C_6 \otimes (C_3 \times C_3)$ is connected. Next we observe that the product graph has 54 vertices and 216 edges. Since $2q = 6r$, where r is the number of regions in our imbedding, r is 72. (We could also use Theorem 2.4.3 to calculate r. There are 18 digons, each with $|R|_\phi = 3$, and two 6-gons that satisfy the KVL. Thus $r = 18\frac{9}{3} + 2\frac{9}{1} = 72$.) The use of Theorem 1.3.1 then shows that $\gamma_i = \frac{1}{2}[q-p-r] + 1 = \frac{1}{2}[216-54-72] + 1 = 46$.

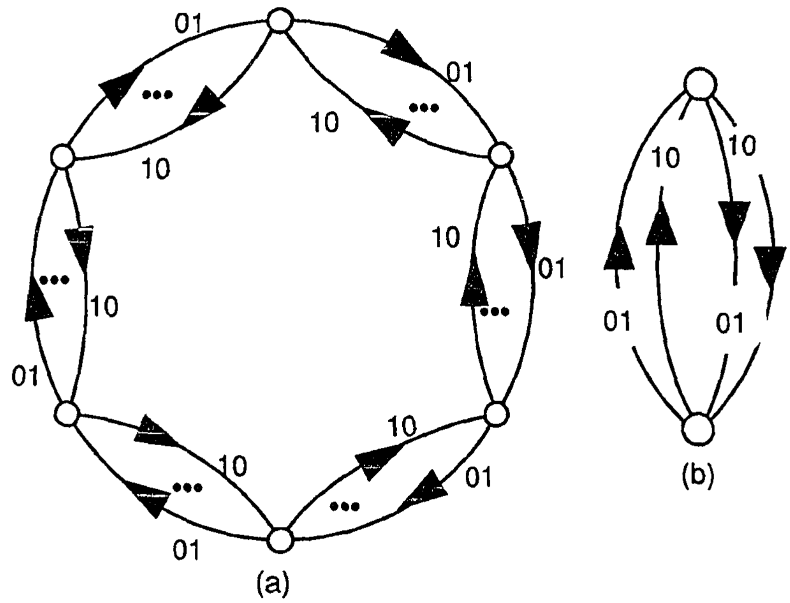


Figure 5.1

Proof of Theorem 5.1.1: The proof of the general case is analogous to that of our example. We take $\Gamma = Z_k^n$ and Δ the standard set of generators so that $G_\Delta(\Gamma) = C_k^n$. We replace every edge of C_{2k} by 2n arcs arranged in a similar arrangement to that of Figure 5.1(b). Since every edge is replaced by 2n edges, C_{2k}^* has 4kn edges, giving $4nk^{n+1}$ edges in the product. All the digons are arranged to satisfy $|R|_\phi = k$, while the two 2k-gons satisfy the KVL; hence we have a 2k-gon imbedding for $C_{2k} \otimes C_k^n$. Since both C_{2k} and C_k^n are connected and C_k^n contains an odd cycle (k is odd), Theorem 3.2.3 tells us that $C_{2k} \otimes C_k^n$ is connected. Also by either observing $2q = 2kr$ or the use of Theorem 2.4.3 we see that $r = 4nk^n$. Of course the order of our product graph is $2k^{n+1}$. The use of Theorem 1.3.1 now yields

$$\gamma_i = \frac{1}{2}[4nk^{n+1} - 2k^{n+1} - 4nk^n] + 1 = k^n[(2n-1)k - 2n] + 1. \square$$

Theorem 5.1.2 Let T_{k+1} denote a tree of order $k+1$. Then for k odd, $T_{k+1} \otimes C_k^n$ has a 2k-gon imbedding on S_{γ_i}, for $\gamma_i = \frac{k^n}{2}[(2n-1)k - 1 - 2n) + 1$.

Proof: Again we take Δ standard for $\Gamma = Z_k^n$, so that $G_\Delta(\Gamma) = C_k^n$. We modify T_{k+1} to get T_{k+1}^* as follows: we start with a drawing of the tree on the sphere. Modify each edge as shown in Figure 5.2(b). (The orientation can be arbitrary.) Figure 5.2(a) shows one way to do this for a tree of order eight. Our modification has two types of regions, these are the digons and the 2k-gon. The former all have $|R|_\phi = k$ and the latter $|R|_\phi = 1$; thus all the regions lift to 2k-gons. To find the genus of the surface where we have this imbedding, we first need a knowledge of the connectedness of $T_{k+1} \otimes C_k^n$, and this follows from Theorem 3.2.3 since both factors are connected and C_k^n has an odd cycle (since k is odd). Furthermore, we note that $T_{k+1} \otimes C_k^n$ has $2nk^{n+1}$ edges, $(k+1)k^n$ vertices, and (using similar

calculations as in the example preceding the proof of Theorem 5.1.1) $2nk^n$ regions. Thus applying Theorem 1.3.1 to this information we have

$$\gamma_i = \frac{1}{2}[2nk^{n+1} - (k+1)k^n - 2nk^n] + 1 = \frac{k^n}{2}[(2n-1)k - 1 - 2n] + 1. \square$$

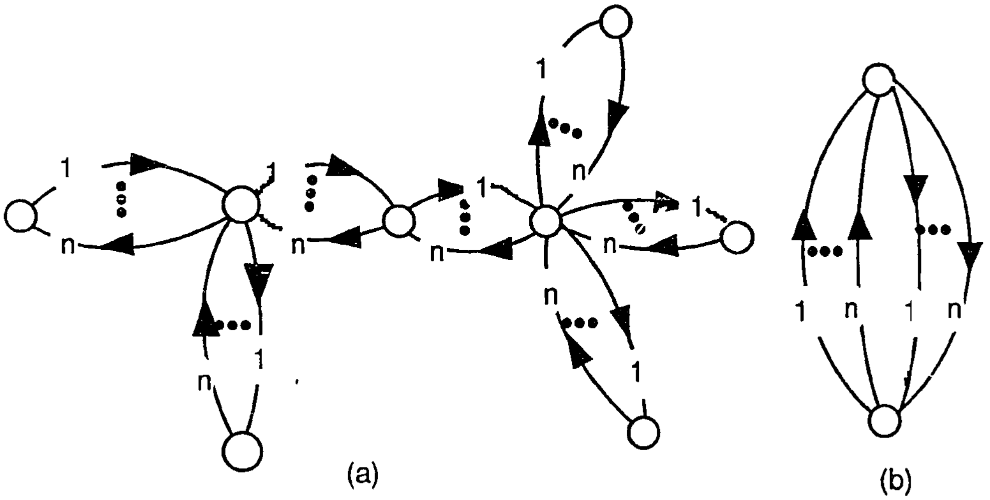


Figure 5.2

Corollary 5.1.3a The graph $P_{k+1} \otimes C_k^n$ for odd k has a 2k-gon imbedding on the surface with genus $\frac{k^n}{2}[(2n-1)k - 1 - 2n] + 1$.

Corollary 5.1.3b The graph $K_{1,k} \otimes C_k^n$ for odd k has a 2k-gon imbedding on the surface with genus $\frac{k^n}{2}[(2n-1)k - 1 - 2n] + 1$.

Theorem 5.1.4 The graph $C_{2k} \otimes' C_k^n$, $n \geq 1$ and $k \geq 2$, has a 2k-gon imbedding on S_{γ_i}, for $\gamma_i = k^n[2n(k-1) - 1] + 1$.

Proof: Let $\Gamma = Z_k^n$ with Δ the standard set of generators, so that $G_\Delta(Z_k^n) = C_k^n$. We modify C_{2k} by replacing each edge with the arrangement shown in Figure 5.3; we do this with consistency as determined by C_{2k} imbedded in the sphere.

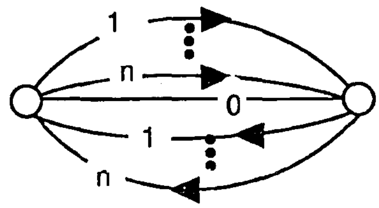


Figure 5.3

Our modification contains two types of regions, the digons and the 2k-gons. The digons have $|R|_\phi = k$ and the 2k-gons, $|R|_\phi = 1$. Thus both types lift to 2k-gons. This means then we have the desired imbedding. We first realize that $C_{2k} \otimes' C_k^n$ is connected, because the two factors are connected, satisfying the conditions of Theorem 3.2.5. Furthermore $C_{2k} \otimes' C_k^n$ has $2(2n + 1)k^{n+1}$ edges, $2k^{n+1}$ vertices, and $2(2n + 1)k^n$ regions, which by Theorem 1.3.1 says $\gamma_i = k^n[\,(2n + 1)k - k - (2n + 1)] + 1 = k^n[2n(k - 1) - 1] + 1$. ❑

The construction above is readily seen to extend from C_{2k} to any bipartite H with a 2k-gon imbedding or a non bipartite H with a 2k-gon imbedding that has a bichromatic dual.

Theorem 5.1.5 Let T_{k+1} stand for a tree of order $k + 1$; then for $k \geq 2$, $T_{k+1} \otimes' C_k^n$ has a 2k-gon imbedding on S_{γ_i}, for $\gamma_i = k^n[n(k - 1) - 1] + 1$.

Proof: Let $\Gamma = Z_k^n$ with Δ the standard set of generators, so that $G_\Delta(Z_k^n) = C_k^n$. We modify T_{k+1} as in the proof of Theorem 5.1.2 to get T_{k+1}^*, except this time we

replace every edge by $2n + 1$ edges arranged as seen in Figure 5.3. Again our modification has two types of regions; these are the digons and the 2k-gon. The former all have $|R|_\phi = k$ and the latter $|R|_\phi = 1$; thus all the regions lift to 2k-gons, giving 2k-gon imbedding for the graph product under discussion. If we find what q, p, and r are for $T_{k+1} \otimes C_k^n$, a routine calculation using Theorem 1.3.1 will establish the genus of the surface where we obtained the 2k-gon imbedding. But these are $(2n + 1)k^{n+1}$, $(k + 1)k^n$, and $(2n + 1)k^n$, respectively. Then since $T_{k+1} \otimes C_k^n$ is connected we get $\gamma_i = k^n[n(k-1) - 1] + 1$. ❑

Corollary 5.1.6a The graph $P_{k+1} \otimes' C_k^n$ has a 2k-gon imbedding on the surface with genus $\gamma_i = k^n[n(k-1) - 1] + 1$.

Corollary 5.1.6b The graph $K_{1,k} \otimes' C_k^n$ has a 2k-gon imbedding on the surface with genus $\gamma_i = k^n[n(k-1) - 1] + 1$.

Theorem 5.1.7 If $H = G_{\Delta'}(\Gamma_1)$, with $\Delta' = \{a, b, a + b\}$ for Γ_1 abelian, then $H \underline{\times} C_4$ has a 4-gon imbedding on the surface of genus $8|\Gamma_1| + 1$.

Proof: Let $\Gamma = Z_2^2$, with $\Delta = \{10, 01\}$; then $G_\Delta(\Gamma) = C_4$. Furthermore, we observe that H imbeds in S_1 as an $|\Gamma|$-fold covering of the one-vertex, three loop, pseudograph imbedded on S_1 as shown in Figure 4.24. We modify this pseudograph so that it lifts to the modification of H. This is shown in Figure 5.4. This figure shows that all the digons have $|R|_\phi = 2$ and the 4-gons satisfy the KVL; hence by Theorem 2.4.3 they all lift to 4-gons in $H^* \times_\phi \Gamma = H \underline{\times} C_4$, giving a 4-gon imbedding. We next count the number of edges and regions for this imbedding. To find the size, we first note that at the beginning we had $3|\Gamma_1|$ edges for H. In the modification H^* each edge was replaced by three edges, giving $9|\Gamma_1|$ edges. In

addition to this we have two half-edges at each vertex of H, giving $|\Gamma_1|$ edges altogether. Thus we have $10|\Gamma_1|$ edges. This means that $H \underline{\times} C_4$ has $40|\Gamma_1|$ edges. It is not hard to see that $H \underline{\times} C_4$ has $4|\Gamma_1|$ vertices. Since the size is $40|\Gamma_1|$ and all the regions are 4-gons, it follows that $r = 20|\Gamma_1|$. Both the factors are connected, giving a connected $H \underline{\times} C_4$ by Theorem 3.2.6. An application of Theorem 1.3.1 gives the genus of the surface of our imbedding to be

$$\gamma_i = \frac{1}{2}[\,40|\Gamma_1| - 4|\Gamma_1| - 20|\Gamma_1|\,] + 1 = 8|\Gamma_1| + 1. \ \square$$

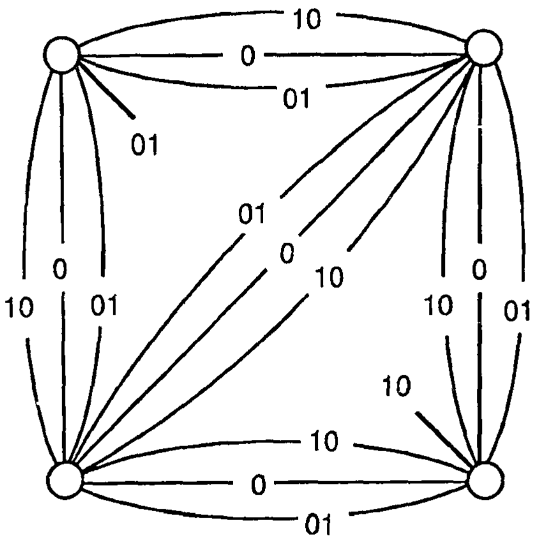


Figure 5.4

Corollary 5.1.8a The graph $K_7 \underline{\times} C_4$ has a quadrilateral imbedding on S_{57}.

Proof: Take $a = 1$ and $b = 2$, $\Gamma_1 = Z_7$ in the theorem. $\square$

Corollary 5.1.8b The graph $K_{4(2)} \underline{\times} C_4$ has a quadrilateral imbedding on S_{65}.

Proof: Take a = 1 and b = 2, $\Gamma_1 = Z_8$ in the above theorem. ❑

Corollary 5.1.8c The graph $K_{3(3)} \underline{\times} C_4$ has a quadrilateral imbedding on S_{73}.

Proof: Take a= 10 and b = 01, $\Gamma_1 = Z_3 \times Z_3$ in Theorem 4.5.2. ❑

Let Δ' be defined as in the statement preceding Theorem 4.5.7, for Γ_1 abelian; then we get the following result that generalizes the above theorem.

Theorem 5.1.9 Let H be regarded as an $|\Gamma_1|$-fold covering of a one-vertex, 6k −3 loop, pseudograph imbedded in S_k. Then for $n = 4k - 2, k \geq 1$, $H \underline{\times} Q_n$ has a 4-gon imbedding on the surface of genus $k2^{4k-1} |\Gamma_1|\left[(3k-2)\right] + 1$.

Proof: We have seen the case k =1 in Theorem 5.1.7. The imbedding of H for cases k = 2, and k = 3 can be found in the proofs of Theorems 4.5.4 and 4.5.5. The higher cases are extensions of these with appropriate generators on corresponding surfaces. If we take $\Gamma = Z_2^n$, with Δ as the standard set of generators, then $G_\Delta(\Gamma) = Q_n$. We modify the pseudograph imbedded on S_k according to Table 2.3(6) and in the fashion of Theorem 4.5.7, except an edge bearing " 0 " is added internally; this will lift to H^*. Since H has $|\Gamma_1|$ vertices and$(6k-3)|\Gamma_1|$ edges, H^* has $|\Gamma_1|$ vertices and $|\Gamma_1|[(6k-3)(n+1) + \frac{n}{2}]$ edges: the (6k-3) edges of H replaced by n + 1 edges each in H^* and the n half-edges at each vertex give $\frac{n}{2}$ edges. This means that $H \underline{\times} Q_n$ has $2^n|\Gamma_1|$ vertices and $2^n|\Gamma_1|[(6k-3)(n+1) + \frac{n}{2}]$ edges. We can also show that the 4-gon regions of H^* satisfy the KVL and the digons $|R|_\phi = 2$. This means all the regions lift to 4-gons and we have a 4-gon imbedding for our product, $H \underline{\times} Q_n$. The fact that each region is a 4-gon tells us that $r = 2^{n-1}|\Gamma_1|[(6k-3)(n + 1) + \frac{n}{2}]$. If we substitute 4k − 2 for n in the above numbers, we get $p = 2^{4k-2}|\Gamma_1|$,

$q = 2^{4k-2}|\Gamma_1|[24k^2 - 16k + 2]$, and $r = 2^{4k-3}|\Gamma_1|[24k^2 - 16k + 2]$. Since $H \underline{\times} Q_n$ is connected, an application of euler's theorem, Theorem 1.3.1, gives the genus of the surface where this 4-gon imbedding of $H \underline{\times} Q_n$ is obtained as follows:

$$\gamma_i = \frac{1}{2}[q - p - r] + 1.$$

$$= \frac{1}{2}[2^{4k-2}|\Gamma_1|(24k^2 - 16k + 2) - 2^{4k-2}|\Gamma_1| - 2^{4k-3}|\Gamma_1|(24k^2 - 16k + 2)] + 1$$

$$= \frac{1}{2}[2^{4k-3}|\Gamma_1|(24k^2 - 16k + 2) - 2^{4k-2}|\Gamma_1|] + 1$$

$$= 2^{4k-4}|\Gamma_1|[24k^2 - 16k] + 1$$

$$= k\ 2^{4k-1}|\Gamma_1|[3k - 2] + 1. \ \square$$

5.2 Application to Genus Upper Bounds

As is often the case, when the genus of a graph is not known, upper and lower bounds are established. Rough bounds could be easy to find; sharper bounds might be hard to calculate. For any graph that we encounter, Theorem 1.3.3 gives a general lower bound, and for bipartite graphs Theorem 1.3.4 gives a sharper lower bound. Depending on the graph, better lower bounds may be found by exploiting certain properties. As for the upper bound the natural method is to imbed a graph on some surface and calculate the genus of this surface. Imbedding a graph on a surface other than the sphere or sometimes the torus is usually not an easy task, unless we have a means of seeing the graph as a covering graph of another graph that could be readily imbedded on surfaces of lower genus. For Cayley graphs, voltage graphs serve as a means of achieving this; for graph products (with the second factor a Cayley graph) the surgical-voltage technique discussed in this dissertation applies. The beauty of the

technique is that it will always give an upper bound and with careful considerations one can make the upper bound more efficient. It is not our aim in this section to give a list of upperbounds for the genus parameter for various graphs, but to give two examples to show how the technique might be used.

It was mentioned in Chapter I that this method was first used in trying to find the genus of the repeated cartesian product of three triangles [15]. A generalization of this problem is to determine $\gamma(C_{2n+1} \times C_{2n+1} \times C_{2n+1})$ for $n \geq 1$. Voltage graph theory readily gives an upper bound of $8n^3 + 4n^2 - 2n$, but using the surgical-voltage technique White [28] showed that this bound can be improved to $4n^3 + 4n^2 + 2n + 1$. The lower bound using Theorem 1.3.4 ($n \geq 2$) is $2n^3 + 3n^2 + \frac{3}{2}n + \frac{5}{4}$. The genus was found to be asymptotically $2n^3$ by the work of Mohar, Pisanski, and White [16].

To see how this technique can be used in giving a good upper bound for other graphs, we are going to use two examples. Let us try to find $\gamma(K_{3(2)} \underline{\otimes} C_4)$, the genus of the strong tensor product of the graph $K_{2,2,2}$ with C_4. The first thing we try to do is to see C_4 as $G_\Delta(\Gamma)$ for some group Γ and generating set Δ. We have two ways of doing this. These are: $C_4 = G_{\Delta'}(Z_4)$, $\Delta' = \{1\}$ and $C_4 = G_{\Delta''}(Z_2 \times Z_2)$, $\Delta'' = \{10, 01\}$.

We will modify $K_{3(2)}$ acording to Table 2.3(4), but before doing that let us establish a lower bound for the genus of $K_{3(2)} \underline{\otimes} C_4$. First we note that $K_{3(2)}$ has 6 vertices and 12 edges. Using Table 3.1 we see that $K_{3(2)} \underline{\otimes} C_4$ has size 120 and of course its order is 24. Since C_4 is bipartite, from Theorem 3.3.4 we see that $K_{3(2)} \underline{\otimes} C_4$ is bipartite, and because $K_{3(2)} \underline{\otimes} C_4$ is connected as a result of Theorem 3.2.4, from Corollary 1.3.5 we get a lower bound of 19 for $K_{3(2)} \underline{\otimes} C_4$.

Now let us modify $K_{3(2)}$ as seen in Figure 5.5, by taking $\Gamma_1 = Z_4$ and $\Delta' = \{1\}$ so that $C_4 = G_{\Delta'}(\Gamma_1)$.

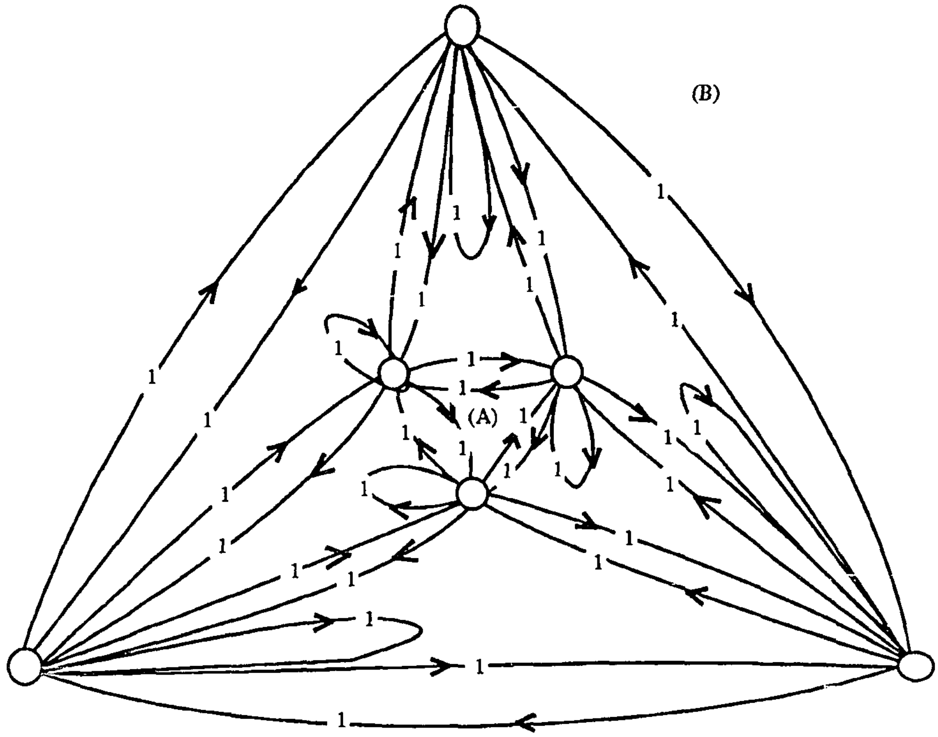


Figure 5.5

In this figure we have four types of regions: 4-gons, 3-gons, digons, and loops. By Theorem 2.4.3 the six 4-gons satisfy the KVL. Thus in $K^*_{3(2)} \times_\phi Z_4$ they will lift to 4-gons (24 in all). The two 3-gons, marked (A) and (B), have $|R|_\phi = 4$; thus they will lift to 12-gons (2 in all). The 12 digons have $|R|_\phi = 2$ and will lift to 4-gons (24 in all), and the six loops have $|R|_\phi = 4$ and they also will lift to 4-gons (6 in all). Hence the total number of regions in our imbedding is $24 + 2 + 24 + 6 = 56$. Thus

using euler's equation (Theorem 1.3.1) we get the genus of the surface where we obtained our imbedding to be $\frac{1}{2}[120 - 24 - 56] + 1 = 21$.

This means that $19 \leq \gamma(K_{3(2)} \underline{\otimes} C_4) \leq 21$. These bounds are fairly sharp, since they are close together, but we would like to improve at least one of them. One way to improve the upper bound is to try to rearrange our loops and edges, but a careful look would show that we cannot do any better than this for $\Gamma_1 = Z_4$. Thus we consider looking at C_4 as $G_\Delta(Z_2 \times Z_2)$, with $\Delta = \{10, 01\}$. Our modification of $K_{3(2)}$ will then look like the one in Figure 5.6.

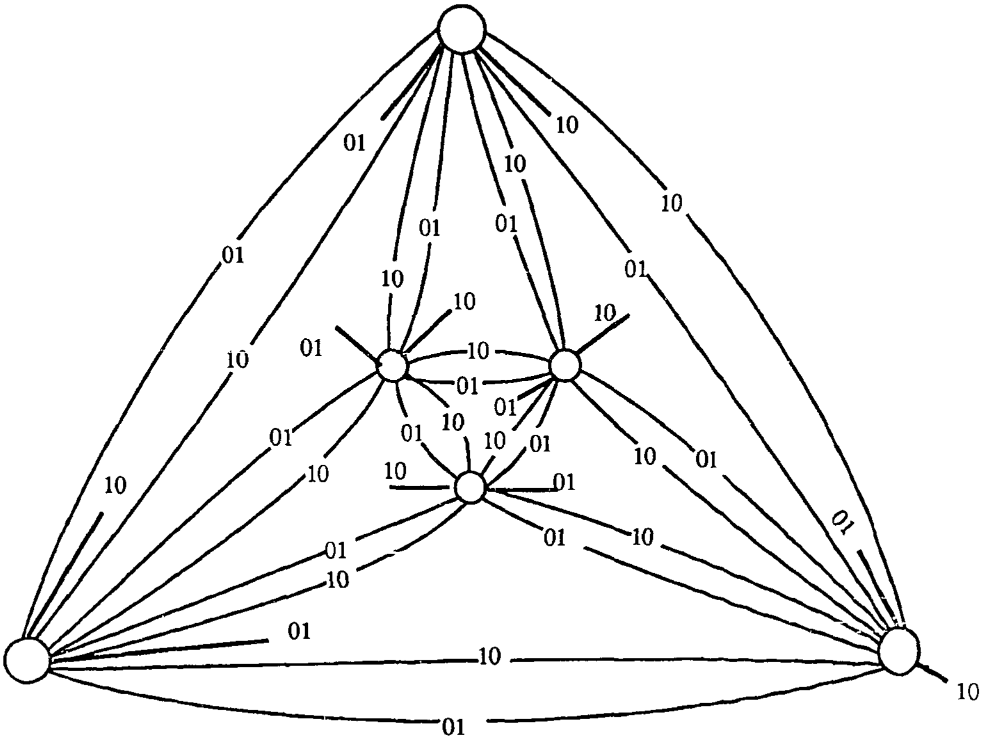


Figure 5.6

This time we have only two types of regions in $K^*_{3(2)}$, the 4-gons and the digons. The 4-gons all satisfy the KVL while the digons have $|R|_\phi = 2$. Thus all of them will lift to 4-gons, giving a 4-gon imbedding for our bipartite graph $K_{3(2)} \underline{\otimes} C_4$, so that the lower bound is achieved. Thus we have a genus imbedding for $K_{3(2)} \underline{\otimes} C_4$; the genus of $K_{3(2)} \underline{\otimes} C_4$ is therefore 19.

This example illustrates two things. First, in trying to modify H we should try to make our choices as efficiently as possible and this we exploited throughout Chapter IV. Second, we often should try another generating set or sometimes another group altogether, in order to get a better upper bound using the surgical-voltage imbedding technique.

For our final example, involving an infinite family of graphs, we note that Theorem 5.1.7 determines that $\gamma(G_{\Delta'}(\Gamma_1) \underline{\times} C_4) \leq 1 + 8|\Gamma_1|$. The lower bound $1 + \frac{14}{3}|\Gamma_1|$ from Theorem 1.3.3 can be improved as follows: (Let $H = G_{\Delta'}(\Gamma_1)$, as before.) Since $H \underline{\otimes} C_4$ is bipartite, any 3-cycle in $H \underline{\times} C_4$ must include at least one type (ii) edge. Since each such edge bounds at most two triangular regions, and there are $12|\Gamma_1|$ such edges in all, $r_3 \leq 24|\Gamma_1|$. Thus, since (by Theorem 1.3.2) $\sum_{i\geq 3} ir_i = 2q = 80|\Gamma_1|$, $r \leq 26|\Gamma_1|$. Then Theorem 1.3.1 (which applies, since a genus imbedding of a connected graph is always a 2-cell imbedding, see White [24, p.61]) shows that $\gamma(H \underline{\times} C_4) \geq 1 + 5|\Gamma_1|$.

Thus we know that $1 + 5|\Gamma_1| \leq \gamma(H \underline{\times} C_4) \leq 1 + 8|\Gamma_1|$. We therefore seek either more efficient constructions, to lower the upper bound; additional arguments involving the structure of the graph $H \underline{\times} C_4$ to further raise the lower bound; or both. These are the challenges of calculating the genus parameter.

CHAPTER VI

OPEN PROBLEMS

In this final chapter we would like to mention some questions that merit further investigations. Since we have defined several new graph products, several properties of these graphs could be investigated depending upon the area of interest of the investigator. The main criterion for introducing the new products was that they seem to provide genus imbeddings. Our method of representing these graph products seems to suggest an easier way of looking at the products and we believe will encourage new methods of proof for many results. Here we raise problems that pertain to graph imbeddings only.

1. As we saw in Chapter IV, most of the genus imbeddings we got were for the graph products that did not use type (i) edges. What classes of graphs use this type of edge but give genus imbeddings by use of the surgical-voltage technique?

2. Throughout this dissertation we have investigated Cayley graphs for abelian groups only. What about graph products that have their second factor as a Cayley graph for a non abelian finite group? Do we have some classes of graphs that give genus imbeddings using this method for the different graph products?

3. Most of the genus imbeddings we obtained came from Cayley graphs where the groups had several involutions. Are there other groups which give rise to genus imbeddings by using a special arrangement of their generators?

4. Our study on isogonal imbeddings was limited to those graph products that were investigated for genus imbeddings but did not lend themselves to one. Hence a further study on other graph products that have isogonal imbeddings is important.

5. All the graph products that gave isogonal imbeddings in Section 5.1 were investigated to see if the imbeddings are symmetrical. It was found by a routine calculation that none was symmetrical. Part of the reason seems to be that the base graph H^* has several types of region, that all lift to the same kind of region above. But is this always the case? In other words, if a voltage graph has more than one type of region that lifts to the same type of region, is the imbedding of the covering graph always not symmetrical?

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