The Enumeration of Graph Imbeddings

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THE ENUMERATION OF GRAPH IMBEDDINGS

by

Robert G. Riepep

A Dissertation
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Faculty of The Graduate College
in partial fulfillment of the
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Graphs can be drawn on surfaces. Here, graphs may have multiple edges or loops (pseudographs), the surfaces are closed orientable 2-manifolds (sphere, torus, etc.) and their generalizations (pseudosurfaces and generalized pseudosurfaces), and the drawings are 2-cell imbeddings. For quite some time it has been known that a connected graph has \( \Pi(\text{degree}(v) - 1)! \) 2-cell imbeddings on surfaces. More detailed information about these imbeddings has been wanting. In addition, many of the imbeddings counted above 'look the same' when all vertex and edge labels are removed. The resulting unlabeled imbeddings are fewer in number and more difficult to enumerate than their labeled counterparts.

This dissertation enumerates the labeled and unlabeled 2-cell imbeddings of two important classes of pseudographs, bouquets (one vertex, many loops) and dipoles (two vertices, multiple edges), on surfaces, pseudosurfaces, and generalized pseudosurfaces. The enumeration is by size, by size and number of regions, by size and region distribution \( (r_k \text{ regions of boundary length } k) \), and, in the case of the bouquet, by symmetry. For example, the fraction of all labeled 2-cell imbeddings of the dipole with \( n \) edges having \( r \) regions on surfaces is twice the fraction of permutations in the symmetric group of degree
$n+1$ having $r$ orbits provided $n$ and $r$ have the same parity.

A correspondence between imbeddings of the bouquet on pseudosurfaces and the totality of all pseudograph imbeddings on surfaces is exploited to enumerate the latter in both the labeled and unlabeled case. For example, there are four connected pseudographs with two edges; this set of pseudographs has a total of twenty labeled and five unlabeled 2-cell imbeddings on surfaces.
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The enumeration of graph imbeddings

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Western Michigan University, 1990
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CHAPTER I

INTRODUCTION

An impossible task would be to study graph theory and never draw a picture of a graph. In fact, many theorems and even more conjectures have been born from these pictures. Since graph theorists subsist on a steady diet of conjectures (so many it is doubtful we will ever go hungry) it is appropriate to begin with one.

Conjecture. Whenever the definition of 'graph' is given a drawing of one will appear almost immediately.

The point to be made is that graph theory is a very visual branch of mathematics and this fact is probably what makes it so enjoyable. A case in point: The theory gained early impetus by the observation that whenever a connected graph was successfully drawn on a plane (a planar graph) the graph partitioned the plane into regions which could always be 4-colored so neighboring regions had different colors. The Four Color Theorem, as it came to be called, is probably the most famous 'oscillating' theorem in the history of mathematics. At various times in its past it has enjoyed the status of truthfullness and at others, doubt. Currently, it is true again.
Many of the incorrect 'proofs' (Kempe [18], for example) were founded on erroneous assumptions about drawings. Harary and Palmer [11] argue that the Four Color Theorem can legitimately be settled by counting all the planar graphs, counting the 4-colorable planar graphs, and comparing the two.

To a graph theorist the sphere and the plane are synonymous and the Four Color Theorem may be briefly stated as: 'the chromatic number of the sphere is four'. Some graphs ('most' in fact) can not be drawn on the sphere (without miscellaneous edge crossings); these graphs require a more complex surface such as a torus or projective plane. The complete graphs, $K_n$ on $n=5,6$ or 7 vertices, for example, can be drawn on the torus but not on the sphere. From the fact that $K_7$ can be drawn on the torus, but $K_9$ can not, we have that the chromatic number of the torus is 7.

The sphere and the torus are two of an infinite collection of topological spaces which are the favorite candidates for graph drawings. A member is characterized as a sphere with $k$ handles, and denoted $S_k$. A generalization of the Four Color Theorem is the Heawood Map-coloring Theorem which gives the chromatic number of $S_k$, $k>0$, as

$$\frac{7 + \sqrt{1 + 48k}}{2}.$$  

Heawood 'proved' this to be the case in 1890 [12] but, like Kempe, his proof was not valid.

A proof of this theorem requires that one show the simplest
surface of the above collection on which the complete graph $K_n$ can be drawn has $\frac{(n-3)(n-4)}{12}$ handles, $n \geq 7$. This number is the genus of the complete graph. It was not until 1963 that Mayer [19] provided the last few details of the extensive work of Ringel, Youngs, and others to complete the proof that the genus of the complete graphs is given by the expression above. The proof is given in its entirety in Ringel’s book [27].

Suppose, for the moment, out of the very large number of different ways of drawing $K_n$ on surfaces, we know how many are on the surface $S_k$, for any $k$. This information is the genus distribution of the graph. In the very least, then, we would know the smallest $k$ such that $K_n$ can be drawn on $S_k$ and, as a result, have a proof of the Heawood Map-coloring Theorem. It is not expected, therefore, that the genus distribution of the complete graphs will come easily. In fact, finding the genus distribution of any nontrivial graph appears to be a difficult task.

If it is agreed that finding the genus distribution of a given graph is difficult then this next problem is all but impossible. The graph drawings we are interested in are characterized as being 2-cell imbeddings. That is, once the graph is drawn on a surface, delete those points of the surface which are the graph. If all the components of the resulting space are homeomorphic to an open unit disk in the plane (a 2-cell) then the graph imbedding is 2-cell.

For a variety of reasons these are the drawings of interest. First, without the 2-cell requirement even the simplest graph,
an isolated vertex, has an infinite number of imbeddings; keep adding handles to a surface imbedding. With the requirement, every connected graph has some maximum genus and, hence, a finite number of imbeddings.

Second, if a graph with $p$ vertices and $q$ edges is 2-cell imbedded on some surface, resulting in $r$ regions, then the number $p-q+r$ is an invariant of that surface, its characteristic. Thus, in the case of 2-cell imbeddings, there is an arithmetic relationship between the surface and the graph.

The third and last reason proffered is that in many instances, it is not the graph which is of interest in an imbedding, but rather the boundaries of the regions. This is the case for example in constructing block designs (see White [29]). Moreover, the useful mathematical concept of a dual becomes possible for a 2-cell imbedding (the regions become the vertices of a new graph, imbedded on the same surface, see [29]).

Given a particular graph, then, it is of interest to determine its region distribution. That is, how many 2-cell imbeddings does it have with, say, two 3-sided regions and a 4-sided region? It has been remarked that this is, in general, a difficult question to answer. If for a given graph, it can be answered, then we would certainly know the number of imbeddings resulting in $r$ regions, and using $p-q+r$, be able to determine the genus distribution of the graph.

If all the vertex and edge labels are removed from the
drawing of a graph, the resulting unlabeled graph represents the isomorphism class of the graph. It is clear that this can be done regardless of the surface that the graph is drawn on, and we have an unlabeled imbedding. For a given graph, there are generally far fewer of these than of the labeled variety, and their number is more difficult to determine as well. Thus, we are interested in finding the unlabeled genus distribution and the unlabeled region distribution of a graph.

At this time, what is known about the number of imbeddings of a graph is the following. From the work of Heffer [13] and Edmonds [6], there is a total of \( \prod (\text{degree}(v) - 1)! \) labeled 2-cell imbeddings of a connected graph on surfaces. Hence, the genus and region distributions are refinements of this number. Recently, Mull, Rieper, and White [20] have shown how to determine this number in the unlabeled case although the calculations are by no means simple. It follows that the distributions by genus and region size in the unlabeled case are very difficult, in general, to find.

A pseudograph is a generalization of a graph where loops and multiple edges are allowed. In this thesis, the number and distribution of 2-cell imbeddings of pseudographs are examined. Specifically, we concentrate on two classes of pseudographs which are simple in their structure but have a rich variety of imbeddings. These are the class of bouquets (one vertex, many loops) and the class of dipoles (two vertices, many edges). By subdividing each edge of a dipole we recognize the result as the
complete bipartite graph $K_{2,n}$.

The topological spaces on which these pseudographs are imbedded are the surfaces introduced earlier and their generalizations to pseudosurfaces and generalized pseudosurfaces. The latter two classes of spaces are of interest because of a correspondence between $2$-cell imbeddings of bouquets and dipoles on these spaces and the totality of pseudograph imbeddings on surfaces (and the generalization to hypergraph imbeddings). Chapter VI is devoted to establishing and exploiting the correspondence.

In Chapter II, pseudographs, the above topological spaces, and imbeddings are defined consistent with the purposes of this thesis. In Chapter III, the enumerative techniques to be used and some known results to be applied in the analysis are given for reference. Chapters IV and V are devoted to the investigation of the imbeddings of the bouquet and dipole, respectively. We conclude in Chapter VI with the enumeration of the imbeddings of all pseudographs (and hypergraphs).
CHAPTER II

PSEUDOGRAPHS AND THEIR IMBEDDINGS

2.1. Pseudographs

A pseudograph $H$ is an ordered triple $(A, E, V)$ where $A$ is a nonempty finite set, $E$ is a partition of $A$ into 2-sets (hence, $|A|$ is even), and $V$ is a partition of $A$. The sets $A$, $E$, and $V$ are referred to as the arc set, edge set, and vertex set, respectively. If more than one pseudograph is present then $A(H)$, $E(H)$, and $V(H)$ denote the above sets for the pseudograph $H$. The members of $A$ are arcs, the cells of the partition $E$ are edges, and the cells of $V$ are vertices.

It is important to note how this definition of pseudograph differs from the more predominant 'graph with multiple edges and self adjacencies allowed.' First, it is the arcs which receive the labels and second, isolated vertices can not occur (and are of little interest in topological graph theory). The definition is motivated by the goal to enumerate imbeddings of graphs on surfaces without becoming overwhelmed by their topology.

For us an imbedding will be a combinatorial object (called a map) with a well-known interpretation as a topological object. Specifically, each pseudograph can be represented as a 1-dimensional CW-complex as shown in Figure 2.1. The vertices are the 0-cells and the edges the 1-cells. Each arc of $A$
corresponds to a directed edge of the complex directed away from the vertex which contains that arc in the pseudograph. If both arcs of an edge are contained in the same vertex then the edge is called a loop. The combinatorial imbeddings of a pseudograph will correspond to equivalence classes of imbeddings of the corresponding CW-complex. The reader is referred to the article by Gross, Robbins, and Tucker [10] for more details.

Arc set $A = \{ a_1, a_2, a_3, a_4 \}$

Edge set $E = \{ \{a_1, a_2\}, \{a_3, a_4\} \}$

Vertex set $V = \{ \{a_1, a_2, a_3\}, \{a_4\} \}$

Pseudograph $H = (A, E, V)$

Figure 2.1. The representation of a pseudograph as a 1-dimensional CW-complex.

To illustrate how a graph without isolated vertices corresponds to a pseudograph consider the complete graph $K_3$ with vertex set $V(K_3) = \{ u, v, w \}$ and edge set $E(K_3) = \{ uv, uw, vw \}$. Define a pseudograph $H$ as follows:

$A(H) = \{ u', v', u'', v'', v''', w''' \}$,

$E(H) = \{ \{ u', v' \}, \{ u'', w'' \}, \{ v''', w''' \} \}$, and

$V(H) = \{ \{ u', u'' \}, \{ v', v''' \}, \{ w'', w''' \} \}$.

Thus, $H$ has six arcs, three edges, and three vertices as expected. $H$ is diagrammed in Figure 2.2 and it is clear that the original labeled graph can be recovered from the diagram.
In topological graph theory it is the connected graphs and their imbeddings which are of interest. To define a connected pseudograph we first define an edge and a vertex to be incident if their intersection is nonempty. Two vertices are adjacent if they are incident with a common edge and are said to be joined by that edge. Similarly, two edges are adjacent if they are incident with a common vertex. A walk from a vertex $u$ to a vertex $w$ is an alternating sequence of vertices and edges beginning at $u$ and ending at $w$, $u=v_1, e_1, v_2, e_2, \ldots, v_n=w$, where consecutive pairs are incident. If a walk exists between every two distinct vertices of a pseudograph then it is connected. We note that if a pseudograph is connected then the CW-complex associated with it is connected as a topological space.

A pseudograph $H=(A,E,V)$ is isomorphic to a pseudograph $H'=(A',E',V')$ if there exists a bijection of their arc sets $f:A\rightarrow A'$ such that $\{\{f(a): a\in e\}: e\in E\} = E'$ and $\{\{f(a): a\in v\}: v\in V\} = V'$. 

Figure 2.2. The representation of a labeled graph as a pseudograph.
That is, \( f \) induces a bijection between the edge sets and vertex sets. If \( f \) is an isomorphism between \( H \) and \( H' \) we write \( f:H \rightarrow H' \) (although the domain and range of \( f \) are the arc sets). The relation 'is isomorphic to' is an equivalence relation on the set of all pseudographs. The equivalence classes are called unlabeled pseudographs and are represented as 1-dimensional CW-complexes with no labels on the vertices, arcs, or edges. If \( H \) and \( H' \) are isomorphic pseudographs we write \( H \cong H' \).

An isomorphism of a pseudograph with itself is an automorphism and the totality of these together with composition of functions form the automorphism group of the pseudograph. If \( H = (A,E,V) \) is a pseudograph we note that each automorphism of \( H \) is a permutation of the arc set \( A \). Denoting the symmetric group of all permutations on \( A \) by \( \Sigma_A \) and the automorphism group of the pseudograph by \( \Gamma(H) \) we have \( \Gamma(H) \leq \Sigma_A \) (\( \Gamma(H) \) is a subgroup of \( \Sigma_A \)). If \( G \) is a graph then members of its automorphism group are vertex mappings while for a pseudograph they are arc mappings; hence, if some graph is also realized as a pseudograph, the automorphism groups will be isomorphic as groups but the degrees of their representations as permutation groups will be different. This is mentioned now to avoid confusion later.

Two important isomorphism classes of pseudographs which will be extensively investigated are bouquets and dipoles. A bouquet with \( n \) loops, denoted \( B_n \), is a single vertex and \( n \) loops as shown in Figure 2.3. A dipole with \( n \) edges, denoted \( D_n \),
has two vertices joined by n edges as shown in Figure 2.3. Both bouquets and dipoles are members of a larger class of pseudographs called pseudostars. A pseudostar is a pseudograph which contains a 'central' vertex incident with every edge. Several examples are also shown in Figure 2.3. A multistar is a pseudostar without loops. Enumerative techniques developed for bouquets and dipoles often apply equally well to pseudostars.

![Figure 2.3. A bouquet, dipole, multistar and pseudostar.](image)

2.2. Imbedding Spaces

A surface is a closed orientable 2-manifold and is characterized as a sphere with handles, where the number of handles gives the genus of the surface. The surface of genus g is denoted $S_g$. Thus, $S_0$ is the sphere and $S_1$ the torus. This class of topological spaces is enlarged to pseudosurfaces as follows. Let X be a partition of a finite set of points of a surface $S_g$. The topological space obtained by identifying the points in each cell of the partition is a pseudosurface of genus g (note that there are an infinite number of distinctly different pseudosurfaces having the same genus). Each point of the pseudosurface
resulting from an identification of the points of a cell of \( X \) (where the cell contains at least two points) is called a **singular point**. These are precisely the points of the space which fail to have a neighborhood homeomorphic to a 2-cell.

If instead of a single surface we take a finite collection of surfaces and identify the points in each cell of a partition of a finite number of points the resulting space is a **generalized pseudosurface**, whenever the space is connected. Deleting the singular points from a generalized pseudosurface may result in a disconnected topological space. The closure of each component obtained in this manner is called a **block** and is easily seen to be a pseudosurface. The genus of the generalized pseudosurface is taken to be the sum of the genera of its blocks. Figure 2.4 illustrates some of these ideas. Thus, a surface is a pseudosurface is a generalized pseudosurface and each of these is a suitable candidate for the 'drawing' of a pseudograph.

If \( x \) is a singular point resulting from the identification of \( m \) points, \( m > 1 \), then \( x \) has **degree of singularity** equal to \( m \). The **characteristic** \( \chi \) of a surface of genus \( g \) is \( 2-2g \). The characteristic of a pseudosurface \( Q \) of genus \( g \) with singular points of degrees \( m_1, m_2, \ldots, m_n \) is defined to be

\[
\chi(Q) = 2 - 2g - \sum_{i=1}^{n} (m_i - 1),
\]

which agrees with the above in case \( Q \) is a surface. If \( Q \) is a generalized pseudosurface of genus \( g \) with \( b \) blocks and singular points of degrees \( m_1, m_2, \ldots, m_n \) then its characteristic is
Figure 2.4. Examples of the imbedding spaces.

defined to be

\[ \chi(Q) = 2b - 2g - \sum_{i=1}^{n} (m_i - 1). \]

The definitions of characteristic are of course motivated by the Euler-Poincare formula \( p - q + r = \chi(Q) \) which relates graphical parameters \( p \) (number of vertices) and \( q \) (number of edges) to imbedding parameters \( \chi \) and \( r \) (number of regions) when a graph is 2-cell imbedded (see Section 2.3) on a surface \( Q \). The formula remains valid for pseudograph imbeddings in case \( Q \) is a generalized pseudosurface.

2.3. Imbeddings and Maps

Given a 1-dimensional CW-complex \( C \), an imbedding of \( C \) into a generalized pseudosurface \( Q \) is a continuous injection \( i: C \to Q \) such that each singular point of \( Q \) is the image of a vertex of
the complex. These vertices are called singular vertices. If every component of \( Q - i(C) \) is a 2-cell then \( i \) is a 2-cell imbedding (this requires that \( C \) be connected).

Two imbeddings of \( C \) in \( Q \), \( i:C \rightarrow Q \) and \( j:C \rightarrow Q \) are equivalent if there is an orientation preserving homeomorphism \( h \) of \( Q \) onto \( Q \) such that \( h \circ i = j \). Equivalent and inequivalent imbeddings of a complex are depicted in Figure 2.5. Take note that the counterclockwise cyclic orders of the arcs directed away from each vertex are identical for the equivalent imbeddings shown and are different for the inequivalent pair. It is now well known that these cyclic lists characterize the equivalence classes of imbeddings.

![Equivalent imbeddings on a sphere](image1)

Equivalent imbeddings on a sphere

![Inequivalent but congruent imbeddings on a sphere](image2)

Inequivalent but congruent imbeddings on a sphere

Figure 2.5. Equivalence and congruence of pseudograph imbeddings.

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Two imbeddings \( i:C \rightarrow Q \) and \( j:C \rightarrow Q \) are **congruent** if there exists an orientation preserving homeomorphism \( h:Q \rightarrow Q \) such that \( h\circ i(C) = j(C); \) that is, the image of \( C \) under \( h\circ i \) is the same as that under \( j \). The inequivalent imbeddings of Figure 2.5 are seen to be congruent. Recognizing \( i(C) \) and \( j(C) \) as the same pseudograph, we give an equivalent definition of congruence as the existence of an automorphism of the pseudograph which extends to an orientation preserving homeomorphism of \( Q \).

Intuitively, two drawings depicting imbeddings of \( C \) into \( Q \) represent congruent imbeddings if after removing all labels the drawings 'look the same'. Hence, we extend congruence to the case where \( C \) and \( C' \) are possibly different complexes as follows. Two imbeddings \( i:C \rightarrow Q \) and \( j:C' \rightarrow Q \) are **congruent** if there exists an isomorphism of the pseudographs which extends to an orientation preserving homeomorphism of \( Q \).

Henceforth, an equivalence class of imbeddings is referred to as a **labeled** imbedding and a congruence class as an **unlabeled** imbedding. For a connected pseudograph \( H \), realized as a 1-dimensional CW-complex, the goal of this thesis is to enumerate the labeled and unlabeled 2-cell imbeddings on generalized pseudosurfaces. Here lies a problem. The number of these imbeddings may be infinite for a fixed pseudograph. To see this consider the imbedding of the bouquet with one loop shown in Figure 2.6. It is clearly 2-cell, but by adding as many blocks as desired, each block homeomorphic to a sphere,
we obtain an infinite collection of 2-cell imbeddings on generalized pseudosurfaces where no two of the spaces are homeomorphic. Moreover, the number of regions can be made arbitrarily large and most of these regions are 0-sided.

![Figure 2.6](image)

Figure 2.6. An infinite collection of inequivalent 2-cell imbeddings of \( B_1 \) on generalized pseudosurfaces.

For this reason we assume that each region boundary of a 2-cell imbedding is at least 1-sided, which is equivalent to requiring each block of a generalized pseudosurface to have some edge imbedded on it. In this manner each of these 2-cell imbeddings corresponds to a rotation scheme, a method introduced by Hefter [13] and refined by Edmonds [6], which naturally leads to the idea of a map. This correspondence replaces the topological imbeddings with a combinatorial object and it is in this setting that we count imbeddings.

To this end let \( H=(A,E,V) \) be a pseudograph. A rotation at a vertex \( v \) is a permutation of the arcs in \( v \) and is denoted \( \rho_v \). A rotation on \( V \) is the permutation on the arc set of \( H \) given by a rotation at each vertex. Denoting this permutation
by \( \rho_v \) we have \( \rho_v(a) = \rho_v(a) \) if \( a \in V \). Similarly, we define a rotation at an edge \( e \) as a permutation of the arcs in \( e \) and denote it by \( \rho_e \). The rotation on \( E \) is the permutation of the arc set of \( H \) given by a rotation at each edge and is denoted \( \rho_E \). Thus, \( \rho_v \) and \( \rho_E \) are members of the symmetric group \( \Sigma_A \) on the arc set. A map \( M \) is an ordered triple \( M = (H, \rho_E, \rho_V) \) where \( H \) is a pseudograph, \( \rho_E \) is a rotation on the edge set of \( H \) and \( \rho_V \) is a rotation on the vertex set of \( H \).

If \( H \) is a connected pseudograph then each map \( M = (H, \rho_E, \rho_V) \) with \( \rho_E \) a fixed-point free involution can be identified with a labeled 2-cell imbedding on a generalized pseudosurface \( Q \) where each region is at least 1-sided. Moreover, summarizing the work of Garman [7], Jacques [15], Petroelje [22], and many others, we have the following, where \( \|\alpha, \beta, \cdots\| \) denotes the number of orbits of the subgroup of \( \Sigma_A \) generated by \( \alpha, \beta, \cdots \).

**Theorem 2.1.**

(i) The region boundaries are the orbits of the permutation \( \rho_V \circ \rho_E \) and each cycle appearing in the disjoint cycle representation of this permutation describes a closed walk along an oriented region boundary.

(ii) A vertex \( v \) occupies a singular point of degree \( m \) if and only if the rotation at \( v \), \( \rho_v \), has \( m \) orbits; hence, \( Q \) is a surface if and only if \( \rho_v \) is cyclic for each \( v \) in \( V(H) \).

(iii) The number of blocks of the generalized pseudosurface \( Q \) is \( \|\rho_E, \rho_V\| \); hence, \( Q \) is a pseudosurface if and only if the
subgroup \langle \rho_E, \rho_V \rangle \text{ of } \Sigma_A \text{ is transitive.}

(iv) The genus \( g \) of \( Q \) is given by

\[
g = \frac{1}{2} (||\rho_E|| - ||\rho_V|| - ||\rho_V \circ \rho_E||).
\]

These ideas are illustrated with an example. Let the pseudograph \( H \) be a bouquet of three loops, \( B_3 \), with arc set \( A = \{1, 2, 3, 4, 5, 6\} \), edge set \( E = \{\{1, 2\}, \{3, 4\}, \{5, 6\}\} \), and vertex set \( V = \{\{1, 2, 3, 4, 5, 6\}\} \). Figure 2.7 depicts a 2-cell imbedding of the corresponding complex. It is evident from the drawing that the vertex is singular of degree three, the generalized pseudosurface \( Q \) has two blocks, one of the blocks has genus 0, and the other block has genus 1 (hence, \( Q \) has genus 1).

Figure 2.7. A 2-cell imbedding of \( B_3 \) and the punctured neighborhoods of its vertex.

To obtain the map corresponding to this imbedding proceed
as follows. Orient the generalized pseudosurface. Each vertex \( v \) has a neighborhood which is a finite number (its degree of singularity) of 2-cells identified at a single point occupied by the vertex. Deleting this point from the neighborhood we note that each component is an oriented punctured disk as shown in Figure 2.7. On each of these we list the counterclockwise order of the arcs and assemble these lists into a rotation at \( v \), which in the example is \( \rho_v = (1)(2)(345) \). Doing this for every vertex present we obtain the rotation on \( V \), \( \rho_V \). For a pseudograph there is but one choice for the rotation on the edge set which in this example is \( \rho_E = (12)(34)(56) \). Hence, the map is \((B_2, (12)(34)(56), (1)(2)(345))\).

We then observe that the orbits of \( \langle \rho_E, \rho_V \rangle \) are \( \{1,2\} \) and \( \{3,4,5,6\} \) corresponding to the arcs imbedded on the two blocks. Also, \( \rho_V \circ \rho_E = (12)(3546) \) so the two regions have boundaries consisting of arcs 1 and 2 (on the block of genus 0), and 3, 5, 4, and 6 (on the block of genus 1). Moreover, if the boundary of each region is traversed while keeping the region on the right (which is possible in the oriented case) then the order of appearance of the arcs is the same as the order appearing in \( \rho_V \circ \rho_E \). The genus \( g \) of \( Q \) according to Theorem 2.1(iv) is \( g = 2 - \frac{1}{2}(3-3+2) = 1 \) and the characteristic \( \chi \) of \( Q \) is determined as \( \chi = 2 \cdot 2 - 2 \cdot 1 - (3-1) = 4 - 2 - 2 = 0 \). Since \( p-q+r = 1 - 3 + 2 = 0 \) the imbedding satisfies \( p-q+r = \chi \).

We now define isomorphic maps whose isomorphism classes correspond to unlabeled 2-cell imbeddings. Two maps \( M=(H, \rho_E, \rho_V) \)
on $H=(A,E,V)$ and $M'=(H',E',V')$ on $H'=(A',E',V')$ are isomorphic if there exists a pseudograph isomorphism $f:H\rightarrow H'$ such that the following diagrams are commutative.

$$
\begin{array}{c}
A \xrightarrow{f} A' \\
\downarrow \rho_E \downarrow \rho_{E'} \\
A \xrightarrow{f} A'
\end{array} \quad \begin{array}{c}
A \xrightarrow{f} A' \\
\downarrow \rho_V \downarrow \rho_{V'} \\
A \xrightarrow{f} A'
\end{array}
$$

If this is the case we say $f$ is a map isomorphism and write $f:M\rightarrow M'$ (although the domain and range of $f$ are the arc sets) and $M \cong M'$. From the diagrams we see that $f\rho_E f^{-1} = \rho_E$ and $f\rho_V f^{-1} = \rho_V$, so that $f\rho_V \circ \rho_E f^{-1} = f\rho_V f^{-1} f\rho_E f^{-1} = \rho_V \circ \rho_E$. and we conclude that the oriented region boundaries of the labeled 2-cell imbeddings corresponding to $M$ and $M'$ are preserved by $f$. Hence, the labeled imbeddings are congruent.

If $H$ and $H'$ are the same pseudograph the isomorphic maps are automorphic. The isomorphism $f$ in this case becomes a map automorphism and the totality of these is easily seen to be a subgroup of the pseudograph automorphism group. Denote the map automorphism group of $M=(H,\rho_E,\rho_V)$ by $\Gamma(M)$ then $\Gamma(M)<\Gamma(H)<\Sigma_A$. Moreover, the index $[\Gamma(H):\Gamma(M)]$ of $\Gamma(M)$ in $\Gamma(H)$ measures the number of labeled 2-cell imbeddings of $H$ congruent to the imbedding corresponding to $M$ (see Biggs and White[3]).

In the previous example $H$ was $B_3$ with pseudograph automorphism group of order $3!\cdot2^3 = 48$ (the group is the...
wreath product of $\Sigma_3$ about $\Sigma_2$). The map
\[ M = ( B_3, (12)(34)(56), (1)(2)(3546) ) \] has map automorphism
group generated by the permutations (12) and (3546). The
order of $\Gamma(M)$ is then 8 and we have $[\Gamma(H):\Gamma(M)] = 48/8 = 6$.
The six inequivalent but congruent imbeddings are shown in
Figure 2.8.

Figure 2.8. The six inequivalent imbeddings of $B_3$ congruent to
the imbedding shown in Figure 2.7.
CHAPTER III

ENUMERATIVE TECHNIQUES

3.1 Counting Series

The emphasis of this thesis is enumeration. Specifically, we determine how many pseudograph imbeddings there are as a function of some parameters such as size and order of the pseudograph or number of regions of the imbedding. Often the results are encoded as a series which can then be manipulated to obtain even more results. The definitions, notation and techniques to be introduced shortly (here and in the next section) can be found in the excellent book on enumeration by Harary and Palmer [11].

Let \( \{a_n\} \) be an infinite sequence of polynomials; the infinite series \( \sum_{n=0}^{\infty} a_n x^n \) is the exponential counting series for \( a_n \) and the series \( \sum_{n=0}^{\infty} a_n x^n \) is the ordinary counting series. The former is often used for labeled problems while the latter is used for unlabeled problems. If \( A(x) \) is a counting series we write \([x^n]A(x)\) as the coefficient of the \(n^{th}\) power of \(x\).

If two exponential counting series are related by
\[ 1 + B = \exp(A) = 1 + A + A^2/2! + A^3/3! + \cdots \]
then their coefficients are related by the recursion equation
\[ a_n = b_n - \sum_{k=1}^{n-1} \binom{n-1}{k-1} a_k b_{n-k} \]  \hspace{1cm} (3.1)

We will have frequent need of this result and illustrate how it occurs.
Suppose for each \( n \), \( b_n \) is a polynomial in \( y \), where the coefficient of \( y^r \) is the number of labeled 2-cell imbeddings with \( r \) regions on generalized pseudosurfaces of the bouquet \( B_n \).

Suppose \( a_n \) is the polynomial enumerating these same imbeddings on pseudosurfaces. Let \( B(x) = \sum_{n=1}^{\infty} b_n \frac{x^n}{n!} \) and \( A(x) = \sum_{n=1}^{\infty} a_n \frac{x^n}{n!} \).

As will often be the case the series \( B(x) \) enumerating imbeddings on generalized pseudosurfaces is much easier to determine than \( A(x) \). To illustrate how the two series are connected, Figure 3.1 shows how \( m \) imbeddings of bouquets on pseudosurfaces (counted by \( A \)) determine an imbedding on a generalized pseudosurface (counted by \( B \)).

![Diagram](image)

**Figure 3.1.** Obtaining a generalized pseudosurface imbedding of the bouquet from pseudosurface imbeddings.

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Moreover, if the $i^{th}$ imbedding has $n_i$ loops and $r_i$ regions, $i=1, 2, \ldots, m$, then the resulting imbedding on the generalized pseudosurface has $\Sigma n_i$ loops and $\Sigma r_i$ regions. It is essential that these index parameters be additive in this manner. It is easy to see that every imbedding on generalized pseudosurfaces is obtained in this way. However, the generalized pseudosurface imbedding shown in Figure 3.1 could have been obtained in $m!$ ways from the same pseudosurface imbeddings by simply rearranging them. Hence, we divide the count by the number of permutations, which is $m!$. In other words, the imbeddings on generalized pseudosurfaces that have $m$ blocks are enumerated by $A^m/m!$. In total then we have

$$B = \frac{A^1}{1!} + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots + \frac{A^m}{m!} + \cdots,$$

or $B+1 = \exp(A)$.

The series $A(x)$ can then be determined from the series $B(x)$ by comparing coefficients in $A = \log(B+1) = \frac{B^1}{1} - \frac{B^2}{2} + \frac{B^3}{3} - \cdots$ or by using the recursion equation given previously.

The technique of obtaining the labeled 2-cell imbeddings on generalized pseudosurfaces of a class of pseudographs from the imbeddings on pseudosurfaces applies equally well to the class of labeled dipoles. The labeled dipole set is taken to be

$$D = \left\{ \begin{array}{c} 1^+ \circlearrowright 1^+ \circlearrowleft 1^+ \circlearrowright 3^+ \\ 1^- \circlearrowleft 1^- \circlearrowleft 1^- \circlearrowleft \end{array} \right\}.$$

Hence, we obtain a generalized pseudosurface imbedding by identifying the '+' vertices as one vertex and the '-' vertices as
one vertex for a set of pseudosurface imbeddings of members of D. Moreover, if a member of D is 2-cell imbedded on a
generalized pseudosurface with m blocks then the pseudograph
imbedded on each block can be identified with a unique member
of D. Thus, every 2-cell imbedding on a generalized pseudosurface
can be obtained by identifying corresponding vertices of dipoles
2-cell imbedded on pseudosurfaces.

If \( A(x) \) and \( B(x) \) are ordinary counting series enumerating
the unlabeled 2-cell imbeddings on pseudosurfaces and
generalized pseudosurfaces, respectively, then \( A \) and \( B \) are
related by \( B(x)+1 = \exp \sum_{k=1}^{\infty} \frac{A(x^k)}{k} \). This has been shown to be the case by Riddel [26] and follows from Polya’s Enumeration
Theorem. The coefficients of \( A(x) \) are found from those of \( B(x) \)
by comparing coefficients in

\[
\sum_{k=1}^{\infty} \frac{A(x^k)}{k} = \log(B(x)+1) = \frac{B_1}{1} - \frac{B_2}{2} + \frac{B_3}{3} - \cdots . \tag{3.2}
\]

Since the bouquet has but one vertex, the unlabeled 2-cell
imbeddings on generalized pseudosurfaces are obtained from the
pseudosurface imbeddings by identifying the vertices and so the
series enumerating the imbeddings satisfy the above relationship.
In the case of the unlabeled dipole however, it is first necessary
to distinguish between the two vertices by rooting one of them.
The series enumerating unlabeled 2-cell imbeddings of rooted
dipoles on generalized pseudosurfaces and pseudosurfaces then
satisfy the above relationship.

Having used the relationship to obtain the number of rooted
unlabeled 2-cell imbeddings on pseudosurfaces from those on generalized pseudosurfaces we are obligated to remove the root to obtain the desired information. However, in anticipation of the results of Chapter VI, it is mentioned now that the rooted imbeddings of the dipole have a very important application. They count the totality of all unlabeled hypermaps; these are natural generalizations of maps and will be defined in Chapter VI.

3.2. Graph Superposition

In Chapters IV and V it will be shown how some imbeddings of the bouquet and dipole correspond to the superposition of digraphs on the same vertex set. The enumeration theorem of Redfield [25] and a generalization provided by Palmer and Robinson [21] lend themselves to the enumeration of the corresponding unlabeled imbeddings. The definitions and notation needed to apply the theorems is now introduced.

Let \( G \) be a permutation group with object set \( X = \{x_1, x_2, \ldots, x_n\} \). Each permutation \( \alpha \) in \( G \) can be written uniquely as a product of disjoint cycles. We let \( j_k(\alpha), k=1,2,\ldots,n \), be the number of cycles of length \( k \) in the disjoint cycle representation of \( \alpha \). The cycle type of \( \alpha \) is the \( n \)-tuple \( J(\alpha) = (j_1, j_2, \ldots, j_n) \) where \( j_k = j_k(\alpha) \) for \( k=1,2,\ldots,n \). Note that \( J(\alpha) \) also gives a partition of the integer \( n \) which has \( j_k \) parts equal to \( k \) satisfying \( \sum k \cdot j_k = n \). We write \( J \upharpoonright n \) to denote a partition \( J \) of the integer \( n \). The notation \( 1^{j_1}2^{j_2}\ldots n^{j_n} \) is also used to denote the partition \( (j_1, j_2, \ldots, j_n) \) where \( k^{j_k} \) is omitted if \( j_k \) is zero. For example,
\( J=(2, 1, 0, 0, 1, 0, 0, 0, 0) \) is a partition of 9 with two parts equal to 1, one part equal to 2, and one part equal to 5, which is also denoted \( 1^2 2^1 5^1 \). The permutation \( \alpha = (x_1)(x_4)(x_2 x_3)(x_5 x_6 x_7 x_8) \) has cycle-type \( J \).

The cycle index of the permutation group \( G \), denoted \( Z(G) \), is the polynomial in the variables \( s_1, s_2, \ldots, s_n \) defined by

\[
Z(G) = \frac{1}{|G|} \sum_{\alpha \in G} \prod_{k=1}^{n} j_k(\alpha). 
\]

If it is necessary to display the variables we write \( Z(G; s_1, s_2, \ldots, s_n) \). Letting \( N_J \) denote the number of permutations in \( G \) of cycle-type \( J \) we have

\[
Z(G) = \frac{1}{|G|} \sum_{J \in \text{perm} \Sigma} N_J \cdot \prod_{k=1}^{n} s_k^{j_k(\alpha)}. 
\]

Three permutation groups of particular importance in this thesis are the symmetric group of all permutations on \( X \), denoted \( \Sigma_X \) (or \( \Sigma_n \) if \( X=\{1, 2, \ldots, n\} \)), the cyclic groups of order \( n \), each denoted \( C_n \), generated by a full cycle, \((x_1 x_2 \cdots x_n)\), in \( \Sigma_X \), and the identity permutation group of degree \( n \), denoted \( E_n \), which is generated by the identity permutation of \( \Sigma_X \).

There are \( \frac{n!}{\prod_{k=1}^{n} k^{j_k(\alpha)}} \) members of \( \Sigma_X \) that have cycle-type \( J=(j_1, j_2, \ldots, j_n) \) (see [11]) and, hence,

\[
Z(\Sigma_X) = \frac{1}{n!} \sum_{J \in \text{perm} \Sigma} \frac{n!}{\prod_{k=1}^{n} k^{j_k(\alpha)}} \prod_{k=1}^{n} s_k^{j_k(\alpha)}. 
\]

For example,

\[
Z(\Sigma_2) = \frac{1}{2}(s_1^2 + s_2) \quad \text{and} \quad Z(\Sigma_3) = \frac{1}{6}(s_1^3 + 2s_1s_2 + 3s_3). 
\]

The cycle index of \( C_n \) was determined by Redfield [25] to be

\[
Z(C_n) = \frac{1}{n} \sum_{d|n} \phi(d) s_d^n, \quad \text{where} \ \phi \ \text{is the euler phi-function}. 
\]

For example, \( Z(C_4) = \frac{1}{4}(s_1^4 + s_2^2 + 2s_4) \). The cycle index of \( E_n \) is \( s_1^n \).

If \( A \) and \( B \) are permutation groups with object sets \( X \) and \( Y \),
respectively, the wreath product of $A$ about $B$, denoted $A[B]$, is a permutation group with object set $X \times Y$ (see [11]). The cycle index of $A[B]$ is determined from those of $A$ and $B$ by a process resembling composition of functions. Explicitly, each variable $s_k$ appearing in $Z(A)$ is replaced by $Z(B; s_k, s_{2k}, s_{3k}, \ldots)$. For example,

$$Z(\Sigma_2[C_2]) = \frac{1}{2} \left( \left\lfloor \frac{1}{2} \left( s_1^2 + s_2 \right) \right\rfloor^2 + \left\lfloor \frac{1}{2} \left( s_2^2 + s_4 \right) \right\rfloor^2 \right)$$

$$= \frac{1}{8} \left( s_1^4 + 2s_1^2s_2 + 3s_2^2 + 2s_4 \right).$$

Redfield introduced two operations, $\cap$ and $\cup$, which operate on cycle indexes. Although he defined these operations for an arbitrary number of cycle indexes we only need to define them for two.

The cap operation is first defined for a pair of monomials $s_1^{i_1} s_2^{i_2} \cdots s_n^{i_n}$ and $s_1^{j_1} s_2^{j_2} \cdots s_n^{j_n}$, by

$$\left( s_1^{i_1} s_2^{i_2} \cdots s_n^{i_n} \right) \cap \left( s_1^{j_1} s_2^{j_2} \cdots s_n^{j_n} \right) = \left\{ \begin{array}{cl} \prod_{k=1}^{n} j_k \cdot k! & \text{if } i_k = j_k \text{ for all } k \\ 0 & \text{otherwise.} \end{array} \right.$$  

The cap operation is then extended linearly. Thus, cap operates on two cycle indexes to produce a number. The cup operation is defined in terms of cap to produce a polynomial as follows. For the monomials above we have

$$\left( s_1^{i_1} s_2^{i_2} \cdots s_n^{i_n} \right) \cup \left( s_1^{j_1} s_2^{j_2} \cdots s_n^{j_n} \right) = \left( s_1^{i_1} s_2^{i_2} \cdots s_n^{i_n} \right) \cap \left( s_1^{j_1} s_2^{j_2} \cdots s_n^{j_n} \right) s_1^{i_1} s_2^{i_2} \cdots s_n^{i_n}.$$

The cup operation is then extended linearly to arbitrary polynomials in the variables $s_1, s_2, \ldots, s_n$. For example,
\[
 Z(C_4) \cap Z(\Sigma_2[C_2]) = \frac{1}{4} (s_1^4 + s_2^2 + 2s_4) \cap \frac{1}{8} (s_1^4 + 2s_1^2s_2 + 3s_2^2 + 2s_4)
 = \frac{1}{32} (s_1^4 \cap s_1^4 + 3s_2^2 \cap s_2^2 + 4s_4 \cap s_4)
 = \frac{1}{32} (24 + 3 \cdot 8 + 4 \cdot 4) = 2 \quad \text{and}
 Z(C_4) \cup Z(\Sigma_2[C_2]) = \frac{1}{32} (24s_1^4 + 24s_2^2 + 16s_4).
\]

The cap and cup operation are used in the enumeration of graph superpositions, a concept we now define. Let \( G_1 \) and \( G_2 \) be two unlabeled graphs (digraphs) which have the same number of vertices, and let \( \Gamma(G_1) \) and \( \Gamma(G_2) \) be their automorphism groups, respectively. Color the edges of \( G_1 \) 'solid' and those of \( G_2 \) 'dashed'. The two graphs can then be drawn on the same set of unlabeled vertices and distinguished by the color of their edges. The resulting pseudograph is a superposition. Redfield [25] and Read [24] observed that the number of different superpositions is \( \Gamma(G_1) \cap \Gamma(G_2) \). The observation is a particular application of Redfield's Enumeration Theorem.

As an example (a preview of what is to come) consider the superpositions of a directed 4-cycle and the digraph with two components, each a directed 2-cycle as shown in Figure 3.2. Their automorphism groups are \( C_4 \) and \( \Sigma_2[C_2] \), respectively. Hence, \( \Gamma(G_1) \cap \Gamma(G_2) = \Gamma(C_4) \cap \Sigma_2[C_2] = 2 \) as determined in a previous example. The two different superpositions are also shown in the figure.
Figure 3.2. The two different superpositions of $G_1$ and $G_2$.

Note that the automorphism group of the superposition on the left when considered as a subgroup of the symmetric group on the four vertices is the group $C_2[E_2]$. The automorphism group of the superposition on the right is $C_4$. The sum of the cycle indexes of these two groups is

$$Z(C_2[E_2]) + Z(C_4) = \frac{1}{2}((s_1^2 + s_2^2) + \frac{1}{4}(s_1^4 + s_2^2 + 2s_4))$$

$$= \frac{1}{4}(3s_1^4 + 3s_2^2 + 2s_4).$$

The cup of $Z(\Gamma(G_1))$ with $Z(\Gamma(G_2))$ was previously determined to be $\frac{1}{32}(24s_1^4 + 24s_2^2 + 16s_4)$ which simplifies to the above sum of cycle indexes. That these two polynomials are the same is a particular instance of Redfield's Decomposition Theorem [25]. This theorem will be used in Chapter IV to enumerate the surface imbeddings of the bouquet by their symmetries.

3.3. Character Theory of the Symmetric Group

In Chapter II it was shown that all pertinent information about the 2-cell imbeddings of a pseudograph is contained in the
pairs of permutations \((\rho_E, \rho_V)\), both permutations being members of the symmetric group \(\Sigma_A\) on the arc set. The product of \(\rho_V\) by \(\rho_E\), denoted \(\rho_R\), contains all the desired information about the 2-cell regions of the imbedding. Hence, knowing \(\rho_V\), \(\rho_E\) and \(\rho_R\) it is possible to determine everything of interest concerning the imbedding.

When asked to determine the number of imbeddings having a particular region distribution, i.e., so many regions of a given size, we are being asked to determine in the group \(\Sigma_A\) the number of occurrences of \(\rho_V \circ \rho_E\) having a particular cycle type under suitable restrictions. Or when asked how many imbeddings have exactly \(r\) regions (a question related to the genus of the imbedding) we are being asked how often \(\rho_V \circ \rho_E\) has exactly \(r\) orbits.

These questions and others are addressed by considering the group algebra and irreducible characters of the symmetric groups. In general, if \(C_i\), \(C_j\), and \(C_k\) are conjugacy classes of a finite group, then the product of the members of \(C_i\) with the members of \(C_j\) can be written in the group algebra as a formal sum over all classes \(C_k\): \(C_i \cdot C_j = \sum n_{ijk} C_k\). The class multiplication coefficients, \(n_{ijk}\), are the numbers we desire to answer the previous questions and are interpreted as giving the number of ways members of \(C_i\) multiply members of \(C_j\) to give a fixed member of \(C_k\).

If \(\chi^{(i)}\) denote the irreducible character of the group \(\Gamma\) associated with the class \(C_i\) and \(f^{(i)}\) the degree of \(\chi^{(i)}\). The
value of $\chi^{(i)}$ on any member of class $C_j$ is denoted by $\chi_{ij}^{(i)}$; in particular, $\chi_{ij}^{(i)}$ is its value on the identity class so that

$$f^{(i)} = \chi_{ij}^{(i)}.$$  

A consequence of the orthogonality relations of irreducible characters is the following (see Curtiss and Reiner [5]).

$$n_{ijk} = \frac{h_i h_j}{|\Gamma|} \sum \frac{\chi_i \chi_j \chi_k}{\chi_1},$$  \hspace{1cm} (3.3)

where $h_i$ is the cardinality of the class $C_i$ and the sum extends over all irreducible characters. The difficulty in using (3.3) is lack of knowledge about the irreducible characters of the symmetric group. There is one case, however, where (3.3) can be used; if at least one of the conjugacy classes is that of an $n$-cycle (hereinafter referred to as a full cycle) in the symmetric group $\Sigma_n$ of degree $n$, then the recent results of Jackson [14] provide the necessary knowledge of the characters.

His results are extensive and can not be reproduced in their entirety here so the information we need is summarized. If one of the conjugacy classes indexed in (3.3) is that of the full cycles in $\Sigma_n$, then only $n$ terms of the sum appearing in (3.3) are nonzero and we have the following.

**Theorem 3.1 (Jackson).**

$$n_{ijk} = \frac{h_i h_j}{n!} \sum_{\lambda=0}^{n-1} \frac{\chi^{(\lambda)}_i \chi^{(\lambda)}_j \chi^{(\lambda)}_k}{\chi^{(\lambda)}_1},$$  \hspace{1cm} (3.4)

where, if $m$ indexes the class of cycle-type $(j_1,j_2,\ldots,j_n)$ then

$$\sum_{\lambda=0}^{n-1} \chi^{(\lambda)}_m y^\lambda = \frac{1}{1+y} \prod_{k=1}^n (1-(-y)^k)^{j_k}.$$  \hspace{1cm} (3.5)
Hence, by comparing coefficients the character values on each class can be calculated and from these the value of $n_{ijk}$. These results will be used to determine how many labeled 2-cell imbeddings on surfaces of the bouquet, dipole, and in general, multistars, have a particular distribution of region sizes.

If we are interested in finding out how many of these imbeddings have exactly $r$ regions then Jackson provides an explicit means for doing so in some cases. Following his notation, $e_r^{(d)}(N)$ denotes how often a fixed $dN$-cycle in $\Sigma_{dN}$ multiplies members of the class with cycle type $(0, \ldots, 0, j_d = N, 0, \ldots, 0)$ to obtain a permutation having exactly $r$ orbits. From these numbers it is possible to determine the number of labeled 2-cell imbeddings on surfaces which have exactly $r$ regions for bouquets and dipoles. The genus distribution of these two pseudographs follows immediately from this information.

The numbers $e_r^{(d)}(N)$ are given in terms of the Stirling numbers of the first and second kinds, which satisfy the respective equations:

$$x(x-1)(x-2)\cdots(x-n+1) = \sum_{m=0}^{n} s_n^{(m)} x^m,$$  \hspace{1cm} (3.6)

$$x^n = \sum_{m=0}^{n} S_n^{(m)} x(x-1)(x-2)\cdots(x-m+1).$$  \hspace{1cm} (3.7)

It is well known [4] that $|s_n^{(m)}| = (-1)^{n-m}$ $s_n^{(m)}$ is the number of permutations in $\Sigma_n$ which have exactly $m$ orbits. We present the following result of Jackson.
Theorem 3.2 (Jackson).

\[
e_r^{(d)}(N) = \frac{1}{(1+dN)d^{N-r}} \sum_{m=N+r}^{1+dN} d^m \binom{m}{r} S_m^{(m)} S_{m-r}^{(N)}.
\]

3.4. Unlabeled 2-cell Imbeddings on Surfaces

Mull, Rieper and White [20] have shown how to determine the number of congruence classes of 2-cell imbeddings on surfaces for an arbitrary connected graph. The technique is applicable to pseudographs by obtaining from the pseudograph a graph through subdivision of the edges. Preceding the four theorems from this reference we present some necessary definitions.

For a graph \(G\), \(V(G)\) is its vertex set, \(N(v)\) is the neighborhood of a vertex \(v\) consisting of all vertices adjacent to \(v\), and \(AutG\) is the group of automorphisms of \(G\). Each 2-cell imbedding of \(G\) on a surface is specified by a rotation scheme similar to that of a pseudograph presented in Section 2.3. At each vertex \(v\) there is given a cyclic permutation of the neighborhood of \(v\). The permutation is referred to as a rotation at \(v\), denoted \(\rho_v\), and the family of these indexed by the vertices of \(G\) is referred to as a rotation on \(G\) and denoted by \(\rho\). Thus, \(\rho = \{ \rho_v \}_{v \in V(G)}\) is the rotation on \(G\).

Two rotations on \(G\), \(\rho\) and \(\rho'\), are defined to be equivalent, if there exists \(\alpha\) in \(AutG\) such that \(\alpha \rho_v \alpha^{-1} = \rho'_v\) for each \(v\) in \(V(G)\). Equivalent rotations on \(G\) describe congruent imbeddings and, hence, the number of congruence classes is the same as

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the number of equivalence classes. Denote the set of congruence classes by \( C(G) \) and the set of rotations on \( G \) by \( R(G) \). We now define an action of \( \text{Aut}G \) on the members of \( R(G) \) with the orbits corresponding to the members of \( C(G) \).

For each \( \alpha \) in \( \text{Aut}G \) and \( \rho \) in \( R(G) \) let \( \alpha(\rho) \) be the rotation on \( G \) given by \( (\alpha(\rho))_\alpha(v) = \alpha\rho \alpha^{-1} \) for each \( v \) in \( V(G) \). Thus, \( \alpha\rho \alpha^{-1} \) is a rotation at \( \alpha(v) \) and, by definition, \( \alpha(\rho) \) is seen to be equivalent to \( \rho \). Hence, the orbits of this action are the equivalence classes of rotations on \( G \) and their number is \( |C(G)| \), which we desire. We now use Burnside's lemma to enumerate the orbits. The fixed set of \( \alpha \), denoted \( F(\alpha) \), is defined by \( F(\alpha) = \{ \rho \in R(G) : \alpha(\rho) = \rho \} \) and we have the following result.

**Theorem 3.3.** \( |C(G)| = \frac{1}{|\text{Aut}G|} \sum_{\alpha \in \text{Aut}G} |F(\alpha)| \).

It is often the case that not every automorphism contributes to the above sum, as the next theorem states.

**Theorem 3.4.** If \( \alpha \in \text{Aut}G \) fixes two adjacent vertices then either \( \alpha \) is the identity permutation or \( |F(\alpha)| = 0 \).

The next two theorems show how \( |F(\alpha)| \) can be determined. For \( \alpha \in \text{Aut}G \) and \( v \in V(G) \) define the fixed set at \( v \) to be the set of rotations at \( v \) fixed by \( \alpha \) under conjugation (the set may be empty). Denote this set by \( F_v(\alpha) \); then \( \rho_v \) is in \( F_v(\alpha) \) if and only if \( \alpha\rho_v\alpha^{-1} = \rho_v \). In the disjoint cycle representation of \( \alpha \) let \( l(v) \) be the length of the cycle containing \( v \); then we have the
Theorem 3.5. If $\alpha \in \text{Aut}G$ then $|F(\alpha)| = \prod_{v \in S} |F_v(\alpha^i(v))|$, where the product extends over a complete set $S$ of orbit representatives for the group $\langle \alpha \rangle$ acting on $V(G)$.

Denoting the restriction of $\alpha^i(v)$ to the neighbors of $v$ by $\alpha^i(v)|_{N(v)}$, Mull, Rieper, and White have shown that the cycle type of this permutation must have a particular form or else $|F_v(\alpha^i(v))|$ and, hence, $|F_v(\alpha)|$, is zero. For simplicity, let $n = |N(v)|$ and $\phi$ denote the euler phi-function; also, refer to any permutation whose orbits all have length $d$ as $d$-regular; then we have the following result.

Theorem 3.6.

$$|F_v(\alpha^i(v))| = \begin{cases} 
\phi(d) \left(\frac{n}{d} - 1\right)! d^{\frac{n}{d} - 1} & \text{if } \alpha^i(v)|_{N(v)} \text{ is } d\text{-regular} \\
0 & \text{otherwise.}
\end{cases}$$

The use of these four theorems usually proceeds by applying Theorems 3.4 and 3.5 to reduce the sum appearing in Theorem 3.3 to a few automorphisms and then calculating $|F(\alpha)|$ for these by using Theorems 3.5 and 3.6. The technique is illustrated in Chapters IV where the number of unlabeled imbeddings of the bouquet are calculated.
CHAPTER IV

2-CELL IMBEDDINGS OF THE BOUQUET

4.1. Introduction

The simplicity of the bouquet belies its importance in topological graph theory. Imbeddings of many other graphs are obtained by lifting an imbedding of the bouquet using voltage graph theory (see Gross [9]). Of special interest are Cayley graphs (see [29]) whose lifts are called Cayley maps by Biggs [2]. Also, by contracting a spanning tree of a connected graph to a single vertex a bouquet is obtained. And, as we will see in Chapter VI, information about the totality of 2-cell imbeddings of all pseudographs on surfaces is contained in the imbeddings of the bouquet on pseudosurfaces. Part of Chapter VI is devoted to establishing this correspondence and using it to enumerate the number of 2-cell imbeddings of all pseudographs on surfaces.

4.2. Labeled 2-cell Imbeddings

The arc set, edge set, and vertex set of the bouquet with n loops, $B_n$, are taken as follows:

$A(B_n) = \{1,2,3,\ldots,2n\}$,

$E(B_n) = \{\{1,2\}, \{3,4\}, \ldots, \{2n-1,2n\}\}$, and

$V(B_n) = \{\{1,2,3,\ldots,2n\}\}$. 

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The rotation on the edge set $E$ is then $(1\ 2)(3\ 4)\cdots(2n-1\ 2n)$. The number of possible rotations on the vertex set $V$ is $(2n)!$ giving the number of labeled 2-cell imbeddings of $B_n$ on generalized pseudosurfaces. From these we obtain the number on pseudosurfaces using the techniques outlined in Chapter 3.

For reference, the unlabeled 2-cell imbeddings and rotation schemes describing the labeled imbeddings of $B_2$ are shown in Figure 4.1. It is helpful to examine these drawings as the numbers are produced in what follows.

4.2.1. Enumeration by Number of Edges

For the labeled problem we use exponential counting series defined as follows.

\[ G_n = n! \left[ x^n \right] G(x) = \text{number of labeled 2-cell imbeddings of } B_n \text{ on generalized pseudosurfaces}, \]
\[ P_n = n! \left[ x^n \right] P(x) = \text{number above on pseudosurfaces}, \]
\[ S_n = n! \left[ x^n \right] S(x) = \text{number above on surfaces}. \]

There are $(2n)!$ possible rotations on $V$ giving

\[ G(x) = \frac{2x}{1!} + 24 \frac{x^2}{2!} + 720 \frac{x^3}{3!} + \ldots + (2n)! \frac{x^n}{n!} + \ldots \]

The series $G$ and $P$ are related by $G+1=\exp(P)$ and hence, using the recursion (3.1) given in Chapter 3,

\[ P_n = G_n - \sum_{k=1}^{n-1} \binom{n-1}{k-1} P_k G_{n-k}, \]

we find that

\[ P(x) = 2 \frac{x}{1!} + 20 \frac{x^2}{2!} + 592 \frac{x^3}{3!} + 33,888 \frac{x^4}{4!} + \ldots. \]
Figure 4.1. The 8 unlabeled 2-cell imbeddings of $B_3$ together with the $24$ rotation schemes describing labeled 2-cell imbeddings.
The labeled 2-cell imbeddings on surfaces occur when the rotation on V is cyclic and there are \((2n-1)!\) such occurrences giving

\[ S(x) = \frac{1}{1!}x + \frac{5}{2!}x^2 + \frac{120}{3!}x^3 + \cdots + \frac{(2n-1)!}{n!}x^n + \cdots. \]

This information is summarized in Table 4.1 below.

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<th>Number of edges</th>
<th>1</th>
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<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>S 1 6 120 5040 362880 39916800 6227020800 1307674368000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

G: Generalized pseudosurfaces
P: Pseudosurfaces
S: Surfaces

Number of labeled 2-cell imbeddings of the bouquet by number of edges.

Table 4.1.

4.2.2. Enumeration by Number of Edges and Number of Regions

Since the edges are labeled but not the regions we use a counting series with two indeterminants which is exponential in the indeterminant counting edges \((x)\) and ordinary in the indeterminant counting regions \((y)\). The series enumerating labeled 2-cell imbeddings of the bouquet with \(n\) edges and \(r\) regions are defined as follows:

\[ G_{n,r} = [y^r]G_n(y) = n![x^n,y^r]G(x,y) = \text{number on generalized pseudosurfaces}, \]

\[ P_{n,r} = [y^r]P_n(y) = n![x^n,y^r]P(x,y) = \text{number on pseudosurfaces}. \]
\[ S_{n,r} = [y^r]S_n(y) = n! [x^n, y^r]S(x, y) = \text{number on surfaces}, \]

where \( G_n(y) \), \( P_n(y) \), and \( S_n(y) \) are polynomials enumerating the number of labeled 2-cell imbeddings of \( B_n \) by number of regions.

To determine \( G(x, y) \) recall that the number of orbits of the permutation \( \rho_v \circ \rho_e \) is the number of regions for the imbedding determined by the map \( \langle B_n, \rho_e, \rho_v \rangle \). In the present case \( \rho_e = (1 \ 2) (3 \ 4) \cdots (2n-1 \ 2n) \). Thus, if \( \rho_R \) is any permutation in \( \Sigma_A \) with \( r \) orbits then there is precisely one rotation on \( V \) satisfying \( \rho_v \circ \rho_e = \rho_R \). Hence, we seek the number of permutations in \( \Sigma_A \) that have exactly \( r \) orbits.

In Section 3.3 was mentioned the well-known result that the number of permutations in \( \Sigma_N \) with exactly \( r \) orbits is the absolute value of the Stirling number of the first kind \( |s_N^{(r)}| \).

These numbers satisfy the equation
\[ z(z+1)(z+2) \cdots (z+N-1) = \sum |s_N^{(r)}| z^r \]
and recurrence
\[ |s_N^{(r)}| = |s_N^{(r-1)}| + (N-1) |s_N^{(r-2)}|. \]

Thus, we have \( G_{n,r} = |s_N^{(r)}| \) and it follows that
\[ n! [x^n] G(x, y) = |s_{2n}^{(1)}| y + |s_{2n}^{(2)}| y^2 + \cdots + |s_{2n}^{(2n)}| y^{2n} \]
\[ = y(y+1)(y+2) \cdots (y+2n-1) + \cdots \]

Using the recurrence (4.2) it is easy to show that the coefficients of \( G \) satisfy the recurrence equation
\[ G_{n,r} = G_{n-1,r-2} + (4n-3)G_{n-1,r-1} + (2n-1)(2n-2)G_{n-1,r} \]  \( (4.3) \)

Since \( P \) and \( G \) are related by \( G+1 = \exp(P) \) we can use
\[ P_n(y) = G_n(y) - \sum_{k=1}^{n-1} \binom{n-1}{k-1} P_k(y) G_{n-k}(y) \]

where

\[ P_k(y) = k! [x^k] P(x,y) \] and \[ G_k(y) = k! [x^k] G(x,y) \] to determine the coefficients of \( P \) from \( G \) quite rapidly. The results are displayed in Table 4.2. The series \( G \) and \( P \) begin

\[ G(x,y) = (y + y^2) \frac{x}{1!} + (6 y + 11 y^2 + 6 y^3 + y^4) \frac{x^2}{2!} + \cdots \]

\[ P(x,y) = (y + y^2) \frac{x}{1!} + (6 y + 10 y^2 + 4 y^3) \frac{x^2}{2!} + \cdots \]

Note that \( P(x,1) \) is the previous series enumerating the imbeddings by number of edges only. Also note that the characteristic of the pseudosurface on which \( B_n \) is 2-cell imbedded can be calculated from \( \chi = p - q + r = 1 - n + r \) and the information contained in the above series. It is unfortunate that the genus of the pseudosurface can not be calculated as easily, for this calculation requires knowledge of the degrees of the singular points.

To determine \( S(x,y) \) it is necessary to find the number of solutions in \( \Sigma_{2n} \) to \( \rho_\mathcal{V} \circ \rho_\mathcal{E} = \rho_\mathcal{R} \) where \( \rho_\mathcal{V} \) is a full cycle,

\( \rho_\mathcal{E} = (1 \ 2 \ (3 \ 4) \cdots (2n-1 \ 2n) \), and \( \rho_\mathcal{R} \) has \( r \) orbits. The solution is found from the work of Jackson summarized in Section 3.3.

Recall that \( e_r^{(d)}(N) \) is the number of solutions in \( \Sigma_{dN} \) to \( \rho_{\mathcal{V}} \circ \rho_{\mathcal{E}} = \rho_{\mathcal{R}} \) where \( \rho_{\mathcal{V}} \) is a fixed full cycle, \( \rho_{\mathcal{E}} \) has all \( d \)-cycles, and \( \rho_{\mathcal{R}} \) has \( r \) orbits. In the present case \( d=2 \) and \( N=n \). Thus,

\[ (\text{number of full cycles in } \Sigma_{2n}) \times e_r^{(2)}(n) \] is the number of solutions in \( \Sigma_{2n} \) to \( \rho_{\mathcal{V}} \circ \rho_{\mathcal{E}} = \rho_{\mathcal{R}} \) where \( \rho_{\mathcal{V}} \) is a full cycle, \( \rho_{\mathcal{E}} \) is a full involution, and \( \rho_{\mathcal{R}} \) has \( r \) orbits. This is the same number as

\[ (\text{number of full involutions in } \Sigma_{2n}) \times S_{n,r} \]. There are \((2n-1)\)
Generalized pseudosurface
Pseudosurface
Surface

Labeled 2-cell imbeddings of the bouquet by number of edges (n) and number of regions (r).

Table 4.2

full cycles in $\Sigma_{2n}$ and $\frac{(2n)!}{2^n \cdot n!}$ full involutions so that

$$(2n-1)! \cdot e^{(2)}_r(n) = \frac{(2n)!}{2^n \cdot n!} S_{n,r} \quad \text{or} \quad S_{n,r} = 2^{n-1} \cdot (n-1)! \cdot e^{(2)}_r(n).$$

From Theorem 3.2 in Section 3.3 the following explicit expression is easily obtained.

**Theorem 4.1.** $S_{n,r} = \frac{(n-1)!}{2n+1} \sum_{m=n+r}^{2n+1} 2^{m-r-1} \binom{m}{r} \binom{m}{n} S_{m-r}$. 

---

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The determination of the numbers $S_{n,r}$ is more easily accomplished using a recursion equation obtained from one found by Jackson for the $e_r^{(2)}(n)$ as follows:

$$(n+1)e_r^{(2)}(n) = (2n-1)(n-1)(2n-3)e_r^{(2)}(n-2) + 2(2n-1)e_{r-1}^{(2)}(n-1)$$

becomes

$$\frac{(n+1)}{2n-1 \cdot (n-1)!} S_{n,r} = \frac{(2n-1)(n-1)(2n-3)}{2n-3 \cdot (n-3)!} S_{n-2,r} + \frac{2(2n-1)}{2n-2 \cdot (n-2)!} S_{n-2,r-1}$$

which reduces to the following result.

**Theorem 4.2.** \((n+1)S_{n,r} = 4(2n-1)(n-1)^2(2n-3)(n-2)S_{n-2,r} + 4(2n-1)(n-1)S_{n-1,r-1}, \)

with boundary conditions $S_{n,r} = 0$ if $r ≤ 0$ or $n ≤ 0$, $S_{1,2} = 1$, and $S_{2,1} = 2$.

The recursion equation was used to generate the numbers appearing in Table 4.2.

4.2.3. Region Distribution

The exponential counting series enumerating labeled 2-cell imbeddings of $B_n$ with $r_k$ regions of size $k$ are as follows:

$$n! \left[ x_1^{n_1} y_1^{r_1} x_2^{n_2} y_2^{r_2} \cdots y_{2n}^{r_{2n}} \right] g(x, y_1, y_2, \cdots) = \text{number on generalized pseudosurfaces},$$

$$n! \left[ x_1^{n_1} y_1^{r_1} x_2^{n_2} y_2^{r_2} \cdots y_{2n}^{r_{2n}} \right] p(x, y_1, y_2, \cdots) = \text{number on pseudosurfaces},$$

$$n! \left[ x_1^{n_1} y_1^{r_1} x_2^{n_2} y_2^{r_2} \cdots y_{2n}^{r_{2n}} \right] s(x, y_1, y_2, \cdots) = \text{number on surfaces}.$$

There are \( \frac{(2n)!}{\prod_{k=1}^{2n} k^k \cdot r_k!} \) permutations in $\Sigma_{2n}$ which have $r_k$
cycles of length \(k\), \(k=1,2,\cdots,2n\), and each of these corresponds to a 2-cell imbedding of \(B_n\) on a generalized pseudosurface with \(r_k\) regions of size \(k\). If \(\rho_R\) is one of these permutations then 
\[\rho_V = \rho_R \circ \rho_E\]
is the rotation on \(V\) giving the region distribution described by \(\rho_R\) when \(\rho_E = (1\,2\,(3\,4)\cdots(2n-1\,2n))\). Thus, if 
\[R=(r_1,r_2,\cdots,r_{2n})\]
is a partition of \(2n\) (denoted \(R|2n\)) then 
\[G(x,y_1,y_2,\cdots) = \sum_{n=1}^{\infty} \frac{x^n}{n!} \sum_{R|2n} \frac{(2n)!}{\prod_{k=1}^{2n} r_k!} y_1^{r_1} y_2^{r_2} \cdots y_{2n}^{r_{2n}}.\]

For each \(n\) let \(G_n\) be the polynomial given by 
\[G_n(y_1,y_2,\cdots,y_{2n}) = \sum_{R|2n} \frac{(2n)!}{\prod_{k=1}^{2n} r_k!} y_1^{r_1} y_2^{r_2} \cdots y_{2n}^{r_{2n}}.\]

We recognize the relationship between \(G_n\) and the cycle index polynomial of the symmetric group \(\Sigma_{2n}\). That is, the coefficient of 
\[y_1^{r_1} y_2^{r_2} \cdots y_{2n}^{r_{2n}}\]
in \(G_n\) is the number of elements in \(\Sigma_{2n}\) having \(r_k\) cycles of length \(k\), \(k=1,2,\cdots,2n\). In terms of these polynomials we have 
\[G = \sum_{n=1}^{\infty} G_n \frac{x^n}{n!}.\]
Since \(G\) and \(P\) are related by \(G + 1 = \exp(P)\) we have from (3.1) that 
\[P_n = G_n - \sum_{k=1}^{n-1} \frac{(n-1)!}{(k-1)!} P_k G_{n-k}.\]
So the series \(P\) begins 
\[P(x) = G_1 x + (G_2 - G_1^2) \frac{x^2}{2!} + (G_3 - 3G_2 G_1 + 2G_1^3) \frac{x^3}{3!} + \cdots.\]

A more detailed listing of the coefficients of \(P(x)\) is given below.
\[
\begin{align*}
P_1 &= G_1 \\
P_2 &= G_2 - G_1^2 \\
P_3 &= G_3 - 3G_2 G_1 + 2G_1^3 \\
P_4 &= G_4 - 4G_3 G_1 - 3G_2^2 + 12G_2 G_1^2 - 6G_1^4 \\
P_5 &= G_5 - 5G_4 G_1 - 10G_3 G_2 + 20G_3 G_1^2 + 30G_2^2 G_1 - 60G_2 G_1^3 + 24G_1^5
\end{align*}
\]
If the number of labeled 2-cell imbeddings having a specified region distribution is desired then we can proceed as follows. Let \( R = 1^1, 2^2, \cdots, (2n)^{2n} \) denote the target region distribution where there are \( r_k \) regions of size \( k, \quad k = 1, 2, \cdots, 2n \); then the number of labeled 2-cell imbeddings of \( B_n \) on generalized pseudosurfaces having this region distribution is

\[
\frac{(2n)!}{\prod_{k=1}^{2n} k^{r_k} \cdot r_k!}
\]

which is the coefficient of \( y_1^1 y_2^2 \cdots y_{2n}^{2n} \) in the polynomial \( G_n \).

To determine how many of these occur on pseudosurfaces we first express \( P_n \) in terms of the \( G_k \) using the recursion equation.

For example, if we want to determine the number of 2-cell imbeddings of \( B_4 \) on pseudosurfaces with regions described by \( R = 1^4 2^2 \) then

\[
P_4 = G_4 - 4G_3 G_1 - 3G_2^2 + 12G_2 G_1^2 - 6G_1^4 \quad (4.4)
\]

and we seek the coefficient of \( y_1^4 y_2^2 \) in \( P_4 \). The polynomial \( G_3 \) contains many terms which when multiplied by \( G_1 \) (the term 4\( G_3 G_1 \) in (4.4) above) make no contribution to the number of imbeddings having regions described by \( 1^4 2^2 \). Hence, when evaluating (4.4) we keep only those terms of the polynomials on the righthand side which make a contribution to the term of interest. To complete the example we have

\[
[y_1^4 y_2^2] (G_4 - 4G_3 G_1 - 3G_2^2 + 12G_2 G_1^2 - 6G_1^4)
\]
= 210 - 4[y_1^4y_2^2](15y_1^4y_2 + 45y_1^2y_2^2)(y_1^2 + y_2)
- 3[y_1^4y_2^2](y_1^4 + 6y_1^2y_2 + 3y_2^2)^2
+ 12[y_1^4y_2^2](y_1^4 + 6y_1^2y_2 + 3y_2^2)(y_1^2 + y_2)^2
- 6[y_1^4y_2^2](y_1^2 + y_2)^4

= 210 - 4(15 + 45) - 3(5 + 36) + 12(1 + 12 + 3) - 6(6)
= 0.

Table 4.3 contains the number of labeled 2-cell imbeddings with a particular region distribution for the bouquet on generalized pseudosurfaces, pseudosurfaces, and surfaces.

To determine the number of 2-cell imbeddings of $B_n$ on surfaces having $j_s$ regions of size $s$, $s=1,2,\ldots,2n$, it is necessary to determine how many solutions to $\rho_\gamma \circ \rho_\xi = \rho_\nu$ exist in $\Sigma_{2n}$ where $\rho_\gamma$ is any 2$n$-cycle, $\rho_\xi$ is (12)(34)\cdots(2n-1\ 2n)$ and $\rho_\nu$ is any permutation having $j_s$ cycles of length $s$, $s=1,2,\ldots,2n$.

Let $i$ index the class of 2$n$-cycles in $\Sigma_{2n}$, $j$ index the class of permutations having cycle-type $(j_1,j_2,\ldots,j_{2n})$, and $k$ index the class of full involutions. The number we seek is the class multiplication coefficient $n_{ijk}$ given in Section 3.3 as

$$n_{ijk} = \frac{h_i h_j}{(2n)!} \sum \frac{x_i x_j x_k}{x_1}$$

where in this case $h_i = (2n-1)!$, $h_j = (2n)!/(\Pi j_s \cdot j_s!)$, and the sum extends over all irreducible characters of $\Sigma_{2n}$. However, in this case one of the classes is that of the full cycles in which case many of the character values are in fact zero. The remaining character values are $\pm 1$ on this class and their values on all the classes are determined using (3.5) given in Section 3.3. Specifically, we have
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<th>G</th>
<th>P</th>
<th>S</th>
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<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Total</td>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
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</table>

<table>
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<th>P</th>
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</tr>
<tr>
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<table>
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<th>P</th>
<th>S</th>
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</tr>
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G: Generalized pseudosurface imbeddings  
P: Pseudosurface imbeddings  
S: Surface imbeddings

The region distribution of labeled 2-cell imbeddings of the bouquet.

Table 4.3
<table>
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<th>Regions</th>
<th>G</th>
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Total 3628800 3134208 362880

G: Generalized pseudosurface imbeddings
P: Pseudosurface imbeddings
S: Surface imbedding

Table 4.3 continued

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\[
\begin{align*}
\chi_1^{(\lambda)} &= [y^\lambda] \frac{1}{1+y} (1-(-y)^{2n})^\lambda = [y^\lambda] \frac{1-(-y)^{2n}}{1+y} = (-1)^\lambda, \\
\chi_j^{(\lambda)} &= [y^\lambda] \frac{1}{1+y} \left( \frac{2n}{s=1} (1-(-y)^s) \right)^j, \\
\chi_k^{(\lambda)} &= [y^\lambda] \frac{1}{1+y} (1-(-y)^2)^n = [y^\lambda] \frac{1}{1+y} (1-y^2)^n \\
&= [y^\lambda] \frac{1-y^2}{1+y} \sum_{m=0}^{n-1} \binom{n-1}{m} (-y^2)^{n-m-1} \\
&= [y^\lambda] (1-y) \sum_{m=0}^{n-1} \binom{n-1}{m} (-1)^{n-m-1} 2n-2m-2 \\
&= [y^\lambda] \left( \sum_{m=0}^{n-1} \binom{n-1}{m} (-1)^{n-m-1} 2n-2m-2 \right) y \\
&= [y^\lambda] \left( \sum_{m=0}^{n-1} \binom{n-1}{m} (-1)^{n-m-1} 2n-2m-2 \right) y + \sum_{m=0}^{n-1} \binom{n-1}{m} (-1)^{n-m} 2n-2m-1 \\
&= \left\{ \begin{array}{ll}
(-1)^{\lambda/2} \binom{n-1}{n-\lambda/2} & \text{if } \lambda \text{ is even} \\
(-1)^{\lambda+1} \binom{n-1}{n-\lambda+1} & \text{if } \lambda \text{ is odd}
\end{array} \right.
\end{align*}
\]

Making the appropriate substitutions in (3.4) and simplifying we have the following result.

**Theorem 4.3.** The number of 2-cell imbeddings of \( B_n \) on surfaces having \( j_s \) regions of size \( s \), \( s=1,2,\ldots,2n \), is

\[
\chi_j^{(\lambda)} = [y^\lambda] \frac{1}{1+y} (1-(-y)^s)^j = [y^\lambda] \frac{1}{1+y} (1+y)^{2n} \\
= [y^\lambda] (1+y)^{2n-1} = \binom{2n-1}{\lambda}.
\]
The region distributions for 2-cell imbeddings on surfaces of $B_n$, $n=1,2,3,$ and 4, are given in Table 4.3. To illustrate we calculate the number of imbeddings of $B_4$ having region sizes given by $(3,1,1,0,\cdots,0)$; that is, three 1-sided regions, a 2-sided region, and a 3-sided region.

\[
\frac{2n-1}{2^n \prod_{s=1}^{s=n} j_s!} \left( \frac{2n-1}{\lambda} \right)^{\lambda} \left( \frac{1(-y)^s}{1+y} \right)^{j_s} = \frac{1+y}{1+y} (1+y)^3 (1-y^2) (1+y^3)
\]

\[= 1 + 2y + 0y^2 - y^3 + y^4 + 0y^5 - 2y^6 - y^7.
\]

The multipliers appearing in the sum are listed in Table 4.4 below.

Thus, there are

\[
\frac{7!}{1^3 \cdot 3! \cdot 2^1 \cdot 1! \cdot 3^1 \cdot 1!} (1 + \frac{2}{7} + \frac{3}{5} + \frac{3}{35} + \frac{2}{7} + 1)
\]

\[= 384 \text{ imbeddings.}
\]

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Table 4.4.
4.3. Unlabeled 2-cell Imbeddings

The unlabeled imbeddings (congruence classes) of the bouquet with $n$ loops can be counted in several ways. We can take a particular labeled bouquet, say the one with arc set $A = \{1, 2, \ldots, 2n\}$, edge set $E = \{\{1, 2\}, \{3, 4\}, \ldots, \{2n-1, 2n\}\}$ and vertex set $V = \{1, 2, \ldots, 2n\}$ and allow its automorphism group to act on the set of rotations. The number of orbits under this action is the number of unlabeled 2-cell imbeddings. It is this approach which will be taken to enumerate the unlabeled 2-cell imbeddings on surfaces so that the results of Section 3.4 can be used.

Alternatively, we could take all bouquets with arc set $A = \{1, 2, \ldots, 2n\}$ and allow the full symmetric group $\Sigma_{2n}$ to act on the set of rotations. The number of orbits under this action is again the number of unlabeled 2-cell imbeddings. We use this approach on generalized pseudosurfaces and pseudosurfaces.

4.3.1. Enumeration by Number of Edges

For the unlabeled problem we use ordinary counting series enumerating the number of unlabeled 2-cell imbeddings of $B_n$ as follows:

$G_n = \left[ x^n \right] G(x) = \text{number on generalized pseudosurfaces}$,

$P_n = \left[ x^n \right] P(x) = \text{number on pseudosurfaces}$,

$S_n = \left[ x^n \right] S(x) = \text{number on surfaces}$.

Let $H$ and $H'$ be bouquets with arc set $A = \{1, 2, \ldots, 2n\}$ and
let $M = (H, \rho_E, \rho_V)$ and $M' = (H', \rho'_{E'}, \rho_{V'})$ be maps. Then from Chapter II, $M$ and $M'$ are congruent maps if there exists a bijection $\alpha$ of $A$ onto $A$ such that $\alpha\rho_E \alpha^{-1} = \rho_E$ and $\alpha\rho_V \alpha^{-1} = \rho_V$. Note that in this case $\alpha$, $\rho_E$ and $\rho_V$ are all permutations in the symmetric group $\Sigma_{2n}$. In addition, $\rho_E$ is always a full involution and $\rho_V$ is unrestricted.

Thus, we are led to consider the group $\Sigma_{2n}$ acting on the set $R = \{(\rho_E, \rho_V) \in \Sigma_{2n} \times \Sigma_{2n}: \rho_E$ is a full involution $\}$ by $\alpha: (\rho_E, \rho_V) \mapsto (\alpha\rho_E \alpha^{-1}, \alpha\rho_V \alpha^{-1}), \alpha \in \Sigma_{2n}$. The number of orbits is the number of congruence classes of 2-cell imbeddings of $B_n$ on generalized pseudosurfaces.

To determine the number of orbits we use Burnside's lemma as follows. For each $\alpha$ in $\Sigma_{2n}$ we define the fixed set of $\alpha$, denoted $F(\alpha)$, as $F(\alpha) = \{(\rho_E, \rho_V) \in R: \alpha\rho_E \alpha^{-1} = \rho_E$ and $\alpha\rho_V \alpha^{-1} = \rho_V\}$. In terms of the cardinality of these sets we have

**Lemma 4.4 (Burnside).**

$$G_n = \frac{1}{(2n)!} \sum_{\alpha \in \Sigma_{2n}} |F(\alpha)|.$$  

To determine the numbers $|F(\alpha)|$ we define sets $F_E(\alpha)$ and $F_V(\alpha)$ similar to $F(\alpha)$ as follows:

- $F_E(\alpha) = \{\rho_E \in \Sigma_{2n}: \rho_E$ is a full involution and $\alpha\rho_E \alpha^{-1} = \rho_E\}$
- $F_V(\alpha) = \{\rho_V \in \Sigma_{2n}: \alpha\rho_V \alpha^{-1} = \rho_V\}$.

Observe that $F(\alpha)$ is the cartesian product of $F_E(\alpha)$ with $F_V(\alpha)$; so we need only determine the latter two cardinalities separately to determine that of $F(\alpha)$. Moreover, since the $\rho_V$ are unrestricted, $F_V(\alpha)$ is the centralizer of $\alpha$ in $\Sigma_{2n}$ and, hence,
we have the following well-known result (see Lemma 4.16 in
Section 4.3.2).

**Lemma 4.5.** If the cycle type of $\alpha$ in $\Sigma_{2n}$ is given by the
$n$-tuple $J(\alpha) = (j_1, j_2, \ldots, j_{2n})$ then $|F_V(\alpha)| = \prod_{k=1}^{2n} j_k \cdot j_k!$.

We also observe the following.

**Lemma 4.6.** $|F_E(\cdot)|$ and $|F_V(\cdot)|$ are constant on conjugacy classes.

**Proof.** If $\beta = \sigma \alpha \sigma^{-1}$ then the mapping $\theta_E \to \sigma \theta_E \sigma^{-1}$ is seen to be a
bijection from $F_E(\alpha)$ to $F_E(\beta)$. Similarly, for $F_V(\alpha)$ and $F_V(\beta)$. ♦

Thus, we write $|F_E(J)|$ and $|F_V(J)|$ to be the values of these class
functions on the class of cycle-type $J = (j_1, j_2, \ldots, j_{2n})$ in $\Sigma_{2n}$.

The problem now simplifies to

**Lemma 4.7.** $G_n = \sum_{J_{12n}} |F_E(J)|$.

**Proof.** The number of permutations in $\Sigma_{2n}$ of cycle-type
$J = (j_1, j_2, \ldots, j_{2n})$ is $\frac{(2n)!}{\prod_{k=1}^{2n} j_k \cdot j_k!}$, so that

$$G_n = \frac{1}{(2n)!} \sum_{\alpha \in \Sigma_{2n}} |F(\alpha)| = \frac{1}{(2n)!} \sum_{\alpha \in \Sigma_{2n}} |F_V(\alpha)| \cdot |F_E(\alpha)|$$

$$= \frac{1}{(2n)!} \sum_{J_{12n}} \frac{(2n)!}{\prod j_k \cdot j_k!} |F_V(J)| \cdot |F_E(J)|$$
To determine \(|\mathcal{F}_E(J)|\) we fix \(J = (j_1, j_2, \ldots, j_{2n})\) and choose a permutation \(\alpha\) of this cycle type. We then seek the number of full involutions that are fixed by \(\alpha\) under conjugation. If \(\rho_E\) is one of these then it is clear that \(\rho_E\) fixes \(\alpha\) under conjugation as well. Writing \(\alpha\) as a product of disjoint cycles we can regard conjugation of \(\alpha\) by members of \(\mathcal{F}_E(\alpha)\) as permuting the cycles of \(\alpha\) among themselves; in particular, a \(k\)-cycle maps to a \(k\)-cycle. For example, \(\rho_E = (12)(34)(56)\) fixes \(\alpha = (135)(246)\) by mapping the 3-cycle (135) to the 3-cycle (246), and vice versa.

Now we restrict our attention to the set of symbols, \(X_k\), appearing in the \(k\)-cycles of \(\alpha\) for some fixed \(k\). Let \(\alpha_k\) be the restriction of \(\alpha\) to this set and let \(\mathcal{F}_E(\alpha_k)\) be the set of full involutions in the symmetric group on \(X_k\) that are fixed by \(\alpha_k\) under conjugation; then

\[
\text{Lemma 4.8. } |\mathcal{F}_E(\alpha)| = \prod_{k=1}^{2n} |\mathcal{F}_E(\alpha_k)| \quad \text{where } |\mathcal{F}_E(\alpha_k)| \text{ is taken to be } 1 \text{ if } j_k = 0 \quad (X_k \text{ is empty in this case}).
\]

\textbf{Proof.} Let \(X_k\) be the \(k \cdot j_k\) symbols appearing in the \(j_k\) \(k\)-cycles of \(\alpha\) then \(\rho_E = \prod \rho_E|_{X_k}\) fixes \(\alpha\) under conjugation if and only if \(\rho_E|_{X_k}\) fixes \(\alpha_k\) under conjugation for each \(k\) with \(j_k\) nonzero. \(\star\)
It remains to determine $|F_E(\alpha_k)|$. Here we have two cases, $k$ odd and $k$ even.

Lemma 4.9. In the case $k$ is odd,

$$|F_E(\alpha_k)| = \begin{cases} 
\frac{j_k!}{2^{j_k/2}(j_k/2)!} & \text{if } j_k \text{ is even} \\
0 & \text{if } j_k \text{ is odd}
\end{cases}$$

Proof. Let $\alpha_k = (a_0 a_1 \cdots a_{k-1}) \cdots$ and $\rho_E \in F_E(\alpha_k)$. Suppose $\rho_E(a_0) = a_1$ where $1 \leq i \leq k-1$; then $\rho_E$ has the disjoint cycle representation $\rho_E = \cdots (a_0 a_i) \cdots$ and since $\rho_E$ is fixed by $\alpha_k$ under conjugation it follows that $\alpha_k \rho_E \alpha_k^{-1} = \cdots (a_i a_{i+1}) \cdots$. This implies $a_{i+1}$ equals $a_0$, which implies $2i$ is congruent to 0 modulo $k$. Since $k$ is odd we must have $k$ dividing $i$ but $1 \leq i \leq k-1$. Hence, $\rho_E$ must map the members of a $k$-cycle to a different $k$-cycle. Furthermore, if $\alpha_k = (a_0 a_1 \cdots a_{k-1}) (b_0 b_1 \cdots b_{k-1}) \cdots$ and $\rho_E = \cdots (a_0 b_0) \cdots$ then for $0 \leq i \leq k-1$ we have $\alpha_k \rho_E \alpha_k^{-1} = \cdots (a_i b_i) \cdots$ and we conclude that $\rho_E = \cdots (a_0 b_0) (a_1 b_1) \cdots (a_{k-1} b_{k-1}) \cdots$. Thus, $\rho_E$ maps each member of a pair of $k$-cycles onto the other so that $j_k$ must be even else $|F_E(\alpha_k)|$ is zero.

There are \( \frac{j_k!}{2^{j_k/2}(j_k/2)!} \) ways to pair the $j_k$ $k$-cycles and for each pair \((a_0 a_1 \cdots a_{k-1}), (b_0 b_1 \cdots b_{k-1})\) there are $k$ possible images of $a_0$ under $\rho_E$ (which then determine the images of $a_i$ for all other $i$). Since there are $j_k/2$ pairs in each
Lemma 4.10. In the case \( k \) is even,

\[
|F_{E}(\alpha_{k})| = \sum_{m=0}^{[j_k/2]} \binom{j_k}{2m} \frac{(2m)!}{2^m \cdot m!} \cdot k^m.
\]

Proof. If \( k \) is even then \( \rho_{E} \) maps members of a pair of \( k \)-cycles onto each other or maps a \( k \)-cycle onto itself. The latter can be done in only one way: If \( \alpha_{k} = \cdots (a_1 a_2 \cdots a_{k/2} b_1 b_2 \cdots b_{k/2}) \cdots \) then \( \rho_{E} = \cdots (a_1 b_1)(a_2 b_2) \cdots (a_{k/2} b_{k/2}) \cdots \). The former, as in the case when \( k \) is odd, can be done in \( k \) ways for each pair of \( k \)-cycles. Hence, \( |F_{E}(\alpha_{k})| \) is

\[
\sum_{m=0}^{[j_k/2]} \binom{j_k}{2m} \frac{(2m)!}{2^m \cdot m!} \cdot k^m.
\]

In summary we present the following theorem.

Theorem 4.11.

\[
|F_{E}(J)| = \prod_{k \text{ odd}} j_k! \frac{j_k/2}{2^{j_k/2} \cdot (j_k/2)!} \prod_{k \text{ even}} \sum_{m=0}^{[j_k/2]} \binom{j_k}{2m} \frac{(2m)!}{2^m \cdot m!} k^m.
\]

To determine \( G_n \) it is necessary to sum the above numbers over all partitions \( J \) of \( 2n \). To accomplish this we define a
sequence of infinite series \( \{E_k(x)\} \) with the property that
\[
G(x) + 1 = \prod E_k(x).
\]
Specifically,
\[
E_k(x) = \begin{cases} 
\sum_{p=0}^{M_8} \frac{(2p)!}{2^m \cdot p!} \left( \frac{p}{2m} \frac{(2m)!}{k^m} \right) x^{p k^2} & \text{if } k \text{ is even} \\
\sum_{p=0}^{M_8} \frac{(2p)!}{2^m \cdot p!} \left( \frac{p}{k} \right) x^p & \text{if } k \text{ is odd}
\end{cases}
\]

In terms of these series we have
\[
G(x) + 1 = (1 + x + 3x^2 + 15x^3 + 105x^4 + 945x^5 + 10395x^6 + 10395x^7 + 135135x^8 + \cdots) \\
\cdot \left(1 + x + 3x^2 + 7x^3 + 25x^4 + 81x^5 + 331x^6 + 1303x^7 + 5937x^8 + \cdots\right) \\
\cdot \left(1 + 3x^3 + 27x^6 + \cdots\right) \cdot \left(1 + 5x^4 + 13x^8 + 73x^8 + \cdots\right) \\
\cdot (1 + 5x^8 + \cdots) \cdot (1 + x^4 + 9x^8 + \cdots) \cdot (1 + 7x^7 + \cdots) \\
\cdot (1 + x^6 + \cdots) \cdot (1 + x^8 + \cdots) \cdot (1 + x^7 + \cdots) \cdot (1 + x^8 + \cdots) \cdot \cdots
\]
\[
= 1 + 2x + 8x^2 + 34x^3 + 182x^4 + 1300x^5 + 12634x^6 + 153598x^7 + 2231004x^8 + \cdots,
\]
giving the number of unlabeled 2-cell imbeddings of the bouquet on generalized pseudosurfaces.

To determine the number of these which are on pseudosurfaces we have that \( G(x) \) and \( P(x) \) are related by
\[
G(x) + 1 = \exp \sum_{k=1}^{\infty} \frac{P(x^k)}{k}
\]
and so the coefficients \( P_n \) are determined from the \( G_n \) by comparison as outlined in Chapter 3. The
counting series for pseudosurfaces begins

\[ P(x) = 2x + 5x^2 + 20x^3 + 107x^4 + 970x^5 + 9436x^6 + 125820x^7 + 1858591x^8 + \ldots \]

These results, in addition to the number on surfaces to be derived shortly, are listed in Table 4.5 at the end of this section.

To determine the number of unlabeled 2-cell imbeddings of the bouquet on surfaces we apply the technique given in Section 3.4 by identifying the pseudograph \( B_n \) with the graph obtained by subdividing each loop twice as shown in Figure 4.2. We denote this graph by \( B_n \) as well and note that \( \text{Aut}(B_n) \cong \Sigma_n \Sigma_2 \) provided \( n > 1 \). If \( n = 1 \) then clearly there is exactly one unlabeled 2-cell imbedding of \( B_1 \) on surfaces. Henceforth it is assumed that \( n > 1 \).

![Figure 4.2. The identification of the pseudograph \( B_n \) with a graph.](image)

If we denote the vertex of large degree by \( v \), then every automorphism \( \alpha \in \text{Aut}(B_n) \) fixes \( v \). Since \( v \) is adjacent to every other vertex, Theorem 3.4 guarantees that if \( \alpha \) fixes another vertex then either \( \alpha \) is the identity automorphism in which case \( |F(\alpha)| = (2n-1)! \) or \( |F(\alpha)| = 0 \). Hence, suppose \( \alpha \) fixes no other vertex.
Since \( l(v) = 1 \) Theorem 3.5 and Theorem 3.6 imply that \( |F(\alpha)| > 0 \) if and only if \( |F_v(\alpha^1)| > 0 \) if and only if

\[
J(\alpha^1|_{N(v)}) = (0, \ldots, 0, j_d = 2n/d, 0, \ldots, 0)
\]

for some divisor \( d \) of \( 2n \), if and only if \( J(\alpha) = (1, 0, \ldots, 0, j_d = 2n/d, 0, \ldots, 0) \). If \( \alpha \) has this cycle type and \( u \) is any vertex other than \( v \) then \( l(u) = d \), the order of \( \alpha \), and we have \( |F_u(\alpha^1(u))| = F_u(\text{identity})| = 1 \) since there is but one possible cyclic rotation at \( u \). For \( v \) we have from Theorem 3.6 that \( |F_v(\alpha^1)| = \phi(d) \left( \frac{2n}{d} - 1 \right)! \frac{2n}{d} - 1 \). Since this reduces to \( (2n-1)! \) in case \( d = 1 \), which agrees with the result for \( \alpha \) being the identity automorphism, we have the following result.

Lemma 4.12. \( S_n = \frac{1}{2^n \cdot n!} \sum \phi(d) \left( \frac{2n}{d} - 1 \right)! \frac{2n}{d} - 1 \) where the sum is over all members of \( \Sigma_n[\Sigma_2] \) having cycle-type \((0, \ldots, 0, j_d = 2n/d, 0, \ldots, 0) \) for some divisor \( d \) of \( 2n \).

It remains to determine the number of elements in \( \Sigma_n[\Sigma_2] \) having cycle-type \((0, \ldots, 0, j_d = 2n/d, 0, \ldots, 0) \). We refer to these \( d \)-regular permutations generically as regular. In general, the number of cycles of a particular type in \( \Sigma_n[\Sigma_2] \) is found from the cycle index polynomial \( Z[\Sigma_n[\Sigma_2]; s_1, s_2, \ldots, s_{2n}] \), which in turn is found by composing the cycle index polynomial of \( \Sigma_n \) with that of \( \Sigma_2 \). Specifically, \( Z[\Sigma_2; s_1, s_2] = \frac{1}{2}(s_1^2 + s_2) \) and

\[
Z[\Sigma_n; t_1, t_2, \ldots, t_n] = \frac{1}{n!} \sum_{\text{cycles}} \prod_{k=1}^{n} \frac{n!}{k^k \cdot j_k!} t_k^{j_k} \quad \text{Thus,} \quad j_k = j_k^1.
\]
\[ Z[\Sigma_n[\Sigma_2]; s_1, s_2, \ldots, s_n] = \frac{1}{n!} \sum_{J \in \mathcal{J}_n} \prod_{k=1}^{n} \frac{n! (s_k^2 + s_{2k})^j_k k^k}{j_k! j_k! 2^k} \]

\[ = \frac{1}{2^n \cdot n!} \sum_{J \in \mathcal{J}_n} 2^n \cdot n! \prod_{k=1}^{n} \frac{(s_k^2 + s_{2k})^j_k}{(2k)^j_k \cdot j_k!} \]

\[ = \frac{1}{2^n \cdot n!} \sum_{J \in \mathcal{J}_n} 2^n \cdot n! \prod_{k=1}^{n} \frac{\sum_{m=0}^{j_k} \binom{j_k}{m} s_k^m s_{2k}^{j_k-m}}{(2k)^j_k \cdot j_k!} \]

by the binomial theorem.

The regular permutations occur when \( m = 0 \) or \( j_k \) in the above sum so we are led to consider

\[ \frac{1}{2^n \cdot n!} \sum_{J \in \mathcal{J}_n} 2^n \cdot n! \prod_{k=1}^{n} \frac{s_k^{j_k} + s_{2k}^{j_k}}{(2k)^j_k \cdot j_k!} \quad (4.5) \]

If the partition \( J \) of \( n \) in the above sum is \( d \)-regular then the product over \( k \) occurring in \( (4.5) \) has only one factor corresponding to \( k = d \). In this case there is a contribution of

\[ \frac{s_{n/d}^{2n/d}}{(2d)^{n/d} \cdot (n/d)!} \quad (4.6) \]

\[ \frac{s_d^{2n/d}}{(2d)^{n/d} \cdot (n/d)!} \quad (4.7) \]

to the sum in \( (4.5) \) for each divisor \( d \) of \( n \). The permutations in \( \Sigma_n[\Sigma_2] \) corresponding to those counted in \( (4.6) \) are \( 2d \)-regular and those counted in \( (4.7) \) are \( d \)-regular. The remaining regular permutations of \( \Sigma_n[\Sigma_2] \) occur when
\( J = (0, \cdots, 0, j_d = \frac{n}{2d} - 2m, 0, \cdots, 0, j_{2d} = m, 0, \cdots, 0) \) is the
partition of \( n \) occurring in (4.5) corresponding to permutations in
\( \Sigma_n \) having \( \left( \frac{n}{d} - 2m \right) \) d-cycles and \( m \) 2d-cycles, \( m = 0, 1, \cdots, \left\lfloor \frac{n}{2d} \right\rfloor \).
For each of these values of \( m \) the contribution to the sum in
(4.5) is
\[
2^n \cdot n! \frac{s_{2d}^{n/2d} n/d}{s_{2d}^{2n/d'}} = 2^n \cdot n! \frac{s_{d'}^{2n/d'}}{s_{d'}^{n/d'} (2d')! (\frac{n}{2d})!} \]
Thus, for each \( m = 0, 1, \cdots, \left\lfloor \frac{n}{2d} \right\rfloor \) we have a contribution of
\[
2^n \cdot n! \frac{s_{2d}^{n/2d}}{(2d) \cdot (\frac{n}{d} - 2m)! \cdot (4d) \cdot m!} \]
to the sum in (4.5). The permutations in \( \Sigma_n [\Sigma_2] \) corresponding
to those counted in (4.8) are also 2d-regular.

If \( m \) is zero in (4.8) then we have the contribution counted
in (4.5). Hence, we need only consider (4.7) and (4.8). If 2d
is a divisor of \( n \) as well as d a divisor of \( n \), then when \( m \) is
\( \frac{n}{2d} \) (an integer in this case) we let \( d' = 2d \) in (4.8) and obtain

\[
2^n \cdot n! \frac{s_{d'}^{2n/d'}}{(4d) \cdot (2d)! (\frac{n}{2d})!} \]
This contribution to the sum in (4.5) has previously been
counted in (4.7) but not every contribution counted in (4.7)
arises in this way (2d and d must both divide \( n \)). To avoid a
double count we allow m to range from 0 to \( \frac{n-1}{2d} \) instead of from 0 to \( \left\lfloor \frac{n}{2d} \right\rfloor \). These are the same for all divisors d of n unless 2d is a divisor of n as well, in which case they differ by one.

In summary, for each divisor d of n there are
\[
\frac{2^n \cdot n!}{(2d)^{n/d} \cdot (n/d)!} \quad \text{d-regular permutations in } \Sigma_n[\Sigma_2] \text{ counted in (4.7)}
\]
and
\[
\sum_{m=0}^{\left\lfloor \frac{n-1}{2d} \right\rfloor} \frac{2^n \cdot n!}{(2d)^{2m} \cdot (4d) \cdot \left( \frac{n}{d} - 2m \right)! \cdot m!} \quad \text{2d-regular permutations that are counted in (4.8) but not in (4.7).}
\]
Using Lemma 4.12 and performing a few simplifications we have the following result.

**Theorem 4.13.** The number of unlabeled 2-cell imbeddings of \( B_n \) on surfaces is
\[
\sum_{d \mid n} \frac{\phi(d)(2d-1)\left( \frac{n}{d} \right)!^{d-1}}{2^{n/d} \cdot \left( \frac{n}{d} \right)!} + \sum_{d \mid n} \sum_{m=0}^{\left\lfloor \frac{n-1}{2d} \right\rfloor} \frac{\phi(2d)(\frac{n}{d} - 1)!^m}{2 \cdot \left( \frac{n}{d} - 2m \right)! \cdot m!}.
\]

It will be shown in Section 4.3.3 that the above sum is the value of \( Z(C_{2n}) \cdot Z(\Sigma_n[C_2]) \). That is, the cap operation of Redfield (see Section 3.2) applied to the cyclic group \( C_{2n} \) and the wreath product of the symmetric group \( \Sigma_n \) about the cyclic group \( C_2 \).
Table 4.5

<table>
<thead>
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<th>Number of edges</th>
<th>( G )</th>
<th>( P )</th>
<th>( S )</th>
</tr>
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<tr>
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<td>1</td>
<td>1</td>
</tr>
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<td>5</td>
<td>2</td>
</tr>
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</tr>
<tr>
<td>10</td>
<td>699699968</td>
<td>32743182</td>
<td>32743182</td>
</tr>
</tbody>
</table>

\( G \): Generalized pseudosurfaces  
\( P \): Pseudosurfaces  
\( S \): Surfaces

Number of unlabeled 2-cell imbeddings of the bouquet by number of edges.

4.3.2. Enumeration by Number of Edges and Number of Regions

For the unlabeled problem we use ordinary counting series enumerating the number of unlabeled 2-cell imbeddings of the bouquet having \( r \) regions defined as follows:

\[
G_n(r) = \left[ y^r \right] G_n(y) = \left[ x^n y^r \right] G(x, y) = \text{number on generalized pseudosurfaces},
\]

\[
P_n(r) = \left[ y^r \right] P_n(y) = \left[ x^n y^r \right] P(x, y) = \text{number on pseudosurfaces},
\]

\[
S_n(r) = \left[ y^r \right] S_n(y) = \left[ x^n y^r \right] S(x, y) = \text{number on surfaces},
\]

where \( G_n(y), P_n(y), \) and \( S_n(y) \) are polynomials enumerating the unlabeled 2-cell imbeddings of \( B_n \) by number of regions.

If \( H \) is a bouquet with arc set \( \mathcal{A} = \{1, 2, \ldots, 2n\} \) and \( M = (H, \rho_{E}, \rho_{V}) \) is a map then \( \rho_{R} = \rho_{V} \circ \rho_{E} \) describes the regions of the 2-cell imbedding corresponding to the map \( M \). The imbedding has \( r \) regions if \( \rho_{R} \) has \( r \) orbits (considered as a permutation in the group \( \Sigma_{\mathcal{A}} \)). To enumerate the unlabeled 2-cell imbeddings...
of the bouquet by number of edges and number of regions we
refine the analysis presented in Section 4.3.1. Hence, we will
use many of the results obtained there.

The members of the set
\[(\rho_{E}, \rho_{V}) \in \Sigma_{2n} \times \Sigma_{2n}: \rho_{E} \text{ is a full involution, } \|\rho_{V} \circ \rho_{E}\|=r\] are
precisely the rotation schemes describing imbeddings of \(B_{n}\) having
\(r\) regions where the set of bouquets is those having arc set
\\{1, 2, \ldots, 2n\}. The action of \(\Sigma_{2n}\) on the above set of rotation
schemes is defined by \(\alpha: (\rho_{E}, \rho_{V}) \rightarrow (\alpha \rho_{E} \alpha^{-1}, \alpha \rho_{V} \alpha^{-1}), \alpha \in \Sigma_{2n}\).
The number of orbits of this action is the number of congruence
classes of 2-cell imbeddings of \(B_{n}\) with \(r\) regions on generalized
pseudosurfaces.

As usual, we say \(\alpha\) fixes \((\rho_{E}, \rho_{V})\) if \(\alpha: (\rho_{E}, \rho_{V}) \rightarrow (\rho_{E}, \rho_{V})\).
Letting \(\rho_{R} = \rho_{V} \circ \rho_{E}\), we observe that \(\alpha\) fixes \((\rho_{E}, \rho_{V})\) if and only
if \(\alpha\) fixes \((\rho_{E}, \rho_{R})\). Moreover, if \(\rho_{E}\) is a full involution in \(\Sigma_{2n}\)
and \(\rho_{R}\) is any permutation in \(\Sigma_{2n}\) with \(r\) orbits such that
\((\rho_{E}, \rho_{R})\) is fixed by \(\alpha\) then \((\rho_{E}, \rho_{R} \circ \rho_{E})\) is a rotation scheme of
the bouquet fixed by \(\alpha\) (let \(\rho_{V} = \rho_{R} \circ \rho_{E}\)). Thus, the members
of the set \(R_{r} = \{(\rho_{E}, \rho_{R}) \in \Sigma_{2n} \times \Sigma_{2n}: \rho_{E} \text{ is a full involution, and } \|\rho_{R}\|=r\}
are interpreted as giving imbeddings of \(B_{n}\) with \(r\) regions. The
action of \(\Sigma_{2n}\) on this set is as above and the number of orbits of
the action is again the number of congruence classes of 2-cell
imbeddings having \(r\) regions. We use Burnside's lemma to
enumerate the orbits as follows.

For \(\alpha \in \Sigma_{2n}\) define the following fixed sets of \(\alpha:\)
\(F(\alpha) = \{(\rho_{E}, \rho_{R}) \in R_{r}: \alpha\text{ fixes } (\rho_{E}, \rho_{R})\},\)
\( F_E(\alpha) = \{ \phi_E \in \Sigma_{2n} : \phi_E \text{ is a full involution and } \alpha \phi_E \alpha^{-1} = \phi_E \} \), and
\( F_r(\alpha) = \{ \phi_R \in \Sigma_{2n} : \| \phi_R \| = r \text{ and } \alpha \phi_R \alpha^{-1} = \phi_R \} \).

As in Section 4.3.1 we observe that \(|F(\cdot)|, |F_E(\cdot)|, \text{ and } |F_r(\cdot)|\) are class functions so we write \(|F(J)|, |F_E(J)|, \text{ and } |F_r(J)|\) to be their values on any member of the class of permutations having cycle-type \( J = (j_1, j_2, \cdots, j_{2n}) \) and observe also that
\( F(\alpha) = F_E(\alpha) \times F_r(\alpha) \). We present the following result.

**Theorem 4.14.** \( G_{n, r} = \sum_{J \in \Sigma_{2n}} \frac{1}{\prod k^j \cdot j_k !} |F_E(J)| \cdot |F_r(J)| \)

**Proof.** This is Burnside's lemma for the present case together with the above observations.

Since a permutation in \( \Sigma_{2n} \) can have at most \( 2n \) orbits we have \( G_n(y) = \sum_{r=1}^{2n} G_{n, r} y^r \) as the polynomial enumerating unlabeled 2-cell imbeddings of \( B_n \) by number of regions. Performing this summation and using the previous theorem we have

**Theorem 4.15.** \( G_n(y) = \sum_{J \in \Sigma_{2n}} \frac{|F_E(J)|}{\prod k^j \cdot j_k !} \sum_{r=1}^{2n} |F_r(J)| y^r \)

Recall from Section 4.3.1 that \( F_\nu(\alpha) \) is the centralizer of \( \alpha \) in \( \Sigma_{2n} \) and has cardinality \( \frac{2n}{\prod k^j \cdot j_k !} \). The polynomial \( \sum_{r=1}^{2n} |F_r(J)| y^r \), whose determination is essential in Theorem 4.15, enumerates the centralizer of \( \alpha \) by number of orbits. Hence, we seek information about the centralizer. If \( J(\alpha) = (j_1, j_2, \cdots, j_N) \)
and $C(\alpha)$ is the centralizer of $\alpha$ then

**Lemma 4.16.** The group $C(\alpha)$ is isomorphic to the group $\Sigma_{j_1}[C_1] \times \Sigma_{j_2}[C_2] \times \ldots \times \Sigma_{j_N}[C_N]$, where $\Sigma_{j_k}[C_k]$ is the wreath product (group composition) of the symmetric group $\Sigma_{j_k}$ about the cyclic group $C_k$.

**Proof.** If $\beta \alpha \beta^{-1} = \alpha$ then the induced action of $\beta$ on the cycles appearing in the disjoint cycle representation of $\alpha$ has two parts. It permutes the $k$-cycles of $\alpha$ among themselves (this is the action of the symmetric group) and composes this with cyclically permuting the elements of any one $k$-cycle (the action of the cyclic group).

Thus, the polynomial which enumerates the members of $C(\alpha)$ by number of orbits is the product of the polynomials enumerating the members of $\Sigma_{j_k}[C_k]$ by number of orbits. Therefore, we are led to consider how many members of $\Sigma_{j_k}[C_k]$ have a specified number of orbits.

For a permutation group $G$, the cycle index $Z[G]$ (see Section 3.2) contains the information desired. If each variable $s_k$ in a cycle index $Z[G; s_1, s_2, \ldots]$ is replaced by a single variable $y$ then the resulting polynomial $Z[G; y]$ has as coefficient of $y^r$ the number of members of $G$ with $r$ orbits divided by $|G|$.

In our case $G$ is the wreath product of $\Sigma_{j_k}$ about $C_k$. In general, for permutation groups $A$ and $B$, the cycle index of their wreath product is given by the composition of their cycle indexes in a manner detailed in Section 3.2. It follows that...
\[ [\Sigma_k[C_k]] \cdot Z[\Sigma_k; y] \cdot Z[C_k; y] \] is the polynomial we seek. We have observed that the coefficient of \( y^r \) in the polynomial
\[ y(y+1)(y+2) \cdots (y+j_k-1) \] is the number of permutations in \( \Sigma_{j_k} \) which have \( r \) orbits and so
\[ Z[\Sigma_k; y] = \frac{1}{j_k!} y(y+1)(y+2) \cdots (y+j_k-1). \] Since
\[ Z[C_k; s_1, s_2, \ldots, s_k] = \frac{1}{k} \sum_{d|k} \phi(d) s_d \] we have \( Z[C_k; y] = \frac{1}{k} \sum_{d|k} \phi(d) y^{k/d} \).
\[ Z[\Sigma_k[C_k]; y] = Z[\Sigma_k; y] \cdot Z[C_k; y] \]
\[ = \frac{1}{j_k!} \left( \frac{1}{k} \sum_{d|k} \phi(d) y^{k/d} \right) \left( \frac{1}{k} \sum_{d|k} \phi(d) y^{k/d} + 1 \right) \cdots \left( \frac{1}{k} \sum_{d|k} \phi(d) y^{k/d} + j_k - 1 \right) \]
\[ = \frac{1}{k^k \cdot j_k!} \left( \sum_{d|k} \phi(d) y^{k/d} \right) \left( \sum_{d|k} \phi(d) y^{k/d} + k \right) \cdots \left( \sum_{d|k} \phi(d) y^{k/d} + (j_k - 1)k \right). \]
Since the order of \( \Sigma_{j_k[C_k]} \) is \( k^k \cdot j_k! \), we have
\[ k^k \cdot j_k! \cdot Z[\Sigma_k[C_k]; y] = \left( \sum_{d|k} \phi(d) y^{k/d} \right) \cdots \left( \sum_{d|k} \phi(d) y^{k/d} + (j_k - 1)k \right) \]
as the polynomial enumerating the members of \( \Sigma_k[C_k] \) by number of orbits. Denoting this polynomial by \( R_{j_k, k}(y) \) and defining \( R_{0, k}(y) = 1 \) for all \( k \), we combine the above result with Lemma 4.16 on the nature of the centralizer of a member of \( \Sigma_N \) and present the following.

**Theorem 4.17.** For \( \alpha \) in \( \Sigma_N \) of cycle-type \( J(\alpha) = (j_1, j_2, \ldots, j_N) \), the number of permutations in the centralizer of \( \alpha \) which have exactly \( r \) orbits is
\[ [y^r] \prod_{k=1}^{N} R_{j_k, k}(y) \] where
\[ R_{j_k, k}(y) = \left( \sum_{d|k} \phi(d) y^{k/d} \right) \cdots \left( \sum_{d|k} \phi(d) y^{k/d} + (j_k - 1)k \right) \] and
\[ R_{0, k}(y) = 1 \] for all \( k \).
Corollary \[ \sum_{r=1}^{2n} |F_r(j)| \cdot y^r = \prod_{k=1}^{2n} R_{j,k}(y). \]

As an illustration consider \( \alpha = (12)(34)(567) \). The cycle type of \( \alpha \) is \( J(\alpha) = (0,2,1,0,0,0) \), so there are \( (2^2 \cdot 2!)(3^1 \cdot 1!) = 24 \) members in the centralizer of \( \alpha \). They are

\[
\begin{align*}
(1)(2)(3)(4) & \cdot (5)(6)(7), \\
(1)(2)(34) & \cdot (5)(6)(7),
\end{align*}
\]

\[
\begin{align*}
(12)(3)(4) & \cdot (5)(6)(7), \\
(12)(34) & \cdot (5)(6)(7),
\end{align*}
\]

Hence, by inspection the polynomial enumerating these elements by number of orbits is \( y^7 + 2y^6 + 5y^5 + 6y^4 + 6y^3 + 4y^2 \). Using Theorem 4.17 we have \( \sum_{k=1}^{2n} R_{j,k}(y) = R_{2,2}(y) \cdot R_{1,3}(y) \), where

\[
\begin{align*}
R_{2,2}(y) &= \left( \sum_{d \mid 2} \phi(d) y^{2/d} \right) \left( \sum_{d \mid 2} \phi(d) y^{2/d} + 2 \right) = (y^2 + y) (y^4 + y^2 + y + 2) \quad \text{and} \\
R_{1,3}(y) &= \left( \sum_{d \mid 3} \phi(d) y^{3/d} \right) = (y^3 + 2y). \quad \text{Their product,} \\
(y^2 + y)(y^4 + y^2 + y) \cdot (y^3 + 2y), \quad \text{is} \quad y^7 + 2y^6 + 5y^5 + 6y^4 + 6y^3 + 4y^2, \quad \text{agreeing with the previous calculation.}
\end{align*}
\]

For a fixed \( k \), coefficients of the polynomials \( R_{m,k}(y) \) can be found recursively as follows.
\[ R_{0,k}(y) = 1, \]
\[ R_{m+1,k}(y) = \left( \sum_{d|k} \phi(d) y^{k/d} + mk \right) \cdot R_{m,k}(y), \]
and hence,
\[ [y^r] R_{m+1,k}(y) = \sum_{d|k} \phi(d) [y^{r-k/d}] R_{m,k}(y) + mk[y^r] R_{m,k}(y). \]

For example, when \( k=2 \) we have
\[ [y^r] R_{m+1,2}(y) = [y^{r-2}] R_{m,2}(y) + [y^{r-1}] R_{m,2}(y) + 2m[y^r] R_{m,2}(y) \]
with boundary condition \( R_{0,2}(y) = 1 \) from which Table 4.6 is generated.

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<td>1</td>
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</tbody>
</table>

The number of permutations in \( \Sigma_m[C_2] \) having \( r \) orbits.

Table 4.6

To determine \( G_n(y) \) using Theorem 4.15 it remains to find \( |F_E(J)| \), but these are precisely the same numbers determined in Section 4.3.1 and hence, we can present the final result.

**Theorem 4.18.** \( G_{n,r} = [y^r]d_n(y) \), where
\[
d_n(y) = \sum_{J \subseteq \{2, \ldots, n\}} |F_E(J)| \prod_{k=1}^{2n} R_{j_k,k}(y) \]
and
\[
G_{n,r} = \prod_{k=1}^{2n} j_k! \]
\[ |F_e(J)| = \prod_{k \text{ odd}} \frac{j_k!}{2^{j_k/2} \cdot (j_k/2)!} \prod_{k \text{ even}} \sum_{m=0}^{[j_k/2]} \frac{(2m)!}{2^m \cdot m!} k^m \]

\[ R_{jk, k}(y) = \left( \sum_{d|k} \phi(d) y^{k/d} \right) \cdots \left( \sum_{d|k} \phi(d) y^{k/d} + (j_k-1)k \right) \]

\[ R_{0, k}(y) = 1. \]

As in Section 4.3.1 the sum over all partitions \( J \) of \( 2n \) is accomplished by defining a sequence of infinite series in \( x, \{E_k(x)\} \), the coefficients being polynomials in \( y \), with the property
\[ G_n(y)+1 = [x^n] \prod E_k(x), \] which implies that \( G(x, y)+1 = \prod E_k(x) \).

\( E_k(x) \) is defined as follows.

\[
E_k(x) = \left\{ \begin{array}{ll}
\frac{\sum_{p=0}^{\infty} \left( \frac{[p/2]!}{[2m]!} \frac{(2m)!}{2^m \cdot m!} \frac{R_{p, k}(y)}{k^p \cdot p!} \right)}{x^{p/2}} & \text{k even} \\
\frac{\sum_{p=0}^{\infty} \left( \frac{(2p)!}{2^p \cdot p!} \frac{R_{p, k}(y)}{k^p \cdot p!} \right)}{x^{p/2}} & \text{k odd}
\end{array} \right.
\]

In terms of these series we have \( G(x, y)+1 = \)

\[
(1 + \frac{R_{2,1}}{2!} x^1 + \frac{3R_{4,1}}{4!} x^2 + \frac{15R_{6,1}}{6!} x^3 + \frac{105R_{8,1}}{8!} x^4 + \frac{945R_{10,1}}{10!} x^5 + \cdots) x
\]

\[
(1 + \frac{R_{1,2}}{2} x^1 + \frac{3R_{2,2}}{8} x^2 + \frac{7R_{3,2}}{48} x^3 + \frac{25R_{4,2}}{384} x^4 + \frac{81R_{5,2}}{3840} x^5 + \cdots) x
\]

\[
(1 + \frac{3R_{2,3}}{18} x^3 + \cdots) x (1 + \frac{R_{1,4}}{4} x^2 + \frac{5R_{2,4}}{32} x^4 + \cdots) x
\]

\[
(1 + \frac{5R_{2,5}}{50} x^5 + \cdots) x (1 + \frac{R_{1,6}}{6} x^3 + \cdots) x (1 + \cdots) x
\]

\[
(1 + \frac{R_{1,8}}{9} x^4 + \cdots) x (1 + \cdots) x (1 + \frac{R_{1,10}}{10} x^5 + \cdots) \cdots =
\]

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1 + (y + y^2)x + (2y + 3y^2 + 2y^3 + y^4)x^2 + (5y + 10y^2 + 10y^3 + 6y^4 + 2y^5 + y^6)x^3 + \ldots$
giving the number of unlabeled 2-cell imbeddings of the bouquet on generalized pseudosurfaces by number of edges \(x\) and number of regions \(y\). Table 4.8 contains this information for up to five edges.

To determine the number of imbeddings which are on pseudosurfaces we have that $G(x, y)$ and $P(x, y)$ are related by

$G(x, y) + 1 = \exp \sum_{k=1}^{\infty} \frac{P(x^k, y^k)}{k}$

and so the coefficients $P_{n, r}$ are determined from the $G_{n, r}$ by comparison. The counting series for unlabeled 2-cell imbeddings of the bouquet on pseudosurfaces by number of edges and number of regions begins

$P(x, y) = (y + y^2)x + (2y + 2y^2 + y^3)x^2 + (5y + 8y^2 + 5y^3 + 2y^4)x^3 + \ldots$

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</tr>
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Generalized pseudosurface
Pseudosurface

Unlabeled 2-cell imbeddings of the bouquet by number of edges \(n\) and number of regions \(r\).

Table 4.7

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4.3.3. Region Distribution

The enumeration of the unlabeled 2-cell imbeddings of the bouquet by number and size of their regions is a difficult task which is met with partial success. In this section the region distribution for unlabeled generalized pseudosurface imbeddings is calculated and from these the pseudosurface imbeddings. The latter have an important interpretation which will be exploited in Chapter VI. The region distribution of unlabeled surface imbeddings, however, remains unknown. Although this is the case, it is shown in the next section that some interesting information about the symmetries of the surface imbeddings can be determined.

To begin, we first show that each congruence class of 2-cell imbeddings of the bouquet on a generalized pseudosurface with a given distribution of regions corresponds to the superposition of two digraphs. Redfield's Enumeration Theorem then provides the means to determine exactly how many unlabeled imbeddings have the given distribution of regions.

We take as the arc set of the bouquet, $B_n$, the set $A = \{1, 2, \cdots, 2n\}$. In this case any permutation $\rho_V$ in $\Sigma_{2n}$ together with any full involution $\rho_E$ in $\Sigma_{2n}$ describe a 2-cell imbedding of a bouquet on a generalized pseudosurface. The region boundaries of the embedding are given by the orbits of the permutation $\rho_R = \rho_V \circ \rho_E$.

Alternatively, any permutation $\rho_R$ in $\Sigma_{2n}$ that has $r_k$ orbits
of length \( k \), \( k = 1, 2, \cdots, 2n \), together with any full involution, \( \rho_E \) in \( \Sigma_{2n} \), also describe a 2-cell imbedding of the bouquet \( B_n \). In this case the rotation on the vertex set is taken to be \( \rho_R \circ \rho_E \) and the pair \( (\rho_R \circ \rho_E, \rho_E) \) describes an imbedding with \( r_k \) regions of size \( k \), \( k = 1, 2, \cdots, 2n \). It follows that the pair \( (\rho_R, \rho_E) \) describes this same imbedding.

Let \( R \) be a partition of \( 2n \) and consider the action of \( \Sigma_{2n} \) on \( \{(\rho_R, \rho_E) \in \Sigma_{2n} \times \Sigma_{2n} \mid \rho_E \text{ is a full involution, } \rho_R \text{ has cycle-type } R\} \) given by \( \alpha : (\rho_R, \rho_E) \to (\alpha \rho_R \alpha^{-1}, \alpha \rho_E \alpha^{-1}) \), \( \alpha \in \Sigma_{2n} \). The number of orbits under this action is the number of unlabeled generalized pseudosurface imbeddings of the bouquet with regions given by the partition \( R \). We now show how each of these orbits corresponds to the superposition of two digraphs.

To a permutation \( \rho_R \) in \( \Sigma_{2n} \) we associate a labeled digraph on the set \( A = \{1, 2, \cdots, 2n\} \) whose components are all directed cycles corresponding in number and size to the cycle representation of \( \rho_R \). The directed edges of this digraph are all 'solid' colored. Similarly, a labeled digraph is associated with a full involution, \( \rho_E \) in \( \Sigma_{2n} \). Its directed edges are 'dashed' colored.

Figure 4.3 depicts a typical pair \( (\rho_R, \rho_E) \) and the labeled superposition of the two digraphs on the same set of vertices. It is clear that the permutations can be recovered from the superposed digraphs. Moreover, if the vertex labels of the superposed digraphs are arbitrarily permuted then the new pair \( (\rho'_R, \rho'_E) \) represented by the labeled superposition is in the same
orbit as \((\rho_R, \rho_E)\) under the action defined above. Thus, by removing the vertex labels (but keeping the edge colors) from the labeled superposed digraphs corresponding to \((\rho_R, \rho_E)\) we obtain a representation of its congruence class. Hence, to determine the number of unlabeled imbeddings of the bouquet that have regions described by the partition \(R\) of \(2n\), we seek the number of different superpositions of two unlabeled digraphs. One of these digraphs has 'solid' colored edges and has components which are directed cycles corresponding in size and number to the parts of the partition \(R\). The other unlabeled digraph has 'dashed' colored edges and each of its components is a directed 2-cycle.

\[
\rho_R = (123)(45)(67)(8) \quad \rho_E = (12)(34)(56)(78)
\]

Figure 4.3. The representation of two permutations as the labeled superposition of two digraphs.

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Observe that if $R$ has $r_k$ parts equal to $k$, $k=1, 2, \cdots, 2n$, then the automorphism group of its associated digraph is isomorphic to $\prod_{k=1}^{2n} \Sigma_k[C_k]$. Similarly, the automorphism group of the digraph associated with the full involution is $\Sigma_n[C_2]$.

Redfield's Enumeration Theorem expresses the number of different superpositions of the two unlabeled digraphs in terms of these two groups and we have the following result.

**Theorem 4.19.** The number of unlabeled 2-cell imbeddings of the bouquet $B_n$ on generalized pseudosurfaces that have $r_k$ regions of size $k$, $k=1, 2, \cdots, 2n$, is $Z(\prod_{k=1}^{2n} \Sigma_k[C_k]) \cap Z(\Sigma_n[C_2])$.

In exactly the same manner as above we can associate the unlabeled surface imbeddings of the bouquet with the different superpositions of two unlabeled digraphs. In this case, one of the digraphs is a $2n$-cycle corresponding to the rotation of arcs at the single vertex, the other has each component a directed 2-cycle. The automorphism group of the first digraph is $C_{2n}$ and we have the next result which will be used in Section 4.4.

**Theorem 4.20.** The number of unlabeled 2-cell surface imbeddings of the bouquet $B_n$ is $Z(C_{2n}) \cap Z(\Sigma_n[C_2])$.

Theorem 4.19 was used to determine the number of unlabeled 2-cell imbeddings on generalized pseudosurfaces by region distribution. From these the number of pseudosurface imbeddings with a given region distribution are determined. The results are tabulated in Table 4.8.
The region distribution of unlabeled 2-cell imbeddings of the bouquet on generalized pseudosurfaces and pseudosurfaces.

Table 4.8

4.4. Enumeration of Surface Imbeddings by Their Symmetries

Although the bouquet has a great deal of symmetry we will see in this section that this property is not shared by most of its surface imbeddings. Here it is shown that all the map-automorphism groups of the bouquet are cyclic and that Redfield's Decomposition Theorem provides a means to determine exactly how many occurrences there are of each cyclic group.
This allows us to enumerate the surface imbeddings, both labeled and unlabeled, by the symmetries they possess.

Theorem 4.20 in Section 4.3.3 states that the number of unlabeled 2-cell imbeddings of the bouquet $B_n$ on surfaces is given by $Z(C_{2n}) \cap Z(\Sigma_n[C_2])$. Let $N$ denote this number. It is clear that congruent maps have isomorphic (as permutation groups) map-automorphism groups. The following is then an application of Redfield's Decomposition Theorem.

**Theorem 4.21.** $Z(C_{2n}) \cup Z(\Sigma_n[C_2]) = \sum_{k=1}^{N} Z(\Gamma(M_k))$ where \{ $M_k$: $k=1, 2, \ldots, N$ \} is a complete set of congruence class representative maps.

One of the shortcomings of cycle indexes is that the decomposition theorem does not, in general, guarantee uniqueness. Thus, the left side of the above equation may be of little help in determining the right side. In our case, however, the computations are straightforward. That is, the number of unlabeled maps that have a prescribed map-automorphism group is readily computable. To see this, we first show that the map-automorphism groups of $B_n$ are isomorphic (as permutation groups) to $C_d[E_{2n/d}]$ for some divisor $d$ of $2n$, where $E_{2n/d}$ is the identity permutation group of degree $2n/d$.

Let $M=(H, \rho_E, \rho_V)$ be a map with $<\rho_V, \rho_E>$ transitive; then the map automorphisms have the following properties.
Lemma 4.22. If an automorphism \( f \in \Gamma(M) \) fixes an arc then \( f \) is the identity automorphism.

**Proof.** Suppose \( fa = a \) for some arc \( a \). From the definition of map automorphism we have \( f \rho_V f^{-1} = \rho_V \) and \( f \rho_E f^{-1} = \rho_E \), or \( f \rho_V = \rho_V f \) and \( f \rho_E = \rho_E f \). Thus, \( f \rho_V a = \rho_V fa = \rho_V a \) and \( f \rho_E a = \rho_E fa = \rho_E a \). Hence, \( f \) also fixes the arcs \( \rho_V a \) and \( \rho_E a \) and must then fix \( a \) for any member \( \rho \) of \( \langle \rho_V, \rho_E \rangle \). The latter group is transitive so \( f \) fixes all arcs. ⊤

Lemma 4.23. Each automorphism \( f \in \Gamma(M) \) is a regular permutation (all its cycles have the same length).

**Proof.** Let \( d \) be the length of the smallest cycle in the disjoint cycle representation of \( f \) and let \( a \) be an arc in an orbit of size \( d \); then \( f^d a = a \). Since \( f^d \) is a map automorphism fixing an arc we conclude from the previous lemma that it is the identity automorphism. From the choice of \( d \) it must be that all the cycles of \( f \) have length \( d \) and, hence, \( f \) is \( d \)-regular. ⊤

The above lemmas are true for any pseudograph 2-cell imbedded on a pseudosurface (transitivity of \( \langle \rho_V, \rho_E \rangle \)). In the case of the bouquet 2-cell imbedded on a surface we have the following.

Theorem 4.24. If \( M = (B_n, \rho_E, \rho_V) \) is a map corresponding to a 2-cell imbedding of a bouquet on a surface then \( \Gamma(M) \cong C_d[E_{2n/d}] \) for some divisor \( d \) of \( 2n \).
Proof. We may assume that \( A(B_n) = \{1, 2, \ldots, 2n\} \). Since the bouquet has a single vertex and \( M \) describes a surface imbedding it follows that \( \rho_V \) is a full cycle in \( \Sigma_{2n} \). From the definition of map automorphism we have \( f^* \rho_V f^{-1} = \rho_V \) for all \( f \in \Gamma(M) \) and, hence, \( \Gamma(M) \) is a subgroup of the centralizer in \( \Sigma_{2n} \) of \( \rho_V \). The centralizer in \( \Sigma_{2n} \) of a full cycle is the subgroup of \( \Sigma_{2n} \) generated by that full cycle and is, therefore, cyclic. Since subgroups of cyclic groups are themselves cyclic, it follows that \( \Gamma(M) \) is cyclic, say \( \Gamma(M) = \langle f \rangle \). By Lemma 4.23 \( f \) is \( d \)-regular for some divisor \( d \) of \( 2n \) and it follows that \( \Gamma(M) = C_d[E_{2n/d}] \).

We refer to a map \( M = (B_n, \rho_E, \rho_V) \) as \( d \)-fold symmetric if \( \Gamma(M) = C_d[E_{2n/d}] \). The next result can be found in [11] and is a consequence of Redfield's Decomposition Theorem.

**Lemma 4.25.** If \( B \) is a permutation group of degree \( m \) and \( \alpha \) is any permutation of the \( m \) symbols which has order \( r \) then

\[
Z(\langle \alpha \rangle) \cup Z(B) = \sum_{k|r} i_k \cdot Z(\langle \alpha^k \rangle),
\]

where the \( i_k \) are unique nonnegative integers.

Choosing \( \alpha \) so that \( C_{2n} = \langle \alpha \rangle \) we have the following.

**Corollary.** \( Z(C_{2n}) \cup Z(\Sigma_n[C_2]) = \sum_{d|2n} i_d \cdot Z(C_d[E_{2n/d}]) \), where the \( i_d \) are unique nonnegative integers.

By combining the corollary with Theorem 4.21 and Theorem 4.24 we present the main theorem of this section.
Theorem 4.25. For the bouquet $B_{2n}$ there are $i_d$ congruence classes of $d$-fold symmetric 2-cell surface imbeddings for each divisor $d$ of $2n$, where the integers $i_d$ are the unique solutions to $Z(C_{2n}) \cup Z(X_n[C_2]) = \sum_{d|2n} i_d \cdot Z(C_d[E_{2n/d}])$.

We illustrate the calculations for the bouquet $B_3$.

$Z(C_2) = \frac{1}{2} (s_1^2 + s_2)$ and $Z(C_3) = \frac{1}{6} (s_1^3 + 3s_1s_2 + 2s_3)$ imply

$Z(X_3[C_2]) = \frac{1}{6} \left[ \left( \frac{1}{2}(s_1^2 + s_2) \right)^3 + 3 \left( \frac{1}{2}(s_1^2 + s_2) \right) \left( \frac{1}{2}(s_2^2 + s_4) \right) + 2 \left( \frac{1}{2}(s_2^2 + s_6) \right) \right]$

$= \frac{1}{48} (s_1^6 + 3s_1^4s_2 + 9s_1^2s_2^2 + 7s_2^3 + 6s_1s_4 + 6s_2s_4 + 8s_4^2 + 8s_6)$. 

$Z(C_6) = \frac{1}{6} (s_1^6 + s_2^3 + 2s_3^2 + 2s_6)$. 

$Z(C_6) \cup Z(X_3[C_2]) = \frac{1}{6 \cdot 48} \left( 720s_1^6 + 7 \cdot 48s_2^3 + 16 \cdot 18s_3^2 + 15 \cdot 6s_6 \right)$

$= \frac{5}{2} s_1^6 + \frac{7}{6} s_2^3 + 1s_3^2 + \frac{1}{3} s_6$

$= i_6 Z(C_6[E_1]) + i_3 Z(C_3[E_2]) + i_2 Z(C_2[E_5]) + i_1 Z(C_1[E_6])$

$= i_6 (\frac{1}{6} s_1^6 + \frac{1}{6} s_2^3 + \frac{1}{3} s_3^2 + \frac{1}{3} s_6) +$

$= i_3 (\frac{1}{3} s_1^6 + \frac{2}{3} s_2^3 ) +$

$= i_2 (\frac{1}{2} s_1^6 + \frac{1}{2} s_2^3 ) +$

$= i_1 (s_1^6 )$. 

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Thus, we are led to solve the system:

\[
\begin{bmatrix}
\frac{1}{6} & \frac{1}{3} & \frac{1}{2} & 1 \\
\frac{1}{6} & 0 & \frac{1}{2} & 0 \\
\frac{1}{3} & \frac{2}{3} & 0 & 0 \\
\frac{1}{3} & 0 & 0 & 0
\end{bmatrix}
\cdot
\begin{bmatrix}
i_6 \\
i_3 \\
i_2 \\
i_1
\end{bmatrix}
= 
\begin{bmatrix}
\frac{5}{2} \\
\frac{7}{6} \\
1 \\
\frac{1}{3}
\end{bmatrix}
\]

Note that the coefficient matrix is triangular and, hence, the solution of the system is straightforward (in fact, these systems are always triangular). The solutions are \(i_6=1\), \(i_3=1\), \(i_2=2\), and \(i_1=1\). Thus, there are five congruence classes of 2-cell surface imbeddings of \(B_3\), one of which is 6-fold, one 3-fold, two 2-fold, and one 1-fold symmetric. Figure 4.4 shows a representative of each class and their properties.

It appears from the example that a great deal of effort is required to determine the symmetries, primarily in calculating \(Z(C_{2n}) \cup Z(\Sigma_n[C_2])\). We have, however, already simplified the calculations considerably in Section 4.3.1. The final result has been presented in Theorem 4.13 which only needs to be reinterpreted as follows.

**Theorem 4.27.** \(Z(C_{2n}) \cup Z(\Sigma_n[C_2]) = \)

\[
\sum_{d|n} \frac{\phi(d)(\frac{2d}{d}-1)!d^{\frac{n}{d}-1}s_{d^2}}{2^{n/d} \cdot (\frac{n}{d})!} + \sum_{d|n} \sum_{m=0}^{\frac{n-1}{2d}} \frac{\phi(2d)(\frac{2d}{d}-1)!d^{m-1}s_{2d^2}}{2 \cdot (\frac{n}{d}-2m)! \cdot m!}.
\]
### Figure 4.4. Symmetries of the five unlabeled surface imbeddings of the bouquet $B_3$.

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<td>(15)(34)(26)</td>
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<td>symmetry</td>
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<td>48</td>
<td>8</td>
<td>24</td>
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No attempt will be made to solve the systems in general beyond the following remarks. It is quite easy to see that in each case there is exactly one congruence class of 2-cell surface imbeddings of the bouquet $B_n$ which has $2n$-fold symmetry (the largest possible map-automorphism group is $C_{2n}$). With only slightly more effort it can be seen that the number of congruence classes of 2-cell surface imbeddings of the bouquet $B_n$ with $n$-fold symmetry is $\left\lfloor \frac{n}{2} \right\rfloor$.

Table 4.9 lists the number of unlabeled 2-cell surface imbeddings of $B_n$ which are $d$-fold symmetric. Note the relatively large values for $d=1$. These are the 1-fold or asymmetric imbeddings. Although the bouquet has a great deal of symmetry most of its unlabeled surface imbeddings have none.

Given a labeled bouquet $B_n$ and a map $M=(B_n, \rho_E, \rho_V)$, the number of labeled 2-cell imbeddings on surfaces congruent to the imbedding described by $M$ is the index $[\Gamma(B_n) : \Gamma(M)]$. If the map $M$ is $d$-fold symmetric then there are $2^n \cdot n! + d$ imbeddings in its congruence class. Hence, from the information contained in Table 4.9 we construct Table 4.10 listing the number of labeled 2-cell surface imbeddings having $d$-fold symmetry.
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Number of unlabeled 2-cell surface imbeddings of the bouquet \( B_n \) which are \( d \)-fold symmetric.

Table 4.9.
Number of labeled 2-cell surface imbeddings of the bouquet $B_n$ which are $d$-fold symmetric.

Table 4.10.

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CHAPTER V

2-CELL IMBEDDINGS OF THE DIPOLE

5.1. Introduction

The 2-cell imbeddings of the dipole are interesting not only in their own right, but in light of the following observations. The dipole, like the bouquet, is useful as a voltage graph (see [29], for example). Moreover, hypermaps, the natural generalization of maps to be defined in Chapter VI, correspond with the 2-cell imbeddings of the dipole.

5.2. Labeled 2-cell Imbeddings of the Dipole

The arc set, edge set, and vertex set of the dipole with n edges, \( D_n \), are taken as follows:

\[
\begin{align*}
A(D_n) &= \{1, 2, 3, \ldots, 2n\}, \\
E(D_n) &= \{ \{1, 2\}, \{3, 4\}, \ldots, \{2n-1, 2n\} \}, \text{ and} \\
V(D_n) &= \{ \{1, 3, 5, \ldots, 2n-1\}, \{2, 4, 6, \ldots, 2n\} \}.
\end{align*}
\]

The rotation on the edge set \( E \) is then \((1 2)(3 4)\cdots(2n-1 2n)\). The number of possible rotations on the vertex set \( V \) is \( n!^2 \) giving the number of labeled 2-cell imbeddings of \( D_n \) on generalized pseudosurfaces. From these we obtain the number on pseudosurfaces using the techniques outlined in chapter 3.

The unlabeled 2-cell imbeddings and rotation schemes describing the labeled imbeddings of \( D_3 \) are shown in Figure 5.1.
Figure 5.1. The 8 unlabeled 2-cell imbeddings of $D_3$ together with the 35 rotation schemes describing labeled 2-cell imbeddings ($\rho_E = (12)(34)(56)$ and $\rho_R = \rho_V \circ \rho_E$).
5.2.1. Enumeration by Number of Edges

For the labeled problem we use exponential counting series defined as follows:

\[ G_n = \frac{n!}{[x^n]} G(x) = \text{number of labeled 2-cell imbeddings of } D_n \text{ on generalized pseudosurfaces}, \]

\[ P_n = \frac{n!}{[x^n]} P(x) = \text{number above on pseudosurfaces}, \]

\[ S_n = \frac{n!}{[x^n]} S(x) = \text{number above on surfaces}. \]

There are \( n!^2 \) possible rotations on \( V \) giving

\[ G(x) = \frac{1x}{1!} + \frac{4x^2}{2!} + \frac{35x^3}{3!} + \cdots + \frac{n!^2 x^n}{n!} + \cdots. \]

The series \( G \) and \( P \) are related by \( G+1=\exp(P) \) and hence, using the recursion given in Chapter 3,

\[ P_n = G_n - \sum_{k=1}^{n-1} \binom{n-1}{k-1} P_k G_{n-k}, \]

we find that

\[ P(x) = \frac{1x}{1!} + \frac{3x^2}{2!} + \frac{25x^3}{3!} + \frac{426x^4}{4!} + \cdots. \]

This information is summarized in Table 5.1.

The labeled 2-cell imbeddings on surfaces occur when the rotation on \( V \) is cyclic and there are \((n-1)!^2\) such occurrences giving

\[ S(x) = \frac{1x}{1!} + \frac{x^2}{2!} + \frac{4x^3}{3!} + \cdots + \frac{(n-1)!^2 x^n}{n!} + \cdots. \]

5.2.2. Enumeration by Number of Edges and Number of Regions

We use a counting series with two indeterminants which is exponential in the indeterminant counting edges \((x)\) and ordinary in the indeterminant counting regions \((y)\). The series enumerating labeled 2-cell imbeddings of the dipole with \( n \) edges
and \( r \) regions are defined as follows:

\[
G_{n,r} = [y^r]G_n(y) = n! [x^n, y^r]G(x,y) = \text{number on generalized pseudosurfaces,}
\]

\[
P_{n,r} = [y^r]P_n(y) = n! [x^n, y^r]P(x,y) = \text{number on pseudosurfaces,}
\]

\[
S_{n,r} = [y^r]S_n(y) = n! [x^n, y^r]S(x,y) = \text{number on surfaces,}
\]

where \( G_n(y), P_n(y), \) and \( S_n(y) \) are polynomials enumerating the number of labeled 2-cell imbeddings of \( D_n \) by number of regions.

When we were calculating \( G(x,y) \) for the bouquet we could choose any permutation, \( \rho_R \) in \( \Sigma_A \), and be certain there was exactly one rotation on \( V \) such that \( \rho_{V^*} \rho_{E} = \rho_R \). This is no longer true for the dipole because of the presence of two vertices. For example, if \( (13)(24) \) is chosen for \( \rho_R \) then \( \rho_{V^*} \rho_{E} = \rho_R \) has no solution when \( V = \{ 1,3 \}, \{ 2,4 \} \) and \( E = \{ 1,2 \}, \{ 3,4 \} \).

To overcome this difficulty we present an alternative means of describing the 2-cell imbeddings of the dipole. Rather than

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G: Generalized pseudosurfaces
P: Pseudosurfaces
S: Surfaces

Number of labeled 2-cell imbeddings of the dipole by number of edges.
describing the rotation of arcs at each vertex we instead
describe the rotation of edges. Hence, the permutations will be
members of $\Sigma_E$ instead of $\Sigma_A$. Once this has been accomplished
we will see that information about the regions of a particular
imbedding is easily obtained.

To illustrate what is to follow consider the dipole $D_3$ with
arc set $A=\{1^+,1^-,2^+,2^-,3^+,3^-\}$,
edge set $E=\{\{1^+,1^\}\},\{2^+,2^-\},\{3^+,3^-\}$, and
vertex set $V=\{\{1^+,2^+,3^+\},\{1^-,2^-,3^-\}\}$. It is clear that each
edge can be identified with one of 1, 2, or 3, and the two
vertices with one of '+' or '-' . If the rotation on $V$ is chosen to
be $\varphi_V=(1^+2^+)(3^+)(1^-2^-3^-)$ and, of course, the rotation on $E$ is
$\varphi_E=(1^+1^-)(2^+2^-)(3^+3^-)$ then in this case $\varphi_R=(1^+2^-)(2^+3^-3^+1^-)$.
Thus, there is a 2-sided region and a 4-sided region in this
imbedding.

Now consider the permutation in $\Sigma_E$ obtained by deleting
the superscripts in the expression for $\varphi_V$ above. Denote this
permutation as $\sigma_V$; then $\sigma_V=(12)(3)(123)$ which is the
composition of the rotation of edges at the '+' vertex with that at
the '-' vertex. Performing the composition we see that
$\sigma_V=(1)(23)$. This permutation (of edges) can no longer be
interpreted as describing the region boundaries but it does retain
some valuable information about the region sizes. Comparing it
to $\varphi_R=(1^+2^-)(2^+3^-3^+1^-)$ we see that $(1)(23)$ has cycles of
precisely half the length of those appearing in $\varphi_R$.

Observe that $\sigma_V$ is not well defined since $\varphi_V$ can also be
written as \((1^-2^-3^-)(1^+2^+)(3^+),\) in which case \(\sigma_V\) would be \((123)(12)(3^-)=(13)(2).\) However, also observe the end result has the same distribution of cycles. Hence, the claim is that the cycle distribution is invariant and corresponds to the region distribution in the manner suggested. The formal statement and proof of this claim is now presented.

**Theorem 5.1.** Let \(D_n\) be a dipole. Each labeled 2-cell imbedding of \(D_n\) is uniquely determined by a pair of permutations of the edge set and the imbedding has \(r_k\) regions of size \(2k\) if and only if the composition of the two permutations has \(r_k\) cycles of length \(k\). Moreover, the imbedding is on a pseudosurface if and only if the pair of permutations generate a transitive subgroup of \(\Sigma_\mathcal{E}\) and is on a surface if and only if both are cyclic.

**Remark.** When traversing a region boundary of the dipole we must alternate between the two vertices so it follows that every region has even length.

**Proof.** In an arbitrary manner, label the edges of the dipole with the integers 1 through \(n\) and label one of the vertices '+', the other '-' . The arc of an edge \(k\) directed away from the '+' vertex is then labeled \(k^+\), the arc directed away from the '-' vertex is labeled \(k^-\). Each labeled 2-cell imbedding of this dipole is uniquely determined by a rotation of arcs at the '+' vertex and one at the '-' vertex. The two together are encoded as a rotation on \(V, \rho_V\). Let \(\sigma_+\) and \(\sigma_-\) be the rotations of edges.
corresponding, in a natural way, to the rotations of arcs at the
'+' and '−' vertices, respectively. Hence, \( \sigma_+ \) and \( \sigma_- \) are
members of \( \Sigma_E \). Since the rotations of arcs can be recovered
from \( \sigma_+ \) and \( \sigma_- \), it follows that each labeled 2-cell imbedding of
the dipole is uniquely determined by this pair. Moreover,
\[ \langle \rho_V, \rho_E \rangle \] is a transitive subgroup of \( \Sigma_A \) if and only if \( \langle \sigma_+, \sigma_- \rangle \)
is a transitive subgroup of \( \Sigma_E \); and \( (\rho_V, \rho_E) \) corresponds to an
imbedding on a surface if and only if the rotations of the arcs
at each vertex are cyclic if and only if \( \sigma_+ \) and \( \sigma_- \) are both
cyclic. It remains to verify the relationship between the length
of the region boundaries and the length of the cycles appearing
in the disjoint cycle representation of the permutation \( \sigma_+ \sigma_- \).

Keep in mind that if \( e = \{ e^+, e^- \} \) is an edge then \( \rho_E e^+ = e^- \) and
\( \rho_E e^- = e^+ \). Also, each region boundary must have length at least
two, and when traversing a boundary we must alternately visit
the '+' vertex and the '−' vertex. It follows that the sequence
of arcs traversed alternates between a '+' arc and a '−' arc
and, hence, every region boundary contains at least one '+'
arc. Figure 5.2 then gives an accurate portrayal of a segment
of oriented region boundary. Shown are the arc-labeled
boundary and the corresponding edge-labeled boundary. From
the figures the following strings of equalities are evident:
\[
(\rho_V \rho_E)^2 e_1^+ = \rho_V \rho_E^2 \rho_V e_1^- = \rho_V \rho_E e_2^- = \rho_V e_2^+ = e_3^+ \quad \text{and}
(\sigma_+ \sigma_-)^1 e_1 = \quad \sigma_+ e_2 = \quad e_3.
\]
It follows that \( (\sigma_+ \sigma_-)^k e = e \) if and only if \( (\rho_V \rho_E)^{2k} e^+ e^- = e^+ \) and,
hence, the imbedding has \( r_k \) regions of size \( 2k \) if and only if the
 permutation $\sigma_1 \sigma_-$ has $r_k$ cycles of length $k$.

Remark. A generalization of the theorem applies to an arbitrary multistar and will be presented in Chapter VI.

Figure 5.2. Boundary segments of a 2-cell imbedded dipole.

It follows from Theorem 5.1 that the number of labeled 2-cell imbeddings on generalized pseudosurfaces with $r$ regions of the dipole $D_n$ is the number of solutions in $\Sigma_n$ to $\sigma_1 \sigma_- \sigma_R$ where $\sigma_R$ has exactly $r$ orbits. Recall that the number of permutations in $\Sigma_n$ with $r$ orbits is the absolute value of the Stirling number of the first kind, $|s_n^{(r)}|$. For each of these there are $n!$ solutions to the above equation. Hence, we have $G_{n,r} = n! |s_n^{(r)}|$ and it follows that

$$n! [x^n] G(x, y) = n! \left( |s_n^{(1)}| y + |s_n^{(2)}| y^2 + \cdots + |s_n^{(n)}| y^n \right)$$

$$= n! y(y+1)(y+2)\cdots(y+n-1) \text{ and so}$$

$$G(x, y) = \sum_{n=1}^{\infty} y(y+1)(y+2)\cdots(y+n-1) x^n.$$

Using the recurrence equation for the Stirling numbers we have the following recurrence equation for the $G_{n,r}$.
\( G_{n,r} = n G_{n-1,r-1} + n(n-1) G_{n-1,r} \) (5.1)

Since \( P \) and \( G \) are related by \( G+1=\exp(P) \) we can use

\[ P_n(y) = G_n(y) - \sum_{k=1}^{n-1} \binom{n-1}{k-1} P_k(y) G_{n-k}(y), \]

where \( P_k(y) = k! [x^k] P(x,y) \) and \( G_k(y) = k! [x^k] G(x,y) \), to determine the coefficients of \( P \) from \( G \). The results are displayed in Table 5.2. The series \( G \) and \( P \) begin

\( G(x,y) = (y)x/1! + (2y + 2y^2)x^2/2! + (12y + 18y^2 + 6y^3)x^3/3! \cdots, \)

\( P(x,y) = (y)x/1! + (2y + 1y^2)x^2/2! + (12y + 12y^2 + 2y^3)x^3/3! \cdots. \)

To determine \( S(x,y) \) it is necessary to find the number of solutions in \( \Sigma_n \) to \( \sigma_+ \sigma_- = \sigma_R \) where \( \sigma_+ \) and \( \sigma_- \) are full cycles and \( \sigma_R \) has \( r \) orbits. The solution is again found from the work of Jackson summarized in Section 3.3.

Recall that \( e_r^{(d)}(N) \) is the number of solutions in \( \Sigma_{dN} \) to \( \sigma_+ \sigma_- = \sigma_R \) where \( \sigma_+ \) is a fixed full cycle, \( \sigma_- \) has all \( d \)-cycles, and \( \sigma_R \) has \( r \) orbits. In the present case \( d=n \) and \( N=1 \). Thus, in this case, \( e_r^{(n)}(1) \) is the number of solutions in \( \Sigma_n \) to \( \sigma_+ \sigma_- = \sigma_R \) where \( \sigma_+ \) is a fixed \( n \)-cycle, \( \sigma_- \) is any \( n \)-cycle and \( \sigma_R \) has \( r \) orbits. Since there are \( (n-1)! \) choices for the fixed \( n \)-cycle it follows that \( S_{n,r} = (n-1)! e_r^{(n)}(1) \).

From Theorem 3.2 in Section 3.3 the following expression is obtained.

\[ e_r^{(n)}(1) = \frac{1}{(1+n) \cdot n^{1+r}} \sum_{m=r+1}^{1+n} \left( \begin{array}{c} m \\ r \end{array} \right) s_{n+1}^{(m)} \cdot S_{m-r}^{(1)}. \]
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Generalized pseudosurface
Pseudosurface
Surface

Labeled 2-cell imbeddings of the dipole by number of edges \((n)\) and number of regions \((r)\).

Table 5.2

The number \( S_{m-r}^{(1)} \) is 1 in all cases so that

\[
(n+1) e_r^{(n)} (1) = \sum_{m=r+1}^{n+1} \binom{n}{m} e^{(r+1)} \binom{m}{r} S_{n+1}^{(m)} \quad \text{or}
\]
\[(n+1)e_r^{(n)}(1) = \sum_{m=r}^{n+1} m^{-(r+1)} \binom{m}{r} s_{n+1}^{(m)} - \frac{1}{n} s_{n+1}^{(r)} \quad \text{and} \]
\[n \cdot e_r^{(n-1)}(1) = \sum_{m=r}^{n} (n-1)^{m-r-1} \binom{m}{r} s_{n}^{(m)} - \frac{1}{n-1} s_{n}^{(r)}. \quad (5.2)\]

The Stirling numbers satisfy the equation
\[x(x-1)(x-2) \cdots (x-n+1) = \sum_{m=0}^{n} s_{n}^{(m)} x^m; \text{ substituting } y+n-1 \text{ for } x \]
and using the binomial theorem we have
\[(y+n-1)(y+n-2) \cdots (y+n-1-n+1) = \sum_{m=0}^{n} s_{n}^{(m)} (y+n-1)^m \]
\[= \sum_{m=0}^{n} s_{n}^{(m)} \sum_{r=0}^{m} \binom{m}{r} y^r (n-1)^{m-r}. \]

The left side of this equation is \(y(y+1)(y+2) \cdots (y+n-1),\) which is recognized to be \(\sum_{r=0}^{n} s_{n}^{(r)} \cdot y^r.\) On the right side we reverse the summations to obtain
\[\sum_{r=0}^{n} s_{n}^{(r)} \cdot y^r = \sum_{r=0}^{n} \sum_{m=r}^{n} \binom{m}{r} y^r (n-1)^{m-r} \cdot s_{n}^{(m)} \]
\[= \sum_{r=0}^{n} y^r (n-1)^{\frac{1}{r}} \sum_{m=r}^{n} (n-1)^{m-r-1} \binom{m}{r} \cdot s_{n}^{(m)}. \]

The sum over \(m\) appearing on the right side is the same as that appearing in (5.2). Making the appropriate substitution we have
\[\sum_{r=0}^{n} s_{n}^{(r)} \cdot y^r = \sum_{r=0}^{n} (n-1)^{\frac{1}{r}} \cdot y^r (n \cdot e_r^{(n-1)}(1) + \frac{1}{n-1} \cdot s_{n}^{(r)}) \]
\[= \sum_{r=0}^{n} n(n-1)^{\frac{1}{r}} e_r^{(n)}(1) y^r + \sum_{r=0}^{n} s_{n}^{(r)} y^r , \quad \text{and} \]
therefore,
\[\sum_{r=0}^{n} (s_{n}^{(r)} - s_{n}^{(r)}) \cdot y^r = \sum_{r=0}^{n} n(n-1)^{\frac{1}{r}} e_r^{(n-1)}(1) y^r. \]

By comparing coefficients of \(y^r\) it follows that
\[n(n-1)^{\frac{1}{r}} e_r^{(n-1)}(1) = |s_{n}^{(r)}| - s_{n}^{(r)} \quad \text{and, hence, the following theorem results.} \]
Theorem 5.2. The number, $S_{n,r}$, of labeled 2-cell imbeddings of the dipole $D_n$ which occur on surfaces with $r$ regions is given by

$$S_{n,r} = \frac{(n-1)!}{n(n+1)} \left( |s_{n+1}^{(r)}| - s_{n+1}^{(r)} \right).$$

Corollary. The fraction of all 2-cell imbeddings on surfaces of the dipole $D_n$ that have $r$ regions (where $r$ has the same parity as $n$) equals twice the fraction of all permutations in the symmetric group of degree $n+1$ that have $r$ orbits.

Proof. There are a total of $(n-1)!^2$ 2-cell imbeddings of $D_n$ on surfaces; the fraction that have $r$ regions is then

$$\frac{S_{n,r}}{(n-1)!^2},$$

which by the previous corollary is

$$\frac{2|s_{n+1}^{(r)}|}{(n+1)(n)(n-1)!},$$

which simplifies to give the latter fraction.

Corollary. $(n+1)S_{n,r} = (2n-1)(n-1)S_{n-1,r-1} + (n-1)^3(n-2)^3S_{n-2,r}$

$$- (n-1)(n-2)^3S_{n-2,r-2}$$

with boundary conditions $S_{n,r} = 0$ if $r \leq 0$, $n \leq 0$, or $n < r$, and $S_{1,1} = 1$.

Proof. Apply the recursion $|s_n^{(r)}| = |s_{n-1}^{(r-1)}| + (n-1)|s_{n-1}^{(r)}|$. 

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of the Stirling numbers to the result of the first corollary.

For a dipole, $D_n$, 2-cell imbedded on a surface with $r$ regions, the genus $g$ of the surface is given by $g = \frac{n-r}{2}$. Hence, we have the last corollary of the theorem.

**Corollary.** The number of labeled 2-cell imbeddings of $D_n$ on a surface of genus $g$ is

$$\frac{2(n-1)!}{(n+1)(n)} \left\{ \left( \begin{array}{c} n-2g \\ n+1 \end{array} \right) \right\}.$$

5.2.3. Region Distribution

The exponential counting series enumerating labeled 2-cell imbeddings of $D_n$ with $r_k$ regions of size $2k$ are as follows:

$n! \left[ x^n y_1^{r_1} y_2^{r_2} \cdots y_n^{r_n} \right] G(x, y_1, y_2, \cdots) = \text{number on generalized pseudosurfaces,}$

$n! \left[ x^n y_1^{r_1} y_2^{r_2} \cdots y_n^{r_n} \right] P(x, y_1, y_2, \cdots) = \text{number on pseudosurfaces,}$

$n! \left[ x^n y_1^{r_1} y_2^{r_2} \cdots y_n^{r_n} \right] S(x, y_1, y_2, \cdots) = \text{number on surfaces.}$

There are $\frac{n!}{\prod_{k=1}^{n} k^{r_k} r_k}$ permutations in $\Sigma_n$ which have $r_k$ cycles of length $k$, $k=1,2,\ldots,n$. Let $\sigma_R$ be one of these; then for each $\sigma_+ \in \Sigma_n$ there is a unique $\sigma_- \in \Sigma_n$ such that $\sigma_+ \cdot \sigma_- = \sigma_R$.

Thus, there are $n! \frac{n!}{\prod_{k=1}^{n} k^{r_k} r_k}$ labeled 2-cell imbeddings of $D_n$ which have $r_k$ regions of size $2k$, $k=1,2,\ldots,n$. Hence, if $R=(r_1,r_2,\ldots,r_n)$ is a partition of $n$ then

$$G(x, y_1, y_2, \cdots) = \sum_{n=1}^{\infty} \frac{x^n}{n!} \sum_{R\in n} \frac{(n!)^2}{\prod_{k=1}^{r_k} k^{r_k} r_k} y_1^{r_1} y_2^{r_2} \cdots y_n^{r_n}.$$
For each $n$ let $G_n$ be the polynomial given by

$$G_n(y_1, y_2, \ldots, y_n) = \sum_{R \in \mathcal{R}_n} \frac{(n!)^2}{\prod_{k=1}^{n} \frac{1}{k^k \cdot r_k!}} y_1^{r_1} y_2^{r_2} \cdots y_n^{r_n}.$$ 

In terms of these polynomials we have $G = \sum_{n=1}^{\infty} \frac{G_n x^n}{n!}$. Since $G$ and $P$ are related by $G+1=\exp(P)$ we have

$$P_n = G_n - \sum_{k=1}^{n-1} \binom{n-1}{k-1} P_k G_{n-k},$$

so the series $P$ begins

$$P(x) = G_1 x_1 + (G_2 - G_1^2) x_2^2 + (G_3 - 3G_2 G_1 + 2G_1^3) x_3^3 + \cdots.$$ 

This is the same result obtained for the bouquet in Section 4.2.3.

Although the relationship between $P_n$ and the $G_k$ is the same for the dipole as the bouquet, the polynomials $G_k$ are different (and, hence, so are the $P_n$).

To determine the number of labeled 2-cell imbeddings having a specified region distribution we proceed as follows. Let $R=r_1^2 r_2^2 \cdots r_n^2$ denote the target region distribution where there are $r_k$ regions of size $2k$; then the number of labeled 2-cell imbeddings of $D_n$ on generalized pseudosurfaces having this region distribution is $\frac{(n!)^2}{\prod_{k=1}^{n} \frac{1}{k^k \cdot r_k!}} y_1^{r_1} y_2^{r_2} \cdots y_n^{r_n}$ in the polynomial $G_n$. To determine how many of these occur on pseudosurfaces the procedure illustrated by example in Section 4.2.3 is followed. Since the procedure remains the same the details are omitted here and we proceed to the determination of the region distribution on surfaces. The results for $D_n$, $n=1,2,\ldots,6$, are given in Table 5.3.

To determine the number of 2-cell imbeddings of $D_n$ on surfaces having $j_s$ regions of size $2s$, $s=1,2,\ldots,n$, it is necessary
to determine how many solutions to $\sigma_+ \circ \sigma_- = \sigma_R$ exist in $\Sigma_n$ where $\sigma_+$ and $\sigma_-$ are n-cycles and $\sigma_R$ is any permutation having $j_s$ cycles of length $s$, $s=1,2,\ldots,n$. Let $i$ and $j$ index the class of n-cycles in $\Sigma_n$ and $k$ index the class of cycle type $(j_1,j_2,\ldots,j_n)$. In this instance the class multiplication coefficient $n_{ijk}$ gives the number of solutions to $\sigma_+ \circ \sigma_- = \sigma_R$ when $\sigma_+$ and $\sigma_-$ are full cycles and $\sigma_R$ is a fixed member of the class $k$. Since the order of this class is $\frac{n!}{\prod_{s=1}^n j_s!}$ the number we seek is $n_{ijk} \frac{n!}{\prod_{s=1}^n j_s!}$.

From (3.3) in Section 3.3 we have

$$n_{ijk} = \frac{h_i h_j}{n!} \sum \frac{\chi_i \chi_j \chi_k}{\chi_1},$$

where in this case $h_i = h_j = (n-1)!$ and the sum extends over all irreducible characters of $\Sigma_n$.

Since at least one of the classes is that of the full cycles we can again use the results of Jackson to determine the character values. Specifically, we have

$$\chi_i^{(\lambda)} = \chi_j^{(\lambda)} = [y^\lambda] \frac{1}{1+y} (1-(-y)^n)^i = (-1)^\lambda,$$

$$\chi_k^{(\lambda)} = [y^\lambda] \frac{1}{1+y} \prod_{s=1}^n (1-(-y)^s)^{j_s},$$

and

$$\chi_1^{(\lambda)} = [y^\lambda] \frac{1}{1+y} (1-(-y)^1)^n = [y^\lambda] \frac{(1+y)^n}{1+y} = (n-1)^\lambda.$$

Substituting these values into the expression for $n_{ijk}$, multiplying by $\frac{n!}{\prod_{s=1}^n j_s!}$ and simplifying, we obtain the following result.

**Theorem 5.3.** The number of labeled 2-cell imbeddings of $D_n$ on surfaces having $j_s$ regions of size $2s$, $s=1,2,\ldots,n$, is
As an example of a typical calculation using the theorem we determine the number of 2-cell imbeddings of $D_5$ with one 2-sided region and two 4-sided regions. The partition of $n$ corresponding to this region distribution is $(1,2,0,0,0)$ and, hence,

\[
\frac{(n-1)!^2}{\prod_{s=1}^{n} \sum_{\lambda=0}^{j_s} \frac{1}{(n-1)!}} \left[ y^\lambda \right] \frac{\prod_{s=1}^{n} (1-(-y)^s)^{j_s}}{(1+y)^{1+y}}
\]

\[
= \frac{1}{1+y} (1-(-y)^1)^{1+y} (1-(-y)^2)^{1+y} (1+y)(1-y)^3
\]

\[
= 1 - 2y^2 + y^4 = 1 + Oy - 2y^2 + Oy^3 + y^4.
\]

The coefficients are multiplied by $1, 1/4, 1/6, 1/4$, and $1$, respectively. Thus,

\[
\frac{(4!)^2}{1^1 \cdot 1! \cdot 2^2 \cdot 2!} (1 + 0 - 2/6 + 0 + 1) = 120 \text{ imbeddings}.
\]

5.3. Unlabeled 2-cell Imbeddings

Let $H$ be a dipole with arc set $A=\{1^+,1^-,2^+,2^-,\ldots,n^+,n^-\}$, edge set $E=\{\{1^+,1^\},\{2^+,2^\},\ldots,\{n^+,n^-\}\}$, and vertex set $V=\{\{1^+,2^+,\ldots,n^+\},\{1^-,2^-,\ldots,n^-\}\}$. In Section 5.2.2 it was shown that each labeled 2-cell imbedding of $H$ corresponds to a pair of edge rotations, $\sigma_+$ and $\sigma_-$, where '+' refers to the vertex $\{1^+,2^+,\ldots,n^+\}$, and '-' refers to the vertex $\{1^-,2^-,\ldots,n^-\}$. Each edge is identified by one of the labels 1 through $n$ in an obvious manner. Thus, $\sigma_+=(12)(3)$ and $\sigma_-=(132)$ is a typical pair for $H=D_4$. The correspondence was established by showing that each pair, $\sigma_+$ and $\sigma_-$, describes a map.
The region distribution of labeled 2-cell imbeddings of the dipole.

Table 5.3

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**D₃**

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**D₆**

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G: Generalized pseudosurface imbeddings
P: Pseudosurface imbeddings
S: Surface imbeddings

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Recall that two maps on $H$, $(H, \rho^+_E, \rho^+_V)$ and $(H, \rho'_E, \rho'_V)$, correspond to congruent imbeddings if there exists an automorphism of $H$ (an arc mapping), say $\alpha \in \Gamma(H)$, such that $\alpha \rho^+_E \alpha^{-1} = \rho'_E$ and $\alpha \rho^+_V \alpha^{-1} = \rho'_V$. For the dipole, $\Gamma(H)$ is isomorphic to the group $\Sigma_2 \times \Sigma_2$. The question is, when do two pairs, $\{\sigma_+, \sigma_\perp\}$ and $\{\sigma'_+, \sigma'\perp\}$, correspond to congruent imbeddings? The answer is when there exists $(\alpha, \beta)$ in the group $\Sigma_2 \times \Sigma_2(-, -)$ such that $\{\alpha \sigma_2(-) \alpha^{-1}, \alpha \sigma_2(-) \alpha^{-1}\} = \{\sigma'_+, \sigma'_\perp\}$. The permutation $\alpha$ is viewed as permuting the edges, and $\beta$ as permuting the vertices.

For example, $\sigma_+ = (12)(3)$ and $\sigma_\perp = (132)$ correspond to an imbedding congruent to the one corresponding to $\sigma'_+ = (123)$ and $\sigma'_\perp = (1)(23)$ via $(\alpha, \beta) = ((13)(2), (+-))$. Hence, edge 1 and 3 have been transposed as well as the two vertices. The corresponding maps and map automorphism are

$$(H, (1+1^-)(2+2^-)(3+3^-), (1^+2^+)(3^+)(1^-3^-2^-)),$$

$$(H, (1^+1^-)(2+2^-)(3+3^-), (1^+2^+3^+)(1^-)(3^-2^-)),$$

and

$$f = (1^+-3^-)(2+2^-)(1^-3^+),$$

respectively.

Thus, to determine the number of unlabeled 2-cell imbeddings of the dipole we let the group $\Sigma_2 \times \Sigma_2$ act on pairs of permutations in $\Sigma_n$ and determine the number of orbits. The problem with this direct approach is that the means to determine the pseudosurface imbeddings from the generalized pseudosurface imbeddings has been lost. Previously, with the bouquet, since there is a single vertex, any number of pseudosurface imbeddings can be combined in a unique way to give a generalized pseudosurface imbedding by identifying the
vertices. This is no longer the case for the unlabeled dipole since it has two vertices. In Figure 5.3 is shown an example of two pseudosurface imbeddings which can be combined in two different ways to give two distinct generalized pseudosurface imbeddings.

![Figure 5.3. Two unlabeled pseudosurface imbeddings of the dipole which give different generalized pseudosurface imbeddings.](image)

The approach to be taken to circumvent this difficulty is to root each unlabeled imbedding at a vertex. Given a set of rooted pseudosurface imbeddings there is a unique rooted generalized pseudosurface imbedding obtained by identifying the rooted vertices and, by default, the unrooted vertices. Once the rooted imbeddings are enumerated we enumerate the unrooted imbeddings by determining in how many ways an imbedding can be rooted (the choices are one or two).

If an unlabeled 2-cell imbedding of the dipole has only one rooting then in every sense the two vertices of the dipole are indistinguishable. Call these imbeddings vertex-symmetric. If
\{\sigma_+, \sigma_-\} corresponds to a labeled 2-cell imbedding whose congruence class is vertex-symmetric, then there must exist a permutation of the edge set, \(\alpha \in \Sigma_E\), such that \(\alpha \sigma_+ \alpha^{-1} = \sigma_-\) and \(\alpha \sigma_- \alpha^{-1} = \sigma_+\).

Necessarily then, \(\sigma_+\) and \(\sigma_-\) must have the same cycle type. However, this condition is not sufficient. It gives a local condition for vertex-symmetry; that is, in a small neighborhood of each vertex, the two vertices are indistinguishable from one another. Figure 5.4 shows such an instance. Globally, however, the two vertices in the figure are distinguishable by considering the blocks. For example, the block with the singular point of degree three occupied by the '+' vertex has the singular point of degree two occupied by the '-' vertex. This is not the case if '+' and '-' are interchanged in the previous sentence.

![Figure 5.4](image)

Figure 5.4. An imbedding of the dipole which is not vertex-symmetric.

Thus, whenever a vertex-symmetric imbedding on a
generalized pseudosurface has a block whose corresponding imbedding (which is also an imbedding of a dipole) is not vertex-symmetric, there must be a duplicate of this block, complete with imbedding, where the vertices have reversed their roles. It is also possible for blocks to be present whose imbeddings are vertex-symmetric. The point to be made here is that all vertex-symmetric imbeddings on generalized pseudosurfaces have this structure; their blocks occur in pairs of identical imbeddings which are not vertex-symmetric, together with a collection of vertex-symmetric imbeddings. Figure 5.5 illustrates how a generalized pseudosurface imbedding which is vertex-symmetric can be constructed.

![identify these vertices](image1)

![identify these vertices](image2)

Figure 5.5. Construction of a vertex-symmetric imbedding of the dipole on a generalized pseudosurface from pseudosurface imbeddings.

We are now in a position to define the counting series needed. G, P, and S denote the ordinary counting series which enumerate unlabeled 2-cell imbeddings of the dipole by number.
of edges on generalized pseudosurfaces, pseudosurfaces, and surfaces, respectively. G*, P*, and S* denote the corresponding series which enumerate rooted imbeddings, and G', P', and S' those which enumerate vertex-symmetric imbeddings. Since the latter count the imbeddings which have exactly one possible rooting (the others two) it follows that G* = G' + 2(G - G') or G* = 2G - G'. Similarly, we have P* = 2P - P'. The surface imbeddings are all vertex-symmetric, so that S = S' = S*.

We first determine the series G* and from it, P*. Having accomplished this, we then calculate the series G and G'. The relationship between the vertex-symmetric imbeddings on generalized pseudosurfaces (counted by G') and imbeddings on pseudosurfaces (counted by P) previously discussed is then used to determine the series P. Since P and P* will then be known we can calculate P' from P' = 2P - P*.

To begin, let G* _n= [x^n] G*(x) and P* _n= [x^n] P*(x). Each labeled 2-cell imbedding of a rooted dipole, D _n, corresponds to an ordered pair of members of Σ _n, where it is agreed that the first entry of the ordered pair is the rotation of edges at the rooted vertex. Denote the rooted vertex by '+' and the other by '-'. Define (σ+, σ-) ∈ Σ × Σ to be equivalent to (σ', σ') ∈ Σ × Σ if there exists α in Σ such that ασ α⁻¹ = σ' + and ασ α⁻¹ = σ' -. The equivalence classes correspond to unlabeled rooted imbeddings of D _n and are enumerated using Burnside's lemma as follows.

For each α in Σ _n define the fixed set of α, F(α), by
Let \( J = (j_1, j_2, \ldots, j_n) \) be a partition of \( n \); then we have the following theorem.

**Theorem 5.4.**

\[
G^* \equiv \sum_{J \in \mathcal{P}_n} \prod_{k=1}^{n} k^{j_k} j_k!
\]

**Proof.** Let \( \alpha \) have cycle-type \( J = (j_1, j_2, \ldots, j_n) \). Observe that \((\sigma_+, \sigma_-) \in F(\alpha)\) if and only if \( \sigma_+ \) and \( \sigma_- \) are both in the centralizer of \( \alpha \), \( C(\alpha) \). Hence, \(|F(\alpha)| = |C(\alpha)|^2 = \left( \prod_{k=1}^{n} k^{j_k} j_k! \right)^2\).

Since \(|F(\cdot)|\) is a class function, we have from Burnside's lemma that

\[
G^* = \frac{1}{n!} \sum_{\alpha \in \Sigma_n} |F(\alpha)| = \frac{1}{n!} \sum_{J \in \mathcal{P}_n} \frac{n!}{\prod_{k=1}^{n} k^{j_k} j_k!} |F(J)|
\]

\[
= \prod_{k=1}^{n} \frac{k^{j_k} j_k!}{\prod_{k=1}^{n} k^{j_k} j_k!}
\]

The sum over all partitions of \( n \) is accomplished by defining a sequence of series, \( \{E_k(x)\} \), with the property that

\[
G^*(x) + 1 = \prod_{k=1}^{n} E_k(x). \quad \text{These series are defined as}
\]

\[
E_k(x) = \sum_{p=0}^{\infty} k^p p! x^{kp}. \quad \text{Thus,}
\]

\[
G^*(x) + 1 = (1 + x + 2x^2 + 6x^3 + 24x^4 + 120x^5 + 720x^6 + 5040x^7 + 40320x^8 + \cdots)
\]

\[
+ 362880x^9 + 3628800x^{10} + \cdots)x(1 + x^2 + 8x^4 + 48x^6 + 384x^8 + \cdots)
\]

\[
+ 3840x^{10} + \cdots)x(1 + 3x^2 + 18x^4 + 162x^6 + \cdots)x(1 + 4x^4 + 32x^8 + \cdots)
\]

\[
+ \cdots)(1 + 5x^5 + 50x^{10} + \cdots)x(1 + 5x^6 + \cdots)x(1 + 7x^7 + \cdots)
\]

\[
\cdots x(1 + 8x^8 + \cdots)x(1 + 9x^9 + \cdots)x(1 + 10x^{10} + \cdots) \cdots
\]

\[
= 1 + x + 4x^2 + 11x^3 + 43x^4 + 161x^5 + 901x^6 + 5579x^7 + \cdots.
\]

Since \( G^* \) and \( P^* \) are related by \( G^*(x) = \exp \sum_{k=1}^{\infty} (\frac{P^*(x^k)}{k}) \), we have...
Table 5.4 at the end of this section lists the results for up to ten edges.

Next, we determine the series $G$ and from it the series $G'$.

Two ordered pairs, $(\sigma_+, \sigma_-)$ and $(\sigma'_+, \sigma'_-)$ in $\Sigma_n \times \Sigma_n$, correspond to congruent imbeddings of $D_n$ if there exists $(\alpha, \beta)$ in $\Sigma_n \times \Sigma_{\{+,-\}}$ such that $\alpha \sigma_\beta(+) \alpha^{-1} = \sigma'_+$ and $\alpha \sigma_\beta(-) \alpha^{-1} = \sigma'_-$. In words, there is some permutation of the edges ($\alpha$) and some permutation of the two vertices ($\beta$) which together map the one imbedding to the other. We again use Burnside's lemma as follows. The group $\Sigma_n \times \Sigma_{\{+,-\}}$ acts on $\Sigma_n \times \Sigma_n$ by

$$(\alpha, \beta) : (\sigma_+, \sigma_-) \mapsto (\alpha \sigma_\beta(+) \alpha^{-1}, \alpha \sigma_\beta(-) \alpha^{-1})$$

The orbits of this action are, of course, the sets of pairs corresponding to congruent imbeddings of $D_n$. For each $(\alpha, \beta)$ in $\Sigma_n \times \Sigma_{\{+,-\}}$ define the fixed set of $(\alpha, \beta)$, denoted $F(\alpha, \beta)$, as

$$F(\alpha, \beta) = \{ (\sigma_+, \sigma_-) \in \Sigma_n \times \Sigma_n : \alpha \sigma_\beta(+) \alpha^{-1} = \sigma_+ \text{ and } \alpha \sigma_\beta(-) \alpha^{-1} = \sigma_- \}.$$

We have the following result.

**Lemma 5.5 (Burnside).** $G_n = \frac{1}{2 \cdot n!} \sum_{(\alpha, \beta) \in \Sigma_n \times \Sigma_{\{+,-\}}} |F(\alpha, \beta)|$.

Let $e$ denote the identity permutation in $\Sigma_{\{+,-\}}$ and observe that

$$\sum_{(\alpha, \beta) \in \Sigma_n \times \Sigma_{\{+,-\}}} |F(\alpha, \beta)| = \sum_{\alpha \in \Sigma_n} (|F(\alpha, e)| + |F(\alpha, (+-))|) \quad \text{From the definition of the fixed set we see that}$$

$$F(\alpha, e) = \{ (\sigma_+, \sigma_-) \in \Sigma_n \times \Sigma_n : \alpha \sigma_+ \alpha^{-1} = \sigma_+ \text{ and } \alpha \sigma_- \alpha^{-1} = \sigma_- \}.$$ Since $\alpha \sigma_+ \alpha^{-1} = \sigma_+$ if and only if $\sigma_+$ is in the centralizer of $\alpha$, $C(\alpha)$, we have $|F(\alpha, e)| = |C(\alpha)|^2$. The other fixed set is given by
\[ F(\alpha, (+)) = \{ (\sigma_+, \sigma_-) \in \Sigma_n \times \Sigma_n : \sigma_+\alpha^{-1} = \sigma_+ \text{ and } \sigma_+\alpha^{-1}\sigma_- \}. \]

Suppose \( (\sigma_+, \sigma_-) \) is in \( F(\alpha, (+)) \); then \( \sigma_+^2\sigma_+\alpha^{-2} = \alpha(\sigma_+\alpha^{-1})\alpha^{-1} = \alpha(\sigma_-)\alpha^{-1} = \sigma_+ \)
which implies \( \sigma_+ \) is in the centralizer of \( \alpha^2 \), \( C(\alpha^2) \). Moreover, if we choose an arbitrary member of \( C(\alpha^2) \), say \( \sigma_+ \), and let \( \sigma_- \) be \( \alpha(\sigma_+\alpha^{-1}) \) then \( (\sigma_+, \sigma_-) \) is in \( F(\alpha, (+)) \). Thus, we conclude that \( |F(\alpha, (+))| = |C(\alpha^2)| \) and we have the following result.

**Lemma 5.6.** \( G_n = \frac{1}{2 \cdot n!} \sum_{\alpha \in \Sigma_n} \left( |C(\alpha)|^2 + |C(\alpha^2)| \right) \).

We have noted many times that \( |C(\cdot)| \) is a class function in \( \Sigma_n \) and if \( \alpha \) has cycle-type \( J = (j_1, j_2, \ldots, j_n) \) then
\[
|C(\alpha)| = \prod_{k=1}^{n} k^{j_k} \cdot j_k!.
\]
We write \( |C(J)| \) to be \( |C(\alpha)| \) for any \( \alpha \) of cycle-type \( J \) and, admittedly an abuse of notation, write \( |C(J^2)| \) to be \( |C(\alpha^2)| \) for any \( \alpha \) of cycle-type \( J \). We then have the following result.

**Theorem 5.7.** The number of unlabeled 2-cell imbeddings, \( G_n \), of the dipole, \( D_n \), is given by
\[
G_n = \frac{1}{2} \sum_{J \in \mathcal{J}_n} \left( |C(J)| + \frac{|C(J^2)|}{|C(J)|} \right).
\]

**Proof.** From the previous lemma and the above observations we have
\[
G_n = \frac{1}{2 \cdot n!} \sum_{\alpha \in \Sigma_n} \left( |C(\alpha)|^2 + |C(\alpha^2)| \right)
= \frac{1}{2 \cdot n!} \sum_{J \in \mathcal{J}_n} \frac{n!}{\prod_{k=1}^{n} j_k!} \cdot \left( |C(J)|^2 + \frac{|C(J^2)|}{|C(J)|} \right)
= \frac{1}{2} \sum_{J \in \mathcal{J}_n} \frac{1}{|C(J)|} \left( |C(J)|^2 + |C(J^2)| \right).
\]
Observe that the sum of the first term in the above expression is \( \sum_{J \in \mathcal{I}} \left( |C(J)| + \frac{|C(J^2)|}{|C(J)|} \right) \), which was previously shown to be \( G^*_n \).

Hence, multiplying both sides of the expression for \( G_n \) by 2 we have \( 2G_n = G^*_n + \sum_{J \in \mathcal{I}} \frac{|C(J^2)|}{|C(J)|} \). Since the equation \( 2G = G^* + G' \) implies \( 2G_n = G^*_n + G'_n \) for all \( n \), we have the following result.

**Theorem 5.8.** The number of vertex-symmetric unlabeled imbeddings, \( G'_n \), of the dipole, \( D_n \), is given by

\[
G'_n = \sum_{J \in \mathcal{I}} \frac{|C(J)|}{|C(J)|}
\]

We next determine the relationship between the series \( G' \) and the series \( P' \). To do this we first define two sets of vertex-symmetric imbeddings from whose members all of the imbeddings counted by \( G' \) are formed. The first set, denoted \( \mathcal{Y}_1 \), consists of all vertex-symmetric unlabeled imbeddings of the dipole on pseudosurfaces. Hence, \( P'(x) \) enumerates the members of \( \mathcal{Y}_1 \) by number of edges. The second set, denoted \( \mathcal{Y}_2 \), consists of all vertex-symmetric unlabeled imbeddings of the dipole on generalized pseudosurfaces, which have exactly two blocks, neither of which is vertex-symmetric (hence, the two blocks must be identical). The members of \( \mathcal{Y}_2 \) are constructed from pseudosurface imbeddings which are not vertex-symmetric by taking two copies of each and identifying 'opposite' vertices as shown in Figure 5.6.
The series which enumerates $Y_2$ by number of edges is determined as follows. The series $P(x)-P'(x)$ enumerates by number of edges the pseudosurface imbeddings which are not vertex-symmetric. Since we need two copies of each to make up the members of $Y_2$, the series which enumerates its members by number of edges is $P(x^2)-P'(x^2)$.

The members of both $Y_1$ and $Y_2$ are vertex-symmetric imbeddings and if we take any number from their union (including duplicates) and identify vertices as previously shown in Figure 5.5, then we obtain a vertex-symmetric imbedding on a generalized pseudosurface. In fact, all of them are obtained in this manner as outlined earlier. Thus, let $Y$ be the union of $Y_1$ and $Y_2$, and let $X={1,2,\ldots,n}$. Let $Y^X$ be the set of all functions from $X$ into $Y$, each of which can be considered as an $n$-tuple of members of $Y$. Define the weight of a member of $Y$
to be the number of edges of the dipole, and the weight of a function in $Y^X$ as the sum of the weights of each image (which is then the total number of edges present in the corresponding dipole).

The series $P'(x) + P(x^2) - P'(x^2)$ is the **figure counting series** (Polya's terminology [23]) which enumerates the members of $Y$ by weight. Given an $n$-tuple of members of $Y$ (i.e., an element of $Y^X$), any permutation of the $n$-tuple will produce the same vertex-symmetric imbedding; hence, we let the power group $E \Sigma_n$ act on $Y^X$, where $E$ is the identity permutation group acting on $Y$ and $\Sigma_n$ acts on $X$. From Polya's Enumeration Theorem the **function counting series** (or configuration counting series in Polya's terminology) which enumerates the number of equivalence classes of functions by weight is $Z[\Sigma_n; P'(x) + P(x^2) - P'(x^2)]$. This series is interpreted as enumerating by number of edges (the weight) the number of vertex-symmetric unlabeled imbeddings on generalized pseudosurfaces that are configured from $n$ of the figures.

Summing over all values of $n$, we have

$$G'(x) + 1 = \sum_{n=0}^{\infty} Z[\Sigma_n; P'(x) + P(x^2) - P'(x^2)].$$

A well-known result [11] states that

$$\sum_{n=0}^{\infty} Z[\Sigma_n; c(x)] = \exp \sum_{k=1}^{\infty} \left( c(x^k)/k \right)$$

for any series $c(x)$ (proof is by comparing coefficients) and thus, we have the following theorem.

**Theorem 5.9.** $G'(x) + 1 = \exp \sum_{k=1}^{\infty} \frac{P'(x^k) + P(x^{2k}) - P'(x^{2k})}{k}$.
Using $P' = 2P - P^*$, we have

\[ P'(x^k) + P(x^{2k}) - P'(x^{2k}) = 2P(x^k) - P^*(x^k) + P(x^{2k}) - 2P(x^{2k}) + P^*(x^{2k}) \]

\[ = 2P(x^k) - P(x^{2k}) - P^*(x^k) + P^*(x^{2k}). \]

Substituting this in the expression appearing in the above theorem, taking logarithms and simplifying we have

**Corollary.** \[ \sum_{k=1}^{\infty} \frac{2P(x^k) - P(x^{2k})}{k} = \sum_{k=1}^{\infty} \frac{P^*(x^k) - P^*(x^{2k})}{k} + \log(G'(x) + 1). \]

Recall that $G^*(x) + 1 = \exp \sum_{k=1}^{\infty} \frac{P^*(x^k)}{k}$, or

\[ \log(G^*(x) + 1) = \sum_{k=1}^{\infty} \frac{P^*(x^k)}{k}. \]

Using this result in the corollary we have the following.

**Corollary.** \[ \sum_{k=1}^{\infty} \frac{2P(x^k) - P(x^{2k})}{k} = \log \left( \frac{(G'(x) + 1)(G^*(x) + 1)}{G^*(x^2) + 1} \right). \]

We have previously determined the series $P^*$, $G^*$, and $G'$ and so $P$ (and, hence, $P'$) can be calculated using either of the two corollaries. The results of the calculations are given in Table 5.4 at the end of this section.

To determine the number of unlabeled 2-cell imbeddings of the dipole on surfaces we could use the results of Mull, Rieper, and White outlined in Section 3.4. There is an alternate means available, however, which uses a generalization of Redfield's Enumeration Theorem (see [11]) applied to the problem of determining the number of nonisomorphic graph superpositions.

To see how this occurs, consider a typical labeled imbedding
of the dipole, $D_n$, on a surface. Both $\sigma_+$ and $\sigma_-$ must be $n$-cycles in this case. For example, $\sigma_+ = (12345)$ and $\sigma_- = (12543)$.

Each of these permutations can be represented as a directed cycle (a digraph) with the same set of vertices, $V = \{1, 2, 3, 4, 5\}$, as shown in Figure 5.7. The imbedding itself is represented as the superposition of these two digraphs, where, for the moment, each digraph is distinguishable by the color of its edges as shown.

![Figure 5.7](image)

$\sigma_+ = (12345)$  \hspace{1cm} $\sigma_- = (12543)$  \hspace{1cm} $\{\sigma_+, \sigma_-\}$

Figure 5.7. Representation of a dipole imbedding on a surface as the superposition of two directed cycles.

The composition of the permutations, $\sigma_+ \circ \sigma_-$, is readily determined from the superposed digraphs by following an alternating directed trail. Continuing the example, consider $\sigma_+ \circ \sigma_-(1)$. From the vertex labeled 1, follow the directed edge whose color is that of the digraph representing $\sigma_-$, 'dashed' in this instance; we have determined $\sigma_-(1)$. Proceed from there along the directed edge whose color is that of the digraph representing $\sigma_+$, the 'solid' color in this case. We are led to vertex 3 which is $\sigma_+ \circ \sigma_-(1)$.

Note that each vertex necessarily has exactly one edge of
each color directed towards it and one directed away from it. Hence, beginning at any vertex and traversing an alternating directed trail as long as possible (without repeating an edge), we eventually return to the initial vertex and stop. To see this, suppose we arrived at the terminal vertex of such a trail via the 'dashed'-colored edge. Since the 'solid'-colored edge directed away from this vertex is not available, we must have previously visited this vertex and departed via the 'solid'-colored edge. This implies we previously arrived at this vertex via the 'dashed'-colored edge, since colors must alternate, or the trail was initiated there. The latter must be the case since there is but one 'dashed'-colored edge directed towards this vertex.

Thus, the subgraphs spanned by these longest alternating directed trails are maximal alternating directed circuits and, not surprisingly, are the region boundaries of the imbedding. This is because in Section 5.2.2 it was shown that the sequence, \( \sigma_-(e), \sigma_+\sigma_-(e), \sigma_+\sigma_+\sigma_-(e), \ldots \), produces a traversal of the region boundary beginning at edge \( e \), and this sequence is a maximal alternating directed circuit.

The superposition of the two directed cycles has a unique decomposition into its maximal alternating directed circuits (and can be represented as their edge sum). The number and length of these circuits gives the region distribution of the corresponding dipole imbedding. Figure 5.8 shows the decomposition for the previous example. Hence, the imbedding has two 2-sided regions and a 6-sided region.
To determine how many unlabeled imbeddings of $D_n$ are on surfaces we need to determine the number of nonisomorphic superpositions of two interchangeable directed $n$-cycles (interchangeable means the colors are interchangeable). This problem was solved, in general, by Palmer and Robinson [21] by generalizing Redfield's Enumeration Theorem. The details are lengthy and a good presentation of the superposition theory can be found in Harary and Palmer [11] so its development is omitted here.

Applying the method to the superposition of two directed $n$-cycles we present the following theorem.

**Theorem 5.10.** The number, $S_n$, of unlabeled 2-cell imbeddings of the dipole, $D_n$, on surfaces is given by

$$S_n = \sum_{d|n} \phi^2(d) \cdot \frac{(\frac{n}{d} - 1)! \cdot \frac{n}{d}^{d-1}}{2n} + \sum_{d \text{ even}} \frac{\phi(d) \cdot (\frac{n}{2d})! \cdot d^{\frac{n}{2d}}}{(\frac{n}{2d})! \cdot 2n \cdot 2^{n/2d}}$$

$$+ \sum_{d \text{ odd}} \phi(d) \cdot \sum_{k=0}^{n/d} \left( \frac{n/d}{2k} \right) \frac{(2k)! \cdot d^k}{2n \cdot 2^k \cdot k!}.$$
Table 5.4 lists the first few values of \( S_n \). In addition, Figure 5.9 shows the seven nonisomorphic superpositions of interchangeable directed 5-cycles, their decompositions into maximal alternating directed circuits, and a representative of the dipole imbedding described by each.

<table>
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<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
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<td>8</td>
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<td>495</td>
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<td>22024</td>
<td>190585</td>
</tr>
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<td>11</td>
<td>43</td>
<td>161</td>
<td>901</td>
<td>5579</td>
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<td>378360</td>
</tr>
<tr>
<td>G'</td>
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<td>2</td>
<td>5</td>
<td>13</td>
<td>31</td>
<td>89</td>
<td>259</td>
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<td>5</td>
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<td>333</td>
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<td>158022</td>
</tr>
<tr>
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<td>7</td>
<td>26</td>
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<td>624</td>
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<td>7</td>
<td>19</td>
<td>71</td>
<td>369</td>
<td>2393</td>
</tr>
</tbody>
</table>

G : Unlabeled 2-cell imbeddings on generalized pseudosurfaces.
G* : Rooted unlabeled 2-cell imbeddings on generalized pseudosurfaces.
G' : Vertex-symmetric unlabeled 2-cell imbeddings on generalized pseudosurfaces.
P : Unlabeled 2-cell imbeddings on pseudosurfaces.
P* : Rooted unlabeled 2-cell imbeddings on pseudosurfaces.
P' : Vertex-symmetric unlabeled 2-cell imbeddings on pseudosurfaces.
S : Unlabeled 2-cell imbeddings on surfaces.

Number of unlabeled 2-cell imbeddings of the dipole by number of edges.

Table 5.4.
Figure 5.9. The 7 nonisomorphic superpositions of two interchangeable 5-cycles and their decompositions.
CHAPTER VI

OTHER IMBEDDINGS

6.1. Pseudograph Imbeddings on Surfaces

6.1.1. Introduction

Historically, it has been the surface imbeddings of pseudographs which have been of greatest interest; the pseudosurface and generalized pseudosurface imbeddings have only recently been introduced. As we have seen, the enumeration of pseudosurface imbeddings of some pseudographs can be determined from the number of generalized pseudosurface imbeddings. This provides us with the motivation to determine the latter. The enumeration of the pseudosurface imbeddings themselves is motivated by the ambitions of this chapter. Here, we exploit a correspondence between 2-cell imbeddings on surfaces of pseudographs and 2-cell imbeddings of the bouquet on pseudosurfaces. The correspondence is one-to-one and onto; hence, once it is established, the numbers we are interested in have all been determined in Chapter IV.

To establish the correspondence, let $H$ be a pseudograph with arc set $A=\{1^+,1^-,2^+,2^-,\ldots,q^+,q^-\}$, and edge set $E=\{\{1^+,1^\-\},\{2^+,2^\-\},\ldots,\{q^+,q^-\}\}$. Let $H$ be 2-cell imbedded on a surface and identify all the points of the surface occupied by $H$. 

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vertices of $H$. Since these points are finite in number we obtain a pseudosurface. While identifying the points of the surface, we also identify the vertices of $H$. The pseudograph obtained is clearly a bouquet. Moreover, the imbedding of the bouquet on the pseudosurface is 2-cell since deletion of the bouquet from the pseudosurface produces the same result as deletion of the pseudograph from the surface. In fact, the number of regions and their size is unaffected by the identification as well. To see this, note that the pair $(\mathcal{O}_v, \mathcal{O}_e)$ describes both imbeddings and, hence, $\mathcal{O}_v \circ \mathcal{O}_e$ is the same in both cases. Figure 6.1 illustrates the process for a particular pseudograph. Observe that this process is reversible. Thus, the number of labeled 2-cell imbeddings of all (connected) pseudographs on surfaces, where the pseudographs have edge set $\{\{1^+, 1^-\}, \{2^+, 2^-\}, \ldots, \{q^+, q^-\}\}$.

Figure 6.1. Obtaining a 2-cell imbedding of the bouquet on a pseudosurface from a 2-cell imbedding of a pseudograph on a surface.
equals the number of 2-cell imbeddings of the bouquet on pseudosurfaces where the bouquet has the same edge set.
Moreover, the enumeration of the former by size and number of regions is the same as the latter.

In Section 6.1.2 we determine the number of connected pseudographs that have edge set \( \{ \{1^+, 1^-, 2^+, 2^-, \ldots, q^+, q^-\} \} \) and, using the imbedding correspondence, enumerate their 2-cell imbeddings on surfaces. In Section 6.1.3 we repeat the analysis in the unlabeled case.

6.1.2. Labeled Pseudograph Imbeddings on Surfaces

We first determine how many pseudographs there are on a fixed edge set and determine the number of these which are connected. Let \( H \) be an arbitrary pseudograph with arc set \( A = \{ 1^+, 1^-, 2^+, \ldots, q^+, q^- \} \) and edge set \( E = \{ \{1^+, 1^-, 2^+, 2^-, \ldots, q^+, q^-\} \} \). Every possible partition of \( A \) is the vertex set of a labeled pseudograph. If \( J = (j_1, j_2, \ldots, j_{2q}) \) is a partition of \( 2q \) then the number of partitions of \( A \) of type \( J \) is \( (2q)!/(\prod k! j_k)! \). Hence, the number of labeled pseudographs with edge set \( E \) is the sum of these numbers over all partitions \( J \) of \( 2q \).

We do not intend to calculate the number of pseudographs using this expression (there are too many partitions of \( 2q \) when \( q \) is large). It is more usefull to recognize that the number of partitions of the set \( A \) into \( p \) nonempty subsets is the number of labeled pseudographs which have edge set \( E \) and \( p \) vertices.
Moreover, this number is the Stirling number of the second kind, $S_{2q}^{(p)}$. Since these numbers satisfy the recursion,

$$S_{2q}^{(p)} = S_{2q-1}^{(p-1)} + p \cdot S_{2q-1}^{(p-2)} ,$$

we can determine the number of labeled pseudographs of size $q$ and order $p$ recursively. From these we determine the number which are connected.

Let $H(x,y)$ and $C(x,y)$ be the counting series enumerating labeled pseudographs defined as follows.

$$H_{q,p} = [y^p]H_q(y) = q! [x^q y^p]H(x,y)$$

= the number of labeled pseudographs of size $q$ and order $p$.

$$C_{q,p} = [y^p]C_q(y) = q! [x^q y^p]C(x,y)$$

= the number above which are connected. Observe that $H$ and $C$ are exponential counting series in $x$ whose coefficients are polynomials in $y$.

From the previous remarks it follows that $H_{q,p} = S_{2q}^{(p)}$.

The recursion equation for the Stirling numbers can then be used to determine the following recursion equation for $H_{q,p}$.

$$H_{q,p} = H_{q-1,p-2} + (2p-1)H_{q-1,p-1} + p^2 H_{q-1,p} .$$

From the relationship $H+1=\exp(C)$ the values of $C_{q,p}$ can be determined. Some results are given in Table 5.1 which follows. The eleven connected pseudographs with two edges are shown in Figure 5.2. Since each of these has $\Pi_v(\text{degree}(v)-1)!$ 2-cell imbeddings on surfaces, a simple computation shows there are twenty labeled 2-cell imbeddings of all these pseudographs.
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Pseudographs
Connected pseudographs

Number of labeled pseudographs by size \((q)\) and order \((p)\).

Table 5.1.

It is not our intention to enumerate the labeled 2-cell imbeddings on surfaces of all pseudographs in this manner. We use the correspondence with bouquet imbeddings introduced in Section 6.1.1. Observe that if a pseudograph with \(q\) edges is 2-cell imbedded on a surface resulting in \(r\) regions then the
Figure 5.2. The eleven connected pseudographs giving twenty labeled 2-cell imbeddings on surfaces.
pseudograph must be connected. Moreover, its dual imbedding is 2-cell, and the pseudograph imbedded there has q edges and r vertices. Hence, the series enumerating labeled 2-cell imbeddings on surfaces of pseudographs by size and order is the same as that enumerating them by size and number of regions of the imbedding. And both of these are the same as the series which enumerates the labeled 2-cell imbeddings on pseudosurfaces of the bouquet by size and number of regions. Thus, we interpret the numbers produced in Chapter IV in this new light. Table 6.2 is a partial list of the number of 2-cell imbeddings on surfaces of pseudographs of size q and order p.

We are now in a position to affirm the combinatorial definition of pseudograph. We have already noted that the number of pseudographs of size q and order p on a fixed edge set is $S^{(p)}_{2q}$. Although some of these are not connected and, hence, can not be 2-cell imbedded on a single surface, it is clear that each of their components can be so imbedded. If this is accomplished, the resulting imbedding is 2-cell but the image space is not connected. However, performing the same identification of vertices introduced earlier we again obtain a 2-cell imbedding of the bouquet. The imbedding is on a generalized pseudosurface which is not a pseudosurface. In Chapter IV we observed that the number of labeled 2-cell imbeddings with p regions of the bouquet with q edges on generalized pseudosurfaces is $|s_{2q}^{(p)}|$. These observations are summarized in the following theorem.
Theorem 6.1. The Stirling number of the second kind, \( S^{(p)}_{2q} \), gives the number of pseudographs of size \( q \) and order \( p \) on a fixed edge set. The absolute value of the Stirling number of the first kind, \( |s^{(p)}_{2q}| \), gives the number of 2-cell imbeddings on spaces whose components are surfaces, of the above pseudographs.

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6.1.3. Unlabeled Pseudograph Imbeddings on Surfaces.

In Section 6.1.2 we determined the number of labeled pseudographs of size $q$ and order $p$; from these we determined the number which are connected. The corresponding enumeration of unlabeled pseudographs is computationally more difficult. Although this is the case, the number of unlabeled pseudographs with a given degree sequence and the number of their unlabeled surface imbeddings can both be expressed in terms of the cap of two cycle indexes; the two expressions have a remarkable resemblance to one another.

Using the theory of superpositions Read [24] has shown that given a partition $J = (j_1, j_2, \ldots, j_{2q})$ of $2q$, the number of unlabeled pseudographs that have $j_k$ vertices of degree $k$, $k = 1, 2, \ldots, 2q$, is $Z(J \cap \Sigma_k [\Sigma_k]) \cap Z(\Sigma_q [\Sigma_2])$. Although the pseudographs counted need not be connected and, hence, cannot be 2-cell imbedded on a single surface, each of their components can be so imbedded. If this has been done for a pseudograph then identifying the points of the space occupied by vertices we obtain an unlabeled 2-cell imbedding of the bouquet on a generalized pseudosurface. Moreover, if the original pseudograph was connected then the generalized pseudosurface is, in fact, a pseudosurface (and vice versa).

The number of unlabeled 2-cell surface imbeddings of all pseudographs having $j_k$ vertices of degree $k$, $k = 1, 2, \ldots, 2q$, is the same as the number of unlabeled 2-cell surface imbeddings
of all unlabeled pseudographs where the imbedding has \( j_k \) regions of size \( k, k=1,2,\ldots,2q \). The correspondence is via the dual. Since region sizes are preserved under the identification described above we have that the number of unlabeled 2-cell surface imbeddings of all pseudographs having \( j_k \) vertices of degree \( k, k=1,2,\ldots,2q \), is the same as the number of unlabeled 2-cell generalized pseudosurface imbeddings of the bouquet having \( j_k \) regions of size \( k, k=1,2,\ldots,2q \). The latter have been determined in Section 4.3.3 (see Theorem 4.19) and we have the following.

**Theorem 6.2.**

(i) The number of unlabeled pseudographs that have \( j_k \) vertices of degree \( k, k=1,2,\ldots,2q \), is \( Z(\prod_k \Sigma_{j_k}[\Sigma_k]) \cap Z(\Sigma_q[C_2]) \).

(ii) The number of unlabeled 2-cell surface imbeddings of the above pseudographs is \( Z(\prod_k \Sigma_{j_k}[C_k]) \cap Z(\Sigma_q[C_2]) \).

If the number of unlabeled pseudographs of order \( p \) and their imbeddings are desired, then the two expressions above are summed over all partitions of \( 2q \) having exactly \( p \) parts. Summing both expressions over all partitions of \( 2q \) we obtain the number of unlabeled pseudographs of size \( q \) and the number of their unlabeled surface imbeddings.

Encoding the information in an ordinary counting series and applying the techniques outlined in Section 3.1 we obtain the number of unlabeled connected pseudographs and the number of their unlabeled 2-cell surface imbeddings. The
results of some of these calculations are given in Table 6.3. In Figure 6.2 are shown the eleven unlabeled connected pseudographs and their twenty surface imbeddings.

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Number of unlabeled surface imbeddings of connected pseudographs.

Table 6.3.
Figure 6.2. Unlabeled connected pseudographs of size 3 and their surface imbeddings.
6.2. Hypergraphs and Hypermaps

A pseudograph has a very natural generalization by removing the restriction on the edge set $E$ (its members are 2-sets). Formally, a hypergraph is an ordered triple, $(A,E,V)$, where $A$ is a finite nonempty set, and $E$ and $V$ are both partitions of $A$. The terms arc, edge, and vertex refer to the members of the sets $A$, $E$, and $V$, respectively. Moreover, the definitions of incidence, adjacency, walks, and connectivity are the same as those for the pseudograph.

Like the term graph, hypergraph has been defined by others in a similar, but not necessarily identical manner. Most notably are the definitions of hypergraph by Berge [1] and that by Graver and Watkins [8]. The latter contains the former as a special case. The definition here corresponds to that of Berge with the exception of isolated vertices and multiple edges. The correspondence is much the same as that between graph and pseudograph given in Chapter II.

To Berge, a hypergraph is a finite nonempty set $V$ together with a set $E$ of subsets of $V$ (the edges). Provided every member of $V$ is in some member of $E$, each hypergraph so defined corresponds to a hypergraph by our definition. We illustrate with an example which is sustained throughout this section.

Let $\{1,2,3,4\}$ be the vertex set and $\{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\}$ be the edge set of a hypergraph according to
Berge. Take as the arc set, edge set, and vertex set the following:

\[ A = \{1, 2, 3, 1', 2', 4', 1'', 3'', 4''', 2'''', 3''''}, \]
\[ E = \{\{1, 2, 3\}, \{1', 2', 4'\}, \{1'', 3'', 4''\}, \{2'''', 3''''', 4''''\}\}, \]
\[ V = \{\{1, 1', 1''\}, \{2, 2', 2''\}, \{3, 3'', 3'''\}, \{4', 4'', 4''''\}\}. \]

Thus, the two hypergraphs have the same number of vertices and edges. In addition, adjacency and incidence are preserved under the correspondence.

Since pseudographs can be imbedded on surfaces, it is natural to ask the same of hypergraphs. Of course, the imbedding of a pseudograph on a surface is actually the imbedding of the 1-dimensional CW-complex associated with it and we have no such association with a hypergraph. However, Walsh [28] has used a bijection between connected hypergraphs (as defined by Berge) and connected bipartite graphs to 'imbed' the former on surfaces. The imbedding is, of course, the 2-cell imbedding of the associated bipartite graph, modified in a manner described by Jungerman, Stahl and White [17] to reflect the cardinality of each edge.

The bipartite graph is determined as follows. The vertices of one partite set correspond to the vertices of the hypergraph. The vertices of the other partite set correspond to the edges of the hypergraph. Vertices in two distinct partite sets are joined by an edge if their corresponding vertex and edge of the hypergraph are incident (a nonempty intersection). Observe, that in the previous example, the same bipartite graph is
obtained regardless of the hypergraph definition used.

Figure 6.3 depicts an imbedding on the sphere of the hypergraph given earlier. The figure is a reproduction of one given in White [29] where the details for modifying an imbedding of the bipartite graph associated with a hypergraph can be found. In the figure, the regions are the shaded areas, and the unshaded areas are the edges. Note that each region is 2-sided. Also note that about each vertex the counterclockwise cyclic order of each edge can be given. For example, at vertex 1 we have \( \{1,2,3\} \{1,2,4\} \{1,3,4\} \). In our definition, vertex 1 has become vertex \( \{1,1',1''\} \) and the three edges in the rotation above have become, in order, the unprimed edge, \( \{1,2,3\} \), the primed edge, \( \{1',2',4'\} \), and the double primed edge, \( \{1'',3'',4''\} \). The rotation of edges about vertex 1 corresponds, then, to the rotation of arcs about vertex \( \{1,1',1''\} \) given by \( (1,1',1'') \). Doing this for every vertex and denoted \( \rho_V \), we have \( \rho_V=(11''1') (22''2') (33''3') (44''4') \).

Figure 6.3. A hypergraph 'imbedding'.
In a similar manner, each edge shown in the figure has a counterclockwise cyclic order of vertices. For example, around the edge \( \{1, 2, 4\} \) we see \((124)\). For us, this rotation of vertices becomes the rotation of arcs, \((1'2'4')\), at the edge \( \{1', 2', 4'\} \).

Assembling all of these into a single permutation of the arc set, denoted, \( \rho_E \), we have \( \rho_E = (132)(1'2'4')(1''4''3'')(2''3''4'') \).

We have already noted that each region of the imbedding is 2-sided. For a pseudograph the region boundaries were given by \( \rho_V \circ \rho_E \). This is still the case here since \( \rho_V \circ \rho_E = (13')(1'2')(1''4'')(2''3')(3''4'') \), and by examining the figure closely one can see the region boundaries given by the above permutation. Thus, it is very natural to generalize the notion of map (a pseudograph imbedding) to that of a hypermap via the permutations \( \rho_V \) and \( \rho_E \); this we do now.

If \( H = (A, E, V) \) is a hypergraph, a hypermap on \( H \) is an ordered triple, \( M = (H, \rho_E, \rho_V) \), where \( \rho_E \) and \( \rho_V \) are permutations of the arc set of \( H \) with the following restriction. \( \rho_E \) restricted to each edge of \( E \) is a permutation of the arcs in each edge (the permutation is the rotation of arcs at the edge) and, likewise, \( \rho_V \) restricted to the arcs of each vertex is a permutation of those arcs. Note that some authors (Jacques [15] for one) require that the subgroup \( \langle \rho_E, \rho_V \rangle \) of \( \Sigma_A \) be transitive. We did not require this for maps; the instances when \( \langle \rho_E, \rho_V \rangle \) is not transitive describe imbeddings on generalized pseudosurfaces which are not pseudosurfaces. Hence, we do not make this a
The genus, \( g \), of a hypermap \( M=(H,\mathcal{E},\mathcal{P}) \) is defined to be 
\[
g = \|\mathcal{E}_\mathcal{P} \mathcal{P}_\mathcal{E} \| + \frac{1}{2} \left( |\mathcal{A}| - \|\mathcal{E}_\mathcal{P}\| - \|\mathcal{P}_\mathcal{E}\| - \|\mathcal{P}_\mathcal{E} \mathcal{P}_\mathcal{E}\| \right).
\]
Jacques [15] has proven that \( g \) is always a nonnegative integer. The definition of genus agrees with that given in Section 2.3 in case \( M \) is a map. The orbits of \( \mathcal{P}_\mathcal{E} \mathcal{P}_\mathcal{E} \) are taken to be the regions of the hypermap.

The aforementioned proof that \( g \) is a nonnegative integer is algebraic, using the properties of the groups alone. The result is confirmed geometrically in light of the following observation: \( \mathcal{P}_\mathcal{E} \) and \( \mathcal{P}_\mathcal{V} \) can be interpreted as the rotation of edges about the two vertices of a dipole with \(|\mathcal{A}|\) edges. The number of blocks of the generalized pseudosurface on which the dipole is imbedded is \( \|\mathcal{E}_\mathcal{P} \mathcal{E}_\mathcal{P}\| \) (hence, the space is a pseudosurface if and only if \( \langle \mathcal{P}_\mathcal{E}, \mathcal{P}_\mathcal{V} \rangle \) is transitive); and the genus of the generalized pseudosurface given in Theorem 2.1 of Section 2.3 is the same as that above.

The bijective correspondence mentioned earlier between a hypergraph and a bipartite graph also leads to the dipole. Simply identify the vertices of one partite set as a single vertex and those of the other partite set as a single vertex. If first the bipartite graph is 2-cell imbedded on a surface and then the identifications are made we obtain the above dipole imbeddings.

The definitions of isomorphic pseudograph (unlabeled pseudographs) and isomorphic maps (unlabeled imbeddings) did not depend on the edge set having as its members 2-sets.
Hence, we extend those definitions to hypergraphs and hypermaps. Our immediate goal is to exploit the correspondence between hypermaps and dipole imbeddings to enumerate the former. Specifically, we have the following theorems.

**Theorem 5.3.** The number of labeled hypermaps, $M=(H, \rho_E, \rho_V)$, of all hypergraphs $H$ with arc set $A=\{1, 2, \ldots, n\}$ equals the number of labeled 2-cell imbeddings of the dipole, $D_n$, on generalized pseudosurfaces. Moreover, $\langle \rho_E, \rho_V \rangle$ is transitive if and only if the dipole imbedding is on a pseudosurface.

Table 5.4 contains a summary of the number of hypermaps on arc sets of cardinality 8 or less.

**Theorem 5.4.** The number of unlabeled hypermaps on arc sets of cardinality $n$ equals the number of rooted unlabeled 2-cell imbeddings of the dipole, $D_n$, on generalized pseudosurfaces. Moreover, for any representative, $M = (H, \rho_E, \rho_V)$, of an unlabeled hypermap, $\langle \rho_E, \rho_V \rangle$ is transitive if and only if the unlabeled dipole imbedding is on a pseudosurface.

**Remark.** The root is necessary in order to distinguish the vertex of the dipole corresponding to the edge set of the hypergraph from that corresponding to the vertex set.

Table 5.5 lists the unlabeled hypermaps on arc sets of cardinality 10 or less.
### 6.3. Multistars

In this last section we show how some of the techniques introduced earlier can be applied to a large class of pseudographs; this is the class of multistars, defined as those loopless pseudographs where all the edges have a vertex in common (called the central vertex). Figure 6.4 shows a few of the many multistars.
If $v$ is a vertex of a multistar and $\rho_v$ is a permutation of arcs at $v$ we let $\sigma_v$ denote the corresponding permutation of edges at $v$. Let $u$ denote a central vertex of the multistar (dipoles have two choices) and let $v_1, v_2, \ldots, v_m$ denote the remaining vertices. The rotation of arcs at each vertex are assembled into a rotation of arcs on the vertex set $V$ and denoted, as usual, by $\rho_V$. Thus, $\rho_V = \rho_u \rho_{v_1} \rho_{v_2} \cdots \rho_{v_m}$. This permutation on the arc set together with the rotation of arcs, $\rho_E$, on the edge set $E$, determine the 2-cell imbedding with region boundaries given by the orbits of $\rho_V \rho_E$. $\rho_E$ is a full involution; our goal is to eliminate the dependence on this permutation, and instead describe the imbeddings using the rotations of edges at each vertex alone. This is an extension of the technique used for the dipole (itself a multistar). To this end, the permutation of edges corresponding to $\rho_V$ above is $\sigma_u \sigma_{v_1} \sigma_{v_2} \cdots \sigma_{v_m}$. Observe that for a multistar $\sigma_{v_i}$ and $\sigma_{v_j}$ are disjoint for distinct $i$ and $j$. We let $\sigma_v = \sigma_{v_1} \sigma_{v_2} \cdots \sigma_{v_m}$; the order in which the permutations on the right side appear is
immaterial. Thus, the permutation of the arc set \( \rho_V \)
corresponds, in a natural way, to the permutation of edges
\( \sigma_u, \sigma_v \).

Each edge \( e \) must (by definition of a multistar) be incident
with the central vertex \( u \) and one of the \( v_i \). Hence, we let \( e^u \)
denote the arc of \( e \) directed away from vertex \( u \), and \( e' \) denote
the other arc.

Now consider the region boundaries. As for the dipole,
every region boundary must have even length and the arcs
visited in a traversal must alternate between those directed
away from \( u \) and those which are not. Thus, every region
boundary contains some \( e^u \). Figure 6.5 depicts a typical
oriented region boundary, both arc labeled and edge labeled.

![Figure 6.5. Boundary segments of a 2-cell imbedded multistar.](image)

From the above figure the following strings of equalities are
evident.
\[
(\rho_V \rho_E)^2 e_1^u = \rho_V \rho_E \rho_V e_1^v = \rho_V \rho_E e_2^v = \rho_V e_2^u = e_2^u, \quad \text{and}
\]
\[
(\sigma_u \sigma_v)^{-1} e_1 = \sigma_u e_2 = e_3.
\]
It follows that the orbits of the permutation of the edges, \( \sigma_u \sigma_v \),
are in one-to-one correspondence with the orbits of the
permutation of arcs, \( \rho_V \rho_E \), and each orbit of the former has

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precisely half the length of its counterpart in the latter.

In the case of surface imbeddings, each of the edge rotations, \( \sigma_u, \sigma_{v_1}, \sigma_{v_2}, \ldots, \sigma_{v_m} \), must be cyclic. In particular, \( \sigma_u \) must be a full cycle. Suppose among the noncentral vertices of a multistar there are \( d_s \) vertices of degree \( s, s=1,2,\ldots,n \).

For a surface imbedding, \( \sigma_v = \sigma_{v_1}\sigma_{v_2}\cdots\sigma_{v_m} \) must be one of the \( \Pi(s-1)!d_s \) permutations of the edges assembled from cyclic permutations of edges at each vertex \( v_i \). Thus, each of the permutations, \( \sigma_v \), has cycle-type \( (d_1,d_2,\ldots,d_n) \).

To determine the number of 2-cell imbeddings of a multistar on \( n \) edges which have \( r_s \) regions of size \( 2s, s=1,2,\ldots,n \), we seek the number of solutions in \( \Sigma_n \) to \( \sigma_u\sigma_v=\sigma_R \) where \( \sigma_u \) is a full cycle, \( \sigma_v \) is one of the \( \Pi(s-1)!d_s \) permutations of cycle-type \( (d_1,d_2,\ldots,d_n) \) corresponding to edge rotations at the vertices \( v_i \), and \( \sigma_R \) has cycle-type \( (r_1,r_2,\ldots,r_n) \). Let \( i \) index the class of full cycles, \( j \) index the class of cycle-type \( (r_1,r_2,\ldots,r_n) \) and \( k \) index the class of type \( (d_1,d_2,\ldots,d_n) \).

The class multiplication coefficient, \( n_{ijk} \), gives the number of solutions to our problem for a fixed permutation of type \( (d_1,d_2,\ldots,d_n) \). Since there are \( \Pi(s-1)!d_s \) of these, we use the character theory summarized in Section 3.3 and present the following result.

**Theorem 6.5.** If a multistar has \( d_s \) vertices of degree \( s, s=1,2,\ldots,n \), in addition to the central vertex of degree \( n \), then the number of labeled 2-cell imbeddings on surfaces which have
Remark. Note that \((n-1)! \Pi (s-1)^{d_s} \) is the total number of 2-cell imbeddings on surfaces of the multistar. Hence, dividing the above expression by this number gives the fraction of all surface imbeddings having the given region distribution.

As an example, we calculate the number of 2-cell imbeddings on surfaces which have three 2-sided regions \((r_1=3)\) and two 4-sided regions \((r_2=2)\) for the multistar shown in Figure 6.6.

Figure 6.6. A multistar having 66 labeled surface imbeddings having three 2-sided regions and two 4-sided regions.

Thus, we have 
\[(r_1, r_2, \ldots, r_n) = (3, 2, 0, 0, 0)\] and 
\[(d_1, d_2, \ldots, d_n) = (0, 2, 1, 0, 0, 0)\]. The character values are 
\[
\chi^{(\lambda)}_j = [y^\lambda] \frac{\Pi (1-(-y)^s)^{d_s}}{1+y}
\]
\[ \begin{align*}
\chi_k^{(\lambda)} &= [y^\lambda] \left( \frac{1 - (\lambda y)^2}{1 + y} \right) \quad \text{and} \\
&= [y^\lambda] \left( 1 - y^2 + 2y^3 - y^4 - y^5 + y^6 \right).
\end{align*} \]

According to the theorem the number of imbeddings is

\[ \frac{6! \cdot 2}{1^3 \cdot 3! \cdot 2^2 \cdot 2!} \left( 1 + \frac{2}{5} + \frac{1}{15} + \frac{8}{20} + \frac{1}{15} + \frac{2}{5} + 1 \right) = 66. \]

In certain cases an explicit expression for the number of surface imbeddings with \( r \) regions can be given. Call a multistar with \( n \) edges \( d \)-regular if all the vertices have degree \( d \) except, possibly, the central vertex \( u \). A \( 3 \)-regular multistar is shown in Figure 6.7.

![Figure 6.7. A 3-regular multistar.](image)

Recall from Section 3.3 that the numbers \( e_r^{(d)} \) measure how often a fixed full cycle multiplies members of the class of \( d \)-regular permutations to give a permutation with exactly \( r \) orbits. It follows that

\[ (n-1)! \cdot e_r^{(d)} + \frac{n!}{d^{n/d} \cdot (n/d)!} \]

measures how often a fixed \( d \)-regular permutation multiplies members of the class of full cycles to give a permutation with exactly \( r \)
orbits. For a given d-regular multistar, there are \((d-1)! \frac{n}{d}\) d-regular permutations corresponding to rotations of edges at the noncentral vertices which give surface imbeddings (when the rotation at the central vertex is a full cycle). With a few simplifications the number of 2-cell imbeddings on surfaces resulting in \(r\) regions is \(\frac{1}{n} (d!)^\frac{n}{d} (n/d)! \epsilon_r^{(d)}\). Applying Theorem 3.2 we have the following result.

**Theorem 6.6.** The number of labeled 2-cell imbeddings on surfaces resulting in \(r\) regions of a d-regular multistar with \(n\) edges is

\[
\frac{(d-1)! \frac{n}{d} \cdot (n/d)!}{n(n+1)} \sum_{m=\frac{n}{d}+r}^{n+1} d^{m-r} \binom{m}{r} \epsilon_r^{(m)} \sigma_{n+1}^{(m)} \sigma_{m-r}^{(n/d)}.
\]

**Remark.** Using \(p-q+r=2-2\)-genus, we can determine the genus distribution of a d-regular multistar. The genus is given by

\[
\frac{1}{2} + \frac{1}{2} (n-r-\frac{n}{d}).
\]
BIBLIOGRAPHY


