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Enumerating the Orientable 2-Cell Imbeddings of Complete N-Partite Graphs

Bruce P. Mull
Western Michigan University

ENUMERATING THE ORIENTABLE 2-CELL IMBEDDINGS OF COMPLETE N-PARTITE GRAPHS

by

Bruce P. Mull

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Bruce P. Mull, Ph.D.

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This dissertation develops formulas for the number of congruence classes of maps of complete, complete bipartite, complete tripartite, and complete n-partite graphs; these congruence classes correspond to unlabeled imbeddings. The method employed for the enumeration is an extension of that used by Mull, Rieper, and White in 1988. We let the automorphism group act on the set of rotations and use Burnside's Lemma to count orbits for these rotations. Compatible permutations are introduced to determine those automorphisms actually contributing to the number of orbits.

The complete n-partite formula is shown to generalize those of the other three families of graphs. These congruence classes are classified using the hierarchy of Kountanis, Mull, and Rashidi (1989). A new parameter is introduced to random topological graph theory: the average number of symmetries of the maps of a graph. This parameter is evaluated for arbitrary connected graphs G, in terms of the degree sequence, the number of graph automorphisms, and the number of congruence classes of maps of G. Finally, the computer program created to classify the congruence classes, and which verified the formulas for small cases, is presented.

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Mull, Bruce Philip, Ph.D.

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U·M·I
300 N. Zeeb Rd.
Ann Arbor, MI 48106

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Bruce P. Mull

TABLE OF CONTENTS

Table of Contents—Continued

LIST OF TABLES

LIST OF FIGURES

CHAPTER I

GRAPHS, SURFACES, AND PERMUTATION GROUPS

1.1 Introduction

It is clear from the title of this dissertation that something will be counted. What that something might be, however, is not so clear. What is a graph? What is a graph imbedding? What is an orientable imbedding? The answers to these and related questions will be given in this chapter. A more complete treatment of these ideas can be found in many texts on graph theory; see, for example, Chartrand and Lesniak [4], Gross and Tucker [7], or White [18].

We will end the chapter by presenting the background material needed to understand the enumerative method that will be used. More information on enumerative techniques in graph theory can be found in Biggs and White [3] and Harary and Palmer [8]. Much of the material in this chapter, and the next, also appears in Mull, Rieper, and White [15].

1.2 Graphs

A <u>graph</u> G is an ordered pair (V,E), where V is a nonempty set of objects and E is a collection of unordered pairs of distinct objects from V. The members of V

are known as the <u>vertices</u> of G; the members of E are the <u>edges</u> of G. When more than one graph is involved, or to emphasize the underlying graph, we denote these two sets V(G) and E(G). We will assume that no two distinct edges of a graph have both of their vertices in common.

Example 1.1

Let V = {1, 2, 3, 4, 5} and E = {{1,2}, {1,3}, {2,3}, {2,4}, {2,5}, {3,4}, {3,5}, {4,5}}; then G = (V,E) is a graph. Figure 1.1 shows a drawing representing G. Points are used to represent the vertices; lines between points represent edges. Though edges cross in this picture, only the points designated 'o' are vertices.

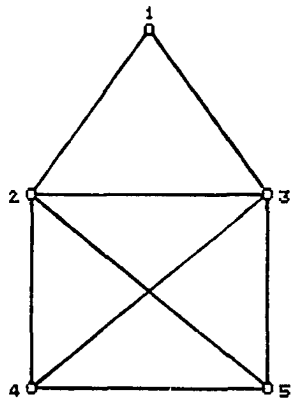


Figure 1.1. A Graph With 5 Vertices and 8 Edges.

If two vertices form an unordered pair in E, we say that they are <u>adjacent</u>. Two edges are adjacent if their unordered pairs have a common vertex. An edge is said to be <u>incident</u> with the two vertices that form its unordered pair.

Example 1.2

Consider the graph of Example 1.1. Since $\{1,3\} \in E$, vertices 1 and 3 are adjacent; vertex 1 is not adjacent with vertex 5 because $\{1,5\} \notin E$. The edges $\{1,2\}$ and $\{2,4\}$ have vertex 2 in common; therefore, these edges are adjacent. There is no vertex in common between $\{1,3\}$ and $\{2,5\}$ so these edges are not adjacent. Finally, the edge $\{2,3\}$ is incident with vertices 2 and 3 but no others.

A graph is said to be <u>complete</u> if E is the collection of all unordered pairs of distinct members of V. If $|V| = n$, then this graph is denoted K_n. A <u>bipartite graph</u> is a graph whose vertex set can be partitioned into two sets, V_1 and V_2, so that every edge of the graph has one vertex from V_1 and one from V_2. If each vertex of V_1 is adjacent with every vertex of V_2, then the graph is a <u>complete bipartite graph</u>. If $|V_1| = m$ and $|V_2| = n$, and the bipartite graph is complete, then the graph is denoted $K_{m,n}$. Finally, an <u>n-partite</u> graph has a vertex set which can be partitioned into $V_1, \ldots, V_n$ so that every edge has vertices from distinct partite sets. If for

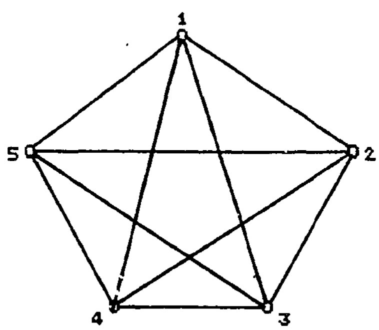


Figure 1.2. The Complete Graph of Order Five, K_5.

every partite set, each vertex is adjacent with every vertex of all other partite sets, then the graph is a <u>complete n-partite graph</u>.

Example 1.3

Consider the graph $G = (V,E)$, where $V = \{1,2,3,4,5\}$, and E is the collection of all 2-element subsets of V; G is the complete graph of order five and is denoted K_5 (Figure 1.2).

Example 1.4

Let $V = \{1,2,3,4,5,6\}$ and partition V into the partite sets $V_1 = \{1,2,3\}$ and $V_2 = \{4,5,6\}$. Suppose that $E = \{\{1,4\}, \{1,5\}, \{2,4\}, \{2,6\}, \{3,4\}, \{3,5\}\}$, then every edge contains a vertex from both partite sets—this is a bipartite graph (Figure 1.3a).

Suppose that the edges $\{1,6\}$, $\{2,5\}$, and $\{3,6\}$ were added to the above graph; then every vertex in V_1 would

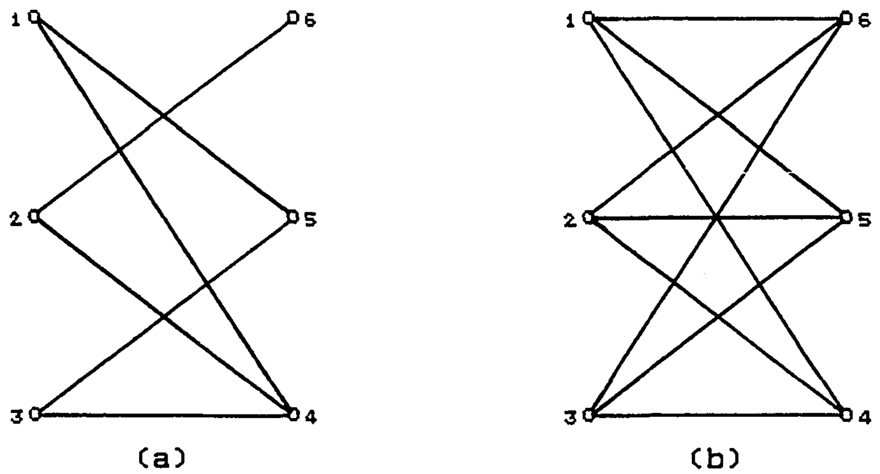


Figure 1.3. A Bipartite Graph and a Complete Bipartite Graph.

be adjacent with every vertex in V_2. The result would be the graph $K_{3,3}$ shown in Figure 1.3b.

Example 1.5

Let $V = \{1,2,3,4,5,6\}$ and partition V into the partite sets $V_1 = \{1\}$, $V_2 = \{2\}$, $V_3 = \{3,4\}$, and $V_4 = \{5,6\}$. A complete 4-partite graph is shown in Figure 1.4. Notice that vertices of different partite sets are always adjacent while vertices in the same partite set never are. The orders of the partite sets are one, one, two, and two; thus the graph shown is $K_{1,1,2,2}$.

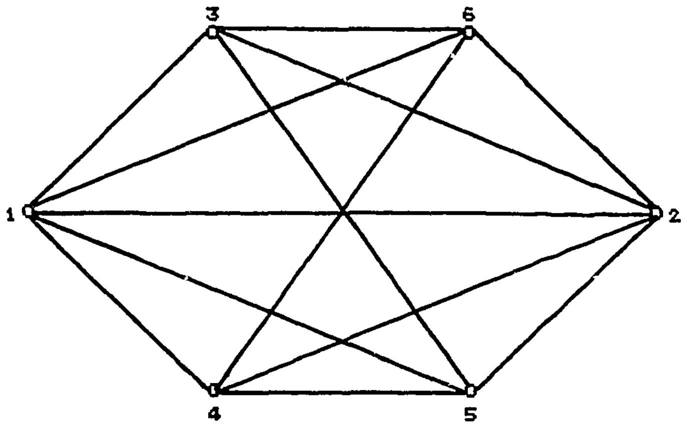


Figure 1.4. The Complete 4-Partite Graph $K_{1,1,2,2}$.

1.3 Surfaces and Imbeddings

A surface is a connected compact 2-manifold that has no boundary; locally, it is homeomorphic to the plane. Let S be a surface and decompose S into 2-cells (the open regions are homeomorphic to an open disk); if it is possible to give the boundary of each 2-cell a direction so that a 1-cell portion of the boundary incident with two

adjacent 2-cells is oppositely directed within those two 2-cells, then S is said to be orientable, otherwise, S is nonorientable. This decomposition, when possible, is called a coherent orientation of S.

Example 1.6

Consider the drawings of Figure 1.5. Figure 1.5a depicts the torus decomposed into 2-cells by K_4. A coherent orientation has been indicated with arrows; thus the torus is an orientable surface. Figure 1.5b depicts the Klein bottle also decomposed into 2-cells by K_4. The directions of the arrows do not form a coherent orientation (both sides of the vertical edge are directed upward). Actually, it is possible to prove that the Klein bottle is nonorientable.

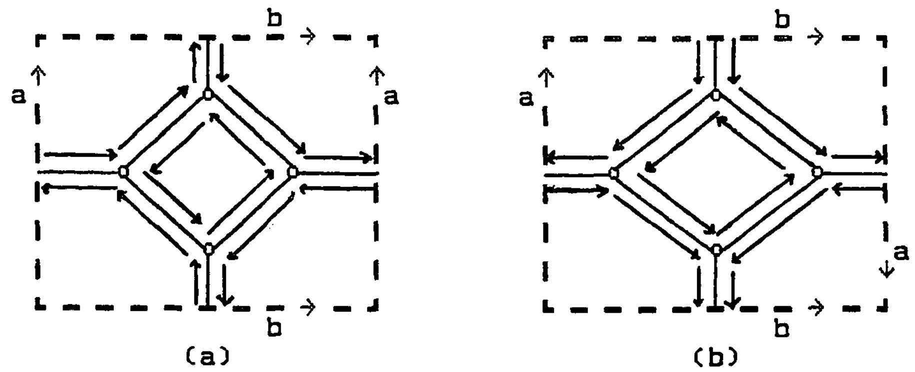


Figure 1.5. Depictions of Two Surfaces.

Given an orientable surface S, S is said to be oriented clockwise (counterclockwise) if the direction of the boundary of every 2-cell, in any decomposition, is such that the 1-cell portions of the boundary, incident

with two adjacent 2-cells, have the boundary always on the right (left), relative to this direction (when viewed from the outside of the surface). The boundaries of a clockwise orientation are actually being traced in a counterclockwise manner. It is called a clockwise orientation because the clockwise ordering of the vertices about every vertex, of the graph inducing the 2-cell decomposition, acting upon the set of arcs of the graph (directed edges—both directions represented), traces out the oriented region boundaries. Therefore, the orientation shown in Figure 1.5a is a clockwise orientation.

The representations of Figure 1.5 are interpreted as follows: edges having the same letter label are identified in such a way that the directions of the arrows on these edges are the same. Frechêt and Fan [5] showed that every surface can be represented as a polygon.

Theorem 1.1. Every surface can be elementarily associated with a polygon whose symbolic representation is one of the following:

(i) aa^{-1}

(ii) $a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}\ldots a_mb_ma_m^{-1}b_m^{-1}$, $m \geq 1$

(iii) $a_1a_1a_2a_2\ldots a_na_n$, $n \geq 1$. ∎

The form (i) corresponds to the sphere; (ii) is a sphere with m handles; and (iii) is a sphere with n crosscaps. Looking at Figure 1.5, we see that the Klein bottle has not been represented as aabb, but rather as $abab^{-1}$—in Problem 5.9, page 55, of [18], we are asked to show that these representations are equivalent.

There are two ways of obtaining new surfaces from old surfaces—adding either a crosscap or a handle. When a crosscap is added to a surface, the interior of a circular region is deleted. Next, every pair of antipodal points along the boundary of this region is then identified; this is equivalent to identifying the points of the circle in a continuous one-to-one and onto fashion with the boundary of a Moebius strip. This identification cannot be done in 3-space (without self intersections), but requires at least four dimensions. Because of this, no attempt will be made to show this operation.

It is possible for a handle to be added to a surface in 3-space without having self intersections. This operation is performed by deleting the interiors of two circular regions; the circular boundaries are then given

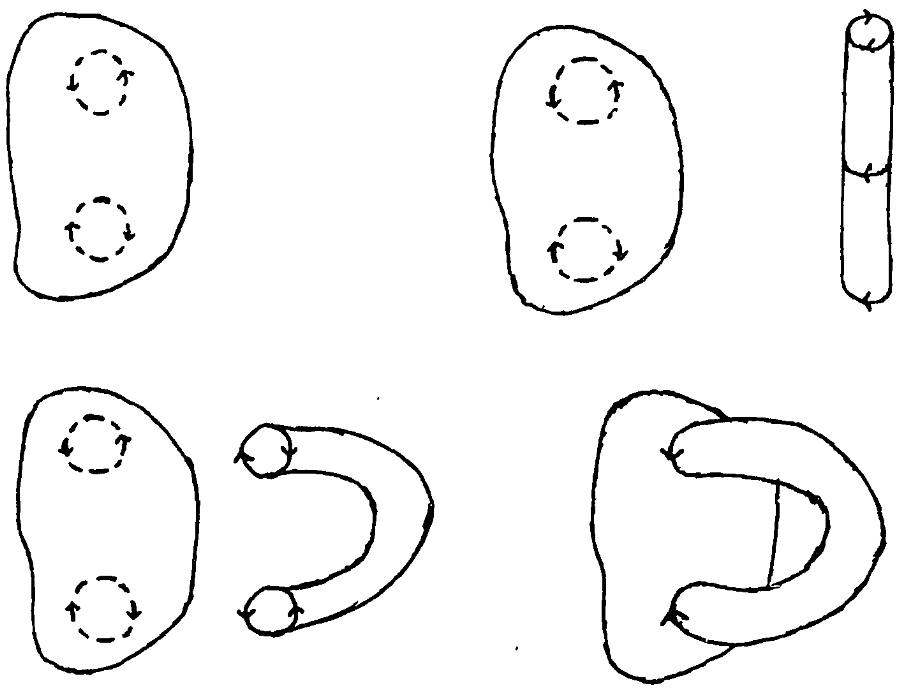


Figure 1.6. Adding a Handle to a Surface.

opposite orientations. A cylinder is then oriented looking at the ends of the cylinder directly, it appears that they have opposite orientation). The ends of the cylinder are now identified in a continuous one-to-one and onto fashion so that the orientations of the ends of the cylinder match those of the two regions. An attempt to show this operation is given in Figure 1.6.

If a surface is orientable, then adding a crosscap will make it nonorientable; adding a handle to a surface never changes the orientation type of the surface—that is, orientable surfaces remain orientable and nonorientable surfaces remain nonorientable. We will focus on orientable surfaces, that is, the sphere and surfaces obtained from the sphere by adding one or more handles. If k handles have been added to the sphere, then the surface is denoted S_k and we say that the surface has genus k. Thus, the sphere is S_0 (and has genus zero). For more information on surfaces, and how they relate to graph theory, the reader may consult [7], [18], or a text on topology such as [13].

When a graph G is represented in a surface S, distinct points of S correspond to the vertices of G. If u and v are vertices of G and {u,v} is an edge, then the points which correspond to u and v are joined by an open arc. These open arcs do not include vertex points—vertex points only occur as end points of arcs. If the arcs are mutually disjoint, then the representation is said to be an <u>imbedding</u> of G in S. A graph G can be thought of as a one dimensional simplicial complex in 3-space. When viewed in this way, an imbedding becomes a continuous

one-to-one function from G into S; we write $i:G \to S$. If a graph is imbedded in an orientable surface then the imbedding is orientable; an imbedding in a nonorientable surface is nonorientable.

The regions of an imbedding, $i:G \to S$, of a graph G in a surface S, are the components of $S - i(G)$, the complement of $i(G)$ in the surface S. If every region is homeomorphic to an open disc, then the imbedding is an open 2-cell imbedding, or more simply, a 2-cell imbedding. Clearly, only connected graphs can have 2-cell imbeddings.

The genus of an imbedding is the genus of the surface in which it occurs. Let G be a graph with m vertices and n edges that is 2-cell imbedded in an orientable surface S. Suppose S has k handles, and that there are r regions in the imbedding; then

$$m - n + r = 2 - 2k.$$

If G is 2-cell imbedded in a nonorientable surface N with r regions, and N has k crosscaps, then

$$m - n + r = 2 - k.$$

The value $m - n + r$ is called the characteristic of the surface. It is well-known (see, for example, [7]) that if two surfaces are both orientable (nonorientable), and they have the same characteristic, then they are homeomorphic.

Example 1.7

Consider the drawings of Figure 1.5—each of these is an imbedding. In both imbeddings the graphs have four vertices, six edges, and two regions (both are K_4). The

surface of Figure 1.5a is orientable; thus we have

$$4 - 6 + 2 = 2 - 2k$$

so that the torus has genus 1. The surface of Figure 1.5b is nonorientable; therefore,

$$4 - 6 + 2 = 2 - k$$

and we compute the genus of the Klein bottle to be two. Note that though both characteristics are zero, since the surfaces are of different orientability type, they cannot be homeomorphic.

An imbedding of a graph G in a surface S is said to be <u>rooted</u> if there is a distinguished edge, vertex, and region of the imbedding. Customarily, the distinguished edge is thought of as having a direction toward the distinguished vertex, in the boundary of the distinguished region. Recently, there have been many papers concerned with counting rooted imbeddings (rooted maps). These papers have fixed the surface and counted all rooted maps having some particular property (see [1], for example). On the other hand, there have been papers that fix the graph and count the number of unlabeled, 2-cell imbeddings for that graph in all of the surfaces in which the graph imbeds (see [15], for example). We have adopted the latter approach in this dissertation.

1.4 Permutation Groups and Congruent Imbeddings

Let G be a graph and suppose that G is imbedded in an orientable surface S (with, say, a clockwise orientation); then the result is an <u>oriented</u> imbedding. Two

oriented imbeddings, $i:G \to S$ and $j:G \to S$, are <u>congruent</u> if they "look alike," that is, there exists an isomorphism of their labeled images, $i(G)$ and $j(G)$, which can be extended to an orientation-preserving autohomeomorphism of the surface S. The congruence classes correspond to the unlabeled imbeddings. We will be concerned with counting these in a manner which ensures that each class is counted exactly once. The method we will use is to count them as algebraic equivalence classes of what we will call graph rotations.

Let G be a connected graph with $V(G)$ and $E(G)$ its vertex and edge sets, respectively. For $v \in V(G)$, we define the <u>neighborhood</u> of v, denoted $N(v)$, to be $N(v) = \{u \in V(G) \mid \{u,v\} \in E(G)\}$. A <u>rotation at</u> v, ρ_v, is a cyclic permutation of the neighborhood of v, so that $\rho_v:N(v) \to N(v)$ is a sequence of length n (assuming that $|N(v)| = n$) that contains all of the neighbors of v, the last member of this sequence is followed by the first member. A <u>rotation scheme</u> is a set of rotations at each vertex of the graph G, $\rho = \{\rho_v\}_{v \in V(G)}$. When no confusion can occur, this will be called a <u>rotation on</u> G. A <u>map</u> M is an ordered pair, $M = (G, \rho)$, where G is a connected graph and ρ is a rotation on G. It is well-known (see, for example, [18]), that M determines an oriented 2-cell imbedding of G in some surface S_k, of genus k, where k can be readily determined from ρ; conversely, by fixing an orientation on S_k, it is easy to see that every oriented 2-cell imbedding of G has a corresponding ρ associated with it. Therefore, we will talk about counting congruence classes of maps.

Example 1.8

Consider the graph $G = (V, E)$, where $V = \{1,2,3,4\}$ and $E = \{\{1,2\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}$ (Figure 1.7). Since vertex 2 is adjacent to vertices 1, 3, and 4, we get $N(2) = \{1,3,4\}$. Without loss of generality, we can assume that a rotation at vertex 2 begins with vertex 1; thus, there are two rotations possible at vertex 2: these are $\rho_2 = (134)$ and $\rho_2 = (143)$.

Figure 1.7 is an imbedding; let us determine the surface. First, we observe that the surface is orientable. Second, we see that there are four vertices (its order), five edges (its size), and three regions in the imbedding (we assume the "outer" region is a 2-cell). Therefore,

$$4 - 5 + 3 = 2 - 2k$$

and we find that k, the genus, is zero. We conclude that the surface is S_0, the sphere.

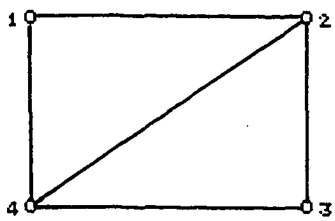


Figure 1.7. A Graph of Order Four and Size Five.

Let $R(G)$ denote the set of all rotations on G; the above remarks establish that $|R(G)| = \prod_{v \in V(G)}(d(v) - 1)!$, where $d(v)$ is the <u>degree</u> of vertex v (the number of edges incident with v). Consider a pair of maps; these maps may be clearly different: perhaps their genera differ, or the region sizes (the number of sides on the boundaries

of its regions) are not the same when the genera agree. Some differences can be more subtle—for example, perhaps both sides of some edge lie in the same region in one imbedding but the other has no such edge. On the other hand, it may be that the two maps are essentially alike; one is merely a relabeling of the other.

Example 1.9

Consider the toroidal imbeddings of K_5 shown in Figure 1.8. Both of these imbeddings have four 3-gons and an 8-gon. These imbeddings cannot be the same, however. In Figure 1.8a we see that the vertex labeled "1" is incident with all four of the 3-gons. No vertex in Figure 1.8b is incident with more than three of them.

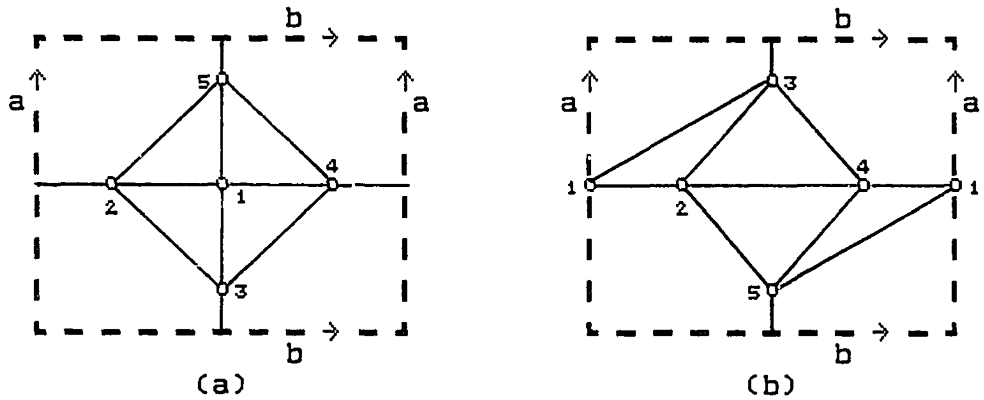


Figure 1.8. Two Toroidal Imbeddings of K_5.

An automorphism of a graph G is a permutation of the vertex set of G which preserves the edge set of G, that is, $\alpha: V(G) \to V(G)$ is an automorphism of G means, for u, v $\in$ V(G), with slight corruption of notation,

$\alpha(\{u,v\}) = \{\alpha(u),\alpha(v)\} \in E(G)$ if and only if $\{u,v\} \in E(G)$.

The set of automorphisms of a fixed graph G forms a group under composition; this group is denoted Aut G.

Biggs [2], see also [3], defined a variety of algebraic equivalence on R(G), using Aut G, which allows us to determine the congruence classes for the maps of G. Two rotation schemes, ρ and σ, are equivalent (and their corresponding maps congruent), if there exists an automorphism $\alpha \in$ Aut G such that, $\forall v \in V(G)$, $\sigma_{\alpha(v)} = \alpha\rho_v\alpha^{-1}$; the common domain for these two permutations is given by $\alpha(N(v)) = N(\alpha(v))$.

A map automorphism for $M = (G, \rho)$ is a graph automorphism of G which preserves the oriented region boundaries of M. Therefore, a map automorphism is a graph automorphism that makes ρ equivalent to itself. The set of map automorphisms for a fixed map M also forms a group under composition; this is a subgroup of Aut G. If Aut M denotes this group, then we have

$$\text{Aut } M = \{\alpha \in \text{Aut } G \mid \rho_{\alpha(v)} = \alpha\rho_v\alpha^{-1},\ \forall v \in V(G)\}.$$

Let Aut G act on R(G) as follows: let $\rho \in R(G)$, $\rho = \{\rho_v\}_{v\in V(G)}$, and $\alpha \in$ Aut G; the action of the resulting permutation group is

$$\alpha(\rho) = \{\alpha\rho_v\alpha^{-1}\}_{v\in V(G)} = \{\sigma_{\alpha(v)}\}_{v\in V(G)} = \{\sigma_v\}_{v\in V(G)} = \sigma,$$

for some $\sigma \in R(G)$. (Formally, a permutation group is a group whose elements are permutations acting on a fixed set, the object set. The group operation is composition of mappings. The order of the object set is called the degree of the permutation group and must be finite.)

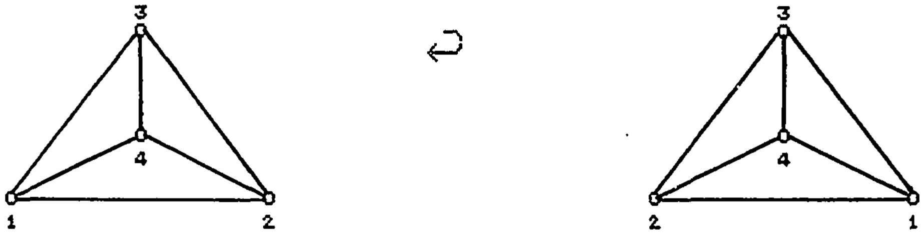
Example 1.10

Let $M = (K_4, \rho)$, where $\rho = \{\rho_1 = (234)$, $\rho_2 = (143)$,

ρ_3 = (124), ρ_4 = (132)} (see Figure 1.9a). Take α = (12) as the automorphism; then

$\alpha(\rho) = \sigma = \{\sigma_1 = (243),\ \sigma_2 = (134),\ \sigma_3 = (142),\ \sigma_4 = (123)\}$.

The resulting map is M' = (K_4, σ) (Figure 1.9b), the "mirror image" of M. We call it that because the counterclockwise rotations of M' are the clockwise rotations of M. The rotation schemes ρ and σ are equivalent (and their corresponding maps are congruent) under α.



(a) (b)

Figure 1.9. Congruent Maps of K_4.

Theorem 1.2. (Biggs) The number of rotation schemes of a graph G which are equivalent to a fixed rotation scheme $\rho \in R(G)$ is the index [Aut G:Aut M], where M = (G, ρ) is the map corresponding to ρ.

Proof: Aut M is the stabilizer of ρ when Aut G acts on R(G). It is well-known (see [3], for example) that the size of the orbit of an object under the action of a permutation group is the index of the stabilizer of the object in the group. ∎

Example 1.11

Consider the map M = (K_4, ρ) given in Example 1.10. Aut K_4 = S_4 (the full symmetric group of degree four) and Aut M = A_4 (the alternating group of degree four). By

Theorem 1.2 we see that there are $[S_4:A_4] = 2$ rotation schemes equivalent to ρ. We conclude that M and M' are the only maps in this congruence class; these are the maps of Figure 1.9.

For $\alpha \in \mathrm{Aut}\ G$, define $F(\alpha) = \{\rho \in R(G) \mid \alpha(\rho) = \rho\}$. Also, define $C(G)$ to be the set of congruence classes induced by the action of Aut G acting on R(G); then

Theorem 1.3. (Biggs) The number of congruence classes of maps $M = (G,\rho)$ having G as the underlying graph is

$$|C(G)| = \frac{1}{|\mathrm{Aut}\ G|} \sum_{\alpha \in \mathrm{Aut}\ G} |F(\alpha)|.$$

Proof: This is just Burnside's Lemma for the present context (see [3], for example). ∎

Example 1.12

We again consider K_4; observe that $|R(G)| = (2!)^4$ so there are sixteen rotations on K_4. Using ad hoc methods (for the moment) we have classified these sixteen rotation schemes (Table 1.1). Note that since cycles of the

Table 1.1
Classifying the Rotations on K_4

Class representative (α)	e	(12)	(123)	(12)(34)	(1234)
Number in the class	1	6	8	3	6
$\lvert F(\alpha)\rvert$	16	0	4	4	2

same length can be transformed into each other, $|F(\alpha)|$ must be invariant on the conjugacy classes of S_4; hence

we need only specify a member for each conjugacy class and the number of rotation schemes it fixes. We use 'e' to specify the identity automorphism for a graph G.

Thus, there are $(1 \cdot 16 + 6 \cdot 0 + 8 \cdot 4 + 3 \cdot 4 + 6 \cdot 2) / 24 = (16 + 0 + 32 + 12 + 12) / 24 = 72 / 24 = 3$ congruence classes for the maps of K_4. Figure 1.9 pictures one of these classes; the other two are shown in Figure 1.10. These maps are unlabeled to emphasize that they are congruence classes of labeled maps.

Figure 1.10a depicts the torus divided into a 3-gon and a 9-gon with Aut $M = \mathbb{Z}_3$ for this class; thus, there are $24 / 3 = 8$ maps in this class. Figure 1.10b has the torus divided into a 4-gon and an 8-gon with Aut $M = \mathbb{Z}_4$ for this class; thus, there are $24 / 4 = 6$ maps in this class. These, together with the two rotation schemes accounted for in Example 1.11, account for all sixteen of them.

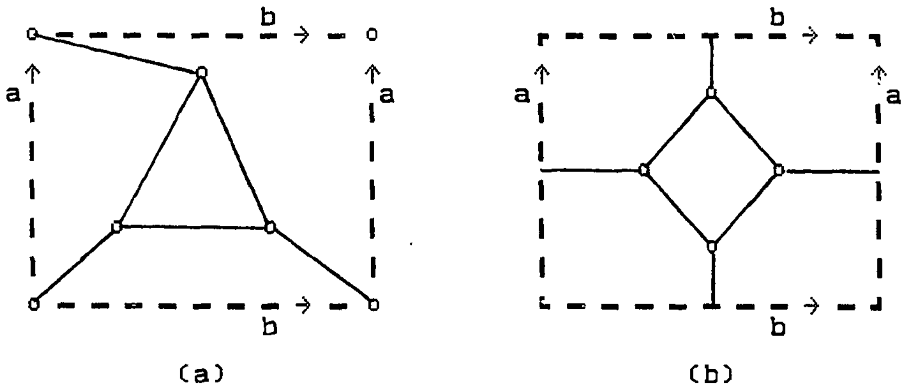


Figure 1.10. The Toroidal Congruence Classes of K_4.

The next few chapters will develop the general theory and establish the counting formula for complete

n-partite graphs. It will be shown that this formula generalizes the formulas for complete, complete bipartite, and complete tripartite graphs (these formulas will be established independently). Asymptotic results will be presented for the complete and complete bipartite cases, as well as implications of these asymptotic results. A new parameter is introduced to random topological graph theory, $\overline{m}(G)$, the average number of symmetries of the maps of the connected graph G. This parameter is evaluated in terms of the degree sequence, the number of graph automorphisms, and the number of congruence classes of maps of G.

We will give the computer program, which classified the congruence classes and verified the formulas for small cases, in an appendix. This program was written in the C programming language and was implemented on an IBM PC clone, the VAX cluster at Western Michigan University, and the Connection Machine at Argonne National Laboratories.

CHAPTER II

THE BASIC METHOD: COMPLETE GRAPHS

2.1 Introduction

In Chapter I we introduced the background needed for the enumerative technique that will be used; in this chapter we will present the method that was originated in [15]. Using this method, we will be able to derive the formula for the number of congruence classes for the maps of complete graphs. Much of the material in this chapter is also in [15].

Theorem 1.3 was not used directly in Example 1.12; instead, a simplification was made which allowed us to reduce the number of terms in the sum. We will see shortly that this is typical—first, however, we need some more background material.

Let α be a permutation of a set with n elements. We define the cycle type of α, $j(\alpha)$, to be an ordered n-tuple, $j(\alpha) = (j_1, j_2, \ldots, j_n)$, where j_k is the number of cycles of length k in the disjoint cycle decomposition of α. Note that $\sum_{k=1}^{n} k \cdot j_k = n$. If only one of the j_k is nonzero, that is, all cycles of α have length k, then α is said to be a <u>uniform</u> (or more precisely, <u>k-uniform</u>) permutation. It is well-known (see [9], for example) that two permutations are equivalent under conjugation if and only if they have the same cycle type.

Theorem 2.1. If α and β are both permutations of an n-element set that are equivalent under conjugation, then for all positive integers k, α^k and β^k are also equivalent under conjugation.

Proof: Since each cycle of α (or β) is disjoint from every other cycle of α (β), we begin by considering a single cycle.

Suppose that $\gamma = (x_0 x_1 \ldots x_{m-1})$ is a cycle of length m and let k be a positive integer; we consider γ^k. We wish to determine the cycle of γ^k containing x_i for $i = 0, \ldots, m - 1$.

Under the action of γ^k, x_i is followed by x_{i+k}, is followed by x_{i+2k}, ..., is followed by $x_{i+ck} = x_i$ (all additions have been taken mod m). Since no restriction has been placed upon x_i, we know that c is a constant ($i = 0, \ldots, m - 1$). We conclude that γ^k is c-uniform.

From number theory (see [17], for example), we know that $c \cdot k = \mathrm{lcm}(m,k)$, where by lcm(m,k) we mean the least common multiple of m and k. Also, if we let gcd(m,k) denote the greatest common divisor of m and k, then we have $\mathrm{lcm}(m,k) = k \cdot m / \gcd(m,k)$. Therefore, $c = m / \gcd(m,k)$, and the cycle of length m, in γ, has become gcd(m,k) cycles of length $m / \gcd(m,k)$, in γ^k.

Now α and β, we have been told, have the same cycle type; thus for each $i = 1, \ldots, n$, there are j_i i-cycles in each. We see from above that each of these i-cycles becomes gcd(i,k) cycles of length $i / \gcd(i,k)$ in the k-th power; therefore, the contribution to the cycle type of α^k (and also β^k) in the entry for the cycles of length $i / \gcd(i,k)$ is thus $j_i \cdot \gcd(i,k)$. We conclude

that α^k and β^k have the same cycle type (and are thus equivalent under conjugation). ∎

Within the proof of Theorem 2.1 there is the proof of a result that will be cited frequently in this dissertation. Because of this, we will state the result as a corollary; this will enable us to refer to the result directly.

Corollary 2.1a. Let γ be a cyclic permutation on an m-element set; then for every positive integer k, γ^k is a c-uniform permutation, where $c = m / \gcd(m,k)$. Furthermore, there are $\gcd(m,k)$ c-cycles in the disjoint cycle decomposition of γ^k.
Proof: See the proof of Theorem 2.1. ∎

Corollary 2.1b. Let G be a connected graph, $\alpha, \beta \in \text{Aut } G$, and $\forall v \in V(G)$, let $\lambda(v)$ denote the length of the cycle containing v in the disjoint cycle decomposition of α. Further, let $\alpha^{\lambda(v)}|_{N(v)}$ denote the restriction of $\alpha^{\lambda(v)}$ to the neighborhood of v. Then, $\forall v \in V(G)$, $\alpha^{\lambda(v)}|_{N(v)}$ and $(\beta\alpha\beta^{-1})^{\lambda(v)}|_{N(\beta(v))}$ have the same cycle type.
Proof: For each cycle $(u_1u_2\ldots u_m)$ in α we know that $(\beta(u_1)\beta(u_2)\ldots\beta(u_m))$ is a cycle in $\beta\alpha\beta^{-1}$. Arbitrarily take $v \in V(G)$; the above shows that the length of the cycle containing v in α is the same as the length of the cycle containing $\beta(v)$ in $\beta\alpha\beta^{-1}$. By Corollary 2.1a, we see that this is also true of v in $\alpha^{\lambda(v)}$ and $\beta(v)$ in $(\beta\alpha\beta^{-1})^{\lambda(v)}$. Now $\beta \in \text{Aut } G$; therefore, β maps N(v) onto $N(\beta(v))$—in fact, β maps $u \in N(v)$ onto $\beta(u) \in N(\beta(v))$.

The theorem now easily follows as u can be picked arbitrarily. ∎

For $\alpha \in \mathrm{Aut}\ G$, define $J(\alpha) = \{\beta\alpha\beta^{-1} \mid \beta \in \mathrm{Aut}\ G\}$ as the structure class of α; the set of structure classes will be denoted $\mathcal{J}(G) = \{J(\alpha) \mid \alpha \in \mathrm{Aut}\ G\}$. Note that if $G = K_n$, then $\mathrm{Aut}\ G \cong \mathbb{S}_n$; in this case, $J(\alpha)$ is the conjugacy class of α. In the next section we shall see that $|F(\alpha)|$ depends upon the cycle type of $\alpha^{\lambda(v)}|_{N(v)}$, $\forall v \in V(G)$. Thus, if $\gamma \in J(\alpha)$, then $|F(\alpha)| = |F(\gamma)|$. Hence, for each $J(\alpha) \in \mathcal{J}(G)$, we need only count how many automorphisms are in $J(\alpha)$ and evaluate $|F(\alpha)|$ for one such α in that structure class.

Theorem 2.2. The number of congruence classes of maps $m = (G,\rho)$ having G as the underlying graph is

$$|C(G)| = \frac{1}{|\mathrm{Aut}\ G|} \sum_{J(\alpha) \in \mathcal{J}(G)} |J(\alpha)| \cdot |F(\alpha)|$$

Proof: Deferred to the next section. ∎

Example 2.1

Consider the graph of Example 1.8—$V = \{1,2,3,4\}$ and $E = \{\{1,2\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}$. There are four rotations on G, $|R(G)| = 1!^2 \cdot 2!^2 = 4$ (Table 2.1). Also, $\mathrm{Aut}\ G = \{e, (13), (24), (13)(24)\}$, where e is the identity element; so $|\mathrm{Aut}\ G| = 4$.

We have $F(e) = \{\rho_{(1)}, \rho_{(2)}, \rho_{(3)}, \rho_{(4)}\}$, $F[(13)] = \emptyset$, $F[(24)] = \{\rho_{(2)}, \rho_{(3)}\}$, and $F[(13)(24)] = \{\rho_{(1)}, \rho_{(4)}\}$. Notice that while (13) and (24) are in the same conjugacy class, they are not in the same structure class.

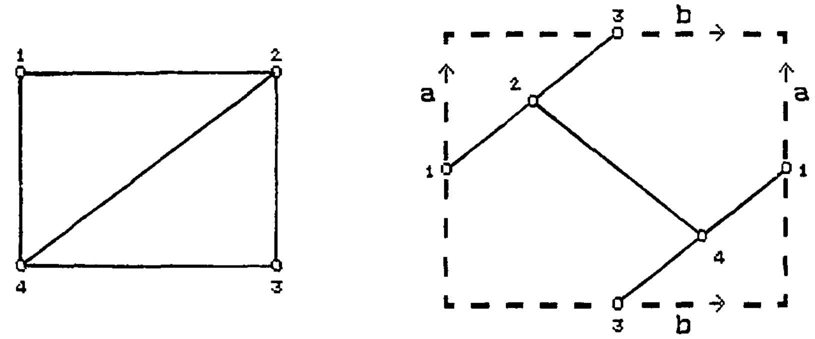
We compute $|C(G)|$; $|C(G)| = (1\cdot4+1\cdot0+1\cdot2+1\cdot2)/4 = 2$.

Table 2.1

The Four Rotation Schemes of Example 2.1

$\rho_{(1)}$	$= \{\rho_1 = (24),\ \rho_2 = (134),\ \rho_3 = (24),\ \rho_4 = (123)\}$
$\rho_{(2)}$	$= \{\rho_1 = (24),\ \rho_2 = (134),\ \rho_3 = (24),\ \rho_4 = (132)\}$
$\rho_{(3)}$	$= \{\rho_1 = (24),\ \rho_2 = (143),\ \rho_3 = (24),\ \rho_4 = (123)\}$
$\rho_{(4)}$	$= \{\rho_1 = (24),\ \rho_2 = (143),\ \rho_3 = (24),\ \rho_4 = (132)\}$

Figure 2.1 shows the two congruence classes for G. Figure 2.1a shows the sphere divided into two 3-gons and a 4-gon—this class contains $\rho_{(1)}$ and $\rho_{(4)}$. Figure 2.1b shows a 10-gon on the torus; this class contains $\rho_{(2)}$ and $\rho_{(3)}$. Notice that Figure 2.1a is like Figure 1.7.



(a) (b)

Figure 2.1. The Congruence Classes of Example 2.1.

Theorem 2.2 allows us to sum over the structure classes instead of over all automorphisms. Many times $|F(\alpha)| = 0$; thus if we can establish when $F(\alpha)$ is nonempty, we can simplify the sum even further.

2.2 The Basic Method

In this section we present the method originated in [15]. We will show that $|F(\alpha)|$ depends only on the

structure type of α, and establish conditions for when $F(\alpha)$ is nonempty. This will allow us to find a formula for the number of unlabeled orientable 2-cell imbeddings of complete graphs.

Theorem 2.3. (Biggs) If $\alpha \in \mathrm{Aut}\ G$ fixes two adjacent vertices, then either α is e (the identity element of the automorphism group) or $F(\alpha)$ is empty.

Proof: This is actually Lemma 5.2.5 of [3]. ∎

Example 2.2

Consider the graph of Example 2.1; $\alpha = (13)$ fixes the adjacent vertices 2 and 4. Therefore, $F(\alpha) = \emptyset$ (and $|F(\alpha)| = 0$).

Let $\alpha \in \mathrm{Aut}\ G$ and $v \in V(G)$; the <u>fixed set of α at v</u> is defined as $F_v(\alpha) = \{\rho_v \mid \alpha(\rho_v) = \alpha\rho_v\alpha^{-1} = \rho_v\}$ (necessarily, α fixes v). That is, it is the set of rotations at v that are fixed by α under conjugation (this set may be empty). Now α is a permutation of $V(G)$, so v appears in some cycle of the disjoint cycle decomposition of α—let $\lambda(v)$ denote the length of this cycle. Furthermore, $\langle\alpha\rangle$ will denote the group generated by α under composition of mappings.

Theorem 2.4. Let $\alpha \in \mathrm{Aut}\ G$ and let S be a complete system of orbit representatives for $\langle\alpha\rangle$ acting on $V(G)$; then

$$|F(\alpha)| = \prod_{v \in S} |F_v(\alpha^{\lambda(v)})|.$$

Proof: By definition of $F(\alpha)$, $\rho \in F(\alpha)$ if and only if

$\alpha(\rho) = \rho$; that is, $\alpha(\rho_v) = \alpha\rho_v\alpha^{-1}$, $\forall v \in V(G)$. So, for all positive integers k and all $v \in V(G)$, $\rho \in F(\alpha)$ if and only if $\alpha^k\rho_v\alpha^{-k} = \rho_{\alpha^k(v)}$. Thus, the rotation at v determines the rotation at each vertex in the orbit of v under the action of $\langle\alpha\rangle$ acting on V(G). Also, if $k = \lambda(v)$ then

$$\alpha^{\lambda(v)}\rho_v\alpha^{-\lambda(v)} = \rho_v,$$

so that $\rho_v \in F_v(\alpha^{\lambda(v)})$. Hence, for each orbit of $\langle\alpha\rangle$, choose $\rho_v \in F_v(\alpha^{\lambda(v)})$; the other rotations are determined by taking $1 \le k < \lambda(v)$ and setting $\rho_{\alpha^k(v)} = \alpha^k\rho_v\alpha^{-k}$. The choices for each orbit are independent of each other, so taking S to be a complete set of orbit representatives gives the theorem. ∎

Example 2.3

Consider the graph K_4 of the previous chapter. Let $\alpha = (12)(34)$. One complete set of orbit representatives for α is $S = \{1,3\}$. In both cases $\lambda(v) = 2$ and α^2 is the identity element. There are two possible rotations at each vertex; both are fixed by the identity element; hence, $|F_1(\alpha^2)| = |F_3(\alpha^2)| = 2$. So, $|F(\alpha)| = 2\cdot2 = 4$, as in Table 1.1.

Recall that a uniform permutation has exactly one nonzero entry in its cycle type. Let φ denote the Euler function and, for $v \in V(G)$, $\alpha^{\lambda(v)}|_{N(v)}$ the restriction of $\alpha^{\lambda(v)}$ to the neighborhood of v. Then,

Theorem 2.5. If $|N(v)| = n$, then

$$|F_v(\alpha^{\lambda(v)})| = \begin{cases} \varphi(c)\cdot c^{(n/c\,-1)}\cdot(n/c\,-1)!, & \text{if } \alpha^{\lambda(v)}|_{N(v)} \text{ is c-uniform,} \\ 0, & \text{otherwise.} \end{cases}$$

Proof: We first show that $F_v(\alpha^{\lambda(v)})$ is nonempty if and only if $\alpha^{\lambda(v)}|_{N(v)}$ is c-uniform, for some positive integer c; for simplicity we will let $\beta = \alpha^{\lambda(v)}|_{N(v)}$.

Every member of $F_v(\alpha^{\lambda(v)})$ is a cyclic permutation of the neighborhood N(v) of v. Let $\rho = (x_0x_1...x_{n-1})$ be in $F_v(\alpha^{\lambda(v)})$. Only the elements of N(v) in $\alpha^{\lambda(v)}$ can affect ρ, so

$$\alpha^{\lambda(v)}\rho\alpha^{-\lambda(v)} = \beta\rho\beta^{-1} = (\beta(x_0)...\beta(x_{n-1})) = \rho.$$

If β is uniform, then this equation has solutions (any ρ such that, for some positive integer k, $\beta = \rho^k$ is a solution); therefore, $F_v(\alpha^{\lambda(v)})$ is nonempty.

Suppose $F_v(\alpha^{\lambda(v)})$ is nonempty and let ρ be one of the rotations in this set. Suppose that we have $\rho = (x_0x_1...x_{n-1})$; then $\rho = \beta\rho\beta^{-1}$ and β commutes with ρ (and thus with all powers of ρ). Suppose that β is such that $\beta(x_0) = x_k$; then

$$\beta(x_i) = \beta\rho^i(x_0) = \rho^i\beta(x_0)$$
$$= \rho^i\rho^k(x_0) = \rho^k\rho^i(x_0) = \rho^k(x_i),$$

for all i = 0, 1, ..., n - 1,; so $\beta = \rho^k$. By Corollary 2.1a, this is a c-uniform permutation, where we know that $c = n / \gcd(n,k)$.

We now know that $F_v(\alpha^{\lambda(v)})$ is nonempty if and only if $\beta = \alpha^{\lambda(v)}|_{N(v)}$ is uniform. We wish to determine the number of rotations that are fixed by such β.

Let R(v) be the set of rotations at v, J_c the set of c-uniform permutations on N(v), and for $\beta \in J_c$, let R_β be the set of rotations at v that are fixed by β under conjugation. Note that since no member of J_c is distinguished over any other member, $|R_\beta|$ is constant; let $|R_\beta| = r$. Then r is the number that we seek, that is,

$|F_v(\alpha^{\lambda(v)})|$. The above remarks show

$$\sum_{\beta \in J_c} |R_\beta| = r \cdot |J_c|. \tag{1}$$

We now count the above sum in a different manner; this will allow us to determine the value of r. Fix $\rho \in R(v)$, we wish to determine the number of $\beta \in J_c$ that fix ρ under conjugation. We have already seen that $\beta = \rho^k$, for some positive integer k. Clearly we need only concern ourselves with $1 \le k \le n$—there are only n elements in $\langle\rho\rangle$. Now β is c-uniform and we have seen that $c = n / \gcd(n,k)$. Thus, $k = m \cdot \gcd(n, k)$ for some positive integer m. Clearly, we must have $1 \le m \le c$, and $\gcd(m,c) = 1$ $(n/c = \gcd(n,k))$. There are $\varphi(c)$ such integers m, so every $\rho \in R(v)$ is fixed by $\varphi(c)$ members of J_c. Therefore,

$$\sum_{\beta \in J_c} |R_\beta| = \varphi(c) \cdot |R(v)|. \tag{2}$$

Recall that if $|N(v)| = n$ then there are $(n - 1)!$ rotations at v; thus $|R(v)| = (n - 1)!$.

Let $\beta \in J_c$; order the sequence of c-cycles in β. Using elementary combinatorics, we see that there are $\left[{n \atop c} \right]$ ways to pick the elements in the first cycle, $\left[{n - c \atop c} \right]$ ways to pick the elements in the second cycle, ..., and so on. Hence, there are $\left[{n \atop c, \ldots, c} \right]$ ways to pick the elements in the ordered sequence of c-cycles. In each of these c-cycles (there are n/c of them), there are $(c - 1)!$ ways to order the elements. Thus, the total number of such β having the sequence of its cycles ordered is

$$\left[{n \atop c, \ldots, c} \right] \cdot (c - 1)!^{(n/c)} = n! \;/\; c^{(n/c)}.$$

Now, each sequence of n/c c-cycles can be ordered in $(n/c)!$ ways; so,

$$(n/c)! \cdot |J_c| = n! \;/\; c^{(n/c)}.$$

Therefore,

$$|J_c| = n! \;/\; [\; c^{(n/c)} \cdot (n/c)! \;]$$

Combining (1), (2), and the formulas for $|R(v)|$ and $|J_c|$, we get

$$|F_v(\alpha^{\lambda(v)})| = r = \varphi(c) \cdot (n-1)! \cdot \frac{c^{(n/c)} \cdot (n/c)!}{n!}$$

$$= \varphi(c) \cdot \frac{c^{(n/c\,-1)}(n/c)!}{n/c}$$

$$= \varphi(c) \cdot c^{(n/c\,-1)} \cdot (n/c-1)!. \qquad \blacksquare$$

Example 2.4

Again we consider K_4; now let $\alpha = (1)(234) \in S_4$. One complete set of orbit representatives for $\langle\alpha\rangle$ is $S = \{1,2\}$. Note that $\lambda(1) = 1$, and $\lambda(2) = 3$; by Theorem 2.4, we get

$$|F(\alpha)| = |F_1(\alpha^1)| \cdot |F_2(\alpha^3)|.$$

Now $\alpha^3 = (1)(2)(3)(4) = e$, the identity, so that $|F_2(\alpha^3)| = 2$; also, $\alpha|_{N(1)}$ is 3-uniform; by Theorem 2.5, we get

$$|F_1(\alpha\ )| = \varphi(3) \cdot 3^{(3/3\,-1)} \cdot (3/3-1)! = 2 \cdot 1 \cdot 1 = 2.$$

Thus $|F(\alpha)| = 2 \cdot 2 = 4$, as in Table 1.1.

We are now ready to provide a proof for Theorem 2.2. By Theorem 2.4, $|F(\alpha)| = \prod_{v \in S} |F_v(\alpha^{\lambda(v)})|$. By Theorem 2.5, $|F_v(\alpha^{\lambda(v)})|$ depends on the cycle type of $\alpha^{\lambda(v)}|_{N(v)}$; this means that (by Corollary 2.1b) $|F(\alpha)|$ depends on the structure class. That is, $|F(\alpha)|$ is constant on the structure class of α, $J(\alpha)$. Since Theorem 1.3 sums over all elements of Aut G, all $|J(\alpha)|$ members of $J(\alpha)$ will be

counted exactly once. Each of these contributes $|F(\alpha)|$ to the sum; hence the sum given in Theorem 2.2 is correct.

Theorems 2.3 - 2.5 allow us to determine $|F(\alpha)|$; in order to use Theorem 2.2 we need to calculate $|J(\alpha)|$. There seems to be no general form for $|J(\alpha)|$ that will work in all cases—if there were, then counting the congruence classes of all graphs would be straight forward. In the next section, we will handle complete graphs. We will determine the structure of those automorphisms for which $|F(\alpha)| > 0$; this will allow us to calculate $|J(\alpha)|$.

2.3 The Formula for the Complete Graph K_n

In this section we use the method developed in the previous section to count the number of unlabeled orientable 2-cell imbeddings of the complete graph K_n. Recall that Aut $K_n \cong S_n$; thus, $\forall\alpha \in$ Aut K_n, $J(\alpha)$ is simply the conjugacy class of α. The next theorem counts the number of elements in the conjugacy class of α.

Theorem 2.6. Let α be a permutation of an n-element set; suppose that $j(\alpha) = (j_1, j_2, \ldots, j_n)$. If $J(\alpha)$ denotes the conjugacy class of α, then

$$|J(\alpha)| = \frac{n!}{\prod_{k=1}^{n} k^{j_k} \cdot j_k!}.$$

Proof: It is easy to tell cycles of different lengths apart so the cycle lengths partition n into discernable types. There are $\left[{n \atop j_1} \right]$ ways to pick the elements of the

first type, the 1-cycles. This leaves $\left[{n - j_1 \atop 2 \cdot j_2} \right]$ ways to pick the second type, $\left[{n - j_1 - 2j_2 \atop 3 \cdot j_3} \right]$ ways for the third type to be picked, and so on. Continuing we find that the number of ways of picking elements of each type is given by

$$\left[{n \atop j_1,\ 2j_2,\ \ldots,\ nj(n)} \right] = \frac{n!}{\prod_{k=1}^{n} (k \cdot j_k)!}. \tag{3}$$

In the proof of Theorem 2.5 we calculated the number of ways that a set can be formed into c-uniform permutations (when the full symmetric group acts on these permuted elements). We will use this result now.

Consider the elements that will form the k-cycles ($k = 1, \ldots, n$)—these are independent of the cycles of any other length, so they can be considered separately. These yield $(k \cdot j_k)! \ / \ (k^{j_k} \cdot j_k!)$ uniform permutations for the elements that form the k-cycles. The independence of the sets of elements gives us

$$\prod_{k=1}^{n} \frac{(k \cdot j_k)!}{k^{j_k} \cdot j_k!} \tag{4}$$

ways of permuting the elements once formed into sets. Therefore, the number of permutations α, whose cycle type is $j(\alpha) = (j_1, \ldots, j_n)$ is given by (3)·(4); that is,

$$|J(\alpha)| = \frac{n!}{\prod_{k=1}^{n} (k \cdot j_k)!} \cdot \prod_{k=1}^{n} \frac{(k \cdot j_k)!}{k^{j_k} \cdot j_k!},$$

which after simplifying gives us the result. ∎

Example 2.5

Consider $\alpha = (1)(234) \in S_4$ (from Example 2.4). Using Theorem 2.6 we find that

$$|J(\alpha)| = \frac{4!}{(1^1 \cdot 1!)\cdot(2^0 \cdot 0!)\cdot(3^1 \cdot 1!)\cdot(4^0 \cdot 0!)} = \frac{24}{3} = 8.$$

We conclude that there are eight permutations of the above cycle type (which is verified in Table 1.1).

In the complete graph, every pair of distinct vertices is adjacent; Theorem 2.3 implies that we need only consider those automorphisms which fix none, one, or all of the vertices. Let $V(K_n) = \{1,2,\ldots,n\}$, then $\mathrm{Aut}\, K_n = \mathfrak{S}_n$; we consider each of these three cases separately. By Theorem 2.2, we need to calculate $|J(\alpha)|$ and $|F(\alpha)|$ for each of these cases.

Let $\alpha \in \mathfrak{S}_n$ and suppose that α fixes no vertex of K_n; then $\lambda(v) \geq 2$, $\forall v \in V(K_n)$. By Theorem 2.6 we need the cycle type of α to calculate $|J(\alpha)|$; we begin by determining this.

Theorem 2.4 implies that $|F(\alpha)| > 0$ if and only if $|F_v(\alpha^{\lambda(v)})| > 0$, $\forall v \in V(G)$. By the proof of Theorem 2.5, this occurs if and only if $\alpha^{\lambda(v)}|_{N(v)}$ is uniform. For $G = K_n$, $\lambda(v) \geq 2$, so Theorem 2.3 implies that $\alpha^{\lambda(v)}$ is the identity element of $\mathfrak{S}_n$. We conclude that α is uniform. The identity element is also uniform, so it can be included in this case.

Suppose that α is d-uniform; then Theorem 2.6 states that

$$|J(\alpha)| = n! \;/\; [d^{n/d}\cdot(n/d)!]$$

and since $\alpha^{\lambda(v)}$ is the identity element, $\forall v \in V(K_n)$, $|F_v(\alpha^{\lambda(v)})| = (d(v)-1)! = (n-2)!$, where $d(v)$ is the degree of the vertex v. There are n/d members in a complete system of orbit representatives, so

$$|F(\alpha)| = (n-2)!^{n/d}.$$

Adding across the conjugacy classes of uniform permutations is just adding across the different cycle lengths which divide n. Therefore, the contribution to $|C(K_n)|$ by uniform permutations is given by Theorem 2.2 as

$$\frac{1}{n!}\sum_{d \mid n} \frac{n!\cdot(n-2)!^{n/d}}{d^{(n/d)}\cdot(n/d)!} = \sum_{d \mid n} \frac{(n-2)!^{n/d}}{d^{(n/d)}\cdot(n/d)!}. \qquad (5)$$

Next suppose that $\alpha \in S_n$ fixes exactly one vertex v; again we wish to determine the cycle type of α. Consider $|F_v(\alpha)|$; by Theorem 2.5, $\alpha|_{N(v)}$ must be uniform. Since only v is fixed, $j(\alpha) = (1,0,\ldots,0,j_d = \frac{n-1}{d},0,\ldots,0)$, for some integer d such that $d \geq 2$. Applying Theorem 2.6 we obtain

$$|J(\alpha)| = \frac{n!}{d^{(n-1)/d}\cdot((n-1)/d)!}.$$

By Theorem 2.5, $|F_v(\alpha)| = \varphi(d)\cdot d^{((n-1)/d-1)}\cdot(\frac{n-1}{d}-1)!$. Also, $\forall u \in V(K_n)$, where $u \neq v$, $\alpha^{\lambda(v)}$ must be the identity element of S_n (Theorem 2.3); therefore, Theorem 2.5 gives $|F_u(\alpha^{\lambda(u)})| = (n-2)!$. There are $(n-1)/d$ d-cycles, and the single 1-cycle in α; hence a complete system of orbit representatives consists of v, and $(n-1)/d$ other members. By Theorem 2.4 we conclude

$$|F(\alpha)| = \varphi(d)\cdot d^{((n-1)/d-1)}\cdot(\frac{n-1}{d}-1)!\cdot(n-2)!^{(n-1)/d}. \qquad (6)$$

Thus, the contribution to $|C(K_n)|$ by automorphisms which fix exactly one vertex is given by Theorem 2.2 as

$$\frac{1}{n!}\sum_{\substack{d \mid (n-1)\\ d \neq 1}} \frac{n!}{d^{(n-1)/d}((n-1)/d)!}\cdot(6)$$

$$= \sum_{\substack{d \mid (n-1)\\ d \neq 1}} \frac{\varphi(d)(n-2)!^{(n-1)/d}}{n-1}. \qquad (7)$$

Theorem 2.7. The number of congruence classes for the complete graph K_n is given by

$$|C(K_n)| = \sum_{d \mid n} \frac{(n-2)!^{n/d}}{d^{n/d} \cdot (n/d)!} + \sum_{\substack{d \mid (n-1) \\ d \neq 1}} \frac{\varphi(d)(n-2)!^{(n-1)/d}}{n-1}.$$

Proof: Combine the formulas (5) and (7). ∎

Example 2.6

Consider the complete graph K_5. Theorem 2.7 states that we need a sum over the divisors of $n = 5$, and a sum over the divisors of $n - 1 = 4$ (that are greater than one). We evaluate each of these sums separately.

The first sum is over the divisors of $n = 5$. The divisors of 5 are 1 and 5. When $d = 1$, the term is

$$\frac{(5-2)!^{5/1}}{1^{5/1} \cdot (5/1)!} = \frac{6^5}{5!} = \frac{7776}{120} = 64.8.$$

When $d = 5$, the term is

$$\frac{(5-2)!^{5/5}}{5^{5/5} \cdot (5/5)!} = \frac{6}{5} = 1.2.$$

Therefore, the value of the first sum is $64.8 + 1.2 = 66$.

The second sum is over the divisors of $n - 1 = 5 - 1 = 4$; these divisors must be greater than one. The divisors of 4, greater than 1, are 2 and 4. When $d = 2$, the term is

$$\frac{\varphi(2) \cdot (5-2)!^{(5-1)/2}}{5-1} = \frac{1 \cdot 6^2}{4} = 9.$$

When $d = 4$, the term is

$$\frac{\varphi(4) \cdot (5-2)!^{(5-1)/4}}{5-1} = \frac{2 \cdot 6}{4} = 3.$$

Therefore, the value of the second sum is $9 + 3 = 12$.

Combining the values of the two sums we get $66 + 12 = 78$. We conclude that there are 78 congruence classes for the maps of K_5.

Using an IBM PC/XT compatible computer, and the program whose listing appears in the appendix, we break down

Table 2.2

The Congruence Classes of K_5 on S_1

5x4	1	6, 2x4, 2x3	2	8, 4x3	3
2x5, 4, 2x3	2	7, 4, 3x3	1	total on S_1:	9

Table 2.3

The Congruence Classes of K_5 on S_2

8, 2x6	1	9, 8, 3	6	11, 6, 3	2
8, 7, 5	1	10, 2x5	1	12, 2x4	2
2x8, 4	2	10, 6, 4	4	12, 5, 3	2
9, 6, 5	1	10, 7, 3	3	13, 4, 3	8
9, 7, 4	3	11, 5, 4	1	14, 2x3	8
		total on S_2:	45		

the congruence classes by face distribution and surface. A summary of this breakdown appears in Tables 2.2 - 2.4. Table 2.2 shows that there are nine toroidal congruence classes. The entry: 2x5, 4, 2x3 means that there are two 5-gons, a 4-gon, and two 3-gons in the imbedding; the 2 which follows means that there are two congruence classes with this face distribution. The other tables show the breakdowns for S_2 and S_3.

Table 2.4

The Congruence Classes of K_5 on S_3

20	24	total on S_3:	24

The program does some further dividing of these congruence classes. In [10] the 2-cell imbeddings of a graph are considered to be of three types: open, strong, and closed. An <u>open</u> 2-cell imbedding is an imbedding in which every region is homeomorphic to an open disk. A <u>strong</u> 2-cell imbedding is an open 2-cell imbedding that has an additional property: the image of every edge of the imbedded graph separates distinct regions of the imbedding. A <u>closed</u> 2-cell imbedding is an imbedding in which the closure of every region in the surface is homeomorphic to a closed disk. We will be considering only graphs whose minimum degree is two; then it follows that every closed 2-cell imbedding is strong, and every strong 2-cell imbedding is open. This minimum degree condition is needed—consider a cycle with a dangling edge; this can be imbedded in the sphere as a closed 2-cell imbedding, but it is not a strong imbedding.

Example 2.7

Consider the 78 congruence classes of maps of K_5. Applying the above hierarchy to these congruence classes we find the following: Only S_1 has closed 2-cell imbeddings; only S_1 and S_2 have strong 2-cell imbeddings. One of the functions of the program in the appendix is to determine which of the congruence classes are closed, which are strong (but not closed), and which are open (but not strong). A summary of these results appears in Table 2.5.

It is interesting to see how quickly the number of congruence classes of maps of complete graphs increases.

In Table 2.6 we see the number of congruence classes for the maps of all complete graphs from order three to order

Table 2.5

Classifying the Congruence Classes of Maps of K_5

	S_1	S_2	S_3	Total by Type
Open (but not Strong)	3	42	24	69
Strong (but not Closed)	3	3	0	6
Closed	3	0	0	3
Total by Surface	9	45	24	78

seven. As one can see from the table, these numbers get large quickly. In Chapter VI we will derive an asymptotic formula for the number of congruence classes of maps of K_n. This asymptotic formula has ramifications to the new twig on the branch of topological graph theory known as random topological graph theory.

Table 2.6

The Numbers of Congruence Classes of Maps of Small K_n

n	$\lvert C(K_n) \rvert$
3	1
4	3
5	78
6	265,764
7	71,095,150,000

In the next chapter we will develop the tools needed to count the number of congruence classes for the maps of complete bipartite graphs. We will see, in later chapters, that the method used in the bipartite case generalizes to complete tripartite graphs and generalizes again to complete n-partite graphs.

CHAPTER III

COMPATIBLE PERMUTATIONS: COMPLETE BIPARTITE GRAPHS

3.1 Introduction

Most of the material in Chapters I and II can be regarded as background for this dissertation—it appears (in a slightly different form) in other sources. This is the chapter that will introduce new ideas. These ideas will extend the method of [15] so that the congruence classes of the maps of complete bipartite graphs (and ultimately of complete n-partite graphs) can be enumerated.

Recall that a complete bipartite graph is a graph $G = (V, E)$ where the vertex set V is partitioned into two partite sets, V_1 and V_2, and where E is the set of all possible unordered pairs of vertices having a vertex from each partite set. Thus, we have

$$V = V_1 \cup V_2 \qquad V_1, V_2 \neq \emptyset \qquad V_1 \cap V_2 = \emptyset$$

$$E = \{\{u, v\} \mid u \in V_1, v \in V_2\}.$$

If $|V_1| = m$ and $|V_2| = n$, then we write $G = K_{m,n}$. If $m \leq n$, then we say that the complete bipartite graph is in <u>standard form</u>.

Let $A(K_{m,n}) = \{\alpha \in \text{Aut } K_{m,n} \mid \exists \alpha_1, \alpha_2, \alpha_i \in \text{Sym } V_i$ $(i = 1, 2), \alpha = \alpha_1\alpha_2\}$; that is, $A(K_{m,n})$ is the set of automorphisms of Aut $K_{m,n}$ which fix the partite sets as sets. Sym V_i is the full symmetric group of the elements of V_i. Let $D(K_{m,n}) = \{\gamma \in \text{Aut } K_{m,n} \mid \exists u_i \in V_i$ $(i = 1, 2)$

$\gamma(u_1) = u_2\}$; that is $D(K_{m,n})$ is the set of automorphisms of Aut $K_{m,n}$ which do not fix the partite sets. This requires $m = n$, and then γ interchanges V_1 and V_2. Then:

Theorem 3.1. The number of congruence classes of maps of $K_{m,n}$ is given by

$$|C(K_{m,n})| = \frac{1}{|\text{Aut } K_{m,n}|} \sum_{\alpha \in A(K_{m,n})} |F(\alpha)| + \frac{1}{|\text{Aut } K_{m,n}|} \sum_{\gamma \in D(K_{m,n})} |F(\gamma)|$$

Proof: Every automorphism of $K_{m,n}$ either fixes the partite sets as sets or it doesn't. Clearly, the conditions are disjoint; thus, the sum expressed in Theorem 1.3 can be rewritten in the above form. ∎

For simplicity, we will write

$$f(m,n) = \frac{1}{|\text{Aut } K_{m,n}|} \sum_{\alpha \in A(K_{m,n})} |F(\alpha)|, \text{ and}$$

$$h(m,n) = \frac{1}{|\text{Aut } K_{m,n}|} \sum_{\gamma \in D(K_{m,n})} |F(\gamma)|.$$

Example 3.1

Consider $K_{2,2}$ where $V = \{0,1,2,3\}$, $V_1 = \{0,2\}$, and $V_2 = \{1,3\}$; $|\text{Aut } K_{2,2}| = 8$. Half of the automorphisms fix the partite sets and half do not. In fact,

$$A(K_{2,2}) = \{(0)(1)(2)(3),\ (02)(1)(3),\ (0)(2)(13),\ (02)(13)\} \text{ and}$$

$$D(K_{2,2}) = \{(01)(23),\ (0123),\ (03)(12),\ (0321)\}.$$

Also, since $d(v) = 2$, $\forall v \in V(K_{2,2})$, there is exactly one

rotation on $K_{2,2}$ (obviously it is fixed by every element in Aut $K_{2,2}$). Therefore,

$$f(2,2) = (1 + 1 + 1 + 1) / 8 = 4/8 = 1/2, \text{ and}$$

$$h(2,2) = (1 + 1 + 1 + 1) / 8 = 4/8 = 1/2.$$

so, $|C(K_{2,2})| = 1/2 + 1/2 = 1$. Figure 3.1 shows the only imbedding for this graph; looking at it one sees why $K_{2,2}$ is also called C_4, the cyclic graph of order four.

Figure 3.1. The Planar (Spherical) Imbedding of $K_{2,2}(C_4)$.

Suppose that $m < n$; then Aut $K_{m,n} \cong S_m \oplus S_n$. Clearly, the full symmetric group of V_i acts on V_i $(i = 1, 2)$ as each vertex of V_i has the same neighborhood as every other such vertex. If $K_{m,n}$ had any other automorphisms then some vertex $u \in V_1$ would map onto a vertex $v \in V_2$. Automorphisms preserve adjacency, therefore, $V_2 = N(u)$ would map onto $N(v) = V_1$. Automorphisms are one-to-one, therefore, $n = |V_2| \leq |V_1| = m$. This contradicts the conditions of this case; thus, when $m < n$, every automorphism fixes the partite sets as sets. We conclude Aut $K_{m,n}$ has the form above.

Now suppose that $m = n$; then Aut $K_{n,n} \cong S_2[S_n]$ (a wreath product). By extending the argument which lead to the contradiction above, we see that the partite sets are either fixed as sets or swapped. Label the vertices of each of the partite sets with the numbers from 1 to n. A

permutation of S_n induces an action on one of the partite sets as follows: if i is mapped onto j then the vertex associated with i is mapped onto the vertex associated with j. Every automorphism of $K_{n,n}$ can be constructed in two parts. The first part is a permutation of S_2 which is interpreted as follows: The identity element of S_2 means that the partite sets are fixed as sets; the other element means that the partite sets are swapped. The second part is a list of two permutations of S_n—the first element acts on V_1 and the second on V_2. It is interpreted as indicated before when the partite sets are fixed; but when they are swapped, then i mapping to j means the vertex associated with i in one partite set is mapped to the vertex associated with j in the other partite set. That is, every automorphism is associated with an element of the wreath product of S_2 about S_n and every such element gives rise to an automorphism. We conclude that the automorphism group is as indicated above.

By Theorem 2.2, the sums for $f(m,n)$ and $h(m,n)$ can be taken over the structure classes of Aut $K_{m,n}$ instead of over $A(K_{m,n})$ and $D(K_{m,n})$. Let $JA(K_{m,n})$ denote the structure classes of $A(K_{m,n})$ and $JD(K_{m,n})$ denote the structure classes of $D(K_{m,n})$; we still have $J(\alpha)$ and $J(\gamma)$ denoting the structure classes of α and γ.

Theorem 3.2. The forms of $f(m,n)$ and $h(m,n)$ are:

$$f(m,n) = \frac{1}{m!\cdot n!\cdot(1+\delta_{m,n})} \cdot \sum_{J(\alpha)\in JA(K_{m,n})} |J(\alpha)|\cdot|F(\alpha)|, \text{ and}$$

$$h(m,n) = \begin{cases} \dfrac{1}{2\cdot(n!)^2} \displaystyle\sum_{J(\gamma)\in JD(K_{m,n})} |J(\gamma)|\cdot|F(\gamma)|, & \text{when } m = n \\ 0 & \text{, otherwise,} \end{cases}$$

where $\delta_{m,n}$ is the Kronecker function ($\delta_{m,n} = 1$, if $m = n$, and $\delta_{m,n} = 0$, otherwise).

Proof: If $m < n$, we have seen that $|\mathrm{Aut}\, K_{m,n}| = |S_m \oplus S_n| = m!\cdot n!$. When $m = n$, $|\mathrm{Aut}\, K_{n,n}| = |S_2[S_n]| = 2!\cdot n!^2 = 2\cdot n!^2$. Also, $h(m,n) > 0$ if and only if $m = n$ (the partite sets are swapped for $\gamma \in D(K_{n,n})$. Putting these into Theorem 2.2 gives the result. ∎

Example 3.2.

Consider the graph $K_{2,4}$; let $V = \{0,1,2,3,4,5\}$, and partition V into $V_1 = \{0,1\}$ and $V_2 = \{2,3,4,5\}$. $\mathrm{Aut}\, K_{2,4} \cong S_2 \oplus S_4$ so that $|\mathrm{Aut}\, K_{2,4}| = 2!\cdot 4! = 2\cdot 24 = 48$. Theorem 3.2 implies that $JD(K_{2,4}) = \emptyset$, as $h(2,4) = 0$. Table 3.1 lists representatives of $J(\alpha)$ for $JA(K_{2,4})$, $|J(\alpha)|$, and $|F(\alpha)|$ for a member of each class.

Table 3.1

The Breakdown of $JA(K_{2,4})$

		(0)(1) acts on V_1			
$\alpha \in J(\alpha)$:	e	(23)	(234)	(23)(45)	(2345)
$\|J(\alpha)\|$:	1	6	8	3	6
$\|F(\alpha)\|$:	36	0	0	4	4
		(01) acts on V_1			
$\alpha \in J(\alpha)$:	e	(23)	(234)	(23)(45)	(2345)
$\|J(\alpha)\|$:	1	6	8	3	6
$\|F(\alpha)\|$:	6	6	0	6	2

Therefore, $f(2,4) = (1\cdot 36 + 6\cdot 0 + 8\cdot 0 + 3\cdot 4 + 6\cdot 4 + 1\cdot 6 + 6\cdot 6 + 8\cdot 0 + 3\cdot 6 + 6\cdot 2)/48 = (36 + 0 + 0 + 12 + 24 + 6 + 36 + 0 + 18 + 12)/48 = 144/48 = 3$.

This means that $|C(K_{2,4})| = 3 + 0 = 3$. That is, there are three congruence classes of maps of $K_{2,4}$. Figure 3.2 shows imbeddings for each of these three classes.

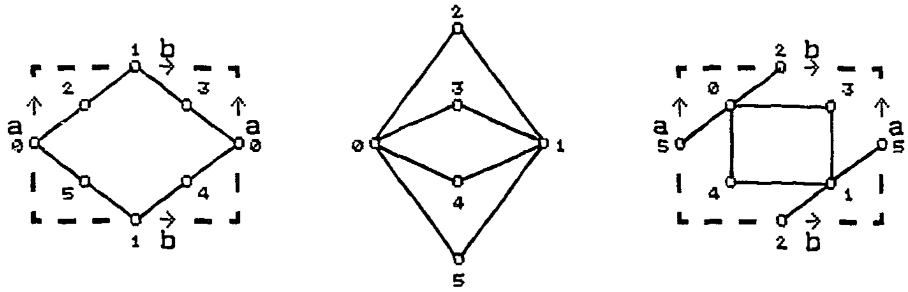


Figure 3.2. The Congruence Classes of $K_{2,4}$.

Example 3.3

Consider $K_{3,3}$; let $V = \{0, 1, 2, 3, 4, 5\}$ with $V_1 = \{0,1,2\}$ and $V_2 = \{3, 4, 5\}$. Aut $K_{3,3} \cong S_2[S_3]$, so, $|\text{Aut } K_{3,3}| = 2! \cdot 3!^2 = 2 \cdot 6^2 = 2 \cdot 36 = 72$. Table 3.2 gives the breakdown for $JA(K_{3,3})$ and Table 3.3 gives the breakdown for $JD(K_{3,3})$.

Table 3.2
The Breakdown of $JA(K_{3,3})$

$\alpha \in J(\alpha)$:	e	(34)	(345)	(01)	(01)(34)
$\|J(\alpha)\|$:	1	3	2	3	9
$\|F(\alpha)\|$:	64	0	16	0	0
$\alpha \in J(\alpha)$:	(01)(345)	(012)	(012)(34)	(012)(345)	
$\|J(\alpha)\|$:	6	2	6	4	
$\|F(\alpha)\|$:	0	16	0	4	

Therefore, $F(3,3) = (1 \cdot 64 + 3 \cdot 0 + 2 \cdot 16 + 3 \cdot 0 + 9 \cdot 0$

Table 3.3

The Breakdown of $JD(K_{3,3})$

$\alpha \in J(\alpha)$:	(03)(14)(25)	(0314)(25)	(012345)
$\lvert J(\alpha)\rvert$:	6	18	12
$\lvert F(\alpha)\rvert$:	8	0	2

$+ 6\cdot 0 + 2\cdot 16 + 6\cdot 0 + 4\cdot 4)/72 = (64 + 0 + 32 + 0 + 0 + 0 + 32 + 0 + 16)/72 = 144/72 = 2.$

Therefore, $h(3,3) = (6\cdot 8 + 18\cdot 0 + 12\cdot 2)/72 = (48 + 0 + 24)/72 = 72/72 = 1.$

This means that $|C(K_{3,3})| = 2 + 1 = 3$. That is, there are three congruence classes of maps of $K_{3,3}$. Figure 3.3 shows imbeddings for each of these three classes.

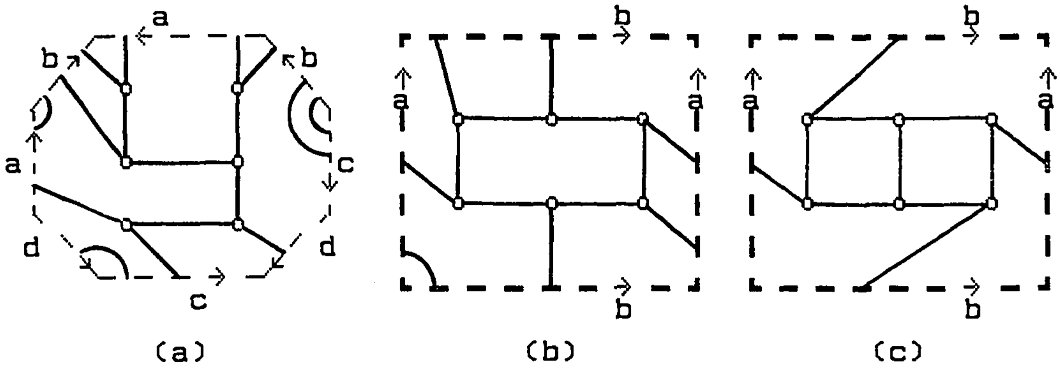


Figure 3.3. The Congruence Classes of Maps of $K_{3,3}$.

In the next section we will calculate $f(m,n)$ in general. We will need to create a definition; this will enable us to determine the structure type of $\alpha \in A(K_{m,n})$ having $|F(\alpha)| > 0$. Once this structure type has been established, we will use Theorems 2.3 - 2.5 to develop the formula for $|F(\alpha)|$.

Theorem 2.6 does not apply in general because the

structure class is only the conjugacy class when the full symmetric group applies. In Section 3.3 we will overcome this deficiency. This will allow us to calculate $h(m,n)$ in all cases. We will end with the formula for the number of congruence classes of maps of $K_{m,n}$.

3.2 Compatible permutations

The fundamental problem of this section is to establish a formula for $f(m,n)$. Theorem 3.2 requires that we pick a representative for $\alpha \in J(\alpha)$, for each $J(\alpha) \in J(G)$, and calculate $|F(\alpha)|$ and $|J(\alpha)|$ for this representative. If we know $j(\alpha)$, then we can determine $J(\alpha)$, and $|F(\alpha)|$ will be given by Theorems 2.4 and 2.5. Therefore, we wish to determine $J(\alpha)$ such that $F(\alpha) \neq \emptyset$. We start our quest with a definition.

The following definition is much more general than needed. The partite sets of a complete bipartite graph are disjoint (hence distinct). This definition allows the concept to be used in contexts which, as yet, we have not had an opportunity to explore.

Let A_1 and A_2 be sets (not necessarily distinct), and for $i = 1, 2$, suppose that α_i is a permutation on A_i. Suppose that α_i has s_i distinct cycle lengths in its disjoint cycle decomposition and denote these cycle lengths λ_{ij}, $j = 1, \ldots, s_i$, where λ_{ij} is the j-th smallest cycle length. Then α_1 and α_2 are said to be <u>compatible</u> if and only if for each λ_{ij} ($i = 1, 2$; $j = 1, \ldots, s_i$) there exists a positive integer c_{ij} such that $\alpha_{3-i}^{\lambda_{ij}}$ is uniform

every cycle has length c_{ij}. Note that if $i = 1$, then $3 - i = 2$, and if $i = 2$, then $3 - i = 1$.

Example 3.4

Let $A_1 = \{1,2,3,4,5,6\}$ and $A_2 = \{1,2,3,4,5,6,7,8,9\}$; take $\alpha_1 = (123456)$ and $\alpha_2 = (123)(456789)$. We have $s_1 = 1$, $\lambda_{11} = 6$; $s_2 = 2$, $\lambda_{21} = 3$, and $\lambda_{22} = 6$. Observe that

$$\alpha_2^{\lambda_{11}} = \alpha_2^6 = (1)(2)(3)(4)(5)(6)(7)(8)(9) \text{ is 1-uniform.}$$

$$\alpha_1^{\lambda_{21}} = \alpha_1^3 = (14)(25)(36) \text{ is 2-uniform, and}$$

$$\alpha_1^{\lambda_{22}} = \alpha_1^6 = (1)(2)(3)(4)(5)(6) \text{ is 1-uniform.}$$

Taking $c_{11} = 1$, $c_{21} = 2$, and $c_{22} = 1$, we see that α_1 and α_2 are compatible permutations. If any of these had not become uniform, then the permutations would not have been compatible.

Example 3.5

Let $A_1 = A_2 = \{1, 2, 3, 4, 5, 6\}$; suppose that we take $\alpha_1 = (123)(456)$ and $\alpha_2 = (12)(3456)$. We now have $s_1 = 1$, $\lambda_{11} = 3$, and $s_2 = 2$, $\lambda_{21} = 2$, $\lambda_{22} = 4$. Since

$$\alpha_2^{\lambda_{11}} = \alpha_2^3 = (12)(3654)$$

is not uniform, α_1 and α_2 are not compatible.

Theorem 3.3. Let $\alpha \in \operatorname{Aut} K_{m,n}$ and suppose that $\alpha = \alpha_1\alpha_2$ for some permutations α_1, α_2 such that α_i permutes the vertices in V_i $(i = 1, 2)$. Then $|F(\alpha)| > 0$ if and only if α_1 and α_2 are compatible.

Proof: Suppose α_1 and α_2 are compatible and that α_i is a permutation of V_i $(i = 1, 2)$; take $\alpha = \alpha_1\alpha_2$. Arbitrarily take $u \in V_1$; then u appears in some cycle of the disjoint

cycle decomposition of α_1; let $\lambda(u)$ be the length of this cycle. Since α_1 and α_2 are compatible, we know that $\alpha_2^{\lambda(u)}$ is uniform. Also, we have $\alpha_2^{\lambda(u)} = \alpha^{\lambda(u)}|_{N(u)}$, and since u is arbitrary, we conclude that $\forall u \in V_1$ $\alpha^{\lambda(u)}|_{N(u)}$ is uniform. A similar argument establishes this $\forall v \in V_2$. Thus, $\forall v \in V$, $|F_v(\alpha^{\lambda(u)})| > 0$ (Theorem 2.5) and therefore $|F(\alpha)| > 0$ (Theorem 2.4).

Now suppose that $|F(\alpha)| > 0$, and that $\alpha = \alpha_1\alpha_2$, where α_i is a permutation on V_i, for i = 1, 2. By Theorem 2.4 we conclude that $\forall v \in V$, $|F_v(\alpha^{\lambda(v)})| > 0$; by Theorem 2.5 we conclude that $\forall v \in V$, $\alpha^{\lambda(v)}|_{N(v)}$ is uniform. Arbitrarily take $u \in V_1$; then u appears in some cycle of the disjoint cycle decomposition of α. Therefore, for each cycle length, λ, in the disjoint cycle decomposition of α_1, α_2^{λ} is uniform. Similarly, for each cycle length, λ, in the disjoint cycle decomposition of α_2, α_1^{λ} is uniform. Therefore, α_1 and α_2 are compatible. ∎

Example 3.6

Let $\alpha_1 = (0)(1)(2)$ and $\alpha_2 = (345)$, where $V_1 = \{0,1,2\}$ and $V_2 = \{3,4,5\}$, as in Example 3.3; set $\alpha = \alpha_1\alpha_2$. Notice that α_2^1 and α_1^3 are both uniform, so α_1 and α_2 are compatible. We conclude that $|F(\alpha)| > 0$, which is confirmed in Table 3.2.

Example 3.7

Let $\alpha_1 = (01)(2)$ and $\alpha_2 = (345)$, V_1 and V_2 as above; set $\alpha = \alpha_1\alpha_2$. Notice that α_1^3 is not uniform, so α_1 and α_2 are not compatible. Therefore, $|F(\alpha)| = 0$, as in Table 3.2.

Theorem 3.2 implies that we need to know something of the structure of compatible permutations to be able to calculate $f(m,n)$. By Theorem 2.5 we see that we need to express λ_{ij} in terms of the c_{ij}. Therefore, we turn our attention to the c_{ij}.

Theorem 3.4. Let α_1 and α_2 be compatible permutations. Suppose that α_i has s_i distinct cycle lengths and that the j-th smallest of these is λ_{ij} ($i = 1, 2$; $j = 1, \ldots, s_i$). Furthermore, suppose $\alpha_{3-i}^{\lambda_{ij}}$ is c_{ij}-uniform. Then $\forall i, k = 1, 2$, where $i \neq k$, $\forall j = 1, \ldots, s_i$, and $\forall \ell = 1, \ldots, s_k$, $c_{ij} \mid \lambda_{k\ell}$, and $\gcd(c_{ij}, c_{k\ell}) = 1$.

Proof: Obviously, $c_{ij} \mid \lambda_{k\ell}$, because $\alpha_k^{\lambda_{ij}}$ is c_{ij}-uniform and all of the cycles are disjoint.

Suppose that $\gcd(c_{ij}, c_{k\ell}) = h$. Since we were told that $\alpha_k^{\lambda_{ij}}$ is c_{ij}-uniform, Corollary 2.1a tells us that $c_{ij} = \lambda_{k\ell} / \gcd(\lambda_{ij}, \lambda_{k\ell})$. Thus, either $h = 1$, or h is a factor of $\lambda_{k\ell}$ one more time than it appears as a factor of λ_{ij}. Also, $\alpha_i^{\lambda_{k\ell}}$ is $c_{k\ell}$-uniform, and Corollary 2.1a says that $c_{k\ell} = \lambda_{ij} / \gcd(\lambda_{ij}, \lambda_{k\ell})$. Thus, either $h = 1$, or h is a factor of λ_{ij} one more time than it is a factor of $\lambda_{k\ell}$. We conclude that $h = 1$. ∎

Example 3.8

Consider α_1 and α_2 of Example 3.4; $\alpha_1 = (123456)$, and $\alpha_2 = (123)(456789)$. We see that $\lambda_{11} = 6$, $\lambda_{21} = 3$, and $\lambda_{22} = 6$; recall that $c_{11} = 1$, $c_{21} = 2$, and $c_{22} = 1$. Observe that $c_{11} \mid \lambda_{21}$, and $c_{11} \mid \lambda_{22}$ ($1 \mid 3$ and $1 \mid 6$), $c_{21} \mid \lambda_{11}$ ($2 \mid 6$), and $c_{22} \mid \lambda_{11}$ ($1 \mid 6$). Also, we know $\gcd(c_{11}, c_{21}) = \gcd(c_{11}, c_{22}) = 1$, as $c_{11} = 1$.

We are now ready to determine the structure of a compatible pair of permutations. Theorem 3.3 states that once we know this, we know the structure of exactly those automorphisms of $K_{m,n}$ that contribute to the value of $f(m,n)$. So once we know the structure we will be able to determine the formula for $f(m,n)$ by applying Theorems 2.4 and 2.5 together with two applications of Theorem 2.6.

Theorem 3.5. Let α_i be a permutation on A_i $(i = 1, 2)$, where α_1 and α_2 are compatible; suppose that $|A_i| = n_i$. Furthermore, suppose that α_i has s_i distinct cycle lengths, λ_{ij}, in its disjoint cycle decomposition, where λ_{ij} is the j-th smallest cycle length $(i = 1, 2;\ j = 1, \ldots, s_i)$. If $\alpha_{3-i}^{\lambda_{ij}}$ is c_{ij}-uniform, then for some positive integer g, where $g \mid \gcd(n_1,n_2)$, we have for i = 1, 2 and $j = 1, \ldots, s_i$:

$$\lambda_{ij} = \frac{g}{c_{ij}} \cdot \prod_{k=1}^{2} \operatorname{lcm}(c_{k\ell};\ \ell = 1, \ldots, s_k).$$

Proof: By Theorem 3.4, for all i, k = 1, 2, where $i \neq k$, we have, for all $j = 1, \ldots, s_i$ and $\ell = 1, \ldots, s_k$, $c_{k\ell} \mid \lambda_{ij}$. Therefore, for each choice of i and j, we conclude that there exists a positive integer, a_{ij}, such that

$$\lambda_{ij} = a_{ij} \cdot \operatorname{lcm}(c_{3-i\,\ell};\ \ell = 1, \ldots, s_{3-i})$$

in particular, for each $j = 1, \ldots, s_1$ and $k = 1, \ldots, s_2$, we have

$$\lambda_{1j} = a_{1j} \cdot \operatorname{lcm}(c_{2x};\ x = 1, \ldots, s_2), \text{ and}$$
$$\lambda_{2k} = a_{2k} \cdot \operatorname{lcm}(c_{1y};\ y = 1, \ldots, s_1).$$

By Corollary 2.1a,

$$c_{1j} = \lambda_{2k} / \gcd(\lambda_{1j},\lambda_{2k}), \text{ and}$$

$$c_{2k} = \lambda_{1j} \,/\, \gcd(\lambda_{1j},\lambda_{2k}).$$

Therefore,

$$c_{1j} \,/\, c_{2k} = \lambda_{2k} \,/\, \lambda_{1j}$$

$$= \frac{a_{2k}\cdot \mathrm{lcm}(c_{1y};\ y = 1,\ \ldots,\ s_1)}{a_{1j}\cdot \mathrm{lcm}(c_{2x};\ x = 1,\ \ldots,\ s_2)};\ \text{thus}$$

$$\frac{a_{2k}}{c_{1j}}\cdot \mathrm{lcm}(c_{1y};\ y = 1,\ \ldots,\ s_1) = \frac{a_{1j}}{c_{2k}}\cdot \mathrm{lcm}(c_{2x};\ x = 1,\ \ldots,\ s_2).$$

Also, by Theorem 3.4, we have, for all $y = 1, \ldots, s_1$ and $x = 1, \ldots, s_2$, $\gcd(c_{1y},c_{2x}) = 1$. Therefore, their least common multiples are also coprime. We conclude there exists a positive integer g such that

$$a_{1j} = g\cdot\frac{\mathrm{lcm}(c_{1y};\ y = 1,\ \ldots,\ s_1)}{c_{1j}},\ \text{and}$$

$$a_{2k} = g\cdot\frac{\mathrm{lcm}(c_{2x};\ x = 1,\ \ldots,\ s_2)}{c_{2k}}.$$

Repeating the argument for $\lambda_{1\ell}$ and λ_{2k}, and for λ_{1j} and λ_{2m}, we conclude that g is independent of the choice of j and k. Combining the formulas for λ_{ij} with that of a_{ij}, we conclude

$$\lambda_{ij} = \frac{g}{c_{ij}} \cdot \prod_{k=1}^{2} \mathrm{lcm}(c_{k\ell};\ \ell = 1,\ \ldots,\ s_k).$$

Suppose that for each $i = 1, 2$ and $j = 1, \ldots, s_i$, there are exactly μ_{ij} cycles of length λ_{ij}; then

$$\sum_{j=1}^{s_i} \mu_{ij}\cdot\lambda_{ij} = n_i\ (i = 1,\ 2),$$

so $g \mid n_1$ and $g \mid n_2$; thus $g \mid \gcd(n_1,n_2)$. ∎

Theorem 3.5 shows that if α_1 and α_2 are compatible, α_i has cycles of length λ_{ij} $(i = 1, 2, j = 1, \ldots, s_i)$, and α_i is a permutation of a set of order n_i, then if

$\alpha_3^{\lambda_{ij}}{}_{-i}$ is c_{ij}-uniform,

$$\lambda_{ij} = \frac{g}{c_{ij}} \cdot \prod_{k=1}^{2} \mathrm{lcm}(c_{k\ell};\ \ell = 1, \ldots, s_k),$$

where g is a common divisor of n_1 and n_2, $c_{1j}|n_2$, and $c_{2\ell}|n_1$. Let c_1 be any set of divisors of n_2 and c_2 be any set of divisors of n_1. Suppose that $c_1 = \{c_{1j};j = 1, \ldots, s_1\}$ and $c_2 = \{c_{2\ell};\ell = 1, \ldots, s_2\}$ and $\gcd(c_{1j},c_{2\ell}) = 1$ $(j = 1, \ldots, s_1,\ \ell = 1, \ldots, s_2)$. Let g be any divisor of $\gcd(n_1,n_2)$ and define λ_{ij} as above. If there exist positive integers $\mu_{ij} \geq 1$ $(i = 1, 2,\ j = 1, \ldots, s_i)$ such that

$$\sum_{j=1}^{s_i} \mu_{ij} \cdot \lambda_{ij} = n_i \quad (i = 1, 2),$$

then it is easy to verify the existence of compatible permutations on sets A_1 and A_2, $|A_i| = n_i$, where α_i is a permutation on A_i and α_i has exactly μ_{ij} cycles of length λ_{ij}. Therefore, we have a strategy for picking the compatible permutations in the sum for $f(m,n)$. We can now evaluate $|F(\alpha)|$ and $|J(\alpha)|$ for $\alpha = \alpha_1\alpha_2$.

Theorem 3.6. Let $c_1 = \{c_{1j};\ j = 1, \ldots, s_1\}$, $c_2 = \{c_{2\ell};\ \ell = 1, \ldots, s_2\}$, where $c_{1j}|n$ and $c_{2\ell}|m$. Suppose that $\gcd(c_{1j},c_{2\ell}) = 1$ and let g be a divisor of $\gcd(m,n)$. Define λ_{ij} as above, let $n_1 = m$ and $n_2 = n$, and suppose $\{\mu_{ij}\}$ is a solution of

$$\sum_{j=1}^{s_i} \mu_{ij} \cdot \lambda_{ij} = n_i \quad (i = 1, 2),$$

with $\mu_{ij} \geq 1$ $(i = 1, 2,\ j = 1, \ldots, s_i)$. Then for α_i a permutation on V_i having exactly μ_{ij} cycles of length

λ_{ij}, and $\alpha = \alpha_1\alpha_2$ we have

$$|F(\alpha)| = \prod_{i=1}^{2} \prod_{j=1}^{s_i} \Phi(c_{ij}, n_{3-i})^{\mu_{ij}},$$

where $\Phi(x,y) = \varphi(x)\cdot x^{(y/x - 1)}\cdot(y/x - 1)!$ and $\varphi(x)$ is the Euler function. Also,

$$|J(\alpha)| = \frac{m! \cdot n!}{\prod_{i=1}^{2} \prod_{j=1}^{s_i} \lambda_{ij}^{\mu_{ij}} \cdot \mu_{ij}!}.$$

Proof: By Theorem 2.4, $|F(\alpha)| = \prod_{v \in S} |F_v(\alpha^{\lambda(v)})|$, where S is a complete system of orbit representatives of $\langle\alpha\rangle$ acting on V. Each of the μ_{ij} cycles of length λ_{ij} needs a representative; we consider one of the λ_{ij}-cycles. For λ_{ij}, $\alpha^{\lambda_{ij}}|_{N(v)}$ is c_{ij}-uniform (where we assume v is a vertex in the λ_{ij}-cycle); thus, by Theorem 2.5,

$$|F_v(\alpha^{\lambda_{ij}})| = \Phi(c_{ij}, n_{3-i}).$$

The independence of the choices for the members of S gives us $|F(\alpha)|$ as above.

To compute $|J(\alpha)|$ we observe that α_1 and α_2 can be picked independently; also, the full symmetric group applies to the individual partite sets so that the structure class can be thought of as a direct product of conjugacy classes. There are μ_{ij} cycles of length λ_{ij} in α_i (for $i = 1, 2$, $j = 1, \ldots, s_i$). Therefore, by Theorem 2.6 there are

$$\frac{n_i!}{\prod_{j=1}^{s_i} \lambda_{ij}^{\mu_{ij}} \cdot \mu_{ij}!}$$

members in $|J(\alpha_i)|$ $(i = 1, 2)$, and

$$|J(\alpha)| = |J(\alpha_1)| \cdot |J(\alpha_2)|.$$

We conclude that $|J(\alpha)|$ has the above form. ∎

Example 3.9

Consider the graph $K_{2,4}$ given in Example 3.2. Suppose $c_{11} = 2$ and $c_{21} = 1$, so that $c_1 = \{2\}$ and $c_2 = \{1\}$. Take $g = 1$ (for simplicity). Then $\lambda_{11} = 1\cdot(1\cdot 2\ /\ 2) = 1$, and $\lambda_{21} = 1\cdot(1\cdot 2\ /\ 1) = 2$. Since $\mu_{11}\cdot\lambda_{11} = 2$, we get $\mu_{11} = 2 \geq 1$. Also, $\mu_{21}\cdot\lambda_{21} = 4$, so $\mu_{21} = 2 \geq 1$. Hence, by the remarks preceding Theorem 3.6 we conclude the existence of an α_1 on $\{0,1\}$ having two 1-cycles, and an α_2 on $\{2,3,4,5\}$ having two 2-cycles. This is confirmed by $\alpha = (0)(1)(23)(45)$ in Table 3.1.

Furthermore, we calculate $|F(\alpha)|$ and $|J(\alpha)|$ for the above α.

$$|F(\alpha)| = \Phi(2,4)^2\cdot\Phi(1,2)^2 = (1\cdot 2^1\cdot 1!)^2\cdot(1\cdot 1^1\cdot 1!)^2$$
$$= 2^2\cdot 1^2 = 4,$$

as in the entry for α given above in Table 3.1. Also,

$$|J(\alpha)| = (2!/(1^2\cdot 2!))\cdot(4!/(2^2\cdot 2!))$$
$$= (2/(1\cdot 2))\cdot(24/(4\cdot 2)) = 1\cdot 3 = 3;$$

this is also confirmed by Table 3.1.

By the remarks preceding Theorem 3.6 we can change the sum of Theorem 3.2 into a sum over g, a sum over $c = \{c_1,c_2\}$, and a sum over μ, where g is a divisor of gcd(m,n), c_1 is a set of divisors of n, c_2 is a set of divisors of m, and μ is the set of all solutions to the set of equations

$$\sum_{j=1}^{s_i} \mu_{ij}\lambda_{ij} = n_i \quad (i = 1,\ 2);\ \mu_{ij} \geq 1,$$

where $n_1 = m$ and $n_2 = n$.

Theorem 3.7. Suppose that $g \mid \gcd(m,n)$, $c = \{c_1, c_2\}$, $c_1 = \{c_{1j}; j = 1, \ldots, s_1\}$, and $c_2 = \{c_{2j}; j = 1, \ldots, s_2\}$. Further suppose that $\gcd(c_{1j}, c_{2\ell}) = 1$, for $j = 1, \ldots, s_1$ and $\ell = 1, \ldots, s_2$; let $n_1 = m$, $n_2 = n$, and define λ_{ij} as

$$\lambda_{ij} = \frac{g}{c_{ij}} \cdot \prod_{k=1}^{2} \operatorname{lcm}(c_{k\ell}; l = 1, \ldots, s_k).$$

If $\{\mu_{ij}\}$ is a solution of

$$\sum_{j=1}^{s_i} \mu_{ij}\lambda_{ij} = n_i;\ \mu_{ij} \geq 1\ (i = 1, 2),\ \text{then}$$

$$f(m,n) = \frac{1}{(1+\delta_{m,n})} \sum_{g} \sum_{c} \sum_{\mu} \prod_{i=1}^{2} \prod_{j=1}^{s_i} \frac{\Phi(c_{ij}, n_{3-i})^{\mu_{ij}}}{\lambda_{ij}^{\mu_{ij}} \cdot \mu_{ij}!}$$

where $\Phi(x,y) = \varphi(x) \cdot x^{(y/x-1)} \cdot (y/x - 1)!$, $\delta_{i,j}$ is the Kronecker function, and $\varphi(x)$ is the Euler function.

Proof: By Theorem 3.2, we see that the $m! \cdot n!$ cancel with those in the formula for $|J(\alpha)|$; thus, taking into consideration the orders of the automorphism groups, we see that the coefficient is correct. The sum over g allows all possible cycle lengths to be considered. The sum over c ensures that no compatible pair of permutation will be overlooked, and the sum over μ actually picks the compatible permutations. Finally, the formulas for $|J(\alpha)|$ and $|F(\alpha)|$ given in Theorem 3.6, when substituted into the triple sum and simplified, give the result. ∎

Example 3.10

Consider the graph $K_{3,5}$. By Theorem 3.2 we know that $h(3,5) = 0$; thus, $|C(K_{3,5})| = f(3,5)$. Also, we know $\gcd(3,5) = 1$, so $g = 1$ (and we can forget about it). The divisors of $n = 5$ are 1 and 5, so $c_1 = \{1\}, \{5\}$, or

$\{1,5\}$. The divisors of $m = 3$ are 1 and 3, so $c_2 = \{1\}$, $\{3\}$, or $\{1,3\}$. Thus, there are $3 \cdot 3 = 9$ possibilities for $c = \{c_1, c_2\}$. We consider each of these cases separately.

(1) Suppose that $c_1 = c_2 = \{1\}$. Then $c_{11} = c_{21} = 1$, and $\lambda_{11} = \lambda_{21} = 1 \cdot (1 \cdot 1/1) = 1$. Solving $\mu_{11} \cdot \lambda_{11} = 3$, for μ_{11}, gives $\mu_{11} = 3$; solving $\mu_{21} \cdot \lambda_{21} = 5$, for μ_{21}, gives $\mu_{21} = 5$. Placing these values into the formula for $f(3,5)$, we get

$$\frac{\Phi(1,5)^3}{1^3 \cdot 3!} \cdot \frac{\Phi(1,3)^5}{1^5 \cdot 5!} = \frac{(1 \cdot 1^4 \cdot 4!)^3}{6} \cdot \frac{(1 \cdot 1^2 \cdot 2!)^5}{120} = \frac{24^3}{6} \cdot \frac{2^5}{120}$$

and the value of this term is 614.4.

(2) Suppose that $c_1 = \{1\}$ and $c_2 = \{3\}$. Then $c_{11} = 1$, $c_{21} = 3$, $\lambda_{11} = 1 \cdot (1 \cdot 3/1) = 3$, and $\lambda_{21} = 1 \cdot (1 \cdot 3/3) = 1$. Solving $\mu_{11}\lambda_{11} = 3$, for μ_{11}, gives $\mu_{11} = 1$; solving $\mu_{21}\lambda_{21} = 5$, for μ_{21}, gives $\mu_{21} = 5$. Placing these values into the formula for $f(3,5)$, we get

$$\frac{\Phi(1,5)^1}{3^1 \cdot 1!} \cdot \frac{\Phi(3,3)^5}{1^5 \cdot 5!} = \frac{(1 \cdot 1^4 \cdot 4!)^1}{3} \cdot \frac{(2 \cdot 3^0 \cdot 0!)^5}{120} = \frac{24}{3} \cdot \frac{2^5}{120}$$

and the value for this term is 32/15.

(3) Suppose that $c_1 = \{1\}$ and $c_2 = \{1,3\}$. Then $c_{11} = 1$, $c_{21} = 3$, $c_{22} = 1$, $\lambda_{11} = 1 \cdot (1 \cdot 3/1) = 3$, $\lambda_{21} = 1 \cdot (1 \cdot 3/3) = 1$, and $\lambda_{22} = 1 \cdot (1 \cdot 3/1) = 3$. Solving $\mu_{11}\lambda_{11} = 3$, for μ_{11}, gives $\mu_{11} = 1$; solving $\mu_{21}\lambda_{21} + \mu_{22}\lambda_{22} = 5$, for μ_{21} and μ_{22} (and recalling that $\mu_{21}, \mu_{22} \geq 1$), gives $\mu_{21} = 2$ and $\mu_{22} = 1$. Placing these values into the formula for $f(3,5)$, we get

$$\frac{\Phi(1,5)^1}{3^1 \cdot 1!} \cdot \frac{\Phi(3,3)^2}{1^2 \cdot 2!} \cdot \frac{\Phi(1,3)^1}{3^1 \cdot 1!}$$

$$= \frac{(1 \cdot 1^4 \cdot 4!)^1}{3} \cdot \frac{(2 \cdot 3^0 \cdot 0!)^2}{2} \cdot \frac{(1 \cdot 1^2 \cdot 2!)^1}{3} = \frac{24}{3} \cdot \frac{2^2}{2} \cdot \frac{2}{3}$$

and the value for this term is 32/3.

(4) Suppose that $c_1 = \{5\}$ and $c_2 = \{1\}$. Then $c_{11} = 5$, $c_{21} = 1$, $\lambda_{11} = 1 \cdot (5 \cdot 1/5) = 1$, and $\lambda_{21} = 1 \cdot (5 \cdot 1/1) = 5$.

Solving $\mu_{11}\lambda_{11}= 3$, for μ_{11}, gives $\mu_{11}= 3$; solving $\mu_{21}\lambda_{21}= 5$, for μ_{21}, gives $\mu_{21}= 1$. Placing these values into the formula for $f(3,5)$, we get

$$\frac{\Phi(5,5)^3}{1^3\cdot 3!}\cdot\frac{\Phi(1,3)^1}{5^1\cdot 1!} = \left(\frac{4\cdot 5^0\cdot 0!}{6}\right)^3\cdot\left(\frac{1\cdot 1^2\cdot 2!}{5}\right)^1 = \frac{4^3}{6}\cdot\frac{2}{5}$$

and the value for this term is 64/15.

(5) Suppose that $c_1 = \{5\}$ and $c_2 = \{3\}$. Then $c_{11}= 5$, $c_{21}= 3$, $\lambda_{11}= 1\cdot(5\cdot 3/5) = 3$, and $\lambda_{21}= 1\cdot(5\cdot 3/3) = 5$. Solving $\mu_{11}\lambda_{11}= 3$, for μ_{11}, gives $\mu_{11}= 1$; solving $\mu_{21}\lambda_{21}= 5$, for μ_{21}, gives $\mu_{21}= 1$. Placing these values into the formula for $f(3,5)$, we get

$$\frac{\Phi(5,5)^1}{3^1\cdot 1!}\cdot\frac{\Phi(3,3)^1}{5^1\cdot 1!} = \left(\frac{4\cdot 5^0\cdot 0!}{3}\right)^1\cdot\left(\frac{2\cdot 3^0\cdot 0!}{5}\right)^1 = \frac{4}{3}\cdot\frac{2}{5}$$

and the value for this term is 8/15.

(6) Suppose that $c_1 = \{5\}$ and $c_2 = \{1,3\}$. Then $c_{11}= 5$, $c_{21}= 3$, $c_{22}= 1$, $\lambda_{11}= 1\cdot(5\cdot 3/5) = 3$, $\lambda_{21}= 1\cdot(5\cdot 3/3) = 5$, and $\lambda_{22}= 1\cdot(5\cdot 3/1) = 15$—but this is larger than both partite sets put together. Clearly, this case cannot occur.

(7) Suppose that $c_1 = \{1,5\}$ and $c_2 = \{1\}$. Then $c_{11}= 5$, $c_{12}= 1$, $c_{21}= 1$, $\lambda_{11}= 1\cdot(5\cdot 1/5) = 1$, $\lambda_{12}= 1\cdot(5\cdot 1/1) = 5$—but this is larger than the partite set it is in. Clearly, this case cannot occur.

(8) Suppose that $c_1 = \{1,5\}$ and $c_2 = \{3\}$. Then $c_{11}= 5$, $c_{12}= 1$, $c_{21}= 3$, $\lambda_{11}= 1\cdot(5\cdot 3/5) = 3$, $\lambda_{12}= 1\cdot(5\cdot 3/1)$—but this is larger than both partite sets put together. Clearly, this case cannot occur.

(9) Suppose that $c_1 = \{1,5\}$ and $c_2 = \{1,3\}$. Then $c_{11}= 5$, $c_{12}= 1$, $c_{21}= 3$, $c_{22}= 1$, $\lambda_{11}= 1\cdot(5\cdot 3/5) = 3$, $\lambda_{12}= 1\cdot(5\cdot 3/1) = 15$—but this is larger than both partite sets put together. Clearly, this case cannot occur.

Combining the values from the five cases that did produce values, we get

$|C(K_{3,5})| = 614.4 + 32/15 + 326/3 + 646/15 + 8/15 = 632.$

That is, there are 632 congruence classes of maps of $K_{3,5}$. We have broken down these 632 congruence classes by face distribution and surface. A summary of this breakdown appears in Tables 3.4 - 3.7.

Table 3.4

The Congruence Classes of $K_{3,5}$ on S_1

6, 6x4	1	Total on S_1:	1

Table 3.5

The Congruence Classes of $K_{3,5}$ on S_2

14, 4x4	11	10, 8, 3x4	18	2x8, 6, 2x4	5
12, 6, 3x4	10	10, 2x6, 2x4	9	5x6	2
		Total on S_2:	55		

Table 3.6

The Congruence Classes of $K_{3,5}$ on S_3

22, 2x4	99	16, 10, 4	82	14, 2x8	11
20, 6, 3x4	46	16, 8, 6	4	2x12, 6	12
18, 8, 4	54	14, 12, 4	42	12, 10, 8	12
18, 2x6	17	14, 10, 6	18	3x10	13
		Total on S_3:	410		

Table 3.7

The Congruence Classes of $K_{3,5}$ on S_4

30	166	Total on S_4:	166

We have classified these 632 congruence classes into

open, strong, and closed 2-cell imbeddings. The results of this classification are given in Table 3.8. We see that closed 2-cell imbeddings occur only on S_1 and S_2;

Table 3.8
Classifying the Congruence Classes of Maps of $K_{3,5}$

	S_1	S_2	S_3	S_4	Total by type
Open (but not Strong)	0	45	407	166	618
Strong (but not Closed)	0	8	3	0	11
Closed	1	2	0	0	3
Total by Surface	1	55	410	166	632

strong 2-cell imbeddings occur only on S_1, S_2, and S_3.

In the next section we will evaluate $h(m,n)$. Theorem 3.2 gives us a partial answer; we will explore the case when $m = n$. We will find that the evaluation of $|F(\gamma)|$ is straightforward—it is the evaluation of $|J(\gamma)|$ that will require the most effort.

3.3 The Evaluation of h(n,n)

In this section we will evaluate $h(n,n)$; we already know that $h(m,n) = 0$ when $m < n$. By Theorem 3.2 we see that we need to find $|F(\gamma)|$ and $|J(\gamma)|$ for $\gamma \in D(K_{n,n})$; we already know that $D(K_{m,n}) = \emptyset$ when $m < n$. We will begin by finding $|F(\gamma)|$; this is a straightforward application of Theorems 2.3 - 2.5. In the process, we will determine the structure type of $\gamma \in D(K_{n,n})$ so that $|F(\gamma)| > 0$. This will allow us to compute the number of automorphisms in the structure class for γ.

We first observe that if some vertex $u \in V_1$ is mapped onto some vertex $v \in V_2$ then the partite sets are

swapped; this was established in the remarks preceding Theorem 3.2. We conclude that, in the disjoint cycle decomposition of $\gamma \in D(K_{n,n})$, vertices of V_1 and vertices of V_2 alternate in each cycle. This implies:

Theorem 3.8. Let $\gamma \in \text{Aut } K_{n,n}$ map a vertex $u \in V_1$ onto a vertex $v \in V_2$ (that is, $\gamma \in D(K_{n,n})$); then $|F(\gamma)| > 0$ if and only if γ is uniform..

Proof: Since vertices in V_1 alternate with vertices in V_2 in each cycle of the disjoint cycle decomposition of γ, every cycle has length $\lambda \geq 2$. In fact, every cycle must have even length. Let $\lambda(u)$ be the length of the cycle of γ containing u; then $\gamma^{\lambda(u)}$ fixes the adjacent vertices u and v. By Theorem 2.3, we conclude that either $\gamma^{\lambda(u)}$is the identity element of Aut $K_{n,n}$ or $F(\gamma) = \emptyset$. Hence, $|F(\gamma)| > 0$ if and only if $\gamma^{\lambda(u)}$ is the identity element of Aut $K_{n,n}$. The argument extends to all cycles because of the alternation of the partite sets in the cycles. Therefore, the result follows. ∎

Example 3.11

Consider the graph $K_{3,3}$ of Example 3.3. Looking at Table 3.3 we observe that Theorem 3.8 is confirmed.

We have observed that every cycle must be of even length and that γ must be uniform for $|F(\gamma)|$ to be greater than zero. We note that if d is a divisor of n then $2 \cdot d$ is an even divisor of $2 \cdot n$ and that every even divisor of $2 \cdot n$ gives a divisor of n. Thus, we have:

Theorem 3.9. Let $\gamma \in D(K_{n,n})$ and suppose that γ is

2d-uniform, where $d \mid n$; then

$$|F(\gamma)| = (n - 1)!^{n/d}.$$

Proof: By Theorem 2.4, $|F(\gamma)|$ is the product of the $|F_v(\gamma^{\lambda(v)})|$ (one for each cycle of γ). Since γ is 2d-uniform, γ^{2d} is the identity element of Aut $K_{n,n}$. Therefore, all $(d(v) - 1)!$ of the rotations at v ($\forall v \in V$) must be fixed by γ^{2d}. We conclude that $|F_v(\gamma^{\lambda(v)})| = (n - 1)!$. There are 2n / 2d = n / d cycles in γ; thus, we have the result. ∎

We see by Theorems 3.8 and 3.9, that each $J(\gamma)$ is determined by a different d, where $d \mid n$. Because the automorphism group of $K_{n,n}$ is a wreath product, we know that $\gamma \in J(\gamma)$ corresponds to the wreath product element in the following way: the first portion is (12)—the partite sets are swapped; the second portion is a list of two permutations of S_n—when i maps onto j then the vertex corresponding to i maps onto the vertex corresponding to j of the other partite set (see the remarks preceding Theorem 3.2). The key is this: the result of these two permutations is a 2d-uniform permutation on the 2n elements of V.

Theorem 3.10. Let d be a positive integer, where $d \mid n$. and suppose that $\gamma \in D(K_{n,n})$ is 2d-uniform, then

$$|J(\gamma)| = \frac{(n!)^2}{d^{(n/d)} \cdot (n/d)!}.$$

Proof: Since vertices from V_1 and V_2 alternate in γ we can create a permutation in $J(\gamma)$ as follows: Create a one-to-one function from V_1 onto V_2; then for each $u \in V_1$, take the u, $\gamma(u)$ pair as a new object. Finally,

create a d-uniform permutation out of these n new objects. Notice that the full symmetric group acts on these new objects so that Theorem 2.6 applies in this case.

There are $n!$ one-to-one functions from V_1 onto V_2. Also, from Theorem 2.6, there are

$$\frac{n!}{d^{n/d}\cdot(n/d)!}$$

d-uniform permutations of an n-element set. Therefore, the result follows. ∎

Example 3.12

Consider the graph $K_{3,3}$ of Example 3.3 and 3.11; take $\gamma = (03)(14)(25) \in D(K_{n,n})$; then $d = 1$. By Theorem 3.10, there are

$$\frac{3!^2}{1^3\cdot 3!} = \frac{36}{6} = 6$$

automorphisms in the structure class of γ. By Theorem 3.9, each of these fixes

$$(3 - 1)!^{3/1} = 2^3 = 8$$

rotation schemes. Both of these values are confirmed by Table 3.3.

Theorem 3.11. The value of $h(n,n)$ is given by

$$h(n,n) = \sum_{d \mid n} \frac{(n-1)!^{n/d}}{2\cdot d^{n/d}\cdot(n/d)!}.$$

Proof: By Theorem 3.2,

$$h(n,n) = \frac{1}{2\cdot n!^2} \sum_{J(\gamma) \in J(G)} |F(\gamma)|\cdot|J(\gamma)|.$$

By the remarks preceding Theorem 3.10, a sum over $J(\gamma)$ is equivalent to a sum over d, where d is a divisor of n. Substituting the formulas for $|F(\gamma)|$ (Theorem 3.9) and $|J(\gamma)|$ (Theorem 3.10) into the above sum, we get the result. ∎

Combining Theorems 3.7 and 3.11, we get a formula for the number of congruence classes of maps of the complete bipartite graph, $K_{m,n}$. This formula is given in Theorem 3.12.

Theorem 3.12. The number of congruence classes of maps of the complete bipartite graph $K_{m,n}$ is given by

$$|C(K_{m,n})| = f(m,n) + h(m,n),$$

where f(m,n) is given by

$$f(m,n) = \frac{1}{(1+\delta_{m,n})} \sum_{g} \sum_{c} \sum_{\mu} \prod_{i=1}^{2} \prod_{j=1}^{s_i} \frac{\Phi(c_{ij}, n_{3-i})^{\mu_{ij}}}{\lambda_{ij}^{\mu_{ij}} \cdot \mu_{ij}!},$$

and

$$h(m,n) = \begin{cases} 0 & , \ m < n \\ \displaystyle\sum_{d \mid n} \frac{(n-1)!^{n/d}}{2 \cdot d^{n/d} \cdot (n/d)!}, & m = n; \end{cases}$$

$\Phi(x,y) = \varphi(x) \cdot x^{(y/x-1)} \cdot (y/x-1)!$, δ is the Kronecker function, and φ is the Euler function; also, $g \mid \gcd(m,n)$, $n_1 = m$, $n_2 = n$, $c = \{c_1, c_2\}$, where $c_i = \{c_{ij} \mid j = 1, \ldots, s_i\}$, $c_{ij} \mid (n_{3-i} / g)$, and $\gcd(c_{ij}, c_{3-i,k}) = 1$, for $j = 1, \ldots, s_1$ and $k = 1, \ldots, s_2$. Furthermore,

$$\lambda_{ij} = \frac{g}{c_{ij}} \prod_{k=1}^{2} \operatorname{lcm}(c_{k\ell};\ \ell = 1, \ldots, s_k), \text{ and}$$

$\{\mu_{ij}\}$ are all sets of solutions ($\mu_{ij} \geq 1$) that satisfy:

$$\sum_{j=1}^{s_i} \mu_{ij}\lambda_{ij} = n_i \quad (i = 1, 2). \qquad ∎$$

Example 3.13

We will again count the congruence classes of $K_{3,3}$; this time we will use Theorem 3.12 and not ad hoc methods. We start by calculating $f(3,3)$.

The $\gcd(3,3) = 3$ so $g = 1$ or $g = 3$. We will consider these cases separately. When $g = 1$, the divisors of $3/1 = 3$ are 1 and 3. Therefore, c_1 and c_2 can be $\{1\}$, $\{3\}$, or $\{1,3\}$. Not both c_1 and c_2 can have 3 in it; thus, instead of nine cases, there are only five.

(1) Suppose that $c_1 = c_2 = \{1\}$. Then $c_{11} = c_{21} = 1$, and $\lambda_{11} = \lambda_{21} = 1\cdot(1\cdot1/1) = 1$. Solving $\mu_{11}\lambda_{11} = 3$, for μ_{11}, we get $\mu_{11} = 3$; similarly, $\mu_{21} = 3$. Placing these values into the formula for $f(3,3)$, we get

$$\frac{\Phi(1,3)^3}{1^3\cdot3!}\cdot\frac{\Phi(1,3)^3}{1^3\cdot3!} = \frac{(1\cdot1^2\cdot2!)^3}{6}\cdot\frac{(1\cdot1^2\cdot2!)^3}{6} = \frac{(2^3)^2}{36} = \frac{16}{9}.$$

(2) Suppose that $c_1 = \{1\}$ and $c_2 = \{3\}$. Then $c_{11} = 1$, $c_{21} = 3$, $\lambda_{11} = 1\cdot(1\cdot3/1) = 3$, and $\lambda_{21} = 1\cdot(1\cdot3/3) = 1$. Solving $\mu_{11}\lambda_{11} = 3$, for μ_{11}, gives $\mu_{11} = 1$; solving $\mu_{21}\lambda_{21} = 3$, for μ_{21}, gives $\mu_{21} = 3$. Placing these values into the formula for $f(3,3)$, we get

$$\frac{\Phi(1,3)^1}{3^1\cdot1!}\cdot\frac{\Phi(3,3)^3}{1^3\cdot3!} = \frac{(1\cdot1^2\cdot2!)^1}{3}\cdot\frac{(2\cdot3^0\cdot0!)^3}{6} = \frac{2}{3}\cdot\frac{2^3}{6} = \frac{8}{9}.$$

(3) If $c_1 = \{3\}$ and $c_2 = \{1\}$, then we also get $8/9$; this due to the symmetry of the formula for $f(3,3)$ which reflects that of $K_{3,3}$.

(4) Suppose that $c_1 = \{1\}$ and $c_2 = \{1,3\}$. Then $c_{11} = 1$, $c_{21} = 3$, $c_{22} = 1$, $\lambda_{11} = 1\cdot(1\cdot3/1) = 3$, $\lambda_{21} = 1\cdot(1\cdot3/3) = 1$, and $\lambda_{22} = 1\cdot(1\cdot3/1) = 3$. Now $\mu_{21}, \mu_{22} \geq 1$; thus, we get $\mu_{21}\lambda_{21} + \mu_{22}\lambda_{22} \geq 4$—but this is larger than the partite set it is in. Clearly, this case cannot exist.

(5) Because of the symmetry of $K_{3,3}$, we see that

$c_1 = \{1,3\}$ and $c_2 = \{1\}$ cannot occur either.

Adding up the the three cases that did produce values, we get: the contribution to $f(3,3)$, when $g = 1$, is

$$(1/2)\cdot(16/9 + 8/9 + 8/9) = (1/2)\cdot(32/9) = 16/9.$$

When $g = 3$, the divisor of $3/3 = 1$ is just 1. Thus, $c_1 = c_2 = \{1\}$. Then $c_{11} = c_{21} = 1$, and $\lambda_{11} = \lambda_{21} = 3\cdot(1\cdot1/1) = 3$. Solving $\mu_{11}\lambda_{11} = 3$, for μ_{11}, gives $\mu_{11} = 1$; similarly, $\mu_{21} = 1$. Placing these values into the formula for $f(3,3)$, we get

$$\frac{\Phi(1,3)^1}{3^1\cdot1!}\cdot\frac{\Phi(1,3)^1}{3^1\cdot1!} = \frac{(1\cdot1^2\cdot2!)^1}{3}\cdot\frac{(1\cdot1^2\cdot2!)^1}{3} = \frac{2}{3}\cdot\frac{2}{3} = \frac{4}{9}.$$

Therefore, the contribution to $f(3,3)$, when $g = 3$, is

$$(1/2)\cdot(4/9) = 2/9.$$

Combining the values for both of the cases gives

$$f(3,3) = 16/9 + 2/9 = 2.$$

We now evaluate $h(3,3)$. The divisors of 3 are 1 and 3; we consider these cases separately. When $d = 1$, we get

$$\frac{(3-1)!^{3/1}}{2\cdot1^{3/1}\cdot(3/1)!} = \frac{2^3}{12} = \frac{2}{3}.$$

When $d = 3$, we get

$$\frac{(3-1)!^{3/3}}{2\cdot3^{3/3}\cdot(3/3)!} = \frac{2}{6} = \frac{1}{3}.$$

Combining these two terms we see that

$$h(3,3) = 2/3 + 1/3 = 1.$$

We conclude that

$$|C(K_{3,3})| = 2 + 1 = 3;$$

this agrees with the evaluations we obtained in Example 3.3.

In the next chapter we will extend the method further. This will enable us to calculate the congruence classes of the maps of complete tripartite graphs. In Chapter V, we will further extend the method so that the congruence classes of maps of complete n-partite graphs can be counted. We will then show that the formula for complete n-partite graphs generalizes those formulas in Chapters II, III, and IV.

CHAPTER FOUR

ANOTHER LOOK AT COMPATIBILITY: COMPLETE TRIPARTITE GRAPHS

4.1 Introduction

In this chapter we will look more deeply at the concept of compatibility than we did in Chapter III. Looking at the definition, we see there is more than one way to extend this concept to three or more permutations. We need to ask ourselves what it is that we wish to accomplish from our definition. In the bipartite case, our definition allowed us to determine the cycle type of those automorphisms that fix the partite sets as sets. This we discovered from Theorem 3.3 which stated that if an automorphism α fixes the partite sets as sets, then $|F(\alpha)| > 0$ if and only if the permutations on the individual partite sets are compatible. This then determines how we should generalize the concept: we want to be able to prove the analog of Theorem 3.3 for complete tripartite graphs from this definition.

Let A_1, A_2, and A_3 be sets (not necessarily distinct) and α_i a permutation on A_i ($i = 1, 2, 3$). Furthermore, suppose that α_i has s_i distinct cycle lengths λ_{ij} in its disjoint cycle decomposition, where λ_{ij} satisfy: $\lambda_{i1} < \lambda_{i2} < \ldots < \lambda_{is(i)}$. Then $\{\alpha_1, \alpha_2, \alpha_3\}$ is a compatible set (multiset) of permutations if and only if for

each λ_{ij}, there exists a positive integer c_{ij} such that for $k \neq i$, $\alpha_k^{\lambda_{ij}}$ is c_{ij}-uniform.

Example 4.1

Let $A_1 = \{1,2,3,4,5,6,7,8,9\}$, $A_2 = \{1,2,3,4\}$, and $A_3 = \{1, 2, 3, 4, 5, 6\}$. Take $\alpha_1 = (123456)(789)$, $\alpha_2 = (12)(34)$, and $\alpha_3 = (123456)$. Then $\lambda_{11} = 3$, $\lambda_{12} = 6$, $\lambda_{21} = 2$, and $\lambda_{31} = 6$ (so $s_1 = 2$, and $s_2 = s_3 = 1$).

$$\alpha_2^3 = (12)(34) \text{ and } \alpha_3^3 = (14)(25)(36);$$

both are 2-uniform.

$$\alpha_2^6 = (1)(2)(3)(4) \text{ and } \alpha_3^6 = (1)(2)(3)(4)(5)(6);$$

both are 1-uniform.

$$\alpha_1^2 = (135)(246)(798) \text{ and } \alpha_3^2 = (135)(246);$$

both are 3-uniform. Finally,

$$\alpha_1^6 = (1)(2)(3)(4)(5)(6)(7)(8)(9) \text{ and } \alpha_2^6 = (1)(2)(3)(4);$$

both are 1-uniform. Take $c_{11} = 2$, $c_{12} = 1$, $c_{21} = 3$, and $c_{31} = 1$. Then $\{\alpha_1, \alpha_2, \alpha_3\}$ is a compatible set of permutations.

Example 4.2

Consider the sets and permutations of Example 4.1, but suppose that α_2 is replaced by $\alpha_2 = (1234)$. Then all λ's and s's are the same, except $\lambda_{21} = 4$. Now

$$\alpha_2^3 = (1432)$$

is 4-uniform, while α_3^3 is 2-uniform.

$$\alpha_2^6 = (13)(24)$$

is 2-uniform, while α_3^6 is 1-uniform. Also,

$$\alpha_1^4 = (153)(264)(789) \text{ and } \alpha_3^4 = (153)(264)$$

are both 3-uniform. Finally, α_1^6 is 1-uniform, while α_2^6 is 2-uniform. Now although $\{\alpha_1, \alpha_2, \alpha_3\}$ is not a compa-

tible set of permutations, taken pairwise the permutations are compatible—this shows another way that compatibility could have been extended (<u>pairwise compatibility</u>). This concept, however, will not be pursued in this dissertation; it may prove useful in the future though.

Having established this definition, we turn our attention to complete tripartite graphs. Recall that a complete tripartite graph has its vertex set partitioned into three partite sets V_1, V_2, and V_3, and that the edge set consists of all possible unordered pairs of vertices from different partite sets. That is,

$$V = V_1 \cup V_2 \cup V_3;\ V_i \neq \emptyset\ (i = 1, 2, 3)$$
$$V_1 \cap V_2 = V_2 \cap V_3 = V_3 \cap V_1 = \emptyset, \text{ and}$$
$$E = \{\{u,v\} \mid u \in V_i,\ v \in V_j,\ (1 \leq i < j \leq 3)\}.$$

If $|V_1| = m$, $|V_2| = n$, $|V_3| = r$, then we write $G = K_{m,n,r}$. If $m \leq n \leq r$, then we say that the complete tripartite graph is in <u>standard form</u>.

Let $A(K_{m,n,r}) = \{\alpha \in \text{Aut } K_{m,n,r} \mid \exists \alpha_i \in \text{Sym } V_i\ (i = 1, 2, 3),\ \alpha = \alpha_1\alpha_2\alpha_3\}$, where Sym V_i is the full symmetric group on V_i. That is, $A(K_{m,n,r})$ is the set of automorphisms that fix the partite sets as sets. Define $B(K_{m,n,r})$ as: $B(K_{m,n,r}) = \{\beta \in \text{Aut } K_{m,n,r} \mid \exists u,v$ (not in the same partite set), $\beta(u) = v$, but one partite set is fixed$\}$. Let $D(K_{m,n,r}) = \{\gamma \in \text{Aut } K_{m,n,r} \mid \exists u, v, w$ (all in distinct partite sets), $\gamma(u) = v$, $\gamma(v) = w\}$. That is, while neither $B(K_{m,n,r})$ nor $D(K_{m,n,r})$ fix the partite sets as sets, $B(K_{m,n,r})$ does fix one of the partite sets.

Theorem 4.1 The number of congruence classes of maps of $K_{m,n,r}$ is given by

$$|C(K_{m,n,r})| = \frac{1}{|\mathrm{Aut}\ K_{m,n,r}|} \sum_{\alpha \in A(K_{m,n,r})} |F(\alpha)|$$
$$+ \frac{1}{|\mathrm{Aut}\ K_{m,n,r}|} \sum_{\beta \in B(K_{m,n,r})} |F(\beta)|.$$
$$+ \frac{1}{|\mathrm{Aut}\ K_{m,n,r}|} \sum_{\gamma \in D(K_{m,n,r})} |F(\gamma)|.$$

Proof: Every automorphism of $K_{m,n,r}$ must fix all, one or none of the partite sets. These are mutually exclusive conditions, so Theorem 1.3 can be rewritten as above. ∎

As in the bipartite case, we will simplify the above sums. For simplicity we will write

$$f(m,n,r) = \frac{1}{|\mathrm{Aut}\ K_{m,n,r}|} \sum_{\alpha \in A(K_{m,n,r})} |F(\alpha)|,$$

$$g(m,n,r) = \frac{1}{|\mathrm{Aut}\ K_{m,n,r}|} \sum_{\beta \in B(K_{m,n,r})} |F(\beta)|,$$

and $$h(m,n,r) = \frac{1}{|\mathrm{Aut}\ K_{m,n,r}|} \sum_{\gamma \in D(K_{m,n,r})} |F(\gamma)|.$$

Example 4.3

Consider the graph $K_{2,2,2}$, where $V = \{0,1,2,3,4,5\}$, $V_1 = \{0, 1\}$, $V_2 = \{2, 3\}$, and $V_3 = \{4, 5\}$. There are $|R(G)| = 3!^6 = 46,656$ rotation schemes; these are clas- in Tables 4.1 - 4.3. We note that $\mathrm{Aut}\ K_{2,2,2} \cong S_3[S_2]$, of order 48.

From Table 4.1 we calculate $f(2,2,2) = (1\cdot46656 + 3\cdot0 + 3\cdot144 + 1\cdot216)/48 = (46656 + 0 + 432 + 216)/48 = 1971/2$. That is, $f(2,2,2) = 985.5$.

From Table 4.2 we calculate $g(2,2,2) = (6\cdot144$

Table 4.1

Classifying Rotations of $K_{2,2,2}$ Using $\alpha \in A(K_{2,2,2})$

α	$\lvert J(\alpha)\rvert$	$\lvert F(\alpha)\rvert$
(0)(1)(2)(3)(4)(5)	1	46,656
(01)(2)(3)(4)(5)	3	0
(01)(23)(4)(5)	3	144
(01)(23)(45)	1	216

+ 6·216 + 6·24 + 6·12) / 48 = (864 + 1296 + 144 + 72) /48 = 99 / 2. That is, $g(2,2,2) = 49.5$.

From Table 4.3 we calculate $h(2, 2, 2) = (8 \cdot 36 + 8 \cdot 6) / 46 = (288 + 48) / 48 = 7$.

Table 4.2

Classifying Rotations of $K_{2,2,2}$ Using $\beta \in B(K_{2,2,2})$

β	$\lvert J(\beta)\rvert$	$\lvert F(\beta)\rvert$
(02)(13)(4)(5)	6	144
(02)(13)(45)	6	216
(0213)(4)(5)	6	24
(0213)(45)	6	12

Table 4.3

Classifying Rotations of $K_{2,2,2}$ Using $\gamma \in D(K_{2,2,2})$

γ	$\lvert J(\gamma)\rvert$	$\lvert F(\gamma)\rvert$
(024)(135)	8	36
(024135)	8	6

Combining, we find $|C(K_{2,2,2})| = f(2,2,2) + g(2,2,2) + h(2,2,2) = 985.5 + 49.5 + 7 = 1042$. That is, there are 1042 congruence classes of maps of $K_{2,2,2}$.

In problem 3.4 of [18], page 21, we are asked to compute Aut $K_{m,n,r}$. We will just give the answers and

allow the reader to supply the proof. When $m < n < r$, then $\mathrm{Aut}\, K_{m,n,r} \cong S_m \oplus S_n \oplus S_r$. When two of the partite sets have the same order and the third has different order (say the orders are m and n, where $m \neq n$), then $\mathrm{Aut}\, K_{m,n,n} \cong S_m \oplus S_2[S_n]$. Finally, when all three partite sets have order n, $\mathrm{Aut}\, K_{n,n,n} \cong S_3[S_n]$.

By Theorem 2.2, the sums for $f(m,n,r)$, $g(m,n,r)$, and $h(m,n,r)$ can all be taken over the structure classes of $A(K_{m,n,r})$, $B(K_{m,n,r})$, and $D(K_{m,n,r})$. We prefix the name of each of these sets with a J to denote the set of structure over it; thus $JA(K_{m,n,r})$ is the set of structure classes in $A(K_{m,n,r})$. We still employ $J(\alpha)$, $J(\beta)$, and $J(\gamma)$ for the structure classes of α, β, and γ, respectively. Let $\xi = [(1+\delta_{m,n}+\delta_{n,r})\cdot(1+\delta_{m,r})]^{-1}$; then

Theorem 4.2. The forms of $f(m,n,r)$, $g(m,n,r)$ and $h(m,n,r)$ are given by:

$$f(m,n,r) = \frac{\xi}{m!\cdot n!\cdot r!} \cdot \sum_{J(\alpha) \in JA(K_{m,n,r})} |F(\alpha)|\cdot|J(\alpha)|$$

$$g(m,n,r) = \begin{cases} 0 & , m < n < r \\ \dfrac{1}{2\cdot n!^2\cdot r!} \cdot \displaystyle\sum_{J(\beta) \in JB(K_{m,n,r})} |F(\beta)|\cdot|J(\beta)| & , m = n < r \\ \dfrac{1}{2\cdot n!^2\cdot m!} \cdot \displaystyle\sum_{J(\beta) \in JB(K_{m,n,r})} |F(\beta)|\cdot|J(\beta)| & , m < n = r \\ \dfrac{1}{6\cdot n!^3} \displaystyle\sum_{J(\beta) \in JB(K_{m,n,r})} |F(\beta)|\cdot|J(\beta)|, & m = n = r \end{cases}$$

and

$$h(m,n,r) = \begin{cases} 0, & \text{not all three } m,n, \text{ and } r \text{ equal} \\ \dfrac{1}{6\cdot n!^3} \displaystyle\sum_{J(\gamma) \in JD(K_{m,n,r})} |F(\gamma)|\cdot|J(\gamma)|, & m = n = r \end{cases}$$

Proof: In the case $m < n < r$ we have seen that Aut $K_{m,n,r} \cong S_m \oplus S_n \oplus S_r$, thus $g(m,n,r)$ and $h(m,n,r)$ are both zero. The value of ξ ensures that the coefficient is correct on the order of the automorphism group (just try all four cases).

Suppose that at least two of the partite sets have the same order. If the other partite set has different order, then $h(m,n,r) = 0$. Also, in this case, we have Aut $K_{m,n,r} \cong S_m \oplus S_2[S_n]$ or $S_r \oplus S_2[S_n]$. Thus, the coefficient for these cases is correct.

Also, if $m = n = r$, then we have seen that Aut $K_{n,n,n} \cong S_3[S_n]$. The coefficients given again ensure that the order of the automorphism group is correct.

Finally, the remarks preceding the theorem show us that the sums are taken correctly. ∎

Example 4.4

Consider the graph $K_{1,2,2}$, with $V = \{1, 2, 3, 4, 5\}$, $V_1 = \{1\}$, $V_2 = \{2, 3\}$, and $V_3 = \{4, 5\}$. Since $1 \neq 2$, we have, from Theorem 4.2, $h(1,2,2) = 0$. The number of rotations is $|R(G)| = 3!\cdot 2!^4 = 96$. These are classified in Tables 4.4 and 4.5.

Table 4.4

Classifying rotations of $K_{1,2,2}$ for $\alpha \in A(K_{1,2,2})$

α	$\lvert J(\alpha)\rvert$	$\lvert F(\alpha)\rvert$
(1)(2)(3)(4)(5)	1	96
(1)(23)(4)(5)	2	0
(1)(23)(45)	1	8

Using Table 4.4, we calculate $f(1,2,2) = (1\cdot 96 + 2\cdot 0$

+ 1·8) / 8 = (96 + 0 + 8) / 8 = 13. Thus, $f(1,2,2) = 13$.

Using Table 4.5, we calculate $g(1,2,2) = (2\cdot 8$ $2\cdot 4) / 8 = (16 + 8) / 8 = 3$. Thus, $g(1,2,2) = 3$.

Table 4.5
Classifying Rotations of $K_{1,2,2}$ Using $\beta \in B(K_{1,2,2})$

β	$\|J(\beta)\|$	$\|F(\beta)\|$
(1)(24)(35)	2	8
(1)(2435)	2	4

Combining, we find $|C(K_{1,2,2})| = f(1,2,2) + g(1,2,2) + h(1,2,2) = 13 + 3 + 0 = 16$. Therefore, there are sixteen congruence classes of maps of $K_{1,2,2}$.

Example 4.5

Consider the graph $K_{1,2,3}$, with $V = \{1,2,3,4,5,6\}$, $V_1 = \{1\}$, $V_2 = \{2, 3\}$, $V_3 = \{4, 5, 6\}$. Since $1 \neq 2$, $1 \neq 3$, and $2 \neq 3$, we know that $g(1,2,3) = h(1,2,3) = 0$. We still need to compute $f(1,2,3)$; $|R(K_{1,2,3})| = 4!3!^2 2!^3 = 24\cdot 36\cdot 8 = 6912$. These rotations are classified in Table 4.6.

Table 4.6
Classifying Rotations of $K_{1,2,3}$

α	$\|J(\alpha)\|$	$\|F(\alpha)\|$
(1)(2)(3)(4)(5)(6)	1	6912
(1)(23)(4)(5)(6)	1	0
(1)(2)(3)(45)(6)	3	0
(1)(23)(45)(6)	3	0
(1)(2)(3)(456)	2	0
(1)(23)(456)	2	0

Using Table 4.6, we compute $f(1,2,3) = (1\cdot 6912 + 1\cdot 0$

$+ 3\cdot0 + 3\cdot0 + 2\cdot0 + 2\cdot0) / 12 = (6912 + 0 + 0 + 0 + 0 + 0) / 12 = 6912 / 12 = 576$. That is, $f(1,2,3) = 576$.

Therefore, $|C(K_{1,2,3})| = 576 + 0 + 0 = 576$. That is, there are 576 congruence classes of maps of $K_{1,2,3}$.

In the next section we will calculate $f(m,n,r)$. We must first determine the cycle types of those automorphisms $\alpha \in A(K_{m,n,r})$ having $|F(\alpha)| > 0$. Once the cycle types have been found we will use Theorems 2.3 - 2.6 to develop the formula.

In those sections which follow we will compute $g(m,n,r)$ and $h(m,n,r)$, explicitly. Once this has been done, we will be ready to calculate $|C(K_{m,n,r})|$—this will be done in the final section.

4.2 Using Compatibility

In this section we will develop a formula for $f(m,n,r)$. Theorem 4.2 requires that we find $|F(\alpha)|$ and $|J(\alpha)|$. The preliminary remarks of Section 4.1 suggest that we refresh our minds with the definition of a compatible set of three permutations. Again we are making the definition more general than it needs to be.

Let A_1, A_2, and A_3 be (not necessarily distinct) sets and α_i a permutation on A_i ($i = 1, 2, 3$). Furthermore, suppose that α_i has s_i distinct cycle lengths λ_{ij} in its disjoint cycle decomposition; where the λ_{ij} satisfy $\lambda_{i1} < \lambda_{i2} < \ldots < \lambda_{is(i)}$. Then $\{\alpha_1, \alpha_2, \alpha_3\}$ is a compatible set (multiset) of permutations if and only if for each λ_{ij}, there exists a positive integer c_{ij} such that

for $k \neq i$, $\alpha_k^{\lambda_{ij}}$ is c_{ij}-uniform.

We will now show that this definition allows us to determine the cycle types of those $\alpha \in A(K_{m,n,r})$ such that $|F(\alpha)| > 0$. We start with an extension of Theorem 3.3.

Theorem 4.3. Let $\alpha \in \text{Aut } K_{m,n,r}$ and suppose that $\alpha = \alpha_1\alpha_2\alpha_3$ for some permutations α_1, α_2, and α_3 such that α_i is a permutation of the vertices in V_i $(i = 1, 2, 3)$. Then $|F(\alpha)| > 0$ if and only if $\{\alpha_1, \alpha_2, \alpha_3\}$ is a compatible set of permutations.

Proof: Suppose that $\{\alpha_1, \alpha_2, \alpha_3\}$ is a compatible set of permutations, where α_i permutes the vertices of V_i; take $\alpha = \alpha_1\alpha_2\alpha_3$. Arbitrarily take $u \in V_1$; then u appears in some cycle of the disjoint cycle decomposition of α_1; let $\lambda(u)$ be the length of this cycle. Since $\{\alpha_1, \alpha_2, \alpha_3\}$ is a compatible set of permutations, there exists a positive integer c such that $\alpha_2^{\lambda(u)}$ and $\alpha_3^{\lambda(u)}$ are both c-uniform. Thus, $\alpha^{\lambda(u)}|_{N(u)}$ is c-uniform $(\alpha^{\lambda(u)}|_{N(u)} = \alpha_2^{\lambda(u)}\alpha_3^{\lambda(u)})$, and since u is arbitrary, we conclude $\forall u \in V_1$, $\alpha^{\lambda(u)}|_{N(u)}$ is c-uniform. A similar argument establishes this for all $v \in V_2$ and $w \in V_3$. Therefore, for all $v \in V$, we have 1) $\alpha^{\lambda(v)}|_{N(v)}$ is uniform, 2) $|F_v(\alpha^{\lambda(v)})| > 0$ (by Theorem 2.5), and 3) $|F(\alpha)| > 0$ (by Theorem 2.4).

Now suppose that $|F(\alpha)| > 0$, and that $\alpha = \alpha_1\alpha_2\alpha_3$, where α_i permutes the vertices of V_i $(i = 1, 2, 3)$. By Theorem 2.4 we conclude that $\forall v \in V$, $|F_v(\alpha^{\lambda(v)})| > 0$; by Theorem 2.5 we conclude that $\forall v \in V$, $\alpha^{\lambda(v)}|_{N(v)}$ is uniform. Arbitrarily take $u \in V_1$; then u appears in some

cycle of the disjoint cycle decomposition of α_1. Hence, for each cycle length, λ, in the disjoint cycle decomposition of α_1, $\alpha^{\lambda}|_{N(u)}$ is uniform. Now, $N(u) = V_2 \cup V_3$; also we note $V_2 \cap V_3 = \emptyset$. By the way we defined α_2 and α_3, we conclude the existence of a positive integer c such that α_2^{λ} and α_3^{λ} are both c-uniform. A similar argument establishes this same result for all $v \in V_2$ and $w \in V_3$. Therefore, we conclude $\{\alpha_1, \alpha_2, \alpha_3\}$ is a compatible set of permutations. ∎

Example 4.6

Consider the graph $K_{1,2,2}$ of Example 4.4. Recall that $V = \{1, 2, 3, 4, 5\}$, $V_1 = \{1\}$, $V_2 = \{2, 3\}$, and $V_3 = \{4, 5\}$. Take $\alpha_1 = (1)$, $\alpha_2 = (23)$, and $\alpha_3 = (45)$; set $\alpha = \alpha_1\alpha_2\alpha_3$. Notice that $\alpha_2^1 = (23)$, and $\alpha_3^1 = (45)$ are both 2-uniform; $\alpha_1^2 = (1)$, and $\alpha_3^2 = (4)(5)$ are both 1-uniform; and α_1^2, and $\alpha_2^2 = (2)(3)$ are both 1-uniform. Therefore, $\{\alpha_1, \alpha_2, \alpha_3\}$ is a compatible set of permutations, and $|F(\alpha)| > 0$ (which is confirmed by the entry for α in Table 4.4).

Example 4.7

Consider again the graph $K_{1,2,2}$ of Example 4.4. This time we take $\alpha_1 = (1)$, $\alpha_2 = (23)$, and $\alpha_3 = (4)(5)$; set $\alpha = \alpha_1\alpha_2\alpha_3$. Though these permutations are pairwise compatible, they do not form a compatible set of permutations. Therefore, $|F(\alpha)| = 0$ (as is verified by the entry of Table 4.4).

We are now ready determine the structure of a

compatible set of permutations; this will allow us to calculate $f(m,n,r)$. As in Chapter III, we will express the λ_{ij} in terms of the c_{ij}. First, we need the structure.

Theorem 4.4. Let $\{\alpha_1, \alpha_2, \alpha_3\}$ be a compatible set of permutations. Suppose that α_i has s_i distinct cycle lengths and that λ_{ij} is the j-th smallest ($i = 1, 2, 3$; $j = 1, \ldots, s_i$). Furthermore, suppose that for $k = 1, 2, 3$, $k \neq i$, $\alpha_k^{\lambda_{ij}}$ is c_{ij}-uniform. Then $\forall\, i = 1, 2, 3$; $\forall j = 1, \ldots, s_i$; $\forall k = 1, 2, 3$, $k \neq i$; and $\forall \ell = 1, \ldots, s_k$ we have, $c_{ij} \mid \lambda_{k\ell}$, and $\gcd(c_{ij}, c_{k\ell}) = 1$.

Proof: Obviously, $c_{ij} \mid \lambda_{k\ell}$, for $\alpha_k^{\lambda_{ij}}$ is c_{ij}-uniform and all of the cycles are disjoint in the decomposition of α_k.

Suppose that $\gcd(c_{ij}, c_{kl}) = h$; since we know that $\alpha_k^{\lambda_{ij}}$ is c_{ij}-uniform, Corollary 2.1a tells us that we must have $c_{ij} = \lambda_{k\ell} / \gcd(\lambda_{ij}, \lambda_{k\ell})$. Hence, either $h = 1$, or h appears as a factor of $\lambda_{k\ell}$ one more time than it is a factor of λ_{ij}. Also since $\alpha_i^{\lambda_{k\ell}}$ is $c_{k\ell}$-uniform, a similar argument establishes that either $h = 1$, or h appears as a factor of $\lambda_{k\ell}$ one less time than it is a factor of λ_{ij}. We conclude that $h = 1$. ∎

Since this theorem is so similar to Theorem 3.4, we will not give an example illustrating it. The reader is referred to Example 3.8 which, though not an example of Theorem 4.4, is close enough to get a feel for what it means.

We now know that the same conditions that were

present in Chapter III also apply in the present chapter; we establish an extension of Theorem 3.5 as well.

Theorem 4.5. Let α_i be a permutation of a set A_i ($i = 1, 2, 3$); suppose that $|A_i| = n_i$, and that $\{\alpha_1, \alpha_2, \alpha_3\}$ is a compatible set of permutations. Further, suppose that α_i has s_i distinct cycle lengths in its disjoint cycle decomposition, and let λ_{ij} denote the j-th smallest cycle length ($i = 1, 2, 3$; $j = 1, \ldots, s_i$). Suppose that for $k = 1, 2, 3$, $k \neq i$, $\alpha_k^{\lambda_{ij}}$ is c_{ij}-uniform. Then for some positive integer g, $g \mid \gcd(n_1,n_2,n_3)$, we have for $i = 1, 2, 3$; $j = 1, \ldots, s_i$

$$\lambda_{ij} = \frac{g}{c_{ij}} \prod_{k=1}^{3} \operatorname{lcm}(c_{k\ell};\ \ell = 1, \ldots, s_k).$$

Proof: By Theorem 4.4, if $k \neq i$, then for each $\ell = 1, \ldots, s_k$, $c_{k\ell} \mid \lambda_{ij}$ ($i, k = 1, 2, 3$; and $j = 1, \ldots, s_i$). Therefore, we must have $\operatorname{lcm}(c_{k\ell};\ \ell = 1, \ldots, s_k) \mid \lambda_{ij}$, when $k \neq i$, and we conclude that there is a positive integer a_{ij} such that

$$\lambda_{ij} = a_{ij} \prod_{\substack{k=1 \\ k \neq i}}^{3} \operatorname{lcm}(c_{k\ell};\ \ell = 1, \ldots, s_k),$$

for each $i = 1, 2, 3$, and $j = 1, \ldots, s_i$.

By Corollary 2.1a, we have

$$c_{1j} = \lambda_{2k} / \gcd(\lambda_{1j},\lambda_{2k}) = \lambda_{3\ell} / \gcd(\lambda_{1j},\lambda_{3\ell})$$

$$c_{2k} = \lambda_{1j} / \gcd(\lambda_{1j},\lambda_{2k}) = \lambda_{3\ell} / \gcd(\lambda_{2k},\lambda_{3\ell})$$

and

$$c_{3\ell} = \lambda_{1j} / \gcd(\lambda_{1j},\lambda_{3\ell}) = \lambda_{2k} / \gcd(\lambda_{2k},\lambda_{3\ell}).$$

Hence, we have

$$\frac{c_{1j}}{c_{2k}} = \frac{\lambda_{2k}}{\lambda_{1j}} = \frac{a_{2k}\cdot\prod_{m\neq 2} \operatorname{lcm}(c_{mn};\ n = 1, \ldots, s_m)}{a_{1j}\cdot\prod_{p\neq 1} \operatorname{lcm}(c_{pr};\ r = 1, \ldots, s_p)}$$

$$= \frac{a_{2k}\cdot \mathrm{lcm}(c_{1n};\ n = 1,\ \dots,\ s_1)}{a_{1j}\cdot \mathrm{lcm}(c_{2r};\ r = 1,\ \dots,\ s_2)}.\quad \text{Thus,}$$

$$\frac{a_{1j}\cdot \mathrm{lcm}(c_{2r}; r = 1,\dots,s_2)}{c_{2k}} = \frac{a_{2k}\cdot \mathrm{lcm}(c_{1n}; n = 1,\dots,s_1)}{c_{1j}},$$

and we conclude that there is a positive integer h_{1j2k} such that

$$a_{1j} = \frac{h_{1j2k}\cdot \mathrm{lcm}(c_{1n}; n = 1,\dots,s_1)}{c_{1j}}$$

and

$$a_{2k} = \frac{h_{1j2k}\cdot \mathrm{lcm}(c_{2r}; r = 1,\dots,s_2)}{c_{2k}}.$$

Also,

$$\frac{c_{1j}}{c_{3\ell}} = \frac{\lambda_{3\ell}}{\lambda_{1j}} = \frac{a_{3\ell}\cdot \Pi_{m\neq 3}\ \mathrm{lcm}(c_{mn};\ n = 1,\ \dots,\ s_m)}{a_{1j}\cdot \Pi_{p\neq 1}\ \mathrm{lcm}(c_{pr};\ r = 1,\ \dots,\ s_p)}$$

$$= \frac{a_{3\ell}\cdot \mathrm{lcm}(c_{1n};\ n = 1,\ \dots,\ s_1)}{a_{1j}\cdot \mathrm{lcm}(c_{3r};\ r = 1,\ \dots,\ s_2)}.$$

Therefore,

$$\frac{a_{1j}\cdot \mathrm{lcm}(c_{3r}; r = 1,\dots,s_2)}{c_{3\ell}} = \frac{a_{3\ell}\cdot \mathrm{lcm}(c_{1n}; n = 1,\dots,s_1)}{c_{1j}},$$

and we conclude that there is a positive integer $h_{1j3\ell}$ such that

$$a_{1j} = \frac{h_{1j31}\cdot \mathrm{lcm}(c_{1n}; n = 1,\dots,s_1)}{c_{1j}}$$

and

$$a_{3\ell} = \frac{h_{1j3\ell}\cdot \mathrm{lcm}(c_{3r}; r = 1,\dots,s_2)}{c_{3\ell}}.$$

Comparing, we find that $h_{1j2k} = h_{1j3\ell}$. Say $h_{1j2k} = g$. By comparing c_{2k} with $c_{3\ell}$, we find that the same value g again appears. Varying j, k, and ℓ, and using an argument similar to the above, we again obtain the same value

g. We conclude that g does not depend upon the choice of i, j, k, or ℓ.

Substituting the form of a_{ij} into the formula of λ_{ij} $(i = 1, 2, 3;\ j = 1, \ldots, s_i)$, and using g for the constant, we obtain

$$\lambda_{ij} = \frac{g}{c_{ij}} \prod_{k=1}^{3} \operatorname{lcm}(c_{k\ell};\ \ell = 1, \ldots, s_k).$$

Suppose that for each $i = 1, 2, 3$, and $j = 1, \ldots, s_i$, there are exactly μ_{ij} cycles of length λ_{ij}. Now,

$$\sum_{j=1}^{s_i} \mu_{ij} \cdot \lambda_{ij} = n_i \quad (i = 1, 2, 3),$$

so $g | n_i$ $(i = 1,2,3)$. That is, $g | \gcd(n_1,n_2,n_3)$ as claimed. ∎

Theorem 4.5 is similar to Theorem 3.5. It shows that if $\{\alpha_1, \alpha_2, \alpha_3\}$ is a compatible set of permutations, and if α_i has cycles of length λ_{ij} $(i = 1, 2, 3;\ j = 1, \ldots, s_i)$, and if α_i is a permutation of a set of order n_i, then if for $k = 1, 2, 3$, $k \neq i$, $\alpha_k^{\lambda_{ij}}$ is c_{ij}-uniform,

$$\lambda_{ij} = \frac{g}{c_{ij}} \prod_{k=1}^{3} \operatorname{lcm}(c_{k\ell};\ \ell = 1, \ldots, s_k),$$

where $g \mid \gcd(n_1,n_2,n_3)$, and $c_{ij} \mid (n_k / g)$. Let c_i be any set of divisors of $\gcd(n_k;\ k \neq i) / g$. Suppose that $c_i = \{c_{ij};\ j = 1, \ldots, s_i\}$ and if $k \neq i$, $\gcd(c_{ij},c_{k\ell}) = 1$ for all $j = 1, \ldots, s_i$ and $\ell = 1, \ldots, s_k$. Let g be a positive integer such that $g \mid \gcd(n_1,n_2,n_3)$ and define λ_{ij} as above. If there exist integers $\mu_{ij} \geq 1$ such that

$$\sum_{j=1}^{s_i} \mu_{ij} \cdot \lambda_{ij} = n_i \quad (i = 1, 2, 3),$$

then it is straightforward to verify the existence of a

set of compatible permutations $\{\alpha_1, \alpha_2, \alpha_3\}$, where α_i is a permutation of a set A_i, $|A_i| = n_i$, and α_i has exactly μ_{ij} cycles of length λ_{ij} ($i = 1, 2, 3$; $j = 1, \ldots, s_i$). Therefore, the strategy of Chapter III extends easily to this case. Using this strategy, and Theorems 2.3 – 2.6, we are now ready to establish the formula for $f(m,n,r)$.

Theorem 4.6. Let $c_i = \{c_{ij};\ j = 1, \ldots, s_i\}$, where the $c_{ij} | n_i/g$, for some positive integer g, $g \mid \gcd(n_1,n_2,n_3)$. Suppose that $\gcd(c_{ij},c_{k\ell}) = 1$, ($i, k = 1, 2, 3$; $i \neq k$; $j = 1, \ldots, s_i$; and $\ell = 1, \ldots, s_k$). Define λ_{ij} as

$$\lambda_{ij} = \frac{g}{c_{ij}} \prod_{k=1}^{3} \operatorname{lcm}(c_{k\ell};\ \ell = 1, \ldots, s_k),$$

and suppose that there exist integers $\mu_{ij} \geq 1$ such that

$$\sum_{j=1}^{s_i} \mu_{ij} \cdot \lambda_{ij} = n_i \quad (i = 1, 2, 3),$$

where $n_1 = m$, $n_2 = n$, and $n_3 = r$. Then for α_i, a permutation on V_i having exactly μ_{ij} cycles of length λ_{ij}, and $\alpha = \alpha_1\alpha_2\alpha_3$, we have

$$|F(\alpha)| = \prod_{i=1}^{3} \cdot \prod_{j=1}^{s_i} [\Phi(c_{ij}, N - n_i)]^{\mu_{ij}},$$

where $N = \sum_{i=1}^{3} n_i$, $\Phi(x,y) = \varphi(x)x^{(y/x-1)}(y/x - 1)!$, and $\varphi(x)$ is the Euler function. Also,

$$|J(\alpha)| = \prod_{i=1}^{3} n_i! \prod_{j=1}^{s_i} \frac{1}{\lambda_{ij}^{\mu_{ij}} \cdot \mu_{ij}!}.$$

Proof: By Theorem 2.4 $|F(\alpha)| = \Pi_{v\in S}|F_v(\alpha^{\lambda(v)})|$, where S is a complete system of orbit representatives. Consider $v \in V_i$ and suppose that v appears in a cycle of length λ_{ij}. Now we note that $|N(v)| = N - n_i$, where $N = n_1 + n_2$

$+ n_3$, so that $|F_v(\alpha^{\lambda_{ij}})| = \Phi(c_{ij}, N - n_i)$, for we know that $\alpha^{\lambda_{ij}}|_{N(v)}$ is c_{ij}-uniform. There are μ_{ij} cycles of length λ_{ij}, so there are μ_{ij} members of S corresponding to these cycles. Combining, and using the fact that these choices for S are independent, we find $|F(\alpha)|$ to be as above.

Each of α_1, α_2, and α_3 can be picked independently of the others as they are permutations of disjoint sets. There are μ_{ij} cycles of length λ_{ij} in α_i ($i = 1, 2, 3$; $j = 1, \ldots, s_i$); therefore, by Theorem 2.6, we find that

$$|J(\alpha_i)| = n_i! \prod_{j=1}^{s_i} \frac{1}{\lambda_{ij}^{\mu_{ij}} \cdot \mu_{ij}!}.$$

The independence of the choices gives $|J(\alpha)|$ as above. ∎

Example 4.8

Consider the graph $K_{1,2,3}$ of Example 4.5, with $V = \{1, 2, 3, 4, 5, 6\}$, $V_1 = \{1\}$, $V_2 = \{2, 3\}$, and $V_3 = \{4, 5, 6\}$. We have already observed that $g(1,2,3) = h(1,2,3) = 0$; therefore, $|C(K_{1,2,3})| = f(1,2,3)$. Note that $\gcd(1,2,3) = 1$; thus, $g = 1$. Also, $\gcd(1,2) = \gcd(1,3) = \gcd(2,3) = 1$; hence $c_1 = c_2 = c_3 = \{1\}$. This means that $c_{11} = c_{21} = c_{31} = 1$, so that $\lambda_{11} = \lambda_{21} = \lambda_{31} = 1$ (hence, $\alpha = e$, the identity). Now Theorem 4.6 gives

$$|F(\alpha)| = \Phi(1,5)^1 \cdot \Phi(1,4)^2 \cdot \Phi(1,3)^3$$
$$= (1 \cdot 1^4 \cdot 4!)^1 (1 \cdot 1^3 \cdot 3!)^2 (1 \cdot 1^2 \cdot 2!)^3 = 24 \cdot 6^2 \cdot 2^3 = 6912.$$

Also, we calculate

$$|J(\alpha)| = \frac{1!}{1^1 \cdot 1!} \cdot \frac{2!}{1^2 \cdot 2!} \cdot \frac{3!}{1^3 \cdot 3!} = 1.$$

And finally, $|\mathrm{Aut}\, K_{1,2,3}| = 1! \cdot 2! \cdot 3! = 2 \cdot 6 = 12$. Combining the above information, and using Theorem 4.2, we find

$$|C(K_{1,2,3})| = (6912 \cdot 1) / 12 = 576.$$

We now see why only the identity element contributed to $|C(K_{1,2,3})|$ (this is verified by Table 4.6).

The strategy given before Theorem 4.6 shows that the sum over the structure classes can be replaced by a triple sum, where the first sum is over the divisors of the greatest common divisor of n_1, n_2, and n_3, the second sum is over $c = c_1 \times c_2 \times c_3$, where $c_i = \{c_{ij}\}$, and where $c_{ij} \mid \gcd(n_k;\ k \neq i)/g$, and the third is over all solutions μ to the possibly underdetermined set of equations

$$\sum_{j=1}^{s_i} \mu_{ij} \cdot \lambda_{ij} = n_i \quad (i = 1, 2, 3),$$

where λ_{ij} is as before.

Theorem 4.7. Let g be a divisor of $\gcd(n_1, n_2, n_3)$, where $n_1 = m$, $n_2 = n$, and $n_3 = r$. Let $c = c_1 \times c_2 \times c_3$, where $c_i = \{c_{ij};\ j = 1, \ldots, s_i\}$, $c_{ij} \mid \gcd(n_k;\ k \neq i) / g$ and $\gcd(c_{ij}, c_{k\ell}) = 1$ $(i, k = 1, 2, 3;\ j = 1, \ldots, s_i;\ i \neq k;$ and $\ell = 1, \ldots, s_k)$. Define λ_{ij} to be

$$\lambda_{ij} = \frac{g}{c_{ij}} \prod_{k=1}^{3} \operatorname{lcm}(c_{k\ell};\ \ell = 1, \ldots, s_k),$$

and suppose $\mu = \{\mu_{ij} \geq 1\}$ is the set of all solutions to the possibly underdetermined set of equations

$$\sum_{j=1}^{s_i} \mu_{ij} \cdot \lambda_{ij} = n_i \quad (i = 1, 2, 3).$$

Then,

$$f(m,n,r) = \xi \sum_{g} \sum_{c} \sum_{\mu} \prod_{i=1}^{3} \prod_{j=1}^{s_i} \frac{[\Phi(c_{ij}, N-n_i)]^{\mu_{ij}}}{\lambda_{ij}^{\mu_{ij}} \cdot \mu_{ij}!},$$

where $\xi = 1 / (1 + \delta_{m,n} + \delta_{n,r})!$, $N = n_1 + n_2 + n_3$, $\Psi(x,y) = \varphi(x)x^{(y/x-1)}(y/x-1)!$, φ is the Euler function, and δ is the Kronecker function.

Proof: By Theorem 4.2, and the formula for $|J(\alpha)|$, we see that $m!n!r!$ cancels; thus, the coefficient before the triple sum is correct. The remarks preceding this theorem show how the sum of Theorem 4.2 can be replaced by the triple sum shown in the theorem. Finally, substitution of the formulas for $|F(\alpha)|$ and $|J(\alpha)|$, given in Theorem 4.6, into Theorem 4.2, gives us the result. ∎

Example 4.9

Consider the graph $K_{2,4,6}$. By Theorem 4.2, $g(2,4,6) = h(2,4,6) = 0$; again we have $|C(K_{2,4,6})| = f(m,n,r) = f(2,4,6)$. Now, $\gcd(2,4,6) = 2$, so $g = 1$ and $g = 2$ are both possible. We consider these cases separately.

Suppose that $g = 1$; then $c_1 = \{1\}, \{2\}, \{1,2\}$ ($\gcd(4,6) = 2$); $c_2 = \{1\}, \{2\}, \{1,2\}$ ($\gcd(2,6) = 2$); and $c_3 = \{1\}, \{2\}, \{1,2\}$ ($\gcd(2,4) = 2$). There appear to be 27 cases, actually since $\gcd(2,2) \neq 1$ (remember the coprime condition), only one of the three can have a two. Therefore, only seven cases need to be considered.

(1) Suppose $c_1 = c_2 = c_3 = \{1\}$; then $c_{11} = c_{21} = c_{31} = 1$. Thus, $\lambda_{11} = \lambda_{21} = \lambda_{31} = 1$ and $\mu_{11} = 2$, $\mu_{21} = 4$, and $\mu_{31} = 6$; by Theorem 4.7, the value of this term is

$$\frac{\Psi(1,10)^2}{1^2 \cdot 2!} \cdot \frac{\Psi(1,8)^4}{1^4 \cdot 4!} \cdot \frac{\Psi(1,6)^6}{1^6 \cdot 6!} = \frac{9!^2}{2!} \cdot \frac{7!^4}{4!} \cdot \frac{5!^6}{6!}$$

$$= \frac{131681894400}{2} \cdot \frac{645241282560000}{24} \cdot \frac{2985984000000}{720}$$

$$= \frac{253708891510192312864997376000000000000}{34560}$$

$= 7{,}341{,}113{,}758{,}975{,}472{,}015{,}769{,}600{,}000{,}000{,}000.$

(2) Suppose $c_1 = c_2 = \{1\}$, and $c_3 = \{2\}$; then $c_{11} = c_{21} = 1$, and $c_{31} = 2$. Therefore, $\lambda_{11} = \lambda_{21} = 2$, and $\lambda_{31} = 1$. Hence, $\mu_{11} = 1$, $\mu_{21} = 2$, and $\mu_{31} = 6$. Theorem 4.7 gives the value of this term as

$$\frac{\Phi(1,10)^1}{1^1\cdot 1!}\cdot\frac{\Phi(1,8)^2}{1^2\cdot 2!}\cdot\frac{\Phi(2,6)^6}{2^6\cdot 6!} = \frac{9!^1}{1}\cdot\frac{7!^2}{2!}\cdot\frac{(2^2\cdot 2!)^6}{2^6\cdot 6!}$$

$$= \frac{362880}{1}\cdot\frac{25401600}{2}\cdot\frac{262144}{64\cdot 720} = \frac{876853541939718389760000}{92160}$$

$$= 95{,}144{,}698{,}561{,}167{,}360{,}000.$$

(3) Suppose $c_1 = c_3 = \{1\}$, and $c_2 = \{2\}$; then $c_{11} = 1$, $c_{21} = 2$, and $c_{31} = 1$. Therefore, $\lambda_{11} = 2$, $\lambda_{21} = 1$, and $\lambda_{31} = 2$. Hence $\mu_{11} = 1$, $\mu_{21} = 4$, and $\mu_{31} = 3$. Theorem 4.7 gives the value of this term as

$$\frac{\Phi(1,10)^1}{1^1\cdot 1!}\cdot\frac{\Phi(2,8)^4}{2^4\cdot 4!}\cdot\frac{\Phi(1,6)^3}{1^3\cdot 3!} = \frac{9!^1}{1}\cdot\frac{(2^3\cdot 3!)^4}{2^4\cdot 4!}\cdot\frac{5!^3}{3!}$$

$$= \frac{362880}{1}\cdot\frac{5308416}{16\cdot 24}\cdot\frac{1728000}{6} = \frac{3328677500682240000}{2304}$$

$$= 1{,}444{,}738{,}498{,}560{,}000.$$

(4) Suppose $c_1 = \{2\}$, and $c_2 = c_3 = \{1\}$; then $c_{11} = 2$, and $c_{21} = c_{31} = 1$. Thus, $\lambda_{11} = 1$, and $\lambda_{21} = \lambda_{31} = 2$. Hence $\mu_{11} = \mu_{21} = 2$, and $\mu_{31} = 3$. Theorem 4.7 give the value of this term as

$$\frac{\Phi(2,10)^2}{2^2\cdot 2!}\cdot\frac{\Phi(1,8)^2}{1^2\cdot 2!}\cdot\frac{\Phi(1,6)^3}{1^3\cdot 3!} = \frac{(2^4\cdot 4!)^2}{2^2\cdot 2!}\cdot\frac{7!^2}{2!}\cdot\frac{5!^3}{3!}$$

$$= \frac{147456}{4\cdot 2}\cdot\frac{25401600}{2}\cdot\frac{1728000}{6} = \frac{6472428473548800000}{96}$$

$$= 67{,}421{,}129{,}932{,}800{,}000.$$

(5) Suppose $c_1 = c_2 = \{1\}$, and $c_3 = \{1, 2\}$; then $c_{11} = c_{21} = 1$, $c_{31} = 2$, and $c_{32} = 1$. Thus, $\lambda_{11} = \lambda_{21} = 2$, $\lambda_{31} = 1$, and $\lambda_{32} = 2$. The following solutions are possible: $(\mu_{11},\mu_{21},\mu_{31},\mu_{32}) = (1,2,2,2), (1,2,4,1)$. In the first case, Theorem 4.7 gives the value of this term as

$$\frac{\Phi(1,10)^1}{1^1\cdot 1!}\cdot\frac{\Phi(1,8)^2}{1^2\cdot 2!}\cdot\frac{\Phi(2,6)^2}{2^2\cdot 2!}\cdot\frac{\Phi(1,6)^2}{1^2\cdot 2!}$$

$$=\frac{9!^1}{1}\cdot\frac{7!^2}{2!}\cdot\frac{(2^2\cdot 2!)^2}{2^2\cdot 2!}\cdot\frac{5!^2}{2!}=\frac{362880}{1}\cdot\frac{25401600}{2}\cdot\frac{64}{4\cdot 2}\cdot\frac{14400}{2}$$

$$=\frac{8495062371532800000}{32}=265{,}470{,}699{,}110{,}400{,}000.$$

In the second case, the value is

$$\frac{\Phi(1,10)^1}{1^1\cdot 1!}\cdot\frac{\Phi(1,8)^2}{1^2\cdot 2!}\cdot\frac{\Phi(2,6)^4}{2^4\cdot 4!}\cdot\frac{\Phi(1,6)^1}{1^1\cdot 1!}$$

$$=\frac{9!^1}{1}\cdot\frac{7!^2}{2!}\cdot\frac{(2^2\cdot 2!)^4}{2^4\cdot 4!}\cdot\frac{5!}{1}$$

$$=\frac{362880}{1}\cdot\frac{25401600}{2}\cdot\frac{4096}{16\cdot 24}\cdot\frac{120}{1}=\frac{4530699931484160000}{768}$$

$$=5{,}899{,}348{,}869{,}120{,}000.$$

Adding the values for these two solutions gives a total of 271,370,047,979,520,000 for the sets c_1, c_2, and c_3.

(6) Suppose $c_1 = c_3 = \{1\}$, and $c_2 = \{1, 2\}$; then $c_{11} = c_{22} = c_{31} = 1$, and $c_{21} = 2$. Thus, $\lambda_{11} = 2$, $\lambda_{21} = 1$, $\lambda_{22} = 2$, and $\lambda_{31} = 2$. Therefore, $\mu_{11} = \mu_{22} = 1$, $\mu_{21} = 2$, and $\mu_{31} = 3$. Theorem 4.7 gives the value of this term as

$$\frac{\Phi(1,10)^1}{1^1\cdot 1!}\cdot\frac{\Phi(2,8)^2}{2^2\cdot 2!}\cdot\frac{\Phi(1,8)^1}{1^1\cdot 1!}\cdot\frac{\Phi(1,6)^3}{1^3\cdot 3!}$$

$$\frac{9!^1}{1}\cdot\frac{(2^3\cdot 3!)^2}{2^2\cdot 2!}\cdot\frac{7!^1}{1}\cdot\frac{5!^3}{3!}=\frac{362880}{1}\cdot\frac{2304}{4\cdot 2}\cdot\frac{5040}{1}\cdot\frac{1728000}{6}$$

$$=\frac{7281482032742400000}{48}=151{,}697{,}542{,}348{,}800{,}000.$$

(7) And finally, suppose $c_1 = \{1,2\}$, and $c_2 = c_3 = \{1\}$; then $c_{11} = 2$, and $c_{12} = c_{21} = c_{31} = 1$. Therefore, $\lambda_{11} = 1$, $\lambda_{12} = 2$—but there are only two vertices in V_1 and these already add up to three! We conclude that this case is not possible.

Therefore, the total of the six cases, when $g = 1$, is 7,341,113,758,975,567,652,401,619,927,040,000.

We now consider when $g = 2$. Since $\gcd(2, 4) = \gcd(4, 6) = \gcd(6, 2) = 2$, dividing these numbers by $g = 2$ gives $c_1 = c_2 = c_3 = \{1\}$. Therefore, $c_{11} = c_{21} = c_{31} = 1$, and $\lambda_{11} = \lambda_{21} = \lambda_{31} = 2$. Hence $\mu_{11} = 1$, $\mu_{21} = 2$, and $\mu_{31} = 3$. Theorem 4.7 gives the value of this term as

$$\frac{\Phi(1,10)^1}{1^1\cdot 1!}\cdot\frac{\Phi(1,8)^2}{1^2\cdot 2!}\cdot\frac{\Phi(1,6)^3}{1^3\cdot 3!} = \frac{9!^1}{1}\cdot\frac{7!^2}{2!}\cdot\frac{5!^3}{3!}$$

$$= \frac{362880}{1}\cdot\frac{25401600}{2}\cdot\frac{1728000}{6} = \frac{15928241946624000000}{12}$$

$$= 1{,}327{,}353{,}495{,}552{,}000{,}000.$$

Adding the totals for $g = 1$ and $g = 2$, we get $f(2,4,6) = 7{,}341{,}113{,}758{,}975{,}568{,}979{,}755{,}115{,}479{,}040{,}000$. Since $|C(K_{2,4,6})| = f(2,4,6)$, this is the number of congruence classes for the maps of $K_{2,4,6}$.

In the next section we shall derive the formula for $g(m,n,r)$. This will allow us to calculate the number of congruence classes of maps of $K_{m,n,n}$, where $m \neq n$. After this, we shall calculate $h(m,n,r)$. This will allow us to find the formula for the number of congruence classes of maps of $K_{m,n,r}$.

4.3 When Exactly One Partite Set is Fixed

In this section we will allow two of the partite sets to have the same order, or if they all have the same order, then one of the partite sets will be fixed. That is, we will derive a formula for $g(m,n,r)$. We already have a formula when $m < n < r$, namely $g(m,n,r) = 0$; thus we are interested in what happens when two of the partite sets are swapped.

We will first calculate $|F(\beta)|$; we will need to know something of the structure of those $\beta \in B(K_{m,n,r})$ for which $|F(\beta)| > 0$. Once this has been accomplished, we will find $|J(\beta)|$; this will allow us to use Theorem 4.2 to find the formula for $g(m,n,r)$.

We will first suppose that $m = n < r$; then we have that $\beta \in B(K_{m,n,r})$ fixes V_3. Observe that if β maps some vertex $u \in V_1$ onto a vertex $v \in V_2$, then β must swap V_1 with V_2. This is because $N(u)$ must be mapped onto $N(v)$, $N(u) = V_2 \cup V_3$, $N(v) = V_1 \cup V_3$, and β fixes V_3 setwise. We conclude that in the disjoint cycle decomposition of $\beta \in B(K_{m,n,r})$, vertices of V_1 alternate with vertices of V_2. Theorem 2.3 can now be used to establish:

Theorem 4.8. Let $\beta \in \text{Aut } K_{n,n,r}$ swap the partite sets V_1 and V_2 and fix V_3 (all setwise); then $|F(\beta)| > 0$ if and only if $\beta|_{V_1 \cup V_2}$ is uniform, and the cycle lengths of $\beta|_{V_3}$ are divisors of the uniform cycle length of $\beta|_{V_1 \cup V_2}$.

Proof: Since vertices of V_1 alternate with those of V_2 in the disjoint cycle decomposition of β, for $v \in V_1 \cup V_2$, $\beta^{\lambda(v)}$ fixes the adjacent vertices v and $\beta(v)$. By Theorem 2.3, $\beta^{\lambda(v)}$ must be the identity element of $\text{Aut } K_{n,n,r}$. Clearly, v can be chosen arbitrarily in $V_1 \cup V_2$, so the theorem follows. ∎

Clearly, a similar theorem could be established for the case $\beta \in \text{Aut } K_{m,n,n}$ which swaps V_2 and V_3, but fixes V_1 (as sets); therefore:

Corollary 4.8. Let $\beta \in B(K_{m,n,r})$ and suppose that V_i and V_j are swapped but V_{6-i-j} is fixed (setwise) by β, where

$1 \le i < j \le 3$. Then $|F(\beta)| > 0$ if and only if β, restricted to $V_i \cup V_j$, is uniform, and the cycle lengths of β restricted to V_{6-i-j} are divisors of the uniform cycle length of β restricted to $V_i \cup V_j$. ∎

Example 4.10

Consider the graph $K_{1,2,2}$ of Example 4.4. We notice in Table 4.5 that both of the entries have $|F(\beta)| > 0$. This is because the conditions forced the uniformity of $\beta|_{V_2 \cup V_3}$, and $|V_1| = 1$ is a divisor of all integers.

Corollary 4.8 alone is not quite enough to establish a formula for $g(m,n,r)$—but it is close. Theorem 2.5 implies that we need to know the uniform cycle lengths of $\beta^{\lambda(v)}|_{N(v)}$. Let $d|n$; then $c = 1$ is the uniform cycle length associated with a cycle length of $2d$ (β^{2d} is the identity element of Aut $K_{m,n,r}$). Also, the other c's must be divisors of $2d$ ($\beta^{2d} = e$). Thus, we must have the following two theorems:

Theorem 4.9. Let $d \mid n$ and suppose that $\beta \in$ Aut $K_{m,n,n}$ is 2d-uniform on $V_2 \cup V_3$. Take $c_1 = \{c_{1j} \mid j = 1, \ldots, s_1\}$, where $c_{1j} \mid 2d$, and $c_2 = \{1\}$; then if there exist positive integers μ_{1j} such that

$$\sum_{j=1}^{s_1} \mu_{1j} \cdot \left[\frac{2d}{c_{1j}} \right] = m,$$

then,

$$|F(\beta)| = (m+n-1)!^{(n/d)} \prod_{j=1}^{s_1} \Psi(c_{1j}, 2n)^{\mu_{1j}}.$$

Proof: By Theorem 2.4, we have $|F(\beta)| = \prod_{v \in S} |F_v(\alpha^{\lambda(v)})|$.

Suppose that $v \in V_1$. Then $\lambda(v) = 2d/c_{1j}$, for some $j = 1, \ldots, s_1$. There are μ_{1j} cycles of this length and Theorem 2.6 gives $|F_v(\beta^{\lambda(v)})| = \Psi(c_{1j}, 2n)$. Now if $v \notin V_1$, then $\lambda(v) = 2d$. Also, β^{2d} is the identity element, and $d(v)$, the degree of v, is $m + n$. We conclude that all $(m+n-1)!$ of the rotations at v are fixed. There are $2n/2d$ such $2d$-cycles. Using the independence of the choices for the members of S we have the result. ∎

Theorem 4.10. Let $\beta \in \mathrm{Aut}\, K_{n,n,r}$ and suppose $d|n$; take $c_1 = \{c_{1j};\ j = 1, \ldots, s_1\}$, where $c_{1j}|2d$, and $c_2 = \{1\}$, then if there exist positive integers μ_{1j} such that

$$\sum_{j=1}^{s_1} \mu_{1j} \cdot \left[\frac{2d}{c_{1j}} \right] = r,$$

then,

$$|F(\beta)| = (n+r-1)!^{(n/d)} \prod_{j=1}^{s_1} \Psi(c_{1j}, 2n)^{\mu_{1j}}.$$

Proof: Similar to the proof of Theorem 4.9. ∎

Nowhere in the proofs of Theorems 4.9 and 4.10 did we use the fact that $m \neq n$ (Theorem 4.9), or $n \neq r$ (Theorem 4.10); therefore the formula for $m = n = r$ must be:

Corollary 4.10. Let $\beta \in B(K_{n,n,n})$ and suppose that $d|n$. Take $c_1 = \{c_{1j};\ j = 1, \ldots, s_1\}$ and $c_2 = \{1\}$, then if there exist positive integers μ_{1j} such that

$$\sum_{j=1}^{s_1} \mu_{1j} \cdot \left[\frac{2d}{c_{1j}} \right] = n, \text{ then,}$$

$$|F(\beta)| = (2n-1)!^{(n/d)} \prod_{j=1}^{s_1} \Psi(c_{1j}, 2n)^{\mu_{1j}}.$$

Proof: Substitute $m = n$ into Theorem 4.9. ∎

We now turn our attention to finding $|J(\beta)|$. Let λ_{ij} denote the cycle length corresponding to c_{ij}; then we must have $\lambda_{ij} = 2d / c_{ij}$ $(j = 1, \ldots, s_i)$. Now, when we have $m < n = r$, then $\mathrm{Aut}\, K_{m,n,n} \cong \mathbb{S}_m \oplus \mathbb{S}_2[\mathbb{S}_n]$. Hence, we can use Theorem 2.6 to count that portion which is attributable to $\beta|_{V_1}$ and Theorem 3.10 to count that portion which is attributable to $\beta|_{V_2 \cup V_3}$. Then we can use the independence of these portions to establish:

Theorem 4.11. Let $d \mid n$ and suppose that $\beta \in \mathrm{Aut}\, K_{m,n,n}$ is 2d-uniform; take $c_1 = \{c_{1j};\ j = 1, \ldots, s_1\}$, where $c_{1j} | 2d$, and let $c_2 = \{1\}$. If there exist positive integers μ_{1j} such that

$$\sum_{j=1}^{s_1} \mu_{1j} \cdot \lambda_{1j} = m,$$

where $\lambda_{1j} = 2d / c_{1j}$ $(j = 1, \ldots, s_1)$, then

$$|J(\beta)| = \frac{m!}{\prod_{j=1}^{s_1} \lambda_{1j}^{\mu_{1j}} \cdot \mu_{1j}!} \cdot \frac{n!}{d^{(n/d)} \cdot (n/d)!}.$$

Proof: For $v \in V_1$, v appears in some cycle of the disjoint cycle decomposition of $\beta|_{V_1}$. Now, $V_1 \cap (V_2 \cup V_3) = \emptyset$, so that $\beta|_{V_1}$ is independent of $\beta|_{V_2 \cup V_3}$. Also, $\beta|_{V_1}$ is a permutation in Sym V_1. The cycle lengths appearing in it are precisely λ_{1j} $(j = 1, \ldots, s_1)$, and there are μ_{1j} cycles of length λ_{1j}. Thus, Theorem 2.6 applies and we conclude that the number of permutations of Sym V_1 corresponding to $\beta|_{V_1}$ is

$$\frac{m!}{\prod_{j=1}^{s_1} \lambda_{1j}^{\mu_{1j}} \cdot \mu_{1j}!}.$$

Now consider $\beta|_{V_2 \cup V_3}$; this is a uniform permutation (by Corollary 4.8). Because we know $\mathrm{Aut}\ K_{m,n,n}|_{V_2 \cup V_3}$ is isomorphic to $S_2[S_n]$, Theorem 3.10 applies and we conclude that the number of elements of $S_2[S_n]$ corresponding to permutations $\beta|_{V_2 \cup V_3}$ is

$$\frac{n!^2}{d^{(n/d)} \cdot (n/d)!}.$$

Using the independence of these choices, we get the theorem. ∎

Theorem 4.12. Let $d \mid n$ and suppose that $\beta \in \mathrm{Aut}\ K_{n,n,r}$ is 2d-uniform on $V_1 \cup V_2$; take $c_1 = \{c_{1j}; j = 1, \ldots, s_1\}$, where $c_{1j} \mid 2d$, and let $c_2 = \{1\}$. If there exist positive integers μ_{1j} such that

$$\sum_{j=1}^{s_1} \mu_{1j} \lambda_{1j} = r,$$

where $\lambda_{1j} = 2d / c_{1j}$ $(j = 1, \ldots, s_1)$, then

$$|J(\beta)| = \frac{r!}{\prod_{j=1}^{s_1} \lambda_{1j}^{\mu_{1j}} \cdot \mu_{1j}!} \cdot \frac{n!^2}{d^{(n/d)} \cdot (n/d)!}.$$

Proof: Similar to the proof of Theorem 4.11. ∎

If $m = n = r$, then things are a little more complicated. This is reflected in the following theorem:

Theorem 4.13. Let $\beta \in B(K_{n,n,n})$; suppose that V_i and V_j are swapped and V_{6-i-j} is fixed (setwise). Suppose that $c_1 = \{c_{1j};\ j = 1, \ldots, s_1\}$, $d \mid n$, and $c_2 = \{1\}$. Let $c_{1j} \mid 2d$, and $\lambda_{1j} = 2d / c_{1j}$. Then

$$|J(\beta)| = \frac{3}{\prod_{j=1}^{s_i} \lambda_{ij}^{\mu_{ij}} \cdot \mu_{ij}!} \cdot \frac{n!^3}{d^{(n/d)} \cdot (n/d)!}.$$

Proof: There are three choices for the fixed partite set; the rest is similar to the proof of Theorem 4.11 ∎

Example 4.11

Consider again the graph $K_{1,2,2}$ of Example 4.10. We calculate $f(1,2,2)$ from Theorem 4.7 as follows: First, $\gcd(1,2,2) = 1$, so $g = 1$. Second, $\gcd(2,2) = 2$, while $\gcd(1,2) = 1$; therefore, $c_1 = \{1\}$, $\{2\}$, or $\{1,2\}$, and $c_2 = c_3 = \{1\}$. We consider these three cases.

(1) Suppose $c_1 = \{1\}$; then $\lambda_{11} = \lambda_{21} = \lambda_{31} = 1$. Thus, $\mu_{11} = 1$, and $\mu_{21} = \mu_{31} = 2$. The term is therefore (before using ξ)

$$\frac{\Phi(1,4)^1}{1^1 \cdot 1!} \cdot \frac{\Phi(1,3)^2}{1^2 \cdot 2!} \cdot \frac{\Phi(1,3)^2}{1^2 \cdot 2!} = \frac{3!^1}{1} \cdot \frac{2!^2}{2!} \cdot \frac{2!^2}{2!} = \frac{6}{1} \cdot \frac{4}{2} \cdot \frac{4}{2} = 24.$$

Now $\xi = 1/(1+0+1)! = 1/2$, so the term is actually 12.

(2) Suppose $c_1 = \{2\}$; then $\lambda_{11} = 1$, and $\lambda_{21} = \lambda_{31} = 2$. Thus, $\mu_{11} = \mu_{21} = \mu_{31} = 1$. The term is therefore

$$\frac{\Phi(2,4)^1}{1^1 \cdot 1!} \cdot \frac{\Phi(1,3)^1}{2^1 . 1!} \cdot \frac{\Phi(1,3)^1}{2^1 \cdot 1!} = \frac{2}{1} \cdot \frac{2!}{2} \cdot \frac{2!}{2} = 2,$$

which actually is 1, since $\xi = 1/2$.

(3) Suppose $c_1 = \{1,2\}$; then $c_{11} = 2$, and $c_{12} = 1$. Hence, $\lambda_{11} = 1$, and $\lambda_{12} = 2$—but there is only one vertex in V_1, so this term cannot exist. We conclude that $f(1,2,2) = 12 + 1 = 13$ (as in Example 4.4).

Third, we calculate $g(1,2,2)$. The divisors of two are one and two. We consider these cases separately.

If $d = 1$, $2d = 2$, so $c_1 = \{1\}$, $\{2\}$, or $\{1,2\}$. But λ_{ij} must be a divisor of m as well as $2d$, that is, an

easier way of obtaining c_1 is to consider $\lambda_1 = \{\lambda_{1j}; j = 1, \ldots, s_1\}$, where $\lambda_{1j} \mid \gcd(m, 2d)$. Therefore, $c_1 = \{2\}$ is the only possibility. Thus, $\lambda_{11} = 1$. We conclude that $\mu_{11} = 1$, and $4/2 = 2$. By Theorem 4.9,

$$|F(\beta)| = 2!^2 \cdot \Phi(2,4)^1 = 4 \cdot 2 = 8$$

(as in Table 4.5). By Theorem 4.11,

$$|J(\beta)| = \frac{1!}{1!^1 \cdot 1!} \cdot \frac{2!^2}{1^{2/1} \cdot (2/1)!} = \frac{4}{2} = 2$$

(as in Table 4.5). Thus the value for this term is $8 \cdot 2 / 8 = 2$. [$|\mathrm{Aut}\, K_{1,2,2}| = 8$.]

Suppose $d = 2$; then $c_1 = \{4\}$, by the above argument. Therefore, $\mu_{11} = 2 / 2 = 1$. Theorem 4.9 yields

$$|F(\beta)| = 2!^1 \cdot \Phi(4,4)^1 = 2 \cdot 2 = 4$$

(as in Table 4.5). Theorem 4.11 gives

$$|J(\beta)| = \frac{1!}{1!^1 \cdot 1!} \cdot \frac{2!^2}{2^1 \cdot 1!} = \frac{4}{2} = 2$$

(as in Table 4.5). Therefore, this term has the value $4 \cdot 2 / 8 = 8 / 8 = 1$. We conclude that $g(1,2,2) = 2 + 1 = 3$. Since $1 \neq 2$, $h(1,2,2) = 0$. Therefore, by Theorem 4.1 we conclude

$$|C(K_{1,2,2})| = 13 + 3 + 0 = 16.$$

This is verified by the calculations of Example 4.4.

By using the insight of Example 4.11 we can obtain the formula for $g(m,n,r)$. Surprisingly, this formula is simpler than Theorem 4.2 led us to believe.

Theorem 4.14. Consider the complete tripartite graph $K_{m,n,r}$, and suppose that $m = n$, or $n = r$, or $m = n = r$. Without loss of generality, we will assume n is the order of the partite sets being swapped and m is the other

order ($m = n$ is possible, but not required). Suppose $d \mid n$ and let $\lambda_1 = \{\lambda_{1j};\ j = 1, \ldots, s_1\}$, where $\lambda_{1j} \mid \gcd(m, 2d)$, $\lambda_2 = \{2d\}$, and $\lambda = \lambda_1 \times \lambda_2$; let $\mu = \{\mu_{1j}\}$ be all solutions $\mu_{1j} \geq 1$ that satisfy:

$$\sum_{j=1}^{s_1} \mu_{1j} \cdot \lambda_{1j} = m.$$

Then,

$$g(m,n,n) = \sum_{d \mid n} \sum_{\lambda} \sum_{\mu} \frac{(m+n-1)!^{(n/d)}}{2 \cdot d^{(n/d)} \cdot (n/d)!} \prod_{j=1}^{s_1} \frac{\Phi\left[\frac{2d}{\lambda_{1j}}, 2n\right]^{\mu_{1j}}}{\lambda_{1j}^{\mu_{1j}} \cdot \mu_{1j}!}$$

Proof: We will first suppose that $m \neq n$. By Theorem 4.9, we have

$$|F(\beta)| = (m+n-1)!^{(n/d)} \prod_{j=1}^{s_1} \Phi(c_{1j}, 2n)^{\mu_{1j}},$$

where $c_{1j} = 2d / \lambda_{1j}$. By Theorem 4.11,

$$|J(\beta)| = \frac{m! \cdot n!^2}{d^{(n/d)} \cdot (n/d)!} \prod_{j=1}^{s_1} \frac{1}{\lambda_{1j}^{\mu_{1j}} \cdot \mu_{1j}!}.$$

Substituting into Theorem 4.2 we obtain the result for this case.

Now suppose that $m = n = r$. Then Corollary 4.10 gives

$$|F(\beta)| = (2n-1)!^{(n/d)} \prod_{j=1}^{s_1} \Phi(c_{1j}, 2n)^{\mu_{1j}},$$

where $c_{1j} = 2d / \lambda_{1j}$. Also Theorem 4.13 states

$$|J(\beta)| = \frac{3n!^3}{d^{(n/d)} \cdot (n/d)!} \prod_{j=1}^{s_1} \frac{1}{\lambda_{1j}^{\mu_{1j}} \cdot \mu_{1j}!}.$$

Again upon substituting into Theorem 4.2 we get the result—the six in the denominator cancels with the three in the numerator. ∎

Notice that by combining Theorem 4.7 with Theorem

4.14 we obtain a formula for the number of congruence classes of maps of $K_{m,n,r}$ where at most two of the partite sets have the same order.

Theorem 4.15. The number of congruence classes of maps of $K_{m,n,n}$, where $m \neq n$ is given by

$$|C(K_{m,n,n})| = f(m,n,n) + g(m,n,n),$$

where

$$f(m,n,n) = \frac{1}{2} \sum_{g} \sum_{c} \sum_{\mu} \prod_{i=1}^{3} \prod_{j=1}^{s_i} \frac{\Phi(c_{ij}, N-n_i)^{\mu_{ij}}}{\lambda_{ij}^{\mu_{ij}} \cdot \mu_{ij}!}$$

and,

$$g(m,n,n) = \sum_{d|n} \sum_{\lambda'} \sum_{\mu'} \frac{(m+n-1)!^{(n/d)}}{2 \cdot d^{(n/d)} \cdot (n/d)!} \prod_{j=1}^{s_1} \frac{\Phi\left[\frac{2d}{\lambda'_{1j}}, 2n\right]^{\mu'_{1j}}}{\lambda'^{\mu'_{1j}}_{1j} \cdot \mu'_{1j}!},$$

where primed and unprimed are probably different (compare the conditions from Theorem 4.7 and Theorem 4.14). ∎

In the next section we will derive the formula for $h(m,n,r)$. We will use this to establish the formula for the number of congruence classes of maps of complete tripartite graphs. We will then combine the formulas for $f(m,n,r)$, $g(m,n,r)$, and $h(m,n,r)$ to do this.

4.4 The Final Piece

In this section we derive a formula for $h(m,n,r)$. We know that if the partite sets do not all have the same order then $h(m,n,r) = 0$; therefore, we are concerned with when they all have the same order. As in all the other cases, we do this by determining $|F(\gamma)|$ and $|J(\gamma)|$ for $\gamma \in D(K_{n,n,n})$.

The first thing that we need to notice is that if none of the partite sets are fixed, then the disjoint cycle decomposition has vertices from the three partite sets cycling. That is, if u, v, and w are three consecutive vertices of any cycle of the disjoint cycle decomposition of γ, then they are in different partite sets.

Theorem 4.16. If $\gamma \in D(K_{n,n,n})$ then γ cycles the partite sets as sets.

Proof: Let u, v, and w be arbitrary vertices of $K_{n,n,n}$, and suppose that $\gamma \in D(K_{n,n,n})$ is such that $\gamma(u) = v$, and $\gamma(v) = w$. We first show that u, v, and w must lie in distinct partite sets. Let V(u), V(v), and V(w) denote the partite sets of u, v, and w. Clearly, V(u) cannot be the same as V(v), since then γ fixes V(u)—a contradiction.

Suppose V(u) = V(w); then V(u) maps onto V(v) and V(v) maps onto V(u). We conclude that γ swaps the partite sets V(u) and V(v), and again one of the partite sets is fixed, another contradiction.

Suppose that V(v) = V(w); then γ maps V(v) onto V(v). Yet again, $\gamma \notin D(K_{n,n,n})$.

We conclude that u, v, and w are in different partite sets. A similar argument using v, w, and $\gamma(w)$, shows that they are in different partite sets. Thus, $V(\gamma(w)) = V(u)$. Continuing, we see that the partite sets cycle, as claimed. ∎

We are now in a position to be able to determine the structure type of $\gamma \in D(K_{n,n,n})$ for which $|F(\gamma)| > 0$.

Theorem 4.17. If $\gamma \in D(K_{n,n,n})$, then $|F(\gamma)| > 0$ if and only if γ is uniform.

Proof: Let $v \in V(K_{m,n,r})$ and let $\lambda(v)$ be the length of the cycle containing v in the disjoint cycle decomposition of γ. From Theorem 4.16 we know that $\gamma^{\lambda(v)}$ fixes the adjacent vertices v and $\gamma(v)$. Theorem 2.3 then says that either $|F(\gamma)| = 0$ or $\gamma^{\lambda(v)}$ must be the identity element of Aut $K_{n,n,n}$. Since no property of v was used other than it being a vertex, the conclusion must hold for all vertices v. ∎

We can now count $|J(\gamma)|$ for $\gamma \in D(K_{n,n,n})$ such that $|F(\gamma)| > 0$. This theorem is an extension of Theorem 3.9. Notice that if $\gamma \in D(K_{n,n,n})$ then for all $v \in V(K_{m,n,r})$, $3|\lambda(v)$ (Theorem 4.16) and thus $\lambda(v) = 3d$ for some divisor of n.

Theorem 4.18. Let $\gamma \in D(K_{n,n,n})$ and $d|n$. Suppose that γ is 3d-uniform; then

$$|J(\gamma)| = \frac{2n!^3}{d^{(n/d)}\cdot(n/d)!}.$$

Proof: The proof is an extension of the method used to prove Theorem 3.9. An automorphism in $D(K_{n,n,n})$ is constructed in stages: first we order the partite sets Theorem 4.16 says the partite sets cycle; we need the order of the cycling; one can assume, without loss of generality, that the ordering begins with V_1; there are two ways that this may be done. Second, construct a one-to-one function from V_1 onto the second ordered partite set; there are $n!$ ways to do this. Similarly, there are $n!$ ways to construct a one-to-one function from the second

to the third. This creates n new objects. Theorem 2.6 now applies to any uniform permutation constructed out of these objects. Thus, there are

$$\frac{n!}{d^{(n/d)}\cdot(n/d)!}.$$

ways that this can be done. Combining the above steps, the theorem follows. ∎

We are now ready to evaluate $|F(\gamma)|$. Once we have done this we will be able to calculate $h(m,n,r)$.

Theorem 4.19. Let $\gamma \in D(K_{n,n,n})$, and let $d \mid n$. Suppose that γ is 3d-uniform; then

$$|F(\gamma)| = (2n-1)!^{(n/d)}.$$

Proof: By Theorem 4.17 we know that γ is uniform. By Theorem 4.16 we know that for some integer d, where $d \mid n$, γ is 3d-uniform. There are $3n/3d = n/d$ cycles. Since γ^{3d} is the identity element, $|F_v(\gamma^{\lambda(v)})| = (d(v) - 1)!$ for all vertices v; thus applying Theorem 2.4 gives the result. ∎

Theorem 4.20. The value of $h(m,n,r)$ is given by

$$h(m,n,r) = \begin{cases} \displaystyle\sum_{d \mid n} \frac{(2n-1)!^{(n/d)}}{3\cdot d^{(n/d)}\cdot(n/d)!} & \text{, if } m = n = r, \\ 0 & \text{, otherwise.} \end{cases}$$

Proof: Combine the formulas for $|J(\gamma)|$ from Theorem 4.18 with the formula for $|F(\gamma)|$ from Theorem 4.19. Then substitute them into Theorem 4.2—the result follows by simplifying. ∎

We can now establish a formula for the number of congruence classes of maps of complete tripartite graphs.

Theorem 4.21. The number of congruence classes for the maps of complete tripartite graphs is given by

$$|C(K_{m,n,r})| = f(m,n,r) + g(m,n,r) + h(m,n,r),$$

where

$$f(m,n,r) = \xi \sum_{g} \sum_{c} \sum_{\mu} \prod_{i=1}^{3} \prod_{j=1}^{s_i} \frac{\Phi(c_{ij},\ N-n_i)^{\mu_{ij}}}{\lambda_{ij}^{\mu_{ij}} \cdot \mu_{ij}!},$$

$n_1 = m$, $n_2 = n$, $n_3 = r$, $N = \sum_{i=1}^{3} n_i$, $c = c_1 \times c_2 \times c_3$, for $i,k = 1, 2, 3$, $c_{ij} \mid \gcd(n_k; k \neq i)/g$, $\gcd(c_{ij}, c_{k\ell}) = 1$, $j = 1, \ldots, s_i$, $\ell = 1, \ldots, s_k$, $g \mid \gcd(n_i;\ i = 1, 2, 3)$, $\xi = 1 / (1 + \delta_{m,n} + \delta_{n,r})!$, where δ is the Kronecker function,

$$\lambda_{ij} = \frac{g}{c_{ij}} \prod_{k=1}^{3} \operatorname{lcm}(c_{k\ell};\ \ell = 1, \ldots, s_k),$$

and $\mu = \{\mu_{ij}\}$, where μ_{ij} are positive integers solutions to

$$\sum_{j=1}^{s_i} \mu_{ij} \cdot \lambda_{ij} = n_i \quad (i = 1, 2, 3), \text{ and}$$

$$g(m,n,r) = \begin{cases} \displaystyle\sum_{d \mid n} \sum_{\lambda} \sum_{\mu} (*), \\ 0, \text{ if all orders are distinct,} \end{cases}$$

where $(*)$ is (assuming $n = r$, no restriction on m)

$$\frac{(m+n-1)!^{(n/d)}}{2 \cdot d^{(n/d)} \cdot (n/d)!} \prod_{j=1}^{s_1} \frac{\Phi\left[\frac{2d}{\lambda_{1j}},\ 2n\right]^{\mu_{1j}}}{\lambda_{1j}^{\mu_{1j}} \cdot \mu_{1j}!}$$

where it might be necessary to relabel m, n, and/or r, $\lambda_i = \{\lambda_{ij};\ j = 1, \ldots, s_i\}$, $\lambda_{ij} \mid \gcd(m, 2d)$, $\lambda_{2i} = \{2d\}$, $\lambda = \lambda_1 \times \lambda_2$, and μ_{ij} are positive integers solutions to

$$\sum_{j=1}^{s_i} \mu_{ij} \cdot \lambda_{ij} = m.$$

and

$$h(m,n,r) = \begin{cases} \displaystyle\sum_{d \mid n} \frac{(2n-1)!^{n/d}}{3 \cdot d^{(n/d)}(n/d)!}, & m = n = r \\ 0 & \text{, otherwise.} \end{cases}$$

Proof: See Theorems 4.7, 4.14, and 4.20 for the individual pieces. ∎

Example 4.12

Consider again the graph $K_{2,2,2}$ of Example 4.3. We will calculate $|C(K_{2,2,2})|$ using Theorem 4.21. We start by calculating $f(2,2,2)$.

First, $\gcd(2,2,2) = 2$, so g is either one or two. We consider each case separately. When $g = 1$, then we find $\gcd(2,2) = 2$, and $2 / 1 = 2$. We conclude that $c_i = \{1\}$, $\{2\}$, or $\{1,2\}$ $(i = 1, 2, 3)$. Now $\gcd(c_{ij}, c_{k\ell}) = 1$, when $i \neq k$, so we see that at most one of the c's can have the two in it. Therefore, instead of there being twenty-seven cases to consider, there are only seven. Each of these will be considered.

(1) Suppose that $c_1 = c_2 = c_3 = \{1\}$, then $c_{i1} = 1$ for $i = 1, 2, 3$. We conclude that $\lambda_{i1} = 1$, and $\mu_{i1} = 2$, for all $i = 1, 2, 3$. Theorem 4.21 gives this term as

$$\frac{\Phi(1,4)^2}{1^2 \cdot 2!} \cdot \frac{\Phi(1,4)^2}{1^2 \cdot 2!} \cdot \frac{\Phi(1,4)^2}{1^2 . 2!} = \frac{[(1 \cdot 1^3 \cdot 3!)^2]^3}{(2!)^3} = \frac{46656}{8} = 5832,$$

before multiplying by $\xi = 1/(1+1+1)! = 1/6$. Thus, the

adjusted value is 972.

(2) Suppose that $c_1 = \{2\}$, but $c_2 = c_3 = \{1\}$. Then we have $c_{11} = 2$, $c_{21} = c_{31} = 1$; so $\lambda_{11} = 1$, and $\lambda_{21} = \lambda_{31} = 2$. We conclude that $\mu_{11} = 2$, $\mu_{21} = \mu_{31} = 1$. Theorem 4.21 gives the value of this term to be

$$\frac{\Phi(2,4)^2}{1^2\cdot 2!}\cdot\frac{\Phi(1,4)^1}{2^1\cdot 1!}\cdot\frac{\Phi(1,4)^1}{2^1\cdot 1!} = \frac{(1\cdot 2^1\cdot 1!)^2}{2}\cdot\frac{(1\cdot 1^3\cdot 3!)^2}{2^2}$$
$$= \frac{4}{2}\cdot\frac{36}{4} = 18.$$

After multiplying by $\frac{1}{6}$, this becomes three (3). By taking advantage of the symmetry of $K_{2,2,2}$, we see that the same value would be obtained for $c_2 = \{2\}$, $c_1 = c_3 = \{1\}$, and $c_3 = \{2\}$, $c_1 = c_2 = \{1\}$. Therefore, we multiply this value by three, getting nine as its total contribution; hence we have done cases 3 and 4 as well.

(5) Suppose that $c_1 = \{1,2\}$, while $c_2 = c_3 = \{1\}$. Then $c_{11} = 2$, $c_{12} = c_{21} = c_{31} = 1$. We calculate that $\lambda_{11} = 1$, $\lambda_{12} = 2$—but this cannot be. There are only two vertices in V_1; we have already accounted for three! We conclude that this case cannot happen. Again employing symmetry, we find that $c_2 = \{1,2\}$, $c_1 = c_3 = \{1\}$, and $c_3 = \{1,2\}$, $c_1 = c_2 = \{1\}$, also cannot happen (cases 6 and 7).

Therefore, the total contribution to $f(2,2,2)$, when $g = 1$, is $972 + 9 = 981$.

Suppose that $g = 2$; we still have $\gcd(2,2) = 2$, but $2 / 2 = 1$, so $c_1 = c_2 = c_3 = 1$. We conclude that $\lambda_{11} = \lambda_{21} = \lambda_{31} = 2$, and thus, $\mu_{11} = \mu_{21} = \mu_{31} = 1$. Substituting these values into the formula, we get

$$\frac{\Phi(1,4)^1}{2^1\cdot 1!}\cdot\frac{\Phi(1,4)^1}{2^1\cdot 1!}\cdot\frac{\Phi(1,4)^1}{2^1\cdot 1!} = \frac{[(1\cdot 1^3\cdot 3!)]^3}{2^3} = \frac{216}{8} = 27.$$

After adjusting by multiplying the value by $\frac{1}{6}$ we get 4.5.

Therefore,

$$f(2,2,2) = 981 + 4.5 = 985.5.$$

This agrees with the value obtained in Example 4.3.

We now evaluate $g(2,2,2)$. We first note that $m = n = 2$ in the formula. The divisors of two are one and two; therefore, $d = 1, 2$; we consider these cases.

(1) If $d = 1$, then $2d = 2\cdot 1 = 2$; $\gcd(2,2) = 2$, so $\lambda_1 = \{1\}$, $\{2\}$, or $\{1,2\}$.

(a) If $\lambda_1 = \{1\}$, then $\lambda_{11} = 1$, and $c_{11} = 2/1 = 2$. Thus, $\mu_{11} = 2$. The formula for $g(2,2,2)$ gives for this case

$$\frac{(2+2-1)!^{2/1}}{2\cdot 1^2\cdot 2!}\ \frac{\Phi(2,4)^2}{1^2\cdot 2!} = \frac{36}{4}\cdot\frac{(1\cdot 2^1\cdot 1!)^2}{2} = 9\cdot 2 = 18.$$

(b) If $\lambda_1 = \{2\}$, then $\lambda_{11} = 2$, and $c_{11} = 2 / 2 = 1$. We conclude that $\mu_{11} = 1$. The formula for $g(2,2,2)$ gives for this case

$$\frac{(2+2-1)!^{2/1}}{2\cdot 1^2\cdot 2!}\ \frac{\Phi(1,4)^1}{2^1\cdot 1!} = \frac{36}{4}\cdot\frac{(1\cdot 1^3\cdot 3!)^1}{2} = 9\cdot 3 = 27$$

(c) If $\lambda_1 = \{1,2\}$, then $\lambda_{11} = 1$, $\lambda_{12} = 2$—but this cannot happen. There are only two vertices in any one partite set; we have accounted for at least three! We conclude that this case cannot happen. Therefore, the total contribution to $g(2,2,2)$, when $d = 1$, is $18 + 27 = 45$.

(2) If $d = 2$, then $2d = 4$. Now $\gcd(2,4) = 2$, so we have $\lambda_1 = \{1\}$, $\{2\}$, or $\{1,2\}$. The same contradiction as before will occur when $\lambda_1 = \{1,2\}$; therefore, this case will not be presented.

(a) Suppose $\lambda_1 = \{1\}$, then $\lambda_{11} = 1$, and $c_{11} = 4 / 1 = 4$. We conclude that $\mu_{11} = 2$. Substituting these values into the formula for $g(2,2,2)$ we get

$$\frac{(2+2-1)!^{2/2}}{2\cdot 2^1\cdot 1!}\ \frac{\Phi(4,4)^2}{1^2\cdot 2!} = \frac{6}{4}\cdot\frac{(2\cdot 4^0\cdot 0!)^2}{2} = 1.5\cdot 2 = 3.$$

(b) If $\lambda_1 = \{2\}$, then $\lambda_{11} = 2$, and $c_{11} = 4 / 2 = 2$. We

conclude that $\mu_{11} = 1$. The formula for g(2,2,2) gives in this case

$$\frac{(2+2-1)!^{2/2}}{2\cdot 2^{1}\cdot 1!}\cdot\frac{\Phi(2,4)^{1}}{2^{1}\cdot 1!} = \frac{6}{4}\cdot\left(\frac{1\cdot 2^{1}\cdot 1!}{2}\right)^{1} = 1.5\cdot 1 = 1.5.$$

Therefore, the contribution to g(2,2,2), when d = 2, is 3 + 1.5 = 4.5. We conclude that g(2,2,2) = 45 + 4.5 = 49.5, which agrees with that calculated in Example 4.3.

We now calculate h(2,2,2). The divisors of two are one and two; we consider these cases separately. Suppose that d = 1; then the formula for h(2,2,2) gives

$$\frac{(2\cdot 2-1)!^{2/1}}{3\cdot 1^{2}\cdot 2!} = \frac{36}{6} = 6.$$

If d = 2, the the formula for h(2,2,2) gives

$$\frac{(2\cdot 2-1)!^{2/2}}{3\cdot 2^{1}\cdot 1!} = \frac{6}{6} = 1.$$

Therefore, h(2,2,2) = 6 + 1 = 7, which agrees with the calculation of Example 4.3. We conclude that $|C(K_{2,2,2})| = 985.5 + 49.5 + 7 = 1042$, as we had calculated in Example 4.3.

In the next chapter, we will derive the formula for the number of congruence classes of maps of complete n-partite graphs. In addition, we will show that the formula for complete n-partite graphs generalizes the formulas given in Chapters II - IV.

The final chapter will be concerned with asymptotic formulas for the number of congruence classes. We will find that these formulas have significance to the new twig on the branch of topological graph theory—random topological graph theory.

CHAPTER V

THE MAIN EVENT: COMPLETE N-PARTITE GRAPHS

5.1 Introduction

In this, the penultimate chapter, we will present the formula for the number of congruence classes of maps of complete n-partite graphs. We will begin by giving the final version of the definition of a compatible set of permutations. Next, we will express Theorem 1.3 as we have done in both the bipartite and tripartite cases. We will calculate the general expressions for each of the three cases into which the problem will be divided. In order to be able to count the congruence classes for the maps of complete n-partite graphs, we will need to generalize the counting argument given in the proofs of Theorems 3.9 and 4.18. Once we have developed the formula, we will end by showing how it generalizes the formulas of the complete, complete bipartite, and complete tripartite cases.

Looking at the definition of a compatible set of permutations, given in Chapter IV, we see how this definition should be extended to more than three permutations. We do this now.

Let A_1, A_2, ..., A_n be (not necessarily distinct) sets and suppose that α_i is a permutation on A_i ($i = 1$, 2, ..., n). Furthermore, suppose that α_i has s_i distinct

cycle lengths λ_{ij} in its disjoint cycle decomposition, where: $\lambda_{i1} < \ldots < \lambda_{is(i)}$. Then $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a (multi-) set of compatible permutations if and only if for each λ_{ij}, there exists a positive integer c_{ij} such that for $k = 1, \ldots, n$, where $k \neq i$, $\alpha_k^{\lambda_{ij}}$ is c_{ij}-uniform.

We now turn our attention to complete n-partite graphs. Recall that a complete n-partite graph has its vertex set partitioned into n partite sets, $V_1, V_2, \ldots, V_n$; its edge set consists of all possible unordered pairs of vertices from different partite sets. That is,

$$V = \bigcup_{i=1}^{n} V_i;\ V_i \neq \emptyset\ (\ i = 1, 2, \ldots, n\ )$$

$$V_i \cap V_j = \emptyset\ (\ 1 \leq i < j \leq n\ ),\ \text{and}$$

$$E = \{\ \{u,v\}\ |\ u \in V_i,\ v \in V_j\ (1 \leq i < j \leq n)\ \}.$$

Let $|V_i| = m_i$ (for $i = 1, 2, \ldots, n$); then we write: $G = K(m_1,m_2,\ldots,m_n)$. If $m_1 \leq m_2 \leq \ldots \leq m_n$, then we say that G is in standard form.

Just as in the tripartite case, an automorphism can be of one of three types—it can fix all of the partite sets, some (but not all) of the partite sets, or none of the partite sets (setwise). We let $A[K(m_1,m_2,\ldots,m_n)]$, $B[K(m_1,m_2,\ldots,m_n)]$, and $D[K(m_1,m_2,\ldots,m_n)]$ denote, respectively, the sets of these three types of automorphisms.

Theorem 5.1. The number of congruence classes of maps of $K(m_1,m_2,\ldots,m_n)$ is given by

$$|C[K(m_1,m_2,\ldots,m_n)]| = \frac{1}{|\mathrm{Aut}\ K(m_1,\ldots,m_n)|} \left\{ \sum_{\alpha \in A[K(m_1,\ldots,m_n)]} |F(\alpha)| \right.$$

$$+ \sum_{\beta \in B[K(m_1,\ldots,m_n)]} |F(\beta)|$$

$$\left. + \sum_{\gamma \in D[K(m_1,\ldots,m_n)]} |F(\gamma)| \right\}.$$

Proof: The three cases are mutually exclusive and exhaustive; thus, this is an obvious way to rewrite Theorem 1.3. ∎

Let $Jx[K(m_1,\ldots m_n)]$ be the set of structure classes over $x[K(m_1,\ldots,m_n)]$, where x = A, B, or D. Then the sums in Theorem 5.1 can be rewritten to be taken over the structure classes (as has previously been done).

Theorem 5.2. The number of congruence classes of maps of $K(m_1,m_2,\ldots,m_n)$ is given by

$$|C[K(m_1,m_2,\ldots,m_n)]| = \frac{1}{|\mathrm{Aut}\ K(m_1,\ldots,m_n)|} \left\{ \sum_{J(\alpha) \in JA[K(m_1,\ldots,m_n)]} |F(\alpha)| \cdot |J(\alpha)| \right.$$

$$+ \sum_{J(\beta) \in JB[K(m_1,\ldots,m_n)]} |F(\alpha)| \cdot |J(\alpha)|$$

$$\left. + \sum_{J(\gamma) \in JD[K(m_1,\ldots,m_n)]} |F(\alpha)| \cdot |J(\alpha)| \right\}.$$

∎

As we have done for both the bipartite and tripartite cases, we will simplify the notation by using functions. For simplicity we will write

$$f(m_1,\ldots,m_n) = \frac{1}{|\mathrm{Aut}\ K(m_1,\ldots,m_n)|} \sum_{J(\alpha) \in JA[K(m_1,\ldots,m_n)]} |F(\alpha)| \cdot |J(\alpha)|,$$

$$g(m_1,\ldots,m_n) = \frac{1}{|\mathrm{Aut}\ K(m_1,\ldots,m_n)|} \sum_{J(\beta)\ \in\ JB[K(m_1,\ldots,m_n)]} |F(\beta)|\cdot|J(\beta)|,$$

$$h(m_1,\ldots,m_n) = \frac{1}{|\mathrm{Aut}\ K(m_1,\ldots,m_n)|} \sum_{J(\gamma)\ \in\ JD[K(m_1,\ldots,m_n)]} |F(\gamma)|\cdot|J(\gamma)|.$$

Example 5.1

Consider the graph $K_{1,1,2,2}$, with $V = \{0,1,2,3,4,5\}$, $V_1 = \{0\}$, $V_2 = \{1\}$, $V_3 = \{2,3\}$, and $V_4 = \{4,5\}$. There are $|R(K_{1,1,2,2})| = 4!^2 \cdot 3!^4 = 746{,}496$ rotation schemes; these are classified in Tables 5.1 - 5.3. Note that the order of the automorphism group is sixteen.

Table 5.1

Classifying Rotations of $K_{1,1,2,2}$ Using $\alpha \in A(K_{1,1,2,2})$

α	$\lvert J(\alpha)\rvert$	$\lvert F(\alpha)\rvert$
(0)(1)(2)(3)(4)(5)	1	746496
(0)(1)(2)(3)(45)	2	0
(0)(1)(23)(45)	1	0

First we calculate $f(1,1,2,2)$; from Table 5.1 we get $f(1,1,2,2) = (1\cdot 746496 + 2\cdot 0 + 1\cdot 0) / 16 = 46656$.

Table 5.2

Classifying Rotations of $K_{1,1,2,2}$ Using $\beta \in B(K_{1,1,2,2})$

β	$\lvert J(\beta)\rvert$	$\lvert F(\beta)\rvert$
(0)(1)(24)(35)	2	0
(0)(1)(2435)	2	0
(01)(2)(3)(4)(5)	1	0
(01)(2)(3)(45)	2	576
(01)(23)(45)	1	864

Second we find $g(1,1,2,2)$. From Table 5.2 we find $g(1,1,2,2) = (2\cdot 0 + 2\cdot 0 + 1\cdot 0 + 2\cdot 576 + 1\cdot 864) / 16 = 126$.

Table 5.3

Classifying Rotations of $K_{1,1,2,2}$ Using $\gamma \in D(K_{1,1,2,2})$

γ	$\lvert J(\gamma)\rvert$	$\lvert F(\gamma)\rvert$
(01)(24)(35)	2	864
(01)(2435)	2	0

Finally, we compute $h(1,1,2,2)$; Table 5.3 yields

$$h(1,1,2,2) = (2\cdot 864 + 2\cdot 0) / 16 = 108.$$

Therefore, $|C(K_{1,1,2,2})| = 46656 + 126 + 108 = 46890$. That is, there are 46,890 congruence classes of maps of $K_{1,1,2,2}$.

If all of the partite sets have different orders, then $\mathrm{Aut}\ K(m_1,\ldots,m_n) \cong S_{m(1)} \oplus \cdots \oplus S_{m(n)}$. If all of the partite sets have the same order, say m, then the graph is denoted $K_{n(m)}$ and $\mathrm{Aut}\ K_{n(m)} \cong S_n[S_m]$. The difficult case occurs when some, but not all, of the partite sets have the same order. Suppose that $K(m_1,\ldots,m_n)$ has q distinct orders of partite sets; suppose there are f_1 of order p_1, f_2 of order p_2, $\ldots$, and f_q of order p_q. Then

$$\mathrm{Aut}\ K(m_1,\ldots,m_n) \cong S_{f(1)}[S_{p(1)}] \oplus \cdots \oplus S_{f(q)}[S_{p(q)}],$$

where we note that $S_1[S_m] \cong S_m$.

In the next section we will find the number of congruence classes of maps of $K(m_1,\ldots,m_n)$ when all of the partite sets have different orders. We will also develop a formula for $f(m_1,\ldots,m_n)$ in general. In the sections that follow we will calculate the number of congruence

classes for the maps of $K_{n(m)}$ (which will help us calculate $h(m_1,\ldots,m_n)$ in general) and finally the general formula for $g(m_1,\ldots,m_n)$.

5.2 Calculating $f(m_1,\ldots,m_n)$

In this section we will find a general formula for $f(m_1,\ldots,m_n)$. This will allow us to calculate the number of congruence classes for $K(m_1,\ldots,m_n)$ when all of the partite sets have different orders. From Theorem 5.2 we see that we need to calculate $|F(\alpha)|$ and $|J(\alpha)|$ for those $\alpha \in A[K(m_1,\ldots,m_n)]$ such that $F(\alpha) \neq \emptyset$. Thus we begin by determining the structure of those α which contribute to $f(m_1,\ldots,m_n)$.

Theorem 5.3. Let $\alpha \in A[K(m_1,\ldots,m_n)]$, $\alpha = \alpha_1\alpha_2\ldots\alpha_n$, where α_i permutes the vertices in V_i $(i = 1, \ldots, n)$. Then $F(\alpha) \neq \emptyset$ if and only if $\{\alpha_1,\ldots,\alpha_n\}$ is a compatible set of permutations.

Proof: The proof is almost identical to the proof in the tripartite case. Let $\{\alpha_1,\ldots,\alpha_n\}$ be a compatible set of permutations on $V_1, \ldots, V_n$ [the partite sets of the complete n-partite graph $K(m_1,\ldots,m_n)$] and suppose that $\alpha = \alpha_1\alpha_2\ldots\alpha_n$. Arbitrarily take $i \in \{1,\ldots,n\}$ and suppose that $u \in V_i$. If $\lambda(u)$ is the length of the cycle of α_i containing u, then (by definition of a compatible set of permutations) there is a single positive integer c such that $\alpha_k^{\lambda(u)}$ is c-uniform, for all $k \in \{1,\ldots,n\}$ such that $i \neq k$. We conclude that $\forall v \in V$, $\alpha^{\lambda(v)}|_{N(v)}$ is uniform. By Theorem 2.5, $|F_v(\alpha^{\lambda(v)})| > 0$; by Theorem 2.4, $|F(\alpha)| > 0$ (so $F(\alpha) \neq \emptyset$).

Now suppose that $F(\alpha) \neq \emptyset$, and that $\alpha = \alpha_1\alpha_2\ldots\alpha_n$, where α_i is a permutation on V_i $(i = 1,\ldots,n)$. By Theorem 2.4, we conclude that $\forall v \in V$, $|F_v(\alpha^{\lambda(v)})| > 0$; by Theorem 2.5, $\forall v \in V$, $\alpha^{\lambda(v)}|_{N(v)}$ is uniform. Suppose that $v \in V_i$; then $N(v) = \bigcup_{j=1, j\neq i}^{n} V_j$. Also, the partite sets are disjoint from each other. Therefore, we conclude that for each cycle length λ in α_i, there exists a positive integer c such that α_j^{λ} is c-uniform (when $j \neq i$). Thus, $\{\alpha_1,\ldots,\alpha_n\}$ is a compatible set of permutations. ∎

Example 5.2

Consider the graph $K_{1,1,2,2}$ of Example 5.1. Recall that $V_1 = \{0\}$, $V_2 = \{1\}$, $V_3 = \{2,3\}$, $V_4 = \{4,5\}$. Suppose that $\alpha_1 = (0)$, $\alpha_2 = (1)$, $\alpha_3 = (23)$, and $\alpha_4 = (4)(5)$. We see that α_2^1 is 1-uniform, while α_3^1 is 2-uniform. Therefore, $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is not a compatible set of permutations. This conclusion is confirmed by Table 5.1 where we see that $|F(\alpha)| = 0$ when $\alpha = (0)(1)(23)(4)(5)$.

Just as Theorem 4.3 generalized to Theorem 5.3, we observe that Theorem 4.4 also generalizes. This is the first step in determining the structure of a compatible set of permutations.

Theorem 5.4. Let $\{\alpha_1,\ldots,\alpha_n\}$ be a compatible set of permutations. Suppose that α_i has s_i distinct cycle lengths λ_{ij} in its disjoint cycle decomposition and that the λ_{ij} satisfy: $\lambda_{i1} < \ldots < \lambda_{in}$ $(i = 1, \ldots, n)$. Furthermore, suppose that for $k = 1, \ldots, n$, where $k \neq i$, $\alpha_k^{\lambda_{ij}}$ is c_{ij}-uniform. Then $\forall i, k = 1, \ldots, n$, where $i \neq k$, $j = 1$,

..., s_i, and $\ell = 1, \ldots, s_k$, we have, $c_{ij} \mid \lambda_{k\ell}$ and $\gcd(c_{ij}, c_{k\ell}) = 1$.

Proof: Obviously, $c_{ij} \mid \lambda_{k\ell}$, for $\alpha_k^{\lambda_{ij}}$ is c_{ij}-uniform and all of the cycles are disjoint in the cycle decomposition of α_k.

Suppose that $\gcd(c_{ij}, c_{k\ell}) = h$; because we know that $\alpha_k^{\lambda_{ij}}$ is c_{ij}-uniform, by Corollary 2.1a, we conclude that $c_{ij} = \lambda_{k\ell} / \gcd(\lambda_{ij}, \lambda_{k\ell})$. Therefore, either $h = 1$, or h appears as a factor of $\lambda_{k\ell}$ one more time than it appears as a factor of λ_{ij}. By a similar argument, we see that $h = 1$, or is a factor of λ_{ij} one more time than it is a factor of $\lambda_{k\ell}$. We conclude that $h = 1$. ∎

We now see that the same conditions that were in the bipartite and tripartite cases also exist in the general case. Therefore, it is no surprise that Theorems 3.5 and 4.5 have an extension to the general case.

Theorem 5.5. Let α_i be a permutation on a set A_i $(i = 1, \ldots, n)$; suppose that $|A_i| = m_i$, and that $\{\alpha_1, \ldots, \alpha_n\}$ is a compatible set of permutations. Further, suppose that α_i has s_i distinct cycle lengths λ_{ij} in its disjoint cycle decomposition, where $\lambda_{i1} < \ldots < \lambda_{is(i)}$. Suppose that for $k = 1, \ldots, n$, where $k \neq i$, $\alpha_k^{\lambda_{ij}}$ is c_{ij}-uniform. Then for some positive integer g, where $g \mid \gcd(m_1, \ldots, m_n)$, we have (for $i = 1, \ldots, n$, and $j = 1, \ldots, s_i$)

$$\lambda_{ij} = \frac{g}{c_{ij}} \prod_{k=1}^{n} \operatorname{lcm}(c_{k\ell};\ \ell = 1, \ldots, s_k).$$

Proof: By Theorem 5.4, if $k \neq i$, then for each $\ell = 1, \ldots, s_k$, $c_{k\ell} \mid \lambda_{ij}$ $(i,k = 1, \ldots, n$, and $j = 1, \ldots, s_i)$.

Therefore, $\operatorname{lcm}(c_{k\ell};\ \ell = 1, \ldots, s_k) \mid \lambda_{ij}$, when $k \neq i$, and we conclude that there exists a positive integer a_{ij} such that

$$\lambda_{ij} = a_{ij} \prod_{\substack{k=1 \\ k \neq i}}^{n} \operatorname{lcm}(c_{k\ell};\ \ell = 1, \ldots, s_k),$$

for each $i = 1, \ldots, n$, and $j = 1, \ldots, s_i$.

Consider $\lambda_{ij} / \lambda_{k\ell}$.

$$\frac{\lambda_{ij}}{\lambda_{k\ell}} = \frac{a_{ij} \prod_{p\neq i} \operatorname{lcm}(c_{pr};\ r = 1, \ldots, s_p)}{a_{k\ell} \prod_{q\neq k} \operatorname{lcm}(c_{qt};\ t = 1, \ldots, s_q)}$$

$$= \frac{a_{ij} \operatorname{lcm}(c_{kr};\ r = 1, \ldots, s_k)}{a_{k\ell} \operatorname{lcm}(c_{it};\ t = 1, \ldots, s_i)}.$$

But, by Corollary 2.1a,

$$c_{ij} = \lambda_{k\ell} / \gcd(\lambda_{ij}, \lambda_{k\ell}), \text{ and}$$

$$c_{k\ell} = \lambda_{ij} / \gcd(\lambda_{ij}, \lambda_{k\ell}).$$

Thus, $\lambda_{ij} / \lambda_{k\ell} = c_{k\ell} / c_{ij}$.

Therefore,

$$a_{ij} \frac{\operatorname{lcm}(c_{kr};\ r = 1, \ldots, s_k)}{c_{k\ell}}$$

$$= a_{k\ell} \frac{\operatorname{lcm}(c_{it};\ t = 1, \ldots, s_i)}{c_{ij}}.$$

Now, $\gcd(c_{kr}, c_{it}) = 1$, hence the same is true for their least common multiples. We conclude that there is a positive integer g such that

$$a_{ij} = g \cdot \frac{\operatorname{lcm}(c_{it};\ t = 1, \ldots, s_i)}{c_{ij}}, \text{ and}$$

$$a_{k\ell} = g \cdot \frac{\operatorname{lcm}(c_{kr};\ r = 1, \ldots, s_k)}{c_{k\ell}}.$$

By varying i, j, k, and ℓ, we find that the value g

is invariant over all possible choices. Substituting the formula for a_{ij} into the formula for λ_{ij} we obtain

$$\lambda_{ij} = \frac{g}{c_{ij}} \prod_{k=1}^{n} \mathrm{lcm}(c_{k\ell};\ \ell = 1, \ldots, s_k).$$

Suppose that for $i = 1, \ldots, n$, and $j = 1, \ldots, s_i$, there are exactly μ_{ij} cycles of length λ_{ij}. Then,

$$\sum_{j=1}^{s_i} \mu_{ij}\lambda_{ij} = m_i \quad (i = 1, \ldots, n),$$

so $g \mid m_i$ $(i = 1, \ldots, n)$; hence $g \mid \gcd(m_1, \ldots, m_n)$ as claimed. ∎

Theorem 5.5 shows that if $\{\alpha_1, \ldots, \alpha_n\}$ is a compatible set of permutations, and if α_i has μ_{ij} cycles of length λ_{ij} $(i = 1, \ldots, n;\ j = 1, \ldots, s_i$, and where α_i permutes a set of order $m_i)$, then if for $k = 1, \ldots, n$, where $k \neq i$, $\alpha_k^{\lambda_{ij}}$ is c_{ij}-uniform,

$$\lambda_{ij} = \frac{g}{c_{ij}} \prod_{k=1}^{n} \mathrm{lcm}(c_{k\ell};\ \ell = 1, \ldots, s_k),$$

where $g \mid \gcd(m_1, \ldots, m_n)$, and where $c_{ij} \mid \gcd(n_k / g;\ k = 1, \ldots, n;\ k \neq i)$.

Let c_i be any set of divisors of $\gcd(n_k;\ k = 1, \ldots, n;\ k \neq i)$; suppose that $c_i = \{c_{ij};\ j = 1, \ldots, s_i\}$ and if $k \neq i$, $\gcd(c_{ij}, c_{k\ell}) = 1$ (for all $i, k = 1, \ldots, n$; $j = 1, \ldots, s_i$; and $\ell = 1, \ldots, s_k$). Let g be a positive integer such that $g \mid \gcd(m_1, \ldots, m_n)$ and define λ_{ij} as above. If there exist integers $\mu_{ij} \geq 1$ such that

$$\sum_{j=1}^{s_i} \mu_{ij}\lambda_{ij} = m_i \quad (i = 1, \ldots, n)$$

then it is not hard to verify the existence of a set of

compatible permutations $\{\alpha_1, \ldots, \alpha_n\}$, where α_i is a permutation on a set A_i, where $|A_i| = m_i$, and α_i has exactly μ_{ij} cycles of length λ_{ij} ($i = 1, \ldots, n$; $j = 1, \ldots, s_i$). Therefore, we again get a strategy for picking the set of compatible permutations. Using this strategy, and Theorems 2.3 - 2.5, we are ready to determine a formula for $f(m_1, \ldots, m_n)$. We begin by determining $|F(\alpha)|$ and $|J(\alpha)|$ for a particular choice of g and c_i ($i = 1, \ldots, n$).

Theorem 5.6. Let g be a positive integer, where g is a divisor of $\gcd(m_1, \ldots, m_n)$, and suppose that for $i = 1, \ldots, n$, $c_i = \{c_{ij};\ j = 1, \ldots, s_i\}$, where $c_{ij} \mid \gcd(n_k / g;\ k = 1, \ldots, n;\ k \neq i)$. Further, suppose that for $i = 1, \ldots, n$, $k = 1, \ldots, n$, where $k \neq i$, $j = 1, \ldots, s_i$, and $\ell = 1, \ldots, s_k$, $\gcd(c_{ij}, c_{k\ell}) = 1$, and define

$$\lambda_{ij} = \frac{g}{c_{ij}} \prod_{k=1}^{n} \operatorname{lcm}(c_{k\ell};\ \ell = 1, \ldots, s_k).$$

Then for $\{\mu_{ij}\}$ a solution of

$$\sum_{j=1}^{s_i} \mu_{ij}\lambda_{ij} = m_i \quad (i = 1, \ldots, n),$$

$M = m_1 + \cdots + m_n$, and α_i a permutation on V_i having exactly μ_{ij} cycles of length λ_{ij}, and $\alpha = \alpha_1 \ldots \alpha_n$, we have

$$|F(\alpha)| = \prod_{i=1}^{n} \prod_{j=1}^{s_i} \Phi(c_{ij}, M - m_i)^{\mu_{ij}}$$

where $\Phi(c, m) = \varphi(c) \cdot c^{(m/c - 1)} \cdot (m/c - 1)!$, and φ is the Euler function. Also,

$$|J(\alpha)| = \prod_{i=1}^{n} \frac{m_i!}{\prod_{j=1}^{s_i} \lambda_{ij}^{\mu_{ij}} \cdot \mu_{ij}!}.$$

Proof: By Theorem 2.4 $|F(\alpha)| = \prod_{v \in S} |F_v(\alpha^{\lambda(v)})|$, where S is a complete system of orbit representatives. Consider $v \in V_i$ and suppose that v appears in a cycle of length λ_{ij}. Now $|N(v)| = M - m_i$, where M is defined as above, so that $|F_v(\alpha^{\lambda(v)})| = \Phi(c_{ij}, M - m_i)$; it is easy to see by the form for λ_{ij} that $\alpha^{\lambda(v)}|_{N(v)}$ is c_{ij}-uniform. There are μ_{ij} cycles of length λ_{ij}, and each of these cycles is independent of the others; thus we see that $|F(\alpha)|$ has the above form.

Each of the α_i is independent of all others as the partite sets are disjoint and α_i is a permutation on V_i ($i = 1, \ldots, n$). For each of the cycle lengths λ_{ij}, there are μ_{ij} such cycles in α_i ($i = 1, \ldots, n$; $j = 1, \ldots, s_i$); therefore, by Theorem 2.6 (and letting $J(\alpha_i)$ denote the structure class of α_i as a permutation of V_i), we find that

$$|J(\alpha_i)| = \frac{m_i!}{\prod_{j=1}^{s_i} \lambda_{ij}^{\mu_{ij}} \cdot \mu_{ij}!}.$$

The independence of the choices gives $|J(\alpha)|$ as claimed above. ∎

Example 5.3

Consider the graph $K_{1,2,3,4}$; $\gcd(1,2,3,4) = 1$, so we see that $g = 1$. Also, $\gcd(2,3,4) = 1$, so $c_1 = \{1\}$; $\gcd(1,3,4) = 1$, so $c_2 = \{1\}$; $\gcd(1,2,4) = 1$, so $c_3 = \{1\}$; and $\gcd(1,2,3) = 1$, so $c_4 = \{1\}$. We conclude that only the identity element of Aut $K_{1,2,3,4}$ contributes to $f(1,2,3,4)$ as $c_{11} = c_{21} = c_{31} = c_{41} = 1$, $\lambda_{11} = \lambda_{21} = \lambda_{31} = \lambda_{41} = 1 \cdot (1 \cdot 1 \cdot 1 \cdot 1 \;/\; 1) = 1$, and $\mu_{11} = 1$, $\mu_{21} = 2$, $\mu_{31} = 3$, and

$\mu_{41} = 4$. Now $|R(K_{1,2,3,4})| = 8!\cdot7!^2\cdot6!^3\cdot5!^4$ (which agrees with the formula for $|F(\alpha)|$ given in Theorem 5.6), and there is only one identity element (which agrees with the formula for $|J(\alpha)|$ given in Theorem 5.6). Since we know $|\mathrm{Aut}\ K_{1,2,3,4}| = 1!\cdot2!\cdot3!\cdot4! = 288$, we conclude

$$f(1,2,3,4) = 79269126001246863360000000000 \cdot 1 / 288$$
$$= 275{,}240{,}020{,}837{,}662{,}720{,}000{,}000{,}000.$$

All of the partite sets have different order, so

$$g(1,2,3,4) = h(1,2,3,4) = 0.$$

Hence, the number given above is the number of congruence classes of maps of $K_{1,2,3,4}$.

The strategy given in and before Theorem 5.6 implies that the sum over the structure classes is equivalent to a triple sum—the first sum is over the divisors of $\gcd(m_1,\ldots,m_n)$, the second sum is over the cartesian product of all sets of divisors of the greatest common divisor of the orders of the neighboring partite sets (that is, the first of these is all possible divisors of the greatest common divisor of the orders of all but the first partite set, etc.), and the third is over all solutions $\{\mu_{ij}\}$ of the equations given in Theorem 5.6 (with the λ_{ij} as given in the theorem).

Theorem 5.7. Let $g \mid \gcd(m_1,\ldots,m_n)$, $c_i = \{c_{ij};\ j = 1, \ldots, s_i\}$ (where $c_{ij} \mid \gcd(n_k;\ k = 1, \ldots, n;\ k \neq i)$), and suppose $\gcd(c_{ij},c_{k\ell}) = 1$ ($i, k = 1, \ldots, n$; $j = 1, \ldots, s_i$; $\ell = 1, \ldots, s_k$, where $k \neq i$). Let $c = c_1 \times \ldots \times c_n$ and define

$$\lambda_{ij} = \frac{g}{c_{ij}} \prod_{k=1}^{n} \mathrm{lcm}(c_{k\ell};\ \ell = 1, \ldots, s_k),$$

and let μ be the set of all solutions $\{\mu_{ij}\}$ satisfying

$$\mu_{ij} \geq 1; \sum_{j=1}^{s_i} \mu_{ij}\lambda_{ij} = m_i \ (i = 1, \ldots, n;\ j = 1, \ldots, s_i).$$

Then,

$$f(m_1,\ldots,m_n) = \$ \sum_g \sum_c \sum_\mu \prod_{i=1}^{n} \prod_{j=1}^{s_i} \frac{\Psi(c_{ij}, M - m_i)^{\mu_{ij}}}{\lambda_{ij}^{\mu_{ij}} \cdot \mu_{ij}!},$$

where $M = \sum_{i=1}^{n} m_i$, $\Psi(c,m) = \varphi(c)\cdot c^{(m/c\ -1)}\cdot(m/c\ -1)!$, and φ is the Euler function. Also, $\$$ is such that it adjusts for the size of the automorphism group. Specifically, if there are q distinct orders of partite sets: f_1 of order p_1, f_2 of order p_2, ..., f_q of order p_q, then we have $\$ = 1 / \prod_{i=1}^{q} f_i!$ (note: $\sum_{i=1}^{q} f_i = n$).

Proof: If $K(m_1,\ldots,m_n)$ has f_1 partite sets of order p_1, f_2 of order p_2, ..., f_q of order p_q, then the automorphism group is Aut $K(m_1,\ldots,m_n) \cong S_{f(1)}[S_{p(1)}] \oplus S_{f(2)}[S_{p(2)}] \oplus \cdots \oplus S_{f(q)}[S_{p(q)}]$. Therefore, we conclude that $|\text{Aut } K(m_1,\ldots,m_n)| = \prod_{i=1}^{q} p_i!^{f_i} \cdot f_i! = \frac{1}{\$} \cdot \prod_{i=1}^{n} m_i!$. We now see that by dividing by the order of the automorphism group, the coefficient is $\$$ and the orders of the partite sets cancel with the expression for $|J(\alpha)|$ given in Theorem 5.6. The remarks preceding Theorem 5.7 show how the sum over the structure classes can be replaced by a triple sum (for Theorem 5.6 shows how the pieces relate to this triple sum). Finally, substitution of $|J(\alpha)|$ and $|F(\alpha)|$ from Theorem 5.6, and simplifying, gives the result. ∎

Example 5.4

Consider the graph $K_{2,4,6,8}$. Again all of the partite sets have different orders; thus, g(2,4,6,8)

$= h(2,4,6,8) = 0$, and $|C(K_{2,4,6,8})| = f(2,4,6,8)$. Now, $\gcd(2,4,6,8) = 2$, so $g = 1, 2$; we consider these cases separately.

When $g = 1$, $\gcd(4,6,8) = 2$, so $c_1 = \{1\}$, $\{2\}$, or $\{1,2\}$. Similarly, $c_2 = c_3 = c_4 = \{1\}$, $\{2\}$, or $\{1,2\}$. It would appear that there are $3^4 = 81$ cases, but remember that members of different sets must be coprime; thus, only one of the c_i can contain a two. Therefore, instead of 81 cases, there are only nine.

(1) Suppose $c_1 = c_2 = c_3 = c_4 = \{1\}$; then $c_{11} = c_{21} = c_{31} = c_{41} = 1$, so $\lambda_{11} = \lambda_{21} = \lambda_{31} = \lambda_{41} = 1\cdot(1\cdot1\cdot1\cdot1 \,/\, 1) = 1$. We conclude that $\alpha = e$ (the identity element), and $\mu_{11} = 2$; $\mu_{21} = 4$, $\mu_{31} = 6$, and $\mu_{41} = 8$. This term is

$$\frac{\Phi(1,18)^2}{1^2\cdot 2!}\cdot\frac{\Phi(1,16)^4}{1^4\cdot 4!}\cdot\frac{\Phi(1,14)^6}{1^6\cdot 6!}\cdot\frac{\Phi(1,12)^8}{1^8\cdot 8!}$$

$$= \frac{17!^2}{2}\cdot\frac{15!^4}{24}\cdot\frac{13!^6}{720}\cdot\frac{11!^8}{40320}$$

$= 99,762,852,921,551,753,579,047,757,328,413,472,867,170,$
$933,209,020,910,698,059,423,756,579,089,428,035,011,$
$870,628,890,025,442,527,711,373,114,345,687,333,648,$
$457,420,742,834,200,763,380,531,200,000,000,000,000,$
$000,000,000,000,000,000,000,000,000,000.$

(2) Suppose $c_1 = c_2 = c_3 = \{1\}$, and $c_4 = \{2\}$; then $c_{11} = c_{21} = c_{31} = 1$, and $c_{41} = 2$, so $\lambda_{11} = \lambda_{21} = \lambda_{31} = 1\cdot(1\cdot1\cdot1\cdot2 \,/\, 1) = 2$, and $\lambda_{41} = 1\cdot(1\cdot1\cdot1\cdot2 \,/\, 2) = 1$. Therefore, $\mu_{11} = 1$, $\mu_{21} = 2$, $\mu_{31} = 3$, and $\mu_{41} = 8$. This term is

$$\frac{\Phi(1,18)^1}{2^1\cdot 1!}\cdot\frac{\Phi(1,16)^2}{2^2\cdot 2!}\cdot\frac{\Phi(1,14)^3}{2^3\cdot 3!}\cdot\frac{\Phi(2,12)^8}{1^8\cdot 8!}$$

$$= \frac{17!}{2}\cdot\frac{15!^2}{8}\cdot\frac{13!^3}{48}\cdot\frac{(1\cdot 2^5\cdot 5!)^8}{40320}$$

$= 224,221,233,828,380,079,700,198,914,028,561,966,962,$
$177,452,083,661,665,982,149,117,870,080,000,000,$
$000,000,000,000,000.$

(3) Suppose $c_1 = c_2 = c_3 = \{1\}$, and $c_4 = \{1,2\}$; then c_{11}

$= c_{21} = c_{31} = c_{42} = 1$, and $c_{41} = 2$, so $\lambda_{11} = \lambda_{21} = \lambda_{31} = \lambda_{42} = 1\cdot(1\cdot1\cdot1\cdot2\ /\ 1) = 2$, and $\lambda_{41} = 1\cdot(1\cdot1\cdot1\cdot2\ /\ 2) = 1$. Therefore, $\mu_{11} = 1$, $\mu_{21} = 2$, and $\mu_{31} = 3$, but the μ_4's are not uniquely determined (hence the sum over μ). The solutions of (μ_{41}, μ_{42}) are (2,3), (4,2), and (6,1).

(a) Take $\mu_{41} = 2$ and $\mu_{42} = 3$; then the term is

$$\frac{\Phi(1,18)^1}{2^1\cdot1!}\cdot\frac{\Phi(1,16)^2}{2^2\cdot2!}\cdot\frac{\Phi(1,14)^3}{2^3\cdot3!}\cdot\frac{\Phi(2,12)^2}{1^2\cdot2!}\cdot\frac{\Phi(1,12)^3}{2^3\cdot3!}$$

$$= \frac{17!}{2}\cdot\frac{15!^2}{8}\cdot\frac{13!^3}{48}\cdot\frac{(1\cdot2^5\cdot5!)^2}{2}\cdot\frac{11!^3}{48}$$

= 1,868,124,586,876,010,350,406,917,183,060,463,028,443, 178,895,032,258,935,626,004,039,938,867,200,000,000, 000,000,000,000,000.

(b) Take $\mu_{41} = 4$ and $\mu_{42} = 2$; then the term is

$$\frac{\Phi(1,18)^1}{2^1\cdot1!}\cdot\frac{\Phi(1,16)^2}{2^2\cdot2!}\cdot\frac{\Phi(1,14)^3}{2^3\cdot3!}\cdot\frac{\Phi(2,12)^4}{1^4\cdot4!}\cdot\frac{\Phi(1,12)^2}{2^2\cdot2!}$$

$$= \frac{17!}{2}\cdot\frac{15!^2}{8}\cdot\frac{13!^3}{48}\cdot\frac{(1\cdot2^5\cdot5!)^4}{24}\cdot\frac{11!^2}{8}$$

= 345,050,428,744,775,360,536,919,768,299,768,062,973, 631,888,259,926,614,372,479,822,672,691,200,000, 000,000,000,000,000,000.

(c) Take $\mu_{41} = 6$ and $\mu_{42} = 1$; then the term is

$$\frac{\Phi(1,18)^1}{2^1\cdot1!}\cdot\frac{\Phi(1,16)^2}{2^2\cdot2!}\cdot\frac{\Phi(1,14)^3}{2^3\cdot3!}\cdot\frac{\Phi(2,12)^6}{1^6\cdot6!}\cdot\frac{\Phi(1,12)^1}{2^1\cdot1!}$$

$$= \frac{17!}{2}\cdot\frac{15!^2}{8}\cdot\frac{13!^3}{48}\cdot\frac{(1\cdot2^5\cdot5!)^6}{720}\cdot\frac{11!}{2}$$

= 16,995,268,832,835,496,353,526,014,561,758,657,839,586, 294,063,403,792,838,740,708,918,558,720,000,000,000, 000,000,000,000.

Combining the totals from (a), (b), and (c) above, we find that the total for case 3 is

2,230,170,284,453,621,207,297,362,965,921,989, 749,256,397,077,355,589,342,837,224,571,530,117, 120,000,000,000,000,000,000,000.

(4) Suppose that $c_1 = c_2 = c_4 = \{1\}$, and $c_3 = \{2\}$; then $c_{11} = c_{21} = c_{41} = 1$, and $c_{31} = 2$, so $\lambda_{11} = \lambda_{21} = \lambda_{41}$

$= 1\cdot(1\cdot1\cdot2\cdot1\ /\ 1) = 2$, and $\lambda_{31} = 1\cdot(1\cdot1\cdot2\cdot1\ /\ 2) = 1$. Therefore, $\mu_{11}= 1$, $\mu_{21}= 2$, $\mu_{31}= 6$, and $\mu_{41}= 4$. This term is

$$\frac{\Phi(1,18)^1}{2^1\cdot1!}\cdot\frac{\Phi(1,16)^2}{2^2\cdot2!}\cdot\frac{\Phi(2,14)^6}{1^6\cdot6!}\cdot\frac{\Phi(1,12)^4}{2^4\cdot4!}$$

$$= \frac{17!}{2}\cdot\frac{15!^2}{8}\cdot\left(\frac{1\cdot2^6\cdot6!}{720}\right)^6\cdot\frac{11!^4}{384}$$

= 3,341,809,802,360,006,565,645,534,042,184,679,192,854,
473,844,055,729,861,782,424,329,912,320,000,000,000,
000,000,000,000.

(5) Suppose that $c_1= c_2= c_4= \{1\}$, and $c_3= \{1,2\}$; then $c_{11}= c_{21}= c_{32}= c_{41}= 1$, and $c_{31}= 2$, so $\lambda_{11}= \lambda_{21}= \lambda_{32}= \lambda_{41} = 1\cdot(1\cdot1\cdot2\cdot1\ /\ 1) = 2$, and $\lambda_{31}= 1\cdot(1\cdot1\cdot2\cdot1\ /\ 2) = 1$. Therefore, $\mu_{11}= 1$, $\mu_{21}= 2$, and $\mu_{41}= 4$, but the μ_3's are not uniquely determined. The solutions of (μ_{31},μ_{32}) are (2,2) and (4,1).

(a) Take $\mu_{31}= 2$ and $\mu_{32}= 2$; then the term is

$$\frac{\Phi(1,18)^1}{2^1\cdot1!}\cdot\frac{\Phi(1,16)^2}{2^2\cdot2!}\cdot\frac{\Phi(2,14)^2}{1^2\cdot2!}\cdot\frac{\Phi(1,14)^2}{2^2\cdot2!}\cdot\frac{\Phi(1,12)^4}{2^4\cdot4!}$$

$$= \frac{17!}{2}\cdot\frac{15!^2}{8}\cdot\left(\frac{1\cdot2^6\cdot6!}{2}\right)^2\cdot\frac{13!^2}{8}\cdot\frac{11!^4}{384}$$

= 1,293,317,021,683,391,781,050,942,665,195,705,173,537,
585,388,868,486,955,433,387,412,265,369,600,000,000,
000,000,000,000,000.

(b) Take $\mu_{31}= 4$ and $\mu_{32}= 1$; then the term is

$$\frac{\Phi(1,18)^1}{2^1\cdot1!}\cdot\frac{\Phi(1,16)^2}{2^2\cdot2!}\cdot\frac{\Phi(2,14)^4}{1^4\cdot4!}\cdot\frac{\Phi(1,14)^1}{2^1\cdot1!}\cdot\frac{\Phi(1,12)^4}{2^4\cdot4!}$$

$$= \frac{17!}{2}\cdot\frac{15!^2}{8}\cdot\left(\frac{1\cdot2^6\cdot6!}{24}\right)^4\cdot\frac{13!}{2}\cdot\frac{11!^4}{384}$$

= 147,003,732,956,353,999,755,374,102,470,907,103,752,
079,857,720,205,421,507,802,054,629,785,600,000,
000,000,000,000,000,000.

Combining the totals from (a) and (b) above, we find the total for case 5 is

1,440,320,754,639,745,780,806,
316,767,666,612,277,289,665,246,

588,692,376,941,189,466,895,155,
200,000,000,000,000,000,000,000.

(6) Suppose $c_1 = c_3 = c_4 = \{1\}$, and $c_2 = \{2\}$; then $c_{11} = c_{31} = c_{41} = 1$, and $c_{21} = 2$, so $\lambda_{11} = \lambda_{31} = \lambda_{41} = 1\cdot(1\cdot2\cdot1\cdot1\ /\ 1) = 2$, and $\lambda_{21} = 1\cdot(1\cdot2\cdot1\cdot1\ /\ 2) = 1$. Therefore, $\mu_{11} = 1$, $\mu_{21} = 4$, $\mu_{31} = 3$, and $\mu_{41} = 4$. This term is

$$\frac{\Phi(1,18)^1}{2^1\cdot1!}\cdot\frac{\Phi(2,16)^4}{1^4\cdot4!}\cdot\frac{\Phi(1,14)^3}{2^3\cdot3!}\cdot\frac{\Phi(1,12)^4}{2^4\cdot4!}$$

$$= \frac{17!}{2}\cdot\frac{(1\cdot2^7\cdot7!)^4}{24}\cdot\frac{13!^3}{48}\cdot\frac{11!^4}{384}$$

= 42,685,528,384,363,531,780,819,739,384,144,877,533,937,
262,389,867,055,726,709,929,936,945,152,000,000,000,
000,000,000,000.

(7) Suppose $c_1 = c_3 = c_4 = \{1\}$, and $c_2 = \{1,2\}$; then $c_{11} = c_{22} = c_{31} = c_{41} = 1$, and $c_{21} = 2$, so $\lambda_{11} = \lambda_{22} = \lambda_{31} = \lambda_{41} = 1\cdot(1\cdot2\cdot1\cdot1\ /\ 1) = 2$, and $\lambda_{21} = 1\cdot(1\cdot2\cdot1\cdot1\ /\ 2) = 1$. Therefore, $\mu_{11} = 1$, $\mu_{21} = 2$, $\mu_{22} = 1$, $\mu_{31} = 3$, and $\mu_{41} = 4$. This term is

$$\frac{\Phi(1,18)^1}{2^1\cdot1!}\cdot\frac{\Phi(2,16)^2}{1^2\cdot2!}\cdot\frac{\Phi(1,16)^1}{2^1\cdot1!}\cdot\frac{\Phi(1,14)^3}{2^3\cdot3!}\cdot\frac{\Phi(1,12)^4}{2^4\cdot4!}$$

$$= \frac{17!}{2}\cdot\frac{(1\cdot2^7\cdot7!)^2}{2}\cdot\frac{15!}{2}\cdot\frac{13!^3}{48}\cdot\frac{11!^4}{384}$$

= 804,730,591,269,665,997,098,364,325,010,660,996,867,
830,908,629,280,772,269,663,278,742,896,640,000,
000,000,000,000,000,000.

(8) Suppose $c_1 = \{2\}$, and $c_2 = c_3 = c_4 = \{1\}$; then $c_{11} = 2$, and $c_{21} = c_{31} = c_{41} = 1$, so $\lambda_{11} = 1\cdot(2\cdot1\cdot1\cdot1\ /\ 2) = 1$, and $\lambda_{21} = \lambda_{31} = \lambda_{41} = 1\cdot(2\cdot1\cdot1\cdot1\ /\ 1) = 2$. Therefore, $\mu_{11} = 2$, $\mu_{21} = 2$, $\mu_{31} = 3$, and $\mu_{41} = 4$. This term is

$$\frac{\Phi(2,18)^2}{1^2\cdot2!}\cdot\frac{\Phi(1,16)^2}{2^2\cdot2!}\cdot\frac{\Phi(1,14)^3}{2^3\cdot3!}\cdot\frac{\Phi(1,12)^4}{2^4\cdot4!}$$

$$= \frac{(1\cdot2^8\cdot8!)^2}{2}\cdot\frac{15!^2}{8}\cdot\frac{13!^3}{48}\cdot\frac{11!^4}{384}$$

= 378,696,748,832,783,998,634,524,388,240,311,057,349,
567,486,413,779,186,950,429,778,231,951,360,000,

000,000,000,000,000,000.

(9) Suppose $c_1 = \{1,2\}$, and $c_2 = c_3 = c_4 = \{1\}$; then $c_{11} = 2$, and $c_{12} = c_{21} = c_{31} = c_{41} = 1$, so $\lambda_{11} = 1\cdot(2\cdot1\cdot1\cdot1\ /\ 2) = 1$, $\lambda_{12} = 1\cdot(2\cdot1\cdot1\cdot1\ /\ 1) = 2$—but $1 + 2 > 2$ (the order of V_1) so this case cannot occur.

Adding the eight occurring cases, we find the contribution to f(2,4,6,8), when g = 1, is

99,762,852,921,551,753,579,047,757,328,413,472,867,170,
933,209,020,910,698,059,423,756,579,089,428,035,011,870,
628,890,025,447,427,881,311,730,714,589,596,382,376,600,
675,033,657,978,013,204,548,126,252,981,598,784,847,872,
000,000,000,000,000,000,000.

We now consider $g = 2$; $\gcd(4,6,8) = \gcd(2,6,8) = \gcd(2,4,8) = \gcd(2,4,6) = 2$. Thus, $c_1 = c_2 = c_3 = c_4 = \{2/2\} = \{1\}$. That is, $c_{11} = c_{21} = c_{31} = c_{41} = 1$, so that $\lambda_{11} = \lambda_{21} = \lambda_{31} = \lambda_{41} = 2\cdot(1\cdot1\cdot1\cdot1\ /\ 1) = 2$. Therefore, $\mu_{11} = 1$, $\mu_{21} = 2$, $\mu_{31} = 3$, and $\mu_{41} = 4$. This term is

$$\frac{\Phi(1,18)^1}{2^1\cdot1!}\cdot\frac{\Phi(1,16)^2}{2^2\cdot2!}\cdot\frac{\Phi(1,14)^3}{2^3\cdot3!}\cdot\frac{\Phi(1,12)^4}{2^4\cdot4!}$$

$$= \frac{17!}{2}\cdot\frac{15!^2}{8}\cdot\frac{13!^3}{48}\cdot\frac{11!^4}{384}$$

= 1,264,267,908,891,674,973,468,743,757,676,661,014,366,
331,029,548,198,674,207,832,812,185,190,400,000,000,
000,000,000,000,000.

Therefore, there are f(2,4,6,8), that is,

99,762,852,921,551,753,579,047,757,328,413,472,867,170,
933,209,020,910,698,059,423,756,579,089,428,035,011,870,
628,890,025,448,692,149,220,622,389,563,065,126,134,277,
336,048,024,309,042,752,746,800,460,814,410,970,038,272,
000,000,000,000,000,000,000

congruence classes of maps of $K_{2,4,6,8}$.

In the next section we will compute $h(m_1,\ldots,m_n)$; this is a deviation from the previous chapters. The reason for this change is because it is easier to calculate

$h(m_1,\ldots,m_n)$ than it is to compute $g(m_1,\ldots,m_n)$.

We will begin by considering $K_{n(m)}$. There is only one order for the partite sets—this we will find is an advantage. Once we have this case under control, we will complicate matters by considering $K(m_1,\ldots,m_n)$, where the orders of all partite sets appear at least twice. Finally, we will consider the general case.

In subsequent sections we will determine the general formula for $g(m_1,\ldots,m_n)$. Again we will find it easier to consider $K_{n(m)}$ first. We will develop the general formula for the number of congruence classes of maps of $K(m_1,\ldots,m_n)$ and show how it generalizes the formulas given in the previous three chapters.

5.3 Calculating $h(m_1,\ldots,m_n)$

In this section we will determine the general formula for $h(m_1,\ldots,m_n)$. We begin by considering $K_{n(m)}$. The automorphism group of $K_{n(m)}$ is isomorphic to $\mathbb{S}_n[\mathbb{S}_m]$. Therefore, the partite sets can cycle in every possible way. If $\gamma \in D(K_{n(m)})$, then all of the partite sets are cycling; that is, the portion of the permutation in $\mathbb{S}_n[\mathbb{S}_m]$ corresponding to the cycling of the partite sets has only cycles of length two or more.

Theorem 5.8. Let $\gamma \in D[K(m_1,\ldots,m_n)]$; then $|F(\gamma)| > 0$ if and only if γ is uniform.

Proof: Suppose $\gamma \in D[K(m_1,\ldots,m_n)]$; then no partite set is fixed. We conclude that $\forall v \in V$, v and $\gamma(v)$ are in different partite sets. Therefore, $\gamma^{\lambda(v)}$ fixes the adja-

cent vertices v and $\gamma(v)$. By Theorem 2.3, either $\gamma^{\lambda(v)}$ is the identity element of Aut $K(m_1,\ldots,m_n)$ or it fixes no rotation scheme of $K(m_1,\ldots,m_n)$. The result now follows. ∎

Theorem 5.8 does not just pertain to $K_{n(m)}$; it applies in general. We now establish conditions on this uniform cycle length.

Theorem 5.9. Let $\gamma \in D(K_{n(m)})$ and suppose that γ cycles the partite sets in cycles of lengths $v_1, v_2, \ldots, v_t$. If γ is d-uniform, then $\operatorname{lcm}(v_1,\ldots,v_t)$ is a divisor of d and d is a divisor of $m \cdot \gcd(v_1,\ldots,v_t)$.

Proof: Consider one of the cycles of partite sets. Suppose that there are v_i partite sets being cycled. Since the partite sets are being cycled, $v_i \mid d$. Since there are $m \cdot v_i$ elements being cycled, $d \mid (m \cdot v_i)$. These relations must apply for all v_i $(i = 1, \ldots, t)$. We conclude that $\operatorname{lcm}(v_1,\ldots,v_t)$ is a divisor of d and that d is a divisor of $\gcd(m \cdot v_1,\ldots,m \cdot v_t) = m \cdot \gcd(v_1,\ldots,v_t)$. ∎

By Theorem 5.9 we see that $\operatorname{lcm}(v_1,\ldots,v_t)$ must be a divisor of $m \cdot \gcd(v_1,\ldots,v_t)$. This suggests a strategy to create the $\gamma \in D(K_{n(m)})$ so that $|F(\gamma)| > 0$. Let $v_1, \ldots, v_t$ be positive integers, with $v_i \geq 2$ $(i = 1, \ldots, t)$. Suppose $\operatorname{lcm}(v_1,\ldots,v_t)$ is a divisor of $m \cdot \gcd(v_1,\ldots,v_t)$ and there exist positive integers $e_1, \ldots, e_t$ such that $\sum_{i=1}^{t} e_i v_i = n$. It is easy to construct a member of $D(K_{n(m)})$. Partition the n partite sets into e_1 sets of v_1, e_2 sets of v_2, ..., e_t sets of v_t; order these sets. Create one-to-one functions between the partite sets that

respect this ordering. Let h be a positive integer which is a divisor of $m \cdot \gcd(v_1, \ldots, v_t)/\mathrm{lcm}(v_1, \ldots, v_t)$. Consider the m objects from one of these e_i ordered sets of v_i partite sets generated by the functions. Create a permutation out of these objects that is $h \cdot \mathrm{lcm}(v_1, \ldots, v_t)/v_i$ uniform. Set $d = h \cdot \mathrm{lcm}(v_1, \ldots, v_t)$; this is a d-uniform automorphism in $D(K_{n(m)})$.

Example 5.5

Consider $K_{6(2)}$; $6 = 6 = 3 + 3 = 2 + 2 + 2 = 2 + 4$. We consider these four cases separately. In the first three cases, we have $t = 1$, and in the last case $t = 2$. When $t = 1$, the partitions of n are over the divisors of n; this is not necessarily true when $t > 1$.

(1) In this case we have $e_1 = 1$ and $v_1 = 6$. We therefore must have $h \mid 2 \cdot 6/6 = 2$, so $h = 1$ or $h = 2$. This means $d = 1 \cdot 6 = 6$, or $d = 2 \cdot 6 = 12$.

(2) In this case we have $e_1 = 2$ and $v_1 = 3$. Again we see that $h = 1$ or $h = 2$ are both possible. This means $d = 3$ or $d = 6$. But notice that these 6-uniform permutations are in a different structure class than those in case 1.

(3) In this case we have $e_1 = 3$ and $v_1 = 2$. Yet again we have $h = 1$ or $h = 2$. This means that $d = 2$ or $d = 4$.

Observe that when $t = 1$, $h \mid m$. This is no accident for $\mathrm{lcm}(v) = \gcd(v) = v$, by convention.

(4) In this case $e_1 = e_2 = 1$, $v_1 = 2$, and $v_2 = 4$. Here it makes sense to take lcm and gcd; $\mathrm{lcm}(2,4) = 4$ and also, $\gcd(2,4) = 2$. Note that $4 \mid 2 \cdot 2 = 4$. The formula for h says that h divides $2 \cdot 2 / 4 = 1$, so $h = 1$. We conclude that $d = 4$. Again note that these 4-uniform permutations

are in a different structure class than those of case 3.

We will continue this example after we develop the the counting argument.

The strategy developed before Example 5.5 actually allows us to determine $|F(\gamma)|$ and $|J(\gamma)|$ for each possible partition of the n partite sets. This we now determine.

Theorem 5.10. Let $\gamma \in D(K_{n(m)})$ cycle the partite sets in cycles of lengths $v_1, v_2, \ldots, v_t$ (that is, if v_i partite sets are cycled, then in every cycle of the disjoint cycle decomposition of γ corresponding to those partite sets every v_i consecutive vertices contains a vertex from each of these partite sets). Suppose that there are e_i cycles of v_i partite sets $(i = 1, \ldots, t)$; let h be a divisor of $m \cdot \gcd(v_1, \ldots, v_t) / \operatorname{lcm}(v_1, \ldots, v_t)$ and set $d = h \cdot \operatorname{lcm}(v_1, \ldots, v_t)$. Then, if γ is d-uniform,

$$|F(\gamma)| = [n \cdot m - m - 1]!^{n \cdot m/d}.$$

Proof: With the conditions above, the remarks preceeding Example 5.5 show how a d-uniform automorphism can be produced. Consider what happens to one of v_i partite cycles of γ. There are $m \cdot v_i$ vertices in the d-cycles of γ that correspond to this partite cycle of length v_i. Thus, we account for $m \cdot v_i/d$ d-cycles of γ. Summing over all of the v_i cycles, there are $m \cdot (e_i \cdot v_i)/d$ d-cycles in these v_i length partite cycles. Therefore, there are $\sum_{i=1}^{t} m(e_i v_i)/d = m \cdot n/d$ d-cycles altogether. By Theorem 2.4, each of these cycles needs a representative. Now γ is d-uniform, so γ^d is the identity automorphism on $K_{n(m)}$. Thus, for

every vertex v, $|F_v(\gamma^d)| = [m(n-1)-1]!$. Combining the above, and using Theorem 2.4 we see that

$$|F(\gamma)| = [n\cdot m - m - 1]!^{n\cdot m/d}. \qquad \blacksquare$$

Theorem 5.11. Let $\gamma \in D(K_{n(m)})$ cycle the partite sets in cycles of lengths $\nu_1, \ldots, \nu_t$; suppose that there are e_i cycles of ν_i partite sets $(i = 1, \ldots, t)$. Let h be a divisor of $m\cdot\gcd(\nu_1, \ldots, \nu_t)/\mathrm{lcm}(\nu_1, \ldots, \nu_t)$, and set $d = h\cdot\mathrm{lcm}(\nu_1, \ldots, \nu_t)$. Then

$$|J(\gamma)| = \frac{n!\cdot m!^n}{\prod_{i=1}^{t}\left[\nu_i\cdot\left[\frac{d}{\nu_i}\right]^{m\nu_i/d}\cdot\left[\frac{m\nu_i}{d}\right]!\right]^{e_i}\cdot e_i!}.$$

Proof: The proof will parallel the strategy given before Example 5.5. We will first count the number of ways that the n partite sets can be partitioned and ordered. Number the partite sets; every ordered partition of the partite sets corresponds (in a natural way) with a permutation of S_n—the cycles of the permutation match the cycles of the partite sets as follows: if x is followed by y in the permutation of S_n then V_x is followed by V_y in the partite cycles of γ; that is, in every cycle of γ having an element of V_x, that element is followed by an element of V_y. In this way, a cycle of length ν_i in the permutation of S_n corresponds to a partite cycle of length ν_i in γ. We conclude that the number of ways to partition the n partite sets in the desired fashion is the same as the number of permutations of S_n having e_1 cycles of length ν_1, e_2 cycles of length ν_2, ..., and e_t cycles of length ν_t. This latter number is given by Theorem 2.6 as

$$\frac{n!}{\prod_{i=1}^{t} \nu_i^{e_i} \cdot e_i!}$$

Now that the partition has been accomplished we need to determine the number of one-to-one functions between the partite sets in each cycle.

Consider a partite cycle of length ν_i; since it is a cycle we can begin with any of the partite sets in the cycle. There are m! one-to-one functions from the first partite set to the second, m! one-to-one functions from the second to the third, ..., and m! one-to-one functions from the next to the last. In all there are $\nu_i - 1$ one-to-one functions being constructed, each independent of the others, so there are $m!^{(\nu_i - 1)}$ ways to create these functions. Observe that each of these constructions can be thought of as producing m new objects to be permuted. We need only determine how many of these new objects to include in each of the cycles to produce a d-uniform automorphism of $K_{n(m)}$ and then apply Theorem 2.6 to do the counting. Each cycle is to be d-uniform on $K_{n(m)}$; there are ν_i vertices in each of the new objects; therefore, we need d / ν_i of the new objects in each cycle of γ. The number of (d/ν_i)-uniform permutations of an m-element set is

$$m! \Big/ \left[\frac{d}{\nu_i}\right]^{\frac{m\nu_i}{d}} \cdot \left[\frac{m\nu_i}{d}\right]!$$

Combining the above with the number of ways to produce the objects being permuted, we find that once the ν_i partite sets have been identified and ordered, there are

$$m!^{\nu_i} \Big/ \left[\frac{d}{\nu_i}\right]^{\frac{m\nu_i}{d}} \cdot \left[\frac{m\nu_i}{d}\right]!$$

ways to produce a d-uniform permutation on these objects.

Each of the partite sets is disjoint from all of the others; hence, each of the partite cycles is independent of all the others. Therefore, the total number of permutations in $J(\gamma)$ is the product of all of the ways of producing the ordered partitions with the numbers of ways of obtaining d-uniform automorphisms in each of the partite cycles. This gives us the value claimed. ∎

Example 5.6

Consider case 4 of Example 5.5. In this case we have $m = 2$, $n = 6$, $e_1 = e_2 = 1$, $v_1 = 2$, $v_2 = 4$, and $d = 4$. By Theorem 5.10,

$$|F(\gamma)| = (6\cdot 2 - 2 - 1)!^{6\cdot 2/4} = 9!^3$$
$$= 47{,}784{,}725{,}839{,}872{,}000.$$

By Theorem 5.11,

$$|J(\gamma)| = \frac{6!\cdot 2^6}{[2\cdot(4/2)^{2\cdot 2/4}\cdot 1!]\cdot 1!\cdot[4\cdot(4/4)^{2\cdot 4/4}\cdot 2!]\cdot 1!}$$
$$= 1{,}440.$$

Let $\mathcal{F}(t,n) = \{\{v_1, v_2, \ldots, v_t\} \mid v_i \geq 2 \ (i = 1, \ldots, t),\ \exists e_1, \ldots, e_t,\ \sum_{i=1}^{t} e_i v_i = n\}$; that is, $\mathcal{F}(t,n)$ is the set of all v_i's that can partition the n partite sets. Given a set of v_i's, let $\mathcal{E}$ be the set of all solutions to

$$\sum_{i=1}^{t} e_i v_i = n.$$

Note that since $v_i \geq 2$, $t \leq \lfloor n/2 \rfloor$, where $\lfloor x \rfloor$ is the floor function (the greatest integer function).

Theorem 5.12. The value of $h(m,m,\ldots,m)$, where there are

n m's, is given by

$$\sum_{t=1}^{\lfloor n/2\rfloor}\sum_{\mathfrak{F}(t,n)}\sum_{\mathcal{E}}\sum_{h}\frac{[n\cdot m - m - 1]!^{\,n\cdot m/d}}{\prod_{i=1}^{t}\left[\nu_i\cdot\left[\frac{d}{\nu_i}\right]^{m\nu_i/d}\cdot\left[\frac{m\nu_i}{d}\right]!\right]^{e_i}\cdot e_i!},$$

where h is a divisor of $m\cdot\gcd(\nu_1,\ldots,\nu_t)/\mathrm{lcm}(\nu_1,\ldots,\nu_t)$, and $d = h\cdot\mathrm{lcm}(\nu_1,\ldots,\nu_t)$.

Proof: The sum over t will actually never get that high in most cases as all of the ν_i are distinct; this will be taken care of by the sum over $\mathfrak{F}(t,n)$. The sum over $\mathfrak{F}(t,n)$ allows for all possible ways to partition the n partite sets; by Theorem 5.9 only those sets of ν_i for which $\mathrm{lcm}(\nu_1,\ldots,\nu_t) \mid [m\cdot\gcd(\nu_1,\ldots,\nu_t)]$ will actually contribute. The sum over $\mathcal{E}$ allows for the possibility of more than one solution to the set of equations given before the theorem. Finally, the sum over h accounts for all of the different possible cycle lengths that can occur for the choice of $\nu_1, \ldots, \nu_t$. The definitions of h and d, together with the formulas for $|F(\gamma)|$ and $|J(\gamma)|$ and allowing for the order of the automorphism group, give the result. ∎

Example 5.7

Consider $K_{6(2)}$ as in Examples 5.5 and 5.6. We see from Example 5.5 that t = 1 or t = 2. We consider these cases separately.

When t = 1; then $\mathfrak{F}(1,6) = \{ \{2\}, \{3\}, \{6\} \}$. Let $\nu_1 = 2$; then $e_1 = 3$. Therefore, $h \mid 2$, so h = 1 or h = 2.

If h = 1 then d = 2. The value of this term is

$$\frac{[6\cdot2 - 2 - 1]!^{6\cdot2/2}}{[2\cdot(2/2)^{2\cdot2/2}\cdot2!]^3\cdot3!} = \frac{9!^6}{4^3\cdot6}$$

$= 5,946,302,144,770,132,332,773,376,000,000.$

If $h = 2$ then $d = 4$. The value of this term is

$$\frac{[6\cdot 2 - 2 - 1]!^{6\cdot 2/4}}{[4\cdot(4/2)^{2\cdot 2/4}\cdot 1!]^{3}\cdot 3!} = \frac{9!^{3}}{8^{3}\cdot 6}$$

$$= 15,554,923,776,000.$$

Let $v_1 = 3$; then $e_1 = 2$. Therefore, $h \mid 2$, so $h = 1$ or $h = 2$.

If $h = 1$ then $d = 3$. The value of this term is

$$\frac{[6\cdot 2 - 2 - 1]!^{6\cdot 2/3}}{[3\cdot(3/3)^{2\cdot 3/3}\cdot 2!]^{2}\cdot 2!} = \frac{9!^{4}}{6^{2}\cdot 2}$$

$$= 240,835,018,232,954,880,000.$$

If $h = 2$ then $d = 6$. The value of this term is

$$\frac{[6\cdot 2 - 2 - 1]!^{6\cdot 2/6}}{[3\cdot(6/3)^{2\cdot 3/6}\cdot 1!]^{2}\cdot 2!} = \frac{9!^{2}}{6^{2}\cdot 2}$$

$$= 1,828,915,200.$$

Let $v_1 = 6$; then $e_1 = 1$. Therefore, $h \mid 2$, so $h = 1$ or $h = 2$.

If $h = 1$ then $d = 6$. The value of this term is

$$\frac{[6\cdot 2 - 2 - 1]!^{6\cdot 2/6}}{[6\cdot(6/6)^{2\cdot 6/6}\cdot 2!]^{1}\cdot 1!} = \frac{9!^{2}}{12}$$

$$= 10,973,491,200.$$

If $h = 2$ then $d = 12$. The value of this term is

$$\frac{[6\cdot 2 - 2 - 1]!^{6\cdot 2/12}}{[12\cdot(12/6)^{2\cdot 6/12}\cdot 1!]^{1}\cdot 1!} = \frac{9!}{24}$$

$$= 15,120.$$

When $t = 2$; then $\mathcal{F}(2,6) = \{\ \{2,4\}\ \}$. Let $v_1 = 2$, and $v_2 = 4$; then $e_1 = e_2 = 1$. Therefore, $h \mid (2\cdot 2\ /\ 4)$; $h \mid 1$, so $h = 1$. We conclude that $d = 4$. The value of this term is

$$\frac{[6\cdot 2 - 2 - 1]!^{6\cdot 2/4}}{[2\cdot(4/2)^{2\cdot 2/4}\cdot 1!]^{1}\cdot 1!\cdot[4\cdot(4/4)^{2\cdot 4/4}\cdot 2!]^{1}\cdot 1!}$$

$$= \frac{9!^{3}}{4\cdot 8} = 1,493,272,682,496,000.$$

So, combining the above,

$h(2,\ldots,2) = 5{,}946{,}302{,}145{,}010{,}968{,}859{,}846{,}739{,}573{,}520$.

We are now ready to generalize this to more than one order for the partite sets. Suppose $K(m_1,\ldots,m_n)$ has q different orders of partite sets: f_1 have order p_1, f_2 have order p_2, ..., f_q have order p_q. Consider the f_i partite sets of order p_i and suppose that these f_i partite sets are cycled in cycles of length v_{ij} for $j = 1, \ldots, t_i$. By the same argument as in Theorem 5.9, we see:

Theorem 5.13. Let $\gamma \in D[K(m_1,\ldots,m_n)]$; suppose that for $i = 1, \ldots, q$, there are f_i partite sets of order p_i and suppose that γ cycles the f_i partite sets of order p_i in cycles of lengths v_{ij} ($j = 1,\ldots,t_i$). Furthermore, suppose there are e_{ij} partite cycles of length v_{ij}. Suppose γ is d-uniform; then $\mathrm{lcm}(v_{i1},\ldots,v_{it(i)})$ is a divisor of d and d is a divisor of $p_i \cdot \gcd(v_{i1},\ldots,v_{it(i)})$. ∎

Using Theorem 5.13, it is easy to prove that we must have $\mathrm{lcm}(v_{ij};\ i = 1,\ldots q,\ j = 1,\ldots,t_i)$ a divisor of d and d a divisor of $\gcd(p_i \cdot \gcd(v_{ij}; j = 1,\ldots,t_i);\ i = 1, \ldots, q)$. Also, we can see that h must be a divisor of $\gcd(p_i \cdot \gcd(v_{ij};\ j = 1,\ldots,t_i); i = 1,\ldots,q)/\mathrm{lcm}(v_{ij}; i = 1, \ldots, q,\ j = 1, \ldots, t_i)$. Since each of the partite sets is disjoint from all others, we can apply the strategy that was developed before Example 5.5 to each of the different orders of partite sets and apply the argument of Theorem 5.10 to each of these orders. Then Theorem 5.10 becomes:

Theorem 5.14. Let $\gamma \in D[K(m_1, \ldots, m_n)]$; suppose that for $i = 1, \ldots, q$, there are f_i partite sets of order p_i and suppose that γ cycles the f_i partite sets of order p_i in cycles of lengths v_{ij} $(j = 1, \ldots, t_i)$. Furthermore, suppose there are e_{ij} partite cycles of length v_{ij}. Let h be a divisor of

$$\frac{\gcd(p_i \cdot \gcd(v_{ij};\ j = 1, \ldots, t_i);\ i = 1, \ldots, q)}{\operatorname{lcm}(v_{ij};\ i = 1, \ldots, q,\ j = 1, \ldots, t_i)}$$

and set $d = h \cdot \operatorname{lcm}(v_{ij}; i = 1, \ldots, q,\ j = 1, \ldots, t_i)$. If γ is d-uniform, then

$$|F(\gamma)| = \prod_{i=1}^{q} [P - p_i - 1]!^{f_i p_i / d},$$

where $P = f_1 p_1 + \cdots + f_q p_q$. ∎

Similarly, Theorem 5.11 becomes:

Theorem 5.15. Let $\gamma \in D[K(m_1, \ldots, m_n)]$; suppose that for $i = 1, \ldots, q$, there are f_i partite sets of order p_i and suppose that γ cycles the f_i partite sets of order p_i in cycles of lengths v_{ij} $(j = 1, \ldots, t_i)$. Furthermore, suppose there are e_{ij} partite cycles of length v_{ij}. Let h be a divisor of

$$\frac{\gcd(p_i \cdot \gcd(v_{ij};\ j = 1, \ldots, t_i);\ i = 1, \ldots, q)}{\operatorname{lcm}(v_{ij};\ i = 1, \ldots, q,\ j = 1, \ldots, t_i)}$$

and set $d = h \cdot \operatorname{lcm}(v_{ij}; i = 1, \ldots, q,\ j = 1, \ldots, t_i)$. If γ is d-uniform, then

$$|J(\gamma)| = \prod_{i=1}^{q} \frac{f_i! \cdot p_i!^{f_i}}{\prod_{j=1}^{t_i} \left[v_{ij} \cdot \left[\frac{d}{v_{ij}} \right]^{p_i v_{ij}/d} \cdot \left[\frac{p_i v_{ij}}{d} \right]! \right]^{e_{ij}} \cdot e_{ij}!}.$$

∎

Let $\mathcal{F}(t_1,p_1,\ldots,t_q,p_q) = \{ \{v_{ij}\} \mid v_{ij} \geq 2, (i = 1, \ldots, q, j = 1, \ldots, t_i), \exists e_{ij}, \sum_{j=1}^{t_i} e_{ij}v_{ij} = p_i \}$; that is, $\mathcal{F}(t_1,p_1,\ldots,t_q,p_q)$ is the set of all possible partitions of the partite sets in the manner being considered. Given a set of v_{ij}'s, let $\mathcal{E}$ be the set of all solutions to

$$\sum_{j=1}^{t_i} e_{ij}v_{ij} = p_i \ (i = 1, \ldots, q).$$

Also, let $\mathcal{T}(p_1, \ldots, p_q) = \{ (t_1,\ldots,t_q) \mid 1 \leq t_i \leq p_i/2 \ (i = 1,\ldots,q) \}$. Note: since $v_{ij} \geq 2$, $t_i \leq \lfloor p_i/2 \rfloor$, where $\lfloor x \rfloor$ is the greatest integer function. Then Theorem 5.12 becomes:

Theorem 5.16. If $K(m_1,\ldots,m_n)$ has q distinct orders of partite sets: f_i of order p_i $(i = 1, \ldots, q)$, then if any of the f_i is equal to one $h(m_1,\ldots,m_n) = 0$; otherwise, $h(m_1,\ldots,m_n)$ is given by

$$\sum_{\mathcal{T}(p_1,\ldots,p_t)} \ \sum_{\mathcal{F}(t_1,p_1,\ldots,t_q,p_q)} \ \sum_{\mathcal{E}} \ \sum_{h} (*)$$

where (*) is

$$\prod_{i=1}^{q} \frac{[P - p_i - 1]!^{f_i p_i/d}}{\prod_{j=1}^{t_i} \left[v_{ij} \cdot [\frac{d}{v_{ij}}]^{p_i v_{ij}/d} \cdot [\frac{p_i v_{ij}}{d}]! \right]^{e_{ij}} \cdot e_{ij}!}. \ \blacksquare$$

Example 5.8

Consider the graph $K_{1,1,2,2}$ of Example 5.1. Here, $p_1 = 1$, $p_2 = 2$ and $f_1 = f_2 = 2$. Therefore, $t_1 = t_2 = 1$, $v_{11} = v_{21} = 2$, and $e_{11} = e_{21} = 1$; $\gcd(2,2) = \mathrm{lcm}(2,2) = 2$, so h is a divisor of $\gcd(2,4)/2 = 1$. We conclude $h = 1$, so $d = 2$. [Observe that only the 2-uniform automorphism of

Table 5.3 has a nonzero value for $|J(\gamma)|$.]

By Theorem 5.14,

$$|F(\gamma)| = (6 - 1 - 1)!^{2/2}\cdot(6 - 2 - 1)!^{4/2} = 4!\cdot 3!^{2} = 864.$$

By Theorem 5.15,

$$|J(\gamma)| = \frac{2\cdot 1!^{2}\cdot 2\cdot 2!^{2}}{[2\cdot(2/2)^{1\cdot 2/2}\cdot 1!]^{1}\cdot 1!\cdot[2\cdot(2/2)^{2\cdot 2/2}\cdot 2!]^{1}\cdot 1!}$$
$$= \frac{2\cdot 8}{2\cdot 4} = 2.$$

Both of these are confirmed by Table 5.3.

Finally, Theorem 5.16 tells us

$$h(1,1,2,2) = \frac{(6 - 1 - 1)!^{2/2}\cdot(6 - 2 - 1)!^{4/2}}{[2\cdot(2/2)^{1\cdot 2/2}\cdot 1!]^{1}\cdot 1!\cdot[2\cdot(2/2)^{2\cdot 2/2}\cdot 2!]^{1}\cdot 1!}$$
$$= \frac{864}{2\cdot 4} = 108.$$

This is confirmed by the calculations of Example 5.1.

In the next section we will develop the general formula for $g(m_1,\ldots,m_n)$. We will again find it easier to begin with $K_{n(m)}$. Once this is under control, we will move on to the general case.

In the final section we will show how the formula for $K_{n(m)}$ generalizes the formula for K_n. We will also show how the formula for $K(m_1,\ldots,m_n)$ generalizes those for $K_{m,n}$ and $K_{m,n,r}$.

5.4 Calculating $g(m_1,\ldots,m_n)$

In this section we will derive the general formula for $g(m_1,\ldots,m_n)$. As we did in the last section, we will begin by considering $K_{n(m)}$—the proofs are easier when all the partite sets have the same order. Once this has been done, we will complicate matters by solving the general case. By the end of this section we will be in a

position to give the formulas for the number of congruence classes of maps of $K_{n(m)}$ and $K(m_1, \ldots, m_n)$. These will be shown to generalize the formulas for K_n, $K_{m,n}$, and $K_{m,n,r}$.

Consider the symmetric complete n-partite graph $K_{n(m)}$. Let $\beta \in \mathrm{Aut}\, K_{n(m)}$; if $\xi(\beta)$ denotes the permutation of S_n corresponding to the partite cycles then $\xi(\beta)$ contains cycles of length one. In particular, if β fixes V_i (setwise) then (i) is a 1-cycle in $\xi(\beta)$. Let $I(\beta)$ be the partite sets corresponding to the 1-cycles in $\xi(\beta)$; that is,

$$I(\beta) = \{ V_i \mid (i) \in \xi(\beta) \}.$$

Further let $H(\beta)$ be the other partite sets; that is,

$$H(\beta) = V - I(\beta),$$

where V is now viewed as $\{ V_i;\ i = 1, \ldots, n \}$ in the above equation.

Theorem 5.17. Let $\beta \in B(K_{n(m)})$; then $|F(\beta)| > 0$ if and only if $\beta|_{H(\beta)}$ is d-uniform, for some positive integer d, and all of the cycle lengths in $\beta|_{I(\beta)}$ are divisors of d.

Proof: Let $v \in V_i$, where $V_i \in H(\beta)$; then $\lambda(v) \geq 2$, and $\beta(v)$ is not in the same partite set as v (how $H(\beta)$ was defined). Hereafter, the proof is similar to that of Theorem 4.8. ∎

We need to determine conditions on d, the cycle length of $\beta|_{H(\beta)}$, in terms of the length of the partite cycles and the order of the partite sets. This is where the fact that only one partite set is involved makes the work easier. Then we need to determine conditions on the

other cycle lengths. Finally we incorporate all of these conditions to produce the final form of $g(m,\ldots,m)$. Later we will see how these generalize for $g(m_1,\ldots,m_n)$.

Theorem 5.18. Let $\beta \in B(K_{n(m)})$ and suppose that $\beta|_{H(\beta)}$ cycles the partite sets in cycles of length $\nu_1, \ldots, \nu_t$. If $\beta|_{H(\beta)}$ is d-uniform, then $\mathrm{lcm}(\nu_1,\ldots,\nu_t)$ is a divisor of d and d is a divisor of $m\cdot\gcd(\nu_1,\ldots,\nu_t)$.

Proof: The proof of this is almost identical to the proof of Theorem 5.9. ∎

Suppose that $|I(\beta)| = r$, and let $\alpha_1, \ldots, \alpha_r$ be the permutations of the individual partite sets of $I(\beta)$; then we will consider the conditions on these permutations. Observe that for every cycle length λ in α_1 then, by Theorem 2.5, $\beta^\lambda|_L$ is uniform, where $L = V - V_{i(1)}$ and i_1 is the number of the partite set on which α_1 is a permutation. Clearly, this also works for $\alpha_2, \ldots, \alpha_r$ and $\beta|_{H(\beta)}$ as well. This means $\{\alpha_1, \ldots, \alpha_r, \beta|_{H(\beta)}\}$ is a compatible set of permutations. Because the condition in Theorem 2.5 is "if and only if," we see that the compatibility condition must be also. Therefore, we have:

Theorem 5.19. Let $\beta \in B(K_{n(m)})$ and suppose $|I(\beta)| = r$. Let $\alpha_1, \ldots, \alpha_r$ denote the permutations on the individual partite sets of $I(\beta)$, where α_k is a permutation on $V_{i(k)}$, and $I(\beta) = \{ V_{i(1)}, \ldots, V_{i(r)} \}$. Then $|F(\beta)| > 0$ if and only if $\{\alpha_1, \ldots, \alpha_r, \beta|_{H(\beta)}\}$ is a compatible set of permutations.

Proof: The proof of this is almost identical to the proof

of Theorem 5.3. ∎

Theorem 5.5 gives conditions on the cycle lengths of compatible permutations. Let $\alpha_0 = \beta|_{H(\beta)}$; by Theorem 5.17, if α_0 is d-uniform, then the notation of Theorem 5.5 says $c_{01} = 1$. Therefore,

$$d = g \prod_{i=1}^{r} \mathrm{lcm}(c_{ik};\ k = 1, \ldots, r),$$

where g is a divisor of gcd(m,d). However, by Theorem 5.18 and the remarks preceding Example 5.5. We have

$$d = h \cdot \mathrm{lcm}(v_1, \ldots, v_t),$$

where h is a divisor of $m \cdot \gcd(v_1, \ldots, v_t)/\mathrm{lcm}(v_1, \ldots, v_t)$. We also know that $\gcd(c_{ij}, c_{k\ell}) = 1$, when $i \neq k$. We need one more observation before we can develop the final strategy that will allow us to develop the formula for $g(m, \ldots, m)$. Consider Theorem 2.3, one implication of this is that at most one of the partite sets can have vertices fixed by $\beta \in B(K_{n(m)})$. Now the strategy:

Pick the partite sets to be fixed (observe that we must have $1 \leq r \leq n - 2$). Let $v_1, \ldots, v_t$ be positive integers, with $v_i \geq 2$ $(i = 1, \ldots, t)$. Suppose that $\mathrm{lcm}(v_1, \ldots, v_t)$ is a divisor of $m \cdot \gcd(v_1, \ldots, v_t)$ and there exist positive integers $e_1, \ldots, e_t$ such that $\sum_{i=1}^{t} e_i v_i = n-r$. Partition the $n - r$ partite sets being cycled into e_1 sets of v_1 partite sets, e_2 sets of v_2 partite sets, ..., e_t sets of v_t partite sets, and order each of these sets of partite sets. Create one-to-one functions between the partite sets that respect this ordering. These functions generate m new objects for each of the e_i sets of v_i partite sets. Let h be a positive integer which is a

divisor of $m \cdot \gcd(\nu_1, \ldots, \nu_t)/\mathrm{lcm}(\nu_1, \ldots, \nu_t)$ and create an $[h \cdot \mathrm{lcm}(\nu_1, \ldots, \nu_t)/\nu_i]$-uniform permutation on these m objects. Set $d = h \cdot \mathrm{lcm}(\nu_1, \ldots, \nu_t)$; then the above creates a d-uniform permutation on the $n - r$ partite sets being cycled. Let $g \mid \gcd(m,d)$ and $c_i = \{c_{ij};\ j = 1, \ldots, s_i\}$ $(i = 1, \ldots, r)$, where $c_{ij} \mid (d/g)$, and $\gcd(c_{ij}, c_{k\ell}) = 1$, when $i \neq k$. [Note that when $g = 1$, at most one of the c_i can have a $c_{ij} = d$.] Also, make sure that

$$d = g \prod_{k=1}^{r} \mathrm{lcm}(c_{k\ell};\ \ell = 1, \ldots, s_k).$$

Set $\lambda_{ij} = d / c_{ij}$ and suppose that $\{\mu_{ij} \mid \mu_{ij} \geq 1\}$ is any solution of the set of equations

$$\sum_{j=1}^{s_i} \mu_{ij}\lambda_{ij} = m \quad (i = 1, \ldots, r).$$

Define $I(\beta) = \{ V_{i(k)} \mid i_k$ is one of the partite sets being fixed $\}$; let α_k be a permutation on $V_{i(k)}$ such that α_k has $\mu_{k\ell}$ cycles of length $\lambda_{k\ell}$ $(k = 1, \ldots, r$, and $\ell = 1, \ldots, s_k)$, where the μ's and the λ's are as above. Further let $H(\beta)$ be the other partite sets and let α_0 be the permutation, on the $n - r$ partite sets that cycle, that was created above. Then $\beta = \alpha_0\alpha_1 \ldots \alpha_r$ is a permutation in $B(K_{n(m)})$. Also, every such β can be formed this way; $\xi(\beta)$ can be used to identify the above $I(\beta)$, $H(\beta)$, d, λ's, and μ's.

The above strategy can be used to determine $|F(\beta)|$ and $|J(\beta)|$ for the above derived β.

Theorem 5.20. Let $\beta \in B(K_{n(m)})$ and suppose $|I(\beta)| = r$, where $I(\beta) = \{ V_{i(k)} \mid (i_k) \in \xi(\beta) \}$. Interpret V to be $\{ V_i;\ i = 1, \ldots, n \}$ and set $H(\beta) = V - I(\beta)$. Suppose

that $\beta|_{H(\beta)}$ is d-uniform and that its partite cycles have lengths $v_1, \ldots, v_t$; further, suppose that there are e_i partite cycles of length v_i $(i = 1, \ldots, t)$. Let h be a divisor of $m \cdot \gcd(v_1, \ldots, v_t)/\mathrm{lcm}(v_1, \ldots, v_t)$ such that the cycle length $d = h \cdot \mathrm{lcm}(v_1, \ldots, v_t)$. Let $g \mid \gcd(d,m)$, and let $c_i = \{c_{ij};\ j = 1, \ldots, s_i\}$, where $c_{ij} \mid (d/g)$; further $\gcd(c_{ij}, c_{k\ell})$ (when $i \neq k$) and

$$d = g \prod_{i=1}^{r} \mathrm{lcm}(c_{ij};\ j = 1, \ldots, s_i).$$

Set $\lambda_{ij} = d / c_{ij}$, and suppose that there are positive integers $\mu_{ij} \geq 1$ that satisfy the equations

$$\sum_{j=1}^{s_i} \mu_{ij}\lambda_{ij} = m \ (i = 1, \ldots, r).$$

Then,

$$|F(\beta)| = [n \cdot m - m - 1]!^{\frac{m(n-r)}{d}} \prod_{i=1}^{r} \prod_{j=1}^{s_i} \Phi(c_{ij},\ n \cdot m - m)^{\mu_{ij}},$$

where $\Phi(c,N) = \varphi(c) \cdot c^{(N-c)/c} \cdot (N/c - 1)!$, and φ is the Euler function.

Proof: If $v \in V_\iota \in I(\beta)$, then v appears in some cycle of length λ_{ij} (where $1 \leq i \leq r$, and $1 \leq j \leq s_i$). From the above conditions, and Theorem 5.19, we see $\beta^{\lambda_{ij}}|_{N(v)}$ is c_{ij}-uniform. Hence, by Theorem 2.5, $|F_v(\beta^{\lambda_{ij}})|$ is equal to $\Phi(c_{ij}, n \cdot m - m)$. There are μ_{ij} λ_{ij}-cycles, each of which needs a representative (by Theorem 2.4). Thus, the contribution of the λ_{ij}-cycles is given by

$$\prod_{i=1}^{r} \prod_{j=1}^{s_i} \Phi(c_{ij},\ n \cdot m - m)^{\mu_{ij}}.$$

We now consider $\beta|_{H(\beta)}$. Let $v \in V_\iota \in H(\beta)$; then $\lambda(v) = d$. By Theorem 5.17, β^d is the identity of

Aut $K_{n(m)}$ so it fixes all of the rotations at v. Now, $d(v) = n \cdot m - m$, $\forall v \in V$, so there are $(n \cdot m - m - 1)!$ rotations at v. Also, there are $m \cdot (n - r)$ vertices in the partite sets of $H(\beta)$; thus, there are $m \cdot (n - r)/d$ cycles. By Theorem 2.4, each of these cycles needs a representative. The contribution by the d-cycles is

$$(n \cdot m - m - 1)!^{(n-r)/d}.$$

Combining the above, using Theorem 2.4, we obtain the result. ∎

Theorem 5.21. Under the conditions of Theorem 5.20, $|J(\beta)|$

$$= \frac{n! \cdot m!^{n}}{r! \prod_{k=1}^{t} \left[v_k \cdot \left[\frac{d}{v_k} \right]^{m v_k / d} \cdot \left[\frac{m v_k}{d} \right]! \right]^{e_k} \cdot e_k! \prod_{i=1}^{r} \prod_{j=1}^{s_i} \lambda_{ij}^{\mu_{ij}} \mu_{ij}!}$$

Proof: Consider $\xi(\beta)$; this has r 1-cycles, e_1 v_1-cycles, e_2 v_2-cycles, ..., e_t v_t-cycles. Since $\xi(\beta)$ is a permutation from S_n, Theorem 2.6 applies. The number of such $\xi(\beta)$'s is therefore, $|J[\xi(\beta)]|$, that is

$$\frac{n!}{r! \prod_{k=1}^{t} v_k^{e_k} e_k!}$$

As in the proof of Theorem 5.11, there are $m!^{(v_k - 1)}$ ways of creating the one-to-one functions (m new objects) in each of the e_k sets of v_k partite sets.

Also, as in the proof of Theorem 5.11, there need to be (d/v_k) new objects in each cycle in order to produce a d-uniform permutation on the m new objects. By Theorem 2.6, the number of (d/v_k)-uniform permutations of a set with m elements is

$$\frac{m!}{\left[\frac{d}{v_k}\right]^{mv_k/d} \cdot \left[\frac{mv_k}{d}\right]!}.$$

Consider the fixed partite set $V_{t(i)}$ and suppose that α_i permutes it. Using the notation of Theorem 5.20, α_i has μ_{ij} λ_{ij}-cycles. Theorem 2.6 implies that the number of ways of producing such an α_i of the above type is

$$\frac{m!}{\prod_{j=1}^{s_i} \lambda_{ij}^{\mu_{ij}} \mu_{ij}!}.$$

Each of the partite cycles of $\xi(\beta)$ is disjoint of all others. Each of the partite sets is also disjoint of all others. We conclude that the number of permutations in $J(\beta)$ is the product of the above numbers given for each of the partite cycles and the number of permutations in the conjugacy class of $\xi(\beta)$. This is exactly the value given for $|J(\beta)|$ in the theorem. ∎

Example 5.9

Consider the graph $K_{6(2)}$ of Examples 5.5 - 5.7. Let $\xi(\beta) = (12)(34)(5)(6)$, $V_1 = \{a_1,b_1\}$, $V_2 = \{a_2,b_2\}$, $V_3 = \{a_3,b_3\}$, $V_4 = \{a_4,b_4\}$, $V_5 = \{a_5,b_5\}$, and $V_6 = \{a_6,b_6\}$. Let $\alpha_1 = (a_5b_5)$, and $\alpha_2 = (a_6b_6)$; further, suppose $\alpha_0 = \beta|_{H(\beta)} = (a_1a_2)(b_1b_2)(a_3a_4)(b_3b_4)$ and take $\beta = \alpha_0\alpha_1\alpha_2$. Note $\xi(\beta)|_{H(\beta)}) = (12)(34)$. Then $\lambda_{11} = \lambda_{21} = 2$, $\mu_{11} = \mu_{21} = 1$, $v_1 = 2$, $e_1 = 2$, $t = 1$, $r = 2$, $s_1 = s_2 = 1$, $n = 6$, $m = 2$, and $d = 2$ (thus, $h = 1$ and $g = 2$).

By Theorem 5.20,

$$|F(\beta)| = (6\cdot 2 - 2 - 1)!^{2\cdot(6-2)/2}\cdot\Phi(1,10)^1\cdot\Phi(1,10)^1$$
$$= 9!^4\cdot 9!\cdot 9! = 9!^6$$

$= 2,283,380,023,591,730,815,784,976,384,000,000.$

By Theorem 5.21,

$$|J(\beta)| = \frac{6!\cdot 2!^6}{2!\cdot(2\cdot(2/2)^{2\cdot 2/2}\cdot 2!)^2\cdot 2!\cdot(2^1\cdot 1!)\cdot(2^1\cdot 1!)}$$

$$= \frac{720}{4} = 180.$$

Let $\mathcal{F}(n,r,t) = \{ \{v_1,\ldots,v_t\} \mid v_i \geq 2\ (i = 1,\ldots,t),$ $\exists e_1,\ldots,e_t,\ \sum_{i=1}^{t} e_i v_i = n - r \}$; that is, $\mathcal{F}(n,r,t)$ is the set of all v_i's that can partition the $n - r$ partite sets in the manner under consideration. Given a set of v_i's let $\mathcal{E}$ be the set of all solutions to

$$\sum_{i=1}^{t} e_i v_i = n - r.$$

Note that $1 \leq r \leq n - 2$, and $1 \leq t \leq \lfloor (n - r)/2 \rfloor$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x. Furthermore, let h be a positive integer which is a divisor of $m\cdot\gcd(v_1,\ldots,v_t)/\mathrm{lcm}(v_1,\ldots,v_t)$ and set d equal to $h\cdot\mathrm{lcm}(v_1,\ldots,v_t)$. Let $g \mid \gcd(m,d)$ and $c = c_1 \times \ldots \times c_r$, where $c_i = \{c_{ij};\ j = 1,\ldots,s_i\}$, and $c_{ij} \mid (d/g)$, and $\gcd(c_{ij},c_{k\ell}) = 1$, $i \neq k$; further, g must be such that

$$d = g\prod_{i=1}^{r} \mathrm{lcm}(c_{ij};\ j = 1,\ldots,s_i),$$

and at most one c_i contains a $c_{ij} = d$. Set $\lambda_{ij} = d / c_{ij}$, and let μ be the set of all solutions $\{\mu_{ij}\}$, $\mu_{ij} \geq 1$, that satisfy the equations

$$\sum_{j=1}^{s_i} \mu_{ij}\lambda_{ij} = m.$$

The above conventions allow us to express the formula for $g(m,\ldots,m)$.

Theorem 5.22. The value of $g(m,\ldots,m)$, where there are n m's, is given by

$$\sum_{r=1}^{n-2} \sum_{t=1}^{\left\lfloor \frac{n-r}{2} \right\rfloor} \sum_{\mathcal{F}(n,r,t)} \sum_{\mathcal{E}} \sum_{h} \sum_{g} \sum_{c} \sum_{\mu} (\S)\cdot(**)$$

where $(\S)$ is

$$\frac{[n\cdot m - m - 1]!^{m(n-r)/d}}{r!\prod_{k=1}^{t}\left[v_k\cdot\left[\frac{d}{v_k}\right]^{mv_k/d}\cdot\left[\frac{mv_k}{d}\right]!\right]^{e_k}\cdot e_k!},$$

and $(**)$ is

$$\prod_{i=1}^{r}\prod_{j=1}^{s_i}\frac{\Psi(c_{ij},\, n\cdot m - m)^{\mu_{ij}}}{\lambda_{ij}^{\mu_{ij}}\mu_{ij}!}.$$

Proof: Since at least one of the partite sets must be fixed, $r \geq 1$; they cannot all be fixed, so $r \leq n - 2$. Therefore, the first sum must be over these values of r. There is at least one partite cycle of length $v \geq 2$; thus, $t \geq 1$. Also, since $v \geq 2$, no more than $\lfloor (n - r)/2 \rfloor$ can cccur. Actually, most of the time t can never reach this maximum value because all of the v's are distinct. The sum over $\mathcal{E}$ accounts for all possible solutions to the equation

$$\sum_{i=1}^{t} e_i v_i = n - r.$$

The value h ensures that all possible cycle lengths d on the partite sets that are cycled are accounted for by the formula. The value g allows for the condition that there is no vertex fixed by β as well as the condition that some might be fixed. The sum over c, together with that

over μ, ensure that no compatible set of permutations is overlooked. Finally, substitution of the values for $|F(\beta)|$, from Theorem 5.20, and $|J(\beta)|$, from Theorem 5.21, into the above eight-fold sum gives us $g(m,\ldots,m)$ as claimed above. ∎

Example 5.10.

Consider the graph $K_{6(2)}$; $r = 1, 2, 3, 4$ are the cases to be considered.

(1) Suppose that $r = 1$; then $n - 1 = 5 = 2 + 3$. We consider these two cases separately.

(a) Suppose that $t = 1$; then $v_1 = 5$, $e_1 = 1$, and $h = 1$ or $h = 2$. If $h = 1$ then $d = 5$. Now, $\gcd(5,2) = 1$, so $g = 1$. Therefore, $c_1 = \{1\}$, $\{5\}$, or $\{1,5\}$. Let $c_1 = \{1\}$; then $c_{11} = 1$. Now, $\lambda_{11} = 5 / 1 = 5 > m = 2$, so this case cannot occur. Since $\{1,5\}$ contains 1, the cycle length corresponding to this value of c would again be 5 and this case also cannot occur. Consider $c_1 = \{5\}$; here, $\lambda_{11} = 5/5 = 1$ and $\mu_{11} = 2$. The value for this term is

$$\frac{(6\cdot 2 - 2 - 1)!^{2\cdot 5/5}\cdot\Phi(5,6\cdot 2-2)^2}{1!\cdot(5\cdot(5/5)^{2\cdot 5/5}\cdot 2!)^1\cdot 1!\cdot 1^2\cdot 2!} = \frac{9!^2\cdot(4\cdot 5^1\cdot 1!)^2}{10\cdot 2}$$
$$= 2{,}633{,}637{,}888{,}000.$$

If $h = 2$ then $d = 2\cdot 5 = 10$. The $\gcd(10,2) = 2$, so $g = 1$ or $g = 2$. Suppose that $g = 1$; then $c_1 = \{1\}$, $\{2\}$, $\{5\}$, $\{10\}$, $\{1,2\}$, $\{1,5\}$, $\{1,10\}$, $\{2,5\}$, $\{2,10\}$, $\{5,10\}$, $\{1,2,5\}$, $\{1,2,10\}$, $\{1,5,10\}$, $\{2,5,10\}$, or $\{1,2,5,10\}$.

Let $c_1 = \{1\}$; then $c_{11} = 1$ and $\lambda_{11} = 10/1 = 10 > 2$, so this case cannot occur. Since 1 is in $\{1,2\}$, $\{1,5\}$, $\{1,10\}$, $\{1,2,5\}$, $\{1,2,10\}$, $\{1,5,10\}$, and $\{1,2,5,10\}$, none of these can occur either.

Let $c_1 = \{2\}$; then $c_{11} = 2$ and $\lambda_{11} = 10/2 = 5$ so this case cannot occur either. Since 2 is in $\{2,5\}$, $\{2,10\}$, and $\{2,5,10\}$, none of these cases can occur either.

Let $c_1 = \{5\}$; then $c_{11} = 5$ and $\lambda_{11} = 10/5 = 2$, so that $\mu_{11} = 1$. The value of this term is

$$\frac{(6\cdot 2 - 2 - 1)!\,2^{\cdot 5/10}\cdot\Phi(5,10)^1}{1!\cdot(5\cdot(10/5)^{2^{\cdot 5/10}}\cdot 1!)^1\cdot 1!\cdot 2^1\cdot 1!} = \frac{9!\cdot(4\cdot 5^1\cdot 1!)^1}{10\cdot 2}$$
$$= 362{,}880.$$

Let $c_1 = \{5,10\}$; then $c_{11} = 10$ and $c_{12} = 5$. Thus, we get $\lambda_{11} = 10/10 = 1$ and $\lambda_{12} = 10/5 = 2$. But $1 + 2 > 2$ so this case cannot occur.

Let $c_1 = \{10\}$; then $c_{11} = 10$. Hence, $\lambda_{11} = 1$, and $\mu_{11} = 2$. The value for this term is

$$\frac{(6\cdot 2 - 2 - 1)!\,2^{\cdot 5/10}\cdot\Phi(10,10)^2}{1!\cdot(5\cdot(10/5)^{2^{\cdot 5/10}}\cdot 1!)^1\cdot 1!\cdot 1^2\cdot 2!} = \frac{9!\cdot(4\cdot 10^0\cdot 0!)}{10\cdot 2}$$
$$= 72{,}576.$$

Suppose that $g = 2$; then $c_1 = \{1\}$, $\{5\}$, or $\{1,5\}$. If $c_1 = \{1\}$ then $c_{11} = 1$, so $\lambda_{11} = 10/1 = 10 > 2$. We conclude that this case cannot occur. Since $\{1,5\}$ contains 1, we see that this case cannot occur either.

If $c_1 = \{5\}$ then $\lambda_{11} = 2$ and $\mu_{11} = 1$. The value of this term is

$$\frac{(6\cdot 2 - 2 - 1)!\,2^{\cdot 5/10}\cdot\Phi(5,10)^1}{1!\cdot(5\cdot(10/5)^{2^{\cdot 5/10}}\cdot 1!)^1\cdot 1!\cdot(2^1\cdot 1!)}$$
$$= \frac{9!\cdot(4\cdot 5^1\cdot 1!)^1}{10\cdot 2} = 362{,}880.$$

Adding up the four subcases that survived in case 1a we find the total is 2,633,638,686,336.

(b) Suppose that $t = 2$. Now, $m = 2$, $\gcd(2,3) = 1$, and $\mathrm{lcm}(2,3) = 6$. Since $2\cdot 1/6 = 1/3$ is not an integer, this case cannot occur.

The total contribution of case 1, when $r = 1$, is 2,633,638,686,336.

(2) Suppose that $r = 2$; then $n - 2 = 4 = 2 + 2$. We conclude that $t = 1$ and there are two cases. We consider these cases separately.

(a) Let $v_1 = 4$; then $e_1 = 1$ and $h = 1$ or $h = 2$. If $h = 1$ then $d = 4$. The gcd(4,2) = 2, so $g = 1$ or $g = 2$. Suppose $g = 1$; then c_1, c_2 = {1}, {2}, {4}, {1,2}, {1,4}, {2,4}, or {1,2,4}. At first glance it would appear there are 49 cases; the coprime condition implies that one of them must be {1}. Suppose $c_1 = \{1\}$; then $\lambda_{11} = 4 > 2$ and we see that none of these cases can actually occur.

If $h = 2$, then $d = 8$. A similar analysis reveals that one of the sets c_i must be {1} and again we see that none of these cases will survive. We conclude that case 2a is not possible for $K_{6(2)}$.

(b) Let $v_1 = 2$; then $e_1 = 2$. Again, $h = 1$ or $h = 2$. If $h = 1$ then $d = 2$. The gcd(2,2) = 2, so $g = 1$ or $g = 2$. Suppose $g = 1$; then c_1, c_2 = {1}, {2}, or {1,2}. The coprime condition implies that one of the sets must be {1}, the other can be anything (except {1}).

Suppose $c_1 = c_2 = \{1\}$. This cannot occur; with $g = 1$ we would have $2 = d = 1\cdot(1\cdot1) = 1$. But $2 = 1$ is absurd!

Suppose $c_1 = \{1\}$ and $c_2 = \{2\}$; then $\lambda_{11} = 2/1 = 2$ and $\lambda_{21} = 2/2 = 1$. Hence, $\mu_{11} = 1$ and $\mu_{21} = 2$. The value of this term is

$$\frac{(6\cdot2 - 2 - 1)!^{2\cdot4/2}\cdot\Phi(1,10)^1\cdot\Phi(2,10)^2}{2!\cdot(2\cdot(2/2)^{2\cdot2/2}\cdot2!)^2\cdot2!\cdot(2!^1\cdot1!)\cdot(1!^2\cdot2!)}$$

$$= \frac{9!^4\cdot9!^1\cdot(1\cdot2^4\cdot4!)^2}{2\cdot4^2\cdot2\cdot2\cdot2}$$

$$= 9{,}438{,}574{,}832{,}968{,}464{,}020{,}275{,}200{,}000.$$

The symmetry of the formula reflects the symmetry of the graph; thus, $c_1 = \{2\}$ and $c_2 = \{1\}$ would yield the same value. We conclude that the total contribution for this case is 18,877,149,665,936,928,040,550,400,000.

Suppose $c_1 = \{1\}$ and $c_2 = \{1,2\}$; then $\lambda_{11} = 2$, $\lambda_{21} = 1$, and $\lambda_{22} = 2$. But $1 + 2 > 2$ so this case cannot occur. By symmetry, we see that $c_1 = \{1,2\}$ and $c_2 = \{1\}$ also cannot occur.

Suppose that $g = 2$; then $c_1 = c_2 = \{1\}$; then $\lambda_{11} = \lambda_{21} = 2$. We conclude that $\mu_{11} = \mu_{21} = 1$. The value of this term is

$$\frac{(6\cdot 2 - 2 - 1)!^{2\cdot 4/2}\cdot\Phi(1,10)\cdot\Phi(1,10)}{2!\cdot(2\cdot(2/2)^{2\cdot 2/2}\cdot 2!)^2\cdot 2!\cdot(2^1\cdot 1!)\cdot(2^1\cdot 1!)}$$

$$= \frac{9!^4\cdot 9!\cdot 9!}{2\cdot 4^2\cdot 2\cdot 2\cdot 2} = \frac{9!^6}{256}$$

$= 8,919,453,217,155,198,499,160,064,000,000.$

If $h = 2$ then $d = 4$. The $\gcd(4,2) = 2$, so $g = 1$ or $g = 2$. Suppose $g = 1$; then c_1, c_2 = $\{1\}$, $\{2\}$, $\{4\}$, $\{1,2\}$, $\{1,4\}$, $\{2,4\}$, or $\{1,2,4\}$. The coprime condition implies that one of the sets must be $\{1\}$. Suppose that $c_1 = \{1\}$; then $\lambda_{11} = 4/1 = 4 > 2$. We conclude that none of these cases can occur.

Similarly, if $g = 2$ then again the coprime condition would imply that one of the sets was $\{1\}$. Thus we see that none of these cases can occur.

Adding up the cases that actually contribute we see that the total for case 2 is

8,938,330,366,821,135,427,200,614,400,000.

(3) Suppose $r = 3$; then $n - 3 = 3$. We conclude $t = 1$, $\nu_1 = 3$, and $e_1 = 1$. Therefore, $h = 1$ or $h = 2$. If $h = 1$ then $d = 3$. The $\gcd(3,2) = 1$, so $g = 1$. Then c_1, c_2,

$c_3 = \{1\}$, $\{3\}$, or $\{1,3\}$. The coprime condition ensures that at least two of them are $\{1\}$. Suppose $c_1 = \{1\}$; then $\lambda_{11} = 3/1 = 3 > 2$ and none of these cases can occur.

If $h = 2$ then $d = 6$. The $\gcd(6,2) = 2$, so $g = 1$ or $g = 2$. The coprime condition guarantees that one of the c_i must be $\{1\}$. Suppose $c_1 = \{1\}$; then $\lambda_{11} = 6/1 = 6 > 2$. We see that none of these cases can happen either.

Combining the above we find that case 3 is not possible.

(4) Suppose $r = 4$; then $n - 4 = 2$. We conclude $t = 1$, $v_1 = 2$, and $e_1 = 1$. Therefore, $h = 1$ or $h = 2$. If $h = 1$ then $d = 2$. The $\gcd(2,2) = 2$, so $g = 1, 2$. Suppose $g = 1$; then c_1, c_2, c_3, $c_4 = \{1\}$, $\{2\}$, or $\{1,2\}$. The coprime condition ensures that at least three of them must be $\{1\}$.

Not all of them can be $\{1\}$, for $g = 1$; then we would have $2 = d = 1\cdot(1\cdot1\cdot1\cdot1) = 1$; $2 = 1$ is ridiculous!

Suppose $c_1 = c_2 = c_3 = \{1\}$ and $c_4 = \{2\}$; then $\lambda_{11} = \lambda_{21} = \lambda_{31} = 2/1 = 2$ and $\lambda_{41} = 2/2 = 1$. We conclude $\mu_{11} = \mu_{21} = \mu_{31} = 1$ and $\mu_{41} = 2$. The value of this term is

$$\frac{9!^{2\cdot2/2}\cdot[\Phi(1,10)^1]^3\cdot\Phi(2,10)^2}{4!\cdot(2\cdot(2/2)^{2\cdot2/2}\cdot2!)^1\cdot1!\cdot(2^1\cdot1!)^3\cdot(1^2\cdot2!)}$$

$$= \frac{9!^2\cdot9!^3\cdot(1\cdot2^4\cdot4!)}{24\cdot4\cdot2^3\cdot2}$$

$$= 1{,}573{,}095{,}805{,}494{,}744{,}003{,}379{,}200{,}000.$$

The symmetry of the graph says that any of them could have been the $\{2\}$; hence, the total contribution of this case is $6{,}292{,}383{,}221{,}978{,}976{,}013{,}516{,}800{,}000$.

Suppose that $c_1 = c_2 = c_3 = \{1\}$ and $c_4 = \{1,2\}$. Then $\lambda_{41} = 2/2 = 1$ and $\lambda_{42} = 2/1 = 2$. But $1 + 2 = 3 > 2$, so this case cannot occur. Symmetry considerations ensure

that none of them can be {1,2}.

Suppose $g = 2$; then $c_1 = c_2 = c_3 = c_4 = \{1\}$. We conclude that $\lambda_{11} = \lambda_{21} = \lambda_{31} = \lambda_{41} = 2 \not| 1\ 2$; therefore, we must have $\mu_{11} = \mu_{21} = \mu_{31} = \mu_{41} = 1$. The value of this term is

$$\frac{9!^{2\cdot 2/2}\cdot\Phi(1,10)^4}{4!\cdot(2\cdot(2/2)^{2\cdot 2/2}\cdot 2!)^1\cdot 1!\cdot(2^1\cdot 1!)^4}$$

$$= \frac{9!^2\cdot 9!^4}{24\cdot 4\cdot 2^4} = \frac{9!^6}{1536}$$

$$= 1,486,575,536,192,533,083,193,344,000,000.$$

If $h = 2$ then $d = 4$. The gcd(4,2) = 2, so $g = 1, 2$. Regardless of the value of g, at least one of c_1, c_2, c_3, or c_4 must be $\{1\}$. Suppose that $c_1 = \{1\}$; then we must have $\lambda_{11} = 4/1 = 4 > 2$. We conclude that this case cannot occur.

Combining all of the subcases of case 4 we find the the total contribution of case 4 is

1,492,867,919,414,512,059,206,860,800,000.

Combining the results of the four cases we find the value of $g(2,2,2,2,2,2)$ is

10,431,198,286,235,647,489,071,113,886,336.

We are now ready to generalize the formulas of Theorems 5.20 and 5.21. Suppose $K(m_1,\ldots,m_n)$ has q distinct orders of its partite sets: f_1 have order p_1, ..., f_q have order p_q. Let $\beta \in B[K(m_1,\ldots,m_n)]$ and suppose that $\xi(\beta)$ is the permutation of S_n corresponding to the partite cycles; then $\xi(\beta)$ has cycles of length one but not all of the cycles have length one.

As in the symmetric complete n-partite case, and let $I(\beta) = \{ V_i \mid (i) \in \xi(\beta) \}$, and $H(\beta) = V - I(\beta)$,

where V is viewed as $\{ V_i \mid i = 1,\ldots,n \}$ in the equation for $H(\beta)$.

Theorem 5.23. Let $\beta \in B[K(m_1,\ldots,m_n)]$; then $|F(\beta)| > 0$ if and only if $\beta|_{H(\beta)}$ is d-uniform, for some positive integer d, and all of the cycle lengths in $\beta|_{I(\beta)}$ are divisors of d.

Proof: The proof is almost identical to the proof of Theorem 5.17. ∎

Just as in the symmetric case, we need to determine conditions on d. The following generalization of Theorem 5.18 parallels the generalization of Theorem 5.9 to Theorem 5.13.

Theorem 5.24. Let $\beta \in B[K(m_1,\ldots,m_n)]$; suppose that for $i = 1,\ldots,q$, there are f_i partite sets of order p_i and suppose that β fixes r_i of the f_i partite sets of order p_i and cycles the other $f_i - r_i$ such partite sets in cycles of lengths v_{ij} ($j = 1, \ldots, t_i$). Furthermore, suppose that β is d-uniform on the partite sets being cycled; then $\mathrm{lcm}(v_{ij};\ j = 1, \ldots, t_i)$ is a divisor of d and d is a divisor of $p_i \cdot \gcd(v_{ij};\ j = 1, \ldots, t_i)$, whenever $f_i - r_i > 0$.

Proof: Clearly, if partite sets of order p_i actually do cycle, then these partite sets must obey Theorem 5.18. The fact that these partite sets are disjoint from all others makes that true. If all of the partite sets of a particular order are fixed by β, there is no reason to assume any relationship holds true. ∎

Let $I_i(\beta)$ be the partite sets of order p_i that are fixed by β and $H_i(\beta) = \mathcal{V}_i - I_i(\beta)$, where $\mathcal{V}_i$ is the set of the f_i partite sets of order p_i. Suppose $|I_i(\beta)| = r_i$, and let α_{ij} $(j = 1,\ldots,r_i)$ be the permutations on the individual partite sets in $I_i(\beta)$; we will consider conditions on these permutations. Observe that for every cycle length λ in α_{11}, then $\beta|_L$ is uniform, where if V_ι is the partite set upon which α_{11} is a permutation then $L = V - V_\iota$. Clearly, this must also be true for all α_{ij} $(i = 1,\ldots,q$, and $j = 1,\ldots,r_i$ having $I_i(\beta) \neq \emptyset)$ as well as for $\beta^\lambda|_{H(\beta)}$, where $H(\beta) = \bigcup_{i=1}^{q} H_i(\beta)$. [This is by Theorem 2.5.] Because the condition in Theorem 2.5 is an "if and only if" condition, we see that this compatibilty condition between all of the α_{ij}'s (for $i = 1, \ldots, q$, $I_i(\beta) \neq \emptyset$) must include the restriction of β on $H(\beta)$. Therefore, we have:

Theorem 5.25. Let $\beta \in B[K(m_1, \ldots, m_n)]$; suppose that $|I_i(\beta)| = r_i$ and that α_{ij} $(j = 1,\ldots,r_i)$ are the permutations on the individual partite sets of $I_i(\beta)$. Then $|F(\beta)| > 0$ if and only if $\{\beta|_{H(\beta)}, \alpha_{ij}; i = 1,\ldots,q, j = 1, \ldots,r_i\}$ is a compatible set of permutations.

Proof: See the remarks preceding this theorem. ∎

Theorems 5.24 and 5.25 are enough to allow us to modify the strategy that was given before Theorem 5.20 so that the determination of $|F(\beta)|$ and $|J(\beta)|$ can be done.

Using Theorem 5.24, it is not hard to show that we must have (assuming $\beta|_{H(\beta)}$ is d-uniform)

$$\mathrm{lcm}(\nu_{ij}; i = 1,\ldots,q,\ H_i(\beta) \neq \emptyset,\ j = 1,\ldots,t_i) \mid d$$

and

$d \mid \gcd(p_i \cdot \gcd(v_{ij}; j = 1,\ldots,t_i); i = 1,\ldots,q,\ H_i(\beta) \neq \emptyset)$.
Then for h a divisor of

$$\frac{\gcd(p_i \cdot \gcd(v_{ij}; j = 1,\ldots,t_i); i = 1,\ldots,q,\ H_i(\beta) \neq \emptyset)}{\operatorname{lcm}(v_{ij}; i = 1,\ldots,q,\ H_i(\beta) \neq \emptyset,\ j = 1,\ldots,t_i)},$$

$d = h \cdot \operatorname{lcm}(v_{ij}; i = 1,\ldots,q,\ H_i(\beta) \neq \emptyset,\ j = 1,\ldots,t_i)$.

Theorem 5.5 gives us conditions on the cycle lengths of a compatible set of permutations. Just as was remarked before Theorem 5.20 we must have (with a slight modification of the notation of Theorem 5.5)

$$d = g \prod_{\substack{i=1 \\ I_i(\beta) \neq \emptyset}}^{q} \prod_{j=1}^{r_i} \operatorname{lcm}(c_{ijk};\ k = 1,\ldots,s_{ij}),$$

where g is a divisor of $\gcd(d,\ p_i; i = 1,\ldots,q, I_i(\beta) \neq \emptyset)$. Now, recall that we just stated

$d = h \cdot \operatorname{lcm}(v_{i\ell};\ i = 1,\ \ldots,\ q,\ H_i(\beta) \neq \emptyset,\ \ell = 1,\ \ldots,\ t_i)$, where h is as given above. We also know, from Theorem 5.5, that $\gcd(c_{ijk},\ c_{\iota\eta\kappa}) = 1$, where $i \neq \iota$, or $j \neq \eta$, or both.

We now have the following strategy: Pick r_i partite sets from each of the f_i partite sets of order p_i to be fixed [observe that at least one of the $r_i \geq 1$, but not all of the $r_i \geq f_i - 1$; $i = 1,\ \ldots,\ q$]. Let v_{ij} $(i = 1,\ \ldots,\ q,\ j = 1,\ \ldots,\ t_i)$ be positive integers, each v_{ij} is at least two, and suppose that there exist nonnegative integers e_{ij} (i,j have same range) which satisfy:

$$\sum_{j=1}^{t_i} e_{ij} v_{ij} = f_i - r_i \quad (i = 1,\ \ldots,\ q).$$

Furthermore, suppose that $\operatorname{lcm}(v_{ij}; i=1,\ldots,q, H_i(\beta)\neq\emptyset, j=1,\ldots,t_i)$ is a divisor of $\gcd(p_i \cdot \operatorname{lcm}(v_{ij}; j=1,\ldots,t_i); i=1,\ldots,q, H_i(\beta)\neq\emptyset)$. For each $i = 1,\ \ldots,\ q$, where $f_i - r_i > 0$,

partition the $f_i - r_i$ partite sets of order p_i into e_{ij} sets of ν_{ij} partite sets each ($j = 1, \ldots, t_i$) and order each of these sets of partite sets. Create one-to-one functions between the partite sets that respect this ordering (these functions generate p_i new objects for each of the e_{ij} sets of ν_{ij} partite sets). Let h be a divisor of

$$\frac{\gcd(p_i \cdot \gcd(\nu_{ij}; j = 1, \ldots, t_i); i = 1, \ldots, q, H_i(\beta) \neq \emptyset)}{\operatorname{lcm}(\nu_{ij}; i = 1, \ldots, q, H_i(\beta) \neq \emptyset, j = 1, \ldots, t_i)};$$

set $d = h \cdot \operatorname{lcm}(\nu_{ij};\ i = 1, \ldots, q,\ H_i(\beta) \neq \emptyset,\ j = 1, \ldots, t_i)$ an create a d/ν_{ij}-uniform permutation on the p_i new objects created from each of the e_{ij} sets of ν_{ij} partite sets of order p_i. This creates a d-uniform permutation on $H(\beta) = \bigcup_{i=1}^{q} H_i(\beta)$. Let $g \mid \gcd(d, p_i; i=1, \ldots, q, I_i(\beta) \neq \emptyset)$ and $c_{ij} = \{c_{ijk}; k = 1, \ldots, s_{ij}\}$ ($i = 1, \ldots, q$, $I_i(\beta) \neq \emptyset$, $j = 1, \ldots, r_i$), where $c_{ijk} \mid (d/g)$, and $\gcd(c_{ijk}, c_{\iota\eta\kappa}) = 1$ (when $i \neq \iota$, or $j \neq \eta$, or both). Also be sure that

$$d = g \prod_{\substack{i=1 \\ I_i(\beta) \neq \emptyset}}^{q} \prod_{j=1}^{r_i} \operatorname{lcm}(c_{ijk};\ k = 1, \ldots, s_{ij}).$$

Set $\lambda_{ijk} = d / c_{ijk}$ and suppose that $\{\mu_{ijk} \mid \mu_{ijk} \geq 1\}$ is any solution of the underdetermined set of equations

$$\sum_{k=1}^{s_{ij}} \mu_{ijk}\lambda_{ijk} = p_i \quad (i = 1, \ldots, q,\ I_i(\beta) \neq \emptyset,\ j = 1, \ldots, r_i)$$

Define $I(\beta) = \bigcup_{i=1}^{q} I_i(\beta)$, where $I_i(\beta) = \{ V_{\iota(i,j)} \mid (\iota_{ij}) \in \xi(\beta),\ |V_{\iota(i,j)}| = p_i \}$; let α_{ij} be a permutation on $V_{\iota(i,j)}$ such that α_{ij} has μ_{ijk} cycles of length λ_{ijk} ($j = 1, \ldots, r_i$, $k = 1, \ldots, s_{ij}$), where the μ's and the λ's are as above. Denote the d-uniform permutation created

earlier as α_{0i}; then $\beta = \alpha_{00} \prod_i \prod_j \alpha_{ij} \in B[K(m_1, \ldots, m_n)]$, where $i = 1, \ldots, q$, $I_i(\beta) \neq \emptyset$, $j = 1, \ldots, r_i$. Also, every such β can be formed this way [$\xi(\beta)$ can be used to identify $I(\beta)$, $H(\beta)$, d, λ's, and μ's].

The strategy developed can be used to determine $|F(\beta)|$ and $|J(\beta)|$ for the above derived β.

Theorem 5.26. Let $\beta \in B[K(m_1, \ldots, m_n)]$ and suppose that $K(m_1, \ldots, m_n)$ has f_i partite sets of order p_i ($i = 1, \ldots, q$). Furthermore, suppose that β fixes r_i partite sets of the f_i partite sets of order p_i and cycles the other $f_i - r_i$ partite sets in cycles of length v_{ij} ($j = 1, \ldots, t_i$). Let $I_i(\beta)$ be the set of partite sets of order p_i that are fixed by β and for $V_{\iota(i,j)} \in I_i(\beta)$, let α_{ij} be β restricted to this partite set. Suppose that α_{ij} has μ_{ijk} cycles of length λ_{ijk}. Let $\mathcal{V}_i$ be the partite sets of order p_i and set $H_i(\beta) = \mathcal{V}_i - I_i(\beta)$. Take $H(\beta) = \bigcup_{i=1}^{q} H_i(\beta)$ and suppose that β is d-uniform on $H(\beta)$. Furthermore, suppose that on $H_i(\beta)$ there are e_{ij} partite cycles of length v_{ij} and let h be a divisor of

$$\frac{\gcd(p_i \cdot \gcd(v_{ij}; j = 1, \ldots, t_i); i = 1, \ldots, q, H_i(\beta) \neq \emptyset)}{\operatorname{lcm}(v_{ij}; i = 1, \ldots, q, H_i(\beta) \neq \emptyset, j = 1, \ldots, t_i)}$$

such that $d = h \cdot \operatorname{lcm}(v_{ij};\ i=1, \ldots, q,\ H_i(\beta) \neq \emptyset,\ j=1, \ldots, t_i)$. Let $g \mid \gcd(d, p_i; i=1, \ldots, q, I_i(\beta) \neq \emptyset)$ and let $c_{ij} = \{c_{ijk}; k = 1, \ldots, s_{ij}\}$ ($i = 1, \ldots, q$, $I_i(\beta) \neq \emptyset$, $j = 1, \ldots, r_i$), where $c_{ijk} \mid (d/g)$ and $\gcd(c_{ijk}, c_{\iota\eta\kappa}) = 1$ (when $i \neq \iota$, $j \neq \eta$, or both). Make sure that

$$d = g \prod_{\substack{i=1 \\ H_i(\beta) \neq \emptyset}}^{q} \prod_{j=1}^{r_i} \operatorname{lcm}(c_{ijk}; k = 1, \ldots, s_{ij}).$$

Set $\lambda_{ijk} = d / c_{ijk}$, and be sure $\lambda_{ijk} \leq p_i$; suppose there are positive integers $\mu_{ijk} \geq 1$ that satisfy

$$\sum_{k=1}^{s_{ij}} \mu_{ijk}\lambda_{ijk} = p_i \quad (i = 1,\ldots,q, I_i(\beta) \neq \emptyset, j = 1,\ldots,r_i).$$

Then

$$|F(\beta)| = \prod_{i=1}^{q} (P - p_i - 1)!^{\frac{p_i(f_i - r_i)}{d}} \prod_{j=1}^{r_i} \prod_{k=1}^{s_{ij}} \Phi(c_{ijk}, P - p_i)^{\mu_{ijk}},$$

where $P = \sum_i f_i p_i$, $\Phi(c,p) = \varphi(c)c^{(p-c)/c}(p/c - 1)!$, φ is the Euler function, and

$$|J(\beta)| = \prod_{i=1}^{q} \frac{f_i!\,p_i!^{f_i}}{(*)\cdot(\$)},$$

where (*) is

$$r_i!\prod_{\ell=1}^{t_i}\left[v_{i\ell}\left(\frac{d}{v_{i\ell}}\right]^{\frac{p_i v_{i\ell}}{d}} \cdot \left[\frac{p_i v_{i\ell}}{d}\right]!\right]^{e_{i\ell}} \cdot e_{i\ell}!,$$

and ($) is

$$\prod_{j=1}^{r_i}\prod_{k=1}^{s_{ij}} \lambda_{ijk}^{\mu_{ijk}} \cdot \mu_{ijk}!.$$

We assume that in a product where $t_i = 0$, or $r_i = 0$, the resulting empty product is 1.

Proof: The proof of $|F(\beta)|$ is almost identical to that of Theorem 5.20. The proof of $|J(\beta)|$ is almost identical to that of Theorem 5.21. The conventions on the product allow the formula to work when all of the partite sets of a single order are fixed or all are cycled. ∎

Example 5.11

Consider the graph $K_{1,1,2,2}$. Let $\xi(\beta) = (12)(3)(4)$. Then $q = 2$, $p_1 = 1$, $p_2 = 2$, $f_1 = f_2 = 2$, $r_1 = 0$, $r_2 = 2$, $t_1 = 1$,

$t_2 = 0$, $v_{11} = 2$, $e_{11} = 1$, $d = 2$, and $h = 1$ are obtainable from $\xi(\beta)$. Suppose $V_1 = \{0\}$, $V_2 = \{1\}$, $V_3 = \{2,3\}$, and $V_4 = \{4,5\}$ and let $\beta = (01)(2)(3)(45)$; then $\lambda_{211} = 1$, $\lambda_{221} = 2$, $\mu_{211} = 2$, and $\mu_{221} = 1$ are discernable. We conclude that $c_{211} = 2/1 = 2$, and $c_{221} = 2/2 = 1$.

The formula for $|F(\beta)|$ gives us

$$(6-1-1)!^{1\cdot(2-0)/2}\cdot 1\cdot(6-2-1)!^{2\cdot(2-2)/2}\cdot\Phi(2,4)^2\cdot\Phi(1,4)^1$$
$$= 4!\cdot(1\cdot 2^1\cdot 1!)^2\cdot 3! = 576.$$

This agrees with the entry of Table 5.2.

The formula for $|J(\beta)|$ gives us

$$\frac{2!\cdot 1^2\cdot 2!\cdot 2^2}{0!\cdot(2\cdot(2/2)^{1\cdot 2/2}\cdot 1!)^1\cdot 1!\cdot 1\cdot 1\cdot(1^2\cdot 2!)\cdot(2^1\cdot 1!)}$$
$$= \frac{2\cdot 2\cdot 2^2}{2\cdot 2\cdot 2} = 2.$$

This again agrees with Table 5.2.

Let $\mathbb{R}(p_1, f_1, \ldots, p_q, f_q)$ be the set of all values of r_i with at least one of the r's nonzero. Let $\mathfrak{T}(\mathbb{R})$ be the set of all t's corresponding to the ways the partite sets can be partitioned and $\mathfrak{F}(\mathfrak{T})$ the set of all v's that show how the partite sets can be cycled. Let $\mathfrak{E}$ be the coefficients that are solutions to the set of equations

$$\sum_{\ell=1}^{t_i} e_{i\ell}v_{i\ell} = f_i - r_i \quad (i = 1,\ldots,q,\ H_i(\beta) \neq \emptyset).$$

Let h be a divisor of

$$\frac{\gcd(p_i\cdot\gcd(v_{i\ell};\ \ell = 1,\ldots,t_i);\ i = 1,\ldots,q,\ H_i(\beta)\neq\emptyset)}{\operatorname{lcm}(v_{i\ell};\ i = 1,\ldots,q,\ H_i(\beta)\neq\emptyset,\ \ell = 1,\ldots,t_i)}$$

and set $d = h\cdot\operatorname{lcm}(v_{i\ell}; i=1,\ldots,q, H_i(\beta)\neq\emptyset, \ell=1,\ldots,t_i)$. Let $g \mid \gcd(d, p_i; i=1,\ldots,q, I_i(\beta)\neq\emptyset)$, $c = c_{11}\times\ldots\times c_{qr(q)}$, where $c_{ij} = \{c_{ijk};\ k = 1,\ \ldots,\ s_{ij}\}$, $c_{ijk} \mid (d/g)$ and $\gcd(c_{ijk}, c_{\iota\eta\kappa}) = 1$ (when $i \neq \iota$, or $j \neq \eta$, or both). Make

sure that

$$d = g_i \prod_{\substack{i=1 \\ H_i(\beta) \neq \emptyset}}^{q} \prod_{j=1}^{r_i} \operatorname{lcm}(c_{ijk}; k = 1,\ldots,s_{ij}).$$

Set $\lambda_{ijk} = d / c_{ijk}$, and be sure that $\lambda_{ijk} \le p_i$; let μ be the set of all solutions $\{\mu_{ijk} \ge 1; k = 1,\ldots,s_{ij}\}$ to

$$\sum_{k=1}^{s_{ij}} \mu_{ijk}\lambda_{ijk} = p_i \quad (i = 1,\ldots,q, I_i(\beta) \neq \emptyset, j = 1,\ldots,r_i).$$

Theorem 5.27. Assume the conditions of Theorem 5.26 and the notation introduced before this theorem; then the value of $g(m_1,\ldots,m_n)$ is given by

$$\sum_{\mathbb{R}} \sum_{\mathfrak{T}} \sum_{\mathfrak{F}} \sum_{\mathfrak{E}} \sum_{h} \sum_{g} \sum_{c} \sum_{\mu} \prod_{i=1}^{q} (\sim)\cdot(**),$$

where $(\sim)$ is

$$\frac{(P - p_i - 1)!^{\,d}{}^{\frac{p_i(f_i - r_i)}{}}}{r_i! \prod_{\ell=1}^{t_i} \left[\nu_{i\ell} \left[\dfrac{d}{\nu_{i\ell}} \right]^{\frac{p_i \nu_{i\ell}}{d}} \cdot \left[\dfrac{p_i \nu_{i\ell}}{d} \right]! \right]^{e_{i\ell}} \cdot e_{i\ell}!}$$

and where $(**)$ is

$$\prod_{j=1}^{r_i} \prod_{k=1}^{s_{ij}} \frac{\Phi(c_{ijk}, P - p_i)^{\mu_{ijk}}}{\lambda_{ijk}^{\mu_{ijk}} \cdot \mu_{ijk}!},$$

where $P = f_1 p_1 + \ldots + f_q p_q$.

Proof: The sum over $\mathbb{R}$ ensures that all possibilities for fixing partite sets and still cycling some others are taken into account. The sum over $\mathfrak{T}$ allows for all possible numbers of distinct partite cycle lengths that could partition the partite sets which cycle. The sum over $\mathfrak{F}$ accounts for all possible lengths of partite cycles that

could partition the cycling partite sets into the appropriate numbers of distinct partite cycle lengths. The sum over $\mathbb{E}$ accounts for all ways in which these given partite cycle lengths can be chosen to add up to the number of partite sets which cycle. The sum over h accounts for all possible cycle lengths d on $H(\beta)$. The sum over g ensures that all possible cycle lengths of β on $I(\beta)$ are explored. The sum over c accounts for all compatible cycle lengths on β to be determined and the sum over μ generates these compatible sets of permutations. The values of $(\tilde{\ })$ and $(^{**})$ are obtained by substitution of the formulas for $|F(\beta)|$ and $|J(\beta)|$ into this eight-fold sum and then dividing by the order of the automorphism group. ∎

Example 5.12.

Consider again the graph $K_{1,1,2,2}$ of Example 5.11. Here we see that $p_1 = 1$, $p_2 = 2$, and $f_1 = f_2 = 2$. There are two possibilities in $\mathbb{R}$: $r_1 = 0$, $r_2 = 2$ and $r_1 = 2$, $r_2 = 0$.

(1) Suppose $r_1 = 0$ and $r_2 = 2$; then $t_1 = 1$, $t_2 = 0$, $v_{11} = 2$, and $e_{11} = 1$. We see that h is a divisor of $1 \cdot 2 / 2 = 1$; that is, $h = 1$. Thus, $d = 1 \cdot 2 = 2$ and g is a divisor of $\gcd(2,2) = 2$, so $g = 1, 2$. Now $c_{11} = \emptyset$ (as $r_1 = 0$) and since $2/1 = 2$, $c_{21} = c_{22} = \{1\}, \{2\}, \{1,2\}$. The coprime condition implies that at least one of the c's must be $\{1\}$.

(a) Suppose $c_{21} = c_{22} = \{1\}$ and $g = 1$; we see that we must have $2 = d = 1 \cdot (1 \cdot 1) = 1$; that is, $2 = 1$, a contradiction.

(b) Suppose $c_{21} = \{1\}$, $c_{22} = \{2\}$, and $g = 1$; then $c_{211} = 2$, $c_{221} = 1$, $\lambda_{211} = 2/2 = 1$, $\lambda_{221} = 2/1 = 2$, $\mu_{211} = 2$, and

$\mu_{221}= 1$. The value of this term is

$$\frac{(6-1-1)!^{1(2-0)/2}\cdot(6-2-1)!^{2(2-2)/2}\cdot 1\cdot[\Phi(2,6-2)^{2}\cdot\Phi(1,6-2)^{1}]}{0![2(2/2)^{1\cdot 2/2}\cdot 1!]^{1}\cdot 1!\cdot 2![1]\cdot(1^{2}\cdot 2!)\cdot(2^{1}\cdot 1!)}$$

$$= \frac{4!^{1}\cdot 5!^{0}\cdot[(1\cdot 2^{1}\cdot 1!)^{2}\cdot(1\cdot 1^{3}\cdot 3!)^{1}]}{[2]^{1}\cdot 2(2)(2)} = \frac{24\cdot 2^{2}\cdot 6}{16} = 36.$$

By symmetry considerations, we obtain the same value when $c_{21}= \{2\}$ and $c_{22}= \{1\}$. Therefore, the total contribution from case 1b is 72.

(c) Suppose $c_{21} = \{1\}$, $c_{22} = \{1,2\}$, and $g = 1$; then $c_{211}= 1$, $c_{221}= 2$, $c_{222}= 1$, $\lambda_{211}= \lambda_{222}= 2$, and $\lambda_{221}= 1$. But $\lambda_{211}+ \lambda_{221}= 1 + 2 > 2 = p_2$, so this case cannot happen.

(d) If $g = 2$; then $2/2 = 1$, so $c_{21}= c_{22}= \{1\}$. We conclude that $\lambda_{211}= \lambda_{221}= 2$ and $\mu_{211}= \mu_{221}= 1$. The value of this term is

$$\frac{(6-1-1)!^{1(2-0)/2}\cdot(6-2-1)!^{2(2-2)/2}\cdot 1\cdot[\Phi(1,6-2)^{1}\cdot\Phi(1,6-2)^{1}]}{0![2(2/2)^{1\cdot 2/2}\cdot 1!]^{1}\cdot 1!\cdot 2![1]\cdot(2^{1}\cdot 1!)\cdot(2^{1}\cdot 1!)}$$

$$= \frac{4!^{1}\cdot 5!^{0}\cdot[(1\cdot 1^{3}\cdot 3!)^{1}\cdot(1\cdot 1^{3}\cdot 3!)^{1}]}{[2]^{1}\cdot 2(2)(2)} = \frac{24\cdot 6\cdot 6}{16} = 54.$$

The contribution from case 1d is 54. We conclude that the total contribution from case 1 is 126.

(2) Suppose $r_1= 2$ and $r_2= 0$; then $t_1= 0$, $t_2= 1$, $\upsilon_{21}= 2$, $e_{21}= 1$. We see that h is a divisor of $2\cdot 2 / 2 = 2$, so $h = 1, 2$.

(a) Suppose $h = 1$; then $d = 1\cdot 2 = 2$. The $\gcd(2,1) = 1$, so $g = 1$. Since $2/1 = 2$, $c_{11}= c_{12}= \{1\}, \{2\}, \{1,2\}$. The coprime condition implies that at least one of the c's is $\{1\}$. Suppose $c_{11}= \{1\}$; then $c_{111}= 1$ and $\lambda_{111}= 2/1 = 2$. But $2 = \lambda_{111} > p_1= 1$, so these cases cannot occur.

(b) Suppose $h = 2$; then $d = 2\cdot 2 = 4$. The $\gcd(4,1) = 1$, so $g = 1$. Since $4/1 = 4$, $c_{11}= c_{12}= \{1\}, \{2\}, \{4\}, \{1,2\}$,

{1,4}, {2,4}, or {1,2,4}. The coprime condition implies that one of the c's is {1}. Suppose $c_{11} = \{1\}$; then $c_{111} = 1$ and $\lambda_{111} = 4/1 = 4$. Now $4 = \lambda_{111} > p_1 = 1$, so none of these cases can occur.

We conclude that case 2 is not possible; hence, the value of g(1,1,2,2) is 126. This value is confirmed by the calculations of Example 5.1.

In the next section we will display the formulas for the symmetric complete n-partite graph and the general complete n-partite graph. We will show how these formulas generalize the formulas for the complete, complete bipartite, and complete tripartite cases. In addition, we will show that by adopting two conventions, the formula for the number of congruence classes of maps of the general complete n-partite graph can be simplified.

5.5 The General Formula

In this section we will give the general formula for the number of congruence class of maps of complete n-partite graphs. We will give explicit formulas for both the symmetric and general cases. We will illustrate how these formulas generalize the formulas for the three cases that have gone before: the complete, complete bipartite, and complete tripartite graphs.

We begin by presenting the formula for the number of congruence classes for the maps of symmetric complete n-partite graphs. We then show how this formula generalizes the formula for the complete graph.

Theorem 5.28. The number of congruence classes of maps of $K_{n(m)}$ is given by

$$|C(K_{n(m)})| = f(m,\ldots,m) + g(m,\ldots,m) + h(m,\ldots,m),$$

where $f(m,\ldots,m)$ is

$$\frac{1}{n!}\sum_{g}\sum_{c}\sum_{\mu}\prod_{i=1}^{n}\prod_{j=1}^{s_i}\frac{\Phi(c_{ij},\, n\cdot m - m)^{\mu_{ij}}}{\lambda_{ij}^{\mu_{ij}}\cdot \mu_{ij}!},$$

where $\Phi(c,m) = \varphi(c)\cdot c^{(m-c)/c}\cdot(m/c - 1)!$, φ is the Euler function, $g \mid m$, $c = c_1 \times \ldots \times c_n$, $c_i = \{c_{ij}; j = 1, \ldots, s_i\}$, $c_{ij} \mid (m/g)$, $\gcd(c_{ij}, c_{k\ell}) = 1$ (where $i \neq k$),

$$\lambda_{ij} = \frac{g}{c_{ij}}\prod_{k=1}^{s_k}\operatorname{lcm}(c_{k\ell};\ \ell = 1, \ldots, s_k),$$

and $\mu = \{\mu_{ij} \geq 1;\ i = 1, \ldots, n,\ j = 1, \ldots, s_i\}$ is the set of all solutions to

$$\sum_{j=1}^{s_k}\mu_{ij}\lambda_{ij} = m \quad (i = 1, \ldots, n);$$

$g(m,\ldots,m)$ is

$$\sum_{r=1}^{n-2}\ \sum_{t=1}^{\left\lfloor\frac{n-r}{2}\right\rfloor}\sum_{\mathfrak{F}}\sum_{\mathfrak{E}}\sum_{h}\sum_{g}\sum_{c}\sum_{\mu}(*)\cdot(**),$$

where $(*)$ is

$$\frac{(m\cdot(n-1) - 1)!^{\frac{m(n-r)}{d}}}{r!\cdot\prod_{k=1}^{t}\left[\nu_k\left[\frac{d}{\nu_k}\right]^{\frac{m\nu_k}{d}}\cdot\left[\frac{m\nu_k}{d}\right]!\right]^{e_k}\cdot e_k!},$$

and $(**)$ is

$$\prod_{i=1}^{r}\prod_{j=1}^{s_i}\frac{\Phi(c_{ij},\, n\cdot m - m)^{\mu_{ij}}}{\lambda_{ij}^{\mu_{ij}}\cdot \mu_{ij}!},$$

where r is the number of partite sets being fixed, t is the number of distinct partite set cycle lengths, $\mathfrak{F}$ is the set of all possible partite cycle lengths having t distinct values, $\mathfrak{E}$ is the set of all possible solutions to the equation

$$\sum_{i=1}^{t} e_i \upsilon_i = n - r,$$

h is a divisor of

$$\frac{m \cdot \gcd(\upsilon_1, \ldots, \upsilon_t)}{\mathrm{lcm}(\upsilon_1, \ldots, \upsilon_t)},$$

$d = h \cdot \mathrm{lcm}(\upsilon_1, \ldots, \upsilon_t)$, g is a divisor of gcd(d,m), c and μ are as above, except now $c_{ij} \mid (d/g)$ $(i = 1, \ldots, r,\ j = 1, \ldots, s_i)$; and $h(m, \ldots, m)$ is

$$\sum_{t=1}^{\lfloor \frac{n-2}{2} \rfloor} \sum_{\mathfrak{F}} \sum_{\mathfrak{E}} \sum_{h} \frac{(m \cdot (n-1) - 1)!^{\frac{m \cdot n}{d}}}{\prod_{i=1}^{t} \left[\upsilon_i \left[\frac{d}{\upsilon_i} \right]^{\frac{m\upsilon_i}{d}} \cdot \left[\frac{m\upsilon_i}{d} \right]! \right]^{e_i} \cdot e_i!},$$

where t, $\mathfrak{F}$, $\mathfrak{E}$, and h and d are as described above.

Proof: The result follows directly from Theorems 5.7, 5.12, and 5.22. ∎

Consider what happens to the above formula when we have m = 1. We first look at $f(1, \ldots, 1)$. Here, $g \mid m$, so g = 1; $c_{ij} \mid (m/g)$, so every $c_{ij} = 1$. We conclude that $\lambda_{ij} = 1$, and $\mu_{ij} = 1$ $(i = 1, \ldots, n,\ j = 1)$; that is, we are talking about the identity element. It is easy to show that $\Psi(1, n-1) = (n-2)!$; hence,

$$f(1, \ldots, 1) = (n-2)!^{n} / n!\,.$$

We now look at $g(1, \ldots, 1)$. The compatibility condition makes us conclude that only r = 1 will contribute to $g(1, \ldots, 1)$. This same compatibility condition also

requires that $t = 1$. Therefore, $v_1 | (n-1)$, $e_1 = (n-1)/v_1$, $h = 1$, so $d = v_1$ and $e_1 = (n-1)/d$. Also, $c_{11} = d$, $\lambda_{11} = 1$, and $\mu_{11} = 1$. We conclude, that for each divisor of $n - 1$, greater than one, the term is

$$\frac{(n-2)!^{(n-1)/d} \cdot \Psi(d,\ n-1)^1}{d^{(n-1)/d} \cdot [(n-1)/d]! \cdot (1^1 \cdot 1!)}$$

$$= \frac{(n-2)!^{(n-1)/d} \cdot \varphi(d) \cdot d^{((n-1)/d\ -1)}((n-1)/d\ -1)!}{d^{(n-1)/d}((n-1)/d)!}$$

$$= \frac{\varphi(d) \cdot (n-2)!^{(n-1)/d}}{n-1}$$

We also note that with so many of the sums taking only one value, the eight-fold sum reduces to a simple sum over $v_1 = d \geq 2$, a divisor of $n - 1$. Thus,

$$g(1,\ldots,1) = \sum_{\substack{d \,|\, (n-1) \\ d \neq 1}} \frac{\varphi(d) \cdot (n-2)!^{(n-1)/d}}{n-1}.$$

Finally we turn to $h(1,\ldots,1)$. The uniformity condition, together with the fact that $m = 1$, implies that $t = 1$. Thus, $v_1 \mid n$, $e_1 = n/v_1$, $h = 1$, so $d = v_1$ and $e_1 = n/d$. We conclude that for each divisor of n, greater than one, the term is

$$\frac{(n-2)!^{n/d}}{d^{n/d} \cdot (n/d)!}$$

We also note that the four-fold sum reduces to a simple sum over $v_1 = d \geq 2$, a divisor of n.

$$h(1,\ldots,1) = \sum_{\substack{d \,|\, n \\ d \neq 1}} \frac{(n-2)!^{n/d}}{d^{n/d} \cdot (n/d)!}.$$

Notice that the form of $f(1,\ldots,1)$ is the same as that of $h(1,\ldots,1)$ above with $d = 1$. Therefore, we obtain

$$|C(K_{n(1)})| = |C(K_n)|$$

$$= \sum_{d \mid n} \frac{(n-2)!^{n/d}}{d^{n/d}\cdot(n/d)!} + \sum_{\substack{d \mid (n-1) \\ d \neq 1}} \frac{\varphi(d)\cdot(n-2)!^{(n-1)/d}}{n-1}.$$

Compare this to the formula for $|C(K_n)|$ given in Chapter II. Thus, the formula for $K_{n(m)}$ does generalize that of K_n.

Before we show how the formula for the general complete n-partite graph generalizes those of complete bipartite and complete tripartite graphs, we show how to simplify the formulas. We have already employed one simplification in the formula; that is, when we would obtain an empty product we have replaced that product with one. We now show how two other conventions will allow us to replace the three functions f, g, and h, with a single function, g.

The first convention that we will employ is that when dealing with an empty sum, the summation symbol and those terms governed by the index of the summation are simply removed from consideration.

Consider the formula for $g(m_1,\ldots,m_n)$ given in the last section. When all $r_i = 0$ $(i = 1,\ldots,q)$, the sum over $\mathfrak{R}$ is dropped (as are those over h, c, g, and μ which depend on it) and the r's are removed from consideration (so are the Φ, λ's, and μ's); the empty products that remain are replaced with one (1). That is,

$$\sum_{\mathfrak{R}} \sum_{\mathfrak{I}} \sum_{\mathfrak{F}} \sum_{\mathfrak{E}} \sum_{h} \sum_{g} \sum_{c} \sum_{\mu} \prod_{i=1}^{q} (\$)\cdot(\$\$),$$

where ($) is

$$\frac{(P - p_i - 1)!^{p_i(f_i - r_i)/d}}{r_i! \prod_{\ell=1}^{t} \left[\nu_{i\ell} \left(\frac{d}{\nu_{i\ell}} \right)^{d} \left(\frac{p_i \nu_{i\ell}}{d} \right)! \right]^{\frac{p_i \nu_{i\ell}}{d}} e_k^{\,e_k}!},$$

and where ($$) is

$$\prod_{j=1}^{r_i} \prod_{k=1}^{s_{ij}} \frac{\Phi(c_{ijk}, P - p_i)^{\mu_{ijk}}}{\lambda_{ijk}^{\mu_{ijk}} \cdot \mu_{ijk}!}$$

becomes

$$\sum_{\mathfrak{T}} \sum_{\mathfrak{F}} \sum_{\mathfrak{E}} \sum_{h} \frac{(P - p_i - 1)!^{p_i f_i/d}}{\prod_{\ell=1}^{t} \left[\nu_{i\ell} \left(\frac{d}{\nu_{i\ell}} \right)^{d} \left(\frac{p_i \nu_{i\ell}}{d} \right)! \right]^{\frac{p_i \nu_{i\ell}}{d}} e_k^{\,e_k}!}.$$

The above is the formula for $h(m_1, \ldots, m_n)$

The second convention is that the conditions on the remaining portions are reinterpreted in terms of the new model.

Consider again the formula for $g(m_1, \ldots, m_n)$. Suppose that all $t_i = 0$ $(i = 1, \ldots, q)$; then $r_i = f_i$ and the sums over $\mathfrak{T}$, $\mathfrak{F}$, $\mathfrak{E}$, and h vanish. The result is

$$\sum_{g} \sum_{c} \sum_{\mu} \prod_{i=1}^{q} \frac{1}{f_i!} \prod_{j=1}^{f_i} \prod_{k=1}^{s_{ij}} \frac{\Phi(c_{ijk}, P - p_i)^{\mu_{ijk}}}{\lambda_{ijk}^{\mu_{ijk}} \cdot \mu_{ijk}!}.$$

This is just a rewriting of $f(m_1, \ldots, m_n)$ when we reinterpret as follows: Originally, g was a divisor of gcd(d, p_i; $i = 1, \ldots, q$, $I_i(\beta) \neq \emptyset$); now it is a divisor of only $\gcd(p_1, \ldots, p_q)$. Originally, the $c_{ijk} \mid (d/g)$; now they are divisors of $\gcd(p_i;\ i = 1, \ldots, q)/g$ [if $f_i = 1$ then it $\gcd(p_k;\ k \neq i)/g$]. Originally, $\lambda_{ijk} = d / c_{ijk}$; now they are

$$\frac{g}{c_{ijk}} \prod_{\iota=1}^{q} \prod_{\eta=1}^{f_\iota} \mathrm{lcm}(c_{\iota\eta\varrho};\ \varrho = 1,\ldots,s_{\iota\eta}).$$

Thus we see that with these two conventions the only formula needed is that of $g(m_1,\ldots,m_n)$.

For the moment, however, we find it easier to work with the three functions. Consider the case where $n = 2$. The formula for $f(m_1,m_2)$ is

$$\xi \sum_{g}\sum_{c}\sum_{\mu} \prod_{i=1}^{2} \prod_{j=1}^{s_i} \frac{\Phi(c_{ij},m_{3-i})^{\mu_{ij}}}{\lambda_{ij}^{\mu_{ij}} \cdot \mu_{ij}!}$$

where $\xi = 1$ $(m_1 \neq m_2)$, or $1/2$ $(m_1 = m_2)$. This is exactly the same as the form given in Chapter III. Similarly, when $n = 3$, we get the same formula as in Chapter IV. Thus, the formula for f generalizes those from before.

The formula for $g(m_1,m_2)$ does not exist—just as we found in Chapter III. When $n = 3$, we have three cases. If all three partite sets have distinct orders, then we note that all of the partite sets must be fixed. Thus, $g(m_1,m_2,m_3) = 0$ (as in Chapter IV). If two of the partite sets have the same order but the third has different order then we can assume the orders are p_1 and p_2 where $f_1 = 1$ and $f_2 = 2$. Also, $r_1 = 1$, $r_2 = 0$, $t_1 = 0$, $t_2 = 1$, $v_{21} = 2$, and $e_{21} = 1$. Now, $h \mid p_2$ and $d = 2h$; $g \mid \gcd(p_1,d)$ and $c_{1jk} \mid d/g$. Putting these into the formula for $g(p_1,p_2,p_2)$ we get

$$\sum_{h}\sum_{g}\sum_{c}\sum_{\mu} \frac{(p_1 + p_2 - 1)!^{p_2/d}}{2 \cdot h^{p_2/d} \cdot (p_2/d)!} \prod_{j=1}^{s_{11}} \frac{\Phi(c_{11j},2p_2)^{\mu_{11j}}}{\lambda_{11j}^{\mu_{11j}} \cdot \mu_{11j}!}$$

The sums over g and c can be combined into a single sum over λ_{11}, where $\lambda_{11j} \mid \gcd(p_1,2h)$. Therefore, we get

$$\sum_{h \mid p_2} \sum_{\lambda_{11}} \sum_{\mu} \frac{(p_1 + p_2 - 1)!^{p_2/h}}{2 \cdot h^{p_2/d} \cdot (p_2/d)!} \prod_{j=1}^{s_{11}} \frac{\Phi(c_{11j}, 2p_2)^{\mu_{11j}}}{\lambda_{11j}^{\mu_{11j}} \cdot \mu_{11j}!}$$

This is the same formula obtained in Chapter IV.

When we have $m_1 = m_2 = m_3 = p_1$ then $f_1 = 3$, $r_1 = 1$, $v_1 = 2$, $e_1 = 1$, and $t_1 = 1$. Now $h \mid p_1$ and $d = 2h$. Also, $g \mid \gcd(d, p_1)$ and we see that substitution into the above is equivalent to replacing p_2 by p_1 in the the above formula. This is exactly what was noted in Chapter IV.

The formula for $h(m_1, m_2)$ is considered now. When $m_1 \neq m_2$ then the partite sets are fixed and $h(m_1, m_2) = 0$. We consider $h(p_1, p_1)$; $t_1 = 1$, $v_1 = 2$, $e_1 = 1$, $h \mid p_1$, $d = 2h$.

Then, $$h(p_1, p_1) = \sum_{h \mid p_1} \frac{(p_1 - 1)!^{p_1/h}}{2 \cdot h^{p_1/h} (p_1/h)!},$$

which is the same as the formula in Chapter III.

When $n = 3$, by similar analysis, we get the same as the result in Chapter IV.

The final chapter is concerned with random topological graph theory. This twig on the branch of topological graph theory was introduced by White and Schwenk in [19]. Further studies in random topological graph theory appear in [11], [12], and [14]. We will end by presenting some questions that have been generated by the topics touched in this dissertation.

CHAPTER VI

RANDOM TOPOLOGICAL GRAPH THEORY: ASYMPTOTIC RESULTS

6.1 Introduction

In this chapter we will relate the work of the previous chapters to a recent offshoot of topological graph theory called random topological graph theory. This topic was introduced by White and Schwenk in [19]. We will introduce a new parameter, the average number of symmetries of the maps of a connected graph G. We will denote this $\overline{m}(G)$.

The way we will relate the previous work to this new parameter is through asymptotic approximations. These approximations will be worked out for the complete graphs and the complete bipartite graphs, but it seems quite likely that they also apply to complete n-partite graphs in general.

We will end this dissertation by presenting some problems for further exploration. These problems will not only relate to enumerative problems in graph theory, but some will also relate to topological problems, and to random topological problems, as well.

6.2 The Average Number of Symmetries

In this section we will derive a formula for the

average number of symmetries of the maps of a connected graph G. This formula will be in terms of the degree sequence (as it relates to the rotations on G), the number of congruence classes for the maps of the graph, and the number of automorphisms of the graph.

Let G be a connected graph of order n. Suppose that $V(G) = \{v_1, v_2, \ldots, v_n\}$ and that $d(v_i) = d_i$ ($i = 1, \ldots, n$). Without loss of generality we may assume that the vertices have been labeled so that $d_1 \leq d_2 \leq \ldots \leq d_n$, but this is not really important to the discussion.

Recall that for all $v \in V(G)$, the number of rotations at v is $(d(v) - 1)!$ and that a rotation on G is an indexed set of rotations, one at each vertex of G. Let R(G) denote the set of all possible rotations on G; then it is clear that $|R(G)| = \prod_{i=1}^{n} (d_i - 1)!$.

The group of automorphisms for a graph G is denoted Aut G. Also, if $M = (G, \rho)$, where $\rho \in R(G)$, then Aut M is the group of automorphisms for the map M. Note that |Aut M| is the number of symmetries of the map M; however, we will introduce a simpler notation:

$$m(\rho) = |\mathrm{Aut}\ M|,$$

where $M = (G, \rho)$ and $\rho \in R(G)$. Furthermore,

$$C(\rho) = \{ \sigma \in R(G) \mid \exists \alpha \in \mathrm{Aut}\ G, \alpha(\rho) = \sigma \}.$$

Note that when $C(\rho)$ is interpreted as a set of maps of G then it consists of those labeled maps that are congruent to the same unlabeled map of G; that is, $C(\rho)$ is a congruence class of the maps of G. Recall that C(G) is the set of congruence classes for the maps of G; the above interpretation shows that we can now express C(G) as

$$C(G) = \{ C(\rho) \mid \rho \in R(G) \}.$$

Let $\overline{m}(G)$ denote the average number of symmetries for all of the maps of G. There are various ways in which an average can be computed. We could refer to the mean of $m(\rho)$, the median of $m(\rho)$, or even the mode of $m(\rho)$; the easiest (and most natural) seems to be the mean. Now the question is what we should we take the mean over: should it be all rotations or should it be just the congruence classes. We have chosen to use Model I (see [19]) to base our definition of average number of symmetries upon. This entire dissertation has been concerned with rotation schemes; it seems most natural, therefore, for the mean to be taken over all rotation schemes. We define $\overline{m}(G)$ as follows:

$$\overline{m}(G) = \frac{1}{|R(G)|} \sum_{\rho \in R(G)} m(\rho)$$

Recall that Bi$\overline{m}(G) = \frac{1}{|R(G)|} \sum_{\rho \in R(G)} m(\rho)$umber of rotations on G that are equivalent to a fixed rotation scheme $\rho \in R(G)$, is [Aut G:Aut M], where $M = (G, \rho)$ (this is Theorem 1.2). With the notation introduced above, this becomes

$$|C(\rho)| = \frac{|\mathrm{Aut}\ G|}{m(\rho)}.$$

Clearly, $m(\rho)$ is a constant over $C(\rho)$. Therefore, we can rewrite the definition of $\overline{m}(G)$ to take the sum over the congruence classes.

$$\overline{m}(G) = \frac{1}{|R(G)|} \sum_{C(\rho) \in C(G)} |C(\rho)| \cdot m(\rho)$$

$$= \frac{1}{|R(G)|} \sum_{C(\rho) \in C(G)} \frac{|\mathrm{Aut}\ G|}{m(\rho)} \cdot m(\rho)$$

$$= \frac{|\mathrm{Aut}\ G|}{|R(G)|} \sum_{C(\rho) \in C(G)} 1.$$

Therefore, we find:

Theorem 6.1. $\overline{m}(G) = \dfrac{|\mathrm{Aut}\ G|\cdot|C(G)|}{|R(G)|}$. ∎

The above shows an application for the determination of the number of congruence classes of maps of a connected graph G, namely, to compute the average number of symmetries for the maps of G. In light of the above, an interesting question is what is the asymptotic behavior for the number of congruence classes. Equivalently, we can ask what is the asymptotic behavior of $\overline{m}(G)$. This will be the topic of the next two sections. For now, let us explore one of the ramifications of the above.

Corollary 6.1. If Aut G is trivial, then

$$|C(G)| = |R(G)|.$$

Proof: For each $\rho \in R(G)$, Aut M is a subgroup of Aut G (where $M = (G,\rho)$). If $|\mathrm{Aut}\ G| = 1$, then $\forall \rho \in R(G)$, $m(\rho) = 1$. Therefore, $\overline{m}(G) = 1$. Substituting for $\overline{m}(G)$ and $|\mathrm{Aut}\ G|$ in Theorem 6.1 gives the result. ∎

Theorem 6.2. Let $G = K(m_1,\ldots,m_n)$, where all of the m_i are distinct $(i = 1,\ldots,n)$ and $n \geq 3$. If $\gcd(m_i,m_j) = 1$, $(1 \leq i < j \leq n)$ then

$$|C(G)| = |R(G)| \Big/ \prod_{i=1}^{n} m_i!\,.$$

Proof: Since $m_i \neq m_j$ $(1 \leq i < j \leq n)$ we know that Aut G $\cong \bigoplus_{i=1}^{n} S_{m_i}$; thus, $|\mathrm{Aut}\ G| = \prod_{i=1}^{n} m_i!$. From the formula for $|C[K(m_1,\ldots,m_n)]$, $g \mid \gcd(m_i; i = 1,\ldots,n)$, so $g = 1$. Also, $c_{ij} \mid \gcd(m_k; k = 1,\ldots,n, k \neq i)$, so $s_i = 1$ and $c_{ij} = 1$ $(i = 1, \ldots, n,\ j = 1, \ldots, s_i)$. Therefore, $\lambda_i = \{1\}$. We

conclude that only the identity element of Aut G fixes any of the rotation schemes. The identity fixes all $|R(G)|$ of them, so the result follows. ∎

Corollary 6.2. Let $G = K(m_1, \ldots, m_n)$, where $n \geq 3$, all of the m_i are distinct $(i = 1, \ldots, n)$, and $\gcd(m_i, m_j) = 1$. Then $\overline{m}(G) = 1$.

Proof: Substitute $|\mathrm{Aut}\ G| = \prod_{i=1}^{n} m_i!$ and $|C(G)|$ from Theorem 6.2 into Theorem 6.1. ∎

In the next two sections we will study $\overline{m}(G)$ using asymptotic approximations of $|C(G)|$. In section 6.2 we will study complete graphs. Section 6.3 will concentrate on complete bipartite graphs. We will interpret the results in light of the two situations described after Theorem 6.1.

6.3 Complete Graphs

In this section we will determine the asymptotic behavior of $\overline{m}(K_n)$. We have seen that this is equivalent to looking at the asymptotic behavior of $|C(K_n)|$. We will interpret the results of this analysis in the light of random topological graph theory.

From Theorem 6.1 we have

$$\overline{m}(G) = \frac{|\mathrm{Aut}\ G| \cdot |C(G)|}{|R(G)|}.$$

We take $G = K_n$. Every vertex of K_n has degree $n - 1$; thus, $|R(K_n)| = (n - 2)!^n$. Also, since $\mathrm{Aut}\ K_n \cong \mathfrak{S}_n$; $|\mathrm{Aut}\ K_n| = n!$. Finally, Theorem 2.7 states that

$$|C(K_n)| = \sum_{d \mid n} \frac{(n-2)!^{n/d}}{d^{n/d}\cdot(n/d)!} + \sum_{\substack{d \mid n-1 \\ d \neq 1}} \frac{\varphi(d)\cdot(n-2)!^{(n-1)/d}}{n-1}.$$

Putting these values into the formula for $\overline{m}(K_n)$ we get

$$\overline{m}(K_n) = \sum_{d \mid n} \frac{n!\cdot(n-2)!^{n/d}}{(n-2)!^{n}\cdot d^{n/d}\cdot(n/d)!}$$

$$+ \sum_{\substack{d \mid n-1 \\ d \neq 1}} \frac{\varphi(d)\cdot(n-2)!^{(n-1)/d}\cdot n!}{(n-1)\cdot(n-2)!^{n}}.$$

Theorem 6.3. $\overline{m}(K_n) \approx 1$, where "$\approx$" means "is asymptotically equal to."

Proof: We will take the limits on each of the sums separately, combining them later.

Consider

$$\lim_{n\to\infty} \sum_{d \mid n} \frac{n!\cdot(n-2)!^{n/d}}{(n-2)!^{n}\cdot d^{n/d}\cdot(n/d)!}.$$

When $d = 1$, the term is

$$\frac{n!\cdot(n-2)!^{n/1}}{(n-2)!^{n}\cdot 1^{n/1}(n/1)!} = 1.$$

Therefore, the above limit is greater than or equal to one and is, in fact, equal to

$$1 + \lim_{n\to\infty} \sum_{\substack{d \mid n \\ d \neq 1}} \frac{n\cdot(n-1)\cdot(n-2)!^{(n+d)/d}}{(n-2)!^{n}\cdot d^{n/d}\cdot(n/d)!}$$

Now, $(n-2)!^{(n+1)/d} \leq (n-2)!^{(n+1)/2}$ (as $d \geq 2$) and there are fewer than n divisors; therefore, the above is

$$\leq 1 + \lim_{n\to\infty} \frac{n\cdot(n-1)^{2}}{2\cdot(n-2)!^{(n-1)/2}} = 1.$$

We conclude that the limit is 1 for the first sum.

Next we consider the second sum; clearly

$$0 \le \lim_{n \to \infty} \sum_{\substack{d \mid n-1 \\ d \ne 1}} \frac{\varphi(d) \cdot (n-2)!^{(n-1)/d} \cdot n!}{(n-1) \cdot (n-2)!^{n}}.$$

Now, $(n-2)!^{(n-1)/d} \le (n-2)!^{(n-1)/2}$ (as $d \ge 2$). Also, $\Sigma_{d \mid n-1} \varphi(d) = n - 1$ (see [17], for example). We conclude that the above limit is

$$\le \lim_{n \to \infty} \frac{n!}{(n-2)!^{(n+1)/2}} = 0.$$

Combining the limits for the two sums we conclude

$$\overline{m}(K_n) \approx 1. \qquad ∎$$

One interpretation of Theorem 6.3 is that if one orientably imbeds the complete graph K_n, in a 2-cell fashion, at random, then the automorphism group of the imbedding, probably, is only the identity element (as n increases without bound). Another interpretation is that when counting the congruence classes of maps of K_n only the identity automorphism need be considered for a good approximation (for n large).

Corollary 6.3. $|C(K_n)| \approx |R(K_n)| \;/\; |\mathrm{Aut}\ K_n|$; that is,

$$|C(K_n)| \approx (n-2)!^{n} \;/\; n!. \qquad ∎$$

One question that comes to mind is how large must n be for Corollary 6.3 to give a good approximation. Table 6.1 indicates that "large" is not all that large.

In the next section we will determine the asymptotic behavior for the average number of symmetries of the maps of the complete bipartite graph $K_{m,n}$. We will relate the result of that analysis to both of the situations shown above.

Table 6.1

An Illustration of Corollary 6.3

n	$\lvert C(K_n)\rvert$	$(n-2)!^n / n!$
3	1	0.17
4	3	0.67
5	78	64.8
6	265,764	265,420.8
7	71,095,150,000	71,094,857,142.86

6.4 Complete Bipartite Graphs

In this section we will determine the asymptotic behavior of $\bar{m}(K_{m,n})$. Once we have done the analysis, we again will interpret the result in terms of both of the situations described in the last section.

We will first consider $K_{p,p}$, where p is a prime. We first compute $f(p,p)$. The $\gcd(p,p) = p$, so $g = 1$ or $g = p$. We consider these two cases separately.

(1) If $g = 1$, then $p/1 = p$, so c_1, c_2 = {1}, {p}, {1,p}. The coprime condition ensures that one of them must be {1}.

(a) Suppose $c_1 = c_2 = \{1\}$; then $\lambda_{11} = \lambda_{21} = 1\cdot(1\cdot 1/1)$. The contribution to $\lvert C(K_{p,p})\rvert$ is

$$\frac{(p-1)!^{2p}}{2\cdot p!^{2}}$$

(b) Suppose $c_1 = \{1\}$ and $c_2 = \{p\}$; then $\lambda_{11} = p$ and $\lambda_{21} = 1$. The contribution to $\lvert C(K_{p,p})\rvert$ is

$$\frac{(p-1)!\cdot(p-1)^{p}}{2\cdot p\cdot p!}$$

By symmetry, we get the same value for $c_1 = \{p\}$ and $c_2 = \{1\}$. Therefore, the total contribution for case 1b is

$$\frac{(p-1)!\cdot(p-1)^p}{p\cdot p!^2}$$

(c) Suppose $c_1 = \{1\}$ and $c_2 = \{1,p\}$; then $\lambda_{11} = p$, $\lambda_{21} = 1$, and $\lambda_{22} = p$. But $1 + p > p$ so this case cannot occur. By symmetry we see that the case $c_1 = \{1,p\}$ and $c_2 = \{1\}$ is also not possible.

(2) If $g = p$, then $p/p = 1$, so $c_1 = c_2 = \{1\}$. Therefore, $\lambda_{11} = \lambda_{21} = p$. The contribution to $|C(K_{p,p})|$ is

$$\frac{(p-1)!^2}{2\cdot p^2}$$

We next compute $h(p,p)$. The divisors of p are 1 and p. If $d = 1$, then the contribution to $|C(K_{p,p})|$ is

$$\frac{(p-1)!^p}{2\cdot p!}.$$

If $d = p$, the the contribution ot $|C(K_{p,p})|$ is

$$\frac{(p-1)!}{2p}.$$

Combining the five terms that survive, we find that the number of congruence classes for the maps of $K_{p,p}$ is

$$|C(K_{p,p})| = \frac{(p-1)!^{2(p-1)}}{2\cdot p^2} + \frac{(p-1)^p}{p^2} + \frac{(p-1)!^2}{2\cdot p^2} + \frac{(p-1)!^{(p-1)}}{2\cdot p} + \frac{(p-1)!}{2\cdot p}$$

Therefore, by multiplying by $|\mathrm{Aut}\ K_{p,p}| = 2\cdot p!^2$ and dividing by $|R(K_{p,p})| = (p-1)!^{2p}$ (as in the formula from Theorem 6.1) we find

$$\overline{m}(K_{p,p}) = 1 + \frac{2\cdot(p-1)^p}{(p-1)!^{2(p-1)}} + \frac{1}{(p-1)!^{(p-4)}} + \frac{p}{(p-1)!^{(p-1)}} + \frac{p}{(p-1)!^{(2p-3)}}$$

Taking limits as p goes through arbitrarily large primes, we see that only the first term survives in the limit. Therefore:

Theorem 6.4. If p is a prime then $\overline{m}(K_{p,p}) \approx 1$. ∎

A similar analysis can easily establish:

Theorem 6.5. If p and q are primes then $\overline{m}(K_{p,q}) \approx 1$. ∎

The general case can be established as follows: First consider when m and n have the following form:

$$p_1 p_2 \cdots p_k$$

That is, when each is a product of primes to the first power. The Euler function is multiplicative (see [17]), so it is easy to establish the form of $\Phi(d,m)$, where d is a divisor of m (similarly for n). This allows the limit to be taken for each term. Also, The number of divisors is easy to establish.

Second allow one of them to be general and the other to have the above form. This case is somewhat harder to establish but it gives one the insight to see what must happen in the general case.

Finally, the general case can be established. The details of this analysis will be presented at a later time; the result, however, will be given:

Theorem 6.6. $\overline{m}(K_{m,n}) \approx 1$. ∎

Theorems 6.4 - 6.6 can be interpreted as follows: If $K_{m,n}$ is randomly 2-cell imbedded on an orientable surface, then if $m \geq 2$ and n is large, or both are large, the automorphism group of the map is probably trivial. Equivalently, only the identity automorphism needs to be

considered in calculating $|C(K_{m,n})|$, when $m \geq 2$ and n is large or when both m and n are large, to obtain a good approximation. The numbers become large so quickly that no table will be given for complete bipartite graphs.

In the next, and final, section we will present topics for future exploration. These questions cover a variety of topics: from algebra and enumeration to topology and imbeddings.

6.5 Open Questions

In this section we will present a few questions and directions for future study. These will be concerned with enumeration, algebra, topological graph theory, and random topological graph theory.

The first questions to be looked at concern enumeration. What other graphical families can be counted using the techniques of this dissertation? For example, the graphical families Q_n (the n-dimensional cube), $C_m \times C_n$, and $K_m \times K_n$ are important in computer science. Can the numbers of congruence classes of their maps be counted?

Algebraic questions concern two topics that have been brought up: structure classes and compatible permutations. What are the general properties of the structure classes for a group? In this setting we have:

Given a group Γ, for $\alpha \in \Gamma$, $J(\alpha) = \{\beta\alpha\beta^{-1} \mid \beta \in \Gamma\}$.

Can $|J(\alpha)|$ be found for arbitrary groups? This would be a large step in determining the number of congruence classes of maps of an arbitrary connected graph.

The automorphisms of a graph have been expressed as products of compatible permutations. The fact that the

partite sets have been disjoint has meant that the structure of the compatible permutations was not altered by this product. What happens if the sets over which the compatible permutations are defined are not disjoint? What happens if they are not all distinct? Can this be used to determine the structures of all automorphisms that will contribute to $|C(G)|$ for an arbitrary connected graph G?

We only presented the definitions for the types of 2-cell imbeddings that a 2-connected graph can possess in this dissertation. For questions concerning topics to be explored about these types of 2-cell imbeddings, the reader should consult [10], when it is published.

The formula for the complete n-partite graph is a generalization of that for the complete bipartite graph. We also know that $\overline{m}(K_{m,n}) \approx 1$. Therefore, we believe Conjecture 1 to reflect the true behavior of $\overline{m}$. The main stumbling blocks to a proof of Conjecture 1 are: the determination of the number of solutions μ and the number of divisors c forming compatible permutations.

Conjecture 1. Let $G = K(m_1, \ldots, m_n)$. If $n \geq 3$, then $\overline{m}(G) \approx 1$ as $m_n \to \infty$. ∎

Lee [11] has shown:

Theorem 6.7. (Lee) If G is a connected graph of order n and size $m \approx cn^{1+\varepsilon}$, where c and ε are positive constants, then $\overline{\gamma}(G) \approx \gamma_M(G)$ (where $\overline{\gamma}(G)$ is the average genus and $\gamma_M(G)$ is the maximum genus of the graph G).

Because of this, and the fact that maximum genus imbeddings tend to have few symmetries, we conjecture:

Conjecture 2. Let G be a connected graph of order n and size $m \approx cn^{1+\varepsilon}$, where c and ε are positive constants. Then $\overline{m}(G) \approx 1$, or equivalently,

$$|C(G)| \approx |R(G)|/|\mathrm{Aut}\ G|. \qquad \blacksquare$$

This is just a sampling of the questions that remain to be answered. The answers to these will likely lead to even more questions.

APPENDIX

Program Listing

The version I will display here is the original version, before it was modified to take into consideration the classification hierarchy of [10]. This version also had the deficiency that it did not divide the congruence classes by surface. The output was just a list of labeled imbeddings (one for each congruence class). The count was then accomplished using a utility program that counts the number of lines in a program. The latest version not only divides by surface, but each surface is divided along the classification hierarchy. I am presently adding a division by region distribution as well as giving the size of the automorphism group of each class. The user will also be able to request summary data for a single surface, or for all surfaces, or ask whether a specified region distribution occurs. A copy of the complete source code for the latest version of this program will be made available to anyone requesting it. For the next year I can be reached by writing:

Bruce P. Mull
Department of Mathematics
Western Washington University
Bellingham, WA 98225

The program stores the matrix as adjacency lists. The adjacency lists are stored as rows of a matrix that is dynamically allocated to be n x (n + 1), where n is the number of vertices in the graph. The code to allocate these lists is listed below:

```
char **alloc_adjacency_list(n)
    int n;          /* the number of vertices */
{
  char **A;         /* pointer to the adjacency lists */
  int    i;         /* loop controller */
```

```
  if ((A = (char **) calloc(n, sizeof(char *))
    == (char **) NULL) {
    fprintf(stderr, "Cannot allocate adjacency matrix.");
    abort();
  }

  for (i = 0; i < n; ++i)
    if ((A[i] = (char *) calloc(n+1, sizeof(char))
      == (char *) NULL) {
      fprintf(stderr, "Out of memory: row %d", i+1);
      abort();
    }

  return A;
}
```

The next thing that must be done depends upon the type of graph; the code below assumes a complete graph.

```
make_complete_graph(n, A)
     int    n;          /* number of vertices in the graph */
     char **A;          /* adjacency list */
{
  char i,j;             /* Loop controllers */
  char pos              /* current position in row i */

  for (i = 0; i < n; ++i) {
    for (pos = j = 0; j < n; ++i)
      if (i % n != j % n)
        A[i][pos++] = (char) j;

      A[i][pos] = MAX_VERTICES;
      A[i][n]   = pos;
  }

  return;
}
```

MAX_VERTICES has been defined by a statement like

```
#define MAX_VERTICES 6
```

appearing somewhere before the function "main" in the program. Note that when each row is filled, pos is the degree of that vertex. MAX_VERTICES is used in "Edmond" to determine the end of the rotation at vertex i.

The following code generates all rotation schemes for a given graph. It does not assume any particular type of graph.

```
rotation(n, v, d, s, R)
     int   n;        /* the order of the graph */
     int   v;        /* the current vertex whose rotations
                        are being generated */
     int   d;        /* the degree of vertex v */
     int   s;        /* the starting position to be
                        modified in the current vertex
                        rotation */
     char *R[];      /* the current rotation scheme */

{
  char   i, j, k;    /* loop controllers */
  char   tmp;        /* used to swap rotation entries */
  char   **P;        /* new rotation scheme being
                        generated */

  if (d < 2 || s == d - 1) {
    if (v == n - 1)
      Edmond(n, R);
    else
      rotation(n, v+1, R[v+1][n], 1, R);
  } else {

  /* Create new rotation scheme */

    P = alloc_adjacency_list(n);

  /* Generate all rotations at vertex v */

    for (i = s; i < d; ++i) {

    /* Base permutation on current graph rotation */

      for (j = 0; j < n; ++j)
        for (k = 0; k <= n; ++k)
          P[j][k] = R[j][k];

    /* Swap starting position with posiiton i */

      tmp = P[v][s], P[v][s] = P[v][i], P[v][i] = tmp;

    /* Generate rotations from next position onward */

      rotation(n, v, d, s+1, P);
```

```
    }

  /* Free space used by new rotation */

    free_adjacency_list(n, P);
  }

  return;
}
```

The source code for "free_adjacency_list" now follows:

```
free_adjacency_list(n, A)
     int    n;         /* order of the graph */
     char **A;         /* pointer to the adjacency list */
{
  char i;              /* loop controller */

  for (i = 0; i < n; ++i)
    free(A[i]);

  free(A);

  return;
}
```

The following function implements Edmond's Algorithm on a given rotation scheme. Before this algorithm can be used there must be a global variable declared, outside of every function, named "imbedding." This variable is declared as:

```
FILE *imbedding;
```

Also, in the function "main," the output file must be opened. The code to do this is just:

```
if ((imbedding = fopen("graph.dat", "w"))
  == (FILE *) NULL) {
  fprintf(stderr, "Cannot open output file.");
  abort();
}
```

The current version of "Edmond" returns the genus of the imbedding; thus, only imbeddings on a particular surface can be generated. This version writes directly to the output file instead of writing to a buffer and testing whether (and where) the output should be outputted.

```
Edmond(n, R)
     int   n;           /* the order of the graph */
     char *R[];         /* the current rotation scheme */
{
  char   v;             /* current vertex being written */
  char   i,j;           /* loop controllers - determine the
                           proper next "arc" of the region */
  char **U;             /* "used corner" matrix - tells
                           whether the next "arc" has been
                           seen before */
  int    r = 0;         /* number of the current region
                           being created */

/* Allocate space for the "used corner" matrix */

  U = alloc_adjacency_list(n);

  /* Find the region boundaries */

  do {

  /* Find starting vertex */

    for (v = -1, i = 0; v < 0 && i < n;  ) {
      for (j = 0; R[i][j] != MAX_VERTICES; ++j)
        if (!U[i][j]) {  /* U[i][j] = 0 is good choice */
          v = i;
          break;         /* stop searching - found v */
        }

      if (R[i][j] == MAX_VERTICES)
        ++i;
    }

  /* If a region exists, then trace out boundary */

    if (v >= 0) {

    /* Place a space between regions */

      if (r)
```

```
      fprintf(imbedding, " ");

    ++r;      /* Update current region */

    fprintf(imbedding, "%d", (int) v);
                         /* assumes MAX_VERTICES <= 10 */

  /* find next corner */

    do {
      U[i][j] = 1;       /* Mark "corner" as used */
      ++j;               /* Determine the next "arc" */

      if (R[i][j]             /* At end of cycle must */
        == MAX_VERTICES)      /* loop around rotation */
        j = 0;

      v = R[i][j];            /* Use the "arc" */

      for (j = 0; R[v][j] != i; ++j)  /* Assumes no */
        ;    /* find next corner */   /* multiple
                                         edges */

      if (i = v, !U[i][j])  /* try doing this in
                               another language -
                               U[i][j] = 1 means that
                               we completed the region
                            */
        fprintf(imbedding, (int) v);

    } while (!U[i][j]);
  }
} while (v >= 0);

fprintf(imbedding, "\n");     /* completed imbedding */

free_adjacency_list(n, U);

return;
}
```

Now that all of the imbeddings have been output to "graph.dat," it is time to filter out the duplicates. This means that "imbedding" must become the input file and another variable, "class," must become the output file; "class" is declared exactly as "imbedding" was earlier. The changeover is accomplished as follows:

```
fclose(imbedding);
if ((imbedding = fopen("graph.dat", "r"))
  == (FILE *) NULL) {
  fprintf(stderr, "Cannot open input file.");
  abort();
}

if ((class = fopen("map.dat", "a+"))
  == (FILE *) NULL) {
  fprintf(stderr, "Cannot open input file.");
  abort();
}

find_congruence_classes();
```

The last line is the key function. The original version used one input file and one output file. Several tests are performed on the imbedding and the class to decide whether they are congruent. If the two imbeddings are not on the same surface, then they cannot be congruent. Notice that "Edmond" creates a string of length 2m + r to represent an imbedding, where m is the number of edges and r is the number of imbedding regions. Thus, we can tell whether two imbeddings are on the same surface by comparing their lengths.

If two imbeddings do not have the same region distribution, then they cannot be congruent. Thus, we need to determine the region sizes and make the comparison. This means that a standard way of looking at the region distributions must be developed. The regions are sorted by region size—largest to smallest. The reason for using largest to smallest is because it makes the next comparison go faster. The next comparison requires some definitions. The first of these is called a vertex skip; the second is called a skip sequence. Both of these require an imbedding to make sense.

Consider an oriented imbedding of a connected graph G in an orientable surface S and suppose that v is a vertex of G; then v appears on the oriented region boundary of at least one region at least one time. Consider some occurrence of v on some region of the imbedding. The <u>vertex skip</u> of this occurrence is the number of arcs along the oriented region boundary of the chosen region between the chosen occurrence of v and its next (possibly same) occurrence on this boundary.

Example A.1

Consider 40123102 032 043 142 134, this represents an imbedding of K_5 on the torus with one 8-gon and four 3-gons. The oriented region boundary 40123102 is interpreted as follows: Start at vertex 4, take the arc (4,0) to the vertex 0, take the arc (0,1) to the vertex 1, etc. Consider the first occurrence of 0 on the 8-gon (the second entry, as listed). The vertex skip of this occurrence is five (arcs 01 12 23 31 10 lie between the occurrences); the vertex skip of the other occurrence is three (arcs 02 24 40 now lie between them).

At a later time several properties of vertex skips will be presented; for now they can just be considered a tool. We now present the second definition (and related ones).

Consider an oriented imbedding of a connected graph G in an orientable surface S; there exists at least one region in this imbedding. Pick a region and a starting

vertex on this oriented region boundary. A skip sequence for the region consists of the list of vertex skips from a chosen occurrence of the starting vertex and continuing once around the region. A major skip is any of the vertex skips of largest size. A minor skip is the vertex skip immediately following a major skip in a skip sequence.

Notice that although all major skips must be the same size, this is not necessarily true of minor skips. Also, if the major skip is the last skip in the sequence, its minor is the first vertex skip (thus, a skip sequence is a cycle). A skip sequence is said to be standardized if it begins at a major skip that is followed by the largest of the minor skips. In the event of a tie, compare each subsequent vertex skip with the corresponding vertex skip starting at the other major skip; start the skip sequence at the major skip that has the first larger corresponding entry in this sequence. If an entire cycle can be made and the sequences are still tied, take either; for example, start at the the lowest (highest) labeled vertex giving the major skip. Skip sequences have many interesting properties; the only one to be explored at this time, however, is this: If two imbeddings are congruent, then there is an ordering of their regions so that the standardized skip sequences of the ordered lists of regions are the same (the converse is not true).

Therefore, we have the following strategy: Given two oriented imbeddings, first determine whether they are on the same surface. If they are, then compare their region

distributions. If those are the same, then decide if there is any ordering of the regions which produces the same skip sequence on each pair of corresponding regions in this ordering. Finally, try to find a permutation of the vertex labels on one imbedding that produces the oriented region boundaries of the other imbedding. This is exactly the strategy of "find_congruence_classes." The code for this function follows:

```
find_congruence_classes()
{
  int  i, j;                       /* loop controllers */
  int  ilen, clen;                 /* lengths of
                                      imbedding and
                                      class */
  int  iskips[MX_RGN+1][MX_LEN];   /* imbedding skip
                                      sequences */
  int  cskips[MX_RGN+1][MX_LEN];   /* class skip
                                      sequences */
  char itest[BUFSIZ+1];            /* imbedding being
                                      tested */
  char ctest[BUFSIZ+1];            /* class being tested
                                      against */
  char irgns[MX_RGN+1][MX_LEN];    /* imbedding region
                                      distribution */
  char crgns[MX_RGN+1][MX_LEN];    /* class region
                                      distribution */
  char found;                      /* while 0, imbedding
                                      is distinct from
                                      current class */

/* Loop through the input file */

  do {

  /* Fetch the next imbedding to test */

    fgets(itest, BUFSIZ, imbedding);
    found = 0;

    if (!feof(imbedding)) {        /* then test this
                                      imbedding */
      ilen = strlen(itest);
      rewind(class);               /* start from first
                                      class */
```

```
    find_faces(itest, irgns);
    fetch_skips(irgns, iskips);

  /* Loop through congruence classes */

    do {
      fgets(ctest, BUFSIZ, class);

      if (!feof(class)) {         /* still something to
                                     compare */
        clen = strlen(ctest);

        if (ilen == clen) {       /* same surface */

        /* Check region distributions */

          find_faces(ctest, crgns);
          found = compare_regions(irgns, crgns);

          if (found) {            /* same regions */

          /* Check skip sequences */

            fetch_skips(crgns, cskips);
            found = compare_skips(iskips,cskips,crgns);
          }

          if (found && crgns[0][0] > 1)
            found = try_to_map(irgns,crgns,cskips);
        }
      }
    } while (!(found || feof(classes)));

    if (!found)               /* new class */
      for (i = 1; i <= irgns[0][0]; ++i) {
        for (ilen = strlen(irgns[i], j = 0;
          j < ilen; ++j)
          fprintf(class, "%c", irgns[i][j]);

        fprintf(class, "%c", (i == irgns[0][0])?
          "\n" : " ");
      }
  }
  } while (!feof(imbedding));

  return;
}
```

The following MACRO constants were defined:

```
#define MX_RGN  70
#define MX_LEN 381
```

The source code of "find_faces" follows. This function also calls a function to sort the regions from largest to smallest.

```
find_faces(embedding, region)
     char embedding[];          /* imbedding whose regions
                                   are sought */
     char region[][MX_LEN];     /* The region distribution
                                */
{
  char  faces[BUFSIZ+1]    /* Strings representing each
                              region */
  char *next_face;         /* pointer to next face */
  int   Rn;                /* number of regions found */
  int   size;              /* number of arcs in current
                              region boundary */
  int   i;                 /* loop controller */

  strcpy(faces, embedding);

/* Determine the first region */

  Rn = 1;
  next_face = strtok(faces, " ");

  for (size = strlen(next_face), i = 0; i < size; ++i)
    if (order <= (next_face[i] - '0'))
      order = next_face[i] - '0' + 1;    /* A global int
                                            defined with
                                            the files */
    else if (next_face[i] == '\n')
      next_face[i] == '\0';

  strcpy(region[Rn], next_face);    /* place the region
                                       */

/* Now find the other faces */

  while (next_face != (char *) NULL) {
    if (++Rn > MX_RGN) {
      fprintf(stderr, "Too many regions!!!");
      abort();
    }

    next_face = strtok(faces, (char *) NULL);
```

```
    if (next_face != (char *) NULL) {
      for (size = strlen(next_face), i = 0;
        i < size; ++i)
        if (order <= (next_face[i] - '0'))
          order = next_face[i] - '0' + 1;
        else if (next_face[i] == '\n')
          next_face[i] == '\0';

        strcpy(region[Rn], next_face);
    }
  }

/* Set initial entry to number of regions found */

  region[0][0] = Rn - 1;

/* Sort, if needed */

  if (region[0][0] > 1)
    sort_regions(region);

  return;
}
```

Next is the code for "sort_regions."

```
sort_regions(faces)
     char faces[][MX_LEN];      /* regions to be sorted */
{
  int  i, j, k;              /* loop controllers */
  char temp;                 /* used to swap regions */

/* This code performs a selection sort of the regions */

  for (i = 1; i < faces[0][0]; ++i) {
    for (j = k = i, ++j; j <= faces[0][0]; ++j)
      if (strlen(faces[k]) < strlen(faces[j]))
        k = j;

    if (k != i)      /* swap is necessary */
      for (j = 0; j < MX_LEN; ++j) {
        temp = faces[k][j];
        faces[k][j] = faces[i][j];
        faces[i][j] = temp;
      }
  }

  return;
}
```

Having found and sorted the regions, the next thing was to determine the skip sequences. The following is the code for "fetch_skips."

```
fetch_skips(regions, skips)
     char regions[][MX_LEN];         /* region distribution
                                        for imbedding */
     int  skips[][MX_LEN];           /* skip sequences of
                                        distribution */
{
  int  i, j, k;      /* loop controllers */
  int  len;          /* length of current region */

/* Loop through the regions */

  for (i = 1; i <= regions[0][0]; ++i)
    for (len = strlen(region[i], j = 0; j < len; ++j) {
      for (k = 1; regions[i][j]
        != regions[i][(j + k) % len];  )
        ++k;

      skips[i][j] = k;

  standardize_skips(skips, regions);      /* make the skips
                                             comparable */

  if (regions[0][0] > 1)
    sort_skips(skips, regions);           /* order the
                                             regions of the
                                             the same size
                                             by skip
                                             sequence */

  return;
}
```

The following code standardizes the skip sequences using the criteria given when they were defined.

```
standardize_skips(skips, regions)
     int  skips[][MX_LEN];           /* skips to be
                                        standardized */
     char regions[]MX_LEN];          /* regions are
                                        adjusted to match
                                        the skips */
{
  int  i, j, k, m;                   /* loop controllers */
```

```
  int  len;                          /* size of current
                                        region */
  int  old_skip[MX_LEN];             /* used for
                                        adjustments */
  char old_face[MX_LEN];             /* used for
                                        adjustments */
  char multiple_majors = 0;          /* if = 1, then need
                                        to test starting
                                        position for
                                        skips */

/* Loop through each skip sequence */

  for (i = 1; i <= regions[0][0]; ++i) {
    len = strlen(regions[i]);

    for (j = k = 0; j < len; ++j) {
      old_skip[j] = skips[i][j];
      old_face[j] = regions[i][j];

      if (skips[i][j] > skips[i][k]) {
        k = j;
        multiple_majors = 0;
      } else if (j != k && (skips[i][j] == skips[i][k]))
         multiple_majors = 1;
    }

    for (j = 0; k > 0 && j < len; ++j) {
      skips[i][j] = old_skip[(j + k) % len];
      regions[i][j] = old_face[(j + k) % len];
    }

    if (multiple_majors) {
      for (j = k = 0; j < len; ++j) {
        old_skip[j] = skips[i][j];
        old_face[j] = regions[i][j];

        if (j != k && (skips[i][j] == skips[i][k])) {
          m = 1;
          while (m < len && (skips[i][(j + m) % len]
            == skips[i][(k + m) % len]))
            ++m;

          if (skips[i][(j + m) % len]
            > skips[i][(k + m) % len])
            k = j;
        }
      }

      for (j = 0; k > 0 && j < len; ++j) {
```

```
          skips[i][j] = old_skip[(j + k) % len];
          regions[i][j] = old_face[(j + k) % len];
        }
      }
    }

    return;
}
```

Next the regions must be sorted by skip sequence, so that it is easy to determine whether two imbeddings have the same skip sequence. Computer analysis shows that by stopping the comparisons at skip sequences, more than 90% of the congruence classes can be found. I believe that as the order of a graph gets larger, the percentage of congruence classes that can be found using only skip sequences approaches 100 (this seems to be true for complete graphs, although the amount of data is small). The following code sorts the regions by skip sequences.

```
sort_skips(skips, regions)
     int  skips[][MX_LEN];              /* key for sorting */
     char regions[][MX_LEN];            /* regions to be
                                           sorted */
{
  int i, j, k, m;     /* Loop controllers */
  int leni, lenj;     /* Lengths of regions whose skips
                         are being compared */
  int tmp;            /* Used to swap the skip sequences */
  char temp;          /* Used to swap the regions */

/* Loop through the regions, comparing skips */

  for (i = 1; i < regions[0][0]; ++i) {
    leni = strlen(regions[i]);

  /* Perform selection sort on regions of the same size
  */

    for (j = m = i, ++j; j < regions[0][0]; ++j) {
      lenj = strlen(regions[j]);
      if (leni == lenj)
        for (k = 0; k < leni; ++k)
```

```
          if (skips[m][k] > skips[j][k])
            break;
          else if (skips[m][k] < skips[j][k]) {
            m = j;
            break;
          }
    }

  /* Order skips from largest to smallest */

    if (m != i)
      for (j = 0; j < leni; ++j) {
        tmp = skips[i][j];
        skips[i][j] = skips[m][j]
        skips[m][j] = tmp;

        temp = regions[i][j];
        regions[i][j] = regions[m][j];
        regions[m][j] = temp;
      }
  }

  return;
}
```

Now that the skips sequences have been determined, the tests begin. The surfaces are tested by comparing the lengths of the imbeddings. If these are found the same then the region distributions are compared. The code for "compare_regions" is given next.

```
char compare_regions(Rns1, Rns2)
     char Rns1[][MX_LEN];      /* 1st imbedding faces */
     char Rns2[][MX_LEN];      /* 2nd imbedding faces */
{
  int  i;            /* Loop controller */
  char result;       /* 1 - same, 0 different */

  if (Rns[0][0] != Rns[0][0]) /* can't happen but there
                                 may be system failure */
    result = 0;
  else {
    for (i = 1; i <= Rns1[0][0]; ++i)
      if (strlen(Rns1[i] != strlen Rns2[i]))
        break;

      result = (i > Rns1[0][0])? 1 : 0;
```

```
  }

  return result;
}
```

If the region distributions are the same, then the skips sequences must be compared. Because they are sorted, this test is much simpler. The following code performs the comparison of the skip sequences of the region distribution.

```
char compare_skips(skps1, skps2, faces)
     int  skps1[][MX_LEN];      /* 1st imbedding skips */
     int  skps2[][MX_LEN];      /* 2nd imbedding skips */
     char faces[][MX_LEN];      /* Needed only for the
                                   region lengths */
{
  int  i, j;           /* Loop controllers */
  int  len;            /* size of the current region */
  char result = 1;     /* 1 - same, 0 - different */

/* Loop through all regions */

  for (i = 1; i <= faces[0][0]; ++i) {

    len = strlen(faces[i]);
    if (len <= 5)          /* Skips must be the same */
      break;

    j = 0;
    while (skps1[i][j] == skps2[i][j] && j < len)
      ++j;

    if (j < len) {
      result = 0;
      break;
    }
  }

  return result;
}
```

If the skip sequences are the same, then one needs to try to map the first imbedding onto the second. That is, one tries to find a permutation of the vertices that

maps the oriented region boundaries of the first imbedding onto the oriented region boundaries of the second imbedding. The code for "try_to_map" is presented now.

```
char try_to_map(rgns1, rgns2, skips)
     char rgns1[][MX_LEN];      /* 1st face distribution */
     char rgns2[][MX_LEN];      /* 2nd face distribution */
     int  skips[][MX_LEN];      /* skip sequences */
{
  int  i, j;                /* Loop indices */
  int  r[MX_RGN+1];         /* r[i] = j means region i of
                               the first imbedding is mapped
                               onto region j of second */
  int  s[MX_RGN+1];         /* s[i] = j means region i had
                               to be shifted j positions for
                               the the mapping to work */
  int  sizes[MX_RGNS+1];    /* sizes of the regions - the
                               region boundaries are
                               converted to index sequences
                               for the permutation to be
                               created; strlen will not work
                               under these conditions */
  char result;              /* 1 - same, 0 - different */
  char mapable;             /* 1 - true, 0 - false */
  char can_add_region;      /* 1 - yes, 0 - no */
  char done[MX_RGNS+1];     /* 1 - r[i] successfully mapped
                               0 - r[i] not yet mapped */
  char r1[MX_RGNS+1][MX_LEN]; /* index sequence
                                 corresponding to 1st
                                 imbedding */
  char r2[MX_RGNS+1][MX_LEN]; /* index sequence
                                 corresponding to 2nd
                                 imbedding */
  char perm[MAX_VERTICES];    /* the permutation that
                                 maps 1st onto 2nd */
  char region[MX_LEN];        /* Test region created from
                                 permutation */

  sizes[0] = rgns1[0][0];     /* set number of regions */

  for (i = 1; i <= sizes[0]; ++i) {

    sizes[i] = strlen(rgns1[i]);    /* set region sizes */
    r[i] = 0;         /* not yet mapped */
    for (j = 0; j < sizes[i]; ++j) {
      r1[i][j] = rgns1[i][j] - '0';
      r2[i][j] = rgns2[i][j] - '0';
    }
```

```
}

r[1] = 1;           /* try to map region 1 to region 1 */
done[1] = 0;        /* not yet mapped */

while(!done[1]) { /* Try to create a map */

/* Set initial conditions */

  s[1] = -1;        /* Avoid special case treatment */
  mapable = 0;      /* No permutation to test yet */
  result = 0;       /* Assume different */

  if (!same_skip(skips[1], skips[r[1]], sizes[1],
    sizes[r[1]]))
    break;          /* No map can exist */

  do {

  /* Wipe out previous attempt */

    for (j = 0; j < order; ++j)
      perm[j] = -1;

  /* Adjust for current attempt */

    for (++s[1];
      !can_shift(skips[1], sizes[1], s[1]);  )
      if (s[1] < sizes[1])
        ++s[1];
      else
        break;          /* Avoid infinite loop */

    if (s[1] == sizes[1])
      break;            /* No map for this r[1] */

  /* Create permutation */

    for (j = 0; j < sizes[1]; ++j)
      perm[r1[1][j]] = r2[r[1][(s[1] + j) % sizes[1]];

  /* Finish creating the permutation */

    for (i = 2; i <= sizes[0]; ++i) {
      done[i] = 0;      /* Get ready to map region i */

    /* Create partial test region for region i */

      for (j = 0; j < sizes[i]; ++j)
        region[j] = perm[r1[i][j]];
```

```
/* Find region for region i to map onto */

  r[i] = 1;            /* Start at beginning */

  while (!same_skip(skips[i], skips[r[i]],
    sizes[i], sizes[r[i]]))
    ++r[i];            /* Avoid the ridiculous */

  while (r[i] <= sizes[0] && sizes[i]
    == sizes[r[i]]) {    /* Try the reasonable */
    do {       /* Ensure one-to-one */
      for (j = 1; j < i; ++j)
        if (r[j] == r[i]) {  /* Already used */
          ++r[i];             /* Try next region */
          break;
        }
    } while (j < i);

    if (!same_skip(skips[i], skips[r[i]],
      sizes[i], sizes[r[i]]))
      break;           /* Region i cannot be mapped -
                          remember how regions are
                          sorted */

    for (s[i] = 0; s[i] < sizes[i]; ++s[i])
      if (can_shift(skips[i], sizes[i], s[i])) {

      /* Test partial region for extendability */

        for (j = 0; j < sizes[i]; ++j)
          if (region[j] < 0)
            continue;           /* No contradiction */
          else if (region[j]
            != r2[r[i]][(s[i] +j) % sizes[i]])
            break;              /* Not one-to-one */

        if (j == sizes[i]) { /* Stop shifting */
          can_add_region = 1;
          break;
        } else {             /* Don't give up */
          can_add_region = 0;
          continue;          /* Shift again */
        }
      }

    if (can_add_region) {     /* Add it */
      done[i] = 1;

      for (j = 0; j < sizes[i]; ++j)
        perm[r1[i][j]]
```

```
                = r2[r[i]][(s[i] + j) % sizes[i]];

            break;          /* Fix r[i] as done */
          } else
            ++r[i];         /* Try mapping to next region */
        }

      /* Test to see if permutation is complete */

        for (j = 0; j < order; ++j)
          if (perm[j] < 0)
            break;                /* Still more to do */

        if (j == sizes[i]) {
          mapable = 1;    /* Possible map discovered */
          break;          /* Stop adding regions */
        }

        if (!can_add_region) {      /* cannot extend */
          for (j = 2; j <= sizes[0]; ++j) {
            done[j] = 0;          /* Undo everything */
            r[j] = 0;
          }

          break;      /* Start over from beginning */
        }
      }

      if (mapable)  /* See if possible map works */
        result = test_perm(r1, r2, sizes, perm);

      if (!result)  /* permutation failed */
        mapable = 0;      /* keep trying */
    } while (!mapable);

    if (!mapable && r[1] < sizes[0]) {  /* still hope */
      for (j = 2; j <= sizes[0]) { /* reinitialize */
        done[j] = 0;
        r[j] = 0;
      }

      ++r[1];  /* region 1 on next region */
    } else
      done[1] = 1;        /* success! - or - hopeless! */
  }

  return result;
}
```

The only thing left to do is supply the code for the

new functions introduced in "try_to_map." These are "same_skip," "can_shift," and "test_perm." The code for "same_skip" appears below.

```
char same_skip(skip1, skip2, size1, size2)
     int skip1[], skip2[];     /* region skip sequences */
     int size1, size2;         /* sizes of the regions */
{
  int  i;              /* Loop controller */
  char result = 1;     /* 1 - same, 0 - different */

  if (size1 != size2)
    result = 0;              /* Not even a chance */

  if (result) {        /* Actually need to check */
    for (i = 0; i < size1; ++i)      /* Test sequences */
      if (skip1[i] != skip2[i])      /* Different -
                                        remember they
                                        are standardized */

    result = (i == size1)? 1: 0;
  }

  return result;
}
```

Next come the code for "can_shift." Actually, this could have been done using the above function; it is faster not to do this, however.

```
char can_shift(skip, size, shift)
     int skip[];                 /* region skip sequence */
     int size;                   /* region size */
     int shift;                  /* amount region boundary
                                    is cycled */
{
  int  i;              /* Loop controller */

  for (i = 0; i < size; ++i)
    if (skip[i] != skip[(i + shift) % size])
      break;

/* 0 - no, 1 - yes */

  return (char) ((i == size)? 1: 0);
}
```

Next is the code for "test_perm."

```
char test_perm(rgns1, rgns2, sizes, perm)
    char rgns1[][MX_LEN];
    char rgns2[][MX_LEN];
    int  sizes[];
    char perm[];
{
  int  i, j, k, m;         /* Loop controllers */
  char region[MX_LEN];     /* image of 1st region i */
  char region_result;      /* 1 - region i has image in 2
                              0 - no image in 2 for i */
  char result;             /* 1 - all regions have image
                              0 - at least one failure */

  for (i = 1; i <= sizes[0]; ++i) {

  /* Create image of region i */

    for (j = 0; j < sizes[i]; ++j)
      region[j] = perm[rgns1[i][j]];

  /* Skip over the absurd */

    j = 1;
    while (sizes[i] != sizes[j])
      ++j;

  /* Check if region is in imbedding 2 */

    while ((j <= sizes[0]) && (sizes[i] == sizes[j])) {

    /* Check if region is shifted region in 2 */

      for (k = 0; k < sizes[j]; ++k) {

      /* Check all shifts k on the region j of 2 */

        for (m = 0; m < sizes[j]; ++m)
          if (rgns2[j][m] != region[(k + m) % sizes[j]])
            break;       /* This region is not image */

        if (m == sizes[j]) {
          region_result = 1;
          break;                /* Found region i image */
        } else {
          region_result = 0;
          continue;             /* Try next shift */
        }
```

```
      }

      if (region_result)
        break;             /* Image found */
      else
        ++j;               /* Try another region of 2 */
    }

  reult &= region_result;      /* Adjust for all regions
                                  seen as of now */
  }

  return result;
}
```

Most C compilers will require header files for the above to work properly. The header files used are declared (usually) at the beginning of the file. This program used the following:

```
#include <stdio.h>
#include <stdlib.h>
#include <ctype.h>
```

Some compilers may require:

```
#include <string.h>
```

The modifications of this program have been extensive. This program, together with a line count utility that was written in assembler, was enough to verify the total numbers of congruence classes. Later versions did the subdivision. The current version is version 6; this version is finally starting to do some of the things that will make it a good tool for research.

BIBLIOGRAPHY

[1] Bender, E. A. and Canfield, E. R. (1988). The asymptotic number of rooted maps on a surface, J. Combin. Theory Ser. A, 43, 244-257.

[2] Biggs, Norman L. (1971). Automorphisms of imbedded graphs, J. Combin. Theory Ser. B, 11, 132-138.

[3] Biggs, Norman L. and White, Arthur T. (1979). Permutation Groups and Combinatorial Structures. Cambridge, MA: Cambridge University Press.

[4] Chartrand, Gary and Lesniak Linda (1986). Graphs & Digraphs (2nd ed.). Monterey, CA: Wadsworth & Brooks/Cole.

[5] Fréchet, M. and Fan, K. (1967). Initiation to Combinatorial Topology. Boston: Prindle, Weber and Schmidt.

[6] Goodwin, W. (1972). Imbedding problems in graph theory. (Senior paper, Western Michigan University, Kalamazoo).

[7] Gross, Jonathon L. and Tucker, Thomas W. (1987). Topological Graph Theory. New York: Wiley Interscience.

[8] Harary, Frank and Palmer, Edgar (1973). Graphical Enumeration. New York: Academic Press.

[9] Jacobson, Nathan (1951). Basic Algebra I. New York: Van Nostrand.

[10] Kountanis, D., Mull, B. P., and Rashidi, R. (1990). A classification hierarchy for 2-cell imbeddings. Manuscript submitted for publication.

[11] Lee, Sang H. (Fall, 1989). An asymptotic result for the average genus of a graph, Graph Theory Newsletter, 18, 2.

[12] Lee, Sang H. and White, Arthur T. (In press). Random topological graph theory, Proc. of the Second International Conference in Graph Theory, Combinatorics, Algorithms and Applications.

[13] Massey, W. S. (1967). Algebraic Topology: An Introduction. New York: Harcourt, Brace & World.

[14] Mull, Bruce P. (Fall, 1989). Expected number of symmetries in 2-cell imbeddings of connected graphs, Graph Theory Newsletter, 18, 2.

[15] Mull, Bruce P., Rieper, Robert G., and White, Arthur T. (1988). Enumerating 2-cell imbeddings of connected graphs, Proc. of the Amer. Math. Soc., 103, 321-330.

[16] Rieper, Robert G. (1990). The enumeration of graph embeddings. (Doctoral dissertation, Western Michigan University, Kalamazoo).

[17] Rosen, K. H. (1974). Elementary Number Theory and Applications. Reading, MA: Addison-Wesley.

[18] White, Arthur T. (1984). Graphs, Groups and Surfaces. New York: North-Holland.

[19] White, Arthur T. and Schwenk, Allan J. (In press). An introduction to random topological graph theory, J. Combin. Ser. B.