Improving Networks Reliability

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IMPROVING NETWORKS RELIABILITY

by

Jamal H. Nouh

A Dissertation
Submitted to the
Faculty of The Graduate College
in partial fulfillment of the
requirements for the
Degree of Doctor of Philosophy
Department of Mathematics and Statistics

Western Michigan University
Kalamazoo, Michigan
December 1990
One of the basic concepts associated with a network is reliability. In this dissertation some new techniques to improve network reliability are introduced. Several new network structures are defined by adding multiple edges. Chapter I gives a brief overview of the history of network reliability and different reliability measures. It also provides a background for the chapters that follow.

In Chapter II a new sequence associated to the edges of a graph $G$ is defined. The traffic vector of an edge of $G$ of order $n$ is defined as

$$TV(e) = (\pi_1(e), \pi_2(e), \ldots, \pi_{n-1}(e))$$

where $\pi_i(e)$ is the number of paths of length $i$ that contain $e$ and is studied in the case in which the graph is a tree.

A probabilistic graph is a graph $G = (V, E)$ together with a probability assignment to the edges and vertices of $G$. The vertices and the edges $G$ are subject to failure with probability $q$, where $0 \leq q \leq 1$. In this dissertation we assume that the vertices of $G$ are absolutely reliable (never fail), but the edges of $E$ are down (i.e., in the fail state) independently with probability $q$.

In Chapter III we introduce pair-connected reliability of a graph $G$. It is the expected number of vertices that are connected in a probabilistic graph $G$. In order to maximize the pair-connected reliability, we use the concept of a traffic vector to characterize those edges in $G$ which are the best choice to be improved, in order to maximize the pair-connected reliability.
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Improving networks reliability

Nouh, Jamal Hussain, Ph.D.
Western Michigan University, 1990
In Chapter IV the global reliability of a graph $G$ is defined as the probability that a graph $G$ is connected. Two methods are described for improving the global reliability of a network $G$.

The first method is multiple edge enhancement and the second is edge improvement or replacement. The first method consists of adding to a given network $G$, multiple edges between vertices that are already joined by an edge in $G$. The second method consists of replacing or improving existing edges in $G$ by more reliable ones.

In Chapter V the K-terminal reliability is defined as the probability that, in a given probabilistic graph, the vertices in the set $K \subseteq E(G)$ are connected. The effect of enhancement or replacement edges on the K-terminal reliability for several classes of graphs are stated. Chapter VI is devoted to possible other extensions of this research.
To the memories of my father Hussein and my mother Nowara.
ACKNOWLEDGMENTS

I wish to thank Professor Alfred Boals, my advisor, for his outstanding guidance, his never-ending encouragement, his sincere interest in the completion of this dissertation, and for being an outstanding teacher. Professor Boals has great enthusiasm for any mathematical concept, and also the unique ability to find any weakness in ideas or proofs. I am grateful for his kindness towards me. He will always be remembered. I am grateful to Professor Gary Chartrand for being my second reader and for sharing his views and mathematical insights with all his students, and for being very friendly to me. I am also thankful to Professor Kenneth Williams for serving as the outside member on my committee. I am also very thankful to Professor Anthony Gioia for being my master degree adviser, and for solving many of my problems with me.

Learning on SunView LATEX software was accomplished with the help of Dr. Ajay Gupta and Dr. Jung Wang. Without their continuous support, knowledge and patience, I would have not been able to finish my dissertation. I am grateful also to both Professor Yousef Alavi and Dr. Joseph McCanna for all the help they gave me. Thanks also to all the faculty with whom I studied at Western Michigan University for the many stimulating years. I am grateful to the Department of Mathematics and Statistics for the assistantships I received throughout my studies. I am also grateful for the support provided by the United States government through the office of the American International Development. I wish to thank everyone in the Mathematic and Statistics Department office and Computer Science Department office for assisting in all the paperwork that was needed to accomplish my goal.
Also I would like to thank Dr. Naveed Sharwani and his research assistants for the help in completing this dissertation. I would also like to thank my best friend Dr. Kannan and his family for their continuous support and their outstanding guidance.

It is extremely unfortunate that my father is not any more in this world to see me complete my Ph.D. dissertation. It was his dream and it was he who encouraged me to go for higher studies. I will always remember him. I am most grateful to my family, especially my wife Amal for her steadfast support and encouragement, my children and my friends who gave their never-ending support, love and understanding throughout my school years.

Finally, I would like to thank all those friends, especially N. Hedroug, for their help in getting this dissertation completed: typing, corrections, and all these things.

Jamal H. Nouh
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CHAPTER I
INTRODUCTION

1.1 Definitions, Notation, and Historical Review

Reliability is concerned with the ability of networks to carry out certain network operations. An important step is the identification of necessary network operations. A widely used model for communication networks in which elements (vertices and edges) are subject to failure is that of a probabilistic graph $G$. Given a graph $G$ with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$, $G$ is called a probabilistic graph, if each element in $V(G)$ and $E(G)$ is assigned a certain probability, say $p(v_i)$ and $p(e_j)$, for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. The number $p(v_i)$ denotes the probability that $v_i$ exists in $V(G)$ and $p(e_j)$ denotes the probability that $e_j$ exists in $E(G)$. If the vertex $v_i$ exists in $V(G)$, then it is considered to be in the up state; otherwise, it is in the failed state. Similarly, if $e_j$ exists in $E(G)$, then it is in the up state; otherwise, it is in the failed state. We call a graph $G$ with $n$ vertices and $m$ edges, an $(n,m)$-graph. Here it is assumed that vertices are fail safe (i.e., never been in the failed state) but that each edge $e \in E(G)$ is down (that is, in a failed state) independently with probability $q$, where $0 \leq q \leq 1$, and $p = 1 - q$ will denote the probability that each edge is in the up state. For $S \subseteq E(G)$, the graph $G$ is said to be in the up state $S$, if the edges that are in the up state are precisely those in $S$. The spanning subgraph of $G$ induced by a set of edges $S$ is denoted by $\langle S \rangle$.

Perhaps the most common operation is communication from a source node $s$ to a target node $t$. 

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For a probabilistic graph $G$, and specified nodes $s, t$, we define the two-terminal reliability to be the probability that there exists at least one $st$-path in $G$ and the probability will be denoted by $R_{s,t}(G, q)$. In the directed case, the problem is called $st$-connectedness (see Colbourn [15]). The $K$-terminal reliability $R_K(G, q)$ is one which measures the probability of having all pairs of vertices in $K$ connected in $G$, where $K \subseteq V(G)$.

Another common operation in networks is broadcasting. In order to model such an operation, we define the all-terminal reliability to be the probability that for any pair $v_1, v_2$ of vertices in $G$, there exists a path from $v_1$ to $v_2$ (equivalently, $G$ has at least a set $S$ in the up state and $< S >$ is spanning tree) and probability will be denoted by $R(G, q)$. If $G$ is a directed graph (digraph), then $R_u(G, q)$ is the probability that the digraph $G$ contains a directed path from a vertex $u$ to every other vertex in $G$. Recently, a new reliability measure called pair-connected reliability has been introduced for graphs (see Siegrist and Slater [6] and Boesch [10]). The pair-connected reliability of a given graph $G$ is the expected number of pairs of connected vertices.

These reliability measures have the following practical application: Assume the graph $G$ models a system in which a subset $K$ of vertices in $G$ represents sufficient processing capability and/or data storage capacity for a processor to execute efficiently. It is important in this case to have the vertices in $K$ connected. The $K$-terminal reliability measures the probability of having all vertices in $K$ connected. On the other hand, if our graph $G$ represents a communication system, in which it is important for each node to communicate with others, in this case, what is important is the expected number of vertices in the probabilistic graph $G$ which stay connected. Pair-connected reliability measures the expected number of pairs of vertices in $G$ which are connected.

Herein, we survey some of the known results, and graph theory notions which are relevant as models to the analysis and synthesis of the network problem.
For the graph-theoretic ideas and reliability notation, we follow the books by Chartrand and Lesniak [13] and Colbourn [14], respectively.

In addition to the above reliability measures, other reasonable measures can be defined. A general mechanism for defining a reliability problem is therefore in order. Given a probabilistic graph $G = (V, E)$, we define

$$R(G, q, f) = \sum_{S \in \Omega} f(S) \cdot R(S)$$

(1.1)

to be a general reliability measure of $G$, where $\Omega$ is the power set of $E$ (that is, the set of all possible states for the system). If $S \in \Omega$, then $R(S)$ is the probability that $G$ is in the up state $S$, which under the assumption of independent and equal probability of failure $q$, means $R(S) = p^{|S|}q^{m-|S|}$ where $|S| = k$ and $m = |E(G)|$.

Note that $(\Omega, R)$ is a probability space, $f$ is a random variable defined in this space, and $R(G, q, f)$ is the expected value of $f$.

Different choices for the function $f$ provide a variety of reliability measures. For the global reliability (all terminal reliability), the formula in (1.1) becomes

$$R(G, q) = \sum_{S \in \Omega} f(S) \cdot R(S)$$

(1.2)

where

$$f(S) = \begin{cases} 
1 & \text{if } < S > \text{ is connected} \\
0 & \text{otherwise.}
\end{cases}$$

The function $f$ is dropped from $R(G, q, f)$ for simplicity. Since $f(S)$ takes the value 0 or 1 and $(\Omega, R)$ is a probability space, the formula in (1.2) measures the expected value that $G$ is connected, namely it is the probability that $G$ contains a spanning tree, in the up state. Observe that $0 \leq R(G, q) \leq 1$.

For the k-terminal reliability, the function $f$ in (1.1) is defined as follows:

$$f(S) = \begin{cases} 
1 & \text{if } < K > \text{ is connected in } G - S \\
0 & \text{otherwise.}
\end{cases}$$

In this case, the function $R(G, q, f)$ is $R_K(G, q)$. If $|K| = 2$, it is called a two-terminal reliability. If $K = \{s, t\}$, then formula (1.1) is written as $R_{s,t}(G, q) = \ldots$
$R(G, q, f)$. Thus $0 \leq R_{st}(G, q) \leq 1$ represents the probability that there exists an $st$-path in $G$.

For the pair-connected reliability of $G$, the function $f(S)$ in (1.1) is denoted by $PC(S)$, and it is equal to the number of pairs of vertices that are connected in $\langle S \rangle$. In the case of the pair-connected reliability, formula (1.1) can be written in the form

$$R(G, q) = \sum_{S \in \Omega} PC(S) \cdot R(S).$$

This is the expected value of the number of pairs of vertices that are connected in $G$.

Ball and Provan [9] showed that computing $R_{st}(G, q)$ is NP-hard, even if $G$ is a planar graph of maximum degree 3. Moreover it can be shown that computing $PC(G, q)$ is NP-hard in the case where $G$ is planar of maximum degree 4.

All the reliability measures we have introduced are number P-complete problems (see Colbourn [15]). Moreover, Gilbert [20], and Frank and Gaul [19] established formulas for the all-terminal and two-terminal reliability of the complete graph $K_n$ of $n$ vertices.

**Theorem 1.1** ([19]) *If $G = K_n$ is the complete graph of $n$ vertices, then*

$$A_n = R(K_n, q) = 1 - \sum_{j=1}^{n-1} C_{j-1}^{n-1} A_j q^{(n-j)}$$

*where $A_j = R(K_j, q)$.\)

Note that $A_n$ has a recurrence relation in terms of $A_j$, for $j < n$.

In the case of the two-terminal reliability, the following formula is obtained for the complete graph.

**Theorem 1.2** ([19]) *For the complete graph $K_n*$

$$R_{st}(K_n, q) = 1 - \sum_{j=1}^{n-1} C_{j-1}^{n-2} A_j q^{(n-j)}$$

*where $A_j = R(K_j, q)$, and $C_m = \frac{n!}{m!(n-m)!}$.*
Similar formulas can obtained for complete bipartite graphs.

Two graphs $G_1$ and $G_2$ are in series connection, if $G_1$ and $G_2$ have only one vertex in common. Such a connection is denoted by $G_1 \cdot G_2$. In general, a set of graphs $G_1, G_2, \ldots, G_r$ is in series connection if no two graphs have more than one vertex in common, and the only cycles are those in $G_i$, $i = 1, 2, \ldots, r$.

The following results can be found in Amin, Siegrist and Slater [2]. Let $A$ be a set of graphs $G_1, G_2, \ldots, G_r$. If $G$ is a graph obtained from $A$ by series connection, then

$$R(G, q) = \prod_{i=1}^{r} R(G_i, q).$$

For pair connected reliability, suppose $G = G_1 \cdot G_2$ is a graph obtained from $G_1$ and $G_2$ by series connection, and let $u$ be the common vertex between $G_1$ and $G_2$, define $PC(G_1(u), q)$ to be the expected number of vertices that are connected to $u$ in $G_1$. This function can be written in the form

$$PC(G_1(u), q) = \sum_{v \in V - \{u\}} E_p(I_{G_1}(u, v))$$

where

$$I_{G_1}(u, v) = \begin{cases} 
1 & \text{if } u \text{ is connected to } v \\
0 & \text{otherwise}
\end{cases}$$

and $E_p(I_{G_1}(u, v))$ is the probability that $u$ and $v$ are connected in the probabilistic graph $G$.

The following result is due to Amin, Siegrist and Slater [3].

**Theorem 1.3** ([3]) If $G = G_1 \cdot G_2$, then

$$PC(G, q) = PC(G_1, q) + PC(G_2, q) + PC(G_1(u), q) \cdot PC(G_2(u), q)$$

For a given tree $T$, the calculation of $R(T, q)$ can be computed easily. Given a tree $T$ of order $n$, then $R(T, q) = p^n$, where $p$ is the probability of having the tree $T$ in state $\{e\}$ for all $e \in E(T)$.

In general the calculation of global and $K$-terminal reliability are not trivial. Moore and Shannon [26] used the following reduction formula in finding $K$-reliability.
Let \( G/e \) be the graph obtained from \( G \) by contraction of the edge \( e \), and let \( G - e \) be the graph obtained from \( G \) after deleting the edge \( e \).

**Theorem 1.4 (Reduction Formula)** If \( G \) is a graph, then

\[
R_k(G, q) = pR_K(G/e, q) + (1 - p)R(G - e, q).
\]

For K-terminal reliability, the above reduction formula uses the following transformation:

If \( e_1 \) and \( e_2 \) are two parallel edges (edges connecting the same vertices), assume \( e_1 \) has probability of failure \( q_1 = 1 - p_1 \), and \( e_2 \) has probability failure \( q_2 = 1 - p_2 \), then \( e_1 \) and \( e_2 \) can be replaced by one edge \( e \) with reliability (probability of being functional) \( p \), where \( p = 1 - q_1 \cdot q_2 \).

On the other hand, if \( e_1 = u_1v \), and \( e_2 = vu_2 \) are incident edges and \( v \) is the common vertex with \( v \not\in K \), then we can replace \( e_1 \) and \( e_2 \) by \( e \), with \( p(e) = p_1 \cdot p_2 \); for \( v \in K \), \( e_1 \) and \( e_2 \) can be replaced by \( e \) with probability \( p = \frac{p_1p_2}{1 - q_1q_2} \).

For global connectivity, the function \( R(G, q) \) in (1.1) can be expressed in terms of the number of induced connected subgraphs of \( G \),

\[
R(G, q) = \sum_{r=0}^{[E]} m_r p^r (1 - p)^{|E| - r}
\]

where \( m_r \) is the number of induced connected subgraph in \( G \) of size \( r \) and \( |E| \) is the size of \( G \).

An \((n,m)\)-graph \( G \) is said to be uniformly optimally reliable if

\[
R(G, q) \geq R(H, q)
\]

for all \((n,m)\)-graphs, and all \( q, \ 0 < q < 1 \). Boesch [11] conjectured that uniformly optimally reliable graphs always exist. In fact, he showed that such graphs exist for classes of graphs with order at most 6.

Opposite to uniformly optimally reliable graphs is the uniformly least reliable graph, it is the one in which every other graph, with the same order and size, is
more reliable. Boesch [10] observed that all trees with \( n \) vertices have the same reliability.

Several authors studying the area of network reliability (see Evans and Smith [17]), have concluded that the synthesis of reliable networks is not a pure graph theory problem, because the decision of which of two graphs is better, is dependent on the probability \( q \). Given two \((n,m)\)-graphs, \( G_1 \) and \( G_2 \), let \( R(G_1, q) \) and \( R(G_2, q) \) be the global reliability polynomials of \( G_1 \) and \( G_2 \), respectively. For different values of \( q \) the two functions \( R(G_1, q) \), and \( R(G_2, q) \) may cross. This can be seen by the following example Kelmans [25] which is the smallest example where two \( R(G, q) \) functions cross.

Example: The graphs \( G_1 \) and \( G_2 \) shown in Figure 1.1 has the global reliability

\[
R(G_1, q) = 4q^2(1-q)^6 + 24q^3(1-q)^5 + \sum_{k=4}^{8} C_k^8 p^k(1-p)^{8-k}
\]

\[
R(G_2, q) = 3q^2(1-q)^6 + 26q^3(1-q)^5 + \sum_{k=4}^{8} C_k^8 q^k(1-q)^{8-k}.
\]

Comparing the two functions \( R(G_1, q) \) and \( R(G_2, q) \), the following can be concluded:

\[
R(G_1, q) > R(G_2, q) \text{ for } 0 < q < \frac{1}{3},
\]
The definition of optimally reliable graphs can be extended to pair-connected reliability. For the pair-connected reliability, Amin, Siegrist, and Slater [6] showed, that the star $K_{1,n-1}$ is the uniformly optimally reliable tree with $n$ vertices. In the same paper, it was shown that the path $P_n$ is the uniformly least reliable tree on $n$ vertices. In [6] the same authors have shown that there do not exist uniformly optimal $(n,m)$-graphs, except in the extreme cases when $m < n - 1$ or $m \geq C_2^n - 1$, where $C_2^n = \frac{n!}{(n-2)!2!}$. When $m \leq n - 1$, it follows easily from the fact that all graphs with $m < n - 1$ have the same reliability; namely 0, and the star $K_{1,n-1}$ is the uniformly optimal graph for $m = n - 1$. On the other hand, when $m \geq C_2^n - 1$, all $(n,m)$-graphs are isomorphic.

For trees, the computation of $PC(T, q)$ is straightforward (see Siegrist [29]). For an arbitrary graph $G$, the distance distribution of $G$ is defined as $D(G) = (d_1(G), d_2(G), ..., d_{n-1}(G))$, where $d_i(G)$ denotes the number of pairs of vertices at distance $i$. The following result shows how to compute the pair-connected reliability of a tree $T$ from its distance distribution.

**Theorem 1.5** The distance distribution $D(T)$ of a tree $T$ completely determines $PC(T, q)$, namely $PC(T, q) = \sum_{i=1}^{n-1} d_i(T)p^i$.

For any graph $G$,

$$R_{s,t}(G, q) \geq p^{\text{dist}(s,t)}.$$  

where $R_{s,t}(G, q)$ is the two-terminal reliability of $G$ and, $\text{dist}(s, t)$ is the distance between $s$ and $t$. The next result follows.

**Theorem 1.6** For any graph $G$,

$$PC(G, q) \geq \sum_{i=1}^{n-1} d_i(G)p^i.$$  

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For trees and series-parallel graphs, efficient algorithms for computing $PC(G, q)$ are described in Amin, Siegrist, and Slater [3]. For certain classes of graphs formulas for $PC(G, q)$ have been determined (see [5], [6]). Here, we present the formula for the $n$-cycle $C_n$ and the wheel $W_{n+1} = K_1 + C_n$ on $n+1$ vertices.

$$PC(C_n, q) = n^2 - np^2 - \frac{n(n-1)}{2} p^n;$$

$$PC(W_{n+1}, q) = n\left[1 - \frac{q^3}{(1-pq)^2}\right][1 - (pq)^n] - n^2 \frac{(pq)^n+1}{(1-pq)^2} + \frac{n(n-1)}{2} \left[1 - \frac{q^3}{(1-pq)^2}\right] + \frac{q^4(1+3p^2) + 2p^4q^3}{(1-pq)^4} \left[n \frac{(pq)-(pq)^n}{1-pq} - \frac{n(n-1)}{2} (pq)^n\right] + \frac{pq^4}{(1-pq)^2} \left[-n^2(pq)^n + n^2(pq)^{n+1} + npq - n(pq)^{n+1} \frac{n(1-n)}{2} (pq)^n\right] + \left(\frac{pq}{1-pq}\right)^2 \frac{n^2(n-1)(n+1)}{12} (pq)^n .$$

A lot of work has been done on network synthesis. The result of such work is important for the design of reliable networks.

If a network is described by its underlying graph $G$, then the cost of building a network could be measured by the number of edges. We assume the cost of each edge is constant. Such an assumption is sometimes valid in practical problems which allow for a simplified model. Various optimization problems are suggested by this model; for example, one might try to find the maximum value of the edge connectivity $\lambda$ over the class of all graphs with prescribed values of $n$ and $m$. This is an example of an extremal graph problem. If $\kappa$ and $\lambda$ represent the vertex and edge connectivity of $G$, respectively, then one may ask for the maximum value of $\kappa$ or $\lambda$ in the class of $(n,m)$-graphs. The first publication related to this topic can be found in Harary [21]. His result is stated in the theorem below. Let $(n, \lambda \geq k)$ denote the size of the graph of order $n$ and edge connectivity at least $k$. Similarly, let $(n, \kappa \geq k)$ denote the size of the graph of a graph of order $n$, size $m$ and vertex connectivity at least $k$. 

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Theorem 1.7 ([21])

\[ \min(n, \lambda \geq k) = \min(n, \kappa \geq k) = \begin{cases} \left\lceil nk/2 \right\rceil & \text{if } \lambda, k \geq 2 \\ n - 1 & \text{if } \lambda = \kappa = 1 \end{cases} \]

Note that \([x]\) denotes the smallest integer not smaller than \(x\); \([x]\) is the largest integer not larger than \(x\). The graph used by Harary [21], to prove the theorem is called the elementary Harary graph and is denoted by \(H(n, k)\). If \(n\) vertices are labeled \(0, 1, 2, \ldots, n - 1\), then \(H(n, k)\) can be constructed by joining each node \(i, 0 \leq i \leq n - 1\), to the node \(i \pm 1, i \pm 2, \ldots, i \pm \left\lfloor \frac{k}{2} \right\rfloor\). The graph \(H(n, k)\) has \(m = \left\lceil \frac{n k}{2} \right\rceil, \kappa = \lambda = k\) and is regular when both \(n, k\) are not odd. For further discussion about optimization on graphical parameters, see Harary [21].

In the following section, we study network synthesis from a different perspective.

1.2 Back-up Links in Networks

As mentioned in Section 1.1, most studies in the area of network synthesis concentrate on finding reliable \((n,m)\)-graphs for specified values of \(n\) and \(m\). In this dissertation, we describe how to optimize the reliability of a given graph \(G\) by means of adding multiple edges to \(G\) or by replacing some edges in \(G\), by more reliable edges.

There are many cases where a network already exists, or the logical design of an \((n,m)\)-graph does not follow an optimal reliable graph. For a given network, one may ask the following question: If the reliability of a set of \(r\) edges in a given \((n,m)\)-graph \(G\) is to be improved, what is the best choice among all subsets of \(r\) edges in \(G\), that should be considered in order to optimize a given a reliability measure of \(G\)?

In this dissertation we introduce two methods to enhance network reliability: (1) multiple edge enhancement, and (2) edge improvement or replacement.

The first method consists of adding in a given network multiple edges to a network with the restriction that edges may only be added between vertices which are
already joined by an edge. The latter method consists of replacing or improving existing edges.

Let $G$ be an $(n,m)$-graph. If some edges in $G$ are replaced by more reliable edges or enhanced by multiple edges, then the new graph has a new probability assignment to its edges. Such an assignment will be denoted by $\Gamma$. If the edges in $G$ are labeled $e_1, e_2, ..., e_m$, then

$$\Gamma = \{(e_1, \Delta p_1), (e_2, \Delta p_2), ..., (e_m, \Delta p_m)\}$$

where the ordered pair $(e_i, \Delta p_i)$ indicates that there is an increase in the reliability of $e_i$ by an amount $\Delta p_i$. Under the assumption that edges in $G$ have the same reliability, namely $p$, the new reliability of the edge $e_i$ will be $p + \Delta p_i$ after enhancement. The restriction of only improving or replacing existing edges or adding multiple edges is to preserve the functionality of the network.

Throughout the discussion, we always assume that edges in the graph $G$ (which represents the network) have the same reliability. Hereafter, the additional edges used to improve reliability of the network are assumed to have the same reliability as the edges in $G$, unless otherwise stated.

In this dissertation we investigate the improvement of three different network reliability measures: the global reliability, K-terminal reliability, and pair-connected reliability. We present some examples to illustrate the above stated measures, and possible ways to improve those measures.

Example 1: In this example we consider the question: Is there an optimal method, with respect to the global reliability, to add two edges to a path of length 4, $P_5$ and is the method independent of $p$?

Let $P_5'[2]$ be any graph obtained from $P_5$ by adding two multiple edges. Let $e_1, e_2, e_3, e_4$ be a labeling of the edges of $P_5$ taken according to their order from one of its end vertices. The answer to the above stated question can be resolved by the following cases:

Case 1: No more than one multiple edge is added between a pair of vertices of $P_5$. Let $G_1$ be a graph obtained from $P_5$ by adding one multiple edge to $e_1$ and
one to \( e_2 \). Then \( R(G_1, \Gamma_1) = p_1^2 \cdot p^2 \), where \( p_1 = 2p - p^2 \), and

\[
\Gamma_1 = \{(e_1, p - p^2), (e_2, p - p^2), (e_3, 0), (e_4, 0)\}.
\]

In this case, the choice of the edges in \( E(T) \) to be improved has no effect on the global reliability.

Case 2: Allow more than one multiple edge between two vertices of \( P_5 \). Let \( G_2 \) be the graph created from \( P_5 \) by adding two multiple edges to the edge \( e_1 \), then \( R(G_2, \Gamma_2) = p_2 \cdot p^3 \), where \( p_2 = 3p - 3p^2 + p^3 \).

\[
\Gamma_2 = \{(e_1, 2p - 3p^2 + p^3), (e_2, 0), (e_3, 0), (e_4, 0)\}
\]

Again, as in Case 1, the choice of the edge \( e_i \) has no effect on the global reliability. To observe which one of the above cases increases the global reliability the most, consider the following difference:

\[
\Delta R(p) = R(G_2, \Gamma_2) - R(G_1, \Gamma_1)
\]

\[
= p_1^2 \cdot p^2 - p_2 \cdot p^3 = p^2(p_1^2 - p_2p)
\]

\[
= p^2[p^2(4 - 4p^2) - p^2(3 - 3p + p^2)]
\]

\[
= p^4[1 - p].
\]

For \( p \in (0, 1) \), the difference function \( \Delta R(p) \) is always positive. Therefore the choice in Case 1 is always better for all values of \( p \).

Example 2: Is there an optimal way with respect to K-terminal reliability to add two multiple edges to the cycle \( C_6 \), with vertex set \( V \) and edge set \( E \), such that the K-reliability for \( K = \{s, t\} \) where \( s \) and \( t \) are two vertices in \( V \) with \( d(s, t) = 3 \), is maximum? (see Figure 1.2). Suppose the vertices of \( C_6 \) are labeled \( v_0, v_1, \ldots, v_5 \) such that \( e_i = v_i v_{(i+1) \mod 6} \) are the edges. Without loss of generality we assume that \( s = v_0 \) and \( t = v_3 \). Let the path from \( s \) to \( t \) containing \( v_1 \) be \( P_1 \) and the path from \( s \) to \( t \) containing \( v_5 \) be \( P_2 \). We proceed by considering two cases:
Case 1: Both additional edges are added to $P_1$. Let $G_1$ be the graph obtained from $C_6$ by adding one multiple edge to both $e_1, e_2$ on $P_1$.

$$R_{s,t}(G_1, \Gamma_1) = 1 - [1 - p_1^2 p][1 - p^3]$$

where

$$p_1 = 2p - p^2, \text{ and } \Gamma_1 = \{(e_1, p - p^2), (e_2, p - p^2)\} \cup \{(e_i, 0)\}$$

for $i = 1, 2, ..., 6$

Case 2: One edge is added to $P_1$ and the other is added to $P_2$.

Let $G_2$ be the graph obtained from $C_6$ by adding one multiple edge to both $e_1$ and $e_4$ on $P_1$ and $P_2$, respectively

$$R_{s,t}(G_2, \Gamma_2) = 1 - [1 - p_1 p^2][1 - p_1 p^2]$$

$$= [1 - p_1 p^2]^2$$

where $p_1 = 2p - p^2$ and

$$\Gamma_2 = \{(e_1, p - p^2), (e_4, p - p^2)\} \cup \{(e_i, 0)\}$$
for all $i = 1, 2, \ldots, 6$. The graphs in Figure 1.3 show the above cases.

To see which case is better, consider the following difference:

$$
\Delta R(p) = R_{s,t}(G_1, \Gamma_1) - R_{s,t}(G_2, \Gamma_2)
= [1 - p_1p_2]^2 - [1 - p_3][1 - P_1^2p]
= [p^8 - 4p^7 + 4p^6 + 2p^4 - 4p^3 + 1] - [1 - 2p^2 + 2p^5 - p^6]
= p^8 - 4p^7 + 5p^6 - 2p^5 + 2p^4 - 4p^3 + 2p^2.
$$

The graph of the function $\Delta R(p)$ is shown in Figure 1.4. Observe that $\Delta R(1) = \Delta R(0) = 0$, and $\Delta R(p) > 0$, for $0 < p < 1$. We conclude choice 1 is the better choice, namely improving one path rather than two paths.

If a probabilistic graph $G$ is given, and $m$ is a positive integer $m \leq |E(G)|$, a natural question is: Which is the best set of edges in $E(G)$ of size $m$ to improve, so we obtain the most reliable graph from $G$ with respect to pair-connected reliability? We consider the following example:

Example 3: Let $P_5$ be the path in Example 1. Let $m$ be the number of extra edges needed to be used in the enhancement. What is the best choice of one edge among $E(P_5)$, so we can increase $PC(P_5, q)$ the most? By Theorem 1.3, the
pair-connected reliability of the tree $T$ is

$$PC(T, q) = \sum_{i=1}^{n-1} D_i p_i$$

where, $D_i$ is the distance distribution of the vertices in $T$. The above formula can be modified as follows

$$(T, \Gamma) = \sum_{i=1}^{n-1} R(P_i).$$

Note that the sum is taken over all paths $P_i$ of length $i$, $R(P_i)$ is the probability that $P_i$ is connected, and $\Gamma$ is the probability distribution of $E$. This modification allows us to find the pair-connected reliability of the tree when the edges have different probability assignment. Now label the edges in $P_5$ as $e_1, e_2, e_3, e_4$ according to their location from one of the end vertices of $T$. By symmetry of the edges in $P_5$, it is necessary to consider the following two cases:

Case 1: Enhancing the edge $e_1$.

Let $G_1$ be the graph obtained from the path $P_5$, by adding new multiple edge on $e_1$. Then

$$PC(G_1, \Gamma_1) = (p_1 + 3p) + (p_1p + 2p^2) + (p_1p^2 + p^3) = (p_1p^3)$$

where

$$\Gamma_1 = \{(e_1, p - p^2), (e_2, 0), (e_3, 0), (e_4, 0)\}$$
and \( p_1 = 2p - p^2 \).

**Case 2: Enhancing the edge \( e_2 \).**

Let \( G_2 \) be the graph obtained from \( P_5 \) by adding one multiple edge to \( e_2 \), then

\[
PC(G_2, \Gamma_2) = (p_1 + 3p) + (2p_1pp + p^2) + (2p_1p^2) + (p_1p^3)
\]

where,

\[\Gamma_2 = \{(e_2, p - p^2), (e_1, 0), (e_3, 0), (e_4, 0)\} .\]

To see the difference between pair-connected reliabilities, consider the following:

\[
PC(G_2, \Gamma_2) - PC(G_1, \Gamma_1) = (p_1p + p^2) + (p_1p^2 + p^3)
\]

\[
= p(p_1 - p) + p^2(p_1 - p)
\]

\[
= (p_1 - p)(p + p^2)
\]

\[
\geq 0
\]

for all \( p \in (0,1) \). From the above analysis, we conclude that improving the edge \( e_2 \) is the best choice, for all values of \( p \).

In Chapter II we define the traffic vector of an edge for a given graph, and we study the traffic vector distributions of the edges of trees. The set \( S \) of traffic vectors is said to be graphic, if there exists a tree \( T \) of order \( |S| + 1 \), such that the set of edges in \( T \) has the set \( S \), as its traffic vector distribution. We prove that the problem of whether a set of traffic vectors is graphic or not is an NP-complete problem.

In Chapter III, we use the traffic vectors in improving the pair-connected reliability. In particular, if \( k \) edges in \( T \) are to be improved, then we use the traffic vector analysis to find a subset \( S \), with \( |S| = k \), and \( S \subseteq E(T) \), such that improving \( S \), increases the pair-connected reliability of \( T \) the most.

In Chapter IV, we study how to improve global reliability of tree networks, unicyclic networks and multi-ring connection networks.
In Chapter V, we present the analysis of improving the two-terminal reliability for parallel and series connection graphs.

In Chapter VI, new reliability measures are presented with some suggestions to improve these reliability measures by using the two methods mentioned in this chapter. In addition, open questions and possible research problems are given.
CHAPTER II

TRAFFIC VECTORS

2.1 Traffic Vectors

A sequence for a graph is simply an invariant which consists of a list of numbers rather than a single number. In this chapter, we would like to introduce the concept of the traffic vector sequence. A number of graph sequences are discussed in literature (see Buckley and Harary [12]). Given a graph \( G \) and a set \( S \) of edges in \( G \), the induced subgraph on \( S \) is denoted by \( \text{Ind}(S) \). The traffic vector of \( S \) will be a sequence of numbers which describes the number of paths of different lengths containing \( S \). In the following chapters, we will use the traffic vector sequence in investigating of improving network reliability. We shall adopt the notations of Harary and Buckley [12].

Definition 1 Let \( G = (V,E) \) be a graph with order \( n \) and let \( E \subseteq S \). The traffic vector distribution of \( S \) is defined as:

\[
TV_G(S) = (\pi_1(S), \pi_2(S), \ldots, \pi_{n-1}(S)),
\]

where \( \pi_i(S) \) denotes, the number of paths of length \( i \) in \( G \) which contain all the edges of \( S \).

We restrict the study of traffic vectors to acyclic graphs.
Remark 1  Given a forest $F$ and a set $S$ of edges in $F$, if $k$ is the length of a minimal path in $F$ which contains $S$, then $\pi_i(S) = 0$ for all $0 \leq i < k$.

We will write $\pi_i(e)$ for $\pi_i(S)$ when $S = \{e\}$. If $F$ is known, we can drop the subscript $F$ in $TV_F(S)$ and simply write $TV(S)$. In a forest $F = (V,E)$, if $E = \{e_1, e_2, \ldots, e_{n-1}\}$ then the set of traffic vectors of the edges in $E$ is called the Traffic Vector Distribution of $F$.

An edge $e = uv$ is called an end edge if one of the vertices $u$ or $v$ has degree one. In a tree $T$, the traffic vector of an edge $e$ in $T$ doesn’t identify $e$ uniquely. In fact, the following example contains two non-isomorphic edges which have the same traffic vector in $T$. (see Figure 2.1). We proceed to construct two non-isomorphic trees $T_1$ and $T_2$, such that the end edges in both have the same traffic vector (see Figure 2.2).

\[
\begin{align*}
TV_T(e_i) &= TV_T(e'_i) = (1,1,4,7,4) \; ; \; i = 1,2,3,4 \\
TV(e_i) &= TV_T(e'_i) = (1,2,4,6,4) \; , \; i = 5,6,7,8 \\
TV(e_9) &= TV(e_{10}) = (1,4,8,4)
\end{align*}
\]
The traffic vector distribution of the set of end edges in $T_1$, and $T_2$ are

$$\{(1,1,4,7,4)^4,(1,2,4,6,4)^4,(1,4,8,4)^2\},$$

where the notation $[TV]^i$ indicates that there are $i$ edges with the same traffic vector $TV$. The two trees $T_1$ and $T_2$ have different traffic vectors distributions. For example, in the tree $T_1$, $TV(e) = (1,5,11,11,4)$ but there are no edge in $T_2$ has this traffic vector. The question of whether the traffic vector distribution of $E(T)$ uniquely determines $T$ remains open.

Let $G = (V,E)$ be a graph of order $n$, for any $v \in V$, the Distant Degree Sequence of $v$ in $G$ is $DDS_G(v) = (d_0(v),d_1(v),\ldots,d_{n-1}(v))$, where $d_i(v)$ is the number of vertices of distant $i$ from $v$. Note that $d_0(v) = 1$ for all $v$. Given a tree $T$ and an edge $e = uv$, the subtrees in $T - e$ which contain $u$ and $v$ will be denoted by $T_u$ and $T_v$ respectively. The Edge Degree Distribution of an edge $e = uv$ in $T$ is the sequence $EDD(uv) = (S_1,S_2)$, where $S_1 = DD_{T_u}(u)$ and $S_2 = DD_{T_v}(v)$.
is sequence \( S_1 \) written in reverse order. Note that the sequence \( EDD(uv) \) is dependent on the order \( uv \).

Example: Consider a tree \( T \), let \( e = uv \) be an edge in \( T \) with

\[
DD_{Tu}(u) = (1,2,3) \quad \text{and} \quad DD_{Tv}(v) = (1,4,6).
\]

The Edge Degree Distribution of \( e = uv \) is \( EDD(uv) = (3,2,1,1,4,6) \)

**Lemma 1** Suppose \( e = uv \) is an edge in a tree \( T \). If

\[
EDD(uv) = (n_i,n_{i-1},\ldots,n_0,m_0,m_1,\ldots,m_k), \quad \text{where} \quad n_0 = m_0 = 1
\]

with \( k \geq l \), then

\[
\pi_i(e) = \sum_{k=0}^{i-1} n_k m_{i-k-1} \quad \forall \ i, \ i = 1,2,\ldots,(l + k - 1).
\]

**Proof:** This follows directly from the fact that any path of length \( i \) containing \( e \) consists of a path of length \( k \) in \( T_u \) and a path of length \( i - k - 1 \) in \( T_v \) together with the edge \( e \). □

**Theorem 2.8** Let \( T \) be a tree of order \( n \) and \( n \geq 2 \), let \( S \) be a non-empty set in \( E(T) \). If \( k \) is the smallest integer such that \( \pi_k(S) \neq 0 \) then

\[
1 \leq \sum_{i=1}^{n-1} \pi_i(S) \leq \left\lfloor \frac{n - k + 1}{2} \right\rfloor \left\lfloor \frac{n - k + 1}{2} \right\rfloor.
\]

**Proof:** Since \( \pi_k(S) \neq 0 \) this implies that there exists a \( uv \)-path \( P \) of length \( k \) which contains all edges of \( S \). If \( k > |S| \), then \( P \) contains edges not in \( S \) and hence \( |E(P)| \geq |S| \).

Let \( T_u \) and \( T_v \) be the subtrees of \( T \) in \( T - E(P) \) which contain \( u \) and \( v \) respectively. The total number of paths containing \( S \) is

\[
\sum_{i=1}^{n-1} \pi_i(S) = |V(T_u)||V(T_v)| = |V(T_u)||n - M - |V(T_u)||
\]  \hspace{1cm} (2.1)
where

\[ M = |V(T)| - |V(T_v)| - |V(T_u)| \]

The fact that \( M \geq |E(P)| \), together with \( |V(T_u)| \geq 1 \) and \( |V(T_v)| \geq 1 \) implies that \( n - 1 \geq M \geq k - 1 \). Observe that \( |V(T)| = |V(T_v)| + |V(T_u)| + M \). Therefore \( |V(T_u)| = |V(T)| - |V(T_v)| - M \). Letting \( |V(T_u)| = x \), equation (2.1) can be written as

\[ f(x) = \sum_{i=1}^{n-1} \pi_i(s) = x(n - M - x). \]

For \( x \geq 1 \), the function \( f(x) \) has a minimum value at \( x = 1 \) and a maximum value at \( x = \left(\frac{n-M}{2}\right) \). Since \( x \) assumes only integer values, \( f(x) \) has a minimum value at \( x = 1 \) and maximum value at \( x = \left\lfloor \frac{n-M}{2} \right\rfloor \) or \( \left\lceil \frac{n-M}{2} \right\rceil \). The fact that \( M \geq k - 1 \) implies that

\[ \left\lfloor \frac{n-M}{2} \right\rfloor \leq \left\lfloor \frac{n-k+1}{2} \right\rfloor \]

and

\[ \left\lceil \frac{n-M}{2} \right\rceil \leq \left\lceil \frac{n-k+1}{2} \right\rceil. \]

Therefore

\[ \sum_{i=1}^{n-1} \pi_i(s) \leq \left\lfloor \frac{n-M}{2} \right\rfloor (n - \left\lfloor \frac{n-M}{2} \right\rfloor) \leq \left\lfloor \frac{n-k+1}{2} \right\rfloor \left\lfloor \frac{n-k+1}{2} \right\rfloor \]

and

\[ \sum_{i=1}^{n-1} \pi_i(s) \geq 1(n - m - 1). \]

The graphs \( G_1 \) and \( G_2 \), shown in Figure 2.3 illustrate that the upper bounds given in the above result are sharp.

**Corollary 1** If \( T \) is a tree of order \( n \) and \( S = \{e\} \subset E(T) \) then

\[ n - 1 \leq \sum_{i=1}^{n-1} \pi_i(e) \leq \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil. \]
Proof: The upper bound follows immediately from Theorem 2.8 with \( k = 1 \) and with the fact that the size of the minimal path which contains \( e \) is one. To show the lower bound, let \( e = uv \) with \( T_u \) and \( T_v \) be the components in \( T - e \) which contains \( u \) and \( v \) respectively. Observe that \( \sum_{i=1}^{n-1} \pi_i(e) = |V(T_u)| \cdot |V(T_v)| \). If \( |V(T_u)| = x \), then \( \sum_{i=1}^{n-1} \pi_i(e) = x(n - x) \). This is a function of \( x \) which has maximum value at \( x = 1 \). Therefore \( \sum_{i=1}^{n-1} \pi_i(e) \geq n - 1 \). \( \square \)

Lemma 2 Let \( T \) be a tree of order \( n \). If \( S \subset E(T) \), then \( \pi_{|S|+1}(S) \leq n - (|S| + 1) \)

Proof: Let \( P \) be a minimal uv-path which contains \( S \). Since any path containing \( S \) must contain \( E(P) \), it follows that \( \pi_k(S) = \pi_k(E(P)) \) for all \( k \). Therefore,

\[
\pi_{|S|+1}(S) \leq \pi_{|E(P)|+1}(E(P)) \leq \text{deg}(u) + \text{deg}(v) \leq n - (|S| + 1)
\]

\( \square \)

Theorem 2.9 An edge \( e = uv \) in a tree of order \( n \) is an end edge, if and only if

\[
\sum_{i=1}^{n-1} \pi_i(e) = (n - 1).
\]
Proof: Let $T$ be a tree of order 1, then $T$ has no edges and $\sum_{i=1}^{n-1} \pi_i(e) = 0 = 1(1-1)$. Assume $n \geq 2$. For any edge $e = uv \in E(T)$,

$$\sum_{i=1}^{n-1} \pi_i(e) = |V(T_u)| \cdot |V(T_v)|$$

where, $T_u$ and $T_v$ are the component of $T-e$ which contain $u$, and $v$ respectively. If $e$ is an end edge, then one of the subtrees $T_u$ or $T_v$ is isomorphic to $K_1$ and

$$\sum_{i=1}^{n-1} \pi_i(e) = 1(n-1).$$

Conversely, let $e$ be an edge in $T$ with $\sum_{i=1}^{n-1} \pi_i(e) = (n-1)$. Thus, $|V(T_u)| \cdot |V(T_v)| = (n-1)$ and $|V(T_u)| + |V(T_v)| = n$. Therefore $e$ is an end edge. □

Definition 2 Let $T$ be a tree of order $n$ and let $S_1$ and $S_2$ be two subsets of $E(T)$ having the same cardinality. The traffic vector $TV(S_1)$ dominates $TV(S_2)$, if $\forall j$, where $j = 1, 2, \ldots, n-1$

$$\sum_{i=1}^{j} \pi_i(S_1) \geq \sum_{i=1}^{j} \pi_i(S_2)$$

$TV(S_1)$ strictly dominates $TV(S_2)$, if $TV(S_1)$ dominates $TV(S_2)$ and there exists $j$ such that

$$\sum_{i=1}^{j} \pi_i(S_1) > \sum_{i=1}^{j} \pi_i(S_2).$$

We will simply denote dominates and strictly dominates by $TV(S_1) \geq TV(S_2)$ and $TV(S_1) > TV(S_2)$, respectively.

An edge $e_0$ in $T$ is called a dominant edge if

$$TV(e_0) > TV(e)$$

for all $e \in E(T)$ An edge set $S$ is called a dominant set, if $\forall j$, $j = 1, 2, \ldots, n-1$, and $I \in E(G)$ with $|S| = |I|$, and

$$\sum_{i=1}^{j} \pi_i(S) \geq \sum_{i=1}^{j} \pi_i(I)$$

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2.2 Analysis of Dominant Edges

**Theorem 2.10** If an end edge in a tree $T$ is a dominant edge, then $T$ is isomorphic to $K_{1,n}$.

**Proof:** Suppose $T$ is a tree which is not isomorphic to $K_{1,n}$, let $e$ be an end edge of $T$. The fact that $T \not\cong K_{1,n}$ implies that there exists an edge $e_0$ in $E(T)$ such that $e_0$ is incident to $e$ and $e_0$ is not an end edge.

Claim: $TV(e_0) > TV(e)$. In order to see this, let $A = \{P_1, P_2, \ldots, P_{\pi_i(e)}\}$ be the set of paths of length $i$ which contain $e$. The set $A$ can be partitioned into two sets $A_1$ and $A_2$. The first set, $A_1$, consists of paths which contain both $e$ and $e_0$ and the second set, $A_2$, consists of paths which contain $e$ but not $e_0$. If $P_i \in A_2$ then $P_i$ can be modified by replacing $e$ by $e_0$ to become a path containing $e_0$. Hence, $\pi_i(e_0) \geq \pi_i(e)$. To show $TV(e_0) > TV(e)$, note that if $e = uv$ and $e_0 = vw$, then

$$\pi_2(e) = \deg(v) - 1$$

and

$$\pi_2(e_0) = \deg(v) + \deg(w) - 2 > \deg(v) - 1.$$

Hence, $\pi_2(e_0) > \pi_2(e)$, and this implies the result. □

**Remark 2** If $P$ is a minimal path containing a set $S$ in a tree $T$, then $TV(S) = TV(E(P))$.

**Remark 3** If $T$ is a tree of order $n$ and $S$ is a set of end edges in $T$, then

$$\sum_{i=1}^{n-1} \pi_i(S) = \begin{cases} 
  n - 1 & \text{for } |S| = 1 \\
  1 & \text{for } |S| = 2 \\
  0 & \text{otherwise.}
\end{cases}$$

Observe that, if $e_0$ is a dominant edge in a tree $T$ then the number of paths containing $e_0$ is maximum, namely

$$\sum_{i=1}^{n-1} \pi_i(e_0) \geq \sum_{i=1}^{n-1} \pi_i(e) \forall e \in E(T).$$
Remark 4 If \( e_1 \) and \( e_2 \) are dominant edges in \( T \), then

\[
\pi_i(e_1) = \pi_i(e_2), \quad \forall i = 1, 2, \ldots, n - 1.
\]

Proof: The fact that \( e_1 \) and \( e_2 \) are dominant edges implies

\[
\sum_{i=1}^{j}(\pi_i(e_1) - \pi_i(e_2)) = 0, \quad \forall j, j = 1, 2, \ldots, n - 1
\]

which implies \( \pi_i(e_1) = \pi_i(e_2) \), \( \forall i \). □

Theorem 2.11 A complete binary tree \( T \) has a dominant edge if and only if the height of \( T \) is at most 2.

Proof: Let \( T \) be a binary tree of order \( n \) (note that \( n \) must be odd), and height \( K, K \geq 3 \). Assume to the contrary that \( e_0 \) is a dominant edge, then

\[
\sum_{i=1}^{n-1} \pi_i(e_0) \geq \sum_{i=1}^{n-1} \pi_i(e), \quad \forall e \in E(T)
\]

Let \( v \) be the root of \( T \) and let \( e_1 \) and \( e_2 \) be the two edges in \( T \) which are incident to \( v \). Necessarily, \( e_0 = e_1 \) or \( e_2 \) see figure 2.4. This follows from the fact that

\[
\sum_{i=1}^{n-1} \pi_i(e_1) = \sum_{i=1}^{n-1} \pi_i(e_2) = \lceil n/2 \rceil \lfloor n/2 \rfloor = \frac{n^2 - 1}{4}
\]

which is an upper bound on the number of paths containing edge of \( T \) when \( n \) is odd. Since \( e_1 \) and \( e_2 \) are symmetric in \( T \), let us assume that \( e_0 = e_1 \). Let \( e_3 \) be an edge incident to \( e_1 \) in \( T \), such that \( e_3 \neq e_2 \). Observe the following:

\[
\sum_{i=1}^{2} \pi_i(e_3) = 4 > \sum_{i=1}^{2} \pi_i(e_1) = 3.
\]

But this contradicts the fact that \( e_1 \) is a dominant edge, therefore \( T \) has no dominant edge. □

A tree \( F_v \) is called a fan, if \( F_v \) has exactly one vertex \( v \) with degree more than two and the vertex \( v \) is called the root.
Theorem 2.12 Given a tree $T$ with edge $e = uv$. Let $T_u$ and $T_v$ be the two components in $T - e$ which contain $u$ and $v$ respectively. If $T_u$ and $T_v$ are two isomorphic fans, then $e$ is a strictly dominant edge.

Proof: Let $T$ be a tree which has the property that $T - e_0$ consists of two identical fans, $F_v$ and $F_u$ and let $e \in E(T) - e_0$. Without loss of generality, we assume $e \in E(F_u)$.

Claim: $e_0$ strictly dominates $e$. Let $A = \{P_1, P_2, ..., P_{\pi_i(e)}\}$ be the set of paths with length $i$ which contain $e$. The set $A$ can be partitioned into two sets, $A_1$ and $A_2$, where $A_1$ consists of paths which contain $e$ and $e_0$ and $A_2$ consists of paths containing $e$ but not $e_0$. By definition, $\pi_i(e) = |A_1| + |A_2|$. Next, we will show that for each path in $A_2$, there exists a path containing $e_0$ but not $e$. If $P \in A_2$, then $P$ contains only edges from $E(F_u)$. By using the second copy $F_v$, one can construct a path $P'$ corresponding to $P$ which contains $e_0$ but not $e$. Therefore, $\pi_i(e_0) \geq \pi_i(e)$. 

Figure 2.4
In order to show that $TV(e_0) > TV(e)$, let $e = xy$ define $T_x$ and $T_y$ to be the two components in $T - e$ containing $x$ and $y$, respectively. Since $e \neq e_0$ either $|V(T_x)| < k$ or $|V(T_y)| < k$, where the order of $F_v$ is $k$. Without loss of generality, let $|V(T_x)| = l < k$.

$$\sum_{i=1}^{n-1} \pi_i(e) = l(n - l) < k^2 = \sum_{i=1}^{n-1} \pi_i(e_0)$$

Therefore $TV(e_0) > TV(e)$. □

The next result shows that there exists a tree $T$ with arbitrary number of dominant edges.

The *Power Star* $K_n^m$ with a tree constructed by identifying every end vertex in $K_{1,n}$ to the center of the star $K_{1,m}$.

Example: The star $K_5^4$ is shown in Figure 2.5.

**Lemma 3** If $n$ and $m$ are two positive integer such that $n \geq 2m + 1$ then there exists a tree $T$ of order $n$, such that $T$ has exactly $m$ strictly dominant edges.
Proof: The proof is by construction. Let \( l = \left\lfloor \frac{n-m-1}{m} \right\rfloor \). Construct a tree \( T \), from the power star \( K^l_m \) by attaching to the center of \( K^l_m \) an extra \( r \) edges, where \( r = (n - m - 1) - m \left\lfloor \frac{n-m-1}{m} \right\rfloor \). It is not difficult to see that the tree \( T \) has exactly \( m \) strictly dominant edges. \( \square \)

Next, exhibit the traffic vector distribution for paths. Let \( P_{n+1} \) be a path of length \( n \). Label the edges in \( P_{n+1} \) according to their location from one of the two end vertices, say \( e_1, e_2, \ldots, e_n \). By simple analysis, it is not difficult to see that, for all \( i = 1, 2, \ldots, \left\lfloor n/2 \right\rfloor \)

\[
TV(e_i) = (1, 2, \ldots, i-1, i, \ldots, i, i-1, \ldots, 2, 1), \forall i \leq \left\lfloor n/2 \right\rfloor
\]

and by the symmetry of the edges on the path

\[
TV(e_i) = TV(e_{n-i}), \forall i \geq \left\lceil n/2 \right\rceil.
\]

Now, if \( n \) is odd then the path \( P_{n+1} \) has an edge \( e_{\left\lceil n/2 \right\rceil} \) with traffic vector

\[
TV(e_{\left\lceil n/2 \right\rceil}) = (1, 2, \ldots, \left\lceil n/2 \right\rceil, \ldots, 2, 1)
\]

which dominates \( TV(e_i), \forall i = 1, 2, \ldots, n \). For even \( n \), the path \( P_{n+1} \) has two dominant edges, namely \( e_{\left\lceil n/2 \right\rceil} \) and \( e_{\left\lfloor n/2 \right\rfloor} \) which have the traffic vector

\[
TV(e_{\left\lceil n/2 \right\rceil}) = TV(e_{\left\lfloor n/2 \right\rfloor}) = (1, 2, \ldots, \left\lfloor n/2 \right\rfloor, \left\lfloor n/2 \right\rfloor, \ldots, 2, 1).
\]

Let \( T = (V, E) \) be a tree and let \( u \) and \( v \in V \). A contraction of \( T \) on \( u, v \) is a tree \( T' \) which is the tree constructed from \( T \) in the following way: If \( P \) represents the path between \( u \) and \( v \) in \( T \), then define \( T_u \) and \( T_v \) to be the two components in \( T - E(P) \) which contain \( u \) and \( v \) respectively. Now construct \( T' \) by joining the vertices \( u \) and \( v \) in the two trees \( T_u \) and \( T_v \) by an edge. If \( uv \) is an edge, then the contraction of \( T \) on \( u \) and \( v \), is \( T \). Figure 2.6 shows a contraction of a tree on two vertices \( u \) and \( v \). The contraction defined above exists and is unique for any two vertices of a tree.
Remark 5 Given a tree $T$, let $P_{k+1}$ be uv-path in $T$, then

$$\pi_{k+j}(E(P_{k+1})) \text{ in } T \text{ is equal to } \pi_{j+1}(uv) \text{ in } T_{uv}$$

where $T_{uv}$ is the tree obtained from $T$, by a contraction on $u$ and $v$.

Remark 6 If $P_{k+1}$ is the minimal path which contains the edge set $S$ in a tree $T$, then $TV(E(P_{k+1})) = TV(S)$.

Theorem 2.13 Let $S_1$ and $S_2$ be two subsets of $E(T)$ with $|S_1| = |S_2|$ and let $P_1$ and $P_2$ be the minimal paths which contain $S_1$ and $S_2$ respectively. If $P_1$ is a subpath of $P_2$, then $TV(S_1) \geq TV(S_2)$.

Proof: Using Remark 6 it is enough to show that $TV(E(P_1)) \geq TV(E(P_2))$. Let $P_1$ be a subpath of $P_2$, then every path of length $i$ which contains $E(P_2)$ must contain $E(P_1)$. Hence, $\pi_i(E(P_1)) \geq \pi_i(E(P_2))$. In fact if $P_1$ is a proper subgraph of $P_2$ then $TV(E(P_1)) > TV(E(P_2))$. To show this, note that $\pi_i(E(P_2)) = 0$ for
all $i$, $0 \leq i < k$ where $k = |E(P_2)|$. Since $P_1$ is a proper subpath of $P_2$ there exists an edge $e$ in $E(P_2) - E(P_1)$. Therefore $|E(P_1)| \leq k - 1$ which implies

$$\pi_{k-1}(E(P_1)) \geq 1 > \pi_{k-1}(E(P_2)) = 0$$

□

**Theorem 2.14** Let $T$ be a tree, with diameter $d$ and let $S \subseteq E(T)$ with $|S| \leq d$. If $\text{Ind}(S)$ is disconnected, then $S$ is not a dominant edge set.

**Proof:** Let $T$ be a tree with $S \subseteq E(T)$. The fact that $T$ has diameter $d$ implies that there exist a path of length $L$ in $T$ for all $L, L \leq d$.

Case 1: There is no minimal path in $T$ which contains $S$. In this case $TV(S)$ is the zero vector, since the traffic vector of a path $P$ of length $|S|$ in $T$ has a nonzero traffic vector, therefore, $TV(P)$ strictly dominates $TV(S)$, and $S$ is not a dominant edge set.

Case 2: There is a minimal $xy$-path $P$ in $T$ which contains $S$. Since $\text{Ind}(S)$ is not connected, there exists an edge $e$ in $E(P)$ such that $e \notin S$ (see Figure 2.7). Let $e_1 = xv$ be the edge in $E(P)$ which is incident to $x$. The set $(S - \{e_1\}) \cup \{e_0\} = S'$ has cardinality equal to $|S|$. If the minimal path which contain $S'$ is $P'$, then $P'$ is a
subpath of $P$. By theorem 2.14, $TV(E(P')) > TV(E(P))$ and $TV(S') > TV(S)$.

Again, this implies that $S$ is not a dominant edge set. □.

**Remark 7** If $S$ is a dominant set, then $\text{Ind}(S)$ contains a path of length $|S|$.

**Caveat lector:** a dominant edge set has different meaning than a set of dominant edges. A set $S_1$ in a graph $G$ is a dominant set if $TV(S_1) \geq TV(I)$, for all $I \subseteq E(G)$, with $|S_1| = |I|$. On the other hand, the set $S_2$ is called a set of dominant edge, if $S_2 \subseteq E(G)$ and for all $e \in S_2$, $e$ is dominant edge.

### 2.3 Characterizing the Set of Dominant Edges

Let $S$ be the set of dominant edges. A natural question to ask is: what is the structure of the induced subgraph on $S$?

**Theorem 2.15** Let $S$ be a subset of the edges of a tree $T$ with the property that $e \in S$ implies that the total number of paths containing $e$, is maximum, then $\text{Ind}(S)$ is a connected subtree.

**Proof:** If $|S| = 1$, then $\text{Ind}(S)$ is connected. Let $|S| > 1$ and assume to the contrary that $\text{Ind}(S)$ is disconnected. There exists two edges $e_1 = xy$ and $e_2 = uv$ such that $e_1$ and $e_2 \in S$ and $\text{Ind}(e_1, e_2)$ is disconnected graph. Without loss of generality let $x$ and $u$ be the two vertices in $\{x, y, u, v\}$ with maximum distance between them. Let $e_0$ be an edge on $xu$-path, incident to $e_1$ such that $e_0 \notin S$. (see Figure 2.8).

Define the following:

- $T_x$ be the component in $T - e_1$ which contain $x$.
- $T_y$ be the component in $T - \{e_1, e_0\}$ which contain $y$.
- $T_c$ be the component in $T - \{e_2, e_0\}$ which contain $u$.
- $T_v$ be the component in $T - e_2$ which contain $v$.
Let $a = |V(T_x)|$, $b = |V(T_y)|$, $c = |V(T_u)|$ and $d = |V(T_v)|$. Since $e_1, e_2 \in S$, 

$$\sum_{i=1}^{n-1} \pi(e_1) = \sum_{i=1}^{n-1} \pi(e_2)$$  \hfill (2.2)

Claim 1: $a = d$.

In order to see this, observe that there are $a \cdot d$ different paths containing both $e_1, e_2$. By using and equation (2.2), we have the following:

$$a(b + c) = d(b + c)$$

This implies $a = d$.

Claim 2: $b \cdot c = 0$.

Assume to the contrary that $b \geq 1$, and $c \geq 1$, then

$$\sum_{i=1}^{n-1} \pi_i(e_0) = (d + c)(a + b)$$

$$\sum_{i=1}^{n-1} \pi_i(e_1) = a(b + c + d).$$
By claim (1), \( a = d \). Therefore, together with the fact that \( e_1 \) is a dominant edge we have:

\[
\sum_{i=1}^{n-1} \pi_i(e_1) - \sum_{i=1}^{n-1} \pi_i(e_0) = (a + c)(a + b) + a(b + c + a) = -bc \leq 0.
\]

Therefore, \( bc = 0 \). Hence, \( e_0 \) does not exist. Therefore, \( \text{Ind}(S) \) is a connected subtree. \( \square \).

**Theorem 2.16** If \( T \) is a tree, and \( S \) is a set of edges in \( T \) with the property that \( e \in S \), then \( e \) is contained in a maximum number of paths, then \( \text{Ind}(S) \cong K_{1,|S|} \).

**Proof:** For \( |S| = 1 \) or \( 2 \), the result follows from Theorem 2.15. Let \( |S| \geq 3 \), by the previous theorem \( < S > \) is connected. Assume to the contrary that \( \text{Ind}(S) \not\cong K_{1,|S|} \). In this case there exists a path \( P \) of length 3 in \( \text{Ind}(S) \), say \( V(P) = \{u_1, u_2, u_3, u_4\} \), and \( E(P) = \{e_1, e_0, e_2\} \). If \( e_0 \) is the edge which is incident to \( e_1 \) and \( e_2 \), then by using exactly the same argument as in Theorem 2.15, we can show that \( \sum_{i=1}^{n-1} \pi_i(e_0) > \sum_{i=1}^{n-1} \pi_i(e_1) \), which contradicts the fact that \( e_0 \) and \( e_1 \) belong to the same total number of paths. Therefore, \( < S > \cong K_{1,|S|} \). \( \square \)

**Corollary 2** If \( T \) is a tree and \( S \) is a set of dominant edges in \( T \), then \( \text{Ind}(S) \cong K_{1,|S|} \).

We will denote the set of dominant edges in a tree \( T \) by \( \text{SDE}(T) \).

**Corollary 3** The only tree \( T \) which has \( \text{SDE}(T) = E(T) \) is the tree \( K_{1,n} \).

**Theorem 2.17** Let \( v \) be a vertex with degree \( m \), in a tree of order \( n \) where \( m \geq 1 \), and let \( N(v) = \{v_1, v_2, \ldots, v_m\} \) be the neighbor set of \( v \). If \( DD_T(v_i) = DD_T(v_j) \) for all \( 1 \leq i, j \leq m \), then for every \( 1 \leq i \leq n \)

\[
\sum_{i=1}^{n-1} \pi_i(vv_i) = \max\{\sum_{i=1}^{n-1} \pi_i(e_i) | e \in T\}
\]

has the property that \( \sum_{i=1}^{n-1} \pi_i(e_i) \) is maximum in \( T \).
Proof:

Let $T_{v_1}, T_{v_2},$ and $T_v$ be the components in $T - \{vv_1, vv_2\}$ which contain $v_1, v_2,$ and $v$, respectively. Let

$$DD_{T_{v_1}}(v_1) = (m_0, m_1, \ldots, m_l)$$

and

$$DD_{T_{v_2}}(v_2) = (n_0, n_1, \ldots, n_k).$$

First: We will show that $DD_{T_{v_1}}(v_1) = DD_{T_{v_2}}(v_2)$. Consider the subtree $T_0$ in $T$ which consists of the subtrees $T_{v_1}$ and $T_{v_2}$, together with the edges $vv_1$ and $vv_2$. (see Figure 2.9).

Claim: $DD_{T_0}(v_1) = DD_{T_0}(v_2)$.

To show the claim, observe that, every vertex is the same distance from $v_1$ as it is from $v_2$. Using this observation together with the fact that $DD_T(v_1) = DD_T(v_2)$, we see that the number of vertices at distance $i$ from $v_1$ in $T_{v_1}$ is the same as the number of vertices at distance $i$ from $v_2$ in $T_{v_2}$, namely $DD_{T_0}(v_1) = DD_{T_0}(v_2)$. 

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Therefore \((m_0, m_1, \ldots, m_l) = (n_0, n_1, \ldots, n_k)\), which implies \(l = k\). Since \(vv_1, vv_2\) are arbitrarily taken from \(N(v)\), the Distance Distribution of \(v_i\) in \(T_v\) is the same for all \(i = 1, 2, \ldots, m\).

Second: Let \(e_0\) be an edge of \(T\) which is not incident to \(v\).

Claim:

\[
\sum_{i=1}^{n-1} \pi_i(vv_1) > \sum_{i=1}^{n-1} \pi_i(e_0).
\]

Let

\[
x = |V(T_{v_1})|
\]

since \(DD_{T_{v_1}}(v_1) = DD_{T_{v_2}}(v_2)\)

\[
|V(T_{v_1})| = |V(T_{v_2})|.
\]

Since \(v_1\) and \(v_2\) are taken arbitrarily from \(N(v)\), it follows that,

\[
|V(T_{v_i})| = |V(T_{v_j})|
\]

for all \(1 \leq i, j \leq m\). Therefore

\[
\sum_{i=1}^{n-1} \pi_i(vv_1) = x[(m - 1)x + 1] = x(n - x) = x(mx - x + 1).
\]

The fact \(e_0 \in E(T) - S\) implies that \(e_0 \in T_{v_i}\) for some \(1 \leq j \leq m\). Let \(e_0 = uw\), let \(T_u\), and \(T_w\) be the two components in \(T - e_0\) which contain \(u, w\) respectively, and \(|V(T_u)| = x_1\) and \(|V(T_w)| = x_2\). Assume \(x_1 \leq x_2\) then \(x_1 \leq x\). Therefore

\[
|V(T_w)| = x_2 > (m - 1)x + 1.
\]

The following is true:

\[
\sum_{i=1}^{n-1} \pi_i(e_0) = x_1x_2 = x_1(n - x_1) = x_1(mx + 1 - x_1).
\]

Suppose \(H(x) = x(n - x)\), then \(H(x)\) is an increasing function of \(x\) in \([0, n/2]\).

Therefore, if \(x_1 < x\), then \(H(x) > H(x_1)\). Thus

\[
\sum_{i=1}^{n-1} \pi_i(vv_1) > \sum_{i=1}^{n-1} \pi_i(e_0).
\]
Hence, $e_i$ has maximum number of paths containing $e_i$ in $T$, for all $i = 1, 2, \ldots, m$.

In the above theorem, the fact that $DD_T(v_i) = DD_T(v_j)$ for all neighbors of a vertex $v$ in $T$ does not imply that the edge $e_i = uv_i$ is the dominant edge in $T$. This can be shown with the example of a complete binary tree.

**Theorem 2.18** Let $T$ be a tree of order $n$ and let $K_{1,m}$ be a subtree in $T$ where $m > 1$. Suppose $E(K_{1,m}) = \{e_i, e_i = vu_i, 1 \leq i \leq m\}$ be a set of dominant edges, then

$$DD_T(u_i) = DD_T(u_j)$$

$\forall i$ and $j; 1 \leq i$ and $j \leq n - 1$.

**Proof:**

Let $T_{u_1}$ be the component in $T - e_1$ which contains $u_1$ and $T_{u_2}$ be the component in $T - e_2$ which contains $u_2$. By an argument similar to the one in the previous theorem, we see that

$$TV_T(e_1) = TV_T(e_2) \implies TV_{T'}(e_1) = TV_{T'}(e_2) \quad (2.3)$$

where $T'$ is the tree consisting of the two subtrees $T_{u_1}$ and $T_{u_2}$ together with the two edges $u_1v$ and $vu_2$. The fact that $e_1$ and $e_2$ are dominant edges, implies that

$$\sum_{i=1}^{n-1} \pi_i(e_1) = \sum_{i=1}^{n-1} \pi_i(e_2).$$

Therefore, $|V(T_{u_1})| = |V(T_{u_2})|$. Let $DD_{T_{u_1}}(u_1) = (n_0, n_1, \ldots, n_k)$ and $DD_{T_{u_2}}(u_2) = (m_0, m_1, \ldots, m_l)$. By using (2.3) we see that $\pi_2(e_1) = \pi_2(e_2)$, and $n_1 + 1 = m_1 + 1$.

We use induction on $j$ to show that $n_j = m_j$ for all $0 \leq j \leq \min(k, l)$. The result is true for $N = 1, 2$. Assume the result is true for all $j; j < N$. In the tree $T'$

$$\pi_{N+1}(e_1) = \sum_{i=0}^{N} n_{N-i}m_i \quad \text{and} \quad \pi_{N+1}(e_2) = \sum_{n=0}^{N} m_{N-n}n_i.$$

Since $\pi_{N+1}(e_1) = \pi_{N+1}(e_2)$ we have

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By the induction hypothesis
\[ \sum_{i=0}^{N-1} n_{N-i}m_i = \sum_{i=0}^{N-1} n_i m_{N-i}. \]
Therefore \( n_N = m_N \).

**Claim:** \( k = l \)

Assume to the contrary that \( k > l \), then
\[ \sum_{k-l} n_{k-l}m_i = \sum_{l-i} n_{l-i}m_i \]
that is \( n_j = 0 \) for all \( K < j < L \) which results in a contradiction. Let \( T_v \) be the component in \( T - \{e_1, e_2\} \) which contains \( v \). By adding the distance degree sequence of \( v \) in the tree \( T_v \), properly, one can see that \( DD_T(u_i) = DD_T(u_j) \forall i, j \).

**Corollary 4** If \( T \) is the tree in the previous theorem \( DD_{T_{ui}}(u_i) = DD_{T_{uj}}(u_j) \) for all \( 1 \leq i, j \leq m \).

**Corollary 5** Let \( S = \{e_1, e_2, ..., e_m\} \) be a set of a dominant edges. If \( e_i = (v, u_i) \)
\( i = 1, 2, ..., m \), then \( EDS(e_i) = EDS(e_j) \), for all \( i, j \), where \( 1 \leq i, j \leq m \).

**Proof:** This follows from Theorem 2.18 together with the definition of an Edge Degree Sequence \( EDS(e) \). □

**Corollary 6** Let \( T \) be a tree of order \( n \), let
\[ S = \{vu_i, i = 1, 2, 3, ..., m\} \]
be a maximal dominant edge set. If \( e_0 = uv \) is an edge in \( T \) then \( \forall i = 1, 2, ..., m \)
\[ |V(T_w)| < |V(T_{ui})| \]
where \( T_w \), and \( T_{ui} \) are the components in \( T - (S \cup \{e_0\}) \) which contain \( w \) and \( u_i \), respectively.
Proof: By Corollary 4, $DD_{Tu_i}(u_i) = DD_{Tu_j}(u_j)$, $\forall i, j : 0 \leq i, j \leq m$. Therefore, $V(T_{u_i}) = |V(T_{u_j})|$ for all $0 \leq i, j \leq m$. If

$$|V(T_w)| = L \text{ and } |V(T_{u_i})| = N$$

then

$$\sum_{i=1}^{n-1} \pi_i(vu_k) = N((m - 1)N + L + 1)$$

for all $1 \leq k \leq m$ and

$$\sum_{i=1}^{n-1} \pi_i(vw) = L(mN + 1).$$

Since $vu_i$ is a dominant edge, then

$$N((m - 1)N + L + 1) > L(mN + 1)$$

$$mN^2 - N^2 + N > mNL - NL + L$$

$$N(mN - N + 1) > L(mN - N + 1).$$

Since $N$ and $m$ are greater than 1, and $mN - N + 1 \neq 0$, it follows that $N > L$.

□

By Theorem 2.14 and Remark 7 we know that a dominant set is always a path, thus, throughout the following discussion we will refer to a dominant set $S$ as a path $P_{|S|+1}$.

2.4 Dominant Edges and the Center of a Tree

The center of a tree is the set of all vertices in $T$ with minimum eccentricity; (see Chartrand and Lesniak [13]). If $\text{Ind}(S(T))$ and $\text{Ind}(C(T))$ denote the induced subtrees of the dominant edge set and the center of $T$, respectively, then we can show that $\text{Ind}(S(T))$ and $\text{Ind}(C(T))$ may be arbitrarily far apart in $T$.

Lemma 4 Let $T$ be a tree constructed by identifying the center of $K_{1,n}$ with an end vertex of the path $P_{m+1}$. If $n \geq m - 3$ then $T$ has a dominant edge.
Proof: Label the vertices in \( E(P_{m+1}) \) according to their distance from the center of the star \( K_{1,n} \), i.e. \( E(P_{m+1}) = \{e_1, e_2, ..., e_m\} \). For any \( e_i, i = 1, 2, 3, ..., m \)

\[
TV(e_i) = (1, 2, ..., i, n + i, ..., n + i - 1, ..., n + 1, n)
\]

\[
TV(e_1) = (1, n + 1, ..., n + 1, n)
\]

where \( TV(e_i) \) and \( TV(e_1) \) are vectors of dimension \( m + 1 \). It is not difficult to see that the following is true, for all \( k, 0 < k < m + 1 \)

\[
\sum_{i=1}^{k} \pi_i(e_1) - \pi_i(e_i) \geq 0
\]

if and only if, \( n > m - 2i + 1 \) (see Figure 2.10). Therefore, \( e_1 \) is a dominant edge in \( T \) whenever \( n > m - 2i + 1 \). \( \square \)

Let \( G_1 \) and \( G_2 \) be two subgraphs of a graph \( G \). Then the distance for \( G_1 \) to \( G_2 \) denoted by \( d(G_1, G_2) \), is defined to be the minimum. \( \{d(u_i, v_i) | u_i \in V(G_1), v_i \in V(G_2)\} \).

**Theorem 2.19** Given a positive integer \( n \in \mathbb{Z}^+ \), there exists a tree \( T \) such that

\[
d(Ind(S(T)), Ind(C(T))) \geq n
\]
Proof Consider the tree constructed in Lemma 4, the center of $T$ can be located with arbitrary distance from the center $v$ of $K_{1,n}$. This can be done simply by choosing $m$ to be large enough, so that the center of $T$ is at distance $n$ from $v$. Now choose $n$ so that $n \geq m - 3$ and Lemma 4 will guarantee that $e_1$ is a dominant edge. Observe that the location of the center at the tree $T$ is independent of $n$ for $n \geq 1$. □

Given a tree $T$, let $v$ be a vertex in $V(T)$. The neighbor vertices of $v$ denoted by $N(v)$ is the set of all vertices adjacent to $v$.

**Theorem 2.20** Let $v$ be a vertex of a tree $T$ of order $n$ and let

$$K_{1,m} = \text{Ind}({v} \cup N(v)).$$

If $m \geq 2$ and $E(K_{1,m})$ is the dominant edge set in $T$, then $v$ is the only center of $T$.

**Proof** Let $T$ be a tree of order $n$ and let $v \in V(T)$, with $N(v) = \{u_1, u_2, ..., u_m\}$. Define $T_{u_i}$ to be the tree in $T - \{v_{u_i}\}$ which contains $u_i$. By Theorem 2.18,

$$DD_{T_{u_i}}(u_i) = DD_{T_j}(u_j)$$

for all $1 \leq i, j \leq m$. Let $e(x)$ be the eccentricity of the vertex $x$. $e(u_i) = e(u_j), \forall i, j, 1 \leq i, j \leq m$. By the symmetric structure of $K_{1,m}$, $e(v) = e(u_1) + 1$. We will show that if $x \in V(T) - \{v\}$, then $e(x) > e(v)$. Since $x \neq v$, it follows that $x \in V(T_{u_i})$ for some $i, 1 \leq i \leq m$. Let $y$ be a vertex in $V(T_{u_i})$, $j \neq i$, such that $d(y, v) = e(v)$. Such a vertex exists, since $m \geq 2$. Now

$$d(x, y) = d(x, v) + d(v, y) = d(x, v) + 1 + e(v) > e(v) + 1.$$

Therefore $e(x) > e(v)$; hence $x$ is not a center vertex. □

2.5 Traffic Sequences

A sequence $(\pi_1, \pi_2, ..., \pi_k)$ is called a traffic sequence, if $\pi_1 = 1$ and $\pi_i$ is a positive integer, for all $i = 1, 2, ..., k$. Such a sequence is denoted by $TV$. Recall
that for a set $S$ of traffic sequence, $S$ is realizable (graphical), if there exists a forest $F$ which has $S$ as its traffic vector distribution.

**Lemma 5** For any traffic sequence $TV$, there exists a tree $T$ such that $T$ has an edge $e_0$ with $TV(e_0) = TV$.

**Proof** Let $TV = (1, \pi_2, \pi_3, \ldots, \pi_k)$. The required tree can be constructed as in Figure 2.11. □

For any set $S$ of traffic vectors, the above lemma suggests a method for constructing a forest $F$ with a subset $A$ in $E(F)$, such that $A$ has the same traffic vector distribution as $S$.

**Remark 8** Given a set of traffic vectors $S = \{TV_1, TV_2, \ldots, TV_n\}$. Where $TV_i = (1, \pi_{i2}, \pi_{i3}, \ldots, \pi_{in})$, $i = 1, 2, \ldots, m$. The following is true:

1. $S$ is not graphic if there exists $TV_i \in S$, such that, $\pi_j > 0$ for some $j \geq m+1$.
2. If $S$ contains $TV_i$, with $\pi_{ij} > (m/2)^2$, then $S$ is not graphic.
Definition 3 Let $e$ be an edge in a directed tree of order $n$. The directed traffic vector is defined to be $TV(e) = (\pi_1(e), \pi_2(e), \ldots, \pi_{n-1}(e))$, where $\pi_i(e)$ is the number of directed path in $T$ which contains $e$.

Remark 9 Let $TV$ be a given traffic sequence. There exists a directed tree $T$ which has an edge $e$, having $TV$ as a traffic vector.

Proof: Let $TV = (\pi_1, \pi_2, \ldots, \pi_{n-1})$. We will construct the directed tree $T$ as shown in Figure 2.12. It is not difficult to see that the edge $e_0$ in $E(T)$ has $TV$ as its traffic vector. □

Theorem 2.21 Given a set of directed traffic sequence $S$, there exists a directed tree $T$, such that $E(T)$ has a subset $E_1$, with $|E_1| = |S|$, and the directed traffic vector distribution of the edges in $E_1$ is the same as $S$.

Proof: We will show that the required set $S$ is graphic by construction. Let $S = \{TV_1, TV_2, \ldots, TV_{|S|}\}$ where $TV_i = (\pi_{i1}, \pi_{i2}, \ldots, \pi_{i(n-1)})$. The graph in Figure 2.13

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illustrates a tree with a subset $S'$ of edges with $|S'| = |S|$ and the set $S' = \{e_1, e_2, \ldots, e_{|S'|}\}$. This can be seen in Figure 2.13. It is not difficult to see that the set of edges $\{e_1, e_2, \ldots, e_{|S'|}\}$ has traffic vector distributions exactly as in $S$. □

Our next result shows that the problem of determining whether a set of traffic sequence is graphic or not is NP-complete. This is accomplished by showing that the known NP-complete partition problem (see [24]) is reducible to a particularization of the decision problem for tree realizability when given a set of traffic sequence.

**Even Partition Problem:**

**Instant:** A finite set $A$ with $|A| = 2k$, and a 'size' $S(a) \in \mathbb{Z}^+$ for each $a \in A$.

**Question:** Is there a subset $A'$ of $A$ such that $|A'| = k$ and $\sum_{a \in A'} S(a) = \sum_{a \in (A - A')} S(a)$?

**The Realization Problem**

**Input:** A set $S = \{\pi_{ij}, \pi_{i2}, \ldots, \pi_{in} | \pi_{ij} \text{ is a positive integer for } i < j < m \text{ and } 1 \leq j \leq n\}$.

**Question:** Does there exist a tree with $S$ as its traffic vector sequence to its edges?

**Theorem 2.22** The Realization Problem is a NP-complete problem.

**Proof:** It is enough to show that the realization for trees of diameter 5 is a NP-complete problem. We reduce the partition problem to the realization problem.

**Instant of Partition Problem:**

A set $A = \{n_i | 1 \leq i \leq 2K\}$ and $n_i$ is a positiv integer, and $S(n_i) = n_i$. Define $2N = \sum_{i=1}^{2K} n_i$ and $n = 2K + 2N + 2$. We define the following set of traffic sequences:

$$C = \{(1, 2K, 2N + K^2, 2NK, N^2) \}
\quad (1, n_i + K, n_iK + K + N - n_i, n_i(K + N - n_i) + N, n_iN), \ 1 \leq i \leq 2k$$
Figure 2.13
\[(1, n_i, K, K + N - n_i, N)^{n_i}, 1 \leq i \leq 2K\]

where \(\sum_{i=1}^{2K} n_i = 2N\) and \(n = 2N + 2K + 2\) and \((TV)^i\) means there are \(i\) occurrence of the traffic vector \(TV\).

We show that the set \(C\) is graphic if and only if the answer to the partition problem is yes. Let \(A\) be the instant of the partition problem with answer no. We show that \(C\) is not graphic.

Assume \(C\) is graphic, and let \(T\) be a tree of order \(n\) and \(C\) is the traffic vectors of its edges. Let \(e_0 = uv\) be the edge in \(T\) with \(TV(e_0) = (1, 2K, 2N + K^2, 2NK, N^2)\) since \(TV(e_0)\) has only five components, there are three possible structures for \(T\).

First structure:
Let \(T_u\) and \(T_v\) be the two components \(T - \{e_0\}\) which contain \(u\) and \(v\) respectively. Let \(A_1\) and \(H_1\) be the set of vertices of distance 1 and 2 from \(u\) respectively and let \(A_2\) and \(H_2\) be the set of vertices with distance 1 and 2 from \(v\) respectively (see Figure 2.14). If \(|A_1| = x\), then \(|A_2| = 2K - x\). Let \(|H_1| = y\), \(|H_2| = z\). By
considering $TV(e_0)$ it follows
\[ y + x(2K - x) + z = 2N + K \]
\[ yz = N^2 \]
\[ y(2K - x) + x(z) = 2NK. \]

Solving the above system implies the following: $x = K$ and $y = z = N$.

**Second structure:**

Define $T_u$ and $T_v$ as in the first structure and let $A_2$, $H_1$ and $H_2$ be the set of vertices in $T_v$ with distance 1, 2 and 3 from $v$. Let $A_1$ be the set of vertices of distance 1 from $u$ in the tree $T_u$ (see Figure 2.15). Let $|A_1| = x$, $|A_2| = 2K - x$ and let $|H_1| = y$, $|H_2| = z$. Since
\[
\sum_{i=1}^{5} \pi_i(e_0) = (N + K + 1)^2 = (n/2)^2.
\]

This implies that $(x + 1)(2K - x + y + z + 1) = (N + K + 1)^2$. Moreover, observe that
\[
\pi_3(e_0) = x(2K - x) + y = 2N + K^2
\]
and $z + xy = 2NK$. It can be shown that the above system of equations implies that $y \leq 0$, which is a contradiction.

**Third structure:**
Define the trees $T_u$ and $T_v$ as above. This structure has $|V(T_u)| = 1$. Let $A_1$, $A_2$, $H_1$ and $H_2$ be the set of vertices with distance 1, 2, 3 and 4 respectively from $v$ (see Figure 2.16).

$$\sum \pi_i(e_0) = (K + N + 1)^2 \neq 1(2K + 2N + 1)$$

for all $K, N \in \mathbb{Z}^+$. Hence, this is impossible.

We consider the first structure: Since $|H_1| = N_1$ and $|A_1| = K$, there are exactly $K$ edges from $H_1$ to $A_1$. Let $E_1 = \{e_1, e_2, \ldots, e_K\}$ be the set of edges from $u$ to $A_1$ and $E_2 = \{e'_1, e'_2, \ldots, e'_N\}$ be the set of edges from $A_1$ to $H_1$. Since edges in $E_1$ are sharing the same vertex $u$, this implies that $\pi_2(e_i) = K + r_i$, where $r_i$ is the number of edges in $E_2$ which are incident to $e_i$. By the structure of the tree $\sum r_i = N$. 

Figure 2.16
Next we claim that the edges in $E_2$ are those with a traffic vector equal to $(1, n, K, K + N - n, N)$. This follows from the fact that

$$1 + n + k + K + N - n + N = 2N + 2N + 1 = 1(n - 1).$$

By the result of Chapter II the edge corresponding to this traffic vector must be an end edge (see Figure 2.17). There are exactly $2N$ end edges in $T$, and exactly

$$\sum_{i=1}^{n-1} n_i = 2N$$

as traffic vectors. Therefore, the end edges in $T$ must have traffic vectors of the form $(1, n, K, K + N - n, N)$. Clearly, if such a tree exists, then the $2K$ traffic vectors which are of the form

$$(1, n, K, N_iK + K + N - n, n_i(K + N - n_i) + N, n_iN)$$

must be assigned to the edges in $E_1$. Since there are $2K$ edges in $E_1$, this assignment is one to one.
Let $e'_n \in E_2$ be an edge connecting a vertex $w_i$ in $H_1$ to a vertex $y_n$ in $A_1$. Let $B_n$ be the set of vertices in $E_2$ which are adjacent to $y_n$, for $e'_n$), \[ \pi_2(e'_n) = n_n \]
where $n_n = n_i$ for some $i$, $1 \leq i \leq 2K$. Therefore, $|B_n| = n_n - 1$; namely, $y_n$ is adjacent to $n_n$ vertices in $H_1$. By a similar argument, we can show that each vertex $y_i$ in $A_1$ is adjacent to exactly $n_i$.

Observe that if $y_n$ is a vertex in $A_1$ which is incident to the edges $e_1, e_2, \ldots, e_{n_i}$ in $H_1$, then
\[ TV(e_s) = TV(e_t), \forall s, t; 1 \leq s, t \leq n_i \]
and $\pi_2(e_s) = n_i \forall 1 \leq s \leq n_i$. This follows from the fact that we have exactly $n_i$ traffic vectors of those. But this implies that the set of traffic vectors $n_i - (1, n_i, k, k + N - n_i, N)$ form a star with a center: say $y_i \in A_1$. There are $k$ sets of those, hence we have $k$ stars, with $k$ distinct centers $\{y_1, y_2, \ldots, y_k\}$ in $A_1$.

By exactly similar analysis, the structure of the edges from $A_2$ to $H_2$ is similar to the one from $A_1$ to $H_1$. It consists of $K$ distinct vertices, each representing a center of a star $K_{1, n_i}$ for some $1 \leq n_i \leq 2K$.

Next, we study the structure of the edges connecting the vertex $u$ to vertices in $A_1$. Let $e_n$ be the edge $uy$. Then we have degree $u = k + 1$, degree $y_n = n_n + 1$. Therefore, $\pi_2(e_n) = n_n + k$ (see Figure 2.18). Thus
\[ TV(e_n) = (1, n_n + K, n_iK + K + N - n_i, n_i(K + N - n_i) + N, n_iN). \]
(2.4)

Next we show that, if there is a partition to $A$, then $C$ is graphic. There are exactly $K$ edges from $u$ to $A_1$, and $K$ edges from $v$ to $A_2$, where each edge has a traffic vector exactly as in 2.4. The fact that $|H_1| = |H_2| = N$ implies that there exist $n_{n_1}, \ldots, n_{n_K}$ such that $\sum_{i=1}^{K} n_i = N$. Therefore the set $\{n_{n_1}, \ldots, n_{n_K}\}$ is partition to the set $A$, which is a contradiction.

Suppose $A = \{n_1, n_2, \ldots, n_{2k}\}$ has a partition. Let
\[ A_1 = \{n_1, n_2, \ldots, n_k\} \]

Figure 2.18

and

\[ A_2 = \{m_1, m_2, \ldots, m_k\} \]

be a partition on the set \( A \) with the following property \( \sum_i n_i = \sum_i m_i = N \). The tree shown in Figure 2.19 has three sets of edges: the set \( E_0 \) which consists of one edge, namely \( uv \) and has

\[ E_0 = \{(1, 2k, 2N + k^2, 2Nk, N^2)\} \]

as its traffic vector. The second set \( E_1 \) consists of edges incident to \( uv \) and has the following traffic vectors:

\[ E_1 = \{(1, n_i + k, n_i k + k + N - n_i, n_i (k + N - n_i) + N, n_i N); i = 1, 2, \ldots, 2k\} \]

The third set is \( E_2 = E(T) - (E_0 \cup E_1) \) and has the following traffic vectors:

\[ E_3 = \{n_i - (1, n_i, k, k + N - n_i, N); i = 1, 2, \ldots, 2k\} \]

The union of the above three sets of traffic vector is equal to \( C \).
Assume $A$ has no partition. We show that such a tree does not exist. Assume to the contrary that such a tree exists, then it should be of the form as shown in Figure 2.20. Moreover, the degree of the vertices in $A_1$ or $A_2$ are of the form $n_i + 1; 1 \leq i \leq 2k$; namely. Each vertex in $A_1$ or $A_2$ is adjacent to $n_i$ vertices in $H_1$ or $H_2$, respectively. By the same analysis in this theorem, $|H_1| = |H_2| = N$. Therefore, there exists a set of elements $n_{a_1}, n_{a_2}, \ldots, n_{a_k}$, such that $\sum_{i=1}^{k} n_{a_i} = N$, which contradicts the fact that $A$ has no partition, and this completes the proof. ∎

Our next result gives an algorithm which allows us to calculate the traffic vector distribution of a tree from the set of degree sequences.

**Theorem 2.23** Let $e = uv$ be an edge in a tree $T$, if $DDS(u)$ and $DDS(v)$ are given, then $TV(e)$ can be calculated.

**Proof** Let $T$ be a tree of order $n$. Let $e = uv$ be an edge with

$$DDS_T(u) = (d_0(u), d_1(u), \ldots, d_{n-1}(u))$$
Let $T_u$ and $T_v$ be the components in $T - e$ which contain the vertices $u$ and $v$, respectively. First, we evaluate $DDS_{T_u}(u)$ and $DDS_{T_v}(v)$ from $DDS_T(u), DDS_T(v)$. If

$$DDT_u(u) = (S_0(u), S_1(u), \ldots, S_{n-1}(u))$$

and

$$DDT_v(v) = (S_0(v), S_1(v), \ldots, S_{n-1}(v))$$

then the following is true:

$$S_K(u) = d_k(v) - S_{k-1}(u) \quad \text{and} \quad S_K(v) = d_k(u) - S_{k-1}(v), \quad \text{where} \quad d_0(v) = 0, d_0(u) = 0.$$

By using the above recursive formula we can now evaluate the traffic vector of $e$. If $TV(e) = (\pi_1(e), \pi_2(e), \ldots, \pi_{n-1}(e))$ then

$$\pi_i(e) = \sum_{k=1}^{n-1} S_k(u) S_{n-k-1}(v), \quad i = 1, 2, \ldots, n - 1.$$
This algorithm can use the known algorithm for finding the distance degree sequence to evaluate the traffic vector.
CHAPTER III

IMPROVING PAIR-CONNECTED RELIABILITY

3.1 Introduction

In this chapter the improvement of pair-connected reliability of trees is discussed. We characterize those edges of a tree whose improvement maximizes the pair-connected reliability.

Let \( G = (V, E) \) be a graph and let \( S \in E \). Define \( PC(S) \) to be the number of connected pairs of vertices in \( < S > \). Recall that the function \( PC(G, q) \) gives the expected number of pairs of vertices that are connected in the random graph (probabilistic graph) \( G \). The function \( PC(G, q) \) seems particularly appropriate for communication networks, in which the goal is to maintain communication between as many pairs of sites as possible. Basic results on algorithms and computational complexity for pair-connected reliability can be found in [3], and [6].

In Chapter I, we defined the pair-connected reliability of a graph \( G = (V, E) \) to be

\[
PC(G, q) = \sum_{S \in E} PC(S)R(S)
\]

(3.1)

where \( R(S) \) is the probability that \( G \) is in the state of \( S \). In general determining the pair-connected reliability for a graph is difficult. In fact, there is no known polynomial time algorithm for finding the pair-connected reliability for a given graph. In order to illustrate the concept of pair-connected reliability, we consider the following example:
Let $G = K_3$ be the complete graph of order 3, and label the vertices in $G$ by $a, b, c$ (see Figure 3.1). The three states corresponding to one edge in a failed condition are: $S_1 = \{ab, bc\}$, $S_2 = \{bc, ca\}$, $S_3 = \{ca, ab\}$. Because $PC(S_1) = PC(S_2) = PC(S_3) = 3$, the three terms in the summation (3.1) are $3p^2q^1$, $3p^2q^1$, and $3p^2q^1$.

In general, the pair-connected reliability polynomial can be written as

$$PC(G, q) = B_1p^mq^{m-1} + B_2p^2q^{m-2} + \ldots + B_{m-1}p^{m-1}q + B_mp^m$$

where each $B_i$ represents the total number of pairs of connected vertices taken over all subgraphs with exactly $i$ edges. For the graph $G$ in Figure 3.1, we have

$$PC(G; q) = 3p^3 + 9p^2q^1 + 3pq^2$$

or recalling that $q = (1 - p)$

$$PC(G; q) = 3p + 3p^2 - 3p^3.$$ 

Recall from Chapter I, that for trees, the computation of $PC(T, q)$ is straightforward. Let $D(G) = (d_1(G), d_2(G), \ldots, d_{n-1}(G))$ be the distance distribution.
of $G$, where $d_i(G)$ denotes the number of pairs of vertices in $G$ with distance $i$ between them. The distance distribution $D(T)$ of a tree $T$ completely determines $PC(T, q)$, namely

$$PC(T, q) = \sum_{i=1}^{n} d_i(T) p^i.$$  

(3.3)

Slater [29] shows that the star $K_{1,n-1}$ is the optimal tree on $n$ vertices, with respect to pair-connected reliability and the path $P_n$ is the least reliable tree on $n$ vertices.

As in Chapter I, we consider a probabilistic graph $G = (V, E)$ in which each edge $e \in E$ fails independently with probability $q$. In this chapter, we will improve the pair-connected reliability of trees by using the edge replacement or improvement and by multiple edge enhancement. For the following discussion, our basic probabilistic graph is tree $T$. If $S = \{e_1, e_2, \ldots, e_k\}$ is the set of edges in $E(T)$ which will be improved, then $S^* = \{(e_1, \Delta_1), (e_2, \Delta_2), \ldots, (e_k, \Delta_k)\}$ denotes the new probability assignment of the edges in $S$, where $(e_i, \Delta_i)$ indicates that the edge $e_i$ has been changed to have new reliability $p + \Delta_i$. Note that the reliability of the edges in $E(T) - S$ remains $p$. Next we find the pair-connected reliability in a tree $T$ when the edges on $T$ have different probability. The pair connected reliability of a graph $G$ may be formulated in terms of two terminal reliability $R_{u,v}(G, q)$ (The probability that $u$ and $v$ are connected in $G$ (see [6])). If $G$ is a graph, then

$$PC(G, q) = \sum_{u,v \in V(G)} R_{u,v}(G, q)$$

(3.4)

where the sum is taken over all undirected pairs of distinct vertices in $V(G)$, and $R_{u,v}(G, q)$ is the probability of having $u$ and $v$ connected in $G$. For a tree $T$ there exists a unique path between every pair of vertices in $V(T)$, therefore if $u$ and $v$ are two vertices in $T$, then the probability that $u$ and $v$ are connected is the same as the probability that no edge in the path from $u$ to $v$ has failed. If $P$ is a uv-path in $T$ with $E(P) = \{e_1, e_2, \ldots, e_i\}$, then

$$R_{u,v}(T, q) = \prod_{j=1}^{i} p_j(e_j)$$
when $p_j(e_j)$ is the probability that the edge $e_j$ is functioning. For the trees we simply denote $R_{u,v}(T,q)$ by $R(I_{i+1})$, where $I_{i+1}$ is the uv-path. For a tree $T$, if $\Gamma$ denotes the probability assignment of $E(T)$, then formula (3.4) becomes

$$PC(T,\Gamma) = \sum_{I_{i+1} \leq T} R(I_{i+1})$$

(3.5)

where the sum is taken over all paths $I_{i+1}$ in $T$, of length $i$ and $R(I_{i+1})$ is the probability of having $I_{i+1}$ connected. If $E(I_{i+1}) = \{e_1, e_2, ..., e_i\}$, and $p(e_j) = p_j$, for $j = 1, 2, ..., i$, then $R(I_{i+1}) = \prod_{j=1}^{i} p_j$. By convention, if $S^*$ (the new probability assignment of the edges in $S$) is known, then $R(T,\Gamma) = R(T, S^*, q)$.

**Lemma 6** Let $e$ be an edge in a tree $T$ of order $n$. If $\{e_1\}^* = \{(e_1, \Delta p)\}$ then

$$PC(T, \{e_1\}^*, q) = \sum_{i=1}^{n-1} [D_i - \pi_i(e_1)]p^i + \pi_i(e_1)p^{i-1}(p + \Delta p)$$

where $D_i$ is the number of the paths of length $i$ in $T$, and $\pi_i(e_1)$ is the number of paths of length $i$ in $T$ which contain the edge $e_1$.

**Proof:** If $A_i$ is the set of paths of length $i$ in $T$, then $A_i$ can be partitioned into two sets, $A_{1i}$ and $A_{2i}$, where $A_{1i}$ consists of paths of length $i$ in $T$ which do not contain $e_1$, and $A_{2i}$ consists of paths of length $i$ in $T$ which contain $e_1$. Necessarily, $|A_i| = |A_{1i}| + |A_{2i}| = D_i$. Let

$$TV(e_1) = (\pi_1(e_1), \pi_2(e_1), ..., \pi_{n-1}(e_1))$$

be the traffic vector of $e_1$, then by definition of $\pi_i(e_1)$, it follows that: $|A_{1i}| = D_i - \pi_i(e_1)$ and $|A_{2i}| = \pi_i(e_1)$. By using the equation in (3.5)

$$R(T, \{e_1\}^*, q) =$$

$$\sum_{i=1}^{n-1} |A_{1i}|p^i + |A_{2i}|p^{i-1}(p + \Delta p) =$$

$$\sum_{i=0}^{n-1} [D_i - \pi_i(e_1)]p^i + \pi_i(e_1)p^{i-1}(p + \Delta p).$$

□ Next we use the concept of traffic vectors, for measuring the effect on pair-connected reliability by edge improvement.

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Theorem 3.24 Suppose $TV(e_i) = (\pi(e_i), \ldots, \pi_{n-1}(e_i))$ and
$TV(e_i) = (\pi_1(e_i), \pi_2(e_i), \ldots, \pi_{n-1}(e_i))$ are traffic vectors of $e_1$ and $e_2$ in tree $T$ respectively. Let $e_1$ and $e_2$ be edges of tree $T$, of order $n+1$. If ${e_1}^* = \{(e_1, \Delta p)\}$, and ${e_2}^* = \{(e_2, \Delta p)\}$ then,

$$PC(T, {e_1}^*, q) - PC(T, {e_2}^*, q) = \Delta p \sum_{i=1}^{n} [\pi_i(e_1) - \pi_i(e_2)] p^{i-1}$$

Proof: Let $D(T) = (D_1, D_2, \ldots, D_n)$ be the distance distribution of $T$. By using Lemma 6 the following is true:

$$PC(T, {e_1}^*, q) = \sum_{i=1}^{n-1} [D_i - \pi_i(e_1)] p^i + \pi_i(e_1) p^{i-1} (p + \Delta p)$$

and

$$PC(T, {e_2}^*, q) = \sum_{i=1}^{n-1} [D_i - \pi_i(e_2)] p^i + \pi_i(e_1) p^{i-1} (p + \Delta p).$$

By taking the difference in the above equations,

$$\Delta PC = PC(T, {e_1}^*, q) - PC(T, {e_2}^*, q) = \sum_{i=0}^{n-1} ([\pi_i(e_2) - \pi_i(e_1)] p^i + [\pi_i(e_2) - \pi_i(e_1)] p^{i-1} (p + \Delta p)) = \sum_{i=0}^{n-1} ([\pi_i(e_2) - \pi_i(e_1)] p^i + [\pi_i(e_1) - \pi_i(e_2)] p^{i-1} (p + \Delta p) \cdot \Delta p = \Delta p \sum_{i=0}^{n-1} [\pi_i(e_1) - \pi_i(e_2)] p^{i-1}.$$ 

□

The above result indicates that the choice between edges is independent of the amount of improvement $\Delta p$ but depends on the traffic vectors of the edges $e_1$ and $e_2$ or the probability $p$.

Corollary 7 Let $e_1$ and $e_2$ be the edges of a tree $T$ of order $(n + 1)$ with

$$TV(e_1) \geq TV(e_2).$$
If \( \{e_1\}^* = \{e_1, \Delta p\} \), and \( \{e_2\}^* = \{e_2, \Delta p\} \), then

\[
PC(T, \{e_1\}^*, q) \geq PC(T, \{e_1\}^*, q)
\]

Proof: Let \( \Delta PC = PC(T, \{e_1\}^*, q) - PC(T, \{e_2\}^*, q) \). By Theorem 3.24

\[
\Delta PC = \Delta p \sum_{i=1}^{n-1} [\pi_i(e_1) - \pi_i(e_2)] p^{i-1}.
\]

Since \( TV(e_1) \geq TV(e_2) \), this implies that

\[
\sum_{i=1}^{j} [\pi_i(e_1) - \pi_i(e_2)] \geq 0
\]

for all \( j = 1, 2, \ldots, n \). Hence for each term in \( \Delta PC \) with negative coefficient, we can associate one or more terms of lower exponents for which the sum of its coefficients equals the absolute value of the negative coefficient. By noting that for \( j > i \) and \( p^j > p^i \), we conclude that \( \Delta PC \geq 0 \), for \( 0 < p < 1 \). □

**Definition 4** An edge \( e_0 \) of a tree \( T \) is called a uniform edge if

\[
PC(T, \{(e_0, \Delta p)\}, q) \geq PC(T, \{(e, \Delta p)\}, q)
\]

for all \( e \in E(T) \) and for all \( 0 \leq \Delta p < 1 - p \).

We now address the relationship between the uniform edge and the dominant edge.

**Corollary 8** If \( e \) is a dominant edge in \( T \), then \( e \) is a uniform edge.

**Proof:** The result follows immediately by using Corollary 7. □

The above result shows that a tree \( T \) with a dominant edge always has a uniform edge.

**Theorem 3.25** The only tree in which an end edge is a uniform edge is the tree \( K_{1,n} \).
Proof: Suppose $e_0 \in E(T)$ is both a uniform and an end edge. Assume that $T \not\cong K_{1,n}$. By Theorem 2.10, there exists an edge $e \in E(T)$ such that $TV(e) > TV(e_0)$. By Theorem 3.24 and Corollary 7, we have the following:

$$PC(T, \{(e, \Delta p)\}) - PC(T, \{(e_0, \Delta p)\}, q) > 0.$$ 

Hence, $e_0$ is not a uniform edge, which is a contradiction. □

If an edge $e_1$ dominates an edge $e_2$, then it follows from Corollary 7, that $PC(T, \{e_1\}^*, q) > PC(T, \{e_2\}^*, q)$. The converse to this statement is not true. This can be seen in the following example. Let $T$ be the tree shown in Figure 3.2.

The tree $T$ has two edges $e_1$ and $e_2$ with the following traffic vectors: $TV(e_1) = (1, 5, 6, 6 + N, h_1, h_2, \ldots)$ and $TV(e_2) = (1, 2, 10, 10, h_1, l_2, \ldots)$ respectively. Since $\sum_{i=1}^{3} \pi_i(e_1) - \pi_i(e_2) = -1$, the edge $e_1$ does not dominate $e_2$. On the other hand, let

$$H(p) = PC(T, \{(e_1, \Delta p)\}^*, q) - PC(T, \{(e_2, \Delta p)\}^*, q).$$

By using Theorem 3.24,

$$H(p) = \Delta p[3p^3 - 4p^2 + (N - 4)p^3 + \sum_{i=5} \pi_i(e_1) - \pi_i(e_2)]p^{i-1}.$$
Figure 3.3

It is not difficult to see that $\sum_{i=5}^{8} [\pi_i(e_1) - \pi_i(e_2)]p^{i-1} \geq 0$ for all $i \geq 5$. Therefore

$$H(p) \geq \Delta p[3 - 4p + (N - 4)p^2].$$

For $p \in (0, 1)$, the sign of $H(p)$ depends upon the sign of the quadratic function $(N - 4)p^2 - 4p + 3$ which has no real root when $N$ satisfies $16 - 12(N - 4) < 0$. Therefore, for $N \geq 6$,

$$(N - 5)p^2 - 4p + 3 > 0.$$ 

For $\Delta p$, $H(p) > 0$ for all $N \geq 6$.

The above example, with the fact that any dominant edge is a uniform edge, suggests the following question: Is there a uniform edge which is not a dominant edge? This question remains open.

In the process of improving network reliability the choice between edges does not depend only on their traffic vectors. The value of the probability $p$ for the edges in $T$ may change the decision. The following example shows an edge $e$ contained in a number of paths which is not a uniform edge.

Example: Consider the tree $T$ (see Figure 3.3). In $T$, $TV(e_1) = (1, 2, 2n +$
$1, 2n, n^2)$ and $TV(e_2) = (1, n + 1, n + 1, 2n, n^2)$.

$$\Delta PC =$$

$$PC(T, \{(e_1, \Delta p)\}^*, q) - PC(T, \{(e_2, \Delta p)\}^*, q) =$$

$$\Delta p \sum_{i=1}^{2n+3} [\pi_i(e_1) - \pi_i(e_2)]p^{i-1} =$$

$$p \cdot \Delta p [np + (1 - n)].$$

For positive values of $\Delta p$ and $p$, we have the following: $np + (1 - n) \geq 0$ if and only if $p \geq \frac{n-1}{n}$. Therefore,

$$\Delta PC = \begin{cases} 
0 & \text{for } p \geq \frac{n-1}{n} \\
< 0 & \text{otherwise.}
\end{cases}$$

Thus, the edge $e_1$ is a better choice for $p \geq \frac{n-1}{n}$, and the edge $e_2$ is a better choice for $p \leq \frac{n-1}{n}$, with respect to improving pair-connected reliability.

**Theorem 3.26** Let $T$ be a tree of order $n+1$. If $e_0$ is a uniform edge in $T$, then

$$\sum_{i=1}^{n} \pi_i(e_0) \geq \sum_{i=1}^{n} \pi_i(e)$$

for all $e \in V(T)$.

**Proof:** Let $TV(e_0) = (\pi_1(e_0), \pi_2(e_0), \ldots, \pi_n(e_0))$ and $TV(e) = (\pi_1(e), \pi_2(e), \ldots, \pi_n(e))$

Thus

$$\Delta PC =$$

$$PC(T, \{(e_0, \Delta p)\}, q) - PC(T, \{(e, \Delta p)\}, q) \Delta PC =$$

$$\sum_{i=1}^{n} [\pi_i(e_0) - \pi_i(e)]p^{i-1} \Delta PC =$$

$$a_1p^0 + a_2p^2 + \ldots + a_np^{n-1} = H(p)$$

where $a_i = \pi_i(e_0) - \pi_i(e)$. Assume to the contrary that $e_0$ is a uniform edge and $\sum_{i=1}^{n} \pi_i(e_0) < \sum_{i=1}^{n} \pi_i(e)$. This implies that $\sum_{i=1}^{n} a_i < 0$, therefore $\lim_{p \to 1} H(p) < 0$. Thus, there exists $p \in (0, 1)$ such that $H(p) < 0$, which contradicts the fact that $H(p) \geq 0$ for all $p$. $\square$
Corollary 9 Let $T$ be a tree of order $n$, and let $S$ be the set of edges in $E(T)$ with the property, $\sum_{i=1}^{n-1} \pi_i(e)$ is maximum. Given $e_0 \in S$ and $e \in E(T)$, there exists $p_0$, $0 < p_0 < 1$ such that $\forall p \in (p_0, 1); 0 \leq \Delta pleq < 1 - p$

\[ PC(T, \{e_0, \Delta p\}, q) > PC(T, \{e, \Delta p\}, q). \]

Proof: Let $e_0$ be an edge in $S$, then

\[ \sum_{i=1}^{n-1} \pi_i(e_0) > \sum_{i=1}^{n-1} \pi_i(e) \quad \forall e \in E(T). \] (3.6)

\[ H(p) = PC(T, \{e_0\}^*, q) - PC(T, \{e\}^*, q) = \Delta p \sum_{i=1}^{n-1} (\pi_i(e_0) - \pi_i(e))p^{i-1} \]

The above expression is a polynomial function of $p$. If $H(p) = a_2 p^1 + a_3 p^2 + \ldots + a_{n-1} p^{n-2}$, then by using (3.6) we have $\sum_{i=2}^{n-1} a_i > 0$. Therefore, $\lim_{p \to 1} H(p) > 0$. Thus there exists $p_0$ with $0 < p_0 < 1$ such that $H(p) > 0$ for all $p > p_0$, which implies the desired result. □

Corollary 10 If $e$ is a uniform edge in a tree $T$, then the number of paths containing $e$ is maximum.

Corollary 11 If $S = \{e_1, e_2, \ldots, e_m\}$ is a set of uniform edges then the $\text{Ind}(S)$ is isomorphic to $K_{1, |S|}$.

Proof: This follows from Theorem 3.26 together with Theorem 2.15 for the result.

3.2 Improving More Than One Edge

Let $T$ be a tree of order $n + 1$. Given $S = \{e_1, e_2, \ldots, e_k\} \subseteq E(T)$ and a positive number $r$, the set $A = \{t_1, t_2, \ldots, t_k\}$ is called $k$-partition to $r$ if $t_i$ is non-negative for all $i = 1, 2, \ldots, k$ and $\sum_{i=1}^{k} t_i = r$. Define the set

\[ S^* = \{(e_1, \Delta p_1), (e_2, \Delta p_2), \ldots, (e_k, \Delta p_k)\} \]
to be a new probability assignment to the edges in \( S \), where \( \Delta p = \sum_{i=1}^{k} \Delta p_i \), and \( \Delta p \leq k(1-p) \). We wish to find a partition of \( \Delta p \), which maximizes the increase in the pair-connected reliability of \( T \). The following result shows the best partition of \( \Delta p \) in the case of \( |S| = 2 \). Throughout the following sections, \( S \) will denote a set of edges in the graph which are to be improved and \( S^* \) will be the new probability assignment for the edges of \( S \).

**Theorem 3.27** Let \( e_1 \) and \( e_2 \) be edges in a tree \( T \) of order \( n + 1 \). Suppose \( S = \{e_1, e_2\} \) and \( S^*(t) = \{(e_1, t\Delta p), (e_2, (1-t)\Delta p)\} \), where \( 0 < t < 1 \), and \( \Delta p \leq 1 - p \) then the function

\[
H(t) = PC(T, S^*(t), q) - PC(T, \{(e_1, \Delta p)\}, q)
\]

has a maximum value when

\[
t = \frac{1}{2} + \frac{\sum[\pi_i(e_1) - \pi_i(e_2)]p^{i-1}}{\sum[\pi_i(S)]p^{i-2}}.
\]

**Proof:** Paths of length \( i \) in \( T \) can be partitioned into three sets:

- **Set 1:** paths containing \( e_1 \) but not \( e_2 \).
- **Set 2:** paths containing \( e_2 \) but not \( e_1 \).
- **Set 3:** paths containing both \( e_1 \) and \( e_2 \). Therefore,

\[
PC(T, S^*(t), q) = \sum_{i=1}^{n} ([D_i - (\pi_i(e_1) + \pi_i(e_2) - \pi_i(S))]p^i + [\pi_i(e_2) - \pi_i(S)])[p + (1-t)\Delta p]p^{i-1} + [\pi_i(e_1) - \pi_i(S)][p + t\Delta p]p^{i-2} + [\pi_i(S)][p + t\Delta p][p + (1-t)\Delta p]p^{i-2})
\]

where, \( D_i \) is the number of ordered pairs of vertices in \( T \) with distance \( i \) between them, \( \pi_i(e_1) \) and \( \pi_i(e_2) \) are the number of paths of length \( i \) which contain \( e_1 \) and \( e_2 \) respectively, and \( \pi_i(S) \) is the number of paths of length \( i \) which contain both edges \( e_1 \) and \( e_2 \). Simplifying the equation in (3.7) gives the following:

\[
PC(T, S^*(t), q) = \sum_{i=1}^{n} [D_i - \pi_i(S)]p^i +
\]

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\[ \left( \pi_i(e_1) \right) \Delta p t + (1 - t) \Delta p \pi_i(e_2) - \Delta p \pi_i(S) \right] p^{i-1} + (\pi_i(S) (\Delta p)^2 \left( t^2 - t^2 \right) p^{i-2} \right]. \\

On the other hand,

\[ PC(T, \{(e_1, \Delta p)\}) = \sum_{i=1}^{n} \left( D_i - \pi_i(e_1) \right] p^i + \pi_i(e_1) p^{i-1} (p + \Delta p). \]

Now the function

\[ H(t) = PC(T, S^*(t), q) - PC(T, \{(e_1, \Delta p)\}, q) = \sum_{i=1}^{n} -\pi_i(S) p^i + \left[ \pi_i(e_1) \Delta p t - \Delta p \pi_i(e_1) + (1 - t) \Delta p \pi_i(e_2) - \Delta p \pi_i(S) \right] p^{i-1} + t(\Delta p)^2 (1 - t) \pi_i(e_1, e_2) p^{i-2}. \]

The function \( H(t) \) has a maximum value in \([0, 1]\) at the critical points. Thus, if \( H'(t) \) is the derivative of \( H(t) \), then

\[ H'(t) = \sum \Delta p \left[ \pi_i(e_1) - \pi_i(e_2) \right] p^{i-1} + \pi_i(S) (\Delta p)^2 \left( 1 - 2t \right) p^{i-2}. \]

Letting \( H'(t) = 0 \), and solving for \( t \) implies

\[ t = t_0 = \frac{1}{2} + \frac{\sum_{i=1}^{n-1} \left[ \pi_i(e_1) - \pi_i(e_2) \right] p^{i-1}}{\sum_{i=2}^{n-1} \pi_i(S) p^{i-2}}. \]

Since \( H''(t) < 0 \) for all \( t \in (0, 1) \), this implies that \( H(t_0) \) has no minimum value in \((0, 1)\) at \( t = t_0. \)

**Corollary 12** In the previous theorem suppose \( TV(e_1) = TV(e_2) \), then the maximum value of \( H(t) \) occurs when \( t = \frac{1}{2} \).

Given \( S = \{e_1, e_2, \ldots, e_k\} \subseteq E(T) \) and

\[ S^*(A) = \{(e_1, t_1 \Delta p), (e_2, t_2 \Delta p), \ldots, (e_k, t_k \Delta p)\} \]

where \( A = \{t_1, t_2, \ldots, t_k\} \) is \( k \)-partition to 1, what is the best \( k \)-partition of 1 which maximizes \( PC(T, S^*(A), q) \) the most? We conjecture the following: If \( A \) contains zero element, then \( PC(T, S^*(A), q) \) is not maximum.
Figure 3.4

The increase in reliability depends not only on the way we distribute $\Delta p$ among the set of edges which are to be improved, but also upon the choice of the edges to be improved. To see this, consider the example:

Let $T$ be the path $P_9$. Label the edges in $E(P_9)$ according to their location from one of the ends, say $E(P_9) = \{e_1, e_2, ..., e_8\}$. Assume we wish to improve two edges in this path, each edge by the amount of $\frac{\Delta p}{2}$ where, $p < \frac{\Delta p}{2} < 1$ (see Figure 3.4). If $S = \{e_1, e_2\}$ then $S^*(A) = \{(e_1, \frac{1}{2} \Delta p), (e_2, \frac{1}{2} \Delta p)\}$ where $A = \{\frac{1}{2}, \frac{1}{2}\}$.

Note that we are taking a fixed partition to 1, namely $\{-5, 5\}$.

Consider the following choices of improving two edges in $E(R_9)$:

1. $e_4$, $e_5$ are the two edges needed to be improved.
   Let $p_0 = p + \Delta p$. $S_1^* = \{(e_4, \Delta p), (e_5, \Delta p)\}$ then
   $$PC(T, S_1^*, q) = (6p + 2p_0) + (4p^2 + 2pp_0 + p_0^2) +
   (2p^3 + 2p^2p_0 + 2pp_0^2) + (2p^3p_0 + 3p^2p_0^2) + (4p^3p_0^2) + 3p^4p_0 + 2p^5p_0^2 + 1p^6p_0^2.$$ 

2. $e_3$ and $e_6$ are the two edges which need to be improved.
   Let $S_2^* = \{(e_3, \Delta p), (e_6, \Delta p)\}$, then
   $$PC(T, S_2^*, q) = (6p + 2p_0) + (3p^2 + 4pp_0) + (6p^2p_0) +
   (4p^3p_0 + p^3p_0^2) + (2p^4p_0 + 2p^3p_0^2 + 3p^4p_0^2 + 2p^5p_0^2 + p^6p_0^2).$$
Consider the difference

$$\Delta PC = PC(T, S^*_1, q) - PC(T, S^*_2, q) =$$

$$(p_0 - p)^2 + 2(p_0 - p)[2pp_0 - 2p^2] + 2p^2p_0(p_0 - p) + 2p^3p_0(p_0 - p).$$

For $p_0 > p$ we get $\Delta PC > 0$. Therefore $e_4$ and $e_5$ is a better choice than $e_6$ and $e_3$.

Next we improve network reliability by adding multiple edges. Let $e$ be an edge in a tree $T$. If $n$ new multiple edges are added to $e$ then the new reliability assignment of $e$ is $\{(e, \Delta_n p)\}$, where $\Delta_n p = (1 - p)(1 - (1 - p)^n)$. Note that $0 \leq \Delta p \leq 1 - p$. If $S^* = \{(e_1, \Delta_n p)(e_2, \Delta_n p), \ldots , (e_k, \Delta_n p)\}$, then we will denote $S^*$ by simply $S^*_{(e_1, e_2, \ldots , e_k)}$.

**Definition 5** An edge set $S = \{e_1, e_2, \ldots , e_m\}$ is called an $m$-uniform set in $T$, if a given $S^* = \{(e_i, \Delta p); i = 1, 2, \ldots , m\}$ then

$$PC(T, S^*, q) \geq PC(T, I^*, q)$$

for all $I = \{e'_1, e'_2, \ldots , e'_m\} \subseteq E(T)$ with $|I| = m$ and $I^* = \{(e'_i, \Delta p); i = 1, 2, \ldots , m\}$.

**Theorem 3.28** Let $e_1$ and $e_2$ be two dominant edges in a tree $T$. If $\{e_i, e_j\}$ is a dominant set, then $\{e_1, e_2\}$ is a $2$-uniform set.

**Proof:** Let $S^* = \{(e_1, \Delta p), (e_2, \Delta p)\}$, and let $e'_1$ and $e'_2$ be two edges in $E(T)$ and $I^* = \{(e'_1, \Delta p), (e'_2, \Delta p)\}$.

Let $x_i$ be the number of paths of length $i$ which contain $e_1$ but not $e_2$, $y_i$ be the number of paths of length $i$ which contain $e_2$ but not $e_1$, and $z_i$ be the number of paths which contain both $e_1$ and $e_2$. Similarly, let $x'_i$ be the number of paths containing $e'_1$ but not $e'_2$, $y'_i$ be the number of paths of length $i$ which contain $e'_2$. 

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but not $e'_1$, and $x'_i$ be the number of paths of length $i$ which contain both $e'_1$ and $e'_2$. If $D_i$ denotes the number of paths of length $i$ in $T$, then the following is true:

$$PC(T, S^*, q) = \sum_{i=1}^{n-1} [D_i - (x_i + y_i + z_i)] p^i + (x_i + y_i) p^{i-1} (p + \Delta p) + z_i p^{i-2} (p + \Delta p)^2$$

and

$$PC(T, I^*, q) = \sum [D_i - (x'_i + y'_i + z'_i)] p^i + (x'_i + y'_i) p^{i-1} (p + \Delta p) + z'_i p^{i-2} (p + \Delta p)^2.$$  

Consider the difference:

$$\Delta PC = PC(T, S^* q) - PC(T, I^*, q) = \sum (x'_i - x_i + y'_i - y_i + z'_i - z_i) p^i + [(x_i - x'_i) + (y_i - y'_i)] p^{i-1} (p + \Delta p) + (z_i - z'_i) p^{i-2} (p + \Delta p)^2.$$  

Observe that

$$x_i + z_i = \pi_i(e_1)$$

$$y_i + z_i = \pi_i(e_2)$$

$$x'_i + z'_i = \pi_i(e'_1)$$

$$y'_i + z'_i = \pi_i(e'_2).$$

Therefore,

$$\Delta PC = \sum [\pi_i(e_1) - \pi_i(e'_1) + \pi_i(e_2) - \pi_i(e'_2)] p^{i-1} \Delta p + (z_i - z'_i) p^{i-2} (\Delta p)^2.$$  

The fact that $e_1$ and $e_2$ are dominant edges and $\{e_1, e_2\}$ is a dominant set implies that $\Delta PC \geq 0$ for all $\Delta p$ and for all $p \in (0,1)$. □

**Theorem 3.29** If $e_1$ and $e_2$ are the only dominant edges in a tree $T$ and the common vertex has degree 2, then $\{e_1, e_2\}$ is a 2-dominant set.
Proof  To the contrary, assume there exist two edges $e'_1$ and $e'_2$ in $T$ such that

$$\sum_{i=1}^{j} \pi_i(e'_1, e'_2) > \sum_{i=1}^{j} \pi_i(e_1, e_2).$$

For some $0 \leq j \leq \text{diam}(T)$ we show that $e_1$ or $e_2$ is not a dominant edge. Without loss of generality, let $\text{Ind}(e'_1, e'_2)$ be the path $u'_1u'_2$. Let $A_1$ be the set of paths of length $L \leq j$ which contain $\{e_1, e_2\}$. Suppose that $T_{u_i}$ and $T_{u_2}$ are the components in $T - \{e_1, e_2\}$, which contain $u_1$ and $u_2$, respectively. Let $T_{u_i}(A_i)$, $i = 1, 2$, be the subtree induced from the two parts of those paths in $A_1$ which are contained in $T_{u_1}$ and $T_{u_2}$, respectively. Let $x_1 = |V(T_{u_1}(A_1))|$, and $y_1 = |V(T_{u_2}(A_1))|$, then

$$\sum_{i=1}^{j} \pi_i(e_1, e_2) = y_1y_2$$

(see Figure 3.5). Similarly, let $A_2$ be the set of all paths containing $e'_1, e'_2$ with length $L \leq j$.

Let $T_{u'_1}$ and $T_{u'_2}$ be the components of $T - \{e'_1, e'_2\}$ which contain $u'_1$ and $u'_2$, respectively and let $x_2 = |V(T_{u'_1}(A_2))|$, and $x_2 = |V(T_{u'_2}(A_2))|$. Then

$$\sum_{i=1}^{j} \pi_i(e'_1, e'_2) = x_1 \cdot x_2.$$

By assumption $x_1 \cdot x_2 > y_1 \cdot y_2$. Without loss of generality, assume $x_1 \leq x_2$, and $y_1 \leq y_2$. The number of paths of length $L$, where $L \leq j$, which contain $e_1$ is $y_1(1 + y_2)$. On the other hand, the number of paths of length $L \leq j$ which contain $e'_2$ is $(x_1 + 1)x_2$. If $x_1x_2 > y_1y_2$, $x_1 \leq x_2$ and $y_1 \leq y_2$, then $x_2 > y_1$ which implies $(x_1 + 1)x_2 > y_1(1 + y_2)$. This contradicts the fact that $e_1$ is a dominant edge. □

Corollary 13 A tree $T$ with exactly two dominant edges $e_1$ and $e_2$ has $\{e_1, e_2\}$ as a 2-uniform set, if the common vertex has degree two.

Proof Using Theorem 3.28 together with Theorem 3.29 will imply the result. □

The improvement of network reliability by changing the probability of more than two edges is not easy to analyze. Our goal is always to come up with the

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Figure 3.5

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Figure 3.6

best choice of a set of edges, and having it remain the best for all values of $p$ with $0 < p < 1$. Unfortunately, for most networks no such choices exist, unless we restrict the value of $p$.

The next result shows that the set of dominant edges contains a $K$-uniform set when the value of $p$ is restricted.

**Theorem 3.30** Let $e_1$ and $e_2$ be two dominant edges in a tree $T$ of order $n$. If $p > \frac{\Delta p}{2}$, then the set $\{e_1, e_2\}$ is a uniform set.

**Proof:** Let $e'_1$ and $e'_2$ be any two edges in $E(T)$. Let $x'_i$ and $y'_i$ represent the number of paths of length $i$, which pass through $e'_1$ but not $e'_2$, and $e'_2$ but not $e'_1$, respectively. Moreover, let $x_i$ and $y_i$ be the number of paths which pass through $e_1$ but not $e_2$ and $e_2$ but not $e_1$, respectively (see Figure 3.6 and 3.7).

Let $\pi_i(e_1, e_2) = x_i$ and $\pi_i(e'_1, e'_2) = z'_i$. By an analysis similar to the one given in the proof of Theorem 3.28, if

$$S^* = \{(e_1, \Delta p), (e_2, \Delta p)\}$$
\[ I^* = \{(e_1', \Delta p), (e_2', \Delta p)\} \]

then

\[ \Delta R = PC(T, S^*, q) - PC(T, I^*, q) \]
\[ = \sum_{i=1}^{n-1} [\pi_i(e_1) - \pi_i(e_1') + \pi_i(e_2) - \pi_i(e_2')]p^{i-1}\Delta p + (z_i - z_i')p^{i-2}(\Delta p)^2 \]
\[ = \sum_{i=1}^{n-1} [\pi_i(e_1) - \pi_i(e_1') + \pi_i(e_2) - \pi_i(e_2')]p + (z_i - z_i')\Delta p]p^{i-2}\Delta p. \]

The fact that \( x_i - x_i' > z_i' - z_i \) and \( y_i - y_i' > z_i' - z_i \) implies the following:

\[ h' = \pi_i(e_1) - \pi_i(e_1') + \pi_i(e_2) - \pi_i(e_2') \geq 2(z_i' - z_i) = 2h \]

Therefore, \( \Delta R \geq 0 \) if and only if \( h' \cdot p - h\Delta p \geq 0 \) or

\[ p > \frac{h}{h'} \Delta p \geq \frac{1}{2} \Delta p \]

\[ \square \]
Theorem 3.31 Let \( e_1, e_2, e_3 \) be three dominant edges in \( T \) of order \( n + 1 \). If \( p > \frac{\Delta p}{2} \), then \( \{e_1, e_2, e_3\} \) is a 3-uniform set in \( T \).

**proof:** Observe that if \( \{e_1, e_2, e_3\} = S \) is a set of three dominant edges, then \( \langle S \rangle \cong K_{1,3} \). Since \( S = \{e_1, e_2, e_3\} \) is a set of dominant edges, then the number of paths which contain exactly one edge in \( S \) is independent of the edge. Moreover the number of paths of length \( i \) which contain exactly two edges in \( S \) is independent of the pair of edges. Let \( x_i \) be the number of paths of length \( i \) which contain exactly one edge in \( S \) and let \( z_i \) be the number of paths of length \( i \) which contain exactly two edges in \( S \). (See Figure 3.8 and 3.9) Now, let \( I = \{e'_1, e'_2, e'_3\} \subseteq E(T) \). For \( j = 1, 2, 3 \), let \( x_i^j \) denote the number of paths of length \( i \) which contain \( e'_j \) but do not intersect \( I - \{e'_j\} \). If \( \pi_i(e'_s, e'_t) = z_i^{st} ; 1 \leq s, t \leq 3 \) then

\[
3z_i - (x_i^1 + x_i^2 + x_i^3) > z_i - z_i^{12} + z_i - z_i^{23} + z_i - z_i^{13}
\]

Let

\[
S^* = \{(e_1, \Delta p), (e_2, \Delta p), (e_3, \Delta p)\}.
\]
\[ I^* = \{(e'_1, \Delta p), (e'_2, \Delta p), (e'_3, \Delta p)\} \]

\[ PC(T, S^*, q) = \sum_{i=1}^{n} \{[D_i - (3x_i + 3z_i)]p^i + 3x_ip^{i-1}(p + \Delta p) + 3z_ip^{i-2}(p + \Delta p)\} \]

and

\[ PC(T, I^*, q) = \sum_{i=1}^{n} \{[D_i - (x_i + x_i^2 + x_i^3 + z_i + z_i^2 + z_i^3)]p^i + (z_i^2 + x_i^3 + z_i^{23} + z_i^{13})p^{i-2}(p + \Delta p)\} \]

If \( \Delta PC = PC(T, S^*, q) - PC(T, I^*, q) \), then

\[ \Delta PC = \]

\[ \sum_{i=1}^{n} \Delta pp^{i-1} \{[3x_i + 6z_i - x_i^1 - x_i^2 - x_i^3 - 2z_i^{12} - 2z_i^{23} - 2z_i^{13}] + \]

\[ (\Delta p)^2 p^{i-2}[3z_i - z_i^{12} - z_i^{23} - z_i^{13}]\} = \]

\[ \sum_{i=1}^{n} \{(\Delta p)^2 p^{i-2}[(3x_i - x_i^1 - x_i^2 - x_i^3 + 6z_i - 2z_i^{12} - 2z_i^{23} - 2z_i^{13})p - \]

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\[ \Delta p(z_i^{12} + z_i^{23} + z_i^{13} - 3z_i) \]

Observe that, for all \( i = 1, 2, \ldots, n \)
\[
\begin{align*}
    x_i + 2z_i &= \pi_i(e_i); \\
    x_i^1 + z_i^{12} + z_i^{13} &= \pi_i(e'_1); \\
    x_i^2 + z_i^{23} + z_i^{12} &= \pi_i(e'_2); \quad \text{and} \\
    x_i^3 + z_i^{13} + z_i^{23} &= \pi_i(e'_3).
\end{align*}
\]

Equation (3.10) can be written as
\[
\Delta PC = \sum \Delta pp_i^{-2}\left[\pi_i(e_1) - \pi_i(e'_1) + \pi_i(e_2) - \pi_i(e'_2) + \pi_i(e_3) - \pi_i(e'_3) \right]p - \Delta p(z_i^{12} + z_i^{23} + z_i^{13} + 3z_i).
\]

Using the fact that \( e_1, e_2, e_3 \) are dominant edges in \( T \), we have
\[
\begin{align*}
h' &= 3\pi_i(e_1) - \pi_i(e'_1) - \pi_i(e'_2) - \pi_i(e'_3) \\
    &> 2[z_i^{12} + z_i^{23} + z_i^{13} - 3z_i] \\
    &= 2h.
\end{align*}
\]

Therefore, \( \Delta PC \geq 0 \) if and only if \( ph' - h\Delta p \geq 0 \), or
\[
p \geq \frac{h}{h'} \Delta p \geq \frac{\Delta p}{2}
\]
\[ \square \]

The above argument can be extended to obtain the following result.

**Theorem 3.32** Let \( S = \{e_1, e_2, \ldots, e_k\} \) be a set of dominant edges in a tree \( T \) of order \( n \), such that \( k < n \). If \( p > \frac{\Delta p}{2} \), then \( S \) is a \( k \)-uniform set.

For some \( k, k \leq |S| \), the following example shows that a set of dominant edges \( S \) does not always contain a \( k \)-uniform set. Consider the tree in Figure 3.10. The set
\[
S = \{e_1, e_2, e_3, e_4, e_5\}
\]
is the maximal set of dominant edges. We show that there are no 2-uniform sets in $S$. Without loss of generality, assume that $\{e_1, e_2\}$ be a 2-uniform set. $TV(e_1) = TV(e_2) = (1,5,8,4)$, and $TV(\{e_1, e_2\}) = (0,1,2,1)$. Let $e'_1$ be an end edge incident to $e_1$, then $TV(e'_1) = (1,1,4,4)$, and $TV(\{e'_1, e_1\}) = (0,1,4,4)$. Let $S^* = \{(e_1, \Delta p), (e_2, \Delta p)\}$ $I^* = \{(e_1, \Delta p), (e'_1, \Delta p)\}$.

Then

$$\Delta PC = PC(T, S^*, q) - PC(T, I^*, q) =$$

$$\sum \Delta p \cdot p^{i-2}[(\pi_i(e_1) + \pi_i(e_2) - \pi_i(e_1) - \pi_i(e'_1))p +$$

$$\Delta p(\pi_i(e_1, e_2) - \pi_i(e_1, e'_1)) =$$

$$\Delta p((4p + 0\Delta p) + p\Delta p((8 - 4)p - 2\Delta p)p\Delta p[1p - 3\Delta p] =$$

$$4p\Delta p + 4p^2\Delta p - 2p(\Delta p)^2 - 3p(\Delta p)^2 + p^2\Delta p =$$
\[2p\Delta p[2 + 2p - \Delta p - 1.5\Delta p + p\Delta p] =
\]
\[2\Delta pp[2 + 2p - 2.5\Delta p + p\Delta p].\]

Letting \(p \to 0\) and \(\Delta p \to 1\) implies there exist \(p\) and \(\Delta p\) for which \(\Delta PC\) is negative.

Observe that for \(p \geq \frac{1}{3}\) and \(0 \leq \Delta p \leq 1 - p\) we have \(p \geq \frac{\Delta p}{2}\). In most network applications, the reliability of the layout of the network is at least \(\frac{1}{3}\), therefore one can apply the above result in improving pair-connected reliability. In general, if \(p > \frac{1}{3}\) then \(p > \frac{\Delta p}{2}\) for all \(0 \leq \Delta p \leq 1 - p\).

### 3.3 k-Reliable Trees

Suppose that we wish to improve the pair-connected reliability of one of two trees of network \(T_1\) or \(T_2\) which have the same order. Let \(k\) be the number of edges whose probability we decide to improve. One can ask whether we should choose the network \(T_1\) or \(T_2\). A tree \(T_0\) of order \(n\) is called a \(k\)-reliable tree, if for any other tree with the same order, there exist \(S = \{e_1, e_2, \ldots, e_k\} \subseteq E(T_0)\) and

\[PC(T_0, \{(e_i, \Delta p)|i = 1, 2, \ldots, k\}, q) \geq PC(T, S^*, q)\]

where \(S^* = \{(e'_i, \Delta p)|i = 1, 2, \ldots, k; e'_i \in S\}\). In this section we investigate the question of finding the \(k\)-reliable tree of given order.

**Theorem 3.33** There exists a 1-reliable tree \(T_0\) of order \(n\) for any \(n \geq 1\).

**Proof:** The proof is by construction. Let \(T_0\) be the tree obtained by identifying the centers of the stars \(K_{\lceil \frac{n-2}{2} \rceil}\) and \(K_{\lfloor \frac{n-2}{2} \rfloor}\) with the end vertices of \(K_2\) (see Figure 3.11). To show that \(T_0\) is a 1-uniform tree, let \(e_0\) be the edge in \(T_0\) which is not an end edge. Note that

\[TV(e_0) = (1, \lceil \frac{n-2}{2} \rceil, \lfloor \frac{n-2}{2} \rfloor).\]
Let $T$ be a tree of order $n$, let $e$ be an edge in $T$. Claim: $TV(e_0) \geq TV(e)$.

Observe that $\pi_2(e_0) = n - 2$ which is an upper bound, and

$$
\sum_{i=1}^{n-1} \pi_i(e_0) = ([\frac{n}{2}]) ([\frac{n}{2}])
$$

which is an upper bound. Therefore $TV(e_0) \geq TV(e)$. The fact that

$$
PC(T_0, \{(e_0, \Delta p)\}, q) - PC(T, \{(e, \Delta p)\}) = \Delta p \sum_{i=1}^{n-1} [\pi_i(e_0) - \pi_i(e)] p^{i-1} \geq 0
$$

implies the result. □
CHAPTER IV

GLOBAL RELIABILITY

4.1 Tree Networks

In this chapter a study of network enhancement is introduced using two methods:

(1) multiple edge replacement, and  
(2) reliability improvement of edges.

The first method consists of adding additional edges to the network, with the restriction that edges are added only between vertices which are already joined by an edge. The latter method consists of replacing edges of the network by more reliable edges. Throughout this chapter, we will only consider global reliability measure.

Recall from Chapter I that the global connectivity of $G$, under the assumption of independent edge failure is given by:

$$R(G,q) = \sum_{S \in \Omega} f(S) \cdot R(S)$$

(4.1)

where $q$ is the probability of failure of any edge in $G$, $\Omega$ is the power set of $E$, and $f(S)$ is defined as

$$f(S) = \begin{cases} 1 & \text{if } <S> \text{ is a connected spanning subgraph of } G \\ 0 & \text{otherwise.} \end{cases}$$

If $G$ is a $\lambda$-edge connected graph, we need at least $\lambda$ edges to disconnect $G$. Therefore $f(S) = 1$ for all $|S| \geq m - \lambda + 1$. 

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For a global reliability, Formula 4.1 can be rewritten as follows

\[ R(G, q) = \sum_{S \in \Omega} m_i R(S) \]  

(4.2)

where \( m_i \) is the number of induced connected subgraphs of size \( i \) in \( G \). Observe that for \((n,m)\)-graph \( G \), \( m_i = 0 \) for all \( i < n - 1 \).

In general, given a graph \( G \) of order \( n \), if \( \Gamma = \{p_1, p_2, \ldots, p_m\} \) is the probability assignment of the edges in \( E(G) \), then Formula 4.2 can be rewritten as

\[ R(G, \Gamma) = \sum_{S \in \Omega} f(S) \cdot R(S). \]

where \( f(S) \) is defined as above and \( R(S) \) is the probability that \( G \) is in state \( S \).

If \( S = \{e_1, e_2, \ldots, e_k\} \), then

\[ R(S) = \prod_{i=1}^{k} p(e_i) \cdot [1 - p(e_i)] \]

where \( p(e_i) \) is the reliability of the edge \( e_i \) (the probability of having \( G \) in state \( \{e_i\} \)).

If the \( p_i \)'s in \( \Gamma \), are equal to \( p \), then \( R(G, \Gamma) = R(G, q) \), where \( q = 1 - p \). The function \( R(G, \Gamma) \) gives the probability that for every pair \( v_1 \) and \( v_2 \) of nodes in \( G \), there is a path from \( v_1 \) to \( v_2 \); equivalently, that is the probability that the graph \( G \) contains at least one spanning tree. If \( G \) is a directed graph and \( s \) is a source node, then in 4.1

\[ f(S) = \begin{cases} 
1 & \text{if } < S > \text{ contains a directed path from } s \text{ to every other vertex in } G \\
0 & \text{otherwise}.
\end{cases} \]

This chapter is devoted to studying the analysis of improving global reliability in tree networks, using the stated methods.

If \( T \) is a tree of order \( n \), then \( T_n[k] \) will denote the class of all graphs obtained from \( T \) by adding \( k \) multiple edges. The graphs in \( T_n[k] \) may have different reliabilities, depending on how the \( k \) extra edges are added. In this section, we investigate the problem of finding an optimally reliable graph with respect to global reliability in \( T_n[k] \).
As in Chapter III, if the only edges in a graph $G$ which are improved by multiple edges or by edge replacement are $S$, then the global reliability of $G$ is denoted by $R(G, S^*, q)$ where $S^*$ is the probability assignment to $S$. If there is no ambiguity, then we simply write $R(G, S^*)$.

Example: Consider the following graphs, $G_1$ and $G_2$ from the class $T_4[3]$ which are shown in Figure 4.1. Let $\{e_1, e_2, e_3\}$ be any labeling of the set $E(T_4)$ and $A = \{e_1, e_2\}$ with $A^* = \{(e_1, 2p - p), (e_2, 2p - p), (e_3, 2p - p)\}$, then

$$R(G_1, A^*) = (2p - p)^3.$$ 

Let $B = \{e_2, e_3\}$ with $B^* = \{(e_2, 2p - p^2), (e_3, 3p - 3p^2 + p^3)\}$ then

$$R(G_2, B^*) = p \cdot (2p - p)(3p - 3p^2 + p^3).$$

We can easily show that $R(G_1, A) > R(G_2, B)$, for all $0 < p < 1$. Therefore, the graph $G_1$ is more reliable than the graph $G_2$. 

Figure 4.1
In general, if $\Gamma = \{p_1, p_2, \ldots, p_{n-1}\}$ is the probability assignment of the edges in a tree $T$ then

$$R(T, \Gamma) = \prod_{i=1}^{n-1} p_i.$$ 

**Definition 6** If $T$ is a tree of order $n$ and the edges in $E(T)$ are labeled $\{e_1, e_2, \ldots, e_{n-1}\}$, then $T_n(t_1, t_2, \ldots, t_{n-1})$, will denote the graph obtained from $T$ by enhancing the edge $e_i$ by $t_i$ extra edges.

Example: In graphs $G_1$ and $G_2$ of the previous example the new graphs are represented by $T_4(1,1,1)$ and $T_4(0,1,2)$ respectively.

Recall from Chapter II that the probability of having two vertices $u$ and $v$ connected, if there are $k$ edges between them, is $1 - (1 - p)^k$. The increase in reliability between $u$ and $v$ is

$$1 - (1 - p)^k - p = (1 - p) - (1 - p)^k = \Delta p.$$ 

Let $A = \{t_1, t_2, \ldots, t_{n-1}\}$ be a partition of a positive integer $k$ and let $F$ be the set of all one-to-one functions from $E(T)$ to $A$. For $f_1$ and $f_2$ in $F$ define

$$G_1 = T_n(f_1(e_1), f_1(e_2), \ldots, f_1(e_{n-1}))$$

$$G_2 = T_n(f_2(e_1), f_2(e_2), \ldots, f_2(e_{n-1})).$$

$G_1$ and $G_2$ have the same global reliability. To see this, consider

$$R(G_1, q) = \prod_{i=1}^{n-1} [1 - (1 - p)^{f_1(e_i)+1}]$$

and

$$R(G_2, q) = \prod_{i=1}^{n-1} [1 - (1 - p)^{f_2(e_i)+1}].$$

The terms in the first product are exactly the same as in the second, except the order may be different, therefore $R(G_1, q) = R(G_2, q)$.

The graphs in Figure 4.2 and Figure 4.3, represent $G_1 = T_4(2,0,0)$, and $G_2 = T_4(0,2,0)$ respectively. The two graphs are not isomorphic, even though $G_1$ and $G_2$ have the same global reliability.

The following combinatorial result will be useful for the next theorem.
Figure 4.2

Figure 4.3

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Lemma 7 Let \( A \) and \( B \) be two sets with \( |A| = k - s \), and \( |B| = k + s \), where \( 0 \leq s \leq k \). Let \( H(s, r) \) be the number of ways of choosing \( r \) element from \( A \cup B \), such that at most \( |A| - 1 \) and \( |B| - 1 \) elements are taken from \( A \) and \( B \) respectively. For fixed \( r \), \( 0 \leq r \leq 2k - 2 \), the function \( H(s, r) \) has a maximum value when \( s = 0 \).

Proof: The number of ways of choosing \( r \) elements from \( A \cup B \) without restriction is

\[
\sum_{i=0}^{r} C_i^{k+s} \cdot C_{r-i}^{k-s} = C_r^{2k}.
\]

If the function \( H(s, r) \) denotes the number of ways of choosing \( r \) elements from \( A \cup B \) when at least one must be left, then \( H(s, r) \) can be described as

\[
H(0, r) = \begin{cases} 
C_r^{2k} & 0 \leq r < k \\
C_r^{2k} - 2C_{r-k}^k & k \leq r \leq 2k - 2
\end{cases}
\]

and for \( s \geq 1 \)

\[
H(s, r) = \begin{cases} 
C_r^{2k} & 0 \leq r < k - s \\
C_r^{2k} - C_{r-k+s}^{k+s} & k - s \leq r \leq k + s \\
C_r^{2k} - [C_{r-k+s}^{k+s} + C_{r-k-s}^{k-s}] & k + s \leq r \leq 2k - 2
\end{cases}
\]

Claim: \( H(0, r) \geq H(s, r) \) for all \( r \).

Case 1: \( 0 \leq r \leq k - s \) then \( H(0, r) = H(s, r) \).

Case 2: \( k - s \leq r < k \)

\[ H(s, r) = C_r^{2k} - C_{r-k+s}^{k+s} \text{, and } H(0, r) = C_r^{2k} \]

hence \( H(0, r) \geq H(s, r) \) for all \( s = 0, 1, \ldots, k \)

Case 3: \( k \leq r < 2k - 2 \)

\[ H(0, r) = C_r^{2k} - 2C_{r-k}^k, \text{ } H(s, r) = C_r^{2k} - C_{r-k+s}^{k+s} \]

It is enough to show that \( C_{r-k+s}^{k+s} \geq 2C_{r-k}^k \). To see this, observe the following:

\[
C_{r-k+s}^{k+s} = \frac{(k + s)!}{(r - k + s)!(2k - r)!}
\]
\[
2C^k_{r-k} = \frac{2k!}{(2k-r)!(r-k)!}
\]

The fact that
\[
(k+s)!(k+s-1), \ldots, (k+1)!(k-1)! > \\
2(r-k+s)(r-k+s-1), \ldots, (r-k+1)(r-k)!k!
\]

implies
\[
(k+s)!(r-k)! > (2k)!(r-k+s)!
\]

Thus
\[
\frac{(k+s)!}{(r-k+s)!} > \frac{(2k)!}{(r-k)!}.
\]

Therefore \(C^k_{r-k+s} \geq 2C^k_{r-k}\) and \(H(0,r) \geq H(s,r)\) for \(k \leq r < k+s\)

Case 4: \(k \leq r \leq 2k-2\).

The fact that
\[
C^k_{r-k+s} + C^k_{r-k-s} \geq C^k_{r-k+s} \geq 2C^k_{r-k}
\]

implies \(H(0,r) \geq H(s,r)\) \(\Box\)

Recall from Chapter III that, if \(S = \{e_1, e_2, \ldots, e_k\} \subseteq E(G)\) is the the set of edges which receive a new probability assignment, namely \(S^* = \{(e_1, \Delta_1), (e_2, \Delta_1), \ldots, (e_k, \Delta_1)\}\), then \(R(G, \Gamma) = G(G, S^*, q)\).

**Lemma 8** Let \(G\) be the graph constructed from the vertex set \(\{x, y, z\}\) by joining \(k+s\) edges of the form \(xy\) and \(k-s\) edges of the form \(yz\) (see Figure 4.4). If \(m_{2k-r}\) is the number of induced connected subgraphs of size \(2k-r\) in \(G\) then \(m_{2k-r} = H(s,r)\), where \(H(s,r)\) is the function defined in Lemma 7.

**Proof:** The fact that any induced connected subgraph of size \(2k-r\) has to use at least one edge of the form \(xy\), and another edge of the form \(yz\) together with Lemma 7 implies the result. \(\Box\)

**Remark 10** Given two \((n,m)\)-graphs \(G_1\) and \(G_2\), if \(m_{N-r}(G_1) > m_{N-r}(G_2)\) for all \(r = 0, 1, 2, \ldots, N\) where \(m_{N-r}(G_1)\) and \(m_{N-r}(G_2)\) are the numbers of the induced connected subgraphs of \(G_1\) and \(G_2\) respectively, then \(R(G_1, q) > R(G_2, q)\).
where,
\[ \Gamma_1\{ (e_1, p(e_1)), (e_2, p(e_2)), \ldots, (e_{|E(G_1)|}, p(e_1)) \} \]
and
\[ \Gamma_2\{ (e_1', p(e_1')), (e_2', p(e_2')), \ldots, (e_{|E(G_2)|}', p(e_1')) \} \]
are the probability assignments of \( E(G_1) \) and \( E(G_2) \) respectively.

Proof: Let \( m_{N-r} \) be the number of connected subgraphs with size \( N - r \), where \( N = n + 2k \).

In \( G_1 \), without loss of generality, let \( e_1 = xy \) and \( e_2 = yz \) be the two edges which receive \( k \) extra edges each. Let \( T_x, T_y \) and \( T_z \) be the components in \( T - \{e_1, e_2\} \) which contain \( x, y \) and \( z \) respectively. If \( r \) is number of the edges which are in the failed state in each of \( G_1 \) and \( G_2 \), then

\[ m_{N-r}(G_1) = H(0, r) \quad \text{and} \quad m_{N-r}(G_2) = H(s, r) \]

where \( H(x, y) \) is the function described in Lemma 7. The fact that \( m_{N-r}(G_1) \geq \)
Figure 4.5

$m_{N-r}(G_2)$ for all $r$, follows from the same lemma; therefore:

$$\sum_{r=0}^{N} m_{N-r}(G_1)p^{N-r} > \sum_{r=0}^{N} m_{N-r}(G_2)p^{N-r}.$$ 

Thus, $R(G_1, \Gamma_1) \geq R(G_2, \Gamma_2)$. □

**Definition 7** The graph $T_{n+1}(t+1, t+1, \ldots, t+1, t, \ldots, t)$ will be denoted by $T_{n+1}\{(t+1, r)\}$, where $r$ is the number of edges in $T_{n+1}$ which receive $t+1$ extra edges.

**Theorem 4.35** Among all enhanced trees in $T_{n+1}[k]$, where $k = tn + r$, $0 \leq r < n$, the most reliable graph is $G_0 = T_{n+1}\{(t+1, r)\}$.

**Proof:** Let $T_{n+1}(t_1, t_2, \ldots, t_n) \in T_{n+1}[k]$ and $T_{n+1}\{(t+1, r)\} \in T_{n+1}[k]$. Observe that

$$A = \{t_1, t_2, \ldots, t_n\}$$

$$B = \{(t+1, t+1, \ldots, t+1, t, \ldots, t)\}$$

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Proof: Let $T_{n+1}(t_1, t_2, \ldots, t_n) \in T_{n+1}[k]$ and $T_{n+1}\{(t+1,r)\} \in T_{n+1}[k]$. Observe that

$$A = \{t_1, t_2, \ldots, t_n\}$$

$$B = \{t + 1, t + 1, \ldots, t + 1, t\}$$

are two partitions to $k$. If $A \neq B$, then there exist $t_i$ and $t_j$ in $A$ such that $|t_i - t_j| \geq 2$. Without loss of generality, let $t_i > t_j$. Assume $e_i = xy$ and $e_j = yz$ are two incident edges in $G_1$ (see Figure 4.5). Let $G_2$ be the graph obtained from $G_1$ by moving one $xy$ edge to $yz$ edge.

Claim: $R(G_1, q) > R(G_2, q)$.

Let $m_{N-r}(G_1)$ and $m_{N-r}(G_2)$ be the count of the induced connected subgraphs with size $N - r$ in $G_1$ and $G_2$ respectively, where $r$ is the number of failed edges in both $G_1$ and $G_2$. The number $m_{N-r}(G_1)$ can be written as $x_1(r) + x_2(r)$ where $x_1(r)$ is the number of the induced connected subgraphs when the failed edges are of the form $xy$ or $yz$ and $x_2(r)$ is the number of the induced connected subgraph when the failed edges are not of the form $xy$ or $yz$. Similarly $m_{N-r}(G_2)$ can be written as $y_1(r) + y_2(r)$ where $y_1(r)$ is the number of failed edges of the form $xy$ or $yz$, and $y_2(r)$ is the number of edges when the failure edges are not of the form $xy$ or $yz$. Necessarily $x_1(r) = y_2(r)$ and by using Theorem 4.35, we have $y_2(r) > x_2(r)$ for all $r = 0, 1, \ldots, 2k$. Therefore, the graph $G_1$ is not an optimally reliable graph in $T_{n+1}[k]$, with respect to the global reliability. □

The above result shows that a tree $T$ with $k$ extra edges is more reliable when the edges are distributed evenly.

Given two trees $T_1$ and $T_2$, if $|V(T_1)| < |V(T_2)|$ then $R(T_1, q) > R(T_2, q)$. This follows from the fact that $p^x > p^y$ if $x < y$, for all $0 < p < 1$.

We will denote the change in global reliability after multiple edge replacement or improvement in the optimal way by $\Delta R(G, S^*)$, where $S^*$ is the probability assignment of the set $S$ which gives the optimal reliability.
Theorem 4.36 Let $T_1$ and $T_2$ be trees of order $n_1$ and $n_2$, respectively and let $m$ be the number of extra edges. For $S \subseteq E(T_1)$ and $I \subseteq E(T_2)$ with $|S| = |I|$, the following is true: If $n_2 > n_1$, then $\Delta R(T_1, S^*) \geq \Delta R(T_2, I^*)$.

Proof: Let $m = nk + r$

\[
\Delta R(T_1, S^*) = p_{n+1} \cdot p_n^{k-r} \cdot p^{n_1-k} - p^{n_1}
\]

\[
\Delta R(T_2, I^*) = p_{n+1} \cdot p_n^{k-r} \cdot p^{n_2-k} - p^{n_2}
\]

where $p_i = 1 - (1 - p)^{i+1}$

\[
\Delta R(T_1, S^*) - \Delta R(T_2, I^*) = p_{n+1} \cdot p_n^{k-r} \cdot [p^{n_1-k} - p^{n_2-k}] - [p^{n_1} - p^{n_2}]
\]

Since $p_{n+1} \cdot p_n^{k-r} > p_n^k$ implies $\Delta R(T_1, S^*) > \Delta R(T_2, I^*)$. □

Enhancing tree networks, can be done by replacing the edges of $T$ by more reliable edges. Let $S = \{e_1, e_2, \ldots, e_k\}$ be the set of edges to be improved in $T$, and

\[S^* = \{(e_1, \Delta p_1), (e_2, \Delta p_2), \ldots, (e_k, \Delta p_k)\}\]

Let $\Delta p = \sum_{i=1}^k \Delta p_i$. The next result shows that the best distribution of $\Delta p$ is the one when $\Delta p_i = \frac{\Delta p}{k}$, for all $i = 1, 2, \ldots, k$. As in Chapter III, let $T_n[\Delta p]$ denote the class of trees of order $n$ and a total increase in edge reliability of $\Delta p$.

Theorem 4.37 Let $\Delta p = \sum_{i=1}^k \Delta p_i$ where $k \leq (n - 1)$. The graph

$T_n(\Delta p_1, \Delta p_2, \ldots, \Delta p_k, 0, \ldots, 0)$ has maximum global reliability when $\Delta p_i = \frac{\Delta p}{k}$, for all $i = 1, 2, \ldots, k$.

Proof: Let $T_n(\Delta p_1, \Delta p_2, \ldots, \Delta p_k, 0, \ldots, 0)$ be a graph in $T_n[\Delta p]$. Without loss of generality, let $\Delta p_1 > \Delta p_2$, we will show that $G_1$ can be modified to a more reliable graph $G_2$, without changing the value $\Delta p$. Let $\Delta p_1 + \Delta p_2 = x$ and consider the graph

\[G_t = T_n(tx, (1-t)x, \Delta p_3, \ldots, \Delta p_k, 0, \ldots, 0)\].
Let

\[ H(t) = R(G_t) = p^{n-k} \cdot (p + \Delta p_3)(p + \Delta p_4) \ldots (p + \Delta p_k) \cdot (p + tx)(p + (1-t)x). \]

The function \( H(t) \) has maximum value at \( t = t_0 \) if and if only if the function \( h(t) = (p + tx)(p + (1-t)x) \) has maximum value at \( t = t_0 \). But latter one is just a quadratic function of \( t \) which has a maximum value at \( t = \frac{1}{2} \). Therefore the function \( H(t) \) has a maximum value at \( t = \frac{1}{2} \). Letting \( G_2 = G_{1/2} = T_n(x/2, x/2, \Delta p_3, \ldots, \Delta p_k, 0, \ldots, 0) \) implies that \( G_2 \) is more reliable than \( G_1 \). This implies that any partition of \( \Delta p \) into \( k \) distinct equal numbers will result in a graph which is not the most reliable. Hence \( \Delta p \) must be partitioned into \( k \) equal numbers, namely

\[ \left\{ \frac{\Delta p_k}{k}, \frac{\Delta p_k}{k}, \ldots, \frac{\Delta p_k}{k} \right\}. \]

\( \square \)

In the above theorem, if the set \( S \) of edges which are to be improved by a total of \( \Delta p \) is \( E(T) \), then it is always better to improve all edges in \( E(T) \), and the distribution of \( \Delta p \) among \( E(T) \) must be done evenly.

### 4.2 Ring Networks

The ring network has been one of the most commonly used network topologies in the design and implementation of local area networks. This is due to its simplicity and expandability. The ring network can be modeled by the cycle \( C_n \) of \( n \) vertices. In this section we analyze the enhancing of ring networks by the two methods mentioned in Section 4.1.

Given a positive integer \( k \) and a cycle \( C_n \), the class of graphs obtained from \( C_n \) by enhancing the edges in \( E(C_n) \) by \( k \) multiple edges, is denoted by \( C_n[k] \). In this chapter we show that \( G \in C_n[k] \) is an optimally reliable graph with respect to the global reliability, if \( k \) extra edges are distributed evenly among \( E(C_n) \).
Definition 8 Let $A = \{G_1, G_2, \ldots, G_n\}$ be a set of graphs. A graph $G_0$ in $A$ is called optimally reliable in $A$, if $G_0$ is the most reliable graph in $A$ with respect to the global reliability.

Throughout this discussion, the term optimally reliable graph refers to the global reliability.

Lemma 9 Assume $C_n$ is a cycle of order $n$. Then $R(C_n, q) = p^n + np^{n-1}q$, where $p = 1 - q$ and $q$ is the probability of failure of the edges in $C_n$.

Proof: By definition of global reliability $R(C_n, q) = \sum_{S \in \Omega} f(S) \cdot p(S)$. Since any two edges in $E(C_n)$ will disconnect $C_n$ therefore, $f(S) = 0$ for all $|S| < n - 1$. This leaves only two cases to consider. If $|S| = n$ then there is only one induced connected subgraph. On the other hand if $|S| = n - 1$ then there are $n$ different ways to chose $S$ so that the induced subgraphs of $S$ is connected. Therefore $R(C_n, q) = 1p^n + (n)p^{n-1}q$. □

Remark 11 If $\Gamma = \{p_1, p_2, \ldots, p_n\}$ is the probability assignment of $G(C_n)$, then

$$R(C_n, \Gamma) = \prod_{i=1}^{n} p_i + \sum_{i=1}^{n} (1 - p_i) \prod_{j \neq i}^{n} p_j.$$ 

In $C_n[k]$, if the edges in $C_n$ are labeled $e_1, e_2, \ldots, e_n$ then the graph $C_n(t_1, t_2, \ldots, t_n)$, denotes the cycle $C_n$ with extra $t_i$ edges are added to the edge $e_i$, $i = 1, 2, \ldots, n$.

Theorem 4.38 If $\gamma$ is any permutation on the set $\{t_1, t_2, \ldots, t_n\}$ then the two graphs

$$C_n(t_1, t_2, \ldots, t_n)$$

$$C_n(t_{\gamma(1)}, t_{\gamma(2)}, \ldots, t_{\gamma(n)})$$

have the same reliability.
Proof: Let $G_1 = C_n(t_1, t_2, ..., t_n)$ and $G_2 = C_n(t_{\gamma(1)}, t_{\gamma(2)}, ..., t_{\gamma(n)})$. By definition if $uv = e_i \in E(G_1)$, then there are $t_i + 1$ edges of the form $uv$. This implies that $p(e_i) = 1 - (1 - p)^{t_i}$, we simply denote $p(e_i)$ by $p_i$. By remark 11, the global reliability of $G_1$ is

$$R(G_1, \Gamma_1) = \prod_{i=1}^{n} p_i + \sum_{i=1}^{n} (1 - p_i) \prod_{j \neq i} p_j$$

where $\Gamma_1$ is the probability assignment of $E(G_1)$. Moreover,

$$R(G_2, \Gamma_2) = \prod_{i=1}^{n} p'_i + \sum_{i=1}^{n} (1 - p'_i) \prod_{j \neq i} p'_j$$

where $p'_i = 1 - (1 - p)^{\gamma(t_i)+1}$ and $p'_j = 1 - (1 - p)^{\gamma(t_j)+1}$ for all $1 \leq i, j \leq n$. $\Gamma_2$ is the probability assignment of $E(G_2)$. Since $\gamma$ is a permutation on the set $\{t_1, t_2, ..., t_n\}$, this implies that for every edge $e_i \in E(G_1)$, there exist an edge $e'_i \in E(G_2)$, such that $p(e_i) = p(e'_i)$ or $p_i = p'_i$. Therefore

$$\prod_{i=1}^{n} p_i = \prod_{i=1}^{n} p'_i$$

and

$$\sum_{i=1}^{n} (1 - p_i) \prod_{j \neq i} p_j = \sum_{i=1}^{n} (1 - p'_i) \prod_{j \neq i} p'_j.$$ 

Thus, $R(G_1, \Gamma_1) = R(G_2, \Gamma_2)$. □

The formula 4.2 can be modified to

$$R(G, \Gamma) = \sum_{r=0}^{N} m_{N-r}(G) \cdot p^{N-r}(1 - q)^r$$

where $m_{N-r}(G)$ is the number of induced connected subgraphs in $G$ after the failure of $r$ edges in $G$.

Theorem 4.39 $R(C_n(k, k, 0, ..., 0)) \geq R(C_n(k + s, k - s, 0, ..., 0))$, for all $s = 0, 1, ..., k$.

Proof: Suppose $G_1 = (C_n(k, k, 0, ..., 0))$ and $G_2 = (C_n(k + s, k - s, 0, ..., 0))$ and let $m_{N-r}(G_1)$ and $m_{N-r}(G_2)$ be the number of connected induced subgraphs
in $G_1$ and $G_2$ with size $N - r$ respectively, $N = n + 2k$ and $r$ is the number of failed edges.

By simple combinatorial analysis we have the following:

$$m_{N-r}(G_1) = (n - 2)C_{r-1}^{2k+2} + C_r^{2k+2}$$

and

$$m_{N-r}(G_2) = (n - 2)\sum_{i=0}^{k-s+1} [C_{r-i-1}^{k-s+1} \cdot C_r^{k+s+1}] + C_r^{2k+2}$$

$$= (n - 2)\sum_{i=0}^{k-s+1} [C_{r-i-1}^{k-s+1} - C_{r-1}^{k+s+1} \cdot C_{r+k+s+1}^{k+s+1}] + C_r^{2k+2}$$

$$= (n - 2)[C_r^{2k+2} - C_{r+k+s+1}^{k+s+1}] + C_r^{2k+2}.$$ 

Since $m_{N-r}(G_1) \geq m_{N-r}(G_2)$ for all $r$, therefore $R(G_1) \geq R(G_2)$. □

**Theorem 4.40** If $G = C_n(t_1, t_2, \ldots, t_n) \in C_n[k]$ is an optimally reliable graph, then $|t_i - t_j| \leq 1$.

**Proof:** Let $G \in C_n[k]$, with $G = C_n(t_1, t_2, \ldots, t_n)$. Assume that there exist $t_i$ and $t_j$ such that $t_i - t_j > 1$.

Claim: $G_1$ is not optimal reliable graph in $C_n[k]$.

In order to show this, let $u_i u_{i+1}$ be the edge $e_i$ and $u_{i+1} u_{i+2}$ be the edge $e_j$ (see Figure 4.6). Let $G'$ be a graph in $C_n[k]$ constructed from $G_1$ by taking an edge of the form $u_i u_{i+1}$ and placing it in parallel between the vertices $u_{i+1}$ and $u_{i+2}$. Suppose that $A_1$ and $A_2$ are the sets of edges of the form $u_i u_{i+1}$ or $u_{i+1} u_{i+2}$ in $G_1$ and $G_2$ respectively. Let $f_j(i)$ represents the count of the induced connected graphs in $G_j - A_j$ with $|E(G_j - A_j)| - i$ edges, $j = 1, 2$ and let $g_j(r - i)$ represents the count of the induced subgraphs with $|A_j| - (r - i)$ edges in $G_i - E(G_j - A_j), j = 1, 2$. Define $m_{N-r}(G_j)$ for $j = 1, 2$ to be the count of a connected subgraphs in $G_j$ with size $N - r$, where $N = n + k$. One can see the following:

$$m_{N-r}(G_1) = \sum_{i=0}^{r} f_1(i) \cdot g_1(r - i)$$

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Observe that
\[ G_1 - A_1 \cong G_2 - A_2. \]
Therefore \( f_1(i) = f_2(i) \) for all \( i = 1, 2, \ldots, r \). Thus, in order to prove that \( m_{N-r}(G_1) \geq m_{N-r}(G_2) \) it is enough to show that \( \sum_{i=0}^{r} g_1(r-i) \geq \sum_{i=0}^{r} g_2(r-i) \), for all \( i \). If \( i \geq 1 \) the result follows from Lemma 8. Therefore \( m_{N-r}(G_1) \geq m_{N-r}(G_2) \) for all \( r \). □

We will denote the optimal ring in \( C_n[k] \) by \( C_n\{(r,t)\} \) which means that there are \( r \) edges in \( C_n \) which receive an extra \( t+1 \) edges and \( n-r \) edges in \( C_n \) which receive \( t \) extra edges. Let \( \Gamma \) be the probability assignment of \( R(C_n\{(r,t)\}) \). Since the notation \( C_n\{(r,t)\} \) describes \( \Gamma \) uniquely, one can replace \( R(G, \Gamma) \) (the global reliability of \( G \)) by \( R(G) \). By convention, if \( r > n \) then \( R(C_n\{(r,t)\}) = 0 \).
Remark 12 For the ring $C_n\{(r,t)\}$, the global reliability is

$$R(C_n(r,t)) = p_{t+1}p_t^{n-r} + C_0^rp_{t+1}p_t^{n-r-1}(1 - p_t) + C_1^rp_{t+1}^{-1}(1 - p_{t+1})p_t^{n-r}.$$ 

where $p_{t+1} = 1 - (1 - p)^{t+2}$ and $p_t = 1 - (1 - p)^{t+1}$

Lemma 10 If $C_n$ and $C_m$ are two cycles and $n < m$, then $R(C_n, q) > R(C_m, q)$.

Proof: Observe that

$$R(C_n, q) = p^n + np^{n-1}(1 - p) \text{ and}$$

$$R(C_m, q) = p^m + mp^{m-1}(1 - p).$$

Letting $m - n = k > 0$ then

$$R(C_n, q) - R(C_m, q) = p^n[1 - p^k] + (1 - p)[n - mp^k]p^{n-1}p$$

$$= (1 - p)p^{n-1}[p + p^2 + p^3 + \ldots + p^k + n - mp^k]$$

$$= [(1 - p)p^{n-1}[p + p^2 + p^3 + \ldots + p^{k-1} + n - (m - 1)p^k].$$

but

$$n + p + p^2 + \ldots + p^{k-1} >$$

$$(k - 1)p^{k-1} + np^{k-1}$$

$$= (k + n - 1)p^{k-1}$$

$$= (m - 1)p^{k-1}$$

$$> (m - 1)p^k.$$ 

Therefore $R(C_n, q) - R(C_m, q) > 0$. □

Let $C_n[\Delta p]$ be the class of all cycles which have a total of $\Delta p$ increase in the reliability of the edges of $C_n$. The following result show that $G \in C_n[\Delta p]$ is an optimally reliable graph if and only if $G = C_n[\Delta p/n, \ldots, \Delta p/n].$

Theorem 4.41 The graph $G = C_n[\Delta p/n, \ldots, \Delta p/n]$ is the optimally reliable graph in $C_n[\Delta p]$.
Proof: Let $G = C_n(\Delta p_1, \Delta p_2, \ldots, \Delta p_n)$ be a graph in $C_n[\Delta p]$. Without loss of generality let $\Delta p_1 > \Delta p_2$.

Claim: $G$ is not an optimally reliable graph.

Let $x = \Delta p_1 + \Delta p_2$ and let $G_t = C_n(tx, (1-t)x, \Delta p_3, \ldots, \Delta p_n)$. First we find the optimal reliable graph in $\{G_t, t \in [0, x]\}$.

$$R(G_t) = H(t) = [p + tx][p + (1 - t)x] \prod_{i=3}^{n}[p + \Delta p_i] +$$

$$[p + tx][1 - p - (1 - t)x] \prod_{i=3}^{n}[p + \Delta p_i] +$$

$$[p + (1 - t)x][1 - p - tx] \prod_{i=3}^{n}[p + \Delta p_i] +$$

$$[p + tx][p + (1 - t)x] \sum_{i=0}^{n}[1 - p - \Delta p_i] \prod_{j \neq i}^{n}[p + \Delta p_j];$$

$$H(t) =$$

$$[x^2(1 - 2t)] \prod_{i=3}^{n}[p + \Delta p_i] +$$

$$[2tx^2 - x^2 + x] \cdot \prod_{i=3}^{n}[p + \Delta p_i] +$$

$$[2tx^2 - x^2 - x] \prod_{i=3}^{n}[p + \Delta p_i] +$$

$$[x^2(1 - 2t)] \sum_{i=1}^{n} \{[1 - p - \Delta p_i] \prod_{j \neq i, j \geq 3}^{n}[p + \Delta p_j]\}.$$

Differentiating with respect to $t$, we get:

$$H'(t) = [x^2(1 - 2t)] \prod_{i=3}^{n}[p + \Delta p_i] +$$

$$\sum_{i}^{n} \{[1 - p - \Delta p_i] \prod_{j \neq i, j \leq 3}^{n}[p + \Delta p_j]\} -$$

$$2x^2[1 - 2t] \prod_{i=3}^{n}[p + \Delta p_i].$$

Solving for $t$ and using the second derivative test, one can show that the function $H(t)$ has a mimium value at $t = \frac{1}{2}$. Therefore, the graph $G_{\frac{1}{2}}$ is the optimally reliable graph in $\{G_t\}$. Observe that $G \in \{G_t\}$, hence $R(G_{\frac{1}{2}}) > R(G)$ and $G$ is not a optimally reliable graph in $C_n[\Delta p]$. □
4.3 Unicyclic Networks

Definition 9 A graph $G$ is called a unicyclic graph if $G$ contains exactly one cycle.

Here we analyze the improvement of unicyclic networks by adding multiple edges.

Given a unicyclic graph $G$, the edges in $E(G)$ can be partitioned into two sets: set $A$, the edges located on the cycle and set $B$, the edges in $E(G) - A$. The induced subgraph $\text{Ind}(A)$ from $A$ is the cycle $C$, and the induced subgraph $\text{Ind}(B)$ is a forest $F$. We denote the family of unicyclic graphs with $|A| = n$ and $|B| = m$ by $\{C_n \cdot F_m\}$ with a cycle $C_n$ and forest $F_m$.

Theorem 4.42 If $G \in \{C_n \cdot F_m\}$ then

$$R(G, q) = p^{n+m} + np^{n+m-1}(1 - p).$$

Proof: The graph $G$ has order $N = n + m$. If $m_{N-r}$ is the number of induced connected subgraph $< S >$ of size $N$, then

$$R(G, q) = \sum_{r=0}^{N} m_{N-r}p^{N-r}(1 - q)^r.$$ 

If $|S| \leq n + m - 2$, then $f(S) = 0$. Therefore, the only cases which need to be considered are $|S| > n + m - 2$. If $|S| = n + m$, then $m_N = 1$ and $R(S) = p^{n+m-1}$. If $|S| = n + m - 1$, then $m_{N-r} = n$ and $R(S) = p^{n+m-1}(1 - p)$. Substituting these values in the formula of $R(G, q)$ implies the result. □

Theorem 4.43 Among all an $(N,N)$-graphs the cycle $C_N$ is the most reliable graph.

Proof: Let $G = C_n \cdot f_m$ be an $(N,N)$-graph, where $C_n$ is the cycle in $G$ and $F_m$ is the induced subgraph $\text{Ind}(E(G) - E(\hat{C}_n))$

$$R(G, q) = p^N + np^{N-1}(1 - p)$$

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and

$$R(C_N, q) = p^N + Np^{N-1}(1 - p)$$

$$R(C_N, q) - R(G, q) = p^{N-1}(1 - p)[N - n]$$

$$\geq 0.$$

for all $N \geq n$. Therefore $C_N$ is the most reliable $(N,N)$-graph. □

Next we improve the unicyclic graph by the method of adding multiple edges. Given a unicyclic graph $G = C_n \cdot F_m$. We denote the family of graphs $C_n \cdot F_m$ which have $k$ extra edges by $C_n \cdot F_m[k]$.

Example: Let $G = C_4 \cdot F_4$. Figure 4.7 and Figure 4.8 show two graphs $G_1$ and $G_2$ respectively from the family $C_4.F_4[4]$. The global reliability of $G_1$ and $G_2$ are

$$R(G_1, q) = p_1^4p^4 + 4p^4p_1^3(1 - p)$$

and

$$R(G_2, q) = p_1^4p^4 + 4p_1^4p_1^3(1 - p),$$

where $p_1 = 2p - p^2$. Since $p_1 > p$, it is not hard to see that $G_2$ is more reliable than $G_1$. 

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Lemma 11 Let $G$ be a graph constructed by identifying one vertex from a graph $G_1$ with a vertex of another graph $G_2$. Then $R(G, q) = R(G_1, q) \cdot R(G_2, q)$.

Proof: Let the graph $G$ be in state $S = S_1 \cup S_2$ where $S_1 = S \cap E(G_1)$ and $S_2 = S \cap E(G_2)$. $< S >$ is connected if and only if $< S_1 >, < S_2 >$ are connected.

$$R(G, q) = \sum_{S \in \Omega} f_G(S) \cdot p(S)$$
$$R(G_i, q) = \sum_{S \in \Omega_i} f_{G_i}(S) \cdot p(S) \text{ for } i = 1, 2$$

where,

$$f_G(S) = \begin{cases} 1 & \text{if } < S > \text{ is connected in } G, \\ 0 & \text{otherwise.} \end{cases}$$

Note that, for $i = 1, 2$ we have:

$$f_{G_i}(S) = \begin{cases} 1 & \text{if } < S > \text{ is connected in } G_i, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $f_G(S) = f_{G_1}(S \cap E(G_1)) \cdot f_{G_2}(S \cap E(G_2))$. Moreover $p(S) = p(S \cap E(G_1)) \cdot p(S \cap E(G_2))$. Thus it follows that $R(G, q) = R(G_1, q) \cdot R(G_2, q)$. □
Definition 10  Given a set of graphs $A = \{G_1, G_2, \ldots, G_k\}$. The graph $G$ is said to be a series connection from $A$, if $G$ is constructed from $A$ in the following way:

1. No two graphs have more than one point in common,
2. $G$ is a connected graph and has a number of cycles equal to the number of cycles in $\cup_{i=1}^{k} G_i$.

Figure 4.9 shows a series connection graph constructed from the set $A = \{K_3, C_4, K_2\}$.

Corollary 14  If $G$ is in series connection from $A = \{G_1, G_2, \ldots, G_k\}$, then

$$R(G, q) = \prod_{i=1}^{k} R(G_i, q).$$

Lemma 11 and Corollary 14 can be extended to any probabilistic graph $G$.

Given a unicyclic graph $G = C_n \cdot F_m$ let $k$ be the number of extra edges. If we label the edges of the cycle $C_n$ by $\{e_1, e_2, \ldots, e_n\}$ and the edges of the forest by $F_m$ by $\{e'_1, e'_2, \ldots, e'_n\}$, then the graph

$$G_0 = C_n(t_1, t_2, \ldots, t_n) \cdot F_m(s_1, s_2, \ldots, s_m)$$

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denotes the graph with \(t_i\) extra edges on the edge \(e_i\) in the circle, and \(s_i\) extra edges on the edge \(e'_i\) of the forest. If \(C_n\) or \(F_m\) has no extra edges, then \(C_n(0,0,\ldots,0)\) and \(F_m(0,0,\ldots,0)\) will simply be replaced by \(C_n\) and \(F_m\), respectively.

**Lemma 12** If \(G_1 = C_1(1,0,\ldots,0) \cdot F_m\) and \(G_2 = C_1 \cdot F_m(1,0,\ldots,0)\), then \(R(G_1) < R(G_2)\).

**Proof:** It follows from Corollary 14 that

\[
R(G_1) = p^n[p^n + p^{n-1}(1 - p_1) + (n - 1)p^{n-2} \cdot p_1(1 - p)]
\]

\[
R(G_2) = p^{m-1} \cdot p_1[p^n + np^{n-1}(1 - p)]
\]

where \(p_1 = 2p - p^2\)

\[
R(G_2) - R(G_1) = p^{m-1}p^{n-1}[n + np_1 + p_1 + pp_1 - np - p - pp]
\]

\[
= p^{m-1}p^{n-1}[n(1 + pp_1 - p_1 - p) + (p_1 - p) + (pp_1 - pp)]
\]

\[
= p^{m-1}p^{n-1}[n(1 - p)(1 - p_1) + (p_1 - p) + p(p_1 - p)]
\]

\[
> 0
\]

for all \(n \geq 3, m \geq 1, 0 < p < 1\). \(\square\)

**Corollary 15** Given a graph \(G = C_n[k_1] \cdot F_m[k_2]\), if \(\Gamma\) is the probability assignment of \(G\), then

\[
R(G, \Gamma) \leq R(C_n\{(t_1, r_1)\}) \cdot F_m\{(t_2, r_2)\}
\]

where \(k_1 = nt_1 + r_1\) and \(k_2 = mt_2 + r_2\).

**Proof:** In the graph \(G\), let \(\Gamma_1\) and \(\Gamma_2\) be the probability assignment of the edges in \(C_n\) and \(F_m\), respectively. We have \(\Gamma = \Gamma_1 \cup \Gamma_2\), therefore

\[
R(G, \Gamma) = R(C_n, \Gamma_1) \cdot R(F_m, \Gamma_2)
\]

\[
R(C_n, \Gamma_1) \leq R(C_n\{(t_1, r_1)\})
\]

and \(R(F_m, \Gamma_2) \leq R(F_m\{(t_2, r_2)\})\).
Therefore,

\[ R(G, \Gamma) \leq R(C_n \{ (t_1, r_1) \} \cdot F_m \{ (t_2, r_2) \}. \]

□

**Lemma 13** Let \( G = C_n(t_1, t_2, \ldots, t_n) \cdot F_m(s_1, s_2, \ldots, s_m) \) and let \( P_{m+1} \) be the path of size \( m \). If \( \Gamma_1 = \{ t_1, t_2, \ldots, t_n \} \) and \( \Gamma_2 = \{ s_1, s_2, \ldots, s_m \} \) are the probability assignments of the edges in \( C_n \) and \( F_m \) respectively, then

\[ R(G, \Gamma) = R(C_n, \Gamma_1) \cdot R(P_{m+1}, \Gamma_2). \]

**Proof:** \( R(G, \Gamma) = R(C_n, \Gamma_1) \cdot R(F_m, \Gamma_2) \). Let \( F = T_1 \cup T_2 \cup \ldots \cup T_t \), where \( T_i \) is a nontrivial component in \( G = C_n \).

\[ R(G, \Gamma) = R(C_n, \Gamma_1) \cdot (\prod_{i=1}^{t} R(T_i, \Gamma'_i)) \]

where \( \Gamma'_i \) is the probability assignment of the edges in \( T_i \) with order exactly the same as in \( F_m \). Since

\[ \prod_{i=1}^{t} R(T_i, \Gamma'_i) = R(P_{m+1}, \Gamma_2). \]

we have

\[ R(G, \Gamma) = R(C_n, \Gamma_1) \cdot R(G, \Gamma_2). \]

□

By using the above lemma we can find the global reliability of graph

\[ G = C_n(t_1, t_2, \ldots, t_n) \cdot F_m(s_1, s_2, \ldots, s_m) \]

by simply considering the graph

\[ G' = C_n(t_1, t_2, \ldots, t_n) \cdot P_{m+1}(s_1, s_2, \ldots, s_m). \]

That is, by replacing the forest \( F_m \) with the path \( P_{m+1} \). This replacement will make the analysis of improving unicyclic graphs much easier.

If the variable in \( R(G, \Gamma) \) is the number of extra edges \( k \), then \( R(G, \Gamma) \) is defined to be \( R(G, \Gamma(t)) \), where \( t = 0, 1, \ldots, k \). The function \( R(G, \Gamma(t)) \) is not continuous

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on \([0, k]\). In the coming discussion we will allow this function to take any value between \([0, k]\). The modified function \(R(G, \Gamma(t))\) is continuous and differentiable for all values of \(t\). If the modified function \(R(G, \Gamma(t))\) is monotonic on \([n, n+1]\) for all \(n = 0, 1, \ldots, k\), then \(R(G, \Gamma(t))\) has maximum or minimum value at \(t_0\), then the original function has maximum or minimum value at \([t_0]\) or \([t_0]\).

**Remark 13** \(R(C_n\{(0, s)\}) = p^s p^{n-s} + (n-s)p^s p^{n-s-1} + sp^{s-1}(1-p)^2 \cdot p^{n-s}\).

**Theorem 4.44** Given a graph \(G = C_n \cdot F_m\), let \(k\) be the number of edges to be used in the enhancement of \(G\). If \(k \leq m\), then \(C_n \cdot F_m\{(0, k)\}\) is the optimally reliable graph in \(C_n \cdot F_m[k]\).

**Proof:** By the previous lemma, it is enough to show that \(C_n \cdot P_{m+1}\{(0, k)\}\) is the optimal graph in \(C_n \cdot P_m[k]\). Let \(t\) be the count of the enhanced edges used to improve \(P_m\). By using Corollary 15, the optimal graph in

\[C_n[t] \cdot P_{m+1}[k-t]\]

is

\[C_n\{(0,t)\} \cdot P_{m+1}\{(0,k-t)\}\].

Let

\[A = \{G_t|G_t = C_n\{(0,t)\} \cdot P_{m+1}\{(0,k-t)\}\}\].

be the set of all graphs with \(t\) and \(k-t\) extra edges distributed evenly among \(E(C_n)\) and \(E(P_{m+1})\) respectively. Next we show that \(G_0\) is the optimally reliable graph in \(A\).

\[
R(G_t) = R(C_n\{(0, t)\}) \cdot R(P_{m+1}\{(0, k-t)\})
= [p^t \cdot p^{n-t} + (n-t)(1-p)p^{n-t-1} \cdot p^t + t(1-p_1)p^{t-1} \cdot p^{n-t} \cdot p^{K-t} \cdot p^{m-K-t}]
= p^k \cdot p^{n+m-K} + (n-t)(1-p)p^{n+m-K-1} \cdot p^K + t(1-p_1)p^{K-1} \cdot p^{m+n-K}.
\]

This is a linear function of \(t\) which has a maximum value at \(t = 0\). Therefore \(G_0\) is the optimally reliable graph in \(A\), and hence in \(C_n \cdot F_m[K]\). \(\square\)
The above result shows an efficient way to improve a given unicyclic graph \( C_n \cdot F_m \), but only if the number of additional edges is not greater than \( m \).

**Corollary 16** In the previous theorem, if \( k > m \) and no more than two edges are allowed between two vertices, then the set

\[
A = \{ G_t | G_t = C_n \{(0,t)\} \cdot P_{m+1}\{(0,k-t)\} \}
\]

is the optimal reliable graph, when \( t = k - m \).

**Proof:** Using Theorem 4.44 we have

\[
R(G_t) = p_1^k \cdot p^{n+m-k} + (n - t)(1 - p)p^{n+m-k-1} \cdot p_t^k + t(1 - p_1)p_1^{k-1} \cdot p^{m+n-k}.
\]

\( R(G_t) \) is a function of \( t \). By finding the second derivative of \( R(G_t) \), we have \( R(G_t)^{''} = 0 \) for all \( t \in [k - m, n] \). Therefore the maximum of \( R(G_t) \) in \([k - m, n]\) is at the end points. By comparing \( R(G_n) \) and \( R(G_{k - m}) \), one can see that \( R(G_{k - m}) \) is the maximum value. Therefore, \( G_{k - m} \) is the optimally reliable graph in \( A \). □

If \( a \) and \( b \) are two vertices connected by \( L \) edges, then the probability of having \( a \) and \( b \) connected will be denoted by \( p_{L-1} \). Notice that \( p_{L-1} = 1 - (1 - p)^L \). By convention \( p_0 = p \). Now, we study the case when the number of extra edges available to enhance the network is \( k \) where \( n + m < k \leq 2n + 2m \).

If each edge in the graph \( G \) is enhanced by \( k - 1 \) edges then the new graph is denoted by \( G^k \).

**Theorem 4.45** Let \( k \leq n \) be the number of extra edges. The graph \( C_n[t] \cdot P_{m+1}^2[k - t] \) is the optimally reliable graph when \( t = k \).

**Proof:** It is enough to find the optimally reliable graph for the following set of graphs

\[
G = C_n\{(0,t)\} \cdot P_{m+1}^2\{(0,k-t)\}.
\]

\[
R(G) = H(t)
\]

\[
= p_2^{k-t} \cdot p_1^{m+k+2t-1} \cdot p^{n-t-1} \cdot [p_t + t(1 - p)^2 \cdot p + (n - t)(1 - p)p - 1].
\]
In order to find the maximum value for $H(t)$, we will first find all critical values of $H(t)$. Let $H'(t)$ be the derivative of $H(t)$ then

$$\frac{H'(t)}{H(t)} = -D \log p^2 + 2D \log p_1 - D \log p + p(p - 1)^3 = 0$$

where

$$D = p_1 p + t(1 - p)^2 \cdot p + (n - t)(1 - p)p_1$$

solving for $t$ we have the following

$$t = n + n(1 - p) + p\left(\frac{2 - p}{1 - p}\right) + \left(\frac{p - 1}{s}\right) = t_0$$

where $s = \log p_2 + \log p_1 + \log p$. By using the second derivative test, one can show that $H''(t) < 0$, therefore, $H(t)$ has a maximum value at $t_0$. For $0 < p < 1$, it can be shown that $t_0 \geq n$, therefore, the function $H(t)$ is increasing in $[0, t_0]$, which implies that $H(t)$ has maximum value at $t = k$. □

4.4 Multi-Ring Graphs

Given a set of cycles $\{C_{t_1}, C_{t_2}, \ldots, C_{t_n}\} = A$, where $t_i$ is the order of the cycle $C_{t_i}$, the graph $G$ constructed from $A$ by series connections (see Chapter I) is called a multi-ring graph. Given a set of cycles $A$, in order to identify the graphs $G$ which result from series connection from $A$, we need to introduce some new notation. If $G$ is a multi-ring graph of $n$ cycles, we construct a labeled tree $T$ of order $n$ from $G$ in the following way: each cycle $C_i$ in $G$ will be replaced by a vertex $v_i$, and two vertices are adjacent in $T$ if the corresponding circles are share a vertex. The order of $T$ is equal to the number of cycles in $A$. The tree $T$ uniquely describes $G$, and $G$ will be denoted by $T(A)$, where $A$ is the set of cycles and $T$ is the labeled tree which describes the connection between the cycles in $G$.

**Remark 14** Let $A = \{C_{t_1}, C_{t_2}, \ldots, C_{t_n}\}$ and let $T$ be any tree of order $n$, then

$$R(T(A), q) = \prod_{C_i \in A} R(C_i, q).$$

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Next we show how to best improve multi-ring networks. Let $T(A)[k]$ be the class of graphs obtained from $T(A)$ by adding $k$ multiple edges.

**Theorem 4.46** The set $B = \{G_t = C_n\{(0,m-t)\} \cdot C_{n+K}\{(0,t)\} : t \in [0,m]\}$ has an optimally reliable graph when

$$t = \frac{m}{2} + \frac{K}{2}p$$

where $m$ is the number of the extra edges.

**Proof:** Let $G_t$ be the graph $T(A)$

$$R(G_t) = R(C_n\{(0,m-t)\}) \cdot R(C_{n+K}\{(0,t)\})$$

$$R(C_n\{(0,m-t)\})$$

$$= \left(\frac{m-t}{2} \cdot \frac{n-m-t}{2}\right) + (n-m+t)\left(\frac{m-t}{2}\right) \cdot \left(\frac{n-m+t}{2}\right) \cdot (1-p)$$

$$+ \left(\frac{m-t}{2}\right)\left(\frac{n-m+t}{2}\right) \cdot \left(\frac{n-m-t}{2}\right) \cdot \left(\frac{n-m-t}{2}\right) \cdot (1-p)$$

$$= \frac{m-t}{2} \cdot \frac{n-m-t}{2} + \left(\frac{m-t}{2}\right)\left(\frac{n-m+t}{2}\right) \cdot \left(\frac{n-m-t}{2}\right) \cdot \left(\frac{n-m-t}{2}\right) \cdot (1-p)$$

Let

$$H_1 = p_1(p + (n-m+t)(1-p)p_1 + (m-t)(1-p)^2p)$$

and

$$H_2 = p_1(p + (n+K-t)(1-p)p_1 + t(1-p)^2p)$$

By Remark 14

$$R(G_t) = p_1^{n-m-t} \cdot p_1^{n-m+t} \cdot \left[H_1(t)\right]\left[H_2(t)\right].$$

Maximizing the function $R(G_t)$ is equivalent to maximizing $[H_1(t)][H_2(t)]$. Let $H(t) = [H_1(t)][H_2(t)]$, then

$$H'(t) = H_1(t)[(1-p)^2p - p_1(1-p)] + H_2(t)[p_1(1-p) - p(1-p)]$$

$$= [H_1(t) - H_2(t)][(1-p)(p(1-p) - p_1)]$$

$$= [H_1(t) - H_2(t)][1-p](-p).$$
Letting $H'(t) = 0$ and solve for $t$ we have that $H_1(t) - H_2(t) = 0$ or

$$2t[p_1 - p(1 - p)] = (m + K)p_1 - mp(1 - p)$$

where $p_1 = 1 - (1 - p)^2 = 2p - p^2$

$$t = \frac{m + 2K - Kp}{2} = \frac{m}{2} + K(1 - \frac{p}{2})$$

$$t = \frac{m}{2} + \frac{K}{2} \cdot p.$$

Calculating $H''(t)$, we find

$$H''(t) = -(1 - p)p[p_1(1 - p) - p(1 - p) - (1 - p)^2p + P_t(1 - p)] = -2p^2(1 - p)^2 < 0$$

for all $0 < p < 1$. Therefore the function $H(t)$ has maximum value at

$$t = \frac{m}{2} + \frac{K}{2} \cdot p = t_0$$

\[\square\]

In the above theorem, note that $t_0$ is a function of $m$, $K$ and $p$. If $p$ is small, then the effect of $K$ becomes small and edges are evenly distributed.

If the two cycles have the same order, the function $H(t)$ has maximum value at $t = \frac{m}{2}$. We will generalize this for any set of cycles of the same order having a series connection.

**Theorem 4.47** Given a set of cycles $A = \{C_{n_1}, C_{n_2}, \ldots, C_{n_k}\}$, where $n_i = n_j$ for all $1 \leq i, j \leq k$, let $m$ be the number of extra edges, where $m \leq \sum_{i=1}^{k} n_i$. Let $\{m_1, m_2, \ldots, m_k\}$ be a partition to $m$ such that $|m_i - m_j| \leq 1$ for all $1 \leq i, j \leq k$. For any tree $T$ of order $k$, the class $T(A)[k]$ has

$$G_0 = C_{n_1}\{(0,m_1)\} \cdot C_{n_2}\{(0,m_2)\} \cdot \ldots \cdot C_{n_k}\{(0,m_k)\}$$

as an optimally reliable graph.

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**Proof:** The proof is by induction on $k = |A|$. The result is true for $k = 1$. Assume the result is true for $|A| < k$. Let $A = \{C_{n_1}, C_{n_2}, \ldots, C_{n_{k+1}}\}$ Let $G \in T(A)[m]$. First we need to show that if $G$ is optimal, then

$$G = T(A'), \quad \text{where} \quad A' = \{C_{n_1}\{(0, \frac{m}{k+1})\}, \ldots, C_{n_{k+1}}\{(0, \frac{m}{k+1})\}\}.$$

Consider the graph

$$G_t = C_{n_1}\{(0, t_1)\} \cdots C_{n_k}\{(0, t_k)\} \cdot C_{n_{k+1}}\{(0, t_{k+1})\}$$

where $\{t_1, t_2, \ldots, t_k, t_{k+1}\}$ is any partition to $m$.

\[R(G_t) = R(C_{n_{k+1}}\{(0, t_{k+1})\}) \prod_{i=1}^{k} R(C_{n_i}\{(0, t_i)\})\]

The function $R(G_t)$ has maximum value when both $\prod_{i=1}^{k} R(C_{n_i}\{(0, t_i)\})$ and $R(C_{n_{k+1}}\{(0, t_{k+1})\})$ are maximum. Assume $t = \sum_{i=1}^{k} t_i$. By induction hypothesis, the $t$ units should be distributed evenly among the $k$ cycles. It is enough to consider the graph

$$G_{t_0} = C_{n_1}\{(0, \frac{t}{k})\} \cdots C_{n_k}\{(0, \frac{t}{k})\} \cdot C_{n_{k+1}}\{(0, m - t)\}.$$ 

\[R(G_{t_0}) = [p_1^{\frac{t}{k}-1} \cdot p^{n-k-1}[p_1 p + (n - \frac{t}{k}) p_1 (1 - p) + \frac{t}{k} (1 - p)^2 p]^{k}] \cdot [p_1^{m-t-1} \cdot p^{n-m+t-1}[p_1 p + (n - m + t) p_1 (1 - p) + (m - t)(1 - p)^2 p]].\]

If $H_1(t) = p_1 p + n - \frac{t}{k} p_1 (1 - p) + \frac{t}{k} (1 - p)^2 p$.

and $H_2(t) = p_1 p + (n - m + t) p_1 (1 - p) + (m - t)(1 - p)^2 p$,

then $R(G_t) = p_1^{m-(k+1)} \cdot p^n (k + 1) \cdot [H_1(t)]^k \cdot H_2(t)$.

The function $R(G_t)$ is maximum at $t_0$, if and only if the function

\[H(t) = [H_1(t)]^k \cdot H_2(t)\]
has a maximum at $t_0$. Differentiating $H(t)$ with respect to $t$, we have

$$H'(t) = k[H_1(t)]^{k-1} \cdot H_1'(t) \cdot H_2(t) + [H_1(t)]^k \cdot H_2'(t).$$

Observe that $H_1(t) \neq 0$ for $0 < p < 1$. Therefore $H'(t) = 0$ implies

$$kH_2(t)H_1'(t) + H_2'(t)H_1(t) = 0$$

or

$$H_2(t)[p(1-p) - p_1] - H_1(t)[p(1-p) - p_1] = 0.$$?

The fact that $p(1-p) - p_1 \neq 0$ for $0 < p < 1$ implies $H_1(t) = H_2(t)$. Namely

$$(n-t/k)p_1(1-p) + t/k(1-p)^2 p =
(n-m+t)p_1(1-p) + (m-t)p(1-p)^2$$

or $\frac{1}{k}(-1-k) = -m$; solving for $t$, we get

$$t = \frac{km}{1+k} = \frac{k}{1+k}m$$

and

$$m - t = \frac{1}{1+k} \cdot m.$$?

The second derivative test shows that at this point $H(t)$ has maximum value.

Since the number of edges takes on only integer value, it follows that the maximum value of $H(t)$ is $\lfloor \frac{m}{1+k} \rfloor$ or $\lceil \frac{m}{1+k} \rceil$ (i.e. if $m = t(k+1) + r$ then $r$ cycles will receive only $t+1$ edges, and $k+1-r$ cycles will receive only $t$ edges). Therefore, if $m_i$ represent the number of extra edges, the cycles $C_{n_i}$ will receive, then for all $1 \leq i, j \leq k+1$, $|m_i - m_j|$ the graph $T(A')$ is the optimal reliable graph in $T(A)[m]$.

**Definition 11** A graph $G$ is said to be $n$-cyclic graph if $G$ has only $n$ cycles and the induced subgraph on those cycles is a multi-ring graph.
Example: The graph in Figure 4.10 shows 3-cycles graph of order 13. If $G$ is an $n$-cyclic graph then $G$ can be written in the form $T(A) \cdot F_m$, where $T$ is the labeled tree which describes the connection between the cycles. The tree $T$ is constructed as follows: If $C_{n_1}$ and $C_{n_2}$ are two cycles in $G$ having the vertex $x$ in common, then we will replace the cycle $C_{n_1}$ by the vertex $v_{n_1}$ and the cycle $C_{n_2}$ by the vertex $v_{n_2}$ and we join the vertex $x$ to the vertices $v_{n_1}$ and $v_{n_2}$ by two edges. The tree corresponding to the graph mention above is shown in Figure 4.10.

**Lemma 14** If $T(A) \cdot F_m$ is an $n$-cyclic graph, then

$$R(T(A) \cdot F_m, q) = p^m t \prod_{i=1}^{k} R(C_{n_i}, q)$$

where $A = \{C_{n_1}, C_{n_2}, \ldots, C_{n_k}\}$.

**Remark 15** If $T(A) \cdot F_m$ is an $n$-cyclic graph, then

$$R(T(A) \cdot F_m, \Gamma) = R(T(A) \cdot P_m, \Gamma)$$
where $P_m$ is the path of length $m$.

**Theorem 4.48** Let $G = T(A) \cdot F_m$ be an $n$-cyclic graph with $A = \{C_{n_1}, C_{n_2}, \ldots, C_{n_k}\}$. Let $M$ be the number of the extra edges with $M \leq m$. $T(A) \cdot F_m \{(0, M)\}$ is the optimally reliable graph in $G[m]$.

**Proof:** We will use induction on $k$ the number of the cycles in $G$. The result is true for $k = 1$ (see Theorem 4.44). Assume the result is true for a graph with a number of cycles less than or equal to $k$. Let $G = T(A) \cdot F_m$ be $(k+1)$-cyclic graph, where $A = \{C_{n_1}, C_{n_2}, \ldots, C_{n_{k+1}}\}$. Suppose $B = \{e_0, t_1, \ldots, t_k, t_{k+1}\}$ is any partition to $M$. By Remark 15, it is enough to consider the graph $G_0 = T(A) \cdot P_m$, where $P_m$ is the path of length $m$. If $t = \sum_{i=1}^k t_i$, then $t_{k+1} = M - t$. We will consider the graph

$$G_A = P_m\{(0, t_0)\} \cdot C_{n_1}\{(0, t_1)\} \cdot C_{n_2}\{(0, t_2)\} \cdots C_{n_k}\{(0, t_k)\} \cdot C_{n_{k+1}}\{(0, t_1)\}.$$  

$$R(G_A) = R(P_m\{(0, t_0)\}) \cdot R(C_{n_{k+1}}\{(0, M - t)\}) \cdot \prod_{i=1}^k R(C_{n_i}\{(0, t_i)\}).$$

By the induction hypothesis

$$R(P_m\{(0, t_0)\}) \cdot \prod_{i=1}^k R(C_{n_i}\{(0, t_i)\})$$

has maximum value when $t_0 = t$. Observe that $C_{n_i}\{(0, 0)\}$ is the cycle $C_{n_i}$ without any change in the reliability of the edges. Thus, we consider the graph

$$G'_A = P_m\{(0, t)\} \cdot C_{n_{k+1}}\{(0, M - t)\} \cdot C_{n_1} \cdot C_{n_2} \cdots C_{n_k}$$

$$R(G'_A) = R(P_m\{(0, t)\}) \cdot R(C_{n_{k+1}}\{(0, M - t)\}) \cdot \prod_{i=1}^k R(C_{n_i}).$$

This is a function of $t$ and the maximum of $R(G'_A)$ is independent of

$$\prod_{i=1}^k R(C_{n_i}\{(0, t_i)\}).$$

Therefore

$$H(t) = R(P_m\{(0, t)\}) \cdot R(C_{n_{k+1}}\{(0, M - t)\}).$$

By an argument analogous to the one used in the proof of Theorem 4.44 we can show that $H(t)$ has maximum value at $t = M$. Therefore $T(A) \cdot F_m\{(0, M)\}$ is an optimally reliable graph in $T(A) \cdot F_m[M]$. □
CHAPTER V

IMPROVING K-TERM NAL RELIABILITY

5.1 Improving K-Terminal Reliability I

Recall from Chapter I that the K-terminal reliability of a graph $G$ is the probability that the vertices in $K$ are connected. We denote this by $R_K(G, q)$ where $q$ is probability of edge failure. The function $R_K(G, q)$ can be written as:

$$R_K(G, q) = \sum_{S \in \mathcal{G}} f(S) \cdot R(S)$$

where $R(S)$ is the probability of having $G$ in state $S$ and

$$f(S) = \begin{cases} 
1 & \text{if } <K> \text{ is connected in } <S>, \\
0 & \text{otherwise}
\end{cases}$$

In case of having only two vertices in $K$, we call such reliability, st-reliability and $R_K(G, q) = R_{st}(G, q)$. If $E(G)$ has $\Gamma$ as its probability assignment, then the st-reliability will be denoted by $R_{st}(G, \Gamma)$. A graph $G$ is called a multistage graph if $V(G)$ can be partitioned into $V_1, V_2, \ldots, V_k$, such that if $e = xy$ is an edge in $E(G)$ then $x$ and $y$ must be located in two consecutive sets, namely $V_i$ and $V_{i+1}$, for some $i, 1 \leq i \leq k - 1$. The sets $V_1, V_2, \ldots, V_k$ are called stages.

Example: The graph $G$ in Figure 5.1 shows a multistage graph of order 6. The set $V(G)$ is partitioned into four different sets, $V_1, V_2, V_3$ and $V_4$. If $|V_1| = |V_{k+1}| = 1$, then such a graph is called an st-multistage graph, and it is denoted by $M_{st}(G)$.

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Since the st-reliability measures the probability that \( s \) is connected to \( t \), it follows that \( R_{\text{st}}(G, q) \) is the probability of having at least one st-path in \( G \). If \( q \) (the probability of edge failure) is known, we drop \( q \) and simply write \( R_{\text{st}}(G) \) instead of \( R_{\text{st}}(G, q) \). In the st-multistage graph, if all st-paths are vertex disjoint, then \( G \) is denoted by \( M_{\text{st}}(k, l) \), where \( k = \text{dist}(s, t) \) and \( l \) is the number of vertex disjoint st-path in \( G \).

**Theorem 5.49** If \( G = M_{\text{st}}(k, l) \) is an st-multistage graph, then

\[
R_{\text{st}}(G, q) = 1 - (1 - p^{k-1})^l.
\]

**Proof:** In the graph \( G \), the two vertices \( s \) and \( t \) are connected if and only if there exists at least one st-path in \( G \). The probability of having an st-path \( P \) in \( G \) is equal to \( p^{k-1} \). Hence, the probability of not having an st-path \( P \) is \( 1 - p^{k-1} \). If \( P_1, P_2, \ldots, P_l \) are the st-paths in \( G \), then the probability of having no st-path is
Therefore, the probability of having an st-path in $G$ is

$$R_{s,t}(G, q) = 1 - (1 - p^{k-1})^i. \text{□}$$

As in Chapter IV, we improve the st-reliability of a graph $G$ by either adding multiple edges or by replacing the edges in $G$ by more reliable ones.

**Remark 16** If $s$ and $t$ are the two end vertices of the path $P_{n+1}$, then

$$R_{s,t}(P_{n+1}, q) = R(P_{n+1}, q) = p^n,$$

where $R(P_{n+1}, q)$ is the global connectivity of the path $P_{n+1}$.

**Proof:** If $s$ and $t$ are the two end vertices of the path $P_{n+1}$, then the probability of having an st-path is the same as the probability of $P_{n+1}$ being connected. □

As in Chapter IV, $G[k]$ denotes the set of all graphs obtained from $G$ by adding $k$ new multiple edges. The graph $G_0 \in G[k]$ is called an optimal st-reliable graph if $R_{s,t}(G_0, q) \geq R_{s,t}(G', q)$ for all $G' \in G[k]$.

**Remark 17** If $s$ and $t$ are the end vertices of the path $P_{n+1}$, then the optimal st-reliability graph is obtained by evenly distributing the $k$ extra edges.

**Proof:** This follows from the fact that $R_{s,t}(G, q) = R(G, q)$ together with the results in Chapter IV. □

Given a multistage graph $G = M_{s,t}(k, l)$, let $G[k]$ be all graphs obtained from $G$ by adding $k$ new multiple edges to $G$. Assume the st-paths of $G$ are labeled $P_1, P_2, \ldots, P_l$. Let $t_i$ be the number of extra edges added on the path $P_i$ of $G$, $i = 1, 2, \ldots, l$; then the new graph is denoted by $G\{t_1, t_2, \ldots, t_l\}$, where $\sum_{i=1}^l t_i = k$. We investigate the question of how to partition $k$ into $l$ numbers $t_1, t_2, \ldots, t_l$, so that the graph $G\{t_1, t_2, \ldots, t_l\}$ is an optimal st-reliable graph in $G[k]$. Throughout our discussion we assume that the $t_i$ edges assigned to the path $P_i$ are distributed evenly on the edges of $P_i$. 

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If the variable in $R_{s,t}(G, \Gamma(t))$ is the number of extra edges, then $R_{s,t}(G, \Gamma(t))$ takes values only at $t = 0, 1, \ldots, k$, where $k$ is the number of the extra edges. As in Chapter IV we will allow the function $R_{s,t}(G, \Gamma(t))$ to take any values of $t$ between 0 and $k$.

**Theorem 5.50** Among all multistage graphs $M_{s,t}(k, 2)$ with $m$ extra edges, where $m \leq k$, the graph $G\{m,0\}$ is the optimal st-reliable graph.

**Proof:** We will consider the graph $G\{m - t, t\}$. The st-reliability $G\{m - t, t\}$ is

$$R_{s,t}(G\{m - t, t\}) = 1 - [1 - p_1^{m-t}p^{k-m+t}](1 - p_t^k p^{k-t})$$

where $p_1 = 2p - p^2$ (the probability that two vertices are connected, when two edges between them are present). Let $R(G\{m - t, t\}) = H(t)$, and consider the function

$$H_0(t) = [1 - p_1^{m-t}p^{k-m+t}][1 - p_t^k p^{k-t}].$$

The function $H(t)$ has maximum value at $t$ if and only if $H_0(t)$ has minimum value at $t$. If $H_0(t)$ represents the first derivative of $H_0(t)$, then

$$H'_0(t) = p_1^{m-t}p^{k-m+t}lnp_1 - p_1^{m-t}p^{k-m+t}lnp - p_t^k p^{k-t}lnp_1 + p_t^k p^{k-t}lnp =$$

$$p_1^{m-t}p^{k-m+t}[lnp_1 - ln p] - p_t^k p^{k-t}[lnp_1 - ln p].$$

Since $lnp_1 - ln p \neq 0$, this implies $H'_0(t) = 0$ if and only if

$$p_1^{m-t}p^{k-m+t} = p_t^k p^{k-t}.$$

Now solving for $t$, we get

$$(k - m + t)ln p + (m - t)ln p = tlnp_1 + (k - t)ln p$$

$$lnp(2t - m) = ln p_1(2t - m)$$

$$(2t - m)(ln p_1 - ln p) = 0.$$  

Since $ln p_1 - ln p \neq 0$ for $0 < p < 1$, implies $t = \frac{m}{2}$. By using the second derivative test, one can show that $H''_0(t) < 0$. Therefore, $H_0(\frac{m}{2})$ is maximum in $(0, m)$. Since
$H_0(t)$ has a maximum value in $(0, m)$, this implies that $H_0(t)$ has a minimum value at the end points of $[0, m]$. By comparing the two values

$$H_0(0) = [1 - p^n p^{k-m}][1 - p^k]$$

and

$$H_0(m) = [1 - p^k][1 - p^n p^{k-m}]$$

one can see that the minimum of $H_0(t)$ occurs at the end points of $[0, m]$. Therefore, $H(t)$ has maximum value at $t = 0$, or $t = m$, namely the optimal reliable graph in $G\{m-t, t\}$ is $G\{0, m\}$ or $G\{m, 0\}$. □

In the process of improving $R_s,t[k, 2]$, the above result shows that it is always better to improve one path rather than two paths.

In a graph $G$, two vertices $s$ and $t$ are connected in parallel if all the st-paths in $G$ are edge disjoint.

Example: Assume $G$ is the circle $C_n$, $n \geq 3$, then any two vertices in $G$ are connected in parallel.

The next results show how to improve st-reliability when the two vertices $s$ and $t$ are in parallel connection. If $G$ is a graph consisting of two vertices $s$ and $t$ together with a set of st-paths $\{P_1, P_2, ..., P_l\}$, then $G$ is denoted by $M_s,t[P_1, P_2, ..., P_l]$. Sometimes, we refer to such a graph as an st-parallel connection graph. If all paths in $A$ have the same length $k$, then $G$ is just the multistage graph $M_s,t(k, l)$.

As in the multistage graph, if $G$ is the graph $M_s,t[P_1, P_2, ..., P_l]$, then $G\{t_1, t_2, ..., t_l\}$ denotes the graph obtained from $M_s,t[P_1, P_2, ..., P_l]$ by adding $t_i$ new edges on the path $P_i$, $i = 1, 2, ..., l$.

The following summary should illustrate the meaning of the different notation used in this chapter.

1. $G[m]$ : a graph with $m$ multiple edges.
2. $M_s,t(k, l)$ : an st-multistage graph with $l$ disjoint st-paths, each of length $k$.
3. $M_s,t[P_1, P_2, ..., P_l]$ : The st-multistage graph with $l$ different st-paths, namely $P_1, P_2, ..., P_l$. Sometimes we refer to such a graph as an st-parallel connection.
graph.

(4) $G\{t_1,t_2,\ldots,t_l\}$: The graph obtained from $M_{st}[P_1,P_2,\ldots,P_l]$, by adding $t_i$ multiple edges on the path $P_i$, for all $i = 1,2,\ldots,l$. Here we assume that the $t_i$ multiple edges are distributed evenly on to the path $P_i$, for $i = 1,2,\ldots,l$.

(5) $R_{st}(G,q)$: The st-reliability of the graph $G$, when all the edges have the probability of failure equal to $q$.

(6) $R(G,q)$: The global reliability of the graph $G$, when all the edges have the probability of failure equal to $q$.

**Theorem 5.51** Let $G = M_{st}[P_{l+1},P_{k+1}]$ and $l \leq k$. For $m \leq l$, the graph $G(m,0)$ is an an optimally reliable graph in $G[m]$.

**Proof:** Since $G$ contains only two st-paths $P_{l+1},P_{k+1}$; we will consider the graph $G\{t,m-t\}$ where $t \in [0,m]$. The st-reliability of $G\{t,m-t\}$ is

$$H(t) = G\{t,m-t\} = 1 - [1 - p_1^{m-t}p^{k-m+t}][1 - p_1^{m-t}]$$

$$= p_1^{m-t}p^{k-m+t} + p_1^{m-t} - p_1^{m-t}p^{k-m+t}$$

where $p_1 = 2p - p^2$. If $H'(t)$ denotes the derivative of $H(t)$, then

$$H'(t) = \ln p_1[p_1^{m-t}p^{k-m+t} - p_1^{m-t}p^{k-m+t}] - \ln p[p_1^{m-t} - p_1^{m-t}p^{k-m+t}]$$

If the function $H(t)$ has a critical point at $t$ then $H'(t) = 0$. This implies that

$$[p_1^{m-t} - p_1^{m-t}p^{k-m+t}][\ln p_1 - \ln p] = 0.$$  

Since $\ln p_1 - \ln p \neq 0$ for all $0 < p < 1$ this implies that

$$p_1^{m-t}p^{k-m+t}. $$

Solving for $t$, we have the following:

$$t = t_0 = \frac{\ln p_1[m] - \ln p[m + l - k]}{2[\ln p_1 - \ln p]}$$
where \( t_0 \) is the critical point for \( H(t) \). By using the second derivative test we have

\[
H''(t) = (p_1^t p^{l-t}) p_1^{m-t} p^{k-m-t}(lnp_1 - lnp)^2.
\]

Since all factors are positive in \( H''(t) \) for \( t \in [0, m] \), then \( H''(t) > 0 \). The function \( H(t) \) has minimum value at \( t_0 \). Thus, \( H(t) \) has maximum value at the end points of \([0, m]\). By inspection:

\[
H(0) = p_1^m p^{k-m} + p^l - p_1^m p^{l+k-m},
\]

\[
H(m) = p_1^m p^{l-m} + p^k - p_1^m p^{l+k-m},
\]

\[
H(m) - H(0) = p^m[p^l - p^k][p_1^m - p^m].
\]

For \( k \geq l \), the difference \( H(m) - H(0) \geq 0 \), which implies \( H(m) \) has a maximum value for \( k > l \); namely it is more efficient to use all \( m \) edges to improve the shortest path \( P_{i+1} \).

The next result shows that it is always best to improve the shortest \( st \)-path for the \( st \)-multistage graph \( M_{st}[P_{i+1}, P_{i+1}, ..., P_{i+1}] \).

**Theorem 5.52** Given \( G = M_{st}[P_{i+1}, P_{i+1}, ..., P_{i+1}] \) with \( l_1 \leq l_2 \leq ... \leq l_N \), where \( m \leq l_1 \), the graph \( G\{m, 0, 0, ..., 0\} \) is the optimal \( st \)-reliable graph in \( G[m] \).

**Proof:** We use induction on \( N \) (the number of \( st \)-paths). The result is true for \( N = 1 \). Assume the result is true for all graphs with the number of \( st \)-paths less than or equal to \( N \).

Let \( G \) be the graph

\[
M_{st}[P_{i+1}, P_{i+1}, ..., P_{i+1}]
\]

(see Figure 5.2). The graph \( G \) can be decomposed into two subgraphs, \( G_1 \) and \( G_2 \).

\( G_1 \) is the subgraph induced by the first \( N \) paths and \( G_2 \) is the graph induced by the path \( P_{i+1} \). Let \( \{t_1, t_2, ..., t_N, t_{N+1}\} \) be a partition to \( m \) and let \( t = \sum_{i=1}^{N} t_i \). Consider the graph \( G_t = G\{t_1, t_2, ..., t_{N+1}\} \). To maximize the \( st \)-reliability of
is distributed evenly among the edges of $P_{N+1+1}$. Therefore, we will consider the
graph $G_t = G\{t, 0, 0, \ldots, m-t\}$.

$$R_{s,t}(G_t) = 1 - (1 - p_i^t p^{1-t})(1 - p_i^{m-t} p^{n+1-m+t}) \prod_{i=2}^{N} (1 - p_i^t).$$

Let

$$H_0(t) = (1 - p_i^t p^{1-t})(1 - p_i^{m-t} p^{n+1-m+t}) \cdot \prod_{i=2}^{N} (1 - p_i^t).$$

The function $R_{s,t}(G_t)$ has a maximum value at $t$ if and only if the function $H_0(t)$
has a minimum value at $t$. Since $\prod_{i=2}^{N} (1 - p_i^t)$ is constant, one can drop it from
$R_{s,t}(G_t)$, for the calculation of the minimum value of $H_0(t)$. Therefore we only
consider the function

$$H(t) = 1 - p_i^t p^{1-t} - p_i^{m-t} p^{n+1-m+t} + p_i^{m} p^{1+n+1}.$$s

Let $H'(t)$ be the derivative of $H(t)$, then

$$H'(t) = -[p_i^t p^{1-t} + p_i^{m-t} p^{n+1-m+t}].$$

As in the proof of the previous theorem, $H(t)$ has a critical point at

$$t = t_0 = \frac{\ln p_i[m] - \ln p[m + l_1 - l_{N+1}]}{2[\ln p_i - \ln p]}.$$

Figure 5.2
As in the proof of the previous theorem, \( H(t) \) has a critical point at
\[
t = t_0 = \frac{\ln p_1[m] - \ln p[m + l_1 - l_{N+1}]}{2[\ln p_1 - \ln p]}.
\]
By testing the second derivative \( H''(t) \), one can see that \( H''(t) < 0 \) for all \( t \in (0, m) \). Therefore, \( H(t_0) \) is a minimum value, which implies that \( R_{s,t}(G_t) \) has maximum value at the end points of \( [0, m] \). By inspection, we have \( H(m) < H(0) \). Therefore, the \( m \) edges should be used to improve \( p_{l_1} \), the shortest st-path. □

Next we generalize the above result to cover the case when only the probability of the paths \( P_1, P_2, \ldots, P_l \) are given. Consider the graph \( G = M_{s,t}[P_1, P_2, \ldots, P_l] \). Assume that edges in each path of \( G \) has the same reliability, but edges in different paths may has different reliability. Let \( p_i \) denote the probability that the path \( P_i \) to be connected (up state). Suppose \( \Delta p \) represents the total increase allowed to be used in improving the st-reliability of the paths. The next result shows how to distribute \( \Delta p \) among the paths of the graph \( G = M_{s,t}[P_1, \ldots, P_l] \), where \( \Delta p < 1 - p_0 \), and
\[
p_0 = \max\{p(P_i) \mid i = 1, 2, \ldots, l\}.
\]
Let \( G[\Delta p] \) represent all graphs obtained from \( G \) by increasing the reliability of the paths in \( G \) by a total amount of \( \Delta p \).

**Theorem 5.53** Given a graph \( G = M_{s,t}[P_1, P_2, \ldots, P_l] \) with
\[
p(P_1) \leq p(P_2) \leq \ldots \leq p(P_l)
\]
and \( \Delta p < 1 - p(P_1) \), the graph \( G\{0, 0, \ldots, \Delta p\} \) is an optimal st-reliable graph in \( G[\Delta p] \).

**Proof:** We use induction on \( l \) (the number of st-paths in \( G \)). The result is true for \( l = 1 \). Assume it is true for \( l \), where \( l \leq N \). Let \( G = M_{s,t}[P_1, P_2, \ldots, P_{N+1}] \) with
\[
p(P_1) \leq p(P_2) \leq \ldots \leq p(P_{N+1})
\]
and \( \Delta p \leq 1 - p(P_{N+1}) \).
Let \( t \) be the total increase in the reliability of the paths \( P_1, P_2, ..., P_N, \) and \( \Delta p - t \) be the total increase in the reliability of \( P_{N+1} \), where \( 0 \leq t \leq \Delta p \). By the induction hypothesis \( t \) must be used to increase the reliability of \( P_N \). Thus, the reliability of \( G \) after increasing \( p(P_N) \) by \( t \) and \( p(P_{N+1}) \) by \( \Delta p - t \) is

\[
H(t) = R_{s,t}(G\{0,0,...,t,\Delta p-t\}) = \\
1 - [1 - (p(P_N) + t)][1 - (p(P_{N+1}) + \Delta p - t)] \prod_{i=1}^{N-1} (1 - p(P_i)).
\]

The function \( H(t) \) has maximum value at \( t_0 \) if and only if

\[
H_0(t) = [1 - (p(P_N) + t)][1 - (p(P_{N+1}) + \Delta p - t)]
\]

has minimum value at \( t = t_0 \). Differentiating \( H_0(t) \) with respect to \( t \), we get

\[
H'_0(t) = [1 - (p(P_N) + t)] - [1 - (p(P_{N+1}) - \Delta p - t)]
\]

For \( H'_0(t) = 0 \),

\[
t = \frac{\Delta p}{2} + \frac{p(P_{N+1}) - p(P_N)}{2} = t_0.
\]

Since \( H''_0(t) = -2 < 0 \), it implies that \( H_0(t_0) \) is maximum in \([0, \Delta p]\). Therefore, \( H_0(t_0) \) has a minimum value at the end points in \([0, \Delta p]\). By inspection, \( H_0(0) < H_0(\Delta p) \). Therefore, the function \( H(t) \) has a maximum value when \( t = 0 \). Hence \( \Delta p \) should be used to improve the most reliable path, namely \( P_{N+1}. \)

**Theorem 5.54** Let \( p_1 \) and \( p_2 \) denote the reliability of the edges in the two st-paths, say \( p_1 \) and \( p_2 \) of the graph \( G = M_{s,t}[P_1, P_2] \) respectively. If \( p_1 \geq p_2 \), then the graph \( G\{m,0\} \) is an optimal reliable graph in \( G[m], 0 \leq m \leq k \).

**Proof:** Consider the graph \( G\{t,m-t\} \), where \( 0 \leq t \leq m \). If an edge \( e = uv \) in \( E(P_1) \) receives an extra edge from the enhanced set, then the probability of having \( u \) and \( v \) connected is \( 1 - (1 - p_1)(1-p) = p_{01} \). Similarly, if an edge \( e = wz \) receives an extra edge, the new reliability is \( 1 - (1 - p_2)(1-p) = p_{02} \). Observe
that \( p_0 > p_2 \), for \( p_1 > p_2 \). The \( st \)-reliability of \( G \) after the above assignment is

\[
R_{s,t}(G(t, m - t)) = H(t) = 1 - [1 - p_0^t p_1^{k-t}] [1 - p_0^{m-t} p_2^{k+t-m}].
\]

Suppose

\[
H_0(t) = [1 - p_0^t p_1^{k-t}] [1 - p_0^{m-t} p_2^{k+t-m}]
\]

\[
= p_0^t p_1^{k-t} p_2^{m-t} p_2^{k+t-m} - p_0^t p_1^{k-t} p_2^{m-t} p_2^{k+t-m}.
\]

Then the function \( H(t) \) has a maximum value at \( t_0 \) if and only if \( H_0(t) \) has a maximum or a minimum value at \( t_0 \). Let \( H'_0(t) \) be the derivative of \( H_0(t) \); then

\[
H'_0(t) = p_0^t p_0^{m-t} p_1^{k-t} p_2^{k+t-m} [\ln p_0 + \ln p_2 - \ln p_1 - \ln p_02] - p_0^t p_1^{k-t} [\ln p_0 - \ln p_1] - p_0^{m-t} p_2^{k+t-m} [\ln p_2 - \ln p_02].
\]

\[
H''_0(t) = p_0^t p_0^{m-t} p_1^{k-t} p_2^{k+t-m} [\ln p_0 + \ln p_2 - \ln p_1 - \ln p_02]^2 - p_0^t p_1^{k-t} [\ln p_0 - \ln p_1]^2 - p_0^{m-t} p_2^{k+t-m} [\ln p_2 - \ln p_02]^2.
\]

Observe that

\[
[\ln p_0 + \ln p_2 - \ln p_1 - \ln p_02]^2 = [(\ln p_0 - \ln p_1) - (\ln p_02 - \ln p_2)]^2
\]

\[
< [\ln p_0 - \ln p_1]^2 + [\ln p_02 - \ln p_2]^2.
\]

Moreover the following is true:

\[
p_0^t p_0^{m-t} p_1^{k-t} p_2^{k+t-m} [(\ln p_0 - \ln p_1) - (\ln p_02 - \ln p_2)]^2 <
\]

\[
p_0^t p_0^{m-t} p_1^{k-t} p_2^{k+t-m} [\ln p_0 - \ln p_1]^2 + p_0^t p_0^{m-t} p_1^{k-t} p_2^{k+t-m} [\ln p_02 - \ln p_2]^2 <
\]

\[
p_0^t p_0^{k-t} [\ln p_0 - \ln p_1]^2 + p_0^{m-t} p_2^{k+t-m} [\ln p_2 - \ln p_02]^2.
\]

This shows that \( H''_0(t) < 0 \), for all \( t \in [0, m] \). Hence, the function \( H_0(t) \) has a minimum value at the end points of the interval \([0, m] \). By inspection, it can be shown that \( H_0(m) < H_0(0) \). Therefore the function \( H(t) \) has a maximum value at \( t = m \), namely the graph \( G\{ (m, 0) \} \) is the optimally reliable graph in \( G[m] \).\( \square \)
**Corollary 17** Given $G = M_s,i[P_1, P_2, \ldots, P_n]$, with $l_1 \leq l_2 \leq \ldots \leq l_N$, if $\Delta p \leq 1 - p^i$, then the graph

$$G\{\Delta p, 0, 0, \ldots, 0\}$$

is the optimal st-reliable graph in $G[\Delta p]$.

**Proof:** For each st-path $P_{i+1}$ in $G$ the probability of having $P_{i+1}$ connected is $p(P_i) = p^i$. Observe that if $l_1 \leq l_2$, then $p(P_{i_1}) \geq p(P_{i_2})$. The result follows immediately by using this observation together with Theorem 5.53 □

Given a set of graphs $A = \{G_1, G_2, \ldots, G_n\}$ where $G_i$ is the st-parallel connection graph $M_{s_i,t_i}[P_{i1}, P_{i2}, \ldots, P_{in}]$ for $i = 1, 2, \ldots, n$, we construct a graph $G$ by identifying the vertex $t_i$ in $G_i$ with $s_{i+1}$ in $G_{i+1}$ for all $i = 1, 2, \ldots, n - 1$. The resulting graph $G$ is in series connection from $A$, and is denoted by $G_1 \cdot G_2 \cdot \ldots \cdot G_n$. The st-reliability of $G$ is defined to be the probability of having an $s_1 t_n$-path in $G$.

The following result will be used in the analysis of improving st-reliability for the series connection of the graphs in $A$. In the next lemma, we assume that $\Delta p$ is used only to improve one edge in the given graph. Given a path $P_{n+1}$, let $E(P_{n+1}) = (e_1, e_2, \ldots, e_n)$ be the labeling of the edges in $P_{n+1}$. Recall from Chapter IV that $P_{n+1}(\Delta p_1, \Delta p_2, \ldots, \Delta p_n)$ denotes the path obtained from $P_{n+1}$ by improving the reliability of the edge $e_i$ by $\Delta p_i$, for all $i = 1, 2, \ldots, n$.

**Lemma 15** Let $P_{n+1}$ be an st-path and $E(P_{n+1}) = (e_1, e_2, \ldots, e_n)$ be the labeling of the edges of $P_{n+1}$. Let $p(e_i)$ denote the reliability of the edge $e_i$ and suppose

$$p(e_1) \leq p(e_2) \leq \ldots \leq p(e_n).$$

If only one edge is to be improved and $\Delta p \leq 1 - p(e_n)$, then $P_{n+1}(\Delta p, 0, \ldots, 0)$ is the optimal st-reliable graph in $P_{n+1}[\Delta p]$. 

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**Proof:** Let $e_i$ be the edge in $E(P_{n+1})$ which receives an increase of $\Delta p$ in its reliability. Since $R_{s,t}(P_{n+1}, \Gamma) = R(P_{n+1}, \Gamma)$, where $\Gamma$ is the probability assignment of $E(P_{n+1})$, it follows that:

$$R(P_{n+1}(0,0,...,\Delta p,0,...,0)) = \prod_{j \neq i} p(e_j)[p(e_i) + \Delta p]$$

$$= \prod_{j=i}^{n} p(e_j) + \Delta p \cdot \prod_{j \neq i} p(e_j)$$

$$= H(p(e_i)).$$

This function of $p(e_i)$ which has a maximum when $\prod_{j \neq i} p(e_i)$ is maximum. The fact that $\prod_{j \neq i} p(e_i)$ is maximum when $p(e_i)$ is minimum implies that $H(p(e_i))$ is the maximum value. Therefore $\Delta p$ should be used to improve the least reliable edge in $E(P_{n+1})$.\(\square\)

If $\Delta p$ is used to improve more than one edge of $E(P_{n+1})$, then the previous result is not true, as can be seen in the following example:

Consider the path $P_3$ with $E(P_3) = \{e_1, e_2\}$. Let $p(e_1) = p_1$, and $p(e_2) = p_2$. Suppose edges $e_1$ and $e_2$ are improved by $\Delta p - t$ respectively, where $\Delta p < 2 - p(e_1) - p(e_2)$. Let $s$ and $t$ be the end vertices in $P_3$ and let $p(s \sim t)$ denote the probability of having an st-path. Then

$$p(s \sim t) = (P_1 + t)(P_2 + \Delta p - t)$$

$$= p_1p_2 + p_1\Delta p + tp_2 + t\Delta p - p_1t - t^2$$

$$= H(t).$$

Let $H'(t)$ be the derivative of $H(t)$, then

$$H'(t) = p_2 + \Delta p - p_1 - 2t = 0.$$ 

Solving for $t$ gives us

$$t = \frac{p_2 - p_1}{2} + \frac{\Delta p}{2} = t_0.$$ 

By testing $H''(t)$, one can show that $H(t_0)$ is maximum. Therefore, it is better to improve both edges $\{e_1, e_2\}$, rather than just one. The best distribution of $\Delta p$
among all the edges on the path $P_{n+1}$ becomes tedious when $n$ is large and the reliability is not the same for all edges.

Given st-multistage graph $G = M_{st}[P_1, P_2, \ldots, P_n]$, let $G[[\Delta p]]$ denote the set of all graphs obtained from $G$ by increasing the st-reliability of the st-paths in $G$ by a total of $\Delta p$. If the path $P_i$ receives an increase $\Delta p_i$ in its st-reliability, then $\sum_{i=1}^{n} \Delta p_i = \Delta p$. The resulting st-reliability depends on the values of $\Delta p_i$'s. Observe that a graph $G_0$ in $G[[\Delta p]]$ is called an optimal st-reliable graph if $R_{st}(G_0, \Gamma_0) > R_{st}(G', \Gamma')$ for all $G' \in G[[\Delta p]]$, where $\Gamma_1$ and $\Gamma_2$ are the probability assignments of $G_0$ and $G'$, respectively.

Assume $A = \{G_1, G_2, \ldots, G_n\}$, where each $G_i$ is the $s_i t_i$-parallel connection graph $M_{s_i t_i}(P_{i1}, \ldots, P_{in})$. Let $G = G_1 \cdot G_2 \cdot \ldots \cdot G_n$ be a graph obtained from $A$ by series connection. We will study the optimally reliable graph in $G[[\Delta p]]$.

We will assume that, if $\Delta p_i$ is the portion of $\Delta p$ to be used on the graph $G_i$, then $\Delta p_i$ should be used in an efficient way to improve the $s_i t_i$-reliability of $G_i$. Namely, we will consider the optimal $s_i t_i$-reliable graph in $G_i[[\Delta p_i]]$.

**Theorem 5.55** Let $G$ be the graph defined above and let $R_{s_i t_i}(G_i) \leq R_{s_i t_i}(G_{i+1})$ for all $i = 1, 2, \ldots, n-1$. Assume $\Delta p < R_{s_i t_i}(G_n)$, if $\Delta p$ is allowed to be used only on the edges of one subgraph $G_i$, $i = 1, 2, \ldots, n$ assume then the optimal graph in $G[[\Delta p]]$ is the one in which $G_i$ has been optimized.

**Proof:** The graph $G$ can be converted to a path $P_{n+1}$ in the following way. Each graph $G_i$ in $G$ is replaced by an edge $e_i = s_i t_i$ with $p(e_i) = R_{s_i t_i}(G_i)$. The new $s_1 t_n$-path has $p(e_1) \leq p(e_2) \leq \ldots \leq p(e_n)$, and the $s_1 t_n$-reliability is the same as the $s_1 t_n$-reliability of $G$. By using Lemma 15, we should use $\Delta p$ to improve $p(e_1)$. Therefore, the best subgraph in $G$ to improve is $G_1$ in $G$. \(\Box\)
5.2 Improving K-terminal Reliability II

In section 5.1, the study of improving st-reliability was restricted to the case of having at most double edges between the adjacent vertices. In this section we allow any number of multiple edges.

Given a multistage graph $G = M_{s,t}(k, l)$, with $m$ multiple edges, where $m \geq k$, one can ask: What is the optimal st-reliable graph in $G[m]$?. The following example illustrates how the distribution of the $m$ edges changes the value of the st-reliability of the graph $G$.

Example: Consider the graph $G = M_{s,t}(4, 2)$. Let $k = 6$ be the number of extra edges. Define the following: $G_1 = G(4, 2)$, $G_2 = G(5, 1)$ and $G_3 = G(6, 0)$. By previous result the edges in each path must be distributed evenly. Consider the following functions:

$$R_1(p) = R_{s,t}(G_1) = 1 - [1 - (1 - p)^2]^4[1 - (1 - p)^3]^2p^3$$
$$R_2(p) = R_{s,t}(G_2) = 1 - [1 - (1 - p)^3][1 - (1 - p)^2]^3[1 - (1 - p)^2]p^3$$
$$R_3(p) = R_{s,t}(G_3) = 1 - [1 - (1 - p)^3]^2[1 - (1 - p)^2]^2[1 - (1 - p)^2]p^3$$

The graph of the three functions $R_1(p)$, $R_2(p)$ and $R_3(p)$ are shown in Figure 5.3. For $0 \leq p \leq 1$, the graph shows that $R_3(p)$ are always greater than $R_1$ and $R_2$. This implies that the best distribution of the 6 extra edges occurs when we use them to improve one path. In fact this result turns out to be true in general.

Let $P_{k+1}(l)$ denote the path of length $k$, with $l$ multiple edges between any two adjacent vertices.

Example: The graph $G = P_3(2)$ is shown in Figure 5.4.

**Theorem 5.56** Let $x + y = l$ be a positive integer. The graphs in the set $A = \{G_x | G_x = M_{s,t}(P_{k+1}(x), P_{k+1}(l-x)) \text{ with } x + y = l\}$ have a st-optimal reliability graph when $x = l$.

**Proof:** The proof is by contradiction. Suppose $x \neq y$ and assume $x \geq y \geq 2$; we will show that the graph $G = M_{s,t}(P_{k+1}(x), P_{k+1}(y))$ is not an st-optimal reliable
Figure 5.3

Figure 5.4
graph in $A$. For the path $P_{k+1}(x)$, the reliability between any two adjacent vertices is $p_{x-1} = 1 - (1-p)^x$ and for path $P_{k+1}(y)$, the reliability between any two adjacent vertices is $p_{y-1} = 1 - (1-p)^y$. For $x \geq y$ we have $p_{x-1} \geq p_{y-1}$. Now construct the graph $G' = M_s,t[P_{k+1}(x + 1), P_{k+1}(y + 1)]$ by removing $k$ edges from $P_{k+1}(y)$ to enhance the path $P_{k+1}(x)$. Theorem 5.54 implies that the graph $G'$ is more reliable with respect to the st-reliability than $G$, which is the required contradiction. □

**Theorem 5.57** If $G = M_{s,t}[k, 2]$, then for any positive integer $m$, $G(0, m)$ is an st-optimal reliable graph in $G[m]$.

**Proof:** Let $A = \{G_i | G_i = G(i, m - i)\}$, we show that $G_0$ is an st-optimally graph in $A$. To the contrary, assume there exists a positive integer $t$ such that $G_t$ is an st-optimal reliable graph in $A$. Let $t = kl_1 + x$ and $m - t = kl_2 + y$; we proceed by case analysis.

**Case 1:** $l_1 = l_2$.

Subcase 1: $x = y = 0$. In this case $G_t = M_{s,t}[P_{k+1}(l_1 + 1), P_{k+1}(l_1 + 1)]$. Since $m$ is positive integer, it follows that $l_1 \geq 1$. By using a similar arguments to those in Theorem 5.56, we can show that the graph $G_0 = M_{s,t}[P_{k+1}(l_1 + 2), P_{k+1}(l_1)]$ is more reliable than $G_t$ with respect to st-reliability. Therefore, $G_t$ is not an st-optimal reliable graph, which is a contradiction.

Subcase 2: $x + y \neq 0$. We consider two possibilities:

1. If $x \neq 0$, then by Theorem 5.54, the graph $G_0 = G(t - x, m - t + x)$ is more reliable with respect to the st-reliability than $G_t$, which again is a contradiction.

2. $x \neq 0$. Construct the graph $G'$ by taking $k$ edges from the path $P_{k+1}\{(l, x)\}$ and use them to improve the path $P_{k+1}\{(l, y)\}$. The fact that the probability of having two adjacent vertices connected in the path $P_{k+1}\{(l, y)\}$ is equal to the probability of having adjacent vertices connected in the path $P_{k+1}\{(l, x)\}$, together with Theorem 5.54, implies that $G'$ is more reliable than $G_t$, which is a contradiction.

**Case 2:** $l_1 \neq l_2$; Without loss of generality, let $l_1 \neq l_2$. For this case the probability
that two adjacent vertices are connected in the path $P_{k+1}\{(l_2, x)\}$ is less than the probability that two adjacent vertices are connected in $P_{k+1}\{(l_2, y)\}$. Construct the graph $G' = M_{s,t}[P_{k+1}\{(l_1 + 1, x)\}, P_{k+1}\{(l_2 - 1, y)\}]$ by taking $k$ edges from $P_{k+1}\{(l_2, y)\}$ and use them to construct $P_{k+1}\{(l_1 + 1, x)\}$. By using Theorem 5.54, it can be shown that $G'$ is more reliable than $G_t$, which is again a contradiction. □

**Lemma 16** The graph $P_{k+1}(2)$ is more reliable than $M_{s,t}[k, 2]$.

**Proof:** Consider the graph $P_{k+1}(2)$ with

$$R(P_{k+1}(2)) = (2p - p^2)^k = p^k(2 - p)^k$$

and the graph $M_{s,t}[k, 2]$ with

$$R(M_{s,t}[k, 2]) = 1 - [1 - p^k]^2 = p^k[2 - p^k].$$

Observe that

$$R(P_{k+1}(2)) - R(M_{s,t}[k, 2]) = p^k[(2 - p)^k - (2 - p^k)].$$

The function $H(x) = (2 - p)^x - (2 - p^x)$ is a positive and increasing function, for all $0 < p < 1$. Therefore $R(P_{k+1}(2)) - R(M_{s,t}[k, 2]) > 0$ for all $p$ and for all $k$. Hence $P_{k+1}(2)$ is more reliable than $M_{s,t}[k, 2]$. □

**Corollary 18** Given a graph $G = M_{s,t}[P_1, P_2]$, let $p_1$ and $p_2$ be the probability of the edges on the paths $P_1$ and $P_2$, respectively. Let $P$ be the path whose vertices are connected by two multiple edges, one with probability $p_1$ and the other with $p_2$, then $R(P) > R(G)$ (see Figure 5.5).

**Theorem 5.58** The graph $P_{k+1}(l)$ is more reliable than $M_{s,t}[k, l]$.

**Proof:** The proof is by induction on $l$. The result is true for $l = 2$. Assume the result is true up to $N$. Suppose $G = M_{s,t}[k, N + 1]$ and let $P_1, P_2, \ldots, P_N, P_{N+1}$ be the set of st-paths of $G$. Let $G_1$ be the induced subgraph by $P_1, P_2, \ldots, P_N$. Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Construct a graph $G'$ from $G$ by replacing $G_1$ by the paths $P_1, P_2, \ldots, P_N$. By the induction hypothesis, the st-reliability of the graph $G'$ is more reliable than $G_1$. Therefore the graph $G'$ is more reliable than $G$, with respect of the st-reliability. Since the graph $G'$ consists of two paths, one has edge reliability equal to $p_{N-1}$ and the other has edge reliability $p$. By using Corallary 18 the graph $P_{k+1}(N + 1)$ is more reliable than $G'$. Therefore, $P_{k+1}(N + 1)$ is more reliable than $G$.

We next study improvement the multistage graphs $M_{s,t}[k,l]$ when $m$ (the number of extra edges ) is more than $k$.

**Theorem 5.59** Let $G = M_{s,t}[k,l]$, the graph $G_1$ in $G[m]$ is an optimal st-reliable graph, when the $m$ edges are used to improve only one path.

**Proof:** The result is true for $k = 1$ or $l = 1$. That can be seen by observing that the graph $M_{s,t}[1, l]$ has just $l$ multiple edge between $s$ and $t$, and the graph $M_{s,t}[k, 1]$ is just the path $P_{k+1}$. Assume $k \geq 2$ and $l \geq 2$. We use induction on $l$ (the number of the disjoint st-path). Assume the result is true for all $l \leq
Consider the graph $G = M_s,t[k,N + 1]$. Let $m = rk + d$; we show that the graph $G[P_1{(r,d)}, P_2, \ldots, P_{N+1}]$ is an optimally reliable graph in $G[m]$. Let $\{t_1, t_2, \ldots, t_N, m - t\}$ be a partition of $m$ which produces the st-optimally reliable graph, where $t = \sum_{i=1}^{N} t_i$. Let $G'$ be the st-optimally reliable graph obtained from $G$ by adding $t_i$ edges to the path $P_i$, for all $i = 1, 2, \ldots, N$ and $m - t$ edges to the path $P_{N+1}$. We show that $G'$ is not an optimally reliable graph. Consider the graph $G'_1$, constructed by adding $t$ extra edges to the path $P_i$ and $m - t$ extra edges to the path $P_{N+1}$. By the induction hypothesis, the graph $G'_1$ is more reliable than $G'$, which is a contradiction. □

Let $\Delta R_m$ denote the increase in st-reliability after the improvement process of the graph; where $m$ is the number of extra edges. The following result measures $\Delta R_m$, when $m$ is given.

**Remark 18** Given a path $P_{n+1}$, let $m$ be the number of extra edges with $s$ and $t$ the two end vertices of $P_{n+1}$. If $m < n$, then the increase in the st-reliability of $P_{n+1}$ after the enhancement is

$$\Delta R_m(P_{n+1}) = p^n[(2 - p)^m - 1].$$

**Proof:** The result follows from the fact that the

$$R_{s,t}(P_{n+1}(0,m)) = \pi^m p^{n-m}$$

and $R_{s,t}(P_{n+1}) = p^n$, where $\pi = 2p - p^2$. □

Given a graph $G_I = P_{n+1}(l)$, we study the increase in $\Delta R_m$ on $G_I$ for different values of $l$.

**Remark 19** For $m < n$, $\Delta R_m(G_I) = [1 - (1 - p)^l]^n[(1 - (1 - p)^{l+1})^m - 1]$.

**Proof:** Let $G_I = P_{n+1}(l)$, then

$$R_{s,t}(G_I) = [1 - (1 - p)^l]^n$$

$$R_{s,t}(G_I(m)) = [1 - (1 - p)^{l+1}]^m[1 - (1 - p)^l]^{n-m}.$$
By taking the difference of $R_{s,t}(G_1)$ and $R_{s,t}(G_1(m))$ we get the result. □

Given a graph $G = M_{s,t}[P_1, P_2]$, let $R_{s,t}(P_1) = p_1$ and $R_{s,t}(P_2) = p_2$ be the probability that the two end vertices of $P_1$ and $P_2$ are connected. Let $\Delta_1$ be the increase in $R_{s,t}(P_1)$ when we improve $P_1$ and $\Delta_2$ be the increase in $R_{s,t}(P_2)$ when we improve $P_2$. Suppose $p_1 > p_2$ and $\Delta_1 > \Delta_2$. If we have only the choice to improve $P_1$ or $P_2$ but not both, we can ask the question: What is the best choice so as to increase the st-reliability of $G$ the most? For the st-reliability, if we choose $P_1$, we have

$$R_{s,t}(P'_1) = p_1 + \Delta_1 + p_2 - p_2(p_1 + \Delta_1)$$

where $P'_{l-1}$ is the path obtained from $P_l$ by increasing its reliability by $\Delta_1$. Similarly,

$$R_{s,t}(P'_2) = p_2 + \Delta_2 + p_1 - p_1(p_2 + \Delta_2)$$

where $P'_{l_2}$ is the path obtained from $P_{l_2}$ by increasing its reliability by $\Delta_2$. Observe that $R_{s,t}(P'_1) - R_{s,t}(P'_2) = \Delta_1(1 - p_2) - \Delta_2(1 - p_1)$. Since $\Delta_1 > \Delta_2$ and $1 - p_2 > 1 - p_1$, it follows that the choice of improving $P_2$ is better to improve $R_{s,t}(G)$.

In improving the multistage graph $G = M_{s,t}[k, 2]$, let $G$ contain the two paths $P_{k+1}(l), P_{k+1}(1)$. The results in this section show that $\Delta R_m(G)$ is maximum when all the $m$ edges are used to improve the path $P_{k+1}(l)$ but not $P_{k+1}(1)$ for $l \geq 1$. 

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CHAPTER VI

CONCLUSION AND FUTURE DIRECTIONS

6.1 Traffic vectors

For simplicity we restrict our discussion to a graph $G$, or digraph $D$ which is free of loops and parallel edges.

Let $G$ be an $(n,m)$-graph. Recall from Chapter II that the traffic vector of a set $S$ of edges in $E(G)$ is a sequence which describes the number of paths of length $i$ which contain $S$, for all $i = 1, 2, \ldots, n - 1$. We extend the definition to a case of digraph $D$. A digraph $D$ is called $(n,m)$-digraph, if the order of $D$ is $n$ and the size is $m$. Given an $(n,m)$-digraph $D$ and $e \in E(D)$, the traffic vector of $e$ is $TV(e) = (\pi_1(e), \pi_2(e), \ldots, \pi_{n-1}(e))$, where $\pi_i(e)$ is the number of the directed paths which contain $e$. Note that the direction of the paths is determined by the direction of the edge $e$.

Example:
Consider the digraph in Figure 6.1. The traffic vector of the edge $e$ is $TV(e) = (1, 3, 1)$.

As in the definition of the dominant edge for the graph, the edge $e_0 \in E(D)$ is a dominant edge, if for a different edges $e \in E(D)$ and for all $j = 1, 2, \ldots, n - 1$ it implies that

$$\sum_{i=1}^{j} \pi_i(e_0) \geq \sum_{i=1}^{j} \pi_i(e).$$

The study of characterizing the set of the dominant edges in a graph or a digraph in general can be interesting problem.
For the case of directed trees, several results have been obtained but are not included in this dissertation.

A digraph $D$ is called a probabilistic digraph if the elementing in $V(D)$ and $E(D)$ are assigned positive real numbers which represent the probability that a given element exists in the set $V(D)$ or $E(D)$. If $v_i$ is assigned the number $p(v_i)$, then the probability of having $v_i$ in $V(D)$ is $p(v_i)$. Observe that $0 \leq p(v_i) \leq 1$. Similarly, $p(e_i)$ denotes the probability of having $e_i$ in $E(D)$. For the following discussion, we assume that the probability assigned to the vertices in $D$ is always 1 (the vertices of $D$ are absolutely reliable). Two vertices in $V(D)$ are connected if there exists a directed path from $u$ to $v$. Define the pair-connected reliability of the digraph $D$ to be the expected number of connected vertices in $D$. The problem of finding the pair-connected reliability of $D$ in general is an open problem. Moreover the study of improving the pair-connected reliability for a general digraph by the methods outlined in this dissertation can be cited as a good research problem. For the case of a directed tree $D(T)$, several results are obtained by using the directed traffic vectors.

Figure 6.1
6.2 K-Terminal Reliability

The study in this dissertation is done only for special classes of graphs, which are commonly used in computer networks and other applications. One may extend the study to other types of networks. For complete graphs and bipartite graphs the study of improving different types of reliability measures is still needed.

Consider for example a graph $G$ of order $n$, and positive integer $k \leq n$. To find the subset $S \subseteq E(G)$ with $|S| = k$ to be improved in $G$ so the increase in the $K$-terminal reliability is maximal has not been analyzed as the complexity goes.

There are many types of reliability measure for networks that can be improved by the two methods mentioned on this dissertation.

To improve network reliability for a probabilistic graph $G = (V, E)$ we assumed that vertices were fail-safe, but each edge $e \in E(G)$ is down (that is, in failed state) independently with probability $q$, $0 < q < 1$. Moreover, assume that the node failures are equal and independent. A natural question is ask: How does one improve the reliability measures associated with $G$ by using the two methods used in this dissertation, where both vertices and edges are subject to failure.

6.3 General Reliability in Probabilistic Graphs

A method of studying reliability in general for $G$ can be extended to find the probability or the expected value of having any property in $G$. Namely, $G$ contains a complete subgraph of order $k$, where $k \leq |V|$; or $G$ has a set of independent number of edges. To improve any of the network reliability measures mentioned above using the methods outlined in this dissertation can be interesting to study.
REFERENCES


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