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Karen S. Holbert
Western Michigan University

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SPECIFIED SUBGRAPHS AND SUBGRAPH-DEFINED PARAMETERS IN GRAPHS

by

Karen S. Holbert

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SPECIFIED SUBGRAPHS AND SUBGRAPH-DEFINED PARAMETERS IN GRAPHS

Karen S. Holbert, Ph.D.
Western Michigan University, 1989

In this dissertation, we consider three well-known subgraphs and three subgraph-defined parameters. The subgraphs we study are maximum matchings, the center and the median. The parameters we investigate are a subgraph-defined degree and two subgraph distances.

Chapter I begins with some preliminary definitions. The remainder of the chapter illustrates the variety of uses of subgraphs by introducing two subgraph distances. The first of these gives a distance between subgraphs \( F \) and \( H \) of the same order in a connected graph \( G \). The associated diameter sequences of several classes of graphs are described. We also introduce a distance for subgraphs having the same size and describe some of the properties of this distance.

In Chapter II maximum matchings in regular graphs with specified edge connectivity are investigated. Several well-known theorems on maximum matchings, including Petersen's 1-factor theorem are generalized.

The center and median subgraphs are studied in Chapter III. First, we give several results considering the "location" of the center and the median in connected graphs. Included is an extension of a result of Hendry which shows that the distance between the center and median can be made arbitrarily large. In addition we introduce two related subgraphs, the \( k \)-median and the pseudocenter. We show that for a given graph \( G \) and certain values of \( k \) (depending on the minimum degree of \( G \)), there exists a connected graph \( H_k \) such that \( G \) is isomorphic to the \( k \)-median of \( H_k \).
Also, we show that not all graphs are pseudocenters and present several sufficient conditions for a graph $G$ to be a pseudocenter of some connected graph $H$.

In Chapter IV, we study a subgraph-defined degree of a vertex. Let $F$ and $G$ be graphs. The $F$-degree of a vertex $v$ in $G$ is the number of subgraphs of $G$ isomorphic to $F$ that contain $v$. A graph $G$ is $F$-regular if every vertex of $G$ has the same $F$-degree. If the vertices of $G$ have distinct $F$-degrees, then $G$ is $F$-irregular. We begin this chapter with several $F$-degree analogs of a well-known result involving degrees. We then show the existence of nonregular, $C_n$-regular graphs. We also produce an infinite class of $P_3$-irregular graphs, $K(1,n)$-irregular graphs for all $n \geq 3$, and $F$-irregular graphs for all the connected graphs $F$ of order 4.
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Specified subgraphs and subgraph-defined parameters in graphs

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Western Michigan University, 1989

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...to my children,

Katie, Jodie and Daniel,

for the joy they give me

and to

Dale,

whose love makes my life complete...
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Karen S. Holbert
CHAPTER I
AN INTRODUCTION TO SUBGRAPHS AND
SUBGRAPH-DEFINED PARAMETERS

In this chapter, we begin with a few preliminary definitions and results. Then in Sections 1.2 and 1.3, we illustrate the use of subgraph-defined parameters by considering two subgraph distances.

1.1 Introduction

We begin by presenting some of the basic definitions and notation which are fundamental to this dissertation. In addition, more specialized definitions will be introduced as needed. All other terms will be defined as in Chartrand and Lesniak [7].

As usual, |S| denotes the cardinality of a set S. We will use V(G) and E(G) to denote the vertex set and edge set of a graph G, respectively. The number |V(G)|, sometimes denoted p(G), is called the order of G and |E(G)| is called the size of G.

A graph H is called a subgraph of a graph G if V(H) ⊆ V(G) and E(H) ⊆ E(G). If v ∈ V(G) and G has order at least 2, then G - v denotes the subgraph with vertex set V(G) - {v} and whose edges are all those of G not incident with v. If e ∈ E(G), then G - e is the subgraph having vertex set V(G) and edge set E(G) - {e}. The graph obtained by the deletion of a set S of vertices or edges, denoted by G - S is defined analogously. If U is a nonempty subset of the vertex set V(G) of a graph G, then the subgraph (U) of G induced by U is the graph having vertex set U and whose edge set consists of those edges of G incident with two elements of U. Similarly, if F is a nonempty subset of E(G), then the subgraph
\(\langle F \rangle\) induced by \(F\) is the graph whose vertex set consists of those vertices of \(G\) incident with at least one edge of \(F\) and whose edge set is \(F\).

In this dissertation, we will be investigating three types of subgraphs. The first of these are the maximum matchings discussed in Chapter II. In Chapter III, we will examine the center and median subgraphs of a connected graph \(G\). The definitions of these subgraphs will be introduced in Chapters II and III.

Given a connected graph \(G\), some of the numbers most frequently associated with \(G\) are the degrees of its vertices and the distances between pairs of vertices. The distance \(d_G(u,v)\) (or simply \(d(u,v)\) if the graph \(G\) is clear) between two vertices \(u\) and \(v\) in a connected graph \(G\) is the length of a shortest \(u-v\) path in \(G\). In this dissertation, we define concepts analogous to the degree of a vertex and the distance between vertices but in terms of subgraphs. A subgraph-defined parameter, the \(F\)-degree of a vertex, will be discussed in Chapter IV. In the remaining two sections of this introductory chapter, we illustrate how subgraphs can be used to generalize distance.

1.2 Subgraph Distance for Subgraphs of the Same Order

The first subgraph distance we will consider was introduced by Johns [10]. We begin with some definitions. For a connected graph \(G\) of order \(p\) and an integer \(n\) such that \(1 \leq n \leq p\), let \(F\) and \(H\) be induced subgraphs of \(G\) of order \(n\). We define a pairing \(\pi\) from the set \(V(F)\), say \(\{v_1, v_2, \ldots, v_n\}\), to the set \(V(H)\) as a one-to-one correspondence that associates a vertex of \(V(F)\) with one of \(V(H)\). The distance induced by \(\pi\) between \(F\) and \(H\) is defined as

\[
d_{\pi}(F, H) = \sum_{i=1}^{n} d(v_i, \pi(v_i))
\]
The subgraph distance between F and H, denoted \( d(F,H) \), is given by

\[
d(F,H) = \min_{\pi} d_{\pi}(F,H).
\]

Observe that if F consists of a single vertex, say u, and H also consists of a single vertex, say v, then \( d(F,H) = d(u,v) \). Thus \( d(F,H) \) is a generalized distance defined in terms of subgraphs. It is shown in [10] that this distance is a metric. As an example, we give in Figure 1.1 a connected graph G, two induced subgraphs F and H of G of order 3, a listing of all the pairings from \( V(F) \) to \( V(H) \), and \( d(F,H) \).

For a subgraph F of a connected graph G with \( p(F) = n \), the subgraph eccentricity \( e(F) \) of F is given by \( e(F) = \max\{d(F,H) | p(H) = n\} \). The \( n \)-radius \( \text{rad}_n G \) of G is defined by \( \text{rad}_n G = \min\{e(F) | p(F) = n\} \), and the \( n \)-diameter \( \text{diam}_n G \) of G is defined as \( \text{diam}_n G = \max\{e(F) | p(F) = n\} \). For a connected graph G, the sequence \( \text{diam}_1 G, \text{diam}_2 G, \ldots, \text{diam}_{p-1} G \) is called the diameter sequence. The following result [10] describes some properties of the diameter sequence of a graph.

**Theorem 1A** Let G be a connected graph of order \( p \).

(i) If \( n \) is an integer such that \( 1 \leq n \leq \left\lfloor \frac{p}{2} \right\rfloor - 1 \), then \( \text{diam}_n G \leq \text{diam}_{n+1} G \).

(ii) If \( n \) is an integer such that \( 1 \leq n \leq p - 1 \), then \( \text{diam}_n G = \text{diam}_{p-n} G \).
### Pairings

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<th>Pairings</th>
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<th>$v_j$</th>
<th>$d(u_i,v_j)$</th>
<th>$d_{\pi_k}(F,H)$</th>
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$d(F,H) = 8$

Figure 1.1

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As an example (from [10]), we compute the diameter sequence of $K_p$. If $p = 2k$ and $n$ is an integer such that $1 \leq n \leq k$, then every subgraph $F$ of $G$ with order $n$ has $e(F) = n$, and therefore $\text{diam}_n K_{2k} = n$. Then by Theorem 1A, part ii), the diameter sequence for $K_{2k}$ is $1, 2, \ldots, k - 1, k, k - 1, \ldots, 2, 1$. Similarly, the diameter sequence for $K_{2k+1}$ is $1, 2, \ldots, k - 1, k, k, k - 1, \ldots, 2, 1$. Also given in [10] are the diameter sequence of $K(2k,2k)$ and a characterization for the diameter sequence of a caterpillar. (A caterpillar is a tree of order $p \geq 3$ whose subgraph obtained by removing its end-vertices is a path.) In this section, we present results on the diameter sequences of several other well-known classes of graphs. Our first result gives the diameter sequence for $C_p$.

**Theorem 1.1** Let $p \geq 3$ be an integer.

(i) If $p$ is odd, then $\text{diam}_n C_p = \frac{np - n^2}{2}$ for $1 \leq n \leq \left\lfloor \frac{p}{2} \right\rfloor$.

(ii) If $p$ is even, then

a) $\text{diam}_n C_p = \frac{np - n^2}{2}$ for $1 \leq n \leq \left\lfloor \frac{p}{2} \right\rfloor$ and $n$ even, and

b) $\text{diam}_n C_p = \frac{np - n^2 + 1}{2}$ for $1 \leq n \leq \left\lfloor \frac{p}{2} \right\rfloor$ and $n$ odd.

(iii) For all $p \geq 3$, $\text{diam}_n C_p = \text{diam}_{p-n} C_p$ for $1 \leq n \leq p - 1$.

**Proof.** Let $G \cong C_p$, and suppose $V(G) = \{v_1, v_2, \ldots, v_p\}$ and $E(G) = \{v_i v_{i+1} | i = 1, 2, \ldots, p - 1\} \cup \{v_1 v_p\}$. We proceed by induction on $n$. Since $\text{diam}_p C_p = \frac{p}{2}$ when $p$ is even and $\text{diam}_p C_p = \frac{p-1}{2}$ when $p$ is odd, (i) and (ii) are satisfied when $n = 1$. For the induction step, assume that (i) and (ii) hold for some arbitrary $n$. We now consider $\text{diam}_{n+1} C_p$. We consider the case when $p$ is odd and the case when $p$ is even separately.

Assume that $p$ is odd. Let $F$ and $H$ be subgraphs of $C_p$ such that $d(F,H) = \text{diam}_{n+1} C_p$. Without loss of generality, we may assume that $v_1 \in V(F)$. Observe that
because $|V(H)| = n + 1$, there exists some vertex $v_i \in V(H)$ such that $d(v_1, v_i) \leq \frac{p-2n-1}{2}$. Therefore,

$$
diam_{n+1} C_p \leq diam_n C_p + \frac{p-2n-1}{2} = \frac{np-n^2}{2} + \frac{p-2n-1}{2} = \frac{(n+1)p-(n+1)^2}{2}.
$$

Further, because of our choice of $F$ and $H$, $diam_{n+1} C_p = \frac{(n+1)p-(n+1)^2}{2}$.

We now consider the case when $p$ is even. Let $F$ and $H$ be subgraphs of $C_p$ such that $d(F,H) = diam_{n+1} C_p$. Without loss of generality, we may assume that $v_1 \in V(F)$. If $n$ is even, then there exists some vertex $v_i \in V(H)$ such that $d(v_1, v_i) \leq \frac{p-2n}{2}$. Therefore, $diam_{n+1} C_p \leq diam_n C_p + \frac{p-2n}{2}$. By the inductive hypothesis, $diam_n C_p = \frac{np-n^2}{2}$. Thus,

$$
diam_{n+1} C_p \leq \frac{np-n^2}{2} + \frac{p-2n}{2} = \frac{(n+1)p-(n+1)^2+1}{2}.
$$

Further, because of our choice of $F$ and $H$, $diam_{n+1} C_p = \frac{(n+1)p-(n+1)^2+1}{2}$. Observe that this satisfies (ii), part (b) since $n+1$ is odd.

If $n$ is odd, there exists some vertex $v_i \in V(H)$ such that $d(v_1, v_i) \leq \frac{p-2n}{2}$. Therefore,

$$
diam_{n+1} C_p \leq diam_n C_p + \frac{p-2n}{2} = \frac{np-n^2+1}{2} + \frac{p-2n}{2} = \frac{(n+1)p-(n+1)^2}{2}.
$$

Further, because of our choice of $F$ and $H$, $diam_{n+1} C_p = \frac{(n+1)p-(n+1)^2}{2}$. Observe that this satisfies (ii), part (b) since $n+1$ is even.

We have already mentioned that the diameter sequence for caterpillars has been characterized. Our next result gives the diameter sequence for another class of trees, namely the full binary trees. Let $T$ be a rooted binary tree having height $h$. Then $T$ is complete if every vertex that is not a leaf has exactly two children. A complete rooted binary tree $T$ is full if every leaf is at level $h$. 

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Theorem 1.2 Let $T$ be a full complete binary tree of height $h$. The diameter sequence for $T$ is given by the following:

(i) For $1 < n < 2^{h+1} - 2$, $diam_n T = diam_{2^{h+1} - 2} T$.

(ii) For $1 < n < 2^h - 2$, $diam_n T = 2nh$.

(iii) For integers $k$ such that $1 < k < h$ and $n$ such that $(2^{h-k} - 1)^2 < n < (2^{h-k+1} - 1)^2k$,

$$diam_n T = \left( \sum_{j=0}^{h-k-1} 2^{h-j}(h-j) \right) + 2(n - (2^{h-k} - 1)^2k)k.$$

Proof. Let $T$ be a full binary tree of height $h$. Then $p(T) = 2^{h+1} - 1$ and (i) follows from Theorem 1A. Let $r$ be the root of $T$ and let $u_{1,1}$ and $v_{1,1}$ be the left and right child of $r$, respectively. We denote by $T_{u_{1,1}}$ the maximal subtree of $T$ rooted at $u_{1,1}$, and we denote by $T_{v_{1,1}}$ the maximal subtree of $T$ rooted at $v_{1,1}$. For $2 < m < h$, let $u_{1,m}, u_{2,m}, \ldots, u_{2^{m-1},m}$ be the ordered vertices of $T_{u_{1,1}}$ at level $m$ of $T$. Also, let $v_{1,m}, v_{2,m}, \ldots, v_{2^{m-1},m}$ be the ordered vertices of $T_{v_{1,1}}$ at level $m$ of $T$. Observe that $d(u_{i,m}, v_{j,m}) = 2m$ ($1 < i < 2^{m-1}, 1 < j < 2^{m-1}, 1 < m < h$).

We now prove (ii). Observe that $diam_1 T = 2h$. Thus, for $1 < n < 2^{h-1}$, $diam_n T < 2nh$. Let $V_1 = \{ u_{1,h}, u_{2,h}, \ldots, u_{n,h} \}$ and let $V_2 = \{ v_{1,h}, v_{2,h}, \ldots, v_{n,h} \}$. Then $d((V_1), (V_2)) = 2nh$. Thus $diam_n T = 2nh$ for $1 < n < 2^{h-1}$.

To prove (iii), we proceed by induction on $h$. For the basis, we consider the case $h = 2$. Let $T$ be a full binary tree of height 2, and let the vertices of $T$ be labeled as described above. Let $L$ be the set of leaves of $T$. Observe that $T - L$ is the tree consisting of $r$, $u_{1,1}$, and $v_{1,1}$. The maximum distance between vertices in $T - L$ is given by $d(u_{1,1}, v_{1,1}) = 2$. This, together with (ii), gives that $diam_3 T = 8 + 2 = 10$ and (iii) is satisfied when $k = 1$. 

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For the induction step, we assume that for some arbitrary $h$, (iii) holds. We now consider a full binary tree $T$ of height $h + 1$. Let $L$ be the set of leaves of $T$. Then $T - L$ is a full binary tree of height $h$. By ii), $\text{diam}_{2h} T = 2^{h+1}(h + 1)$. Further, if $F$ and $H$ are subgraphs of order $2^h$ of $T$ such that $d(F,H) = \text{diam}_{2h} T$, then $V(F) \cup V(H) = L$. Let $F_n$ and $H_n$ be subgraphs of order $n$ of $T$ such that $d(F_n,H_n) = \text{diam}_n T$ ($1 \leq k \leq h - 1$, $(2^{h-k+1} - 1)2^k < n \leq (2^{h-k+2} - 1)2^k - 1$). Then $L \subseteq V(F_n) \cup V(H_n)$ and $[V(F_n) \cup V(H_n)] \setminus L \subseteq V(T - L)$. Thus, $\text{diam}_n T = \text{diam}_{2h} T + \text{diam}_{n-2h} T - L$. But $\text{diam}_{2h} T = 2^{h+1}(h + 1)$. We consider two cases. First, if $1 \leq n - 2^h \leq 2^{h-1}$, $\text{diam}_{n-2h} T - L = 2(n - 2^h)h$ by (ii). Thus, $\text{diam}_n T = 2^{h+1}(h + 1) + 2(n - 2^h)h$, satisfying (iii) when $k = h$.

We now consider the case that $n - 2^h > 2^{h-1}$. Since $2^{h-k+1} - 1)2^k < n \leq (2^{h-k+2} - 1)2^k - 1$, $(2^{h-k} - 1)2^k < n - 2^h \leq (2^{h-k+1} - 1)2^k - 1$. Therefore,

$$\text{diam}_{n-2^h} T = \left(\sum_{j=0}^{h-k-1} 2^h - j(h - j)\right) + 2(n - 2^h - (2^{h-k} - 1)2^k)k$$

by the inductive hypothesis. Then, since $\text{diam}_n T = \text{diam}_{2h} T + \text{diam}_{n-2^h} T - L$,

$$\text{diam}_n T = 2^{h+1}(h + 1) + \left(\sum_{j=0}^{h-k-1} 2^h - j(h - j)\right) + 2(n - 2^h - (2^{h-k} - 1)2^k)k$$

$$= \left(\sum_{j=0}^{h-k} 2^h - j+1(h - j + 1)\right) + 2(n - 2^h - (2^{h-k} - 1)2^k)k$$

$$= \left(\sum_{j=0}^{h-k} 2^h - j+1(h - j + 1)\right) + 2n - 2^h(2^{h-k}) - (2^{h-k} - 1)2^k)k$$

$$= \left(\sum_{j=0}^{h-k} 2^h - j+1(h - j + 1)\right) + 2(n - (2^{h-k+1} - 1)2^k)k.$$

Thus, (iii) is satisfied, and the theorem is proved. $\square$
The problem of determining the diameter sequence for trees in general remains unsolved.

Our next theorem on diameter sequences gives a characterization of the diameter sequence for complete bipartite graphs.

**Theorem 1.3** Let \( p \geq 2 \) be an integer. A sequence \( S: a_1, a_2, \ldots, a_{p-1} \) of positive integers is the diameter sequence of a complete bipartite graph of order \( p \) if and only if \( S \) satisfies:

(i) \( a_1 = 2; \)

(ii) there exists some integer \( k, \left\lfloor \frac{p}{2} \right\rfloor \leq k \leq \left\lfloor \frac{p-1}{2} \right\rfloor \) such that
   a) \( a_{n+1} = a_n + 2 \) for \( 1 \leq n \leq k-1 \), and
   b) either \( a_{k+1} = a_k \) or \( a_{k+1} = a_k + 1; \)

(iii) \( a_{n+1} = a_n \) for \( k+1 \leq n \leq \left\lfloor \frac{p}{2} \right\rfloor - 1 \); and

(iv) \( a_n = a_{p-n} \) for \( 1 \leq n \leq p-1. \)

**Proof.** Let \( G \) be a complete bipartite graph of order \( p \). Then \( G = K(p_1,p_2) \) where \( p_1 \leq p_2 \) and \( p_1 + p_2 = p \). Let \( V_1 = \{u_1, u_2, \ldots, u_{p_1}\} \) and \( V_2 = \{v_1, v_2, \ldots, v_{p_2}\} \) be the partite sets of \( G \). Let \( S \) be the diameter sequence for \( G \). We show that \( S \) satisfies (i) through (iv). Because the diameter of \( K(p_1,p_2) = 2 \), (i) is satisfied. Also, (iv) follows from Theorem 5A. We now verify (ii).

Observe that \( d(u_i,u_j) = d(v_k,v_m) = 2 \) for \( 1 \leq i < j \leq p_1 \) and \( 1 \leq k < m \leq p_2 \). Also, \( d(u_i,v_k) = 1 \) for \( 1 \leq i \leq p_1 \) and \( 1 \leq k \leq p_2 \). Thus, for \( 1 \leq n \leq \left\lfloor \frac{p_2}{2} \right\rfloor \), \( \text{diam}_n G \leq 2n \). Let \( F = \{v_1, v_2, \ldots, v_n\} \) and \( H = \{v_{n+1}, v_{n+2}, \ldots, v_{2n}\} \). Then \( d(F,H) = 2n \) so \( \text{diam}_n G = 2n \), and letting \( k = \left\lfloor \frac{p_2}{2} \right\rfloor \) we see that a) is verified. If we can show that \( \text{diam}_n G = p_2 \) for \( k \leq n \leq \left\lfloor \frac{p_2}{2} \right\rfloor \), then if \( p_2 \) is even, we will have \( \text{diam}_{k+1} G = \text{diam}_k G + 1 \). Thus
showing that $\text{diam}_G = p^2$ for $k \leq n \leq \left\lfloor \frac{p}{2} \right\rfloor$ will verify b). Further, this will verify (iii).

To see that $\text{diam}_G = p^2$ for $k \leq n \leq \left\lfloor \frac{p}{2} \right\rfloor$, suppose first that $F$ and $H$ are subgraphs of $G$ of order $n$ such that $d(F,H) = \text{diam}_G$. Without loss of generality, we may assume that $V(F) \cap V(H) = \emptyset$. Thus, not all the vertices of $V(F) \cup V(H)$ lie in $V_2$. Let $|V(F) \cap V_1| = k_1$, $|V(F) \cap V_2| = m_1$, $|V(H) \cap V_1| = m_2$ and $|V(H) \cap V_2| = k_2$. Then, $k_1 + m_1 = k_2 + m_2 = n$. Observe also that $|V_1 \cap (V(F) \cup V(H))| = k_1 + m_2$ and $|V_2 \cap (V(F) \cup V(H))| = k_2 + m_1$. Let $k = \min\{k_1, k_2\}$ and $m = \min\{m_1, m_2\}$. Note that if $k = k_1$, then $m = m_2$, and if $k = k_2$, then $m = m_1$. Let $V(F) = \{w_1, w_2, \ldots, w_n\}$ and let $V(H) = \{x_1, x_2, \ldots, x_n\}$. Without loss of generality, suppose $w_1, w_2, \ldots, w_k$ are $k$ vertices in $V_1$ and $w_{k+1}, w_{k+2}, \ldots, w_{m+k}$ are $m$ vertices in $V_2$. Suppose further that $x_1, x_2, \ldots, x_k$ are $k$ vertices in $V_1$ and $x_{k+1}, x_{k+2}, \ldots, x_{m+k}$ are $m$ vertices in $V_2$. Then for $1 \leq i \leq k + m$, the pairing of $w_i$ in $V(F)$ and $x_i$ in $V(H)$ contributes $k + m$ to $d(F,H)$. Since the other vertices of $V(F) \cup V(H)$ all lie in $V_1$ or all lie in $V_2$, there are at most $p_2 - (k + m)$ vertices in $(V(F) \cup V(H)) \setminus (\{w_1, w_2, \ldots, w_{k+m}\} \cup \{x_1, x_2, \ldots, x_{k+m}\})$. Thus there are at most $\frac{p_2 - (k + m)}{2}$ pairs of vertices each contributing 2 to $d(F,H)$. Thus $d(F,H) \leq k + m + 2\left(\frac{p_2 - (k + m)}{2}\right) = p^2$. Further, because of our choice of $F$ and $H$ such that $d(F,H) = \text{diam}_G$, we have that $\text{diam}_G = d(F,H) = p^2$.

For the converse, suppose $S$ is a sequence satisfying (i) through (iv). If $a_{k+1} = a_k$, then by the above discussion, $S$ is the diameter sequence for $K(2k, p - 2k)$. If $a_{k+1} = a_k + 1$, then again by the above discussion, $S$ is the diameter sequence for $K(2k + 1, p - 2k - 1)$. $\square$

It would be desirable to generalize this subgraph distance to describe the distance between subgraphs of different orders. The generalization that we feel is most natural
fails, however. We think it is informative to describe this generalization and explain why it fails.

For a connected graph $G$ of order $p$, let $F$ and $H$ be induced subgraphs of $G$. If $F$ and $G$ have the same order, any pairing $\pi$ from the set $V(F)$ to the set $V(H)$ is a subset of $V(F) \times V(H)$. Thus one natural generalization would be to allow $\pi$ to be any subset of $V(F) \times V(H)$ such that the domain of $\pi$ is $V(F)$, and the range of $\pi$ is $V(H)$. Therefore $\pi$ would be defined even if $F$ and $H$ have different orders. Then the distance induced by $\pi$ between $F$ and $H$ could be defined as

$$d_{\pi}(F,H) = \sum_{(u,v) \in \pi} d(u,v).$$

Then the subgraph distance between $F$ and $H$ could be given by

$$d(F,H) = \min_{\pi} d_{\pi}(F,H).$$

The problem with this definition of a generalized subgraph distance is that it fails to be a metric. Consider the graph $G$ of Figure 1.2. Suppose we let $F = \langle \{v_1, v_2, v_3\} \rangle$, $H = \langle \{u_1, u_2\} \rangle$ and $K = \langle \{w_1, w_2, w_3, w_4\} \rangle$. Observe that $d(u_i,v_j) = 1$ for all $i$ and $j$ ($1 \leq i \leq 2, 1 \leq j \leq 3$), $d(u_i,w_k) = 2$ for all $i$ and $k$ ($1 \leq i \leq 2, 1 \leq k \leq 4$) and $d(v_j,w_k) = 1$ for all $j$ and $k$ ($1 \leq j \leq 3, 1 \leq k \leq 4$). Thus, the above definition would yield $d(F,H) = 3$, $d(H,K) = 8$ and $d(F,K) = 12$. Since $d(F,K) \neq d(F,H) + d(H,K)$ the triangle inequality fails to hold. Thus this definition fails to be a metric, and therefore we are unable to compute the subgraph distance for graphs of different orders, via this "natural" definition.
1.3 Subgraph Distance for Subgraphs of the Same Size

In this section, we introduce another distance between subgraphs. In this case, the subgraphs are required to have the same size. This is based on the edge rotation metric (see [8]).

Let \( G \) be a connected graph and let \( F \) be a subgraph of \( G \). We say that \( F \) can be transformed into another subgraph \( F_1 \) of \( G \) by a subgraph edge rotation if \( G \) contains distinct vertices \( u, v, \) and \( w \) such that \( uv \in E(F), uw \in E(G)\setminus E(F) \) and \( F_1 = F - uv + uw \). In this case, \( F \) is transformed into \( F_1 \) by "rotating" the edge \( uv \) of \( F \) into the edge \( uw \) of \( F_1 \). Figure 1.3 shows a connected graph \( G \) and subgraphs \( F \) and \( F_1 \) of \( G \). Note that \( F \) can be transformed into \( F_1 \) by a subgraph edge rotation (\( vw \) is rotated into \( wa \)).
We say that a subgraph $F$ of $G$ can be *transformed* into a subgraph $H$ of $G$, written $F \xrightarrow{G} H$, if either (1) $F = H$, or (2) there exists a sequence $F = F_0, F_1, F_2, \ldots, F_n = H$ ($n \geq 1$) of subgraphs of $G$ such that $F_i$ can be transformed into $F_{i+1}$ by a subgraph edge rotation for $i = 0, 1, \ldots, n - 1$. In Figure 1.4, we give an example of a connected graph $G$ and subgraphs $F$ and $H$ of $G$ such that $F \xrightarrow{G} H$. We show in the figure, the sequence $F = F_0, F_1, F_2, \ldots, F_8 = H$ of subgraphs of $G$ such that $F_i$ can be transformed into $F_{i+1}$ by a subgraph edge rotation for $i = 0, 1, \ldots, 7$. 

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Figure 1.4

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It is immediate that a subgraph $F$ of a connected graph $G$ can be transformed into a subgraph $H$ of $G$ if and only if $H$ can be transformed into $F$. Further, it is clear that $F \xrightarrow{G} H$ only if $F$ and $H$ have the same size. They need not have the same order as illustrated in the example of Figure 1.4. Our first result of this section shows that if $F$ and $H$ are any two subgraphs of a connected graph $G$ of the same size, then $F$ can be transformed into $H$.

**Theorem 1.4** Let $F$ and $H$ be subgraphs of a connected graph $G$ having the same size. Then $F \xrightarrow{G} H$.

**Proof.** Suppose to the contrary that there exists a connected graph $G$ and subgraphs $F$ and $H$ of $G$ having the same size such that $F$ cannot be transformed into $H$. Let $F = F_0, F_1, F_2, \ldots, F_n$ be a sequence of subgraphs of $G$ such that $F_i$ can be transformed into $F_{i+1}$ by a subgraph edge rotation for $i = 0, 1, \ldots, n-1$ and such that $|E(F_n) \cap E(H)|$ is as large as possible. Since $F_n \neq H$, there must exist an edge $a \in E(F_n) \setminus E(H)$ and an edge $b \in E(H) \setminus E(F_n)$. Since the graph $G$ is connected, there must exist a path $P: v_0, v_1, v_2, \ldots, v_m$ such that $a = v_0v_1$ and $b = v_m v_{m-1}$. Without loss of generality, we may assume that there exists no other edge $c$ in $E(F_n) \setminus E(H)$ on $P$. If there exists no edge $e \in E(F_n) \cap E(H)$ on $P$, we may perform $m - 1$ subgraph edge rotations, each time "rotating" the edge $v_i v_{i+1}$ into the edge $v_{i+1} v_{i+2}$ ($i = 0, 1, \ldots, m-2$) giving a subgraph $F'$ of $G$ such that $|E(F') \cap E(H)| > |E(F_n) \cap E(H)|$, producing a contradiction. Thus, we may assume there exists an edge of $E(F_n) \cap E(H)$ on $P$. Let $j$ be the largest integer such that $v_{j-1} v_j \in E(F_n) \cap E(H)$. We now perform $m - j$ subgraph edge rotations, each time "rotating" the edge $v_i v_{i+1}$ into the edge $v_{i+1} v_{i+2}$ ($i = j-1, j, \ldots, m-2$) producing a subgraph $F_{n+1}$ of $G$ such that $|E(F_{n+1}) \cap E(H)| = |E(F_n) \cap E(H)|$. Note that $v_{j-1} v_j \in E(H) \setminus E(F_{n+1})$. 

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Now, if there exists no edge \( e \in E(F_{n+1}) \cap E(H) \) on \( P' \): \( v_0, v_1, v_2, \ldots, v_j \), we may perform \( j - 1 \) subgraph edge rotations, each time "rotating" the edge \( v_i v_{i+1} \) into the edge \( v_{i+1} v_{i+2} \) \( (i = 0, 1, \ldots, j - 2) \) giving a subgraph \( F'' \) of \( G \) such that \( |E(F'')| \cap E(H) > |E(F_{n+1}) \cap E(H)| \), producing a contradiction. Therefore, we may assume there exists an edge of \( E(F_{n+1}) \cap E(H) \) on \( P' \). Let \( k \) be the largest integer such that \( v_k v_{k-1} \in E(F_{n+1}) \cap E(H) \). We now perform \( j - k \) subgraph edge rotations, each time "rotating" the edge \( v_i v_{i+1} \) into the edge \( v_{i+1} v_{i+2} \) \( (i = k - 1, j, \ldots, j - 2) \) producing a subgraph \( F_{n+2} \) of \( G \) such that \( |E(F_{n+2}) \cap E(H)| = |E(F_{n+1}) \cap E(H)| = |E(F_n) \cap E(H)| \).

Continuing in this manner, we must eventually produce a subgraph \( F_{n+t} \) of \( G \) such that \( |E(F_{n+t}) \cap E(H)| = |E(F_n) \cap E(H)| \) and such that there exists no edge \( e \in E(F_{n+t}) \cap E(H) \) on \( P'' \): \( v_0, v_1, v_2, \ldots, v_t \) and where \( v_t v_{t-1} \in E(H) \setminus E(F_{n+t}) \). Thus, we may perform \( t - 1 \) subgraph edge rotations, each time "rotating" the edge \( v_i v_{i+1} \) into the edge \( v_{i+1} v_{i+2} \) \( (i = 0, 1, \ldots, t - 2) \) producing a subgraph \( H' \) of \( G \) such that \( |E(H') \cap E(H)| > |E(F_{n+t}) \cap E(H)| \). But then \( |E(H') \cap E(H)| > |E(F_n) \cap E(H)| \), contradicting our choice of \( F_n \). \( \square \)

Let \( F \) and \( H \) be subgraphs of a connected graph \( G \) having the same size. We define the distance \( d(F, H) \) between \( F \) and \( H \) as 0 if \( F = H \) and otherwise, as the smallest positive integer \( n \) for which there exists a sequence \( F_0, F_1, F_2, \ldots, F_n \) of subgraphs of \( G \) such that \( F_0 = F, F_n = H \) and \( F_i \) can be transformed into \( F_{i+1} \) by a subgraph edge rotation for \( i = 0, 1, \ldots, n - 1 \). Thus the distance between \( F \) and \( H \) is the smallest number of subgraph edge rotations that will transform \( F \) into \( H \). By Theorem 1.4, this distance is a well-defined concept. We show that if \( \mathcal{G}_m \) is the set of all subgraphs of \( G \) of size \( m \) \( (1 \leq m \leq q(G)) \), then \( (\mathcal{G}_m, d) \) is a metric space.
Let \( F, H \) and \( K \) be elements of \( \mathcal{G}_m \). First, \( d(F,H) \geq 0 \) and \( d(F,H) = 0 \) if and only if \( F = H \). Second, \( d(F,H) = d(H,F) \). To see that the triangle inequality holds, suppose \( d(F,H) = n \). Then there exists a sequence \( F_0, F_1, F_2, \ldots, F_n \) of subgraphs of \( G \) such that \( F = F_0, F_n = H \) and \( F_i \) can be transformed into \( F_{i+1} \) by a subgraph edge rotation for \( i = 0, 1, \ldots, n - 1 \). Also if \( d(H,K) = k \), there exists a sequence \( H = H_0, H_1, H_2, \ldots, H_k = K \) of subgraphs of \( G \) such that \( H_j \) can be transformed into \( H_{j+1} \) by a subgraph edge rotation for \( j = 0, 1, \ldots, k - 1 \). Then \( F = F_0, F_1, F_2, \ldots, F_n = H = H_0, H_1, H_2, \ldots, H_k = K \) is a sequence of subgraphs of \( G \) such that \( F_i \) can be transformed into \( F_{i+1} \) by a subgraph edge rotation for \( i = 0, 1, \ldots, n - 1 \) and \( H_j \) can be transformed into \( H_{j+1} \) by a subgraph edge rotation for \( j = 0, 1, \ldots, k - 1 \). Thus \( d(F,K) \leq n + k \), and \( d(F,K) \leq d(F,H) + d(H,K) \).

As an example, consider again the connected graph \( G \) and subgraphs \( F \) and \( H \) of \( G \) given in Figure 1.4. The sequence \( F = F_0, F_1, F_2, \ldots, F_8 = H \) shows that \( d(F,H) \leq 8 \). We show that it requires at least 8 subgraph edge rotations to transform \( F \) into \( H \). Thus we show that \( d(F,H) = 8 \) for this example. We consider the number of subgraph edge rotations necessary to transform \( F \) into \( H \). Observe that because the distance between vertices \( u \) and \( b \) in \( G \) is 3, it will require at least four subgraph edge rotations to "move" the edge \( uv \) to the edge \( bc \) or the edge \( bd \). Since \( d(u,w) = 2 \), it will require at least two subgraph edge rotations to "move" the edge \( uv \) to the edge \( wa \). Similarly, because \( d(v,b) = d(x,b) = 3 \), it will require at least three subgraph edge rotations to "move" \( vw \) or \( wx \) to either the edge \( bc \) or \( bd \). Finally, it will require at least one subgraph edge rotation to "move" the edge \( vw \) or the edge \( wx \) to \( wa \). Since we must "move" some edge of \( F \) to each edge of \( H \), we need to consider all six one-to-one correspondences from \( E(F) \) to \( E(H) \) and the total number of edge rotations required to "move" each edge of \( F \) to its corresponding edge of \( H \).
for all six of these one-to-one correspondences. It is straightforward to verify that
each of the six one-to-one correspondences yields a total of eight subgraph edge
rotations to transform $F$ into $H$. Thus $d(F, H) = 8$.

Before presenting our next result, we need the following definition. Let $G$ be a

graph and let $F$ be a subgraph of $G$. The complement of $F$ relative to $G$, denoted
$\overline{F}_G$ is the subgraph of $G$ given by $\overline{F}_G = G - E(F)$. Observe that if $F$ and $H$ are
subgraphs of $G$ having the same size, then $\overline{F}_G$ and $\overline{H}_G$ are subgraphs of $G$ having
the same size.

**Theorem 1.5** Let $F$ and $H$ be subgraphs of a connected graph $G$ having the same

size. Then $d(F, H) = d(\overline{F}_G, \overline{H}_G)$.

**Proof.** If $d(F, H) = 0$, then $F = H$, implying that $\overline{F}_G = \overline{H}_G$ and $d(\overline{F}_G, \overline{H}_G) = 0$.
Assume then that $d(F, H) = n \geq 1$. Then there exists a sequence $F = F_0, F_1, F_2, \ldots ,$
$F_n = H$ of subgraphs of $G$ such that $F_i$ can be transformed into $F_{i+1}$ by a subgraph
edge rotation for $i = 0, 1, \ldots , n - 1$, where say $F_{i+1} = F_i - u_i v_i + u_i w_i$. Observe that
$\overline{F}_{i+1} = \overline{F}_G - u_i w_i + u_i v_i$, that is, $\overline{F}_{i+1}$ can be transformed into $\overline{F}_G$ by a subgraph
edge rotation. Thus the sequence

$$\overline{F}_G = \overline{F}_0 , \overline{F}_1 , \ldots , \overline{F}_n = \overline{H}_G$$

implies that $d(\overline{F}_G, \overline{H}_G) \leq d(F, H) = n$.

Now by applying the above technique to (1), we have $d(F, H) = d(\overline{F}_G, \overline{H}_G) \leq$
d(\overline{F}_G, \overline{H}_G)$ or

$$n = d(F, H) \leq d(\overline{F}_G, \overline{H}_G) \leq n,$$
producing the desired result. □

Our next result gives an upper bound for $d(F,H)$ for subgraphs $F$ and $H$ of a connected graph $G$ of the same size $n$ in terms of $n$ and the diameter of $G$.

**Theorem 1.6** Let $G$ be a connected graph and let $F$ and $H$ be subgraphs of $G$ of size $n$. Then $d(F,H) \leq n(diam G + 1)$.

**Proof.** Let $F$ and $H$ be subgraphs of $G$ having size $n$. If $F = H$, then clearly $d(F,H) \leq n(diam G + 1)$. We assume, then, that $F \neq H$. Then there exists an edge $a \in E(F) \setminus E(H)$ and there exists an edge $b \in E(H) \setminus E(F)$. Since $G$ is connected, there exists a path $P: v_0, v_1, \ldots, v_m$ of length $m \leq diam G$ such that $a = v_0u$ and $b = v_mw$ for some vertices $u$ and $w$ in $V(G)$. Thus $P': u, v_0, v_1, \ldots, v_m, w$ is a path in $G$ of length $m + 2$. By the process described in the proof of Theorem 1.4, it will take $m + 1 \leq diam G + 1$ subgraph edge rotations to produce a graph $F'$ such that $|E(F') \cap E(H)| = |E(F) \cap E(H)| + 1$. Since this is true for all pairs of edges $a$ and $b$ such that $a \in E(F) \setminus E(H)$ and $b \in E(H) \setminus E(F)$ and there are at most $n$ such pairs, $d(F,H) \leq n(diam G + 1)$. □

This bound cannot be improved in general. For $k \geq 1$ and $n \geq 1$, we define a graph $G_{n,k}$ as follows:

$$V(G_{n,k}) = \{v_1, v_2, \ldots, v_{n+1}\} \cup \{u_1, u_2, \ldots, u_{n+1}\} \cup \{w_1, w_2, \ldots, w_{k-1}\}$$

and

$$E(G_{n,k}) = \{v_iw_1 \mid i = 1, 2, \ldots, n + 1\} \cup \{u_iw_{k-1} \mid i = 1, 2, \ldots, n + 1\} \cup \{w_iw_{i+1} \mid i = 1, 2, \ldots, k - 2\} \cup \{v_iv_{i+1} \mid i = 1, 2, \ldots, n\} \cup \{u_iu_{i+1} \mid i = 1, 2, \ldots, n\}.$$
The graph $G_{4,5}$ is shown in Figure 1.5. Observe that $\text{diam } G_{n,k} = k$, and for $F = (\{v_1, v_2, \ldots, v_{n+1}\})$ and $H = (\{u_1, u_2, \ldots, u_{n+1}\})$, $d(F, H) = n(k + 1)$.

$G_{4,5}$:

$F$: \begin{align*}
v_1 & \quad v_2 \\
v_3 & \quad v_4 \\
v_5 & \quad v_6
\end{align*}

$H$: \begin{align*}
u_1 & \quad u_2 \\
u_3 & \quad u_4 \\
u_5 & \quad u_6
\end{align*}

Figure 1.5
CHAPTER II
MAXIMUM MATCHINGS IN REGULAR GRAPHS WITH SPECIFIED EDGE CONNECTIVITY

A set of pairwise nonadjacent edges in a graph $G$ is called a matching in $G$. A matching of maximum cardinality is a maximum matching. In a graph $G$ of order $p$, a matching of cardinality $\frac{p}{2}$ is called a perfect matching. Graphs with perfect matchings were characterized by Tutte [13].

**Theorem 2A** A graph $G$ has a perfect matching if and only if for every proper subset $S$ of $V(G)$, the number of odd components of $G - S$ does not exceed $|S|$.

Much research has centered around matchings in regular graphs. The following result on this subject is a generalization of Tutte's Theorem due to Berge [1].

**Theorem 2B** Let $G$ be a graph of order $p$ and let $l$ be an integer with $0 \leq l \leq \frac{p}{2}$ edges.

a) Let $p$ be even. Then every maximum matching of $G$ contains at least $(p - 2l)/2$ edges if and only if the number of odd components of $G - S$ does not exceed $|S| + 2l$ for every proper subset $S$ of $V(G)$.

b) Let $p$ be odd. Then every maximum matching of $G$ contains at least $(p - 2l - 1)/2$ edges if and only if the number of odd components of $G - S$ does not exceed $|S| + 2l + 1$ for every proper subset $S$ of $V(G)$.

In particular, many results concern maximum matchings in cubic graphs. A well-known result on this subject is due to Petersen [11].
Theorem 2C  Every cubic graph with at most two bridges contains a perfect matching.

The following theorem (see [5]) generalizes Theorem 2C, providing a bound for the number of edges in a maximum matching in a connected cubic graph G.

Theorem 2D  Every maximum matching in a connected cubic graph of order p with fewer than 3(l + 1) bridges (l ≥ 0) has at least (p − 2l)/2 edges.

Our first result of this chapter generalizes Theorem 2D. We first introduce some definitions. The edge connectivity $\kappa_1(G)$ of a graph G is the minimum number of edges whose removal from G results in a disconnected or trivial graph. A graph G is $m$-edge-connected, $m ≥ 1$, if the edge connectivity of G is greater than or equal to m. Equivalently, G is $m$-edge-connected if the removal of fewer than m edges from G results in neither a disconnected graph nor a trivial graph. An edge-cutset of cardinality k in a connected graph G is a set of k edges whose removal from G results in a disconnected or trivial graph.

Theorem 2.1  Let l be an integer with 0 ≤ l ≤ p/2 and let m be an odd positive integer. If G is an $m$-edge-connected, $(m + 2)$-regular graph of order p containing less than $(m + 2)(l + 1)$ edge-cutsets of cardinality m, then every maximum matching of G has at least $(p − 2l)/2$ edges.

Proof. Suppose, to the contrary, that G contains a maximum matching with fewer than $(p − 2l)/2$ edges. By Theorem 2B, there exists a proper subset S of V(G) such that the number n of odd components of $G - S$ exceeds |S| + 2l. Let |S| = k. Since p is even, n and k are of the same parity, so that
\[ n \geq k + 2l + 2. \]  \hspace{1cm} (1)

Denote the odd components of \( G - S \) by \( G_1, G_2, \ldots, G_n \). Since \( G \) is \( m \)-edge-connected, there are at least \( m \) edges joining vertices of each \( G_i \) (\( 1 \leq i \leq n \)) to vertices of \( S \). Suppose, without loss of generality, that \( G_1, G_2, \ldots, G_t \) are the odd components of \( G - S \) for which there exist exactly \( m \) edges joining vertices of \( G_i \) (\( 1 \leq i \leq n \)) to vertices of \( S \). For \( i = t + 1, t + 2, \ldots, n \), then, there are at least \( m + 2 \) edges joining vertices of \( G_i \) to vertices of \( S \).

Because \( G \) contains less than \((m + 2)(l + 1)\) edge-cutsets of cardinality \( m \),

\[ t < (m + 2)(l + 1). \]

Since at least \( mt + (m + 2)(n - t) = (m + 2)n - 2t \) edges join vertices of \( \bigcup \{ V(G_i) \mid i = 1, 2, \ldots, n \} \) to vertices of \( S \), it follows that

\[ (m + 2)n - 2t \leq (m + 2)k \]

so that by (1),

\[ (m + 2)(2l + 2) \leq 2t, \]

that is,

\[ (m + 2)(l + 1) \leq t \]

producing the desired contradiction. \( \Box \)

Observe that Theorem 2D is obtained from Theorem 2.1 when \( m = 1 \), and Petersen's Theorem follows from Theorem 2.1 when \( m = 1 \) and \( l = 0 \).
The bounds in Theorems 2D and 2.1 depend only on the number of edge-cutsets of cardinality \( m \) in \( G \). Another result on maximum matchings in cubic graphs considers the "location" of the bridges in the graph \( G \). That theorem (see [6]) is stated next.

**Theorem 2E** If the bridges of a connected cubic graph \( G \) lie on \( r \)-edge-disjoint paths of \( G \), then each maximum matching of \( G \) contains at least

\[
\frac{p}{2} - \left\lfloor \frac{2r}{3} \right\rfloor
\]

edges.

We wish to provide a generalization of Theorem 2E by establishing another lower bound on the cardinality of a maximum matching in an \( m \)-edge-connected, \((m + 2)\)-regular graph \( G \) of order \( p \). As in Theorem 2E, for this result we consider where the edge-cutsets of cardinality \( m \) lie in \( G \). We will need a few definitions.

Let \( G \) be an \( m \)-edge-connected graph. We form an edge-cutset representation for \( G \) by choosing exactly one edge from each cutset of cardinality \( m \). By an edge-cutset proper path, we mean a path in \( G \) that contains at most one edge from any cutset of cardinality \( m \). In Figure 2.1, we show a 3-edge-connected graph \( G \), an edge-cutset representation \( R \) for \( G \) and an edge-cutset proper path \( P \) of \( G \).
Theorem 2.2 Let $G$ be an $m$-edge-connected, $(m+2)$-regular graph of even order $p$. If the edges of some edge-cutset representation $R$ for $G$ lie on $r$ edge-cutset proper paths of $G$, then each maximum matching for $G$ contains at least \( \frac{p}{2} - \left\lfloor \frac{2r}{m+2} \right\rfloor \) edges.

Proof. Suppose, to the contrary, that $G$ contains a maximum matching with fewer than \( \frac{p}{2} - \left\lfloor \frac{2r}{m+2} \right\rfloor \) edges. By Theorem 2B, there exists a proper subset $S$ of $V(G)$ such that the number $n$ of odd components of $G - S$ exceeds $|S| + 2 \left\lfloor \frac{2r}{m+2} \right\rfloor$. Let $|S| = k$. Since $p$ is even, $n$ and $k$ are of the same parity, so that

\[ n \geq k + 2 \left\lfloor \frac{2r}{m+2} \right\rfloor + 2 \quad (1) \]
Denote the odd components of \( G - S \) by \( G_1, G_2, \ldots, G_n \). Since \( G \) is connected, every component \( G_i \) \((1 \leq i \leq n)\) contains vertices that are adjacent to vertices of \( S \). We denote by \( E_i \) the set of edges joining vertices of \( G_i \) to vertices of \( S \). Without loss of generality, let \( G_1, G_2, \ldots, G_t \) denote the odd components of \( G - S \) for which \( |E_i| = m \) \((1 \leq i \leq t)\). Thus each \( E_i \) \((1 \leq i \leq t)\) is an edge-cutset of cardinality \( m \). For \( i = t + 1, t + 2, \ldots, n \), then there are at least \( m + 2 \) edges in \( E_i \).

Let \( P_1, P_2, \ldots, P_r \) denote the \( r \) edge-cutset proper paths containing all the edges of \( R \). Then for each \( i \) \((1 \leq i \leq r)\) at most two edges of \( \bigcup \{E_j \cap R \mid j = 1, 2, \ldots, t\} \) lie on \( P_i \). Hence \( t \leq 2r \). Since \( \bigcup \{E_i \mid i = 1, 2, \ldots, n\} \) contains at least \( mt + (m + 2)(n - t) = (m + 2)n - 2t \) edges, it follows that

\[
(m + 2)n - 4r \leq (m + 2)n - 2t \leq (m + 2)k.
\]

Thus,

\[
(m + 2)(n - k) \leq 4r
\]

so that by (1),

\[
(m + 2)(2 \left\lfloor \frac{2r}{m+2} \right\rfloor + 2) \leq 4r,
\]

that is,

\[
(m + 2) \left\lfloor \frac{2r}{m+2} \right\rfloor + (m + 2) \leq 2r.
\]

However,

\[
2r + 1 = (m + 2) \left\lfloor \frac{2r-(m+1)}{m+2} \right\rfloor + (m + 2) \leq (m + 2) \left\lfloor \frac{2r}{m+2} \right\rfloor + (m + 2) \leq 2r,
\]

producing the desired contradiction. \( \square \)
If the edges of some edge-cutset representation \( R \) of an \( m \)-edge-connected, \((m + 2)\)-regular graph \( G \) of even order \( p \) lie on sufficiently few edge-cutset proper paths, then the bound provided in Theorem 2.2 on the number of edges in a maximum matching is an improvement on the bound provided in Theorem 2.1. This observation leads to the following result.

**Corollary 2.3** Let \( m \) be an odd integer \((m \geq 3)\) and let \( G \) be an \( m \)-edge-connected, \((m + 2)\)-regular graph of order \( p \) having \( n \) edge-cutsets of cardinality \( m \) where \((m + 2)l \leq n \leq (m + 2)(l + 1)\) for some integer \( l \geq 0 \). If the edges of some edge-cutset representation for \( G \) lie on \( r \) edge-cutset proper paths where 
\[
\left\lfloor \frac{2r}{m+2} \right\rfloor < l,
\]
then the number of edges in a maximum matching of \( G \) is at least 
\[
\frac{p}{2} - \left\lfloor \frac{2r}{m+2} \right\rfloor.
\]

The result in Corollary 2.3 can be shown to be sharp. Consider \( l \geq 1 \) to be given and choose the maximum \( r \) with \( r \equiv 0 \mod (m + 2) \), say \( r = (m + 2)s \) such that 
\[
\left\lfloor \frac{2r}{m+2} \right\rfloor < l.
\]

Then
\[
r = \begin{cases} 
\frac{(m+2)(l-2)}{2} & \text{if } l \text{ is even} \\
\frac{(m+2)(l-1)}{2} & \text{if } l \text{ is odd}
\end{cases}
\]

We show that there exists an \( m \)-edge-connected, \((m + 2)\)-regular graph \( G \) of order \( p \) having \( t = (m + 2)l + j \) edge-cutsets of cardinality \( m \) \((j = 0, 1, 2, \ldots, m+1)\) such that for any edge-cutset representation \( R \) of \( G \), the edges of \( R \) lie on \( r \) edge-cutset proper paths but no fewer and such that each maximum matching of \( G \) contains exactly 
\[
\frac{p}{2} - \left\lfloor \frac{2r}{m+2} \right\rfloor\text{ edges.}
\]

We will require some definitions and the following results. The **connectivity** \( \kappa(G) \) of a graph \( G \) is the minimum number of vertices whose removal from \( G \) results in a
disconnected graph or the trivial graph. A graph $G$ is said to be $m$-connected, $m \geq 1$, if $\kappa(G) \geq m$. Observe that if a graph $G$ is $m$-connected, then $G$ is necessarily $m$-edge-connected. The following result [2] gives a sufficient condition for a graph to be $m$-connected.

**Theorem 2F** Let $G$ be a graph of order $p \geq 2$ and let $m$ be an integer such that $1 \leq m \leq p - 1$. If $\deg v \geq \left\lceil \frac{p+m-2}{2} \right\rceil$ for every vertex $v$ of $G$, then $G$ is $m$-connected.

**Corollary 2.4** Let $G$ be an $(m - 3)$-regular graph of order $m$. Then $H = C_4 + G$ is $m$-edge-connected.

**Proof.** Observe first that $\delta(H) = m + 1$ and $p(H) = m + 4$. Thus for all $v \in V(H)$,

$$\deg_H v \geq m + 1 \geq \left\lfloor \frac{p+m-2}{2} \right\rfloor.$$

By Theorem 2F, $H$ is $m$-connected and hence $m$-edge-connected. □

**Theorem 2.5** For all odd integers $m$ ($\geq 3$) and all positive integers $s$, there exists an $m$-regular, $m$-edge-connected bipartite graph $H$ of order $4ms$.

**Proof.** Let $H$ be the graph consisting of the $4ms$ cycle $C_{0}: v_1, v_2, \ldots, v_{4ms}, v_1$ together with the edges $\{v_{2r-1}v_{2r+2n} \mid 1 \leq r \leq 2ms, 1 \leq n \leq m - 2\}$, where the subscripts are taken modulo $4ms$. Observe that $H$ is $m$-regular and bipartite of order $4ms$. We show that $H$ is $m$-edge-connected.

Denote by $E_n$ ($1 \leq n \leq m - 2$) the set of edges $\{v_{2r-1}v_{2r+2n} \mid 1 \leq r \leq 2ms\}$. Observe that each $E_n$ is a 1-factor of $H$, and $E_n$ together with $E_{n+1}$ ($1 \leq n \leq m - 3$) form a hamiltonian cycle for $H$. Denote by $C_j$ ($1 \leq j \leq \frac{m-3}{2}$) the hamiltonian cycle.
formed by the edges $E_{2j-1} \cup E_{2j}$. This produces $\frac{m-3}{2}$ edge disjoint hamiltonian cycles for $H$, and considering these cycles together with the hamiltonian cycle $C_0$, we obtain $\frac{m-1}{2}$ edge disjoint hamiltonian cycles of $H$. In fact, $H = C_0 \oplus C_1 \oplus \ldots \oplus C_{m-3} \oplus (E_{m-2})$. Since at least two edges must be removed from each $C_i$ ($0 \leq i \leq \frac{m-3}{2}$) to disconnect $H$, $\kappa_1(H) \geq m - 1$. Suppose to the contrary that $\kappa_1(H) = m - 1$. Then there exists some set $E' \subseteq E(H)$ such that $H - E'$ is disconnected. From the above discussion, we know that $E'$ contains exactly two edges from each hamiltonian cycle $C_i$ ($0 \leq i < m - 2$) and no edges from $E_{m-2}$. Observe that $C_0 - E'$ consists of two paths, say $P_1$ and $P_2$, and assume without loss of generality that $p(P_1) \leq p(P_2)$. Thus $p(P_1) \leq 2ms$. By the symmetry of $H$, we may assume that $v_1$ lies on $P_1$. Then, because $E' \cap E_{m-2} = \emptyset$, $v_{2m-2}$ lies on $P_1$ and each $v_j$ such that $1 \leq j \leq 2m - 2$ lies on $P_1$. But since $v_{2m-1}$ lies on $P_1$ and $E' \cap E_{m-2} = \emptyset$, $v_{4m-4}$ lies on $P_1$ and each $v_j$ such that $1 \leq j \leq 4m - 4$ lies on $P_1$. But then because $v_{4m-3}$ lies on $P_1$ and $E' \cap E_{m-2} = \emptyset$, $v_{6m-6}$ lies on $P_1$ and each $v_j$ such that $1 \leq j \leq 6m - 6$ lies on $P_1$. Continuing in this manner, we find that $v_{4ms-4s}$ lies on $P_1$ and each $v_j$ such that $1 \leq j \leq 4ms - 4s$ lies on $P_1$. But then $p(P_1) \geq 4ms - 4s > 2ms$ for $m \geq 3$, producing the desired contradiction. Thus $\kappa_1(H) \geq m$ and $H$ is m-edge-connected.

The graph $H$ constructed in the proof of Theorem 2.5 when $m = 3$ and $s = 1$ is given in Figure 2.2.
Theorem 2.5 has the following corollary which we need to prove the sharpness of Corollary 2.3.

**Corollary 2.6** For all odd integers \( m \geq 3 \) and all positive integers \( s \), there exists an \( m \)-edge-connected bipartite graph \( H' \) of order \( 4(m + 1)s \) with partite sets \( V_1 \) and \( V_2 \) of cardinality \( 2(m + 2)s \) and \( 2ms \) respectively such that each vertex of \( V_1 \) has degree \( m \) and each vertex of \( V_2 \) has degree \( m + 2 \).

**Proof.** Let \( H \) be the graph constructed in the proof of Theorem 2.5. Then we define \( H' \) by

\[
V(H') = V(H) \cup \{v_{4ms+2j} \mid 1 \leq j \leq 4s\}
\]

and
\[ E(H') = E(H) \cup \{v_{4ms+2}\} v_k \mid 1 \leq j \leq 4s, k = mj - m + 1 + 2i, i = 0, 1, 2, \ldots, m - 1 \text{ if } j \text{ is odd and } k = mj - 2m + 1 + 2i, i = 0, 1, 2, \ldots, m - 1 \text{ if } j \text{ is even}. \]

Then \( H' \) is \( m \)-edge-connected since \( H \) is \( m \)-edge-connected. Further, \( H' \) is bipartite with partite sets \( V_1 = \{v_{2r} \mid 1 \leq r \leq 2(m + 2)s\} \) and \( V_2 = \{v_{2r-1} \mid 1 \leq r \leq 2ms\} \) satisfying the desired conditions. \( \square \)

The graph \( H' \) produced in the proof of Corollary 2.6 when \( m = 3 \) and \( s = 1 \) is shown in Figure 2.3.

The final preliminary result we need before proving the sharpness of Corollary 2.3 is given next.

**Lemma 2.7** For all odd integers \( m \) (\( \geq 3 \)) there exists an \( m \)-edge-connected, \( (m + 1) \)-regular graph \( F \) of order \( 2m \).

**Proof.** Because \( K_{2m} \) is the edge sum of \( m - 1 \) hamiltonian cycles and a 1-factor, we can produce an \( (m + 1) \)-regular graph \( F \) of order \( 2m \) as the edge sum of \( \frac{m+1}{2} \) of the hamiltonian cycles of \( K_{2m} \). Certainly \( F \) is \( m \)-edge-connected since at least two edges from each of the \( \frac{m+1}{2} \) hamiltonian cycles must be removed to disconnect \( F \). \( \square \)
We are now prepared to begin the construction of the desired graph $G$. We begin by constructing a graph $P_n^*$ ($n \geq 1$) consisting of graphs $H_1, H_2, \ldots, H_n$, where $H_i$ ($1 \leq i \leq n - 1$) is the $m$-edge-connected, $(m + 1)$-regular graph of order $2m$ constructed in the proof of Lemma 2.7 and $H_n = C_4 + H''$ where $H''$ is an $(m - 3)$-regular graph of order $m$. By Lemma 2.7 and Corollary 2.4, each $H_i$ ($1 \leq i \leq n$) is $m$-edge-connected. Let $V(H_i) = \{u_{i,1}, u_{i,2}, \ldots, u_{i,m}\} \cup \{v_{i,1}, v_{i,2}, \ldots, v_{i,m}\}$, and
denote the \( m \) vertices of \( H_n \) of degree \( m + 1 \) by \( \{u_{n,1}, u_{n,2}, \ldots, u_{n,m}\} \). Then \( P_n^* \) is produced by joining \( v_{i,j} \) and \( u_{i+1,j} \) (\( 1 \leq i \leq n - 1 \), \( 1 \leq j \leq m \)). Observe that each \( P_n^* \) (\( n \geq 1 \)) has odd order and is \( m \)-edge-connected since each \( H_i \) (\( 1 \leq i \leq n \)) is \( m \)-edge-connected. Further, \( P_n^* \) contains exactly \( n - 1 \) edge-cutsets of cardinality \( m \). Also, exactly \( m \) vertices of \( P_n^* \) have degree \( m + 1 \) and all the other vertices of \( P_n^* \) have degree \( m + 2 \).

Consider next the graphs \( G_1, G_2, \ldots, G_{2(m+2)s-1} \) each isomorphic to \( P_1^* \) and \( G_{2(m+2)s} \) where

\[
G_{2(m+2)s} \equiv \begin{cases} 
P_{2m+5+j}^* (j = 0, 1, 2, \ldots, m+1) & \text{if } l \text{ is even} \\
P_{m+3+j}^* (j = 0, 1, 2, \ldots, m+1) & \text{if } l \text{ is odd}
\end{cases}
\]

Label the \( m \) vertices of \( G_i \) of degree \( m + 1 \) by \( x_{i,1}, x_{i,2}, \ldots, x_{i,m} \) (\( 1 \leq i \leq 2(m + 2)s \)). Next, consider the graphs \( G'_1, G'_2, \ldots, G'_{2ms} \) each isomorphic to \( K(m+1,m+2) \), and label the vertices of degree \( m + 1 \) in \( G'_j \) by \( y_{j,1}, y_{j,2}, \ldots, y_{j,m+2} \) (\( 1 \leq j \leq 2ms \)). Observe that each \( G_i \) (\( 1 \leq i \leq 2(m + 2)s \)) is \( m \)-edge-connected, and each \( G'_j \) (\( 1 \leq j \leq 2ms \)) is \( m \)-edge-connected since \( K(m+1,m+2) \) is \( m \)-edge-connected.

Let \( H' \) be the \( m \)-edge-connected bipartite graph of order \( 4(m+1)s \) constructed in the proof of Corollary 2.6. Recall that \( H' \) has partite sets \( V_1 \) and \( V_2 \) of cardinalities \( 2(m + 2)s \) and \( 2ms \) respectively, such that each vertex of \( V_1 \) has degree \( m \) and each vertex of \( V_2 \) has degree \( m + 2 \). Let \( V_1 = \{x_1, x_2, \ldots, x_{2(m+2)s}\} \) and let \( V_2 = \{y_1, y_2, \ldots, y_{2ms}\} \).

We now construct \( G \) from \( G_1, G_2, \ldots, G_{2(m+2)s}, G'_1, G'_2, \ldots, G'_{2ms} \) by joining \( y_{i,j} \) to \( x_{k,f} \) if and only if \( y_{i}x_{k} \in E(H') \) and \( y_{t,w}x_{k,f} \in E(G) \) for \( t < i, 1 \leq w \leq m + 2 \) or for \( t = i \) and \( w < j \) (\( 1 \leq i \leq 2ms, 1 \leq j \leq m + 2, 1 \leq k \leq 2(m + 2)s, 1 \leq f \leq m + 2 \)).
m). We show in Figure 2.4 the graph $G$ when $m = 3$ and $s = 1$. Denote by $E_i$ the edges joining the vertices $x_{i1}, x_{i2}, \ldots , x_{im}$ of $G_i$ to vertices of $\bigcup \{ G'_j \mid j = 1, 2, \ldots , 2m \}$. Then each $E_i$ $(1 \leq i \leq 2(m + 2)s)$ is an edge-cutset of $G$ of cardinality $m$. Because $H', G_1$ and $G_j$ $(1 \leq i \leq 2(m + 2)s, 1 \leq j \leq 2ms)$ are $m$-edge-connected, $G$ is $m$-edge-connected. Also, $G$ is $(m + 2)$-regular. Observe that $G$ contains the $2(m + 2)s = 2r$ edge-cutsets of cardinality $m$, namely $E_1, E_2, \ldots , E_{2(m+2)s}$, and $G_{2(m+2)s}$ contains $2(m + 2) + j$ or $m + 2 + j$ edge-cutsets of cardinality $m$ depending on whether $l$ is even or odd respectively. So $G$ contains $(m + 2)l + j$ edge-cutsets of cardinality $m$. Observe also that the edges in any edge-cutset representation for $G$ lie on $(m + 2)s = r$ edge-cutset proper paths. Thus by Corollary 2.3, every maximum matching of $G$ contains at least $\frac{p}{2} - \left\lfloor \frac{2r}{m+2} \right\rfloor$ edges.

It remains to be shown that every maximum matching of $G$ contains at most $\frac{p}{2} - \left\lfloor \frac{2r}{m+2} \right\rfloor$ edges. We use Theorem 2B to prove this statement. Let $S = \{ x_{ij} \mid 1 \leq i \leq 2(m + 2)s, 1 \leq j \leq m \}$. Then $|S| = 2m(m + 2)s$ and

$$G - S = \begin{cases} [2m(m + 1)s]K_1 \cup [2(m + 2)s - 1]P_1^* \cup P_{2m+5+j}^* & \text{if } l \text{ is even} \\ [2m(m + 1)s]K_1 \cup [2(m + 2)s - 1]P_1^* \cup P_{m+3+j}^* & \text{if } l \text{ is odd} \end{cases}$$

Therefore, $G - S$ contains $2m(m + 2)s + 4s = |S| + 4s$ odd components. Theorem 2B now states that every maximum matching of $G$ contains at most $\frac{p}{2} - 2s = \frac{p}{2} - \left\lfloor \frac{2r}{m+2} \right\rfloor$ edges. Hence every maximum matching of $G$ contains exactly $\frac{p}{2} - \left\lfloor \frac{2r}{m+2} \right\rfloor$ edges.

The cases where $r \equiv i \,(\text{mod } m+2)$ for $i = 1, 2, \ldots , m + 1$ can be handled in a similar manner.
For $n = 6 + j$ ($j = 0, 1, 2, 3, 4$),

$$P_n^* = \overline{K}_2 + C_4 \ldots \overline{K}_2 + C_4 \overline{K}_3 + C_4$$

Figure 2.4
CHAPTER III

THE CENTER, MEDIAN AND RELATED SUBGRAPHS

In this chapter we explore some results on the center and median of a graph and introduce several new subgraphs related to these well-known subgraphs.

3.1 The Center and Median

The distance $d_H(u,v)$ between two vertices $u$ and $v$ in a connected graph $H$ is the length of a shortest $u-v$ path in $H$. The distance of a vertex $v$ in $H$ is defined by

$$d_H(v) = \sum \{d_H(u,v) \mid u \in V(H)\}$$

A vertex of minimum distance is called a median vertex of $H$. The median is the subgraph of $H$ induced by its median vertices and is denoted by $M(H)$. For subgraphs $F$ and $G$ of $H$, the distance between $F$ and $G$ is defined by

$$d_H(F,G) = \min \{d_H(u,v) \mid u \in V(F), v \in V(G)\}.$$ 

The eccentricity of a vertex $v$ in $H$ is defined by $e_H(v) = \max \{d_H(u,v) \mid u \in V(F)\}$. A vertex of minimum eccentricity is called a central vertex and the center $C(H)$ of $H$ is the subgraph induced by its central vertices. The radius $\text{rad } H$ is defined as $\min \{e_H(v) \mid v \in V(H)\}$.

There exist infinitely many graphs $G$ whose center and median are the same subgraph of $G$. Hendry [9] showed that for any graph $G$, there exists a connected graph $H$ such that $C(H) = M(H) \cong G$. He showed that the condition $C(H) = M(H)$ is

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not necessary by constructing graphs \( H \) for which \( C(H) \equiv M(H) \equiv G \) but \( C(H) \) and \( M(H) \) are disjoint. Hendry extended this result further by showing that for any two graphs \( F \) and \( G \), there exists a connected graph \( H \) such that \( C(H) \equiv F \), \( M(H) \equiv G \) and where \( M(H) \) and \( C(H) \) are disjoint. The graph \( H \) so constructed by Hendry had the property that \( d_H(C(H),M(H)) = 1 \). These results might very well suggest that \( C(H) \) and \( M(H) \) must be "close" for each graph \( H \). However, such is not the case as we extend Hendry's result by showing that the distance between \( C(H) \) and \( M(H) \) can be made arbitrarily large.

**Theorem 3.1** Let \( G \) be a graph of order \( n \) and minimum degree \( \delta \) and let \( F \) be a graph of order \( p \). For all integers \( k \geq 2 \), there exists a connected graph \( H \) of order \( 2n + 2p - \delta + 4k \) such that \( C(H) \equiv F \), \( M(H) \equiv G \) and \( d_H(C(H),M(H)) = k \).

**Proof.** Let \( V(G) = \{v_1, v_2, \ldots, v_n\} \) with \( \deg_G v_n = \delta \) and \( N(v_n) = \{v_1, v_2, \ldots, v_{\delta}\} \). We first construct a graph \( H_0 \) with \( V(H_0) = V(G) \cup X \), where \( X = \{x_{\delta+1}, x_{\delta+2}, \ldots, x_{n+1}\} \). The edge set of \( H_0 \) consists of the edges of \( G \) together with several additional edges. First we join each vertex of \( \{x_n, x_{n+1}\} \) to all the vertices of \( G \), and for \( \delta + 1 \leq j \leq n - 1 \), we add the edges from \( x_j \) to those vertices of \( G \) not adjacent to \( v_j \). In particular, \( x_j v_j \in E(H_0) \). (At this point, \( \deg_{H_0} v_i \leq n + 1 \), \( 1 \leq i \leq n \).) We complete the construction of \( H_0 \) by arbitrarily adding edges between \( X \) and \( V(G) \) to increase the degree of each vertex \( v_i \) to exactly \( n + 1 \). Observe that each vertex \( x_j \) \( (\delta + 1 \leq j \leq n + 1) \) is adjacent to some vertex \( v_i \) and that \( e_{H_0}(v_i) \leq 2 \).

We now define the graph \( H \) as follows:

\[
V(H) = V(H_0) \cup V(F) \cup \{w_1, w_2, \ldots, w_{p+2k-1}\} \cup \{t_1, t_2, \ldots, t_{k-1}\} \cup \{z_1, z_2, \ldots, z_{k+1}\}
\]

and
\[ E(H) = E(H_0) \cup E(F) \cup \{v_iw_j \mid 1 \leq i \leq n, 1 \leq j \leq p + 2k - 1\} \cup \{v_it_1 \mid 1 \leq i \leq n\} \cup \{t_jt_{j+1} \mid 1 \leq i \leq k - 2\} \cup \{t_{k-1}u \mid u \in V(F)\} \cup \{uz_1 \mid u \in V(F)\} \cup \{z_jz_{j+1} \mid 1 \leq j \leq k\} \]

Note that the order of \( H \) is \( 2n + 2p - \delta + 4k \) and that \( H \) is connected. Observe also that \( e_H(u) = k + 1 \) for all \( u \in V(F) \). Further, this is the minimum eccentricity among all the vertices of \( H \) and no other vertex has eccentricity \( k + 1 \). Therefore \( C(H) \equiv F \).

We show \( M(H) \equiv G \). Consider \( d_H(v_i), \ (i \leq i \leq n) \). Recall that
\[ d_{H_0}(v_i) = n + 1 + 2(n - \delta - 1) = 3n - 2\delta - 1. \]
Observe that
\[ \sum_{j=1}^{p+2k-1} d_H(v_i, w_j) = p + 2k - 1, \]
\[ \sum_{j=1}^{k-1} d_H(v_i, t_j) + \sum_{j=1}^{k+1} d_H(v_i, z_j) = \left( \sum_{j=1}^{2k+1} j \right) - k = 2k^2 + 2k + 1, \]
and
\[ \sum_{u \in V(F)} d_H(v_i, u) = pk. \]
Therefore, \( d_H(v_i) = 3n + (k + 1)p - 2\delta + 2k^2 + 4k - 1 \). We now compute \( d_H(t_j) \).

Since
\[ \sum_{i=1}^{n} d_H(v_i, t_j) = nj, \]
\[ \sum_{i=1}^{p+2k-1} d_H(w_i, t_j) = (j + 1)(p + 2k - 1) = (j + 1)p + 2jk + 2k - j - 1, \]
\[ n+1 \sum_{i=\delta+1}^{n+1} d_H(x_i, t_j) = (j + 1)(n - \delta + 1) = (j + 1)n - (j + 1)\delta + j + 1, \]

\[ \sum_{i=1}^{j-1} d_H(t_i, t_j) = \sum_{i=1}^{j-1} (j - i) = \frac{j(j - 1)}{2}, \]

\[ \sum_{i=j+1}^{k-1} d_H(t_i, t_j) + \sum_{i=1}^{k+1} d_H(z_i, t_j) = \left( \sum_{i=1}^{2k-j+1} i \right) - (k - j) = \frac{(2k - j + 1)(2k - j + 2) - k + j}{2}, \]

and

\[ \sum_{u \in V(F)} d_H(u, t_j) = (k - j)p, \]

we have that

\[ d_H(t_j) = (2j + 1)n + (k + 1)p - (j + 1)\delta + 2k^2 + 4k + j^2 - j + 1. \quad (1) \]

Since (1) attains its minimum value when \( j = 1, \)

\[ d_H(t_j) > d_H(t_1) = 3n + (k + 1)p - 2\delta + 2k^2 + 4k + 1 > d_H(v_i) \quad (1 \leq i \leq n). \]

Observe that

\[ d_H(z_i) > d_H(u) > d_H(t_1) \quad \text{for all } u \in V(F), \quad 1 \leq i \leq k + 1. \]

Also,

\[ d_H(w_j) > d_H(v_i) \quad \text{and} \quad d_H(x_m) > d_H(v_i) \quad \text{for all } 1 \leq j \leq p + 2k - 1, \quad 1 \leq i \leq n, \quad \delta + 1 \leq m \leq n + 1. \]
Therefore, $M(H) \equiv G$. We conclude the proof by noting that $d_H(C(H), M(H)) = k$.

Recall that Hendry showed that the center and median of a graph can coincide or be disjoint. Our next result considers a situation between these two extremes. We show that the center and median of a graph can be different subgraphs yet have some vertices (and hence a nonempty induced subgraph) in common. Let $F$ and $G$ be subgraphs of a graph $H$ such that $V(F) \cap V(G) \neq \emptyset$. Then the intersection of $F$ and $G$ denoted $F \cap G$ is the subgraph of $H$ induced by $V(F) \cap V(G)$.

**Theorem 3.2** Let $G$ be a graph of order $n$ and minimum degree $\delta$ and let $F$ be a graph of order $p$. Let $K$ be a nonempty graph of order $k \leq p - 1$ isomorphic to an induced subgraph of both $F$ and $G$. Then there exists a connected graph $H$ of order $3p + 3n - 2k - \delta + 2$ such that $C(H) \equiv F$, $M(H) \equiv G$, and $C(H) \cap M(H) \equiv K$.

**Proof.** Let $K' = \langle \{v_1, v_2, \ldots, v_k\} \rangle \equiv K$, $((i_j \in \{1, 2, \ldots, n\}, 1 \leq j \leq k)$. Let $V(F) = \{u_1, u_2, \ldots, u_p\}$ and without loss of generality, suppose $K'' = \langle \{u_1, u_2, \ldots, u_k\} \rangle \equiv K$. We construct preliminary graphs $H_0, H_1, \text{and } H_2$ needed for the construction of $H$.

We first define a graph $H_0$ as in the proof of Theorem 3.1. Recall that $V(H_0) = V(G) \cup X$, where $X = \{x_{\delta+1}, x_{\delta+2}, \ldots, x_{n+1}\}$. The edge set of $H_0$ consists of the edges of $G$ together with several additional edges. First we join each vertex of $\{x_n, x_{n+1}\}$ to all the vertices of $G$, and for $\delta + 1 \leq j \leq n - 1$, we add the edges from $x_j$ to those vertices of $G$ not adjacent to $v_j$. In particular, $x_jv_j \in E(H_0)$. (At this point, $deg_{H_0}v_i \leq n + 1$, $1 \leq i \leq n$.) We complete the construction of $H_0$ by arbitrarily adding edges between $X$ and $V(G)$ to increase the degree of each vertex $v_i$ to exactly
n + 1. Observe that each vertex \( x_j \) \((\delta + 1 \leq j \leq n + 1)\) is adjacent to some vertex \( v_i \) and that \( e_{H_0}(v_i) \leq 2 \).

Next, we construct the preliminary graph \( H_1 \). Let \( t = \max\{\deg_{F} u_i - \deg_{K''} u_i | u_i \in V(K'')\} \) and without loss of generality, suppose \( \deg_{F} u_1 - \deg_{K''} u_1 = t \). Let \( V(H_1) = V(F) \cup Z \) where \( Z = \{z_1, z_2, \ldots, z_{t+1}\} \). The edge set of \( H_1 \) consists of the edges of \( F \) together with some additional edges. First, we join each vertex of \( \{u_{k+1}, u_{k+2}, \ldots, u_p\} \) to all the vertices of \( Z \). Next, for \( 1 \leq j \leq k \), we join \( u_i \) to each vertex of \( \{z_1, z_2, \ldots, z_s\} \) where \( s = t - \deg_{F} u_i - \deg_{K''} u_i + 1 \). Finally, we add the edges \( \{z_iz_j | 1 \leq i < t + 1, 1 \leq j \leq t + 1\} \). Observe that each \( u_i \in V(K'') \) is adjacent to exactly \( t + 1 \) vertices in \( V(H_1) - V(K'') \). Further, \( z_1 \) is adjacent to each vertex \( u_i \) \((1 \leq i \leq p)\) and \( \text{rad} \ H_1 \leq 2 \).

Recall that \( K' = \langle\{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\}\rangle \cong K \cong K'' = \langle\{u_1, u_2, \ldots, u_k\}\rangle \), \((ij \in \{1, 2, \ldots, n\}, 1 \leq j \leq k)\). Thus there exists an isomorphism \( \varphi : V(K') \to V(K'') \). Without loss of generality assume \( \varphi(v_{ij}) = u_j \) for \( 1 \leq j \leq k \). We form a new graph \( H_2 \) from \( H_0 \) and \( H_1 \) by "identifying" vertex \( v_{ij} \in V(H_0) \) with vertex \( u_j \in V(H_1) \). That is,

\[
V(H_2) = V(H_0) \cup V(H_1) - \{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\} \quad \text{and}
\]

\[
E(H_2) = E(H_0) \cup E(H_1) - E(K') \cup \\
\{u_jv_i | v_i \in E(H_0), 1 \leq j \leq k, \ v_i \notin \{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\}\} \cup \\
\{u_jx_m | v_j \in E(H_0), 1 \leq j \leq k, \ \delta + 1 \leq m \leq n + 1\}.
\]

Observe that for \( 1 \leq j \leq k \), \( \deg_{H_2} u_j = \deg_{H_0} v_{ij} + (t + 1) = (n + 1) + (t + 1) = n + t + 2 \). Also, observe that \( e_{H_2}(u_j) \leq 2 \) for \( 1 \leq j \leq k \).

We now define the graph \( H \) as follows:
\[ V(H) = V(H_2) \cup \{w_1, w_2, \ldots, w_{p+n+1}\} \cup \{y_1, y_2, \ldots, y_{p-k-t-1}\} \]

\[ E(H) = E(H_2) \cup \{v_iw_l \mid v_i \in V(H_2), k+1 \leq l \leq p\} \cup \{v_iw_r \mid v_i \in V(H_2), 1 \leq r \leq p+n+1\} \cup \{u_jw_r \mid 1 \leq j \leq k, 1 \leq r \leq p+n+1\} \cup \{z_iy_c \mid 1 \leq c \leq p-k-t-1\}. \]

Note that \( H \) is connected of order \( 3n + 3p - 2k - \delta + 2 \) and \( e_H(u_j) = 2 \) for \( 1 \leq j \leq p \). Further \( \text{rad} \ H = 2 \) and no other vertex of \( H \) has eccentricity 2. Thus, \( C(H) \equiv F \).

We show \( M(H) \equiv G \). We first consider \( d_H(u_j) \) for \( 1 \leq j \leq k \). Recall that \( \text{deg}_H(u_j) = n + t + 2 \); so

\[ \text{deg}_H(u_j) = (n + t + 2) + (p + n + 1) = 2n + p + t + 3. \]

Since \( e_H(u_j) = 2 \), \( u_j \) is distance 2 from the remaining \( (3n + 3p - 2k - \delta + 2) - (2n + p + t + 3) = n + 2p - 2k - \delta - t - 2 \) vertices of \( H \). Thus,

\[ d_H(u_j) = (2n + p + t + 3) + 2(n + 2p - 2k - \delta - t - 2) = 4n + 5p - t - 4k - 2\delta - 1. \]

We now consider \( v_i \in V(H) \). Recall that \( d_{H_0}(v_i) = n + 1 = 2(n - \delta - 1) = 3n - 2\delta - 1 \). Observe that

\[ d_H(v_i, w_r) = 1 \quad \text{for } 1 \leq r \leq p + n + 1 \]

\[ d_H(v_i, u_l) = 1 \quad \text{for } k + 1 \leq l \leq p \]

\[ d_H(v_i, z_b) = 2 \quad \text{for } 1 \leq b \leq t + 1, \text{ and} \]

\[ d_H(v_i, y_c) = 3 \quad \text{for } 1 \leq c \leq p - k - t - 1. \]

Thus,
\[d_H(v_j) = (3n - 2\delta - 1) + (p + n + 1) + (p - k) + 2(t + 1) + 3(p - k - t - 1)\]

\[= 4n + 5p - t - 4k - 2\delta - 1 = d_H(u_j) \text{ for } 1 \leq j \leq k.\]

We now compute \(d_H(u_j).\) For \(k + 1 \leq l \leq p,\) \(\deg_{F} u_j \leq p - 1.\) Also

\[d_H(u_l, v_i) = 1 \quad \text{for all } v_i \in V(H)\]

\[d_H(u_l, x_m) = 1 \quad \text{for } \delta + 1 \leq m \leq n + 1\]

\[d_H(u_l, z_b) = 1 \quad \text{for } 1 \leq b \leq t + 1\]

\[d_H(u_l, w_r) = 2 \quad \text{for } 1 \leq r \leq p + n + 1 \quad \text{and}\]

\[d_H(u_l, y_c) = 2 \quad \text{for } 1 \leq c \leq p - k - t - 1.\]

Therefore,

\[d_H(u_j) \geq (p - 1) + (n - k) + (n - \delta + 1) + (t + 1) + 2(p + n + 1)\]

\[2(p - k - t - 1)\]

\[= 4n + 5p - t - 3k - \delta + 1 > 4n + 5p - t - 4k - 2\delta - 1\]

\[= d_H(u_j) \quad (1 \leq j \leq k),\]

since \(k > 0,\) \(\delta > 0.\)

Observe that

\[d_H(x_m) > d_H(v_i) \text{ for all } v_i \in V(H), \delta + 1 \leq m \leq n + 1.\]

Also, observe that

\[d_H(w_r) > d_H(v_i) \text{ for all } v_i \in V(H), 1 \leq r \leq p + n + 1.\]

Further,
dn(yc) > dH(zb) > dH(uj) > dH(uz)

for 1 \leq c \leq p - k - t - 1, 1 \leq b \leq t + 1, k + 1 \leq l \leq p, 1 \leq j \leq k.

Therefore,

\[ M(H) = (\{u_1, u_2, \ldots, u_k\} \cup \{v_i \mid 1 \leq i \leq n, i \notin \{i_j \mid 1 \leq j \leq k\}\}) \equiv G \]

We conclude the proof by noting that \( M(H) \cap C(H) = (\{u_1, u_2, \ldots, u_k\}) \equiv K. \]

3.2 The k-Median

In this section, we consider an induced subgraph closely related to the median. Let \( d_1, d_2, \ldots, d_l \) be the distinct distances of the vertices of a connected graph \( H \) where \( d_1 < d_2 < \ldots < d_l \). Then the median \( M(H) \) is the subgraph of \( H \) induced by the vertices of distance \( d_1 \). For \( 1 \leq k \leq l \), we define the \( k \)-median, denoted \( M_k(H) \), as the subgraph of \( H \) induced by the vertices of distance \( d \leq d_k \). Thus the median \( M(H) \) is \( M_1(H) \). As an example, Figure 3.1 shows a connected graph \( H \), a table of the distances of the vertices of \( H \) and the 3-median \( M_3(H) \) of \( H \).

It has been shown (see [12]) that every graph \( G \) is the median of some connected graph \( H \). We now present a similar result for the \( k \)-median. We show that for a given graph \( G \) and certain values of \( k \) (depending on the minimum degree of \( G \)), there exists a connected graph \( H_k \) such that \( M_k(H_k) \equiv G. \)
Theorem 3.3 Let $G$ be a graph of order $n$ and minimum degree $\delta$. Then for all integers $k$ such that $1 \leq k \leq \delta + 1$, there exists a connected graph $H_k$ of order $2n + k - \delta$ such that $M_k(H_k) \cong G$.

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ with $\deg_G v_n = \delta$ and $N(v_n) = \{v_1, v_2, \ldots, v_\delta\}$. For $k$ such that $1 \leq k \leq \delta + 1$, we construct the graph $H_k$ with $V(H_k) =$
V(G) ∪ X, where X = {x₈₊₁, x₈₊₂, ..., xₙ₊ₖ}. The edge set of Hₖ consists of the edges of G together with several additional edges. First we join the vertex of xₙ to all the vertices of G, and for 1 ≤ j ≤ k, we join xₙ+j to each vertex of {vⱼ, vⱼ₊₁, ..., vₙ}. Also, for δ + 1 ≤ m ≤ n - 1, we add the edges from xₘ to those vertices of G not adjacent to vₘ. In particular, xₘ is joined to vₘ. (At this point, deg_Hₖ vᵢ ≤ n + i, 1 ≤ i ≤ k - 1, and deg_Hₖ vₗ ≤ n + k for k ≤ l ≤ n) We complete the construction of Hₖ by arbitrarily adding edges between X and V(G) to increase the degree of each vertex vᵢ (1 ≤ i ≤ k - 1) to exactly n + i and to increase the degree of each vertex vₗ (k ≤ l ≤ n) to exactly n + k. Observe that each vertex xⱼ (δ + 1 ≤ j ≤ n + k) is adjacent to some vertex vₗ (1 ≤ t ≤ n) and that rad Hₖ ≤ 2. Note also that the order of Hₖ is 2n + k − δ.

We show that Mₖ(Hₖ) ≡ G. First for 1 ≤ i ≤ k − 1,

d_Hₖ(vᵢ) = n + i + 2(n + k − δ − i − 1) = 3n + 2k − 2δ − 2 − i.

Also, for k ≤ l ≤ n + k,

d_Hₖ(vₗ) = n + k + 2(n − δ − 1) = 3n + 2k − 2δ − 2 − k.

For δ + 1 ≤ m ≤ n + k,

d_Hₖ(xₘ) ≥ d_Hₖ(xₙ) = n + 2(n + k − δ − 1) = 3n + 2k − 2δ − 2 > d_Hₖ(v₁).

Therefore, Mₖ(Hₖ) ≡ G. □

We have shown in Theorem 3.1 that there exist connected graphs whose center and median are arbitrarily far apart. We extend this result, showing that for some values of k even the distance between the center and the k–median can be made arbitrarily large.
Theorem 3.4 Let $G$ be a graph of order $n$ and minimum degree $\delta$ and let $F$ be a graph of order $p$. For all integers $k$ such that $1 \leq k \leq \delta + 1$ and for all integers $l \geq 2$, there exists a connected graph $H'_k$ such that $C(H'_k) \equiv F$, $M_k(H'_k) \equiv G$, and $d_{H'_k}(C(H'_k), M_k(H'_k)) = l$.

Proof. We begin by constructing the graph $H_k$ from $G$ as in the proof of Theorem 3.3. We then define the graph $H'_k$ as follows:

$V(H'_k) = V(H_k) \cup V(F) \cup \{w_1, w_2, \ldots, w_{p+2l-1}\} \cup \{t_1, t_2, \ldots, t_{-1}\} \cup \{z_1, z_2, \ldots, z_{l+1}\}$

and

$E(H'_k) = E(H_k) \cup E(F) \cup \{v_1w_j \mid 1 \leq i \leq n, 1 \leq j \leq p + 2l - 1\} \cup \{v_1t_{i+1} \mid 1 \leq i \leq n\} \cup \{v_{i+1}t_{i+1} \mid 1 \leq i \leq l - 2\} \cup \{t_{k-1}u \mid u \in V(F)\} \cup \{uz_1 \mid u \in V(F)\} \cup \{z_jz_{j+1} \mid 1 \leq j \leq l\}$.

The verification that $C(H'_k) \equiv F$, $M_k(H'_k) \equiv G$, and $d_{H'_k}(C(H'_k), M_k(H'_k)) = l$ is similar to the proof of Theorem 3.1 and is omitted. □

3.3 The Pseudocenter

In this section, we consider a subgraph closely related to the center. Let $a_1, a_2, \ldots, a_k$ where $a_1 < a_2 < \ldots < a_k$, be the distinct eccentricities of the vertices of a connected graph $H$. It is well-known that $a_2 = a_1 + 1$. We define the pseudocenter of $H$, denoted $P(H)$, as the subgraph induced by the vertices of $H$ of eccentricity $a_1$ or $a_2$. Figure 3.2 shows a connected graph $G$, its center $C(G)$ and its pseudocenter $P(G)$.
It is known (see [3]) that for any graph $G$ there exists a connected graph $H$ such that $C(H) \cong G$. The following theorem shows however that not every graph $G$ is a pseudocenter for some connected graph $H$.

**Theorem 3.5** For $p \geq 1$, there exists no connected graph $H$ such that $P(H) \cong K_p$.

**Proof.** Let $G \cong K_p$ and let $V(G) = \{v_1, v_2, \ldots, v_p\}$. The result is obvious for $p = 1$, so we assume that $k \geq 2$. Suppose to the contrary that there exists a connected graph $H$ such that $P(H) = G$. Without loss of generality, suppose $e_H(v_1) = a_1 = \text{rad } H$ and $e_H(v_2) = a_2 = a_1 + 1$. Then there exists some $w \in V(H)$ such that $d_H(v_2, w) = a_2$. Observe that $d_H(v_1, w) = a_1$ because $d_H(v_1, w) \leq a_1$ and if $d_H(v_1, w) < a_1$, then...
d_\text{H}(v_2,w) < a_1 + 1 = a_2, contradicting our choice of w. Note that v_1 is adjacent in H only to vertices of G, because if there exists some u \in V(H)\setminus V(G) adjacent to v_1 in H, then e_\text{H}(u) \leq a_1 + 1 = a_2. But this implies u \in P(H) contradicting the fact that P(H) = G. Let P: v_1 = u_0, u_1, \ldots, u_{a_1} = w be a v_1-w path in H of length a_1. Then u_1 \in V(G) and d_\text{H}(u_1,w) \leq a_1 - 1. But v_2u_1 \in E(G) because G \cong K_p. Thus P': v_2, u_1, u_2, \ldots, u_{a_1} = w is a v_2-w path in H of length a_1, contradicting the fact that d_\text{H}(v_2,w) = a_2. □

The following theorem provides a sufficient condition for a graph G to be a pseudocenter of some connected graph H.

**Theorem 3.6** Let G be a graph such that V(G) can be partitioned into three nonempty subsets V_1, V_2, and V_3, satisfying the following conditions:

i) every vertex of V_1 \cup V_3 is adjacent to some vertex of V_2,

ii) no vertex of V_1 is adjacent to a vertex of V_3, and

iii) every vertex of V_2 is adjacent to at least one vertex of V_1 and at least one vertex of V_3.

Then there exists a connected graph H such that P(H) \equiv G.

**Proof.** Let G be a graph such that V(G) can be partitioned into three nonempty subsets V_1, V_2, and V_3 satisfying the conditions above. We define the graph H by

\[ V(H) = V(G) \cup \{u_1, u_2, u_3, w_1, w_2, w_3\} \quad \text{and} \]

\[ E(H) = E(G) \cup \{u_1v \mid v \in V_1\} \cup \{w_1x \mid x \in V_3\} \cup \\
\{u_1u_2, u_2u_3, w_1w_2, w_2w_3\}. \]

We show P(H) \equiv G. We first consider e_\text{H}(y) for y \in V_2. Let v \in V_1. Then because y is adjacent to some vertex t in V_1, d_\text{H}(y,v) \leq 3 as y, t, u_1, v is a y-v path in H
of length 3. Similarly, \( d_H(y,x) \leq 3 \) for \( x \in V_3 \). Observe that \( d_H(y,u_3) = 4 \) because \( y, t, u_1, u_2, u_3 \) is a shortest \( y-u_3 \) path in \( H \). Similarly, \( d_H(y,w_3) = 4 \). Also, \( d_H(y,u_2) = 3 \), \( d_H(y,w_2) = 3 \), \( d_H(y,u_1) = 2 \) and \( d_H(y,w_1) = 2 \). Further, if \( a \in V_2 \), \( a \neq y \), then \( a \) is adjacent to some \( t' \in V_1 \), so \( y, t, u_1, t', a \) is a \( y-a \) path of length 4 in \( H \). Thus \( d_H(y,a) \leq 4 \) and \( e_H(y) = 4 \) for \( y \in V_2 \).

Let \( v \in V_1 \). We show that \( e_H(v) = 5 \). Observe that \( d_H(v,u_3) = 3 \), \( d_H(v,u_2) = 2 \), and \( d_H(v,u_1) = 1 \). Let \( y \in V_2 \). Since \( v \) is adjacent to some \( s \in V_2 \) by condition i) and \( s \) is adjacent to some \( z \in V_3 \), while \( y \) is adjacent to some \( b \in V_3 \), \( v, s, z, w_1 \), \( b, y \) is a \( v-y \) path in \( H \) of length 5. Thus \( d_H(v,y) \leq 5 \) for all \( y \in V_2 \). Observe that for \( x \in V_3 \), \( d_H(v,x) \leq 4 \), since \( v, s, z, w_1, x \) is a \( v-x \) path in \( H \) of length 4. Also, \( d_H(v,w_1) = 3 \), \( d_H(v,w_2) = 4 \), and \( d_H(v,w_3) = 5 \). Finally, for \( t \in V_1 \), \( t \neq v \), \( d_H(v,t) \leq 2 \) as \( v, u_1, t \) is a \( v-t \) path in \( H \) of length 2. Thus \( e_H(v) = 5 \).

By a similar argument, \( e_H(x) = 5 \) for \( x \in V_3 \). We conclude the proof by noting that \( e_H(u_1) = 6 \) as \( d_H(u_1,w_3) = 6 \), \( e_H(u_2) = 7 \) since \( d_H(u_2,w_3) = 7 \), and \( e_H(u_3) = 8 \) because \( d_H(u_3,w_3) = 8 \). Similarly, \( e_H(w_1) = 6 \), \( e_H(w_2) = 7 \), and \( e_H(w_3) = 8 \).

Using Theorem 3.6, we can show that several classes of well-known graphs, \( K(m,n) \), \( C_p \), and \( P_k \) are pseudocenters (for most values of \( m, n, p, \) and \( k \)).

**Corollary 3.7** For \( 1 \leq m \leq n \), \( n \geq 2 \), there exists some connected graph \( H \) such that \( P(H) \equiv K(m,n) \).

**Proof.** Let \( G \equiv K(m,n) \) (\( 1 \leq m \leq n \), \( n \geq 2 \)), and let \( V(G) = \{v_1, v_2, ..., v_m, u_1, u_2, ..., u_n\} \), where \( \{v_1, v_2, ..., v_m\} \) and \( \{u_1, u_2, ..., u_n\} \) are the partite sets of \( G \). Let \( V_1 = \{u_1, u_2, ..., u_{n-1}\} \), \( V_2 = \{v_1, v_2, ..., v_m\} \), and let \( V_3 = \{u_n\} \). It is straightforward to verify that \( V_1, V_2, \) and \( V_3 \) satisfy the conditions of Theorem 3.6. Therefore there exists some connected graph \( H \) such that \( P(H) \equiv G \equiv K(m,n) \).
Corollary 3.8 For $p \geq 4$, $p \neq 7$, there exists a connected graph $H$ such that $P(H) \equiv C_p$.

Proof. Let $G \equiv C_p$ ($p \geq 4$, $p \neq 7$). Let $V(G) = \{v_1, v_2, \ldots, v_p\}$ and $E(G) = \{v_iv_{i+1} \mid 1 \leq i \leq p-1\} \cup \{vpv_1\}$. We show that $V(G)$ can be partitioned into three nonempty subsets $V_1$, $V_2$, and $V_3$ satisfying the hypotheses of Theorem 3.6. We consider four cases.

CASE 1. $p \equiv 0$ (mod 4).

Since $p \equiv 0$ (mod 4), $p = 4k$ for $k \geq 1$. Let

$V_1 = \{v_i \mid i \equiv 2$ (mod 4), $2 \leq i \leq 4k - 2\}$,

$V_2 = \{v_{2j+1} \mid 0 \leq j \leq 2k - 1\}$, and let

$V_3 = \{v_l \mid l \equiv 0$ (mod 4), $4 \leq l \leq 4k\}$.

CASE 2. $p \equiv 1$ (mod 4).

Then $p = 4k + 1$ for $k \geq 1$. Let

$V_1 = \{v_i \mid i \equiv 2$ (mod 4), $2 \leq i \leq 4k - 2\}$,

$V_2 = \{v_{2j+1} \mid 0 \leq j \leq 2k - 1\}$, and let

$V_3 = \{v_l \mid l \equiv 0$ (mod 4), $4 \leq l \leq 4k\} \cup \{v_{4k+1}\}$.

CASE 3. $p \equiv 2$ (mod 4).

Thus, $p = 4k \neq 2$ for $k \geq 1$. Let

$V_1 = \{v_i \mid i \equiv 2$ (mod 4), $2 \leq i \leq 4k - 2\} \cup \{v_{4k-1}\}$,

$V_2 = \{v_{2j+1} \mid 0 \leq j \leq 2k - 1\} \cup \{v_{4k}\}$, and let

$V_3 = \{v_l \mid l \equiv 0$ (mod 4), $4 \leq l \leq 4k - 4\} \cup \{v_{4k+1}, v_{4k+2}\}$.

CASE 4. $p \equiv 3$ (mod 4).

Since $p \equiv 3$ (mod 4) and $p \neq 7$, $p = 4k + 3$ for $k \geq 2$. Let

$V_1 = \{v_i \mid i \equiv 2$ (mod 4), $2 \leq i \leq 4k - 6\} \cup \{v_{4k-1}, v_{4k}\}$,

$V_2 = \{v_{2j+1} \mid 0 \leq j \leq 2k - 2\}$, and let
\[ V_3 = \{ v_l \mid l \equiv 0 \pmod{4}, 4 \leq l \leq 4k - 4 \} \cup \{ v_{4k+2}, v_{4k+3} \}. \]

The reader can verify that in each of the four cases, \( V_1, V_2, \) and \( V_3 \) satisfy the conditions of Theorem 3.6. \( \square \)

**Corollary 3.9** For \( k \geq 3, \; k \neq 4 \), there exists a connected graph \( H \) such that \( P(H) = P_k \).

**Proof.** Let \( G \cong P_k \) (\( k \geq 3, \; k \neq 4 \)), and let \( V(G) = \{ v_1, v_2, \ldots, v_k \} \) and \( E(G) = \{ v_iv_{i+1} \mid 1 \leq i \leq k - 1 \} \). We again use Theorem 3.6, partitioning \( V(G) \) into nonempty subsets \( V_1, V_2, \) and \( V_3 \) satisfying the conditions of the theorem. We consider two cases.

**CASE 1.** \( k \) is even.

Since \( k \geq 3, k \neq 4 \), and \( k \) is even, \( k = 2m \) for \( m \geq 3 \). Let
\[ V_1 = \{ v_1 \} \cup \{ v_l \mid l \equiv 2 \pmod{4}, 6 \leq l \leq k \}, \]
\[ V_2 = \{ v_2 \} \cup \{ v_{2j+1} \mid 2 \leq j \leq m - 1 \}, \] and let
\[ V_3 + \{ v_3 \} \cup \{ v_i \mid i \equiv 0 \pmod{4}, 4 \leq i \leq k \}. \]

**CASE 2.** \( k \) is odd.

If \( k \) is odd, then \( k = 2m + 1 \) for \( m \geq 1 \). Let
\[ V_1 = \{ v_1 \mid i \equiv 1 \pmod{4}, 1 \leq i \leq k \}, \]
\[ V_2 = \{ v_{2j} \mid 1 \leq j \leq m \}, \] and let
\[ V_3 = \{ v_l \mid l \equiv 3 \pmod{4}, 3 \leq l \leq k \}. \]

Note that in each case, \( V_1, V_2, \) and \( V_3 \) satisfy the conditions of Theorem 3.6. \( \square \)

Corollaries 3.7, 3.8, and 3.9 handle all cases of complete bipartite graphs, cycles and paths except for \( K(1,1), C_3, C_7, P_1, P_2, \) and \( P_4 \). Since \( P_1 \cong K_1, K(1,1) \cong P_2 \cong K_2 \) and \( C_3 \cong K_3 \), we know from Theorem 3.5, that \( P_1, K(1,1), P_2, \) and \( C_3 \) are not pseudocenters of any connected graphs. Therefore, the only cases remaining to

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be investigated are \( C_7 \) and \( P_4 \). The following theorem provides another sufficient condition for a graph to be a pseudocenter. We conclude this chapter by using this result to show that there exist connected graphs \( H_1 \) and \( H_2 \) such that \( P(H_1) \equiv C_7 \) and \( P(H_2) \equiv P_4 \).

**Theorem 3.10** Let \( G \) be a graph such that \( V(G) \) can be partitioned into four nonempty subsets \( V_1, V_2, V_3 \) and \( V_4 \), satisfying the following conditions:

i) every vertex of \( V_1 \) is adjacent to some vertex of \( V_2 \),

ii) every vertex of \( V_4 \) is adjacent to some vertex of \( V_3 \),

iii) no vertex of \( V_1 \) is adjacent to a vertex of \( V_3 \cup V_4 \),

iv) no vertex of \( V_4 \) is adjacent to a vertex of \( V_1 \cup V_2 \),

v) every vertex of \( V_2 \) is adjacent to at least one vertex of \( V_1 \) and at least one vertex of \( V_3 \), and

vi) every vertex of \( V_3 \) is adjacent to at least one vertex of \( V_4 \) and at least one vertex of \( V_2 \).

Then there exists a connected graph \( H \) such that \( P(H) \equiv G \).

**Proof.** Let \( G \) be a graph satisfying the hypotheses of the theorem. We define the graph \( H \) as follows.

\[
V(H) = V(G) \cup \{u_1, u_2, u_3, w_1, w_2, w_3\} \quad \text{and} \\
E(H) = E(G) \cup \{u_1v \mid v \in V_1\} \cup \{w_1x \mid x \in V_4\} \cup \\
\{u_1u_2, u_2u_3, w_1w_2, w_2w_3\}.
\]

The verification that \( P(H) \equiv G \) is similar to the proof of Theorem 3.5 and is omitted. \( \square \)
Corollary 3.11 There exist connected graphs $H_1$ and $H_2$ such that $P(H_1) \cong C_7$ and $P(H_2) \cong P_4$.

Proof. Let $G_1 \cong C_7$, $V(G_1) = \{v_1, v_2, \ldots, v_7\}$ and $E(G_1) = \{v_iv_{i+1} \mid 1 \leq i \leq 6\} \cup \{v_1v_7\}$. Let $G_2 \cong P_4$, $V(G_2) = \{u_1, u_2, u_3, u_4\}$ and $E(G_2) = \{u_iu_{i+1} \mid i = 1, 2, 3\}$.

We partition $V(G_1)$ into four nonempty subsets $V_1, V_2, V_3$ and $V_4$ satisfying the conditions of Theorem 3.10, and we partition $V(G_2)$ into four nonempty subsets $V'_1, V'_2, V'_3$ and $V'_4$ also satisfying the conditions of Theorem 3.10. Let $V_1 = \{v_2, v_3\}$, $V_2 = \{v_1, v_4\}$, $V_3 = \{v_5, v_7\}$, $V_4 = \{v_6\}$; and $V'_i = \{u_i\}$ for $i = 1, 2, 3, 4$. □
CHAPTER IV

SUBGRAPH-DEFINED DEGREES

In this chapter we introduce our first subgraph-defined parameter and explore some results concerning this parameter.

4.1 Introduction

We begin with the definition of this subgraph-defined parameter. Let $F$ and $G$ be graphs. The $F$-degree of a vertex $v$ in $G$, denoted $F_{\text{deg}} v$, is the number of subgraphs of $G$ isomorphic to $F$ that contain $v$. Thus, the $F$-degree is a generalized degree defined in terms of the subgraph $F$. Observe that the $K_2$-degree of a vertex $v$ in any graph $G$ is simply the degree of $v$ in $G$. In Figure 4.1, a graph $G$ is shown together with the $P_3$-degrees of its vertices. The $P_3$-degree of a vertex $v$ in any graph $G$ is given by

$$P_3{\text{deg}} v = \binom{\text{deg} v}{2} + \sum_{u \in N(v)} (\text{deg} u - 1)$$

where $\binom{\text{deg} v}{2}$ counts those subgraphs of $G$ isomorphic to $P_3$ containing $v$ as a central vertex, while $\sum_{u \in N(v)} (\text{deg} u - 1)$ counts those subgraphs of $G$ isomorphic to $P_3$ where $v$ is an end-vertex. Here, $N(v)$ represents the set of all vertices in $G$ adjacent to $v$, and is called the neighborhood of $v$. 

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We begin with several well-known results involving degrees and their F-degree analogs. "The First Theorem of Graph Theory" states that the sum of the degrees of the vertices in a graph \(G\) equals twice the size of \(G\). An analogous theorem involving F-degrees [4] can be stated as follows.

**Theorem 4A** Let \(F\) and \(G\) be graphs. Then

\[
\sum_{v \in V(G)} F\deg v = p(F) \cdot n(F)
\]

where \(n(F)\) is the number of subgraphs of \(G\) isomorphic to \(F\).
If the order of $F$ is even, then the right hand side of the equation in Theorem 4A is an even number. Therefore $\sum_{v \in V(G)} \deg v$ must be even. Thus, we have the following result [4].

**Corollary 4B** Let $G$ be a graph. If $F$ is a graph of even order, then $G$ contains an even number of vertices with odd $F$-degree.

This corollary is, of course, a generalization of the well-known result that in any graph there is an even number of odd vertices. The following theorem is another well-known result involving degrees.

**Theorem 4C** Let $n$ be a nonnegative integer and let $m$ be an even nonnegative integer. Then there exists a graph $G$ of order $m + n$ having $n$ vertices of even degree and $m$ vertices of odd degree.

Since the usual degree is equivalent to the $K_2$-degree and $K_2$ is connected of order two, the following is a generalization of Theorem 4C involving $F$-degrees.

**Theorem 4.1** Let $F$ be a connected graph of order $p > 2$. Let $n$ be a nonnegative integer and $m$ a nonnegative multiple of $p$. Then there exists a graph $G$ of order $m + n$ having $n$ vertices of even $F$-degree and $m$ vertices of odd $F$-degree.

**Proof.** Suppose that $m = k \cdot p$ where $k$ is a nonnegative integer. Let $G \equiv kF \cup nK_1$. Then $G$ contains $k \cdot p = m$ vertices of $F$-degree 1. Further, since $p \geq 2$, the $n$ isolated vertices of $G$ each have $F$-degree 0. □

We wish to present other $F$-degree analogs of Theorem 4C in the cases where $F \equiv P_n$ for odd $n$ and $F \equiv K(1,3)$. However, we first require a definition and some
preliminary results. A graph $G$ is $F$-regular of degree $k$ if every vertex of $G$ has $F$-degree $k$. Of course, $K_2$-regularity is equivalent to regularity. The following result will prove to be useful.

**Lemma 4.2** For $k \geq 3$ and $p \geq k$, the graph $C_p$ is $P_k$-regular of degree $k$.

**Proof.** Let $G \cong C_p$, and suppose $V(G) = \{v_0, v_1, \ldots, v_{p-1}\}$ and $E(G) = \{v_iv_{i+1} \mid 0 \leq i \leq p - 2\} \cup \{v_{p-1}v_0\}$. For $i = 0, 1, \ldots, p - 1$ and $j = 1, 2, \ldots, k$, let $P_{ij}$ be the path $v_{i-j+1}, v_{i-j+2}, \ldots, v_{i-j+k}$ (where the subscripts are taken modulo $p$). Then each $P_{ij}$ is a distinct subgraph of $G$ isomorphic to $P_k$ containing $v_i$. Further, these are the only subgraphs of $G$ isomorphic to $P_k$ containing the vertex $v_i$. Thus $P_k\deg v_i = k$ for each $v_i \in V(G)$. ☐

We are now prepared to present a result analogous to Theorem 4C when $F \cong P_n$ for odd $n$.

**Theorem 4.3** Let $k \geq 3$ be an odd integer, let $n$ be a nonnegative integer and let $m \geq k$ be an integer. Then there exists a graph $G$ of order $m + n$ having $n$ vertices of even $P_k$-degree and $m$ vertices of odd $P_k$-degree.

**Proof.** Let $G \cong C_m \cup nK_1$. The $m$ vertices of the subgraph of $G$ isomorphic to $C_m$ all have $P_k$-degree $k$ by Lemma 4.2. Also, the $n$ isolated vertices of $G$ all have $P_k$-degree $0$. Since $k$ is odd and the order of $G$ is $m + n$, the theorem is proved. ☐

The next result is a variation of Theorem 4.3 in the case when $k = 3$. 

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**Theorem 4.4** Let \( n \geq 4 \) and \( m \geq 3 \) be integers. Then there exists a graph \( G \) of order \( m + n \) having \( n \) vertices of positive even \( P_3 \)-degree and \( m \) vertices of odd \( P_3 \)-degree.

**Proof.** Recall that the \( P_3 \)-degree of a vertex \( v \) in any graph \( G \) is given by

\[
P_3\deg v = \frac{\deg v}{2} + \sum_{u \in N(v)} (\deg u - 1).
\]

Using (1), it is straightforward to verify that if \( G \) is a regular graph of degree 4, then \( G \) is \( P_3 \)-regular of degree 18. Since for every positive integer \( t \), the graph \( K_{2t+1} \) is the edge sum of \( t \) hamiltonian cycles, and \( K_{2t} \) is the edge sum of \( t - 1 \) hamiltonian cycles and a 1-factor, there exists a 4-regular graph \( G_n \) of order \( n \) for all \( n \geq 5 \). Let \( G \equiv C_m \cup G_n \). Then \( G \) is a graph of order \( m + n \) containing \( n \) vertices of \( P_3 \)-degree 18 and \( m \) vertices of \( P_3 \)-degree 3. \( \Box \)

We now turn our attention to \( K(1,3) \)-degrees. Observe that the \( K(1,3) \)-degree of a vertex \( v \) in a graph \( G \) is given by

\[
K(1,3)\deg v = \binom{\deg v}{3} + \sum_{u \in N(v)} \binom{\deg u - 1}{2},
\]

where \( \binom{\deg v}{3} \) counts those subgraphs of \( G \) isomorphic to \( K(1,3) \) containing \( v \) as a central vertex. The number \( \sum_{u \in N(v)} \binom{\deg u - 1}{2} \) counts those subgraphs of \( G \) isomorphic to \( K(1,3) \) where \( v \) is an end-vertex. The following result is an analog to Theorem 4C involving \( K(1,3) \)-degrees.
Theorem 4.5 Let $n$ be a nonnegative integer and let $m \geq 4$ be an even integer. Then there exists a graph $G$ of order $m + n$ having $n$ vertices of even $K(1,3)$-degree and $m$ vertices of odd $K(1,3)$-degree.

Proof. Since $m \geq 4$ is even, either $m \equiv 0 \pmod{4}$ or $m \equiv 2 \pmod{4}$. We consider these two cases separately. First, if $m \equiv 0 \pmod{4}$, then $m = 4k$ for some integer $k \geq 1$. In this case, let $G \equiv kK(1,3) \cup nK_1$. Then $G$ contains $4k = m$ vertices of $K(1,3)$-degree 1 and $n$ vertices of $K(1,3)$-degree 0.

Now suppose $m \equiv 2 \pmod{4}$. Then $m = 4j + 2$ for some integer $j \geq 1$. Let $H$ be the graph defined by $V(H) = \{v_1, v_2, \ldots, v_6\}$ and $E(H) = \{v_i v_{i+1} \mid i = 2, 3, 4, 5, 6\} \cup \{v_i v_j \mid i, j \in \{2, 3, 4, 5, 6\}\}$. Computing the $K(1,3)$-degrees of the vertices of $H$ using (2), we find that $K(1,3)\deg v_i = 15$ and $K(1,3)\deg v_i = 9$ for $i = 2, 3, 4, 5, 6$. Thus all six vertices of $H$ have odd $K(1,3)$-degree. Let $G \equiv (j-1)K(1,3) \cup H \cup nK_1$. Then $G$ contains $4(j-1) + 6 = 4j + 2 = m$ vertices of odd $K(1,3)$-degree and $n$ vertices of $K(1,3)$-degree 0. □

When $n \geq 4$, we can extend Theorem 4.5, producing a graph having positive $K(1,3)$-degrees.

Theorem 4.6 Let $n \geq 4$ be an integer and let $m \geq 4$ be an even integer. Then there exists a graph $G$ of order $m + n$ having $n$ vertices of positive even $K(1,3)$-degree and $m$ vertices of odd $K(1,3)$-degree.

Proof. We consider the case when $n = 4$ first. Suppose first that $m \equiv 0 \pmod{4}$. Then $m = 4k$ for some integer $k \geq 1$. Using (2), we find that $K_4$ is $K(1,3)$-regular of degree 4. In this case, let $G \equiv kK(1,3) \cup K_4$. Then $G$ contains $4k = m$ vertices of $K(1,3)$-degree 1 and 4 vertices of $K(1,3)$-degree 4. If $m \equiv 2 \pmod{4}$, then
\( m = 4j + 2 \) for some integer \( j \geq 1 \). Let \( H \) be the graph defined in the proof of Theorem 4.5. Recall that all six vertices of \( H \) have odd \( K(1,3) \)-degree. Let \( G \equiv (j-1)K(1,3) \cup H \cup K_4 \). Then \( G \) contains \( 4(j - 1) + 6 = 4j + 2 = m \) vertices of odd \( K(1,3) \)-degree and \( 4 \) vertices of \( K(1,3) \)-degree \( 4 \).

Now suppose \( n \geq 5 \). As explained in the proof of Theorem 4.4, there exists a 4-regular graph \( G_n \) of order \( n \) for all \( n \geq 5 \). Using (2), we find that \( G_n \) is \( K(1,3) \)-regular of degree 16. First, if \( m \equiv 0 \pmod{4} \), then \( m = 4k \) for some integer \( k \geq 1 \). In this case, let \( G \equiv kK(1,3) \cup G_n \). Then \( G \) contains \( 4k = m \) vertices of \( K(1,3) \)-degree 1 and \( n \) vertices of \( K(1,3) \)-degree 16. Now suppose \( m \equiv 2 \pmod{4} \). Then \( m = 4j + 2 \) for some integer \( j \geq 1 \). Let \( G \equiv (j-1)K(1,3) \cup H \cup G_n \). Then \( G \) contains \( 4(j - 1) + 6 = 4j + 2 = m \) vertices of odd \( K(1,3) \)-degree and \( n \) vertices of \( K(1,3) \)-degree 16. \( \square \)

Our last result of this section concerns \( F \)-regularity. It has been shown in [4] that there exist nonregular, \( F \)-regular graphs for \( F \equiv K(1,n) \) and \( F \equiv K_p \) \( (n \geq 2, p \geq 3) \). We show that there exist nonregular, \( F \)-regular graphs for \( F \equiv C_n \) \( (n \geq 3) \).

**Theorem 4.7** For each pair \( r, s \) of positive integers and each integer \( n \geq 3 \), there exists a connected nonregular, \( C_n \)-regular graph \( G \) such that \( C_n \text{deg} \ v \geq r \) for every \( v \in V(G) \) and \( \Delta(G) - \delta(G) = s \).

**Proof.** Observe first that for all positive integers \( p, K_p \) is \( F \)-regular for all subgraphs \( F \) of \( K_p \). In particular, \( K_p \) is \( C_n \)-regular for all \( n \) such that \( 3 \leq n \leq p \). For \( n \geq 3 \), there are \( \frac{n!}{2} \) subgraphs isomorphic to \( C_n \). Thus \( K_n \) is \( C_n \)-regular of degree \( \frac{n!}{2} \). Now let \( v \in V(K_p) \) for \( p \geq n \geq 3 \). Since \( v \) is contained in \( \binom{p-1}{n} \) subgraphs of \( K_p \) isomorphic to \( K_n \), and \( K_n \) is \( C_n \)-regular of degree \( \frac{n!}{2} \), \( v \) is
Let \( r \) and \( s \) be given. Then there exists a \( k \) such that \( \binom{k-1}{n-1} \binom{n}{2} \geq r \). For \( i = 0, 1, \ldots, s \), let \( G_i \cong K_k \) and let \( v_i \in V(G_i) \). Let \( G \) be obtained from \( G_0, G_1, \ldots, G_s \) by joining \( v_0 \) to each of \( v_1, v_2, \ldots, v_s \). Observe that for \( i = 1, 2, \ldots, s \), the edge \( v_0 v_i \) is a bridge and is therefore not contained in any subgraph of \( G \) isomorphic to \( C_n \). Thus \( G \) is \( C_n \)-regular of degree \( \binom{k-1}{n-1} \binom{n}{2} \geq r \). Observe that \( G \) is connected. Further, \( \delta(G) = k - 1 \) and \( \Delta(G) = \deg v_0 = k - 1 + s \). Thus, \( \Delta(G) - \delta(G) = s \). □

4.2 F-Irregular Graphs

A graph \( G \) is \( F \)-irregular if the \( F \)-degrees of the vertices of \( G \) are distinct. For \( F \cong K_2 \), the following result is well-known.

**Theorem 4D** There exists no nontrivial \( K_2 \)-irregular graph.

The following conjecture was made in [4].

**Conjecture 4E** There exists an \( F \)-irregular graph for every connected graph \( F \) of order at least 3.

P. Erdős, G. Székely and W.T. Trotter have shown that there are infinitely many \( K_3 \)-irregular graphs. Since the \( P_3 \)-degrees of the graph \( G \) of Figure 4.1 are distinct, \( G \) is a \( P_3 \)-irregular graph. Thus the conjecture is verified for connected graphs \( F \) of order 3. In [4] it was shown that the graph \( G \) of Figure 4.1 is one of exactly two unicyclic \( P_3 \)-irregular graphs. In this section, we show that there exists an infinite
class of $P_3$-irregular graphs and present $K(1,n)$-irregular graphs for $n \geq 3$. In addition, we verify Conjecture 4E for all the connected graphs of order 4.

Our first result of this section produces an infinite class of $P_3$-irregular graphs. This result together with the infinite class of $K_3$-irregular graphs produced by Erdős, Székely and Trotter not only verifies Conjecture 4E for graphs $F$ of order 3, but extends it, by showing that there exist infinitely many $F$-irregular graphs for all connected graphs $F$ of order 3.

**Theorem 4.8** For $n \geq 3$, there exists a $P_3$-irregular graph of order $2n + 2$.

**Proof.** For each $n \geq 3$, we define a graph $G_n$ by

$$V(G_n) = \{w_1, w_2\} \cup \{v_1, v_2, \ldots, v_{n-1}\} \cup \{u_1, u_2, \ldots, u_{n+1}\}$$

and

$$E(G_n) = \{u_1w_1, u_1w_2, u_2w_2\} \cup \{u_kv_j | 1 \leq j \leq n - 1, 1 \leq k \leq j + 2\} \cup \{v_iv_j | 1 \leq i < j \leq n - 1\}.$$ 

Observe that the order of $G_n$ is $2n + 2$. We show that $G_n$ is $P_3$-irregular. Observe that $\deg w_i = i$ ($i = 1, 2$), $\deg v_j = n + j$ ($1 \leq j \leq n - 1$) and $\deg u_k = n - k + 2$ ($1 \leq k \leq n + 1$). Recall that

$$P_3\deg v = \left(\frac{\deg v}{2}\right) + \sum_{u \in N(v)} (\deg u - 1) \quad (3)$$

We first note that since $N(w_1) = \{u_1\}$ and $N(w_2) = \{u_1, u_2\}$,

$$P_3\deg w_1 = n < 2n = P_3\deg w_2.$$ 

We now consider $P_3\deg v_j$ ($1 \leq j \leq n - 1$). By (3),
\[ P_{3\deg v_j} = \left( \frac{\deg v_j}{2} \right) + \left[ \sum_{i=1}^{n-1} (\deg v_i - 1) \right] - (\deg v_j - 1) + \left[ \sum_{k=1}^{j+2} (\deg U_k - 1) \right]. \]

Substituting, we obtain the following:

\[ P_{3\deg v_j} = \left( \frac{n+j}{2} \right) + \left[ \sum_{i=1}^{n-1} (n+i-1) \right] - (n+j-1) + \left[ \sum_{k=1}^{j+2} (n-k+1) \right]. \]  

Thus for \( 1 \leq j \leq n-2 \),

\[ P_{3\deg v_{j+1}} = P_{3\deg v_j} + (n+j) + (n+j-1) - (n+j) + (n-j-2) \]
\[ = P_{3\deg v_j} + 2n - 3. \]

Since \( 2n-3 > 0 \) for \( n \geq 3 \) and \( j \geq 1 \), \( P_{3\deg v_{j+1}} > P_{3\deg v_j} \).

Observe that for \( j = 1 \), (4) gives \( P_{3\deg v_1} = 2n^2 - 2 \). Now consider \( P_{3\deg u_1} \).

By (3),

\[ P_{3\deg u_1} = \left( \frac{\deg u_1}{2} \right) + \left[ \sum_{i=1}^{n-1} (\deg v_i - 1) \right] + \left[ \sum_{k=1}^{2} (\deg w_k - 1) \right] \]
\[ = \left( \frac{n+1}{2} \right) + \left[ \sum_{i=1}^{n-1} (n+i-1) \right] + 1 \]
\[ = 2n^2 - 2n + 2 \]
\[ = P_{3\deg v_1} - 2n + 4. \]

Because \( -2n + 4 < 0 \) for \( n \geq 3 \), \( P_{3\deg u_1} < P_{3\deg v_1} \).

Also, by (3),

\[ P_{3\deg u_2} = \left( \frac{\deg u_2}{2} \right) + \left[ \sum_{i=1}^{n-1} (\deg v_i - 1) \right] + \deg w_2 - 1 \]
\[ = \left( \frac{n}{2} \right) + \left[ \sum_{i=1}^{n-1} (n+i-1) \right] + 1 \]
For $3 \leq k \leq n + 1$,

\[
P_{3}\text{deg } u_{k} = \left( \frac{\text{deg } u_{k}}{2} \right) + \left[ \sum_{i=k-2}^{n-1} (\text{deg } v_{i} - 1) \right]
\]

\[
= \left( \frac{n-k+2}{2} \right) + \left[ \sum_{i=k-2}^{n-1} (n + i - 1) \right].
\] (5)

Therefore,

\[
P_{3}\text{deg } u_{3} = \left( \frac{n-1}{2} \right) + \left[ \sum_{i=1}^{n-1} (n + i - 1) \right]
\]

\[
= P_{3}\text{deg } u_{2} - n.
\]

Also, for $3 \leq k \leq n$, since vertex $u_{k+1}$ is adjacent to all the neighbors of $u_{k}$ except vertex $v_{k-2}$ which has degree $n + k - 2$,

\[
P_{3}\text{deg } u_{k+1} = P_{3}\text{deg } u_{k} - (n - k + 1) - (n + k - 3)
\]

\[
= P_{3}\text{deg } u_{k} - 2n + 2.
\]

So $P_{3}\text{deg } u_{k+1} < P_{3}\text{deg } u_{k}$ for $1 \leq k \leq n$.

Finally, we observe that by (5),

\[
P_{3}\text{deg } u_{n} = 4n - 4 \quad \text{and} \quad P_{3}\text{deg } u_{n+1} = 2n - 2.
\]

Since, for $n \geq 3$, $n < 2n - 2 < 2n < 4n - 4$, we have that

\[
P_{3}\text{deg } w_{1} < P_{3}\text{deg } u_{n+1} < P_{3}\text{deg } w_{2} < P_{3}\text{deg } u_{n}.
\]

Thus,
$P_3\deg w_1 < P_3\deg u_{n+1} < P_3\deg w_2 < P_3\deg u_n <$
$P_3\deg u_{n-1} < \ldots < P_3\deg u_1 < P_3\deg v_1 < P_3\deg v_2 < \ldots < P_3\deg v_{n-1}$. □

It turns out not only that $G_n$ is a $P_3$-irregular graph for all $n \geq 3$, but that $G_n$ is $K(1,n)$-irregular for $n \geq 3$. Before proving this result, we consider an example. Figure 4.2 shows the graph $G_3$ together with both the $P_3$-degrees of its vertices and the $K(1,3)$-degrees of its vertices. The $K(1,n)$-degree of a vertex $v$ in a graph $G$ is given by

$$K(1,n)\deg v = \binom{\deg v}{n} + \sum_{u \in N(v)} \binom{\deg u - 1}{n - 1}$$

where $\binom{\deg v}{n}$ counts those subgraphs of $G$ isomorphic to $K(1,n)$ containing $v$ as a central vertex, while $\sum_{u \in N(v)} \binom{\deg u - 1}{n - 1}$ counts those subgraphs of $G$ isomorphic to $K(1,n)$ where $v$ is an end-vertex.
We now present the general result.

**Theorem 4.9** For $n \geq 3$, there exists a $K(1,n)$-irregular graph.
Proof. As in the proof of Theorem 4.1, for \( n \geq 3 \) we define a graph \( G_n \) by

\[
V(G_n) = \{ w_1, w_2 \} \cup \{ v_1, v_2, \ldots, v_{n-1} \} \cup \{ u_1, u_2, \ldots, u_{n+1} \}
\]

and

\[
E(G_n) = \{ u_{i}w_{i}, u_1w_2, u_2w_2 \} \cup \{ u_{k}v_{j} \mid 1 \leq j \leq n-1, 1 \leq k \leq j + 2 \} \cup \{ v_{i}v_{j} \mid 1 \leq i < j \leq n-1 \}.
\]

We show that \( G_n \) is \( K(1,n) \)-irregular. Observe that \( \deg w_i = i \) (\( i = 1, 2 \)), \( \deg v_j = n + j \) \((1 \leq j \leq n-1)\) and \( \deg u_k = n-k+2 \) \((1 \leq k \leq n+1)\). Recall that

\[
K(1,n)\deg v = \left( \frac{\deg v}{n} \right) + \sum_{u \in N(v)} \left( \frac{\deg u - 1}{n-1} \right).
\]

We first note that since \( N(w_1) = \{ u_1 \} \) and \( N(w_2) = \{ u_1, u_2 \} \),

\[
K(1,n)\deg w_1 = n < n + 1 = K(1,n)\deg w_2.
\]

We now consider \( K(1,n)\deg v_j \) \((1 \leq j \leq n-1)\). By (6),

\[
K(1,n)\deg v_j = \left( \frac{\deg v_j}{2} \right) + \left[ \sum_{i=1}^{n-1} \left( \frac{\deg v_{i-1}}{n-1} \right) \right] - \left( \frac{\deg v_{j-1}}{n-1} \right) + \left[ \sum_{k=1}^{j+2} \left( \frac{\deg u_{k-1}}{n-1} \right) \right].
\]

Substituting, we obtain the following:

\[
K(1,n)\deg v_j = \left( \frac{n+j}{2} \right) + \left[ \sum_{i=1}^{n-1} \left( \frac{n+i-1}{n-1} \right) \right] - \left( \frac{n+j-1}{n-1} \right) + n + 1.
\]

Thus for \( 1 \leq j \leq n-2 \),

\[
K(1,n)\deg v_{j+1} = K(1,n)\deg v_j + \left( \frac{n+j+1}{n} \right) - \left( \frac{n+j}{n} \right) - \left( \frac{n+j}{n-1} \right) + \left( \frac{n+j-1}{n-1} \right)
\]

\[
= K(1,n)\deg v_j + \frac{j+1}{n-1} \left( \frac{n+j-1}{n-2} \right).
\]
Since \( \frac{j+1}{n-1} \left( \frac{n+j-1}{n-2} \right) > 0 \), for \( j \geq 1 \),

\[ K(1,n) \deg v_{j+1} > K(1,n) \deg v_j. \]

Observe that for \( j = 1 \), (7) gives

\[ K(1,n) \deg v_1 = n + 2 + \left[ \sum_{i=1}^{n-1} \left( \frac{n+i-1}{n-1} \right) \right]. \]

Now consider \( K(1,n) \deg u_1 \). By (6),

\[
K(1,n) \deg u_1 = \binom{\deg u_1}{2} + \left[ \sum_{i=1}^{n-1} \left( \frac{\deg v_i - 1}{n-1} \right) \right] \\
= n + 1 + \left[ \sum_{i=1}^{n-1} \left( \frac{n+i-1}{n-1} \right) \right]. \\
= K(1,n) \deg v_1 - 1.
\]

Also, by (6),

\[
K(1,n) \deg u_2 = 1 + \left[ \sum_{i=1}^{n-1} \left( \frac{\deg v_i - 1}{n-1} \right) \right] \\
= K(1,n) \deg u_1 - 1.
\]

For \( 3 \leq k \leq n + 1 \),

\[
K(1,n) \deg u_k = \left[ \sum_{i=k-2}^{n-1} \left( \frac{\deg v_i - 1}{n-1} \right) \right].
\]

Therefore,

\[
K(1,n) \deg u_3 = \left[ \sum_{i=1}^{n-1} \left( \frac{\deg v_i - 1}{n-1} \right) \right]
\]
\[ = K(1,n)\deg u_2 - 1. \]

Also, for \(3 \leq k \leq n\), since vertex \(u_{k+1}\) is adjacent to all the neighbors of \(u_k\) except vertex \(v_{k-2}\) which has degree \(n + k - 2\),

\[ K(1,n)\deg u_{k+1} = K(1,n)\deg u_k - \left\lfloor \frac{n + k - 3}{n - 1} \right\rfloor. \]

So \(K(1,n)\deg u_{k+1} < K(1,n)\deg u_k\) for \(1 \leq k \leq n\).

Finally, we observe that

\[ K(1,n)\deg u_{n+1} = \left\lfloor \frac{2n - 2}{n - 1} \right\rfloor > n + 1 = K(1,n)\deg w_2, \]

for \(n \geq 3\). Thus we have

\[ K(1,n)\deg w_1 < K(1,n)\deg w_2 < K(1,n)\deg u_{n+1} < K(1,n)\deg u_n < \ldots < K(1,n)\deg u_1 < K(1,n)\deg v_1 < K(1,n)\deg v_2 < \ldots < K(1,n)\deg v_{n-1}. \]

There are six nonisomorphic connected graphs of order 4. They are \(P_4, C_4, K_4, K_4 - e, K(1,3)\) and another which we shall call \(H\). These six graphs are given in Figure 4.3. We have already shown in Theorem 4.9 that there exists a \(K(1,3)\)-irregular graph. The following result [4] verifies that there exists a \(K_4\)-irregular graph.
Figure 4.3

Theorem 4F. For $n \geq 3$, there exists a $K_n$-irregular graph.

We show that there exists an $F$-irregular graph for the other four connected graphs of order 4. Our first result considers $F = P_4$.

Theorem 4.10. There exists a $P_4$-irregular graph.

Proof. We define a graph $G$ by

$V(G) = \{v_1, v_2, \ldots, v_{10}\}$ and $E(G) = \{v_1v_2, v_2v_3, v_2v_8, v_3v_4, v_3v_8, v_4v_5, v_4v_8, v_4v_9, v_5v_6, v_6v_7, v_8v_9\}$.
In Figure 4.4, we show the graph $G$ together with its 29 subgraphs isomorphic to $P_4$. It is straightforward to verify that these are all the subgraphs of $G$ isomorphic to $P_4$. From Figure 4.4, we see that

\[
\begin{align*}
P_4\text{deg } v_1 &= 5 \\
P_4\text{deg } v_2 &= 17 \\
P_4\text{deg } v_3 &= 20 \\
P_4\text{deg } v_4 &= 24 \\
P_4\text{deg } v_5 &= 9 \\
P_4\text{deg } v_6 &= 4 \\
P_4\text{deg } v_7 &= 1 \\
P_4\text{deg } v_8 &= 23 \\
P_4\text{deg } v_9 &= 6 \\
P_4\text{deg } v_{10} &= 7.
\end{align*}
\]

Since all the $P_4$-degrees of $G$ are distinct, $G$ is $P_4$-irregular. □
Our next result considers the case when $F$ is isomorphic to the graph $H$ of Figure 4.3.

**Theorem 4.11** Let a graph $H$ be defined by

$V(H) = \{y_1, y_2, y_3, y_4\}$ and
\[ E(H) = \{y_1y_2, y_1y_3, y_2y_3, y_3y_4\}. \]

Then there exists an \( H \)-irregular graph.

**Proof.** We define a graph \( G \) by

\[ V(G) = \{u_1, u_2, u_3, v_1, v_2, w, x\} \quad \text{and} \quad \]

\[ E(G) = \{u_1u_2, u_1u_3, u_1v_1, u_1w, u_2v_1, u_2v_2, u_2w, u_3v_2, u_3w, u_3x, v_2x, wx\}. \]

The graph \( G \) is given in Figure 4.5. We show that \( G \) is \( H \)-irregular. Observe that there exactly five 3–cycles in \( G \). These 3–cycles are \( T_1: u_1, u_2, w, u_1; \ T_2: u_1, u_2, v_1, u_1; \ T_3: u_3, w, x, u_3; \ T_4: u_3, v_2, x, u_3; \) and \( T_5: u_1, u_3, w, u_1. \) Since \( u_1 \) is contained in six subgraphs of \( G \) isomorphic to \( H \) containing \( T_1 \), four subgraphs of \( G \) isomorphic to \( H \) containing \( T_2 \), two subgraphs isomorphic to \( H \) containing \( T_3 \), one subgraph of \( G \) isomorphic to \( H \) containing \( T_4 \) and six subgraphs of \( G \) isomorphic to \( H \) containing \( T_5 \), \( H\deg u_1 = 19. \) Observe that \( H\deg u_2 = 14 \) since \( u_2 \) is contained in six subgraphs of \( G \) isomorphic to \( H \) containing \( T_1 \), four subgraphs of \( G \) isomorphic to \( H \) containing \( T_2 \), one subgraph isomorphic to \( H \) containing \( T_3 \), one subgraph of \( G \) isomorphic to \( H \) containing \( T_4 \) and two subgraphs of \( G \) isomorphic to \( H \) containing \( T_5 \). Note that \( H\deg u_3 = 18 \) because \( u_3 \) is contained in two subgraphs of \( G \) isomorphic to \( H \) containing \( T_1 \), one subgraph of \( G \) isomorphic to \( H \) containing \( T_2 \), five subgraphs of \( G \) isomorphic to \( H \) containing \( T_3 \), and four subgraphs of \( G \) isomorphic to \( H \) containing \( T_4 \) and six subgraphs of \( G \) isomorphic to \( H \) containing \( T_5 \). Because \( v_1 \) is only contained in two subgraphs of \( G \) isomorphic to \( H \) containing \( T_1 \), four subgraphs of \( G \) isomorphic to \( H \) containing \( T_2 \) and one subgraph of \( G \) isomorphic to \( H \) containing \( T_5 \), \( H\deg v_1 = 7. \) The vertex \( v_2 \) is contained in one subgraph of \( G \) isomorphic to \( H \) containing \( T_1 \), one subgraph of \( G \) isomorphic to \( H \) containing \( T_2 \), two subgraphs of \( G \)
isomorphic to $H$ containing $T_3$ four subgraphs isomorphic to $H$ containing $T_4$, and one subgraph isomorphic to $H$ containing $T_5$. Therefore, $H_{\text{deg}} v_2 = 9$. Because there are six subgraphs of $G$ isomorphic to $H$ containing $T_1$ and $w$, two subgraphs of $G$ isomorphic to $H$ containing $T_2$ and $w$, five subgraphs of $G$ isomorphic to $H$ containing $T_3$ and $w$, two subgraphs of $G$ isomorphic to $H$ containing $T_4$ and $w$, and six subgraphs of $G$ isomorphic to $H$ containing $T_5$ and $w$, $H_{\text{deg}} w = 21$. Finally, $H_{\text{deg}} x = 12$ since $x$ is contained in one subgraph of $G$ isomorphic to $H$ containing $T_1$, five subgraphs of $G$ isomorphic to $H$ containing $T_3$ four subgraphs of $G$ isomorphic to $H$ containing $T_4$ and two subgraphs of $G$ isomorphic to $H$ containing $T_5$. □

![Graph G](image)

**Figure 4.5**

Our next result gives a $C_4$-irregular graph.

**Theorem 4.12** There exists a $C_4$-irregular graph.

**Proof.** We define a graph $G$ by
V(G) = \{u_1, u_2, u_3, v_1, v_2, v_3, w, x\} and
E(G) = \{u_1u_2, u_1u_3, u_1v_1, u_1w, u_2u_3, u_2v_1, u_2v_2, u_2w, u_3v_1, u_3v_2, u_3v_3,
\quad u_3w, v_2x, v_3x, wx\}.

The graph G is shown in Figure 4.6. We show that G is C₄-irregular. Let G₁ = \langle u_1, u_2, u_3, w \rangle, G₂ = \langle u_1, u_2, u_3, v_1 \rangle and note that G₁ ≅ G₂ ≅ K₄. Because K₄ contains three subgraphs isomorphic to C₄, each of G₁ and G₂ contain three subgraphs isomorphic to C₄. Let G₃ = \langle u_1, u_2, u_3, v_2 \rangle, G₄ = \langle u_1, u_2, v_1, w \rangle, G₅ = \langle u_1, u_3, v_1, w \rangle, G₆ = \langle u_2, u_3, v_1, w \rangle, G₇ = \langle u_2, u_3, v_2, w \rangle and G₈ = \langle u_2, u_3, v_1, v_1, v_2 \rangle. Observe that G₃ ≅ G₄ ≅ G₅ ≅ G₆ ≅ G₇ ≅ G₈ ≅ K₄ – e. Since K₄ – e contains one subgraph isomorphic to C₄, for i = 1, 2, ..., 8 each Gᵢ contains exactly one subgraph isomorphic to C₄. Let G₉ = \langle u_2, v_2, w, x \rangle, G₁₀ = \langle u_3, v_2, w, x \rangle, G₁₁ = \langle u_3, v_3, w, x \rangle and G₁₂ = \langle u_3, v_2, v_3, x \rangle. Observe that G₉ ≅ G₁₀ ≅ G₁₁ ≅ G₁₂ ≅ C₄.

![Graph G](image)

Figure 4.6

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We first calculate $C_4\deg u_1$. Since $u_1 \in G_i$ for $i = 1, 2, 3, 4$ and 5, $C_4\deg u_1 = 3 + 3 + 1 + 1 + 1 = 9$. Observe that $C_4\deg u_2 = 12$ because $u_2 \in G_i$ for $i = 1, 2, 3, 4, 6, 7, 8$ and 9. Similarly, $C_4\deg u_3 = 14$ because $u_3 \in G_i$ for $i = 1, 2, 3, 5, 6, 7, 8, 10, 11$ and 12. Because $v_1 \in G_i$ for $i = 2, 4, 5, 6,$ and 8, $C_4\deg v_1 = 3 + 1 + 1 + 1 + 1 = 7$. Also, $C_4\deg v_2 = 6$ because $v_2 \in G_i$ for $i = 3, 7, 8, 9, 10$ and 12. Further, $C_4\deg v_3 = 2$ because $v_3 \in G_i$ for $i = 11$ and 12, and $C_4\deg x = 4$ because $x \in G_i$ for $i = 9, 10, 11$ and 12. Finally, because $w \in G_i$ for $i = 1, 4, 5, 6, 7, 9, 10$ and 11, $C_4\deg w = 3 + 7 = 10$. □

Our last result of this section and the chapter considers the case when $F \equiv K_4 - e$. This result completes the verification of Conjecture 4E for all the connected graphs $F$ of order 4.

**Theorem 4.13** Let $F \equiv K_4 - e$. Then there exists an $F$-irregular graph.

**Proof.** We define a graph $G$ by

$$V(G) = \{u_1, u_2, u_3, v_1, v_2, v_3, w, x\} \text{ and}$$
$$E(G) = \{u_1u_2, u_1u_3, u_1v_1, u_1w, u_2u_3, u_2v_1, u_2w, u_3v_1, u_3v_2, u_3v_3, u_3w, u_3x, v_2x, v_3x, wx\}.$$ 

The graph $G$ is shown in Figure 4.7. We show that $G$ is $F$-irregular. Let $G_1 = \langle u_1, u_2, u_3, w\rangle$, $G_2 = \langle u_1, u_2, u_3, v_1\rangle$ and note that $G_1 \equiv G_2 \equiv K_4$. Because $K_4$ contains six subgraphs isomorphic to $F$, each of $G_1$ and $G_2$ contain six subgraphs isomorphic to $F$. Let $G_3 = \langle u_1, u_2, u_3, v_2\rangle$, $G_4 = \langle u_1, u_2, v_1, w\rangle$, $G_5 = \langle u_1, u_3, v_1, w\rangle$, $G_6 = \langle u_2, u_3, v_1, w\rangle$, $G_7 = \langle u_2, u_3, v_2, w\rangle$, $G_8 = \langle u_2, u_3, v_1, v_2\rangle$, $G_9 = \langle u_3, v_2, w, x\rangle$, $G_{10} = \langle u_3, v_3, w, x\rangle$ and $G_{11} = \langle u_3, v_2, v_3, x\rangle$. Observe that $G_3 \equiv G_4 \equiv G_5 \equiv G_6 \equiv G_7 \equiv G_8 \equiv G_9 \equiv G_{10} \equiv G_{11} \equiv F$. 

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We now calculate the $F$-degrees of the vertices of $G$. Since $u_1 \in G_i$ for $i = 1, 2, 3, 4$ and $5$, $Fdeg\ u_1 = 6 + 6 + 1 + 1 + 1 = 15$. Observe that $Fdeg\ u_2 = 17$ because $u_2 \in G_i$ for $i = 1, 2, 3, 4, 6, 7$ and $8$. Similarly, $Fdeg\ u_3 = 20$ because $u_3 \in G_i$ for $i = 1, 2, 3, 5, 6, 7, 8, 9, 10$ and $11$. Because $v_1 \in G_i$ for $i = 2, 4, 5, 6$, and $8$, $Fdeg\ v_1 = 6 + 1 + 1 + 1 + 1 = 10$. Also, $Fdeg\ v_2 = 5$ because $v_2 \in G_i$ for $i = 3, 7, 8, 9$ and $11$. Further, $Fdeg\ v_3 = 2$ because $v_3 \in G_i$ for $i = 10$ and $11$, and $Fdeg\ x = 3$ because $x \in G_i$ for $i = 9, 10$ and $11$. Finally, because $w \in G_i$ for $i = 1, 4, 5, 6, 7, 9$ and $10$, $Fdeg\ w = 6 + 6 = 12$. □
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