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Effect of Computer Graphics Use on Student Understanding of Calculus Concepts

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EFFECT OF COMPUTER GRAPHICS USE ON STUDENT UNDERSTANDING OF CALCULUS CONCEPTS

by

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EFFECT OF COMPUTER GRAPHICS USE
ON STUDENT UNDERSTANDING
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Charlene E. Beckmann, Ph.D.

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Student understanding of selected calculus concepts as developed through use of a Cartesian coordinate graphical representation system were investigated. Subjects ($N = 163$) enrolled in first-semester calculus sections at Western Michigan University participated in one of four treatment conditions: Graphics (G), exposure to a computer-graphically-developed conceptual course; Graphics Plus (G+), exposure to the same course as G subjects plus provision of computer graphics software and related supplemental assignments; Standard 1 (S1), exposure to a graphically-developed, conceptual course; and Standard 2 (S2), exposure to a traditional skill-oriented course.

Two investigations were undertaken. In Investigation 1, comparisons were made between G and G+ sections on student: (a) understanding of Cartesian graphs, including the ability to use graphs in understanding calculus concepts, and (b) attitudes toward the use of graphs. In Investigation 2, comparisons were made between G, G+, S1, and S2 sections on student: (a) performance on routine applied, routine symbolic, and nonroutine symbolic questions; (b) performance

on the departmental final exam and its subscales; (c) changes in attitudes toward mathematics; and (d) attitudes toward the course. Prior calculus experience was used as a blocking variable for cognitive measures on two levels, prior and no prior experience. Multivariate analyses with covariates, precalculus competency and attitudes toward mathematics, were performed for cognitive variables. χ^2 tests were conducted for affective variables.

For Investigation 1, no significant differences $(p < .05)$ were detected between the G and G+ sections. G+ subjects' scores on cognitive variables were slightly higher than those of G subjects, suggesting that further study is warranted. Attitudes pertaining to the use of graphs were overwhelmingly positive.

For Investigation 2, significant differences favored G subjects over S2 subjects on nonroutine symbolic questions. Questionable significant differences were detected for routine questions. Attitudes toward the course and mathematics were generally positive. Retention rates were much higher for the conceptually-developed sections than for the technique-oriented section.

Results suggest that developing calculus concepts through the use of a graphic representation system, especially as presented through computer graphics, can positively affect student understanding and interest without necessarily, negatively influencing skill acquisition.

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Effect of computer graphics use on student understanding of calculus concepts

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Dedicated...

To Ben for his lighthearted antics and warm hugs,

To Colleen for her sweetness and thoughtfulness,
and beautiful flowers,

To Melanie who always believed this could be done
even when I wasn't so sure,

To Dave who's love means more than life to me....

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11 to God to thank Him for all of the wonders in my life. The prayer began, "Dear Dave...".

Charlene E. Beckmann

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CHAPTER I

INTRODUCTION

Calculus is ranked among the top 5 college courses in enrollment. Nearly 700,000 college and 300,000 high school students take calculus each year (Steen, 1988). However, student withdrawal and failure rates for the first semester course exceed 50% in many large universities (Petersen, 1986). Those students who do complete the course are often unable to solve problems that vary only slightly from those in the course text (Lochhead, 1983). In its present state, calculus is perceived as not meeting the needs of the students, the client disciplines – those academic areas whose students will use calculus ideas in future courses, or the mathematics community itself (Douglas, 1986; Kolata, 1988; Petersen, 1986; Ralston, 1984; Roberts, 1984; Steen, 1988b).

In a perusal of the current literature concerning college mathematics in the first two years, the dissatisfaction with the current calculus course expressed by the mathematics community and the client disciplines becomes apparent. Empirical research pertaining to calculus and current literature on the psychological foundations of mathematics education and on the use of representations in understanding mathematics suggest viable options for change. The present study outlines a possible direction for change in the teaching of the first course of calculus based on this empirical and theoretical framework.

The suggestion for change is presented in the form of an experimental course.

The emphasis of the experimental course is conceptual, based on the use of a Cartesian coordinate graphic representation system, especially as developed through the use of computer graphics software. For students who have completed such a course, answers to the following questions concerning students abilities and attitudes with respect to calculus and the use of graphs are of interest:

1. Does exposure to a graphically-developed, conceptual calculus course where skill development is de- emphasized have a negative effect on student acquisition of calculus-related manipulative skills?

2. Does exposure to such a course increase student conceptual understanding of calculus as measured by performance on nonroutine questions?

3. Does exposure to such a course influence student attitudes towards the usefulness of graphs?

4. Does exposure to such a course influence student attitudes towards mathematics?

5. For students who have had prior calculus experience – at least half of a semester – are the answers to questions 1 and 2 different from those obtained for students without such experience?

The following pages present an overview of: (a) the perceived difficulties with the way calculus is presently taught; (b) suggestions for change and alternative approaches to teaching calculus; (c) the psychological foundations of mathematics supporting a conceptual approach to calculus, including the use of representations in understanding mathematics; and (d) a brief description of the course delivered

and the study undertaken to test its effectiveness.

Calculus: The Need for Change

The content of the calculus course is being questioned for a variety of reasons. One of the most prominent is the lack of conceptual emphasis in the course. While conceptual introduction and proof are often given in current calculus textbooks, the majority of textbook space is devoted to sample and practice exercises. These exercises are designed, for the most part, to provide a guide for student practice of techniques (Lyons, 1988). Assuming that instructors follow the course text, this emphasis on manipulative skill building will also be reflected in their classroom teaching and in the subsequent evaluation of student understanding. In an extensive nationwide sample of calculus final exams, Steen (1988b) found that 90% of the questions reflect this technique-orientation. These final exam questions required students to "solve, sketch, find, evaluate, determine, calculate, graph, integrate, and differentiate." Since computer and advanced calculator routines are available to perform many, if not all, of these techniques, it becomes important to question the necessity of teaching these skills for their mastery.

The current discontent with the calculus content is also attributed to its sheer volume. The content of the first semester course has grown to include topics of interest to several client disciplines. Topics of importance to biology and economics have been introduced in the last 10 years, adding to existing topics of importance to engineering and the physical sciences (Lyons, 1988). There have been few corresponding efforts to delete topics from calculus texts.

Although applications to other disciplines have been included in texts, new applications to real-world models are minimally reflected (Petersen, 1986; Roberts, 1984). Problem sets on related rates and optimization contain all-too-familiar, unmotivating word problems such as those concerning ladders sliding down walls, vehicles travelling toward or away from each other at 90° angles, and spherical balloons inflating at constant rates. The problem conditions of "real-life" problems are often simplified to the point of making them unrealistic. One example of such a problem is a tanker leaking oil at a constant rate in a spherical pattern; no wind, no waves, and the oil is leaking uniformly from all sides of the ship (Hurley, 1987, p. 146).

Another factor necessitating the questioning of topics in the current calculus course is the widespread availability of computers and computer software. The computer and programmable and/or graphing calculator have recently become viable alternatives as a tool in the calculus classroom (Tucker, 1987). These tools are relatively inexpensive and widely available. Well-written, useful software has become increasingly available, including software packages specifically written for calculus instruction and learning (Cunningham & Smith, 1986, 1987, 1988; Frantz, 1986; Heid, 1983; Waits & Demana, 1987a; Wilf, 1982). An increase in such realistic uses of computers, calculators, and available software would allow for the inclusion of real situations modelled by complicated functions and real data with "messy" numbers. Computers can be utilized to solve, sketch, find, evaluate, determine, calculate, graph, integrate, and differentiate. Student effort can then

be concentrated on understanding the concepts of calculus, translating problem situations into a mathematical form which can be entered into the computer, and interpreting computer-produced information in terms of the original problem settings. Currently, however, the technology available for numerical, symbolic, and graphical enhancement/interpretation of calculus concepts remains virtually untapped.

Calculus: Current Reform Movements and Suggestions for Change

A national initiative, sponsored by the National Science Foundation (NSF), has begun in response to the above concerns. Calculus is the "gateway to all areas of science and engineering" (Kolata, 1988, p. 89) and needs to be part of the college core curriculum. The NSF initiative is an effort to encourage widespread reform of the calculus curriculum, reform that has begun at an increasing number of institutions around the country.

The existing efforts at redefining the calculus sequence suggest that the content of the first semester course is fairly tight (Douglas, 1986; Larson, Smith, & Zorn, 1987). It has been suggested that conceptual understanding be emphasized and that manipulative skill be delegated a lesser role (Fey et al., 1984; Heid, 1984; Larson, Smith, & Zorn, 1987; Petersen, 1986). It has been further suggested that functions given in graphical and tabular form, as well as symbolic form, play a major role in the development of calculus ideas (Douglas, 1986, p. ix). Moreover, the graphical and numerical capabilities of technological tools should be fully

utilized. Until recently, few projects or studies had focused on ideas related to these suggestions.

Empirical Research

Studies concerned with student misconceptions and errors in calculus suggest that student difficulties are often due to their lack of conceptual understanding (Davidson, 1980; Geuther, 1987; Orton, 1983a, 1983b). Students seemed to have an especially difficult time with understanding the concept of limit (Confrey, 1980; Davis & Vinner, 1986; Orton; Tall & Vinner, 1981). It was suggested that students, upon entry into calculus, have had few mathematical experiences with continuous ideas and that the transition from a discrete to a continuous concept of number was especially difficult for them (Confrey, Orton). It was further suggested that the use of Cartesian coordinate graphs might provide a continuous representation to aid students in the transition from discrete to continuous (Orton).

In studies concerned with student understanding of Cartesian graphs, it was suggested that student difficulties in constructing and interpreting graphs also pertained to a discrete concept of number. Students tended to interpret graphs point-wise rather than globally (Janvier, 1978; Kerslake, cited in Janvier, 1978; Ponte, 1984; Wagner, Rachlin, & Jensen, 1984). Suggestions for remediation of such difficulties included use of computer graphics to provide an environment in which to study global as well as local properties of graphs (Wagner et al.,1984). Further, Janvier (1978) listed several global features of Cartesian graphs, most of

which are studied in a first course in calculus.

Studies concerning the use of graphs in calculus suggested that certain types of problems were especially suited to graphical interpretation and solution (Patterson, 1983). Also, visual training might enhance understanding of concepts, especially for females students (Ferrini-Mundy, 1987).

Few studies have been conducted exploring the use of computer graphics in aiding student understanding of calculus concepts. While not specifically related to use of graphs, Heid (1984) found that full use of the computer as a tool, to generate results otherwise obtained by hand, allowed student attention to be focused on conceptual understanding. She suggested that the use of multiple representations might contribute to students' deep conceptual understanding of calculus ideas. Using a computer graphic environment to develop the concept of derivative, Tall (1986a) found that such an environment helped promote deep conceptual understanding. In both of these studies, skill development was not negatively affected by a conceptual approach through the use of the computer. However, student difficulties with the concept of limit did not seem to be alleviated in either approach.

Psychological Foundations

The curricular suggestions – (a) that conceptual understanding be emphasized, (b) that manipulative skill be delegated a lesser role, (c) that functions given in graphical, tabular, and symbolic form be used to develop calculus ideas, and (d) that technological tools be fully utilized – find support in the theoretical

literature.

Theoretical foundations for the present study are derived from various sources. Among these are: (a) the constructivist theory of learning (Piaget, 1967; Skemp, 1987; Von Glasersfeld, 1987); (b) the cognitive science/information processing point of view as applied to mathematics education (Davis, 1984); and (c) theory concerning the role of the development of understanding through the use of multiple representations (Davis, 1984, 1986; Goldin, 1987a, 1987b; Hiebert, 1987; Skemp, 1987).

The constructivist point of view, as first presented by Piaget (1967), and extended by Von Glasersfeld (1987) and Skemp (1987), describes the learner as the builder of his/her own knowledge. Learning occurs through maturation, socialization, experience and equilibration. All learning is based on a person's existing cognitive structures. When presented with new experiences, these structures are either updated to include the new experiences, or altered to make sense of the world. Motivation to learn comes from within, driven by the learner's need to maintain or achieve equilibrium.

The cognitive science/information processing point of view as described by Davis (1984) maintains that students learn by basing new information and experiences on existing schema. These existing schema have been built on primitive ideas that are often visual. These primitive ideas – space, time, size, weight, etc. – have existed in the learner, for the most part, since his or her third or fourth year of life. A person's memory is described metaphorically as similar to the memory

banks of a computer. Information is stored in a vast warehouse to be recalled when needed. Recall memory is small, allowing only brief outlines of information to be recalled at any one time. Efficient coding and linking of information to existing schema must occur in order for recall to take place.

In mathematics, coding of information takes the form of various representations such as tables, graphs, natural language, symbols, and pictures. Each of these representations carries with them salient properties of the signified world, those ideas being represented (Kaput, 1987). Representations aid in storage and retrieval of information by coding information into a form that is easier to manipulate, or by simplifying complex relationships, removing the "irrelevant" aspects of the problem. In this way, use of representations reduces memory load and increases storage capacity (Davis, 1984).

Procedural knowledge of these representations supports conceptual understanding by decreasing the amount of mental attention and energy that must be employed to use the representation. Mental effort can then be focused on understanding the concept being developed. The amount of practice required to routinize a procedure is often considerable. Time spent on practice must be justified in terms of the importance of mastery of the procedure to further the understanding of new concepts.

The Problem Area

Calculus is the study of how quantities change with respect to each other. Variables can be assigned to these quantities and their interrelationships repre-

sented in various forms such as graphs, tables of values, algebraic expressions, and pictures. The Cartesian coordinate graph provides a natural illustration of how these variables change with respect to each other (Janvier, 1978, 1987b). Graphs then provide a natural representation system in which to develop the concepts of calculus (Goldin, 1987b; Janvier, 1987b; Kaput, 1987; Sawyer, 1961).

In a perusal of calculus texts (Goldstein, Lay & Schneider, 1980; Grossman, 1988; Hurley, 1987; Purcell & Varberg, 1984; Swokowski, 1983; Thomas & Finney, 1986), one finds many concepts of calculus that are intuitively developed through graphs. The use of computer and calculator graphics can provide a dynamic display of these and many other ideas.

Students at the level of calculus have constructed various schema of relationships between physical quantities. They also have had many experiences with mathematical representations. In the present study, the intent was to build upon students' existing schema by developing calculus concepts through graphically modelling familiar relationships between physical quantities. Developing calculus concepts using a graphic representation which is, in turn, based on familiar real-life situations, allows the interplay of a variety of representations: graphic, symbolic, and natural language. Each representation contributes toward the overall understanding of the concepts of calculus. The symbolic representation provides local information point-by-point; the graphic representation gives global information; and natural experiences provide a meaningful base upon which the others make sense.

Statement of the Problem

The present study was designed to investigate the extent to which use of a Cartesian coordinate graphical representation system, developed and extended through the use of computer graphics software, influences students: (a) conceptual understanding of the content in a first calculus course; (b) attitudes toward the usefulness of graphs; and (c) attitudes toward mathematics. As used in this discussion, graphic representation implies the use of Cartesian coordinate planar graphs to represent the concepts of calculus. These graphs may be generated on the computer or drawn by hand.

The study was designed to broaden the knowledge base related to student understanding, especially through the use of visual representations, in conceptualizing mathematics. The purpose was to compare the relative effectiveness of calculus instruction varying in the level of the use of a graphic representation in developing the concepts of limit, continuity, and derivative on: (a) student facility with the use of a graphic representation system in calculus; (b) student competence with routine symbolic calculus skills; (c) student ability to solve nonroutine, symbolic calculus problems; (d) student ability to solve applied calculus problems; (e) student attitude toward mathematics; and (f) student attitude toward the use and usefulness of graphs.

To investigate the effects of the use of a graphic representation on student understanding in calculus, it was first necessary to develop a conceptual calculus course in which the concepts of limit, continuity and derivative were developed

graphically, relying heavily on dynamic and static graphic displays generated by the computer. Written protocols giving detailed descriptions of this experimental course are given in Appendix A. The protocols describe the development of the concepts of limit, continuity, and derivative, and related generalizations as these were presented graphically, often based on familiar real-world applications. Included in the protocols are: (a) any in-class presentations whose graphic and/or conceptual development did not coincide with the course outline in "Calculus" by Hurley (1987) and the accompanying instructor's manual; and (b) descriptions of commercial software utilities and investigator-written computer programs used in the course to illustrate these ideas. Listings of the investigator-written computer programs are given in Appendix G. Related supplemental assignments are given in Appendix F.

Research Goals

Based on the empirical and theoretical framework previously described, the goals of the present study are:

1. To determine if the level of exposure to a graphical representation of concepts in a conceptual calculus course and/or prior calculus experience influences student understanding of the Cartesian coordinate system, including the ability to use such a representation system in understanding calculus concepts;

2. To determine if the level of exposure to a graphical representation of concepts in a conceptual calculus course influences student attitude toward the use and usefulness of a Cartesian coordinate system in understanding calculus

concepts;

3. To compare the performance of students exposed to a graphically-developed conceptual calculus course with that of students exposed to more traditional calculus instruction on routine applied, routine symbolic, and nonroutine symbolic calculus problems.

4. To compare changes in attitude toward mathematics of students exposed to a graphically-developed conceptual calculus course with those of students exposed to more traditional calculus instruction.

5. To develop written protocols outlining the development of the concepts of limit, continuity, and derivative, and their related generalizations as developed graphically in a conceptual calculus course.

6. To develop graphic-based computer software which can be used to illustrate the concepts of limit, continuity, and derivative and their related generalizations.

7. To develop conceptual, graphical, and situation-based homework assignments which aid student discovery or understanding of selected calculus concepts through the use of a graphic representation system.

The Study

Four treatment sections were defined for the study: Graphics (G), Graphics Plus (G+), Standard 1 (S1), and Standard 2 (S2). The Graphics and Graphics Plus sections were taught by the investigator. For these sections, calculus ideas were developed graphically, often through the use of computer graphics software, based on student experiences with familiar physical quantities. These two sections

differed in that subjects in the G+ section were provided the computer software *Master Grapher* (Waits & Demana, 1987a) and were given six supplemental assignments, four of which required use of the software. In-class discussion differed between these sections only in the questions students asked and in the amount of direction necessary to aid G+ students in the use of the computer.

The Standard sections were each taught by instructors other than the investigator. The S1 instructor's college teaching experience was comparable to that of the investigator. The S2 instructor had more experience teaching upper-division mathematics courses than either the investigator or the S1 instructor. The S1 and S2 instructors essentially followed the departmental syllabus which paralleled the course text.

The S1 instructor, a graduate teaching assistant, often used graphs drawn on the blackboard to introduce and develop concepts. The emphasis was less skill-oriented than the typical course, but slightly more skill- oriented than the course presented to the G and G+ sections. The S2 instructor, a second year assistant professor, was more traditional in her approach with a greater emphasis on skill development. Subjects in all four sections were given the same textbook assignments.

From observations made during the pilot study, it was suspected that prior calculus experience could be a significant factor influencing student performance and attitude. Prior calculus experience, defined to be experience with at least one-half of a semester of calculus, including the rules for finding derivatives, was

therefore used as a blocking variable. Two levels were defined for the blocking variable: (1) prior calculus (P), and (2) no prior calculus (N).

Subjects participating in the study were those who enrolled in four of seven sections of Calculus I at Western Michigan University, Kalamazoo, Michigan during the Winter semester of 1988. The treatments G, G+, and S1 were randomly assigned to the 9:00 a.m., 10:00 a.m., and 12:00 p.m. sections respectively. Subjects enrolled in the remaining 10:00 a.m. section participated in the S2 treatment. Table 1 presents the initial assignment of subjects to cells in the design matrix.

Table 1

Initial Assignment of Subjects Indicating Cell Sizes

Prior Calculus Experience	Treatments G	G+	S1	S2	TOTAL
Prior Calculus	13	17	14	14	58
No Prior Calculus	18	19	19	14	70
Total	31	36	33	28	128

Under the constraints of Western Michigan University course registration procedures, it was not possible to randomly assign students to sections. It was as-

sumed that the lack of random assignment of subjects to sections introduced no bias. To determine the correctness of this assumption and to account for pre-existing variation in subjects, a pretest of subject precalculus knowledge and a mathematics attitude survey were given during the first week of class to all subjects participating in the study.

The following data were gathered for each subject participating in the study:

1. Precalculus pretest score;

2. Pre-attitude scores, pertaining to: (a) interest in mathematics, (b) interest in taking more courses in mathematics, and (c) perceptions of the nature of mathematics;

3. Prior calculus experience;

4. Scores on selected exam and quiz questions common to all sections, categorized into the following subscales: (a) applied, (b) symbolic routine, and (c) symbolic nonroutine;

5. Scores on a uniform comprehensive departmental final exam further categorized as: (a) questions concerning limits, continuity, and derivatives, and (b) questions concerning precalculus concepts and skills, antidifferentiation, and integration; and

6. Post-attitude scores pertaining to: (a) interest in mathematics, (b) interest in taking more courses in mathematics, and (c) perception of the nature of mathematics.

Additional data was gathered for each subject participating in the G and G+

sections:

7. Scores for graphic questions categorized as: (a) the development of symbol manipulation procedures, (b) the use of symbols and rules as referents for building more abstract systems further categorized as applied, symbolic routine, and symbolic nonroutine; and (c) post-attitude scores pertaining to subject attitudes toward the use and usefulness of a graphic representation system.

Each exam and quiz question was examined for content validity by a group of five faculty members in the Department of Mathematics and Statistics at Western Michigan University. The questions were graded using the Focused Holistic Scoring Point Scale (adapted from Charles, Lester, & O'Daffer, 1987; Geuther, 1986; Malone, Douglas, Kissane, & Mortlock, 1986) described in Appendix C.

Hypotheses

To accomplish the aforementioned goals, the following hypotheses were investigated:

1. There is no significant difference between treatments on: (a) subject performance on exam and quiz questions requiring use of the Cartesian coordinate graphic representation system, including the use of such a representation system in understanding calculus concepts; (b) subject attitudes toward the use and usefulness of the Cartesian coordinate graphic representation system in understanding calculus concepts; (c) subject performance on routine applied, routine symbolic, and nonroutine symbolic calculus exam and quiz questions; (d) subject performance on the departmental final exam and on subscales of questions fur-

ther categorized as: (i) questions pertaining to the concepts of limit, continuity, and derivative, and (ii) questions pertaining to precalculus, antiderivative, and integral concepts; or (e) pre to post changes in subject interest in mathematics, and interest in taking more courses in mathematics.

2. There is no significant difference between subjects with no prior calculus experience and subjects with prior calculus experience on: (a) subject performance on exam and quiz questions requiring use of the Cartesian coordinate graphic representation system, including the use of such a representation system in understanding calculus concepts; (b) subject performance on routine applied, routine symbolic, and nonroutine symbolic calculus exam and quiz questions; or (c) subject performance on the departmental final exam questions and on subscales of questions further categorized as: (i) questions pertaining to the concepts of limit, continuity, and derivative, and (ii) questions pertaining to precalculus, antiderivative, and integral concepts;

3. There is no significant interaction between treatment and prior calculus experience on: (a) subject performance on exam and quiz questions requiring use of the Cartesian coordinate graphic representation system, including the use of such a representation system in understanding calculus concepts; (b) subject performance on routine applied, routine symbolic, and nonroutine symbolic calculus exam and quiz questions; or (c) subject performance on the departmental final exam and on subscales of questions further categorized as: (i) questions pertaining to the concepts of limit, continuity, and derivative, and (ii) questions pertaining

to precalculus, antiderivative, and integral concepts;

To test the hypotheses, two investigations were undertaken, Investigation 1 and Investigation 2.

Investigation 1

Investigation 1 was designed to make comparisons between the G and G+ treatments to which overt instruction of the graphic representation system was given. Since no such instruction was given in the S1 and S2 treatments, they were not included in this investigation. Separate analyses were performed for the cognitive dependent variables, and the affective variables. To test Hypotheses 1a, 2a, and 3a, a two-factor multivariate analysis of covariance was performed. The factors were treatment (G and G+) and prior calculus experience (P and N). To test Hypothesis 1b, χ^2 tests were performed to detect differences in treatment frequency distributions. The pretest of precalculus symbolic and graphic competencies and pre- attitude scales of level of interest in mathematics and interest in taking more courses in mathematics were used as covariates. The dependent variables were:

1. Scores for graphic questions categorized as: (a) the development of symbol manipulation procedures, and (b) the use of symbols and rules as referents for building more abstract systems further categorized as applied, symbolic routine, and symbolic nonroutine; and

2. Post-attitude scores pertaining to subject attitudes toward the use and usefulness of a graphic representation system.

Investigation 2

Investigation 2 was designed to test Hypotheses 1c, 1d, 1e, 2b, 2c, 3b and 3c. Separate analyses were conducted for: (a) exam and quiz subscales; (b) the departmental final exam and related subscales; and (c) the affective dependent variables. To test the hypotheses concerning cognitive variables, two-factor multivariate analyses of covariance were performed. The factors were treatment (G, G+, S1, and S2), and prior calculus experience (P and N). The pretest of precalculus symbolic and graphic competencies and pre-attitude scales of level of interest in mathematics and interest in taking more mathematics were used as covariates. The dependent variables were:

1. Scores on selected exam and quiz questions common to all sections, categorized into routine applied, routine symbolic, and nonroutine symbolic subscales;

2. Scores on the uniform comprehensive departmental final exam and subscales categorized as: (a) questions concerning limits, continuity,and derivatives; and (b) questions concerning precalculus concepts and skills, antidifferentiation, and integration.

To test the hypothesis concerning affective variables, χ^2 and LSD tests were conducted on the data by treatment (G, G+, S1, and S2). The affective dependent variables were:

3. Differences between pre-and post-attitude scores pertaining to: (a) interest in mathematics, and (b) interest in taking more courses in mathematics.

Limitations

Eight limitations of the study were identified:

1. The G, G+, and S1 treatments were randomly assigned to the 9:00 a.m, 10:00 a.m., and 12:00 p.m. sections. No such random assignment occurred with the S2 section.

2. Student selection of calculus sections was not completely random. Some subjects in sections S1 and S2 chose those sections on the basis of prior information about the course instructor.

3. The S1 section met during the noon hour. Nontraditional students tend to enroll more often in noon and evening sections.

4. A description of the nature of the S1 and S2 treatments was based solely on self-report data from the instructors of each section. No observations of these sections were made by the investigator.

5. Irregularities in the administration of the pretest for section S2 may have decreased the validity of the test as a measure of pretreatment differences.

6. Differences in a grading policies among instructors resulted in a higher incidence of missing data for the S1 and S2 sections as compared to the G and G+ sections.

7. Student responses to all exam and quiz questions used in the study were photocopied to allow independent grading by the investigator. The copying process was not completed on one occasion causing missing data for 4 subjects in the S1 and S2 sections.

8. Exam and quiz questions were not administered to subjects in all sections on the same day.

9. Homework was assigned but not graded. Some G+ subjects did not complete the supplemental assignments.

Significance of the Study

The state of calculus course content and instruction has been called into question, due to perceived student difficulties in understanding concepts and also due to the ready availability of computers and advanced calculators. A call for reform of the calculus curriculum has gone out from national professional mathematics and science organizations. Response to this call has taken the form of various projects that focus on the use of the computer as a tool for numerical methods, symbolic manipulation, and graphics.

The literature concerning student difficulties in understanding calculus concepts suggest that students enter the course having a discrete concept of number. This concept of number is incompatible with the study of continuous phenomena. A graphic representation, especially as displayed on a computer, shows promise for the development of understanding of continuous processes.

This study proposes to broaden the knowledge base related to student use of a graphic representation system in understanding the calculus concepts of limit, continuity, and derivative as this representation system and these concepts are developed through the use of computer graphics. Projects of this sort are necessary to determine the best balance of potential uses of the computer as a tool in

mathematics, and especially in the calculus curriculum.

CHAPTER II

REVIEW OF THE LITERATURE

The present study was undertaken to investigate student understanding of the calculus concepts of limit, continuity, and derivative as these are developed through the use of a graphic representation, and in particular, as this representation is presented on the computer. To study student conceptual understanding of calculus concepts, especially as such could be enhanced graphically, a perusal of the current research was undertaken to seek answers to the following questions:

1. What difficulties do students experience in attempting to understand the concepts of calculus?

2. What implications can be drawn from the literature concerning the use of graphs, in enhancing the teaching and learning of mathematics, and, in particular, calculus?

3. What implications can be drawn from the literature concerning the use of computer graphics technology in enhancing the teaching and learning of calculus?

4. What support does the theoretical literature lend to a conceptual approach to mathematics in general, and calculus in particular, especially as such an approach is taken through the use of Cartesian coordinate graphs?

In the following pages, a discussion of the related literature provides partial answers to these questions.

Student Difficulties in Understanding Calculus

Studies were reviewed pertaining to student understanding of calculus concepts, which included descriptions of common errors, misconceptions and difficulties experienced by students. Although the intent of the studies reviewed were often quite different, common threads were evident across studies. Difficulties experienced by students in understanding the concepts of limit, continuity, and derivative are discussed.

As a result of several years of observation of calculus students, Tall and Vinner (1981) reported on common difficulties of university students with calculus. The terms concept definition, concept image, and potential conflict factor were used to discuss the results of these observations. A concept definition is the formal mathematical definition of a concept. Concept image is the particular meaning of a concept perceived by an individual. For a particular concept, the student's concept image and the formal concept definition can be quite different. Potential conflict factors are factors contributing to a student's concept image that conflict with the formal concept definition. They are termed "potential" conflict factors since such factors might not be evoked at times which would cause actual cognitive conflict.

Difficulties cited by Tall and Vinner (1981) in student understanding of calculus arise from discussions of the concepts of limit and continuity. Limits most often are presented in an intuitive manner with informal language as a dynamic process where x approaches a, causing $f(x)$ to approach c." With such develop-

ment, difficulties arise in cases in which $f(x) = c$ since "$f(x)$ approaches c often is interpreted by students as not ever actually reaching c, but rather $f(x)$ is getting as close to c as one would like.

Difficulties with understanding the concept of continuity seemed to arise from the use of particular examples or nonexamples with which students were familiar upon entry into a college-level calculus course. Students indicated that continuous functions could be written with one formula and that the graphs of such functions were all in one piece. Students also confused the ideas of continuity and gradient. All such ideas were part of the students' concept images that were not consistent with the formal concept definition and which were determined to be potential conflict factors.

For the present study, Tall and Vinner's (1981) findings suggest that concepts be developed intuitively but in such a way that they will not cause conflict in future refinements of the students' concept images. Suggestions for such an approach to the concept of limit are given in Tall and Schwarzenberger (1978) in which the limited accuracy of measurement is used as a starting point for the discussion of limit. Such an approach was taken in the experimental study to introduce the idea of tolerance or error ϵ. The mathematical idea of limit then follows from being able to specify any sufficiently small error.

In a related study, Davis and Vinner (1986) reported the results of a study of student conceptions of the concept of limit of an infinite sequence. Fifteen students attending the University High School of the University of Illinois at

Urbana-Champaign participated in a 2-year calculus sequence in which deep understanding of concepts was developed prior to introduction to manipulative techniques. In the study, students had been led through several in-depth discovery sessions to define the concept of limit of a sequence. Several months after such careful instruction, it was determined from each student's responses on a written examination, that the concept definition did not coincide with the student's concept image. Student errors were described as: (a) "naive" misconceptions – misconceptions resulting from resorting to an intuitive concept image that existed in the student before instruction, and (b) oversight or omission errors – errors resulting when some aspect of a concept definition that was known to the student was omitted. Unlike Tall and Vinner (1981), Davis and Vinner (1986) suggest that, regardless of how carefully instruction takes place and how well a student appears to understand a concept, that a parallel incorrect concept image, dating to previous understanding and experiences, exists in the student's mind. These previous concept images might be recalled as readily as a newer concept image that is more consistent with the concept definition.

As also suggested in Tall and Vinner (1981), concept images develop as the result of experiences with mathematics and with non-mathematical situations. Too often, concept images, created as the result of mathematical training, exist separately from those that existed before such training. The two images may not cause conflict for the student because they are recalled at different times and inconsistencies are not discovered.

Davis' and Vinner's (1986) findings suggest that one should attempt to build student concept definitions on concept images that they already possess, if such can be determined. They go a step further than Tall and Vinner (1981) when they suggest that conceptual "noise" – an association of an aspect of the concept image to an idea or quality that is not consistent with the concept definition – can be expected and is probably too resilient to be eliminated. Davis and Vinner agree with Tall and Vinner in their suggestion that attempts can be made to minimize such noise by using language and examples that do not evoke strong non-mathematical images or experiences that are inconsistent with the mathematical definition. Davis and Vinner also suggest using varied examples to minimize students developing mathematical concept images that are incorrectly generalized.

A different approach is taken by Confrey (1980) who discusses the theory of conceptual change as it applies to mathematics, and in particular, as it applies to students' conception of number in the transition from algebra, geometry, and trigonometry to calculus. While the approach to calculus taken by Tall and Vinner (1981) and Davis and Vinner (1986) treats calculus as a fixed set of truths, the theory of conceptual change contends that concepts are not static or final. They change over time as more information about these concepts becomes known. Students' knowledge about a particular concept undergoes the same sort of growth, from naive conceptualizations to more specific.

If one accepts the theory of conceptual change as it applies to mathematics, then Confrey (1980) suggests that a theory-proof or inductive approach to teach-

ing mathematics is inadequate. The development of knowledge, rather than its transition or acquisition, needs to be emphasized. In contrast, the approach taken by Tall and Vinner (1981) suggests that teaching of mathematics is to promote acquisition of knowledge.

To illustrate an application of the theory of conceptual change to mathematics, Confrey (1980) identified the development of student understanding of the concept of number as an area in which students experience great difficulty. Students' concept of number as a result of algebra, geometry, and trigonometry courses is discrete; the analogy given is that numbers are like an infinite deck of cards, it is always possible to insert one between two others. Although students have had various experiences with continuous quantities such as time, motion, and area, these experiences are generally non-mathematical or are often discretized when studied as mathematical entities. The first time that most students are confronted with the need for a continuous concept of number is in calculus when the concept of limit is introduced.

To study student understanding of the concept of number, as it evolved from discrete to continuous, Confrey (1980) used a clinical interview method with 11 student volunteers from a college mathematics course at Cornell University. Four of the students were chosen for the final analysis. The study took place over a 3 week time period. Subjects met with Confrey three times each week, once in individual clinical interviews, and twice in group sessions. At these times, subjects were asked to respond to a total of eight problems that were designed

to investigate their understanding of infinite processes and the concept of number resulting from such processes.

Confrey (1980) determined in her investigation that responses to problems were consistent by subject. Each subject, recognizing contradictions across problems, was determined to resolve the contradictions. Subjects differed in their ability to recognize such contradictions. Some subjects attacked each problem as if it were new, not seeing interrelationships among the problems. Such subjects did not use the results of one problem to make sense of another. Subjects varied in their use of experiences and relevant knowledge from outside of mathematics in solving the problems.

Confrey (1980), and Tall and Vinner (1981), and Davis and Vinner (1986) each identified difficulties with student understanding of continuous processes as these were exhibited through difficulties with understanding the concept of limit. Confrey's study diverged from the others in that the language and symbolism of limit were not used and the study did not take place in a calculus setting. Confrey's results also had somewhat different implications for the present study.

Confrey's (1980) findings suggest that student difficulties with a continuous concept of number might be remediated through the use of a continuous representation of number. For example, students might be aided in developing a less discrete concept of number through the use of graphs. Computer-generated graphs might be particularly useful since familiar function such as $f(x) = \frac{x^2}{x}$ could be graphed on the computer and magnified around the point of disconti-

nuity. A discussion of the computer's response to division by zero – resulting in error – coupled with students' discovery that a hole is not visible in a computer graph of the function even after several magnifications might be are appropriate starting point in developing students' ideas about the limiting process and about the concepts of limit and continuity.

Confrey's (1980) findings also suggest that students vary greatly in the amount of concrete experiences that they recall and relate to ideas being studied in mathematics. Some students see mathematics as related to their world outside of mathematics and call upon such experiences to make sense of the mathematics they are studying. Other students see mathematics as sterile and unrelated to their experiential world outside of mathematics. Such findings suggest that the instructor should make overt attempts to relate mathematical ideas to familiar experiential settings and to draw upon such rich associations in the development of mathematical ideas. Students also vary in their attempts to relate experiences and knowledge from within mathematics and tend to solve problems in isolation of other related problems. For this study, an attempt should be made to call students' attention to the relationships between and among concepts and processes encountered in problems in calculus.

In a research study conducted by Orton (1983a, 1983b), a clinical interview method was used to investigate student understanding of elementary calculus. The study involved 110 students across the age range of 16 to 22 years. All subjects were students whose major course of study was mathematics, either in

sixth forms (comparable to high school) or in college. Subjects were interviewed on two separate occasions for one hour each session. Subjects responded to 38 items pertaining to various skills and concepts of calculus.

Orton (1983a, 1983b) discovered that many of the students' difficulties with differentiation and integration were based in their difficulties with limits. He observed that the discussion of limits in calculus was the first real experience subjects had with a continuous number system, consistent with the observations of Confrey (1980). He observed that in previous mathematics courses, experiences with continuous ideas like time and area were reduced to discussions of discrete ideas. Orton suggested that a gradual build-up of continuous processes leading to limit ideas in school mathematics might remedy some of the difficulties students experienced with the limit concept in calculus.

Orton also discovered that subjects appeared to be learning techniques without understanding the underlying meaning of the result of the technique, or why a particular technique is used in a given problem. He described subject errors as most often arising from lack of deeper understanding of the concepts involved. Further, subjects' lack of facility with algebraic symbols and lack of familiarity with calculus symbols caused difficulties in their performance of calculus techniques.

To remedy student difficulties with understanding concepts of calculus, Orton (1983a, 1983b) suggested that graphs, based on data from real-life situations, should be used and that ideas such as slope of a tangent, rate of change, and area

under a curve be developed and explained in terms of graphs and their applied interpretations. As a result of the study Orton stated that, "rules without reasons cannot be justified" (1983a, p. 10) and "it is wrong to attempt to introduce calculus without a long and persistent study of graphs and rate of change" (1983b, p.243).

Orton's (1983a, 1983b) findings are consistent with those of Tall and Vinner (1981), Davis and Tall (1986) and Confrey (1980). They suggest that student understanding of calculus should be based on their previous experiences with real-life situations. Orton goes further to suggest that these real-life situations should be represented with graphs. He also suggests that students should not be taught calculus techniques void of the conceptual understanding behind them. Use of algebraic symbols in introductory stages should be minimized to prevent student difficulties with algebraic manipulations from interfering with deeper conceptual understanding.

One of the goals of a research study conducted by Geuther (1987) was to develop an error classification scheme to categorize student errors in a first semester calculus course. The error classification scheme presented by Geuther was an extension and adaptation of several such schemes found in the literature, including that used by Orton (1983a, 1983b). Geuther analyzed 120 calculus unit tests, categorizing the 600 student solutions using the existing schemes as a framework. Eight student volunteers were interviewed concerning their solutions to comparable exams to determine their solution processes and errors. The error categories

determined through this process are:

1. "Wrong reconstruction of detail" – some piece of an appropriate algorithm is not reconstructed correctly;

2. "Wrong analogy type" – student invents an algorithm that is appropriate for one situation and generalizes it incorrectly to another;

3. "Misidentification" – student misapplies an algorithm, ignoring a constraint of the algorithm;

4. "Lack of connections" (piecework)" – student approaches a problem in various ways giving very little evidence of an overall plan, student tends to include everything he/she can think of without determining if the information written is relevant to the problem;

5. "Misinterpretation of symbols" – student gives the symbol a different meaning from its conventional meaning; and

6. "Faulty computation" – errors in arithmetic or algebra are made in the solution process. (Geuther, 1986, pp. 116-118)

The categories above were used to aid students in self-remediation of errors by having them reflect on the solution processes they used. The exams for which the error classification scheme was devised were generally algorithmic in nature. Error patterns noted for such exams tended to be errors in performing techniques.

Geuther (1986) observed that errors made by subjects were often overgeneralizations. Subjects applied techniques in situations where they were inappropriate due to a lack of attention paid to: (a) the conditions given in the problem or (b)

the hypotheses or initial conditions required by the algorithm. Calculus symbols were often used almost arbitrarily. Subjects lacked commitment to their solution processes. Geuther reported that the slightest question concerning the subject's solution process resulted in the subject changing his/her mind. Consistent with Orton (1983a, 1983b), Geuther suggested that each of the above conditions indicated that students were performing techniques without the benefit of underlying conceptual understanding. For the present study, Geuther's and Orton's findings suggest that understanding should be based on conceptual foundations that allow students to determine where an algorithm is appropriate and when the conditions of a theorem have been met. Meaning should be created for symbols with respect to possible origins and uses.

Geuther's (1986) error classification scheme provided insight into the nature of algorithmic calculus errors. In the present study, this scheme together with a problem-solving evaluation scale, provided a useful scale for rating student progress and understanding of calculus concepts and techniques.

Another study pertaining to student errors and misconceptions in calculus was conducted by Davidson (1980). Davidson reported on a pilot study conducted to determine the effects of a small group guided- discovery approach to calculus on student understanding. Twelve students participated in the study which took place in a first and second semester calculus course. Davidson provided questions to be investigated by subjects. He talked with the entire class for a maximum of 10 minutes each day. Subjects worked in groups of 4. With limited guidance from the

teacher, subjects "formulated definitions, stated theorems, provided the theorems, constructed examples and counterexamples, and developed techniques for solving classes of problems" (Davidson, 1980, pp. 34-35). Skills were developed through problem situations and were based on conceptual understanding. Subjects were provided hints when they appeared to need help.

Davidson (1980) reported a variety of subject difficulties with calculus, many of which have also been identified by Orton (1983a, 1983b) and Geuther (1987). Common errors included: (a) misuse or omission of symbols, (b) computation or algebraic errors, (c) incorrect applications of basic formulas, (d) overgeneralizations, and (e) logical errors. Subjects tended to forget definitions, even those used often. As also noted by Confrey (1980), subjects often treated problems as new problems, not relating them to previous work. In trying to solve problems or in doing proofs using the definition of the limit of a function, "students encountered great conceptual and technical difficulties" (Davidson, 1980, p. 46). Subjects' graphical examples were usually continuous, differentiable functions with a finite number of optimal points. They indicated that they believed discontinuous functions were contrived.

Subjects in Davidson's (1980) study experienced many of the same errors reported by Davis and Vinner (1986), Orton (1983a, 1983b), and Geuther (1987). For the present study, Davidson's (1980) error identifications suggest the following: (a) a broad range of continuous, discontinuous, differentiable, and non- differentiable functions would be useful in helping students devise test cases to determine

the truth value of theorems and identities, especially as such functions are represented graphically; and (b) similarities and differences between new ideas and previous work should be called to students' attention to prevent them from "reinventing the wheel". As also noted by Davis and Vinner (1986), Tall and Vinner (1981), and Confrey (1980), the concept of limit was difficult for students. Students pre-existing intuitive ideas interfered with the formal concept definition.

Summary

From the studies discussed above, several difficulties were repeatedly observed in student understanding of calculus. Geuther (1987) and Orton (1983a, 1983b) reported that students seemed to be approaching problems with very little understanding of the basis for their techniques or the results of their solutions. In all of the other studies, conceptual bases were provided for student understanding. Errors resulting from applying rules without understanding were not as prevalent, although Davidson (1980) noted incidences of misapplications of formulas and overgeneralizations (Confrey, 1980; Davis & Vinner, 1986; Tall & Vinner, 1981). Davidson (1980), Geuther and Orton, both observed that students' use of symbols did not seem to be based in understanding. Notation was used almost arbitrarily. Davidson and Geuther noted that students tended to overgeneralize and use concepts and processes where they were not appropriate. Confrey and Davidson observed that students often failed to see the relatedness of similar problems and processes, attacking each one in isolation. Confrey further suggested that students varied greatly in the amount of non-mathematical experiences that they

were able to relate to the mathematics being studied, with many students seeing no relationship at all.

In all of the reported studies, indications of student difficulty with limit were evident. Confrey (1980) and Orton (1983a, 1983b) suggested that such was due to students' discrete sense of number upon entry into calculus. Davis and Vinner (1986) and Tall and Vinner (1981) suggested that student difficulties arise from their individual concept images, created through mathematical and non-mathematical experiences, that are inconsistent with the formal concept definition. Davis and Vinner suggest further that correct concept images can be formed and can exist in an individual's memory without ever conflicting with a parallel incorrect concept image. On occasions when the concept definition is to be recalled, the older concept image is as likely to be recalled as the more correct concept image even when the former has been very carefully built.

Suggestions for the present study include a conceptual approach to calculus that is based on familiar non- mathematical and mathematical experiences. Understanding of a variety of functions should be developed and represented graphically to provide examples and nonexamples for theorems and definitions, as well as to help prevent overgeneralizations of theorems and processes to situations in which they are not appropriate. The use of graphs should be considered to provide a continuous representation in which to develop the concept of number, and subsequently, the concept of limit. Relationships between and among processes and concepts should be called to students' attention to aid in a coherent development

of theory since such a development does not seem to occur naturally.

The Use of Graphs in Understanding Calculus

Studies concerning student understanding and difficulties with calculus suggest that the Cartesian coordinate graphic representation system is an appropriate symbol system with which to represent calculus ideas. It was noted that student difficulties are often based in their concept of number. Upon entry into calculus, a student's concept of number is most often discrete (Confrey, 1980; Orton, 1983a, 1983b). It is not until the introduction of the concept of limit that students are presented with infinite processes requiring understanding of a continuous number system. Graphs were suggested as useful tools in modelling the continuous number system of calculus (Davidson, 1980; Orton, 1983b).

To ascertain the advantages and disadvantages associated with the use of graphs in understanding calculus, research studies pertaining to student understanding of Cartesian graphs in school mathematics and calculus were reviewed. The results of these studies are discussed, especially as they pertain to the teaching and learning of calculus.

Use of Graphs in Mathematics

The results of the National Assessment of Educational Progress conducted in 1986 indicate that 17 year old students experienced difficulty in using symbolic and graphic representations (Brown et al., 1988b; Dossey, Mullis, Lindquist, & Chambers, 1988). Only two-thirds of 11th-grade students could identify the graph

of the solution of the inequality $3x \geq 15$. One-third of the 11th-grade students could identify the inequality associated with the graph of $x \geq y$. Over 75% of the 17-year-olds could identify a point in the third quadrant of a coordinate system. However, performance dropped 20% for students who had completed at least one algebra course, when asked to identify the coordinates 2 units to the right and 3 units up from a given point. Only 51% of the 17-year-olds performed at or above Level 300 – Moderately Complex Procedures and Reasoning – which included use of graphs for more sophisticated numerical reasoning. In geometry, results indicated that a student performance was better if an item could be solved visually than if it required more abstract thinking. These results indicate that, in general, 11th-grade students have not developed competence with a Cartesian graphical representation system, although the potential for understanding seems to be enhanced when visual tools are available.

Few studies have focused on determining student understanding of Cartesian coordinate graphs. A study conducted to investigate the difficulties that students experience in learning algebra (Wagner, Rachlin, & Jensen, 1984) found deficiencies in student understanding of graphs. Eight 9th-grade algebra students were each interviewed eight times. In determining students' abilities with generalization and reversibility, results showed that although students were able to graph linear equations using various methods, they were unable to find values for x and y that satisfied the equation $3x - 5y = 2$ using the graph of this function. "Students had no real notion what the graph of a function represents" (Wagner et

al., 1984, p. 35). In general, student performances were poor on all tasks involving the interpretation of graphs. The suggestion was made that graphs generated on a computer would be useful in allowing students to examine the effects that changes in the parameters m and b of the equation $y = mx + b$ have on the graph of this function. Also, the computer's capacity to plot many points quickly was indicated as an aid in developing the concept of a line graph as an infinite collection of points. The latter suggestion was consistent with suggestions made by Orton (1983a, 1983b) and Davidson (1980) in aiding student development of a continuous concept of number. Difficulties in graphing recognized at the college level, by Orton and Davidson, were evident in 9th-grade students.

Kerslake's study of student understanding of graphs (cited in Janvier, 1978) investigated student performance in the following five categories: (1) reading graphs and plotting points, (2) continuity or density of the real number system, (3) rate of change and gradient, (4) straight lines and equations, and (5) interpretation. Data provided from administration of a 17-item questionnaire to 600 students in the age range of 12 to 15 years indicated that students were able to read graphs of integer data (90 to 95% mastery), and interpolate data from graphs (70% mastery) but had a difficult time plotting points and reading graphs with decimal data (30 to 35% mastery). Students tended to view graphs of lines or curves as full of points, consistent with the discrete concept of number observed elsewhere (Confrey, 1980; Orton, 1983a, 1983b; Wagner et al., 1984). Rate of change and gradient ideas were only tested by one item and were generally mis-

understood. Students were unsuccessful in determining equations given graphs of lines, and graphs were misinterpreted more often than they were interpreted correctly. Kerslake's findings were consistent with those alluded to in Wagner et al. (1984).

In an extension of Kerslake's work (cited in Janvier, 1978), Janvier (1978) conducted a study to determine student difficulties with interpretation of graphs. In this study, he presented an extensive list of global features of graphs – qualities displayed by graphs as they were viewed through intervals rather than point-by-point. The global features identified by Janvier (1978) are: (a) maximum, minimum; (b) intervals over which a function is increasing or decreasing; (c) intervals over which a function is greater than, less than, or equal to a specified value; (d) drops and rises of curves between two given points; (e) the family of curves obtained by translating a given curve upward, downward, or sideways and the consequences of such translations; (f) discontinuities; (g) cyclic patterns; (h) constant rates of change; (i) the nature of non- constant rates of change; (j) symmetry; (k) extrapolation and interpolation, asymptotic behavior; (l) dispersion of a graph and differences in shape measured by area; and (m) many curves of the same graph. Most of these ideas are studied in calculus, suggesting that a graphic symbol system is a natural system with which to represent calculus ideas.

In Janvier's (1978) study of student interpretations of graphs, 20 students in the age range of 12 to 15 were interviewed to determine their understanding of information represented by Cartesian graphs. Janvier identified several difficulties

experienced by subjects as they interpreted graphs in terms of situations. Subjects tended to interpret graphs point-wise rather than globally. Such subjects "discretized" graphs, viewing them as a collection of points. Their interpretation of the real number line was consistent with that observed by Confrey (1980), Kerslake (cited in Janvier, 1978), and Orton (1983a, 1983b).

Subjects who were beginning to view interval feature of graphs experienced other difficulties. They confused the ideas of increasing (decreasing) with higher (lower) values. For example, in interpreting a distance vs. time situation, "faster" was interpreted as "higher" rather than "steeper" with respect to the graph modeling the situation.

In interpreting a graph with respect to the situation it modeled, students were more successful for situations with which they were most familiar. Situations regarding distance vs. time were generally easiest for students to interpret graphically, although visual distractors often prevented students from interpreting graphs correctly. For example, a concave up, increasing graph would be interpreted as modeling a cyclist's distance vs. time function as he or she ascends a hill, disregarding the interpretation of steepness as velocity.

Janvier (1978) noted that interval interpretations of graphs are rarely fostered in school texts. He suggested that students would benefit from inclusion of the study of global features of graphs in addition to the point- wise features that are already addressed.

For the present study, Janvier's (1978) findings suggest that graphs provide a

natural representation with which to present the ideas of rate of change, especially non-constant rate of change. He suggests that students' graphing experiences are often in interpreting local rather than global features of graphs, and that visually distracting features are prone to misinterpretation especially when based on unfamiliar situations. For the present study, the use of graphs will require remediation of the difficulties with a discrete vs. continuous number system, and a concentration on the global vs. local properties of functions as represented by graphs. Distance vs. time situations are familiar to students and seem to have promise for aiding the interpretation and subsequent understanding of graphs and the ideas they embody.

A study was conducted by Ponte (1984) to identify difficulties of high school and preservice teachers in functional reasoning and in reading, constructing, and interpreting Cartesian graphs. The difficulties reported are similar to those identified by Janvier (1978). Subjects for the study were 179 eleventh grade students, 52 preservice elementary school teachers, and 31 preservice secondary school teachers. Subject responses ($N = 262$) to a multiple-choice test and interview tasks ($n = 26$) concerning graph reading, interpretation of variation, and interpretation of variation in variation were analyzed.

Problems involving variation in variation – rate of change, smoothness, and continuity – were the most difficult for all subjects. Subjects displayed a pointwise thinking process which hampered their approach to such problems, a difficulty noted by Janvier (1978), Kerslake (cited in Janvier, 1978), and Orton (1983a,

1983b). Interpretation of variation problems was more difficult than graph-reading problems for 11th-graders, but not for preservice teachers. Subjects successfully used graphs to gather numeric data, although often inaccurately. However, they experienced difficulty in interpreting underlying relationships.

Ponte's (1984) findings reiterated those of Janvier (1978). His suggestions for classroom teaching were similar. Implications for the present study are those indicated earlier, although reinforced by this study.

Use of Graphs in Calculus

The use of graphs has been suggested as being important in the development of conceptual understanding of calculus ideas (Davidson, 1980; Janvier, 1978; Orton, 1983a, 1983b). Professional mathematical organizations have also suggested use of a graphical representation in the teaching and learning of calculus (Douglas, 1986; National Council of Teachers of Mathematics, 1987). However, very few studies have been conducted that specifically address student understanding of calculus concepts through the use of graphs.

Patterson (1983) conducted a study to explore the problem-solving behavior of calculus students on problems requiring the use of graphs. Subjects for the study were 34 Calculus II and Calculus III students from two universities. Each student was interviewed as he or she solved six mathematics problems requiring precalculus graphing skills and knowledge of derivatives and integrals at the level of a first course in calculus.

One of the integration problems serves as an illustration of the research

method employed. In this problem, subjects were asked to find the area between the curves of the functions $y = x^2$ and $y = x^3$ between $x = -1$ and $x = 2$. These functions intersect twice on the given interval. Solution to the problem requires determination of the intersection points and of which function dominates on the resulting intervals. Subjects were observed as they solved the problem. In their solution process, subjects were prompted to draw a graph if they did not do so spontaneously. Patterson (1983) observed that none of the given problems were solved without the use of a graph. If the graph drawn was not accurate, and the solution was still incorrect, subjects were prompted to draw a more accurate graph. If the subject's graph was still not accurate and the subject was not able to solve the problem, then the investigator provided the subject with an accurate graph.

The problems in the study were well-suited to graphic solution. Patterson (1983) found that, an average of 72% of the subjects drew graphs spontaneously. However 38% of the subjects drew graphs that were not sufficiently accurate to solve the problem, even when prompted to do so. On the average, only 32% of these second and third semester calculus students solved the problems correctly, although the problems were Calculus I and precalculus problems.

Patterson (1983) indicated that subjects did not often plot points to produce an accurate graph, even when prompted to do so. Several subjects indicated that they had been told not to plot points in the production of graphs for use in calculus courses. Without accurate graphs on which to rely, subjects (wisely) tended not to

refer to their graphs in obtaining solutions. Subjects who showed understanding of the information given in graphs tended to construct them accurately and to use the information given. Such subjects were often able to determine, from the problem, the amount of accuracy needed to solve the problem correctly and then draw graphs accordingly.

Patterson (1983) suggested that two methods of graphing had been "learned" by students, one in which points were plotted to produce a graph – algebra method, and one in which rough sketches were produced without plotting points – calculus method. Students did not coordinate the two methods, and seemed to view them as appropriate only in the context in which they had been learned. For the present study, efficient methods of graphing that allow rough sketches to be drawn guided by a few plotted points should be presented and their use encouraged. Patterson (1983) suggested that students observe graphs being drawn by teachers, and view graphs in their textbooks that represent calculus ideas, but tend not to use them themselves. If taught explicitly, Patterson's findings suggest that graphs are potentially beneficial in helping students solve certain types of calculus problems.

A tangentially-related study by Ferrini-Mundy (1987) was conducted to determine the effect of spatial training on student understanding of integration concepts related to area and solids of revolution. Subjects for the study were 250 students randomly drawn from the Calculus I enrollment at the University of New Hampshire. Subjects were randomly assigned to three treatments: (1) audio-visual

(AV) spatial training; (2) audio-visual, tactile (AVT) spatial training; and (3) a control. Subjects in the AV and AVT training groups participated in individual training sessions in which they responded to six modules involving two- and three-dimensional training tasks. Ferrini-Mundy determined that male subjects outperformed female subjects on a spatial posttest, although initial spatial differences were not detected. She suggested that the final differences might have been due to a combination of the initial spatial training that occurred while taking the spatial pretest and concurrent coursework in physics and engineering in which graphs were used. The spatial training more directly influenced the calculus understanding of female subjects. Females outperformed males on an exam pertaining to work, area, and solids of revolution.

Ferrini-Mundy (1987) did not find differences in ability between males and females. She did suggest that females seemed to benefit more from the spatial training than did males. For the present study, Ferrini- Mundy's findings suggest that the use of visual aids might positively influence student understanding of calculus concepts.

Summary

Many of the qualities of functions that are studied in calculus can be embodied with Cartesian coordinate graphs (Janvier, 1978; Ponte, 1984). Graphs have been suggested as a natural representation system with which to develop and enhance understanding of the concepts of calculus (Davidson, 1980; Orton, 1983a, 1983b). However, student experiences with graphs do not often go beyond plotting

and connecting points (Brown et al., 1988a; Dossey et al., 1988; Janvier, 1978; Kerslake, cited in Janvier; 1978; Ponte, 1984; Wagner et al., 1984). Students are unable to determine the equations of even the simplest linear functions given graphically, and are often unable to interpret graphs in terms of the situations they represent. In school mathematics, graphs are introduced, developed, and produced point-by-point. Students consequently perceive graphs as discrete collections of points (Kerslake, cited in Janvier, 1978; Ponte, 1984; Wagner et al., 1984). In interpreting graphs, they often fail to see graphs globally, as possessing interval qualities (Janvier, 1978). Such concrete views cause difficulties in the use of graphs in calculus where they represent continuous ideas (Confrey, 1980; Davidson, 1980; Orton, 1983a, 1983b).

The Effect of Use of Computer Graphics Technology in Calculus

It has been suggested that student difficulties in understanding the concepts of calculus seemed to be based in their discrete concept of number (Confrey, 1980; Orton, 1983a, 1983b). Because of their capacity to represent continuous phenomena, Cartesian coordinate graphs were suggested as a natural symbol system with which to represent calculus ideas, thereby alleviating some of the difficulties arising through use of other representation systems (Davidson, 1980; Orton, 1983a, 1983b; Ponte, 1984). However, difficulties in using and interpreting graphs also seemed to be based in students' discrete number concept. Students often interpreted graphic information point-wise rather than globally (Janvier, 1978;

Kerslake, cited in Janvier, 1978; Ponte, 1984). It was indicated that such difficulties probably arise from student experiences with graphs in school mathematics. Students are taught to plot points, to connect these points to sketch graphs of functions, and to determine coordinate of points (Janvier, 1978; Ponte, 1984; Wagner et al., 1984). Each of these processes reinforces the point-wise interpretation of graphs. Global properties of graphs are rarely addressed in school mathematics where graphs are, too often, produced by hand. As a result, graphs have been the end-product of an exercise, rather than a tool to be used in analyzing problems and understanding concepts (Heid, 1984).

In a computer graphic environment, the difficulties that arise from a point-wise interpretation of graphs might possibly be alleviated. If graphs are perceived as continuous, then perhaps use of graphs will promote understanding of a continuous concept of number, and in turn alleviate some of the difficulties students encounter in the transition from the study of a discrete number system to the study of a continuous number system.

To determine the potential benefits and uses of computer graphs in a first course in calculus for developing understanding of concepts, both computer graphic curriculum projects and studies concerning student understanding through the use of computer graphs were reviewed.

Computer Graphic Curriculum Projects

A notable curriculum project undertaken by Waits and Demana (1987b) is the Ohio State University Calculator and Computer Precalculus Project (C^2PC).

Instructional materials incorporating full use of computer and calculator graphics capabilities in developing the concepts of function, analytic geometry, and trigonometry have been developed and are currently in the piloting stage.

Objectives of the curriculum project (Waits & Demana, 1987b) include: (a) to study the behavior of functions and relations through their graphical representations, (b) to graphically determine solutions to systems of equations and inequalities, (c) to graphically determine maxima and minima of functions, (d) to graphically explore solutions to real world problems not accessible to students through paper-pencil algorithms, and (e) to provide a graphic as well as algebraic representation of concepts while exploring the connections between these.

Waits and Demana (1987a) have written very powerful computer graphics software, *Master Grapher*, which allows the behavior of functions and classes of functions to be studied, interrelating graphic and symbolic representations. They further suggest several investigations which can be undertaken through the use of the software: (a) determining the behavior of a function on small intervals of x, (b) determining end behavior of a function, (c) determining extrema and zeroes graphically, (d) solving inequalities, and (e) solving realistic problems with "messy" formulas. While these suggestions are made in a precalculus context, they concern a subset of calculus ideas. Determining behaviors of functions on small intervals and as $|x|$ grows large provide an intuitive background for the discussion of limits at a point and limits at infinity. Determining zeroes and extrema are also calculus concerns. The call for incorporation of more realistic problems into

the curriculum is aided by use of the computer to handle messy numbers.

Another curriculum project, which makes heavy use of computer graphs is being undertaken in Sweden (Brolin & Greger, 1987). The ADM Project, *Analysis of the Role of the Computer in Mathematics Teaching* was initiated to determine: (a) how computers can be employed in the teaching of mathematics, (b) to what extent students should learn to use the computer as a tool, (c) the goals and teaching methods for mathematics courses if the computer is used fully, and (d) the best method of introducing students to the limitations of the computer.

Suggested goals for teaching calculus are: (a) to develop the concept of function, (b) to familiarize students with properties of certain functions and classes of functions, and (c) to encourage use of functions in formulating, preparing, and solving certain kinds of problems (Brolin & Greger, 1987). Skills taught to achieve these goals, usually seen as central to the teaching of calculus, are: table building, curve sketching, determining zeroes of functions, investigating growth properties, determining extrema, and evaluating areas. In a computer environment, the necessity of mastering these skills by-hand can be reduced in favor of computer-generation of all but basic techniques. The results of these techniques can then be means through which higher level problem solving is pursued rather than the goal of calculus instruction (Brolin & Greger, 1987). Problems can be less focused, permitting various interpretations, each of which might be explored through computer use. One such problem suggested by Brolin and Greger, is a general form of the familiar box problem as follows: "From a square sheet of

metal, measuring 7m × 7m, four congruent squares are removed at the corners and a box is formed. Investigate this situation!" (Brolin & Greger, 1987, p. 8)

Several investigations can take place from this single problem statement, including determining maximum volume or minimum surface area or a combination of both given a cost estimation for materials and/or intended contents. Problems in a course which assumes full use of the computer do not have to be as precisely focused as in traditional courses where the computer is not used. Such situations allow for more realistic uses of calculus ideas (Brolin & Greger, 1987).

In a report of a conference concerning use of information technology in mathematics education sponsored by the National Science Foundation (Fey et al., 1984), the impact on curricula of full use of computer technology was discussed. In particular, goals and recommendations for the teaching of calculus were addressed. The intended calculus curriculum, identified in the report, seeks to: (a) provide students with problem-solving tools that are applicable in many other disciplines, (b) extend geometric and algebraic intuition and skills acquired in earlier study, and (c) trace the history of critical mathematical ideas. The focus of the course is on functions. Limits were recognized as difficult for students to grasp and the suggestion was made to build intuition with limits through an intuitive approach to derivative. In this approach, rates of change would be investigated over intervals of decreasing size through applications and graphs. It was suggested that calculations of limits be delayed and that the concepts of derivative and integral be developed through applications. It was suggested that derivatives be intro-

duced as rates of change which can be modelled graphically. This should lead to a discussion of secants over small intervals, and finally to the definition of the derivative as the slope of a tangent and as instantaneous rate of change. Integrals should be motivated as a search for areas under various curves followed by a discussion of Riemann Sums. The Fundamental Theorem of Calculus should be introduced before proceeding with discussion of techniques of integration. Graphs and applications should be used often.

The uses of computer-generated graphs suggested by Fey et al. (1984) were similar to those described by Brolin and Greger (1987), Waits and Demana (1987b), and Demana and Waits (1987, 1988b). With computer-generated graphs, the focus can move to interpretation of a curve's intercepts, asymptotes, slopes, and concavity, and the interpretations of those of related functions (Fey, et al., 1984).

Lucas (1987) suggests several feasible entry points for incorporating computer graphics into the usual calculus course. Suggestions include use of graphs to: (a) observe the behavior of functions at a point and as $|x|$ grows large – to graphically support the development of the concepts of limit and continuity, (b) display the slopes of tangents and the relationships between a function and its first and second derivatives, (c) display extrema in a discussion of optimization, (d) solve inequalities, (e) display functions with "nasty" coefficients, (f) determine roots and solutions to systems of equations, (g) display the results of use of Riemann sums in estimating area under curves, (h) demonstrate the closeness of fit of func-

tions with their Taylor polynomial approximations, (i) display parametric and polar functions, and (j) display various periodic functions.

Summary of Curricular Suggestions

Curricular suggestions indicate that the availability of computer graphics software allows a rethinking of the way graphs can be used in promoting understanding of mathematical concepts. The global features of graphs become prominent when graphs are no longer painstakingly produced point-by-point by hand (Tall, 1986a). The global features of graphs listed by Janvier (1978) can be displayed in either a static or dynamic mode with computer graphics. Basic functions are easily and efficiently displayed graphically on the computer. Global characteristics of functions such as maximums, minimums, domain, range, periodicity, symmetry, and rate of change can be discussed for each function (Brolin & Greger, 1987; Demana & Waits, 1987, 1988a, 1988b; Fey et al., 1984; Lucas, 1987; Waits & Demana, 1987b). Relationships between families functions can be studied through changing a single parameter several times, and observing the effect of such a change on the graph of the parent function (Douglas, 1987; Lucas, 1987; Wagner et al., 1984). The characteristics of the parent functions can then be generalized to other members of the family (Fey et al., 1984; Lucas, 1987). Studies employing a subset of these and other uses of computer Cartesian graphs in calculus were reviewed.

Studies Implementing Use of Computer Graphics in Calculus

A landmark study incorporating full use of the computer was conducted by

Heid (1984) to determine the extent to which conceptual development was aided by previous or concurrent mastery of skills. Heid exploited full use of the computer to teach a resequenced applied calculus course in which conceptual understanding was emphasized, and skill development was delegated to the last 3 weeks of the course. During the first 12 weeks of the course, the instructors and students in two small experimental sections ($n = 19$ and $n = 20$) of applied calculus at the University of Maryland used a microcomputer to perform most of the algorithms that were the focus of the regular course. Experimental sections varied slightly in instructional approach. In one section, basic examples of each of the algorithms were demonstrated during the course presentation. In the other section, no such symbolic illustration concurred until the last 3 weeks of the course. Comparative data were collected for subjects in both of the experimental sections and in a large lecture ($n = 100$) section of the same course. Data consisted of audio-tapes, copies of student assignments, exams, and quizzes, observation notes, and questionnaire responses. Interview data were gathered from 5 large lecture subjects and 15 subjects from the experimental groups.

Results from interviews and exam and quiz questions indicated that students in the experimental sections displayed much deeper conceptual understanding than large lecture students. All students, experimental and large lecture, displayed lack of understanding of the concept of limit and the idea of derivative as modelling rate of change. Performances on a uniform departmental final exam which, for the most part, tested skill development, were comparable for experimental and

large lecture students. Prior or concurrent skill acquisition did not seem to be a necessary factor in development of conceptual understanding. On the other hand, conceptual understanding did appear to contribute to skill acquisition for students participating in this study.

In Heid's (1984) study, graphs played a central role. Students viewed a large variety of computer-generated graphs, studying similarities and differences between related functions and connections between graphs and other representations, including applied interpretations of graphs. Heid suggested that at least some of the deep conceptual understanding exhibited by students in experimental sections might be attributed to their ability to reason from multiple representations. Her results suggest that further research to determine the contribution of the use of graphs to develop conceptual understanding is warranted.

Such a study was conducted by Tall (1986a). Tall studied the effectiveness of use of interactive computer graphics to develop understanding of the concept of derivative. He describes the use of the computer as providing a "generic organizer" – "An environment enabling the learner to explore examples to mathematical processes and concepts, providing cognitive experience to assist in the abstraction of higher order concepts embodied by the organizer" (Tall, 1986a, p. vii). The generic organizer is the computer equivalent of manipulative materials such as Dienes' blocks and Cuisenaire rods. It provides an environment which allows the learner to play with various aspects of a concept, and to begin to distinguish aspects which are relevant from those that are not. In his study, Tall investigated

student understanding of the concept of derivative as it was introduced through use of the computer graphic programs – generic organizers – MAGNIFY and GRADIENT written by Tall. MAGNIFY allows the behavior of graphs to be investigated under high magnification. GRADIENT has two subroutines. One of these demonstrates the numerical limit of the gradient of secants tending to the tangent of a curve. The second subroutine draws successive "tangents"– actually secants containing the points $(x,\ f(x)), (x+c,\ f(x+c))$ for a small fixed value of c – along a user-chosen curve while concurrently plotting the gradient function.

Tall's (1986a) study was conducted with 112 sixteen-year-old students from various sixth form institutions. Forty-three students participated in the experimental sections in which the computer was used. Subjects were exposed to the computer graphics software in class and in small group sessions. The amount of computer exposure among subjects varied widely, both in class and during small group sessions.

Results of Tall's (1986a) study suggest that subjects in the experimental sections experienced fewer difficulties in understanding the concept of derivative. Experimental subjects performed significantly better than control subjects on the following tasks: (a) sketching the derivative for a given graph; (b) given several graphs of functions, recognizing the graph of the derivative of one of these; (c) specifying a non-differentiable function; and (d) relating the derivative to the gradient, and to the gradient function. Experimental subjects often spoke of the gradient in dynamic terms, such as finding a gradient of the tangent by deter-

mining the gradient of a secant as two points move closer together. Subjects still exhibited difficulty in understanding limiting processes, even though they had performed such processes when magnifying graphs to discuss the gradient of a graph at a point. Performance of manipulative techniques was not significantly different for control and experimental subjects.

Tall's (1986a) findings suggest that the use of graphs and in particular computer graphs, provides a rich environment in which the various aspects of calculus concepts can be explored deeply. Student understanding through such exploration was enhanced significantly without a decrease in skill development. Tall's use of MAGNIFY and GRADIENT show promise in aiding student understanding of the concept of derivative. Similar programs would be useful for the present study.

Summary

Curricular suggestions for implementation of computer graphic technology in calculus indicate that computer graphs might be useful in developing the concepts of function, limit, continuity, derivative, and integral. They allow rapid and efficient generation of graphs of many functions, permitting global properties of functions to be studied and used in developing calculus ideas.

Results of two studies implementing the use of computer graphs, indicate that such use could potentially contribute to student understanding of calculus concepts. Heid's (1984) study suggests that using the computer as a tool, to generate results normally obtained by hand, allows student attention to be focused on conceptual understanding. Her results also suggest that the use of multiple

representations might have contributed to students' deep conceptual understanding of calculus ideas. Graphs, as used in Heid's study, did not seem to alleviate student difficulties with understanding the concepts of limit and rate of change.

Tall (1986a) investigated the use of computer graphs in developing a conceptual understanding of derivative. Tall's results suggest that a computer graphic environment can promote deep student understanding of the concept of derivative, but might not have as great an effect on their understanding of the concept of limit.

Psychological Foundations

A theoretical foundation for a conceptual, graphical development of calculus can be found in the literature. In particular, the constructivist viewpoint as expressed by Piaget (1967, 1973), Von Glasersfeld (1987), and Skemp (1987), and the cognitive science/information processing point of view as presented by Davis (1984) support the use of a conceptual approach to calculus.

Implications for a conceptual approach to the teaching and learning of mathematics are given by the above as well as by Bruner (1973), Ausubel (1968, 1973), and Dienes (1973). Support for the use of multiple representations, and in particular, use of a Cartesian coordinate graphic representation in developing the concepts of calculus can be found in a theory for developing competence with written mathematical symbols as presented by Hiebert (1987), and discussions of the use of representations in mathematics (Davis, 1984; Goldin, 1987a, 1987b; Janvier, 1987; Kaput, 1987; Lesh, Behr, & Post, 1987; Lesh, Post, & Behr, 1987;

Skemp, 1987).

Theories of Learning Mathematics

The prevailing psychologies of mathematics education are similar. They each focus on the learner as the constructor of his or her own knowledge, as initially proposed by Piaget (1967). Refinements of Piaget's theory have been made by von Glasersfeld (1987), Skemp (1987), and Davis (1984).

Piaget (1967) explains the learning process as being similar to the process of adaptation of an organism to its environment. An organism survives (or doesn't survive), depending on its ability to adapt to its surroundings. If placed in an environment similar to that with which it is already accustomed, it adapts to the environment fairly easily, behaving similarly as it did while in its former surroundings. If changes are more drastic and, perhaps, actually life-threatening, the organism attempts to change in some manner necessary to survive in the new surroundings. While evolution – moderate to drastic change of an organism in adapting to its environment – often takes centuries, the same is not true of the adaptation that is known as learning.

Piaget suggests that all learning takes place as a result of maturation, experience, social transmission, and equilibration (Mathematics Resource Project [MRP], 1978). A person's mental activity is organized into structures which have been formed through constant interaction with the learner's mental, physical, and emotional environment. Just as the organism adapts when presented a new environment, Piaget suggests that these structures change as a result of the

learner's innate need to make sense of new experiences. The learner adapts existing mental structures to be consistent with new experiences, either through assimilation – fitting the new experience into an existing structure, or through accommodation – reorganizing and/or altering existing structures to make them consistent with new experiences.

A child's first learning experiences are with the physical world. Very strong mental images are developed regarding concepts such as distance, space, size, time, and oneness. These mental, visual images are generally developed at a very young age, 3 or 4 years, and are accessed throughout adulthood. It is these mental images that become the bases of mathematical learning. When one studies formal mathematics, Piaget suggests that one must begin with these qualitative concrete levels, reserving formalization for a later time as a systematization of the notions already acquired (MRP, 1978). For the learner's mental structures to adapt to the new environment of formal mathematics, the new environment must be attached, in some way, to what the learner already knows of his "concrete" world.

The role of the teacher, to aid the learner who constructs his own knowledge, is to provide an environment that builds on what the learner already knows and that moves the learner toward an understanding that is more consistent with formal mathematics. The teacher must provide an environment where the learner is faced with situations in which existing structures experience a sense of disequilibrium. In the learner's search for equilibrium, existing structures will be required to adapt, either by fitting the new coherently into what is already known, or by

restructuring the old to fit the new observations.

Based on the Piagetian theory that the learner constructs his own knowledge through self-organization of his or her experiences, various suggestions have been made for the teaching of mathematics. Based on the Piagetian principle that "real comprehension of a notion or a theory implies the reinvention of this theory by the subject" (MRP, 1978, p. 85), Bruner espouses discovery as the method of learning. Bruner (1973) suggests that students learn by recognizing basic structures from a range of instances. He suggests that students not be exposed to formalism prematurely, since early introduction of formalism prevents students from realizing that they have been thinking mathematically all along.

As students grow in maturity and experience with the physical and social world, Ausubel (1973) suggests that they are able to assimilate more information without having to "reinvent the wheel". The mature student is able to reflect on ideas and experiences transmitted by a teacher. The student is able to mentally manipulate these ideas and discover their deeper meaning through the process of reflection.

Two factors were identified by Ausubel (1968) as necessary for meaningful learning to take place:

1. The learner must have the proper mental set. He or she must determine that the concepts, principles, or skills of the learning situation are important enough for him or her to put forth the mental effort necessary to internalize new information in a nonarbitrary and nonverbatim way.

2. The learner must have the prerequisites for the new learning situation already in his or her cognitive domain so that these existing conceptual structures may be called upon to allow the learner to act in a meaningful way (Ausubel, 1968). By anchoring new learning in existing conceptual structures, both Bruner (1973) and Ausubel (1968, 1973) suggest that new learning is resilent to forgetting and is also available for use in situations that vary considerably from the situation in which they were learned.

Dienes (1973) describes mathematics as the study of the structure of relationships. Use of formal symbolism allows a way of communicating parts of the structure from one person to another. In order for students to become familiar with concepts expressed formally, Dienes suggests the following four principles of concept learning:

1. Dynamic Principle: Students should progress through three stages of exposure to a concept: (a) the play stage in which students begin to develop an awareness of cause and effect, (b) the structural stage in which a slow realization of the concept's place in a coherent whole begins to evolve, and (c) the practice stage in which the concept is applied, even incorrectly, until it becomes an object for further conceptual development.

2. Constructivity Principle: Children develop constructive thinking long before analytical thinking. In the structuring of the activities in the dynamic principle, construction should always precede analysis.

3. Mathematical Variability Principle: The constancy between qualities of

a concept can be discovered if irrelevant mathematical properties are varied as much as possible.

4. Perceptual Variability Principle: Play as many variations as possible in different media on the same conceptual theme to allow the concept to be viewed in as many contexts as possible to allow for individual differences.

Dienes (1973) states that the problem of learning is how to find the "best fit" between the structure of the task and the structure of the person's thinking. Piaget would agree that the varied environment suggested by Dienes would allow students to call upon existing mental structures to make sense of new experiences.

Von Glasersfeld (1987) extends Piaget's theory, describing learning as a constructive activity. He suggests that knowledge does not represent a picture of the "real" world but provides structure and organization to experience." (von Glasersfeld, 1987, p. 5). To have learned means" to have drawn conclusions from experience and to act accordingly" (p. 8).

Von Glasersfeld (1987) speaks of conceptual structures, as did Piaget (1967). The adequacy of these structures lie in: (a) their predictive value with which the learner may respond to future experiences, (b) how well they fit the learner's past experiences, and (c) how viable they are for solving problems. Mathematical understanding is exhibited not by the retrieval of "facts," but by reflecting on what one knows in order to produce new results. Von Glasersfeld suggests that the emphasis in teaching and learning, shifts from students correctly replicating what the teacher does to students successfully organizing their own experiences.

From a constructivist perspective, learning occurs if an experiential setting familiar to the learner is altered, either physically or through questions, to cause conflict or at least surprise for the learner. As also noted by Piaget (1967) the learner has an inherent need for equilibrium. Altering conceptual structures to be consistent with the new experience can occur only if the learner perceives an inconsistency. Inconsistencies will not be perceived if new experiences are based on ideas not already familiar to the learner.

The learner's motivation for consistency comes from within. Von Glasersfeld (1987) suggests that the stimulus-response method of rewarding students does not provide true motivation for learning. Motivation is intrinsic. It is derived from the individual's need to make sense of the world. Correctness is judged individually. Something is correct if it fits the individual's established conceptual structures.

Von Glasersfeld discusses the role of communication in teaching and learning. If one's conceptual structures need only be individually consistent, making sense of a personal world for the learner, then what is the role of communication in teaching mathematics, and in the learner's exhibiting what he or she knows? Von Glasersfeld suggests that the verbal language or symbol system with which mathematical ideas are represented does not have to be isomorphic to the ideas it represents. It is possible for two persons to communicate if their individual conceptual structures are compatible. That is, communication occurs if the meaning conveyed through the verbal or symbol system used to represent ideas are known to both persons and these represent ideas that are similar and do not clash. In

the constructivist view, knowledge is a personal entity, formed within, not derived from outside the learner. It is through communication of ideas or interpretations of the world that one begins to perceive how consistent his or her ideas are with those of another or with another's view of formal mathematical concepts.

Skemp's (1987) description of student learning processes are consistent with the constructivist viewpoint as described by Piaget (1967) and von Glasersfeld (1987). Skemp outlines schema construction – the construction of conceptual structures. Schemas are built and tested in three modes. Modes of schema building are: (a) through experience, (2) through communication with others, and (3) through creativity. The three modes of schema testing are: (1) through experiment, (2) through discussion with others, and (3) through comparisons with one's own existing beliefs. Any mode of building can be used with any mode of testing, and more than one mode can be used at a time. Skemp's description of schema building is similar to that described by Piaget (MRP, 1978). Modes 1 and 3 of schema building and testing occur through maturation and experience. Mode 2 of schema building and testing occurs through social transmission. The need for equilibration explains the schema testing process in all three modes.

In mathematics, the same or similar symbols are used to represent very different mathematical objects. Skemp (1987) gives the following example. The ordered pair (2,3) can represent a point in a plane, a rational number, or a vector. The sum of two ordered pairs, (2,3) + (4,5), is meaningless in the first case, is $(2 \cdot 5 + 3 \cdot 4,\ 3 \cdot 5)$ in the second, and $(2 + 4,\ 3 + 5)$ in the last. One is able to

determine which method is correct by the context in which the ordered pair is used if one has done more than memorize the rules for operations with ordered pairs.

Skemp (1987) describes two kinds of understanding, instrumental and relational. Instrumental understanding is that which can be described as "rules without reasons." Relational understanding refers to the understanding developed meaningfully, where rules are based in deeper understanding, related to conceptual schemas already familiar to students. In the example with ordered pairs, a student who has instrumental understanding of operations with ordered pairs is as likely to find the sum in an inappropriate way as in an appropriate way because the understanding is not based on the reasons behind the rules. The student with relational understanding can operate with ordered pairs in the context in which they are being used.

To further describe understanding, Skemp (1987) explains deep versus surface structures. In communicating ideas, one relies on verbal language or symbolic representation systems to convey the desired message. The underlying ideas that one tries to convey form the deep structures of mathematics. The verbal language or symbol system that one uses to convey meaning, or through which one receives meaning, are the surface structures. The surface structures of the three meanings associated with the ordered pairs discussed earlier are identical. The deep structures are all quite different. Without relational understanding, the symbols (a, b) could evoke any one of the three conceptual schemas associated with ordered

pairs. Skemp suggests that if relational understanding does not occur, this sort of difficulty will prevail. If one practices rules without reasons, the products of instrumental understanding, then the deep structures never become attractors for stimuli provided by surface structures, and further work with symbols remains with the surface structure.

In the teaching and learning of mathematics, Skemp's (1987) recommendations are consistent with those given by von Glasersfeld (1987) and Piaget (1967). Students learn by constructing their own knowledge through the building of schemas that organize their experiences. To aid students in learning formal mathematics, new experiences, including work with the symbol systems of mathematics, must be based in existing deep conceptual structures. Students will then use these structures to update their schema through the building and testing modes described, allowing these ideas to be represented and transmitted through appropriate surface structures based in deeper meaning.

The cognitive science/information processing point-of-view of learning mathematics presented by Davis (1984) also extends the theory of Piaget (1967). The cognitive science approach is based on the use of the computer to model how humans process information. In this approach, the basic unit of mental activity – the conceptual structure of von Glasersfeld (1987) and the schema of Skemp (1987) – is the frame (Davis, 1984). All information is stored in frames which are built on subframes. Each level of subframes is in turn built on other subframes. As also mentioned by Piaget (1967), von Glasersfeld, and Skemp, all human experience

is an extension of a relationship with prior knowledge. However, Davis explains student errors differently.

Davis (1984) observed that students sometimes exhibit deep understanding of concepts in particular situations, but when tested on the same information at a later date, they often resort to their naive, pre-training responses (see also Davis & Vinner, 1986). In information-processing theory, no frame is destroyed. When a frame is accessed for a particular reason, a copy of it is made, updated by the new experience, and stored as an instantiation of the original or of another related frame. The original is left intact. This theory explains, at least in part, why it is so difficult to change a person's mental set; frames are resistent to change.

Davis (1984) further hypothesizes that all experiences are stored in memory in some form. The storage of new information is automatic. It is the retrieval of this stored information that causes difficulty for the learner. If information is to be retrieved from memory at appropriate times, it must be coded in some way to promote its retrieval. New information must be attached to existing knowledge in a way that causes it to be retrieved when needed. Understanding, then, can be viewed as meaningfully relating new ideas/experiences to existing schema. Those ideas that are not based in existing schema are soon "forgotten" – i.e. lost in the learner's vast warehouse storage called passive memory. When the connections are made to existing schema, these connections provide access routes for recall.

To cause the correct frame to be recalled, Davis (1984) describes the use of a "critic." A critic is a decision-making frame that determines whether or not a

particular frame is useful for a particular purpose: ie. if condition X exists, use frame A; if not, use frame B. The critic must be inserted before the frame is called upon to act.

Davis' (1984) use of a critic is similar to Dienes' (1973) Mathematical and Perceptual Variability principles. In concept formation it is necessary to discriminate between similarities and differences so as to call upon the proper frame when using the concept.

Summary and Implications

The psychological foundations of mathematics education as presented by Piaget (1967), von Glasersfeld (1987), Skemp (1987), and Davis (1984); and the implications for teaching and learning mathematics given by each of the above as well as by Bruner (1973), Ausubel (1968, 1973), and Dienes (1973); contain many similarities. Each of these theories of learning suggests that the learner constructs his or her own knowledge through interaction of existing conceptual structures with new experiences. New experiences can be the result of maturation, social transmission, daily experiences with the world, and/or creativity. It is the learner's need to make sense of the world, the need for equilibration, that causes him or her to assimilate new experiences into existing structures or to accommodate new experiences by changing existing conceptual structures.

Von Glasersfeld (1987) and Skemp (1987) address the role of communication in transmitting and receiving information. Von Glasersfeld suggests that communication between two persons can occur, even though each person has built

individual conceptual structures to organize his or her own experiences. Communication is only hampered if the individuals' conceptual structures contain conflicting information. Skemp suggests that it is through surface rather than deep structures that communication takes place. If surface structures are not based in deep structures, then the underlying meanings interposed by an individual as conveyed by verbal language or symbol systems supported by surface structures can be quite different within and among individuals.

Davis (1984) explains the resilience of errors to remediation by suggesting that no conceptual structure – frame – is ever destroyed. If a learner does not develop new frames with an underlying meaning that allows him or her to call upon these frames at appropriate times, then errors will occur and recur regardless of subsequent training.

For the present study, some implications for teaching and learning are: (a) new experiences must be firmly anchored in a student's existing conceptual structures, if these can be determined; and (b) symbols and verbal descriptions must be built relationally, with deep conceptual foundations to give them contextual meaning. All of the learning theories reviewed here support a conceptual development of calculus ideas. To provide such an environment to promote the development of deep conceptual understanding, the concepts of limit and derivative are developed first through the use of familiar distance versus time situations. Such situations are then modeled graphically and the interrelationships between these representations are discussed. Formalization is delayed. For the most part, definitions, theorems,

and algorithms are not presented symbolically until substantial motivation and conceptual development has occurred.

To promote students' construction of their own understanding, the investigator uses questioning techniques that encourage students to become active participants in class sessions. Students are often asked to provide evidence of their understanding of course concepts. They are asked to determine examples of functions that satisfy definitions, hypotheses of theorems, and initial conditions of algorithms. They are also asked to provide examples of functions that do not have certain suggested properties. To encourage a high level of participation, functions are developed graphically through the use of computer graphics. Such an environment makes available a large repertoire of functions to be used as examples, counterexamples, and test cases in investigations of calculus ideas.

Use of Mathematical Symbols

Several studies were reviewed concerning student misconceptions, use of graphs, and especially use of computer graphs, in calculus. The results of many of these studies suggest that a Cartesian coordinate graphic representation system, through its potential to display continuous ideas, might be useful in representing and in providing conceptual support for calculus ideas. The case was made for the necessity of a conceptual course built on students' existing conceptual structures through a discussion of the theoretical literature. Further theoretical support for the use of a graphic representation is presented in the following pages.

A mathematics symbol representation system is a symbol scheme representing

a mathematical structure with a specified correspondence (Kaput, 1987). Often, mathematical symbol systems are based on referents that are other mathematical symbol systems.

It has been suggested that part of what is meant by "a student understands an idea" is that:

> "(1) He or she can recognize the idea embedded in a variety of quantitatively different representational systems,
>
> (2) He or she can flexibly manipulate the idea within given representational systems, and
>
> (3) He or she can accurately translate the idea from one system to another." (Lesh, Post, & Behr, 1987, p. 36)

The power of symbols lie in their succinctness. Representations carry with them important properties of the referent they represent. Use of symbol representations systems aid in storage and retrieval of information in at least two ways: (a) information can be coded in a form that is easier to manipulate, and (b) complex relationships can be simplified, removing irrelevant aspects of a problem. Each of these uses of symbol representation systems can reduce memory load and increase storage capacity (Davis, 1984).

Because of their succinctness in representing ideas, three difficulties arise through the use of symbols: (1) symbols can be very similar in appearance while representing very different ideas; (2) since symbols are compact and succinct, decoding can be demanding; and (3) there is often no physical or perceptual cor-

respondence between a symbol and its referent, which further hampers decoding (paraphrase of Hiebert, 1987). Two further difficulties are realized when visual symbols are used. Students have strong perceptual conceptual structures based on their observations of and interactions with the physical world. Perceptual cues given through visual symbols could evoke conceptions that are inconsistent with, if not contradictory to, the meaning represented by the visual symbols (Kaput, 1987). Also, visual symbols require another level of abstraction. A drawing of an object meant to represent a class of objects satisfying a given property is often interpreted by students as representing the specific object with the dimensions and qualities of the drawing (Kaput; Skemp, 1987).

Since symbols can represent ideas succinctly without giving perceptual cues as to which ideas these might be, or with giving perceptual cues which might distract correct interpretation, it is imperative that they be developed meaningfully, based in a referent that is familiar to the learner (Davis, 1984, 1986; Hiebert, 1987; Skemp, 1987). Suggestions for developing competence with symbol systems come from historical development of such systems.

Goldin (1987) suggests that, historically, the order of evolution of mathematical symbols is as follows: (a) symbols were created to represent referents, (b) the structure of the symbol system was developed, and (c) the symbol system, once developed meaningfully, was separated from the associated referent and was then available for use as a referent in the building of more abstract symbol systems.

Consistent with Goldin's (1987) observations, Hiebert outlines five major pro-

cesses leading to competence with written mathematical symbols that must be engaged sequentially.

> The five major processes are: (a) connecting individual symbols with referents; (2) developing symbol manipulation procedures; (3a) elaborating procedures for symbols; (3b) routinizing the procedures for manipulating symbols: and (4) using the symbols and rules as referents for building more abstract symbol systems. The third and fourth processes are numbered together because they operate concurrently. (Hiebert, 1987)

As observed by Davidson (1980), Geuther (1986), and Orton (1983a, 1983b), calculus students experienced serious difficulty in working with symbols. Geuther noted that symbol use was almost arbitrary. Davis (1986) suggests that failure of current school mathematics lies in the early formalization of concepts, such as when students are forced to display manipulative facility with symbols before a firm attachment to a meaningful referent has been formed. Skemp (1987) would describe such shallow use of symbols as using surface structures that are not based in deep structures. Students have instrumental but not relational understanding of such symbols. Davis further suggests that symbols developed without deeper understanding cannot be used as a base for higher level symbol systems. Procedural understanding of representations supports further conceptual understanding in that it decreases the amount of mental attention and energy that must be employed to use a representation, allowing the mental effort to be focused on the concept being developed. However, as suggested earlier, procedural understanding can support conceptual understanding only if it is developed conceptually.

For the present study, the Cartesian coordinate graphic symbol system, often in parallel with familiar experiences, is used to represent and develop understanding of the concepts of limit, continuity, and derivative. To use such a system in calculus, the graphic representation system itself is developed through the use of computer-generated graphs. An attempt is made to circumvent the perceptual difficulties identified by Kaput (1987) and Skemp (1987). The movement of a spider, observed as his distance versus time graph is plotted, is used to relate the ideas of steepness and velocity. Also, classes of graphs are viewed and their relationships discussed. Following Hiebert's suggestions, the five major processes are used sequentially to develop competence with graphic and then with calculus symbols. These are elaborated more fully in Appendix B.

Summary

The empirical research and the theoretical framework discussed in this chapter are complimentary. They suggest that development of concepts and procedures must be based in students' existing conceptual structures if student difficulties are to be alleviated. Students experience conceptual difficulties in learning calculus ideas since this is probably the first time that they have been required to understand continuous processes. There is some support, both empirical and theoretical, to suggest that a graphic environmental might be useful in building continuous ideas.

CHAPTER III

DESIGN AND METHODOLOGY

The purpose of the present study is to broaden the knowledge base concerning student conceptual understanding of calculus, especially as developed through the use of a graphic representation system. The theoretical and empirical framework described in Chapter II provides the foundation upon which the present study is based.

Setting

The setting for the present study was the first course in a three-semester calculus sequence offered at Western Michigan University (WMU) during the winter semester of 1988. WMU is a state-supported Carnegie Doctoral I university with an enrollment of approximately 20,000 students, 16,000 of which are undergraduates.

The first course in the calculus sequence is a 4-credit hour course. Calculus sections generally meet 4 days per week for 50 minutes each day for a duration of 14 weeks. During the Winter 1988 semester, calculus sections met a total of 54 class periods.

The Calculus I syllabus and the course text, "Calculus" by J. F. Hurley (1987), are determined by the Undergraduate Committee of the Department of Mathematics and Statistics. Calculus instructors at WMU are obligated to cover

the material in the syllabus so as to prepare students for a comprehensive uniform departmental final exam. This exam is administered to all students enrolled in the course during the final exam week, the fifteenth week of the semester. The Department mandates that the final exam score comprise at least 30% of the student's grade.

Individual instructors determine their own grading policies for the remaining part of the course grade. Daily lesson plans, content and number of exams and quizzes, and office hours are all the responsibility of the individual instructors. A calculus tutorial lab, staffed by upper-division mathematics majors and mathematics graduate students, is made available to students for 12 hours each week.

Subjects

Subjects for the present study were a subset of those students enrolled in the first semester calculus course at WMU during the winter semester of 1988. Subjects in 4 out of 7 sections of Calculus I participated in the study. The 4 sections participating in the study were chosen for their close proximity in time. These sections met at 9:00 a.m., 10:00 a.m. (2 sections), and 12:00 p.m. All 4 sections met on Monday, Tuesday, Thursday, and Friday for 50 minutes each day.

All subjects enrolled in a given section of Calculus I received the same treatment. Three of the treatments; Graphics (G), Graphics Plus (G+) and Standard 1 (S1); were randomly assigned to the 9:00 am., one of the 10:00 a.m., and the 12:00 p.m. sections respectively. Due to large course enrollments, a second 10:00

a.m. section was created after this random assignment had been made. The instructor assigned to this section was then asked to participate in the study. This section was designated as Standard 2 (S2). It was assumed that there would be no bias introduced by including the S2 section in the study.

Under the constraints of WMU course registration procedures, it was not possible to randomly assign students to sections. It was assumed that no section was more attractive to subjects than any of the others. However, 1 subject in the S1 section and 8 subjects in the S2 section indicated they had chosen those particular sections on the basis of prior information obtained about the course instructor.

In the final tally, 303 students enrolled in Calculus I during the Winter 1988 semester, 239 of whom completed the course. Of these, 163 students enrolled in the sections participating in the study. Students who met the following criteria were retained for the final analysis:

1. Subject completed all course exams including the departmental final.
2. Subject responded to the pre-treatment measures.
3. Subject had not been previously enrolled in the investigator's pilot study.

In all, 128 subjects met these criteria and were considered in the final analyses for the study. Table 1 indicates the number of subjects deleted from each section participating in the study along with the reasons for their deletion.

Table 1

Summary of Participants

Description of Subject Enrollment	Treatment G	G+	S1	S2
Initially Enrolled in Section	37	40	40	46
Dropped Course	6	4	5	14
Did not complete pre-experimental measures	0	0	1	1
Enrolled in investigator's pilot study	0	0	1	3
Retained for final analysis	31	36	33	28

Course Content

The study covered the material in the first calculus course up to and including differential calculus – the concepts of limit, continuity, and derivative and related generalizations – with respect to the course text, *Calculus* by Hurley (1987). Chapters 1 through 3 were covered with the exception of Sections 3.8:

Antidifferentiation and 3.9: Simple Differential Equations. A list of topics covered in the study is given in Table 2. Instructors did not necessarily cover topics in the order listed.

Table 2

Topics Covered in the Study

Section	Title
Chapter 1	
1.1	Absolute Value and Inequalities
1.2	Coordinates, Lines, and Circles
1.3	Functions and Graphs
1.4	Limits: A Measure of Change
1.5	Properties of Limits
1.6	One-Sided Limits
1.7	Continuous Functions
Chapter 2	
2.1	Instantaneous Rates of Change and Tangent Lines
2.2	Basic Rules of Differentiation
2.3	Trigonometric Functions and Limits
2.4	Differentiation of Trigonometric Functions
2.5	The Tangent Approximation and Differentials
2.6	The Chain Rule
2.7[a]	Differentiation of Inverse Functions and Roots
2.8	Implicit Differentiation
2.9	Repeated Differentiation
2.10	The Mean Value Theorem for Derivatives

Table 2 – Continued

Section	Title
Chapter 3	
3.1	Related Rates
3.2	Extreme Values of a Function Over a Closed Interval
3.3	Increasing and Decreasing Functions/The First-Derivative Test
3.4	Concavity/The Second-Derivative Test
3.5	Applications Involving Extreme Values
3.6	Limits Involving Infinity
3.7	Curve Sketching
3.8	Newton's Method

[a] Section was covered very briefly. Theory was not covered.

The material for the study comprised approximately 75% of the course. It was presented to the subjects during the first 40 periods of the 54 class periods of the course. In addition questions pertaining to this material appeared on Quiz 7 given on Day 43[1] , (see cross-reference in Appendix D) on Exam 4 given on Day 53, and on the departmental final exam given during the semester final exam week. With the exception of review sessions held by the investigator and the S1 instructor on the day before Exam 4 and by each instructor on the day before the final exam, no further specific instruction over the material covered in the study was given after Day 40, unless the instruction was necessary in explaining ideas or in solving problems related to antidifferentiation or integration. Both Exam 4

[1]For reporting purposes, each day of the course is referred to by number beginning with Day 1 .

and the departmental final exam were comprehensive. During the review sessions for these exams, material from the entire course was discussed.

Treatments

There were four treatment sections that participated in the study:

1. Graphics (G) – exposure to a graphically-developed, conceptual calculus course heavily dependent on the use of computer graphics for in-class exposition;

2. Graphics Plus (G+) – exposure to the same course as the G treatment plus the provision of use of computer graphics software and supplemental assignments for further exploration or reinforcement of course content;

3. Standard 1 (S1) – exposure to a graphically-developed, conceptual calculus course where the computer was not used; and

4. Standard 2 (S2) – exposure to a non-graphical calculus course where skill development was emphasized.

Three instructors taught the four sections. The investigator taught the G and G+ sections and the other instructors each taught one of the S1 and S2 sections.

Before the study began, the three instructors jointly made the following administrative decisions concerning their respective sections. All instructors agreed that:

1. Approximately 30 exam and quiz questions categorized as applied, symbolic routine, and symbolic nonroutine would be common to each section.

2. Ten quizzes and four exams would be given. A common exam and quiz schedule was determined.

3. Questions on the first few assessments were to be dissimilar across sections to prevent subjects from suspecting a collaborative effort among instructors.

4. Homework assignments from the textbook would be uniform across all sections, and determined during weekly meetings.

5. No homework assignments from the text would be collected or graded.

6. No other assignments or supplemental problems would be given to subjects in the G, S1, or S2 sections unless all subjects were to receive the same assignments or supplemental problems.

7. The pretest and pre-attitude survey would be given on Day 3 after instructors had covered sections 1.1 through 1.3 of the text.

8. The attitude survey would be administered again near the end of the semester.

9. All student responses on exam and quiz questions common to each section would be photocopied, on the day the exam or quiz was given, before the instructor graded them, to allow for all subjects' work to be graded independently by the investigator.

10. Variations in course presentation from that given in the departmental syllabus would be reported to the investigator.

The treatments involved in the study are described in more detail below; including descriptions of instructors, schedules, physical environments, grading policies, office hours, instructional styles, instructional emphasis, and assignments.

The Graphics Sections

Two of the calculus sections were defined as the graphics treatments: Graphics (G), and Graphics Plus (G+). Both sections received essentially the same in-class treatment. They differed with respect to the level of exposure to a graphical representation as computer software was available for graphic display. Each subject in the G+ section was provided with the computer graphics software, *Master Grapher* (Waits & Demana, 1987a), to be used outside of class on assignments and for individual exploration. Subjects in the G section were neither provided the software, nor were they encouraged to use the computer outside of class. A description of the features common to the G and G+ sections follows:

Common Features

Instructor. Both the G and G+ sections were taught by the investigator. The investigator's prior college teaching experience included teaching 10 sections of remedial mathematics , 2 sections of Mathematics for the Elementary Teacher, 1 section of an upper division mathematics education course, and 2 sections of calculus.

Schedule. The G treatment was randomly assigned to the 9:00 a.m. section. The G+ treatment was randomly assigned to a 10:00 a.m. section. Both sections met in the same classroom. Subjects were told to attend the section in which they enrolled because classroom capacity and class size did not permit otherwise. They were asked to inform the investigator in advance of special circumstances

that prevented their attendance during their scheduled sections. The investigator then allowed such subjects to attend her other section being careful not to mix in-class treatments on those days.

Physical Environment. The classroom was outfitted with two ceiling-mounted 27- inch NEC RGB monitors. One monitor was mounted in the right front corner of the classroom. The second was mounted approximately one-third of the way back from the front of the classroom on the right side of the room. The investigator connected a Zenith 181-Z laptop (IBM-compatible) computer to the projection system prior to the 9:00 a.m. class on days during which the computer was to be used. The computer was available during most class sessions.

A projection screen was mounted over the blackboard on the left side of the classroom. An overhead projector was available during most class sessions. For class sessions in which the computer was used, it was necessary to turn off the fluorescent lights to maximize visibility of the computer output. The overhead projector, rather than the chalk board, was then used in conjunction with the computer. Enough light was emitted from the overhead projector and from 3 narrow classroom windows to allow for student note-taking. The classroom itself contained 5 rows of tables with 8 chairs in each row.

Grading Policy. Four 100-point exams, ten 15-point quizzes, and a 225-point departmental final exam were given during the semester. The quizzes were averaged over 125 points, rather than 150, to encourage students to attempt each. Exams comprised approximately 53%, quizzes approximately 17%, and the de-

partmental final 30% of the students' grades. No homework assignments were graded.

<u>Office Hours</u>. The investigator was available for scheduled office hours for 4 hours each week – 2 hours per week for each section taught. The investigator was also available by appointment for students who could not make the regularly scheduled office hours.

Of the scheduled office hours, two hours were held in the investigator's office where a laptop computer was available. The remaining 2 hours were held in a computer lab where 9 computers were available. The computer was used during office hours, if necessary, to repeat demonstrations that had occurred in class. During the first 4 weeks of class, the computers in the lab were occasionally used by students from the G+ section who needed further assistance on assignments. Care was taken to prevent subjects in the G section from observing G+ subjects while they were using the computer. Also, the instructor never encouraged or permitted subjects in the G section to operate the computer in class or during office hours. One subject, Student 1 in the G section, and another subject, Student 2 in the G+ section, were dorm roommates. Early in the semester, Student 1 asked the investigator when the G section would be provided the software and assignments given to the G+ section. Student 1 had just been in an open computer lab completing the assignments. As a result, the investigator explained to all subjects in the G section that they could not be provided the *Master Grapher* software because the Department of Mathematics and Statistics could not afford

the additional expense or the computer hardware necessary to handle a large number of students.

Since Student 1 was a highly-motivated individual, it is likely that he completed the assignments given to Student 2. Student 2 eventually dropped the course, but not until all of the assignments had been provided. Student 1 turned in Student 2's software shortly after Student 2 dropped.

It is believed that the above was an isolated occurrence. No other subjects in the G section mentioned having access to the software.

Instructional Style. The instructional style used in the G and G+ sections was interactive. The investigator asked many questions of subjects. Initially, the investigator waited several seconds for a response from subjects before asking a simpler question or before rephrasing the question asked. Subjects appeared to be uncomfortable with the silence, but they learned, early in the semester, that they were expected to answer and to ask questions. When a subject asked a question about a concept being discussed, the investigator often responded with another question to encourage the subject to call upon past experiences or related ideas to answer the initial question. On rare occasions were subjects' questions answered outright. As a result of the instructional method, many subjects began asking and answering questions in class. They not only responded to questions posed by the investigator, but they also began answering each others' questions. On one occasion, a friendly debate ensued between subjects as they tried to understand concavity in terms of acceleration as it was modeled graphically on the computer.

Such an environment was developed to encourage constructive activity on the part of the subjects.

Instructional Emphasis. The in-class presentation of calculus concepts was the same for both the G and G+ sections. The concepts of limit, continuity, and derivative were developed through the use of a Cartesian coordinate planar graphic representation system. Commercial graphics software and investigator-written graphics programs were used by the investigator to illustrate calculus concepts. The investigator's use of graphics displays on the computer and on the overhead allowed: (a) quick generation of many graphs from which students could generalize common properties; (b) dynamic illustration of a moving object which provided subjects with an environment where changes in the position of an object over time could be compared to the related Cartesian graph; (c) a medium through which subjects could develop efficient methods of drawing graphs by hand; (d) a medium through which subjects could develop mental estimation skills with graphs; (e) a medium through which subjects could question the reasonableness of results found symbolically; and (f) a medium through which subjects could view global, as well as point-wise, properties of functions.

The emphasis in this experimental instructional approach was on conceptual understanding, especially as developed through a graphic medium. Emphasis on skill-development was minimal. For example, when rules for finding limits and derivatives were to be presented, these were each motivated graphically, often with the help of the computer. Subjects discovered appropriate rules from what

they knew about graphs of functions. On rare occasion, rules were presented outright. Two notable exceptions to the graphical discovery approach were: (a) in the introduction of the chain rule, and (b) in the introduction of the rule for finding the derivative of inverse functions. In the first case, the motivation was not graphic, but rather through an application. In the second case, the motivation was graphic, but the rule was presented without it first being discovered by subjects. Once subjects had discovered or had been given a rule, two or fewer examples were presented with subjects suggesting the appropriate steps. Subjects were often asked to determine if the symbolic results were reasonable in a graphic context. After such initial presentations, symbolic manipulations were only performed if: (a) subjects requested that the instructor review the solution to a specific homework exercise, (b) the manipulations were necessary in the context of solving an applied problem, or (c) a problem required further use of the result of a manipulative technique – eg. determining the concavity of the graph of a function at a given point which requires finding and using the first and second derivatives.

The concepts of limit, continuity, and derivative were first motivated through applications familiar to the subjects, then the applications were represented graphically. Observations were made regarding the interpretation of the application with respect to its graph. Graphs were then interpreted symbolically and symbols graphically. This translation between symbolic, graphic, and natural language representation systems was also encouraged through assignments, quizzes, and

exams. Specific detailed descriptions of the conceptual, graphically-developed approach used by the investigator in this study are included in the protocols of Appendix A. As an example of the investigator's conceptual emphasis through the use of computer graphics, a description of the evolution of the derivative concept as developed in this study follows.

Evolution of the Concept of Derivative
as Developed Through Computer Graphics

The graphical development of the concept of derivative was begun using the situation described by Sawyer (1961). Subjects were asked to imagine that they were taking pictures of a spider as he climbed a wall. They were asked to imagine the result of several pictures taken of him at uniform time intervals. (Figure 1 was displayed on the overhead.) Subjects were asked to imagine further that a narrow vertical strip containing the spider was cut from each of these pictures, and that these strips were lined up in sequence. (Figure 2 was displayed.) Subjects noted that these pictures formed a graph with time as the independent variable and position as the dependent variable. The spider's progress as he climbed the wall was simulated by using an overhead projector, with the investigator pulling various continuous graphs under a sheet of cardboard. The cardboard had a vertical slit in the center so that only a small part of the graph was visible. A discussion of the spider's progress as he moved "up the wall" (or "down the wall") followed. Subjects observed that his velocity varied depending on the shape of the graph. The discussion continued with the use of an investigator-written program, SPIDER. This program allowed the subjects to observe a spider climbing a wall while at the same time a graph of his position with respect to time was drawn on the screen.

The spider situation was used to develop the idea of rate of change graphically and to imbed the concept of derivative in a graphical referent. Subjects were made aware that a graph can represent the relationship between two quantities and that the shape of the graph tells them something about how the two quantities change with respect to each other.

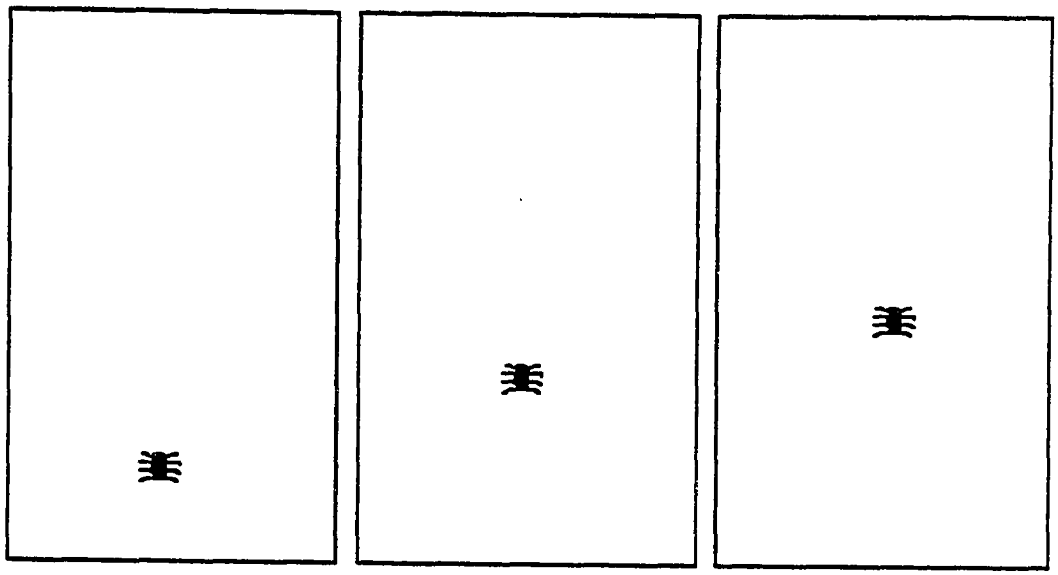


Figure 1. Pictures of a spider climbing a wall.

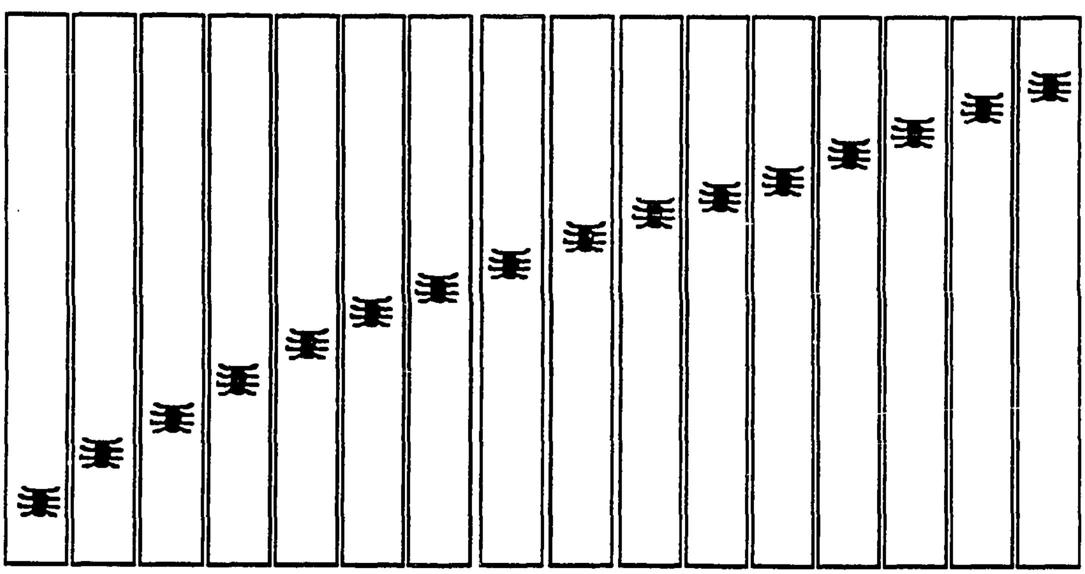


Figure 2. Several pictures of a spider climbing a wall taken at uniform time intervals.

The spider situation was followed by a discussion of another familiar situation, commuting via automobile. The investigator presented the following scenario: "Often, one of the major concerns of a commuter as he travels is how fast he must travel to get from one place to another given the amount of time he has to complete the journey and the distance he must travel. The policeperson, John Law, patrolling a road over which the commuter must travel is concerned with something much different. John Law's concern is focused on the short time interval during which the commuter is travelling the segment of the road he patrols. The commuter determines the speed at which he must travel by dividing the distance he must travel by the time he has available, where distance travelled is determined by the difference of the final mileage and the initial mileage, and time is determined by the difference in the expected arrival time (final time) and the departure time (initial time). John Law determines the commuter's velocity similarly, observing his positions at initial and final times. John Law's reading improves – more closely estimates the commuter's actual speedometer reading – as the width of the time intervals over which the observations are made decrease (no time for the commuter to hit the brakes!)."

The commuter situation was represented graphically by the investigator. The average velocity was discussed with respect to this referent first globally, as a secant to the curve that represents the commuter's position with respect to time travelled, then locally, as secants to very small parts of the graph near a point of interest (where John Law observed the commuter's velocity). Subjects made the observation that the expression:

$$\text{Average Velocity} = \frac{\text{final position - initial position}}{\text{final time - initial time}}$$

is modeled by the expression:

$$\text{Slope, } m = \frac{y_2 - y_1}{x_2 - x_1}$$

which represents the slope of the line segment with endpoints (x_1, y_1) and (x_2, y_2).

Using *Master Grapher* (Waits & Demana, 1987a), subjects observed as a variety of functions were investigated. The investigator graphed a function then magnified the graph several times around a chosen point. Subjects observed that most continuous functions are locally straight on sufficiently small intervals. To preserve intellectual honesty, they also observed that functions exist that cannot be made locally straight at given points. The discussion that followed ended with the realization

> that the slope of a secant on a very small interval can be a very good approximation of the rate of change of two related quantities.
>
> Using an investigator-written program, SECANT, subjects chose a fixed value c and another x-value, x_1. The resulting interval $[c, x_i]$ was halved iteratively, keeping the fixed point and computing a new x-value x_n. The function $f(x) = 2 \sin x$ was drawn as well as the secant between the resulting coordinates $(x, f(x))$ and $(x_n, f(x_n))$. Subjects suggested that, as the length of the interval between x and x_n decreased, the resulting line containing the secant appeared to intersect the curve in one point. Subjects recalled that they had seen a line with such a property related to circles. The line was defined to be a tangent. The investigator generalized the definition intuitively to include all curves.

As illustrated above, the derivative evolved from a discussion of a graph representing the rate of change of the quantities position and time. The rate at which these quantities changed determined the shape of the graph of the function representing them. The rate was represented graphically by the slope of a secant to the curve to be used as a good estimate of the actual rate of change. As the width of the interval decreased, the observation was made that the resulting lines containing these secants appeared to approximate the tangent to the curve at a given point.

This development used graphic and natural language representations in parallel to build meaning for symbolic representations. It illustrates the usefulness of a graphic representation in developing the concepts of calculus. The development is characteristic of the instructional method employed in the course presented to the G and G+ sections.

To allow the additional in-class time necessary to provide such a deep conceptual development, the time usually devoted to discussion of previous assign-

ments was greatly reduced and often eliminated. Only on a few occasions was in-class time used to discuss questions pertaining to textbook assignments. Solutions to such assignments were most often available in the back of the text. Instead, subjects with questions over such assignments were encouraged to make use of the investigator's office hours.

Instructional Time-Frame. Instruction of calculus concepts covered by the study were completed on Day 40 for both the G and G+ sections. Review sessions over subsets of the material presented were held on Day 52 and on the day preceding the departmental final exam.

Although the in-class instructional approach was the same for both G and G+ sections, there were differences in the treatments given these two. The availability of computer software to the G+ subjects for use outside of class and on course assignments comprised the major difference between these sections. The in-class treatment varied only in : (a) the discussion of questions raised by subjects pertaining to assignments and in- class presentations, and (b) the amount of instruction necessary to direct G+ subjects in the use of the computer. Specific differences in the G and G+ treatments are discussed below.

Treatment 1: Graphics

The G treatment was designed to emulate conditions typical to a university setting in which computers are being introduced. The calculus text used at WMU infrequently includes questions that are well-suited for solution in a computer environment. In particular, conceptual, graphical questions are rarely

found in the text or in the accompanying student study guide. It is recognized that the time commitment required to write supplemental assignments is considerable. It is also recognized that many colleges and universities do not have the facilities and software necessary to provide mathematics students access to computers on a regular basis outside of the classroom. The G treatment operated under the above constraints. While in-class development often included use of the computer, software was not provided for out-of-class use by subjects and they were not encouraged to use the computer outside of class.

Assignments. The assignments for the G treatment were taken directly from the course text with the emphasis on these assignments being conceptual as much as possible under that constraint (see Appendix F). An attempt was made to keep the time commitment for assignments to approximately two hours outside of class for each hour in class.

Treatment 2: Graphics Plus

Subjects in the G+ section were provided individual copies of the computer graphics software, *Master Grapher* (Waits & Demana, 1987a). The software is a powerful function grapher that allows subjects to graph functions on any viewing rectangle chosen, to within 8 decimal places of accuracy. It contains menu-driven options that allow users to zoom-in, zoom-out, redraw graphs, transform graphs, draw graphs of inverses, rotate graphs, draw moving or fixed vertical or horizontal lines, overlay graphs, change the plotting mode and speed, etc. A

full description of this software is included with the supplemental assignments (see Appendix F).

Assignments. Assignments for the G+ treatment combined exercises from the course text and investigator-compiled supplemental problem sets. The textbook assignments were identical to those given to the subjects in the G treatment. The supplemental assignments were conceptual and graphic, often requiring the use of the computer graphics software. Most supplementary assignments were designed to encourage the exploration and/or solidification of graphic ideas. Some of these explorations preceded in-class discussion. Others provided a means for further investigation of ideas discussed in class. These assignments are included in Appendix F. Supplemental assignments were given to subjects in the G+ treatment at times when the exposition in the course varied considerably from that given in the text. Textbook assignments either did not exist or were not appropriate during these times. Subjects in the G treatment received no assignments when G+ subjects received supplemental assignments. Solutions to supplemental assignments were discussed in class for questions that required subjects to discuss the observations made through their interaction with the computer. As with the G section, an attempt was made to keep the time commitment for assignments to two hours for each hour in class.

The Standard Sections

Two treatment sections are defined as standard: Standard 1 (S1) and Stan-

dard 2 (S2). The in-class treatments were very different for these sections. The S1 section was a graphically-developed, conceptual calculus course that did not use the computer. The S2 section was a traditional calculus course where skill development was emphasized.

Common Features

The computer was not used in class with either section, and software was not provided to students for use outside of class. The use of the computer was not mentioned in either section.

Neither of these sections was taught by the investigator. The physical environments were identical for these two sections, as were their assignments. Office hours were similar. All other features - schedule, grading policy, instructional style, and instructional emphasis - were unique to each treatment.

Physical Environment. The S1 and S2 sections met in the same theatre-style classroom. The room contained 60 seats arranged in 5 rows of 12 seats each. The room was not equipped for computer demonstration. While a projection screen was mounted in the front of the room, it was not used by either the S1 or the S2 instructor.

Office Hours. Each of the S1 and S2 instructors was available for scheduled office hours for 2 hours per week for each section taught. The S1 instructor taught no other classes. The S2 instructor taught one other class and held additional office hours for students in that class. Both the S1 and S2 instructors

were available by appointment for subjects who could not make their regularly scheduled office hours.

Assignments. The textbook assignments for both the S1 and S2 sections were identical to those given to the G and G+ sections (see Appendix F). No other assignments were given. The time commitment for assignments was kept to approximately two hours outside of class for each hour in class. Solutions to assignments were often found in the text. Both the S1 and S2 instructor spent 15 to 20 minutes of each class period answering questions concerning the assignments.

Treatment 3: Standard 1

Of the Standard sections, the S1 treatment was most like that of the G and G+ treatments. The S1 instructor's self-report data was used to describe the features unique to the S1 section.

Instructor. Treatment S1 was taught by a mathematics graduate assistant whose college-level mathematics teaching experience was comparable to the investigator's. The S1 instructor's prior college teaching experience included teaching 4 sections of remedial mathematics, 5 sections of finite mathematics, 4 sections of Mathematics for the Elementary Teacher, 5 sections of precalculus mathematics, 1 section of an upper-division mathematics course, and 1 section of calculus.

The S1 instructor and the investigator shared an office at one point during

their graduate work. There had been many occasions when the investigator shared her interest and insights concerning graphs before the beginning of the present study. On a few occasions, the S1 instructor remarked that the investigator's work with Cartesian graphs had influenced her own teaching. During the study and in the 3 months preceding the study, the investigator was careful not to share her particular use of graphs in the teaching of calculus with the S1 instructor, so as to not further influence the S1 instructor's use of graphs in her own teaching.

Schedule. The S1 treatment was randomly assigned to the 12:00 p.m. section.[2]

Grading Policy. Four 100-point exams, ten 15-point quizzes, and a 225-point departmental final exam were given during the semester in the S1 section. Subjects were allowed to drop the score on one quiz. This policy had the effect of allowing students to miss one quiz completely, causing a high incidence of missing data for this section. Exams comprised approximately 53%, quizzes approximately 17%, and the departmental final 30% of the students' grades.

[2]The 12:00 p.m. section was included in the present study due to its close proximity in time to the 9 and 10 o'clock sections. In retrospect, it might have been better to avoid the noon hour section. WMU is located in Kalamazoo, Michigan. Kalamazoo's work force is heavily white collar. Many companies encourage their employees to continue their educations by providing them time off to attend classes, reimbursing them for tuition and fees if grades are sufficiently high – usually a C or better for undergraduate courses, and/or awarding them raises depending on their levels of education. Such students have added incentive for performing well. These students most often enroll in evening and noon-hour sections. Data for class levels, curriculum, and majors are given in Chapter IV in the section entitled "Profile of Subjects" for Investigation 2, Analysis 2. No data was gathered concerning employment or company benefits. It is not known if any of the subjects in this section satisfied the above criteria.

No homework assignments were collected or graded.

Instructional Style. Based upon the S1 instructor's self-report, her teaching style is best described as interactive. She asked many questions of subjects. On each occasion, at least one, but usually more than one, subject answered the instructor's question, even if the answer was not correct. The S1 instructor waited a short while for the correct answer. However, if the answer was too slow in coming, she reported to having answered her own question.

At the beginning of each class session, subjects requested that the instructor solve specific problems from previous assignments. About 15 to 20 minutes of each class period were devoted to such review. During presentation of new material, students rarely asked questions. On a few occasions, subjects asked the instructor to explain again an idea that had just been presented.

Instructional Emphasis. The concepts of limit, continuity, and derivative were developed through the use of Cartesian coordinate graphs as these could be drawn by hand by the S1 instructor on the blackboard. Simple graphs such as lines, parabolas, absolute value of x, and rational functions such as $f(x) = \frac{x^2-4}{x+2}$ were often used to illustrate calculus ideas. On the few occasions where they were used, more complicated functions were drawn by plotting points. Trigonometric functions other than sine and cosine were not graphed. The S1 instructor's use of graphs on the blackboard provided students a medium through which concepts could be studied globally as well as pointwise. The S1 instructor used graphs more often in presentation of concepts than

was suggested in the course text and the accompanying instructor's manual. However, whenever the course text used graphs in concept development; such as the graphic development of the Bisection Method, Tangent Approximations and Differentials, and Newton's Method; the S1 instructor's presentation followed that given in the text.

For the most part, the S1 instructor followed the course syllabus, which coincided with the sequence of the course text. There were three exceptions: (1) Section 2.6: The Chain Rule, was introduced after Section 2.2: Basic Rules of Differentiation; (2) the definition of inverse functions were reviewed briefly but the rest of Section 2.7: Derivatives of Inverse Functions, was omitted; and (3) Section 3.3: Increasing and Decreasing Functions/The First-Derivative Test, and Section 3.7: Curve Sketching, appeared in the course syllabus, but were covered very lightly since ideas for these sections had been covered in detail through the graphical presentation of the concept of derivative.

The emphasis in the S1 instructor's approach was on conceptual understanding, often developed through a graphic medium. Skill development, while considered important, was delegated a lesser role. Class instruction and exam and quiz items reflected this emphasis. At the beginning of each 50-minute class session, a minimum of 15 minutes was devoted to review of previous assignments. Subjects' questions tended to pertain to exercises requiring manipulative skill. During this time, an effort was made to explain why the steps followed in solving each problem were sensible and why the result was reasonable. Emphasis

in class, while presenting new material, was on conceptual understanding.

To illustrate the conceptual, graphical instructional approach used with the S1 section, several examples of content presentations are given. In some cases, graphs of functions were used to motivate theorems. In others, theorems were first stated then graphically illustrated.

The following examples illustrate the use of graphs to motivate theorems:

1. When determining $\lim_{x\to 3} f(x)$ where $f(x) = \frac{x^2-9}{x-3}$, the S1 instructor asked subjects to describe the appearance of the graph of $f(x)$. Subjects suggested a few points on $f(x)$. The instructor suggested that subjects simplify $f(x)$. They observed that $f(x) = x + 3$ for $x \neq 3$. The instructor graphed $f(x)$ drawing an open circle at the point where $f(x)$ is undefined. Subjects then observed that

$$\lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} x + 3 = 6$$

from the graph. The theorem concerning limits of equivalent functions was then stated.

2. The graph of $f(x) = 2$ was drawn on the board. Subjects were asked to determine the slope of $f(x)$ at various points. The instructor then stated the rule for finding derivatives of constant functions. A similar approach was used to determine a rule for finding the derivative of $f(x) = mx + b$. Symbolic proofs were not given. Subjects were reminded that they could read the proofs given in the text.

3. The graphs of sine and cosine were drawn on the board. Subjects were asked to suggest values for the slopes of tangents at a few points on each curve. The instructor connected the dots and subjects determined that the derivatives of the sine and cosine function are the cosine and negative sine functions respectively.

In the following examples, theorems were first stated then illustrated graphically:

1. The properties of limits of the functions $f(x) = a$, $f(x) = x$, and $g(x) = a \cdot f(x)$ were stated. An appropriate graph was drawn to illustrate each property.

2. Once the concepts of limit, continuity, and derivative had been presented; the graphs such as $f(x) = |x|$ and $f(x) = \frac{x^2-4}{x-2}$ were used to illustrate the relationships among these.

Examples of conceptual development that were not presented graphically follow:

1. The Power Rule was stated and the symbolic proof was then given by the instructor.

2. The Chain Rule was presented through the use of an example of a food chain (Hurley, 1987, p. 106). In each problem requiring the use of the chain rule in determining the derivative of a composite function $h(x)$, subjects were asked to determine the functions $f(x)$ and $g(x)$ where $h(x) = f(g(x))$. For such functions, subjects were often reminded that the change in h depends

on both the change in f and the change in g.

3. In finding limits, in determining continuity, and in finding derivatives of equations involving the trigonometric functions other than sine and cosine, graphs were not drawn. All work and proofs were presented symbolically.

Instructional Time-Frame. The S1 instructor completed instruction of the calculus concepts covered by the study on Day 40. Review sessions over a subset of the material presented were held on Day 52 and on the day preceding the departmental final exam.

Treatment 4: Standard 2

The course presentation in the S2 treatment was typical of many college mathematics courses. The description of the features unique to the S2 section was derived from comments made by the S2 instructor during weekly meetings with the investigator, and through interviews of a colleague who worked closely with the S2 instructor. The S2 instructor began a leave of absence from WMU on Day 44 of the course after she had completed her presentation on the material relevant to the study.[3] The colleague mentioned above gave the remaining course presentations.

Instructor. Treatment S2 was taught by a second-year assistant professor of mathematics. Her prior college-level teaching experience included teaching 2

[3]The S2 instructor was not available for further interviews and could not check the accuracy of the final description of her section. The final description was verified, during an interview, by the colleague who completed teaching the S2 section in her absence.

sections of remedial mathematics, 2 sections of precalculus, 3 sections of Calculus I, 2 sections of Calculus II, 1 section of elementary linear algebra, and 4 sections of upper division mathematics courses.

Schedule. A second 10:00 a.m. section created as a result of large course enrollments, was designated as the S2 section. The random assignment of treatments to sections had taken place prior to the creation of this section. The instructor assigned to this section agreed when asked to participate in the study. Since one of the experimental sections also met at 10:00 a.m., it was assumed that there would be no bias introduced by including the S2 section in the study.

Grading Policy. Four 100-point exams, ten 15-point quizzes and a departmental final exam were given during the semester. Scores of the eight highest quizzes were averaged over 120 points. The cumulative quiz score was converted to a percent and counted as a fifth exam. Subjects were allowed to drop their lowest exam score, which included the score composed from their eight highest quizzes. This grading policy had the effect of allowing subjects to miss 2 quizzes and 1 exam completely, causing a high incidence of missing data for this section. Exams and quizzes comprised 70%, and the departmental final 30%, of the subjects' grades. No homework assignments were collected or graded.

Instructional Style. The S2 instructor used a lecture-discussion mode in her teaching. At the beginning of each class session, subjects requested that the instructor solve specific problems from previous assignments. The first 15 to 20

minutes of each 50 minute class period were devoted to such review. During these sessions, the instructor showed subjects the solution steps and provided an explanation of each.

The remaining 30 to 35 minutes of class were used to present new material. During this presentation, questions were often used as a way of motivating the need for an idea that was to be presented. These questions were often rhetorical, not requiring subject response.

Subjects in the S2 section were described as reserved by the interviewed colleague. The S2 instructor indicated that subjects did not respond readily to those questions for which a response was expected, although some subject(s) usually responded. Subjects did not often ask questions other than, "Would you explain that again?"

Instructional Emphasis. The S2 instructor followed, with few exceptions, the departmental syllabus whose sequence coincided with the course text. The course was technique-oriented although some conceptual development was presented. Time in class and questions unique to the S2 exams and quizzes reflected the manipulative skill emphasis of the text. Use of graphs in conceptual development was also typical of that presented in the text.

There were three exceptions to the S2 instructor's adherence to the course syllabus: (1) the applications presented as motivation or examples in the text were often omitted to save class time; (2) Section 2.6: The Chain Rule, was introduced immediately after Section 2.2: Basic Rules of Differentiation; and

(3) Section 2.7: Derivatives of Inverse Functions, was covered briefly, omitting the theory.

Instructional Time-Frame. Instruction of calculus concepts pertaining to the study was completed on Day 40. A quiz over this material was given by the S2 instructor on Day 43. On Day 44, her colleague began teaching the remainder of the course. He administered Exam 4, the post-attitude survey, and the departmental final exam.

A review session to prepare students for the comprehensive final exam was held on the day preceding the exam. A subset of the material presented in the study was reviewed during this session.

Prior Calculus Experience

Two levels of prior calculus experience were defined for the present study: (1) Prior calculus experience (P), defined to be experience with at least one-half of a semester of calculus which included the rules for finding derivatives; and (2) No prior calculus experience (N), defined to be either (a) no previous experience with calculus, or (b) experience only with the part of the course preceding the use of rules for finding derivatives.

During the pilot study, on a number of occasions, student comments led the investigator to suspect that prior calculus experience might be a significant factor influencing performance. Three examples of such occasions are given:

1. Early in the pilot study, to encourage student discovery of the rules for

finding limits, each rule was developed using graphs of functions presented on the computer. On a quiz where students were asked to find

$$\lim_{x \to -\infty} \frac{x}{\sqrt{x^2 - 9}}$$

a number of students attempted to find the limit by dividing all terms of $f(x)$ by the highest powered term in the denominator, a rule they had not learned in the presentation given in the course. Common errors included dividing by x^2 and dividing by x. Such errors occurred most often in students who had taken calculus previously.

Error patterns for students who have had prior calculus experience could be well-embedded, inhibiting retraining efforts.

2. A student, who seemed to have a particularly negative attitude toward the pilot course, expressed dissatisfaction saying that the present course was not at all like his former course. He queried as to when the "real calculus" would begin.

Student pre-conceived attitudes and beliefs about the course could affect their willingness to participate in a course with the same name but which is presented in a manner students do not readily recognize as familiar.

3. A student indicated that she had dropped Calculus I late in the previous semester because she felt that she did not understand what she was learning even though she had been receiving acceptable grades – As and Bs – throughout the semester. She remarked that the use of graphs had really helped her understand

the calculus. She was familiar with the rules from her former course, but now she knew why they worked and when to properly apply them.

Those students who had been successful in their previous course, but dissatisfied with their self-assessment of what they had learned, might benefit more from a different instructional approach.

Based upon these and other student comments and errors, the investigator decided to include student level of prior calculus experience as a blocking variable to account for some of the variability in their performance on exams and quizzes.

The investigator did not gather information as to whether the previous calculus experience had been in high school – usually reserved for the best and brightest students – or in college – usually indicating an unsuccessful first attempt. In retrospect, the investigator believes the additional information would have been more helpful in accounting for differences in student performance.

The Study

The purpose of the present study was to investigate the extent to which use of a Cartesian coordinate graphical representation system, developed and extended through the use of computer graphics software, aids student understanding of the concepts in a first calculus course. The study was organized into two parts, Investigation 1 and Investigation 2.

In Investigation 1, comparisons were made between the G and G+ treatment sections which differed in the level of exposure to a graphical representation

in a conceptual calculus course as computer software was available for graphic display. Of interest were comparisons of: (a) levels of student understanding of the Cartesian coordinate graphical representation system, including the use of such a representation system in understanding calculus concepts, and (b) student attitudes toward the use and usefulness of a graphical representation system in understanding calculus concepts.

In Investigation 2, comparisons were made between the G, G+, S1, and S2 treatment sections which differed in the level of exposure to a graphically-developed conceptual calculus course – varying from heavily graphic to a more traditional course. Of interest were comparisons of student: (a) performance on routine applied, routine symbolic, and nonroutine symbolic calculus questions; (b) performance on department final exam questions; and (c) attitudes toward mathematics.

The two investigations were similar in that much of the pre- and post-treatment data gathered was uniform for all sections. The investigations differed in the exam and quiz questions chosen for each analysis. The G and G+ sections were also asked to respond to an Exit Survey soliciting their opinions concerning the use and usefulness of graphs.

Investigation 1

To make the desired comparisons for Investigation 1, a 2×2 fixed effects factorial design was used. The design incorporated 2 factors: (1) a randomly assigned treatment variable – treatments varying with respect to the amount of

exposure to computer graphics software in a graphically-developed, conceptual calculus course; and (2) a fixed blocking variable – prior calculus experience.

The treatment variable had two levels: Graphics (G) and Graphics Plus (G+). The blocking variable had two levels: prior calculus experience (P), and no prior calculus (N). The design matrix with initial cell sizes is given in Table 3.

Table 3

Investigation 1: Design with Initial Cell Sizes

Prior Calculus Experience	Treatments	
	G	G+
P	13	17
N	18	19

Investigation 2

Investigation 2 was exploratory in nature, designed to generate questions for further study. Although statistical procedures were used to generate information about the relative effect of different instructional methods, adherence to strict experimental controls was not possible. While the results from this investigation

cannot be viewed as conclusive, they could provide information on which to base future research.

To compare treatment sections differing in the level of exposure to a graphically-developed, conceptual calculus course, a 2 × 4 fixed effects factorial design was used. The design incorporated 2 factors: (1) a randomly assigned treatment variable – treatments varying with respect to the amount of exposure to a conceptually-developed graphic calculus course; and (2) a fixed blocking variable – prior calculus experience.

The treatment variable had 4 levels: Graphics (G), Graphics Plus (G+), Standard 1 (S1), and Standard 2 (S2). The blocking variable had two levels: prior calculus experience (P), and no prior calculus (N). The design matrix with initial cell sizes is given in Table 4.

Table 4

Investigation 2: Design with Initial Cell Sizes

Prior Calculus Experience	Treatment			
	G	G+	S1	S2
P	13	17	14	14
N	18	19	19	14

Instrument Design and Procedures

The following data was collected for the subjects participating in the investigation listed:

1. Covariates for both Investigation 1 and 2: (a) score on the Pretest of precalculus symbol manipulation competence and precalculus facility with a Cartesian graphic representation system (PRETEST), (b) pre- attitude scores pertaining to student: level of interest in mathematics (PREIIM), and interest in taking more courses in mathematics (PREITM);

2. Dependent variables for Investigation 1: (a) scores on subscales of graphic exam and quiz questions categorized as: (i) the development of symbol manipulation procedures of a graphic representation (GLEVEL2); (ii) the use of graphic symbols and rules as referents for building more abstract systems, further categorized as applied (GAPP); symbolic routine (GSR); and symbolic nonroutine (GSN); (b) post- attitude responses pertaining to student perceptions of the use and usefulness of a Cartesian coordinate graphic representation in understanding calculus concepts (ESn; $n = 3, 4a, 4b, 4c, 4d, 4e$);

3. Dependent Variables for Investigation 2: (a) scores on subscales of exam and quiz questions categorized as: applied (APP), symbolic routine (SR), and symbolic nonroutine (SN); (b) post-attitude scores pertaining to student level of interest in mathematics (POSTIIM), and interest in taking more courses in mathematics (POSTITM); (c) pre- and post-attitude responses pertaining to subjects' perceptions of the nature of mathematics (PREPNMn and POSTPNMn;

$n = 1,2,3,4,5$); (d) scores on a uniform comprehensive departmental final exam (FINAL), further categorized into the subscales: (i) questions pertaining to limits, continuity, and derivatives (FDERIV), and (ii) questions not pertaining to the above, which included questions pertaining to precalculus, antiderivatives, and integral ideals which were not covered in the study (FANTI); and (e) responses to evaluation survey items pertaining to course content, processes, assignments, text, materials, exams, and time commitment (EVAL$n : n = 16,17,18,19,20,21,22$).

Covariates

Students entering Calculus I vary greatly in mathematical ability and attitudes toward mathematics. Differences in student performance on the instruments comprising the dependent variables were expected to reflect these pre-existing conditions.

To determine pre-existing differences in student attitude and ability, the following instruments were used: (a) an attitude survey concerning subject interest in mathematics and interest in taking more courses in mathematics, and (b) a pretest measuring subject precalculus symbol manipulation and graphing competencies. Student scores on these instruments were used as covariates in subsequent analyses.

Pretest. The pretest is a multiple-choice assessment of subject precalculus symbol manipulation competencies and facility with a Cartesian coordinate

graphic representation system. The pretest was used to assess pre-experimental differences among treatments, and as a covariate in further analyses. It was administered to all subjects on Day 3 of the study.

The pretest consists of 20 items that were either investigator-written or adapted from one of several sources. The instrument and a complete description of its construction is given in Appendix D.

An item analysis performed on the pretest given to all subjects ($N = 127$) in the study indicated low discrimination indices for 4 of the items. These were subsequently removed from the instrument. The reliability of the resulting pretest was calculated as .56 using the Kuder-Richardson Formula 20. This reliability coefficient is on the low end of the acceptable range.

The pretest was examined for content validity by a group of five faculty members in the Department of Mathematics and Statistics at WMU. The questions used in determining covariate scores and pre-experimental differences were judged as valid by this group.

Attitude Survey. The Attitude Survey consists of 15 statements designed to assess student attitudes toward mathematics. The attitude survey is composed of two subscales, each containing 5 items, designed to assess subject attitudes toward: (a) interest in mathematics, and (b) interest in taking more courses in mathematics. The remaining 5 items were designed to assess various student perceptions of the nature of mathematics.

The survey was administered to all subjects on Day 3 and again on Day 49 of

the study. The pre-attitude assessments of interest in mathematics (PREIIM), and interest in taking more courses in mathematics (PREITM) were used: (a) to assess pre-experimental differences among treatments; and (b) as covariates in further analyses.

Items for the survey were either taken directly or adapted from items in the *Scale of Attitudes toward Mathematics* (Aiken, 1979) or items on a survey given by Heid (1984). All items were Likert-type with 5 possible responses ranging from strongly agree to strongly disagree. The responses were scored from 1 to 5. For responses to PREIIM, PREITM, POSTIIM, and POSTITM items, 5 indicated the most positive response toward mathematics. The instrument and a complete description of its construction is given in Appendix E.

Internal reliabilities were determined for the pre-attitude subscales for subjects ($N = 127$) involved in the study on Day 3 using Cronbach's Alpha. Internal reliabilities using Cronbach's Alpha were also determined for the post-attitude subscales for subjects ($N = 126$ for POSTIIM and $N = 127$ for POSTITM) involved in the study on Day 49. These are given in Table 5.

Table 5

Internal Reliabilities for Pre-attitude Covariates

	Cronbach's Alpha	
Subscale	PRE	POST
IIM	0.73	0.77
ITM	0.82	0.75

Dependent Variables: Cognitive Measures

Exam and Quiz Subscales. Exam and quiz questions were free response assessments of subject understanding of: (a) the calculus concepts of limit, continuity, and derivative; and/or (b) a Cartesian graphic representation system. These questions were categorized into several subscales. The subscales were used as cognitive dependent variables in comparisons made in Investigations 1 and 2. Each of the questions comprising the subscales was administered to subjects by their respective instructors on one of 4 course exams or 7 quizzes.

The exam and quiz questions that comprise each of the subscales were either written by the instructors participating in the study or adapted from other sources. The instructors made joint decisions concerning the selection of

questions appropriate for all treatment sections. The individual subscales, a list of the sources used, and a description of the selection process are given in Appendix D.

Questions used to assess student understanding in calculus were categorized as follows:

1. Applied vs. Symbolic – questions are considered to be applied if they required application to a real-world model or interpretation of a real-world model. Symbolic questions required no such real-world interpretation or application.

2. Routine vs. Nonroutine – symbolic questions were further categorized as routine or nonroutine. Questions are considered nonroutine if a student attempting such a problem possesses neither a known answer nor a previously established (routine) procedure for finding one (Malone, Douglas, Kissane, and Mortlock, 1980). Questions are considered routine if they appeared in the same or only slightly different form in the presentation of the text and/or on homework assignments.

Subject exam and quiz questions used in the study were photocopied and then graded by the investigator using the Focused Holistic Scoring Point Scale. Subjects earned 0 through 4 points for their responses to questions on exams and quizzes. If a subject missed an assessment, no scores were given to the subject on questions which appeared on that assessment. The Focused Holistic Scoring Point Scale, including a full description of its construction and implementation, is contained in Appendix C.

Individual exam and quiz questions were examined for content validity by a group of five faculty members in the Department of Mathematics and Statistics. Minor revisions in the wording of some questions were suggested by this group. Three members of this group also examined the final collection of common questions and found them to be appropriate in terms of breadth and depth for a first course in calculus.

Exam and Quiz Subscales: Investigation 1. Subscales comprised of exam and quiz questions that required the use of a graphic representation were used in comparisons between the G and G+ sections. These subscales were used to assess subject level of understanding of a Cartesian coordinate graphical representation system, including the ability to use such a representation system in understanding calculus concepts. They included exam and quiz questions that required: (a) translation from a graphic representation to another representation, (b) translation from another representation to a graphic representation, or (c) transformation between graphic representations. Such questions were categorized into the two components of symbol systems identified below (and elaborated in Appendix B):

1. Level 2: The development of symbol manipulation procedures for a graphic representation (GLEVEL2), and

2. Level 4: The use of graphic symbols and rules as referents for building more abstract systems (Hiebert, 1987).

The Level 4 questions were further categorized as applied (GAPP), symbolic

routine (GSR), and symbolic nonroutine (GSN). Once the categorizations had been made, these were verified by a group of four faculty members in the Department of Mathematics and Statistics.

Internal reliabilities for each subscale in Investigation 1 were calculated using Cronbach's Alpha. Of the 67 subjects participating in Investigation 1, only scores for subjects with complete data sets for the particular subscale were included in these calculations. The reliabilities are presented in Table 6.

Table 6

Internal Reliabilities of Exam and Quiz Subscales, Investigation 1

Subscale	Number of items	N	Cronbach's Alpha
GLEVEL2	5	62	0.61
GAPP	8	66	0.64
GSR	15	65	0.70
GSN	11	58	0.70

<u>Exam and Quiz Subscales: Investigation 2</u>. Questions common to the exams and quizzes of sections G, G+, S1 and S2 were used to construct subscales to assess subject understanding of calculus concepts.

Questions used in the Investigation 2 subscales were chosen by the three instructors participating in the study to be fair and appropriate for subjects

in all sections. The instructors administered the questions to their individual sections on exams or quizzes at times when these were most appropriate for the emphasis in their classes. A cross-reference listing each question number and the day it was administered to each section, is given in Appendix D. The questions used in the subscales for Investigation 2 were categorized as applied (APP), symbolic routine (SR), and symbolic nonroutine (SN) as indicated earlier. Once the categorizations had been made, these were verified by a group of four WMU mathematics professors.

Internal reliabilities for each subscale used in Investigation 2 were calculated using Cronbach's Alpha. Of the 128 subjects participating in Investigation 2, only scores for subjects with complete data sets for a particular subscale were included in the reliability calculation. The reliabilities are presented in Table 7.

Table 7

Internal Reliabilities of Exam and Quiz Subscales, Investigation 2

Subscale	Number of Items	*N*	Cronbach's Alpha
APP	5	119	0.70
SR	37	107	0.85
SN	16	107	0.77

Departmental Final Exam. The uniform comprehensive departmental final exam (FINAL) is a multiple-choice assessment of subject academic progress in the Calculus I course. Questions for the exam were submitted by the six WMU instructors who taught Calculus I during the Winter 1988 semester. Another mathematics faculty member selected the questions for the final from those submitted, rewriting those that were inappropriate or ambiguous. The final was then approved by a committee to be administered in three parallel forms to all students taking Calculus I during the Winter 1988 semester.

Two subscales were identified for the final exam (see Appendix D):

1. Questions over the material covered in the present study – limits, continuity, and derivatives (FDERIV), and

2. Questions over the remaining material – precalculus, antidifferentiation, and integration (FANTI).

Internal reliabilities for the final exam and each of the subscales were calculated using the Kuder-Richardson Formula 20. Of the 128 subjects participating in Investigation 2, five subjects responded to Form C of the departmental final. One of the questions on this form was not similar to any question on Forms A and B. These subjects' scores were not used to calculate reliabilities for FINAL and FANTI. The reliabilities are presented in Table 8.

Table 8

Internal Reliabilities for the Departmental Final Exam

Scales	Number of Items	N	Kuder-Richardson-20
FINAL	30	123	0.68
FDERIV	18	128	0.51
FANTI	12	123	0.59

Dependent Variables: Affective Measures

Attitudes towards Graphs: Investigation 1. An exit survey containing multiple-choice and free-response questions concerning subjects (a) uses of graphs, and (b) attitudes toward the use and usefulness of graphs, was developed for use in the G and G+ sections. Through the exit survey, subjects in the G and G+ sections were asked to report their post-treatment attitudes toward the usefulness of hand-drawn and computer-generated graphs, and their opinions of the computer graphics demonstrations used throughout the course (see Appendix E).

Two separate forms of the survey were given - one to the G treatment and one to the G+ treatment. The G+ survey contained all of the questions on the G survey. In addition, the G+ survey also solicited information concerning student use of the computer software, *Master Grapher* (Waits & Demana, 1987) on assignments and subsequent coursework.

These surveys were investigator-written and piloted. Subjects were given 2 weeks in which to respond to the survey at the end at the semester. Subjects were informed that the surveys would not be read or tallied until final grades had been turned in to the records office. They were also awarded 5 points toward their grade if they turned in their responses by the final exam day.

Post-Attitude Survey: Investigation 2. The post-attitude survey was designed to assess subject attitudes toward mathematics upon completion of the study. The attitude survey given on Day 3 was re-administered to all subjects on Day 49. Post-attitude measures assessed subject attitudes toward: (a) interest in mathematics (POSTIIM), (b) interest in taking more courses in mathematics (POSTITM), and (c) perceptions of the nature of mathematics (POSTPNMn; $n = 1,2,3,4,5$).

A large number of subjects were absent on Day 49. Each instructor followed-up on these "missing" students by having as many of them as possible complete the post-attitude survey. Data is not available for the full sample. Comparisons of pre- and post-attitudes across treatments were made for those subjects completing both surveys.

Evaluation Survey: Investigation 2. The evaluation survey is an assessment of subject opinions concerning the course content and/or processes, assignment selection, text and materials, and examinations. It also solicited subject general opinions concerning the course, the amount of understanding gained, and the amount of time invested. The survey items were used in comparisons across

treatments.

The Evaluation Survey was administered to subjects in all sections. Most of the subjects in sections G, G+, and S1 responded to these statements anonymously via the departmental student evaluation survey, immediately following their response to the post-attitude survey. Subjects in the S2 Section and subjects in the G, G+, and S1 sections who were not in class when the departmental student evaluations were administered, responded to the seven statements of the Evaluation Survey (see Appendix E) after they responded to the post-attitude survey.

Limitations

The following limitations are identified for the study:

1. The G, G+, and S1 treatments were assigned at random to 3 sections of Calculus I meeting early in the day during the Winter 1988 semester. A second 10:00 a.m. section was added due to large enrollments. The instructor assigned to this section was asked to participate in the study after the random assignments were made. Since one of the experimental sections met at 10:00 a.m. and the S2 section also met at 10:00 a.m., it was assumed that no bias would be introduced by including the S2 section in the study.

2. Subjects participating in the study were those students who enrolled in the four sections identified as G, G+, S1, and S2. It was assumed that each section was equally attractive to all subjects, ie. no pre- experimental bias was introduced by subject selection of calculus sections. Subjects' selections were not

completely random however. While most students indicated that they had chosen the section in which they were enrolled because it fit their schedule, one subject in the S1 treatment and eight subjects in the S2 treatment indicated that they had chosen those particular sections on the basis of prior information obtained about the course instructor.

3. The S1 section met during the noon hour. Nontraditional students tend to enroll more often in noon and evening sections.

4. The descriptions of the S1 and S2 treatments were based on self-report data from the instructors of each of these sections. The final description of the S1 section was verified by the S1 instructor. The final description of the S2 section was verified by the colleague of the S2 instructor who completed the instruction of the course. In neither case were outside observations made by the investigator nor any other objective observer. Such observations were not made for two main reasons: (a) the environments of each section were preserved to be as close to an intact classroom setting as possible without outside interference; and (b) subjects were not informed that they were participating in a study. Outside observations might have caused them to suspect such participation.

5. The pretest was given on the third day of the semester at the request of one of the instructors who expected to be out of town that day. All instructors agreed to cover the same textbook material in the preceding two days as prescribed by the course syllabus. Subjects in all sections were given identical assignments throughout the semester and in particular prior to the pretest. Subjects were told

to expect a quiz on Day 3 to assure their attendance in class for the pretest. Three limitations arise from the administration of the pretest: (a) subjects in Section S2 were told which subset of the pretest questions would be graded as a quiz. Subjects in G, G+, and S1 sections were told only that a subset of the pretest questions would be graded as a quiz with no information as to which these would be; (b) the graphic approach which characterized the G and G+ treatments was delayed until Day 4; and (c) the results on the pretest, intended as a measure of pretreatment abilities, were influenced by the first two days of instruction since the material of the first two days included a precalculus review.

6. Grading policies for each section were determined independently by their instructors. Each instructor gave 10 quizzes, 4 exams, and the departmental final. The S1 and S2 instructors each allowed subjects to drop their lowest quiz score(s). Such policies resulted in a high incidence of missing data for subjects in these sections.

7. All exam and quiz questions used in the study were graded by the investigator. This required that all exams and quizzes be photocopied before the instructors graded them. The copying process was not completed on one occasion resulting in data being lost for three subjects in the S2 section and one subject in the S1 section.

8. Exam and quiz questions were not always given on the same day across all sections. During the first two months of the study the pace of the S2 instructor was much faster than that of the other two instructors. Subjects in this section

responded to certain questions before the subjects in sections G, G+, and S1. It is not known if subjects participating in the four sections of the study were aware that many exam and quiz questions were common to these sections.

Two comments were made in this regard: (a) a student in the S1 section had a girlfriend in S2 section. He mentioned to his instructor late in the semester that her tests and quizzes seemed to have a lot of the same questions as his. His grades did not reflect that he had been aware of such a coincidence before preparing for the fourth exam. (b) A student in the S1 section observed the investigator returning a folder of old exams to the S1 instructor late in the semester. He mentioned to his instructor that he hoped she wasn't writing the fourth exam with the investigator since he had heard about the difficulty of the investigator's exams.

The first comment leads one to wonder if other subjects were aware of the level of collaboration on exams and quizzes. The second comment assures the investigator that at least some subjects were unaware of the level of collaboration between instructors participating in the present study.

9. Subjects in the G+ section were aware that they were being required to complete assignments that were not given to the subjects in the G section. Some subjects did not see the availability of the software, *Master Grapher* (Waits & Demana, 1987a), as a privilege. In some cases, they saw the use of the software as an added burden to an already difficult course. Since these assignments were not collected or graded, it is likely that some of the G+ subjects did not complete

the supplemental assignments.

CHAPTER IV

DATA ANALYSIS

The analyses conducted in the present study are reported in two parts: (1) Investigation 1 – comparisons relative to the G and G+ treatments only, and (2) Investigation 2 – comparisons relative to all four treatment sections. In each case, the following are given:

1. the hypotheses,
2. the variable names and descriptions,
3. a description of the analyses performed for each investigation including sample populations for each,
4. descriptive statistics for each variable,
5. inferential statistics pertaining to each of the hypotheses, and
6. other descriptive data.

Investigation 1

Hypotheses

The following hypotheses were investigated to make comparisons between treatment sections differing in the level of exposure to a graphical representation in a conceptual calculus course:

1. There is no significant difference between treatments in: (a) subject performance on exam and quiz questions requiring use of the Cartesian coordinate graphic representation system, including the use of such a representation system in understanding calculus concepts; and (b) subject attitudes toward the use and usefulness of the Cartesian coordinate graphic representation system in understanding calculus concepts.

2. There is no significant difference between subjects with no prior calculus experience and subjects with prior calculus experience in performance on exam and quiz questions requiring use of the Cartesian coordinate graphic representation system, including the use of such a representation system in understanding calculus concepts.

3. There is no significant interaction between treatment and prior calculus experience in subject performance on exam and quiz questions requiring use of the Cartesian coordinate graphic representation system, including the use of such a representation system in understanding calculus concepts.

Variable Names and Descriptions

The abbreviations for the variables used throughout the following discussions are given below:

Covariates:

PRETEST: The score on the precalculus symbol manipulation and graphing abilities pretest consisting of eight symbol manipulation questions and eight graphic questions.

PREIIM: The score on the pre-attitude survey for the five statements pertaining to interest in mathematics.

PREITM: The score on the pre-attitude survey for the five statements pertaining to interest in taking more mathematics courses.

Cognitive Dependent Variables:

GLEVEL2: The score on the five exam and quiz questions pertaining to development of symbol manipulation procedures for a graphic representation system.

GAPP: The score on the eight applied exam and quiz questions requiring use of a graphic representation as a referent for building more abstract symbol systems.

GSR: The score on the 15 symbolic routine exam and quiz questions requiring use of a graphic representation as a referent for building more abstract symbol systems.

GSN: The score on the 11 symbolic nonroutine exam and quiz questions requiring use of a graphic representation as a referent for building more abstract symbol systems.

Affective Dependent Variables:

ESn: The score on the exit survey item concerning attitudes towards the use and usefulness of graphs (n = 3, 4a, 4b, 4c, 4d, 4e).

Factors:

TRTMNT: Treatment

G: Graphics (conceptual course with computer graphics in class).

G+: Graphics Plus (same course as G plus software and supplemental assignments).

PRICALC: Prior calculus experience

N: less than half a semester.

P: at least half a semester.

Two separate analyses were performed for Investigation 1:

1. Analysis 1: To test hypotheses 1a, 2 and 3, a multivariate analysis of covariance was performed on the cognitive dependent variables, GLEVEL2, GAPP, GSR, and GSN; using PRETEST, PREIIM, and PREITM as covariates.

2. Analysis 2: To test hypothesis 1b, a χ^2 test was performed to determine treatment differences in the distributions of subject responses.

Analysis 1

The Final Sample

Of the subjects participating in Investigation 1, 30 of the G subjects and 31 of the G+ subjects were retained in Analysis 1. Retained subjects for this analysis are defined to be subjects for whom data sets were: (a) complete for subscales GLEVEL2, GAPP, and GSR, and (b) missing no more than one entry for the GSN subscale. The entry for subjects missing one GSN data point was replaced with the subject's average score for the remaining GSN responses. Such adjusted scores were used in this analysis. Removed subjects are subjects whose data sets did not meet the above criteria.

For each of the G and G+ treatments, comparisons were made between retained subjects and removed subjects on the covariates, PRETEST, PREIIM, and PREITM, and on the final exam score, FINAL.

In the G treatment, only one subject was removed from the analysis. Descriptive statistics for the retained subjects and scores for the removed subject are given in Table 9. The highest possible scores for each variable is listed to provide a basis for comparison.

Table 9

G Treatment: Retained Subjects vs. Removed Subject, Investigations 1 and 2, Analysis 1.

Variables	Highest Possible Score	Retained ($n = 30$) *M*	*SD*	Removed ($n = 1$) Score
PRETEST	16	11.77	2.36	12
PREIIM	25	19.07	2.36	16
PREITM	25	18.37	2.95	14
FINAL	30	12.53	3.84	8

With the exception of the PRETEST, the removed subject's scores were 12% to 17% lower than the scores of subjects retained in the analysis. PRETEST scores were comparable.

Five of the G+ subjects were removed from the full sample for Analysis 1. The descriptive statistics for the 31 retained subjects and 5 removed subjects for the variables, PRETEST, PREIIM, PREITM, and FINAL are given in Table 10. Means for the variables PRETEST, PREIIM, and PREITM are comparable. The FINAL mean for the removed subjects is somewhat lower than that of the retained subjects.

Table 10

Descriptive Statistics for G+ Treatment: Retained vs. Removed Subjects, Investigation 1, Analysis 1.

	Retained Subjects ($n = 31$)		Removed Subjects ($n = 5$)	
Variables	*M*	*SD*	*M*	*SD*
PRETEST	11.81	2.15	11.20	2.28
PREIIM	17.26	3.50	18.20	2.05
PREITM	17.81	4.05	18.00	3.74
FINAL	13.23	4.19	9.60	2.61

For the G+ treatment, to compare retained subjects with removed subjects on PRETEST, PREIIM, PREITM, and FINAL, the folded form of F-statistic[1], F', was used to determine equality of variances for each variable. All variances were determined to be equal ($p < .10$). T-tests were conducted to determine equality of means for removed vs. retained G+ subjects. The results of the t-tests are presented in Table 11. No significant differences were detected between groups for any of the variables.

[1] $F' =$ (larger of s_1^2, s_2^2)/ (smaller of s_1^2, s_2^2).

Table 11

G+ Treatment: Comparisons of Retained vs. Removed Subjects, Investigation 1, Analysis 1.

Variable	*t*	*df*	*p*
PRETEST	-0.58	34	0.565
PREIIM	0.58	34	0.565
PREITM	0.10	34	0.921
FINAL	-1.87	34	0.071

Profile of Subjects. To further describe subjects retained in the final sample for Analysis 1, data pertaining to sex, class, and college are displayed in Figures 3, 4, and 5 respectively.

The highest percentage of males were enrolled in the G+ section. The highest percentage of freshmen –class level 1 – were enrolled in the G section. College enrollments were similar, with subjects in the G section favoring the College of Arts and Sciences more heavily than subjects in the G+ section.

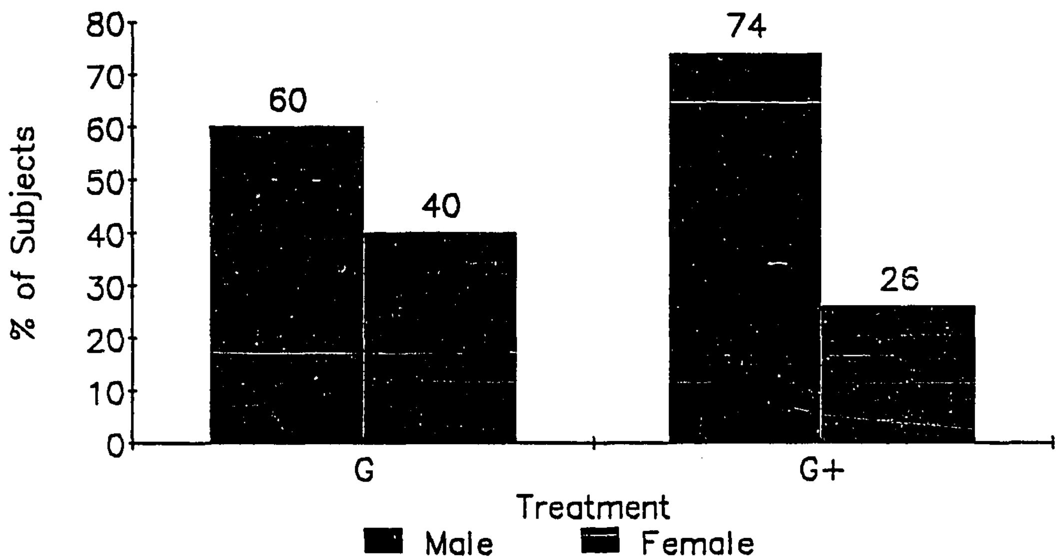


Figure 3. Males versus females, Investigation 1, Analysis 1.

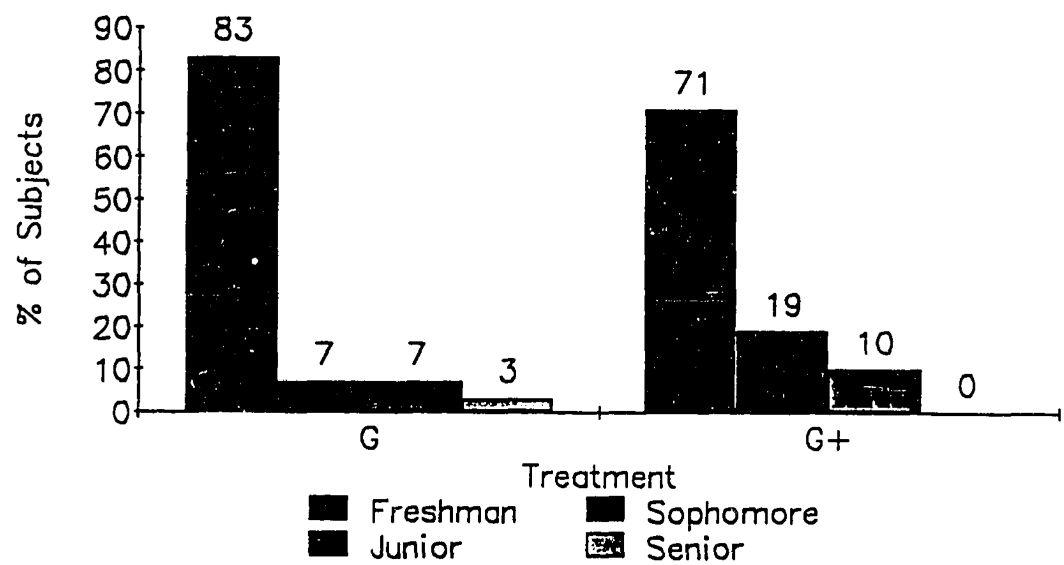


Figure 4. Class level, Investigation 1, Analysis 1.

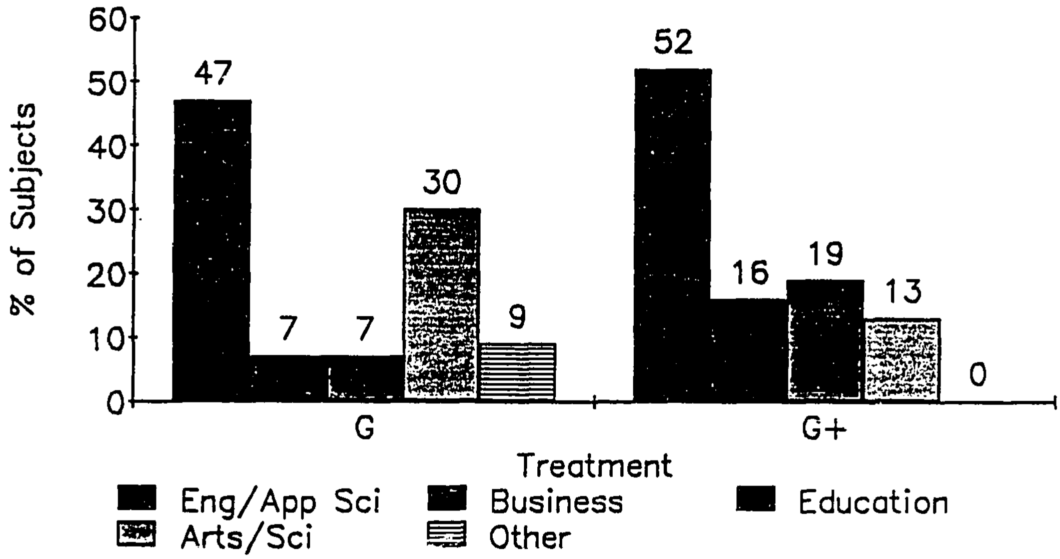


Figure 5. College enrollment, Investigation 1, Analysis 1.

Pre-Investigation Differences. Random assignment of subjects to treatments was not possible under WMU course registration procedures. Treatment sections were assumed to be equal in mathematical ability and attitudes prior to treatment. To test this assumption, PRETEST, PREIIM, and PREITM scores were compared for subjects in the G and G+ sections. Sample means and standard deviations for the variables are listed in Table 12.

Table 12

Descriptive Statistics for Pre-Treatment Measures,
Investigation 1, Analysis 1

	TRTMNT			
Variable	G ($n = 30$)		G+ ($n = 31$)	
	M	*SD*	*M*	*SD*
PRETEST	11.77	2.36	11.81	2.15
PREIIM	19.07	2.36	17.26	3.50
PREITM	18.37	2.95	17.81	4.05

Treatment means are similar for PRETEST and PREITM. The PREIIM mean for the G+ treatment is somewhat lower than for the G treatment. This difference suggest the possibility of pre-experimental treatment bias.

To test the hypothesis that sections did not differ significantly in pre-treatment ability and attitude, one-way analyses of variance were performed for each of the variables, PRETEST, PREIIM, and PREITM. A summary of these analyses appears in Table 13.

Table 13

Analysis of Variance Results for PRETEST, PREIIM, and PREITM, Investigation 1, Analysis 1

Variable	*F*	*df*	*p*
PRETEST	0.000	1,59	.945
PREIIM	5.554	1,59	.022*
PREITM	0.379	1,59	.541

* $p < .05$

The analysis of variance supports the hypothesis of the equality of treatments on PRETEST and PREITM measures. Subjects in the G and G+ sections did not vary significantly in either precalculus symbolic and graphic competencies, or in their interest in taking more mathematics. They differed significantly in level of interest in mathematics (PREIIM) with subjects in the G treatment displaying a more positive interest level toward mathematics than subjects in the G+ treatment.

Design

For Analysis 1, Investigation 1, a 2×2 unbalanced fixed-effects factorial design was used with four dependent variables. The factors were treatment, G

and G+; and prior calculus experience, N and P. The four dependent variables were the graphic subscales, GLEVEL2, GAPP, GSR, and GSN. The design matrix indicating factors and corresponding cell sizes is given in Table 14.

Table 14

The Experimental Design with Final Cell Sizes,
Investigation 1, Analysis 1

PRICALC	TRTMNT G	G+
N	17	16
P	13	15

A 2 × 2 multivariate analysis of covariance was conducted for the graphic subscales, GLEVEL2, GAPP, GSR, and GSN. Although treatments did not differ significantly with respect to PRETEST and PREITM, these variables were used as covariates together with PREIIM to account for some of the variability in the model. The analysis was conducted using the SAS (1986) MANOVA program.

Raw cell means with standard deviations and cell means adjusted for the covariates with standard errors are presented in Table 15 for each of the graphic subscales.

Table 15

Raw and Adjusted Cell Means for GLEVEL2, GAPP, GSR, and GSN

Variable	TRTMNT							
	G				G+			
	PRICALC				PRICALC			
	N		*P*		*N*		*P*	
	Raw	Adj.	Raw	Adj.	Raw	Adj.	Raw	Adj.
GLEVEL2								
M	16.41	15.93	12.85	13.77	15.63	15.45	16.20	16.14
SD[a]	3.02	0.76	4.14	0.87	4.01	0.77	2.54	0.79
GAPP								
M	22.71	22.04	22.08	22.59	23.44	23.61	24.40	24.53
SD[a]	4.71	1.07	4.65	1.23	4.35	1.09	4.27	1.12
GSR								
M	38.06	36.68	36.92	37.22	36.00	36.83	39.40	39.81
SD[a]	10.05	2.11	9.08	2.43	8.79	2.15	7.73	2.20
GSN								
M	23.64	22.20	23.15	24.64	23.43	23.57	25.32	25.49
SD[a]	9.06	1.69	7.80	1.94	7.03	1.72	6.27	1.76

[a] Note: Standard errors are given for adjusted means.

Within treatment groups, adjusted means for subjects with prior calculus experience were higher than adjusted means for subjects with no prior calculus experience for all dependent variables except GLEVEL2. This suggests the possibility of a significant main effect for prior calculus experience. Adjusted means for subjects in the G treatment are slightly lower than for subjects in the G+ treatment, suggesting the possibility of a treatment main effect.

Profile curves of adjusted cell means by subscale for individual cells, G x N, G x P, G+ x N, and G+ x P, are presented in Figure 6. Profile curves are nearly parallel, suggesting no significant interaction of treatment by prior calculus experience. A summary of the multivariate analysis of covariance is presented in Table 16.

Hypothesis 3, there is no significant treatment by prior calculus experience interaction, was not rejected; $F(4, 51) = 1.07$, $p < .381$. The statistical analysis supports the observation made that profile curves appeared parallel indicating no interaction between treatment and prior calculus experience.

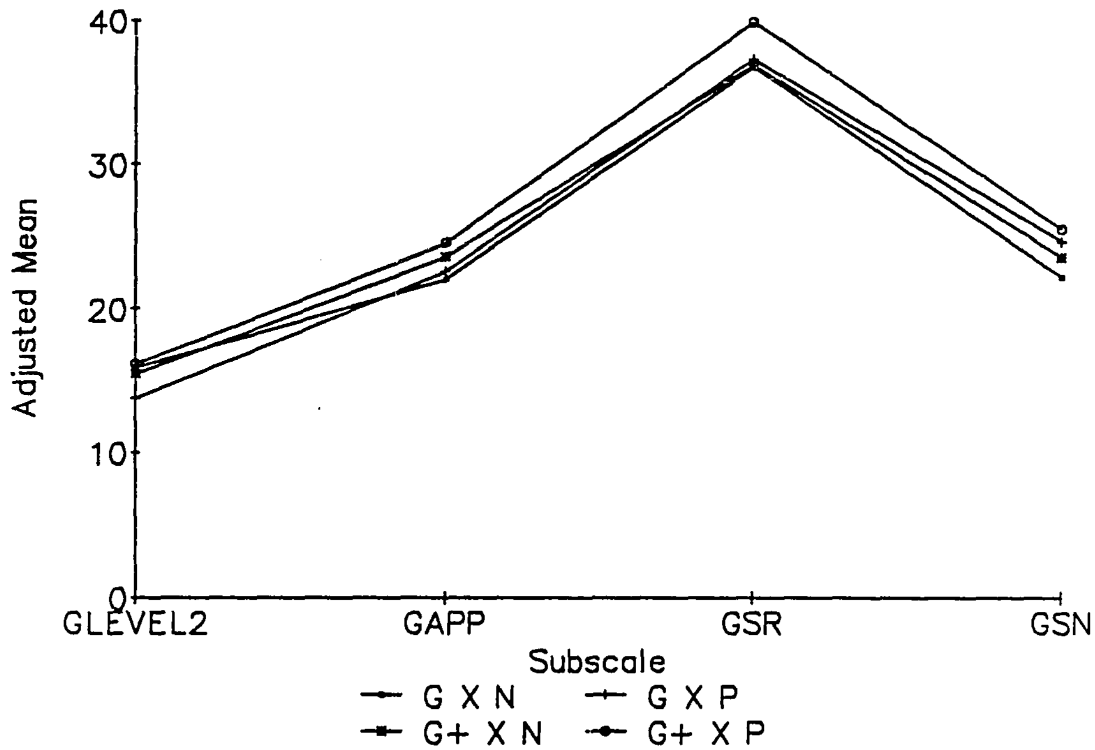


Figure 6. Profile curves, TRTMNT by PRICALC.

Table 16

Multivariate Analysis of Covariance for GLEVEL2, GAPP, GSR, and GSN

Source	*df*	*F*	*p*
TRTMNT × PRICALC	4,51	1.07	.381
TRTMNT	4,51	0.69	.600
PRICALC	4,51	0.85	.502

The multivariate analysis did not show a significant main effect for the treatment variable, $F(4,51) = 0.69$, $p < .60$. Subjects in treatments G and G+ did not vary significantly in their performance on the graphic subscales GLEVEL2, GAPP, GSR, and GSN. Hypothesis 1a, there is no significant difference between treatments, was not rejected.

The multivariate analysis also failed to show a significant main effect for the level of prior calculus experience, $F(4,51) = 0.85$, $p < .50$. While marginal means for subjects with prior calculus experience were slightly higher than for those subjects with no prior calculus experience, with one exception, these differences are not significant. Hypothesis 2, there is no significant difference between subjects with prior calculus experience and without prior calculus experience, was not rejected.

Analysis 2

Analysis 2 was conducted to test Hypothesis 1b, there is no significant difference between treatments on subject attitudes toward the use and usefulness of the Cartesian coordinate graphic representation system in understanding calculus concepts. Subject responses to Exit Survey items, ESn (n = 3, $4a$, $4b$, $4c$, $4d$, $4e$) were used as dependent measures for this analysis. Responses ranged from 1, strongly disagree, to 5, strongly agree.

The Final Sample

Of the subjects participating in Investigation 1, 27 of the G subjects, and 33 of the G+ subjects were retained in Analysis 2. Retained subjects for this analysis

are those subjects who responded to the Exit Survey item being analyzed.

Design

A one-factor, fixed-effects design was used for Investigation 1, Analysis 2. The factor was treatment, G and G+. For each of the six Exit Survey items soliciting subject attitudes toward the use and usefulness of Cartesian graphs, the frequencies of the subject responses were tabulated. χ^2 tests were then conducted for each item to determine if the frequency distributions were equivalent by treatment. Item statements, frequencies of responses, and χ^2 tests of significance are given for each of the six Exit Survey items.

ES3: How often do you find yourself using graphs when solving calculus problems?

The distributions of subject responses to this item, by treatment, are presented in Table 17. The mean response of the G treatment ($M = 3.07$) and that of the G+ treatment ($M = 3.03$) are very close, indicating that subjects in both treatments perceived themselves as using graphs in solving calculus problems about half of the time.

The null hypothesis, no significant difference between frequencies of responses by treatment, was not rejected; $\chi^2(4) = 4.789$, $p < .310$. Subject perception of the use of Cartesian graphs in solving calculus problems did not differ significantly between treatments.

Table 17

Frequency of ES3 Responses by Treatment

Response	TRTMNT G ($n = 27$)	G+ ($n = 33$)
1. Almost always		
Count	0	4
Column %	0.0	12.1
2. Most of the time		
Count	8	7
Column %	29.6	21.2
3. About half of the time		
Count	10	9
Column %	37.0	27.3
4. Some of the time		
Count	8	10
Column %	29.6	30.3
5. Almost never		
Count	1	3
Column %	3.7	9.1

ES4a: The computer graphs were useful in helping me understand calculus ideas.

The distributions of subject responses to this item are presented in Table 18. The distributions for both treatments appear to be similar. The mean response for the G treatment ($M = 3.93$) was slightly higher than that of the G+ treatment ($M = 3.73$) indicating that the subjects in the G section found the computer graphs slightly more useful in understanding calculus ideas than subjects in the G+ section. Subjects in both sections agreed that computer graphs were useful in helping them understand calculus ideas.

Table 18

Frequency of ES4a Responses by Treatment

Response	TRTMNT	
	G ($n = 27$)	G+ ($n = 33$)
2. Disagree		
Count	1	5
Column %	3.7	15.2
3. Undecided		
Count	5	6
Column %	18.5	18.2
4. Agree		
Count	16	15
Column %	59.3	45.5
5. Strongly Agree		
Count	5	7
Column %	18.5	21.2

The null hypothesis of no significant difference between treatment frequency distributions was not rejected for ES4a, $\chi^2(3) = 2.549$, $p < .467$. Subject perceptions regarding the usefulness of computer graphs in understanding calculus ideas were not significantly different between treatments.

ES4b: I would like to continue using graphs in other math classes.

The distributions of subject responses to this item are given in Table 19. Distributions are similar for both treatments. The mean response for the G treatment $(M = 3.59)$ was only slightly higher than that of the G+ treatment $(M = 3.52)$.

The null hypothesis of no significant difference between frequency distributions by treatment was not rejected, $\chi^2(4) = 1.425$, $p < .840$. Subjects in each treatment tended to agree that they would like to continue to use graphs in other mathematics courses.

Table 19

Frequency of ES4b Responses by Treatment

Response	TRTMNT G ($n = 27$)	G+ ($n = 33$)
1. Strongly Disagree		
Count	1	2
Column %	3.7	6.1
2. Disagree		
Count	3	3
Column %	11.1	9.1
3. Undecided		
Count	5	9
Column %	18.5	27.3
4. Agree		
Count	15	14
Column %	55.6	42.4
5. Strongly Agree		
Count	3	5
Column %	11.1	15.2

ES4c: I find graphs useful in helping me think about functions.

The distributions of subjects responses to this item are given in Table 20. The mean response for the G treatment ($M = 4.01$) was slightly higher than the mean response for the G+ treatment ($M = 3.97$).

Table 20

Frequency of ES4c Responses by Treatment

Response	TRTMNT	
	G ($n = 27$)	G+ ($n = 33$)
2. Disagree		
Count	1	1
Column %	3.7	3.0
3. Undecided		
Count	3	6
Column %	11.1	18.2
4. Agree		
Count	17	19
Column %	63.0	57.6
5. Strongly Agree		
Count	6	7
Column %	22.2	21.2

The null hypothesis of no significant difference in frequency distributions by treatment was not rejected, $\chi^2(3) = 0.594$, $p < .898$. There are no significant differences between treatments concerning subject attitudes toward the usefulness of graphs in helping them think about functions. Subjects in both treatments tended to agree that graphs are useful in helping them think about functions.

ES4d: I find I have to draw a graph to understand the problem.

The distribution of subject responses to this item are given in Table 21. The responses for the G+ section appear to be more evenly distributed than those of the G section. The G+ subjects' responses indicate a slightly higher use of drawn graphs in understanding problems than the G subjects' responses. The mean responses for the G and G+ treatments were 3.19 and 3.27 respectively. The differences between sections, while more pronounced for this item, do not appear to be significant.

Table 21

Frequency of ES4d Responses by Treatment

Response	TRTMNT	
	G (n = 27)	G+ (n = 33)
2. Disagree		
Count	6	9
Column %	22.2	27.3
3. Undecided		
Count	11	9
Column %	40.7	27.3
4. Agree		
Count	9	12
Column %	33.3	36.4
5. Strongly Agree		
Count	1	3
Column %	3.7	9.1

The null hypothesis of no significant difference in frequency distributions by treatment was not rejected, $\chi^2(3) = 1.645$, $p < .649$. Subjects did not feel the need to draw graphs to understand problems more frequently in one section than in the other. Subject responses indicate that overall they were undecided as to their need to draw graphs to understand problems. This pattern of response may have been due to the wording of the items. One subject, when explaining his response to this item, wrote "I find I have to draw 'or visualize' a graph to understand the problem." Perhaps if the item had been stated as this student suggested, responses might have been more definitive.

ES4e: I find that I use graphs to solve math problems more often now than I did in previous courses.

The distribution of subject responses to this item are given in Table 22. The distributions are similar for both of the G and G+ sections. The mean responses for the G and G+ treatments were 3.85 and 3.79 respectively. Overall, subjects' current tendencies to use graphs increased over their previous tendencies. Over 70% of all subjects agreed or strongly agreed that they used graphs in solving math problems more often now than in previous courses.

Table 22

Frequency of ES4e Responses by Treatment

Response	TRTMNT	
	G ($n = 27$)	G+ ($n = 33$)
1. Strongly Disagree		
Count	1	0
Column %	3.7	0.0
2. Disagree		
Count	3	6
Column %	11.1	18.2
3. Undecided		
Count	3	4
Column %	11.1	12.1
4. Agree		
Count	12	14
Column %	44.4	42.4
5. Strongly Agree		
Count	8	9
Column %	29.6	27.3

The null hypothesis of no significant difference in frequency distributions by treatment was not rejected, $\chi^2(4) = 1.773$, $p < .777$. Subject tendencies to use graphs in current vs. previous courses was not significantly different between treatments.

Additional Exit Survey Data

The Exit Survey contained several questions designed to solicit information concerning subjects': (a) individual uses of graphs, ESn $(n = 1, 2, 5)$; (b) opinions of the use of computer-generated graphs, ES6 and ES7; and (c) comparisons of previous with current calculus courses for subjects who had had prior experience with Calculus I, ES8a, b, c, and d. Items ES1 and ES2 were multiple-choice questions. The remaining items, ESn $(n = 5, 6, 7, 8a, b, c, d)$, asked for yes/no and open-ended responses. To compare sections G and G+ on the objective parts of each of the Exit Survey items, χ^2 tests were conducted to determine if the treatment frequency distributions were equivalent. For the open-ended items, common responses were tallied.

The Final Sample

Of the subjects participating in Investigation 1, 27 of the G subjects, and 33 of the G+ subjects responded to the Exit Survey. Retained subjects for the anecdotal data are those who responded to the Exit Survey item being reported.

The Survey and Responses

Item statements, frequencies of responses and either χ^2 tests of significance or a tally of the open-ended responses are given for each of the items ESn $(n = 5, 6, 7, 8a, b, c, d)$. For items ES1 and ES2, given only to the G+ subjects, frequencies of responses and percentages are reported with item statements.

ES1: How often did you use the computer on assignments where its use was required/suggested?

Table 23

Frequency of ES1 Responses

Response	Count	%
Very often	2	6.1
Often	7	21.2
Some of the time	8	24.2
Rarely	14	42.4
Never	2	6.1
Total	33	100.0

Item ES1 appeared only on the Exit Survey given to subjects in the G+ section. Subject responses are summarized in Table 23. Subject responses indicate that nearly half of the G+ subjects rarely or never used the computer on assignments requiring its use. Assignments were not collected or graded, a policy that probably contributed to the low level of student computer use.

G+ subjects who rarely or never used the computer on assignments where it was required or suggested received essentially the same treatment as subjects in

the G section. This might help to account for the lack of significant differences between these treatments in earlier analyses.

ES2: How often did you use the computer for the course when its use was not required/suggested?

Table 24

Frequency of ES2 Responses

Response	Count	%
Very often	1	3.0
Often	1	3.0
Some of the time	9	27.3
Rarely	7	21.2
Never	15	45.5
Total	33	100.0

Item ES2 appeared only on the Exit Survey given to subjects in the G+ section. Subject responses to this item are summarized in Table 24. Just over 33% of the subjects used the computer very often, often, or some of the time when it was not required or suggested. The remaining 66.7% of the G+ subjects rarely or never used the computer when its use was not required.

It is not surprising, viewing the results of ES1 and ES2, that the G and G+ sections did not differ significantly on any of the cognitive measures or affective

post-measures. Subject use of computer graphics outside of class did not seem to vary enough by section to cause a difference due to treatment.

ES5: Do you think there is any difference in the frequency with which you tend to use graphs now from work you have done in mathematics previous to this course?

Subject responses to this question are summarized in Table 25.

Table 25

Frequency of ES5 Responses by Treatment

TRTMNT	Response	
	Yes	No
G (n = 27)		
Count	23	4
Row %	85.2	14.8
G+ (n = 33)		
Count	27	6
Row %	81.8	18.2

The distributions of yes and no responses are similar for each section. Based upon a χ^2 test of this data, the null hypotheses of no significant difference between

treatment frequency distributions was not rejected; $\chi^2(1) = 0.121, \; p < .728$.

All subjects who responded "yes" to ES5 indicated through free response that their use of graphs had increased; in many cases this increased use of graphs was considerable. Five of the 10 negative responses indicated that these subjects already valued the use of graphs. The remaining five negative responses were not further qualified. In all, 26 of the 27 G subjects, and 29 of the 33 G+ subjects responded positively to this item in favor of the use of graphs. Over 90% of the G and G+ subjects indicated that they either made fairly heavy use of graphs in previous and current mathematics courses, or increased their use of graphs from previous courses to the current course.

Subjects were asked to express their opinions of the effect of the graphic approach to the course on their views of the usefulness of graphs through part 2 of question ES5:

Has the graphics approach to this course caused you to change your view of the usefulness of graphs in any way?

Subjects in both the G and G+ sections were in general agreement that graphs are very useful in the following ways: (a) helping the mind accept a lot of information at once (13 respondents); (b) building understanding of the relationships between graphs and symbols (5 respondents); (c) visualizing and explaining why certain definitions, theorems, and techniques are sensible and work the way they do (20 respondents); (d) finding a solution and/or determining the reasonableness of the solution (20 respondents); and (e) understanding functions and relation-

ships between similar functions such as $y = \sin 2x$ and $y = 2 \sin x + 6$ (14 respondents). Subjects generally agreed that the computer graphs aided them in their own efficient graphing of functions by hand (9 respondents). Negative responses indicated generally that subjects appreciated the use of graphs in class but did not care to use them in their outside work.

ES6: Are there any concepts that were developed with the computer or any programs that were demonstrated on the computer that you found particularly interesting or useful? If yes, what are they? Describe your reaction/reason.

Table 26

Frequency of ES6 Responses by Treatment

TRTMNT	Response	
	Yes	No
G ($n = 25$)		
Count	21	4
Row %	84.0	16.0
G+ ($n = 31$)		
Count	20	11
Row %	64.5	35.5

The responses for the subjects, given in Table 26, in the G section are somewhat more positive than those given by the subjects in the G+ section. However, based upon a χ^2 a test of these responses, no significant difference was detected between treatment frequency distributions; $\chi^2(1) = 2.679$, $p < .102$.

Of the 15 subjects who answered negatively, only 6 elaborated on their responses. Four of the six subjects indicated that they did not prefer a particular computer demonstration over all others. Had question ES6 been worded differently to allow for a response favoring the use of the computer in developing concepts in general rather than a particular concept developed through the use of the computer, the frequency distributions might have indicated an even stronger positive trend.

Positive responses indicate that subjects appreciated the use of the computer as a visual aid in the classroom. Three subjects did not mention specific concepts, but rather that the use of the computer was helpful overall in developing intuition and understanding of concepts. Most subjects mentioned a variety of concepts that they recalled as being particularly well developed through the use of the computer. Concepts mentioned specifically were: (a) operations and transformations with functions (13 respondents); (b) limits (4 respondents); (c) introduction to derivatives (4 respondents); (d) tangent approximations (2 respondents); and (e) Newton's method for approximating roots (2 respondents). Subjects also indicated which computer programs they found to be particularly useful in classroom demonstrations. These are: (a) SPIDER, for demonstrating velocity and accel-

eration graphically (13 respondents); (b) the zoom-in option of *Master Grapher* (Waits and Demana, 1987a) (4 respondents); (c) FASTAN for drawing derivative curves (6 respondents); (d) DIFFERENTIALS, for illustrating how closely successive tangent approximations estimate the curve of a function (2 respondents); and (e) NEWTON, for illustrating Newton's method for approximating roots (2 respondents).

Subjects responses to the open-ended part of question ES6 were overwhelmingly positive. Even those subjects who chose not to use the computer and/or graphs outside of class felt that the use of graphs as a visual aid in classroom instruction was generally very helpful.

ES7: There were a variety of programs that were instructor-written which were not made available to students outside of class. Do you think it might have been helpful to have been able to use any of these programs on your own?

If yes, please discuss specifically what program(s) you might have found useful outside of class and why?

Table 27

Frequency of ES7 Responses by Treatment

TRTMNT	Response	
	Yes	No
G (n = 25)		
Count	16	9
Row %	64.0	36.0
G+ (n = 29)		
Count	14	15
Row %	48.3	51.7

A summary of responses for ES7 are presented in Table 27. The responses for subjects in the G section were more positive than for subjects in the G+ section. Many subjects in the G section would have liked to have had the computer software made available to them that had been provided the G+ subjects. Subjects in the G+ section had available the software *Master Grapher* and nearly half of these subjects rarely or never used it. G+ subjects were more likely to view the use of the extra programs as requiring more work on their part.

Based upon a χ^2 test of the responses to ES7, no significant difference between treatment frequency distributions were detected, $\chi^2(1) = 1.344, p < .246$.

The observed differences in frequencies between treatments were not significant.

Eight of the subjects, who did not think outside use of the investigator-written programs would have been helpful, elaborated on their negative responses. Subjects indicated that: (a) the point had been made in class and outside use of the software was unnecessary (2 respondents); (b) teacher explanations were necessary to understand the graphs (3 respondents); and (c) students should be able to produce graphs without computers (1 respondent). One subject indicated that he had written his own programs to mimic those demonstrated in class.

Subjects who had given positive responses to question ES7 further described or named the software and/or investigator-written programs that they thought might have been helpful to use: (a) *Master Grapher* (8 respondents from the G section); (b) PLOT, which demonstrated the result of operations performed graphically on two functions (2 respondents); (c) SPIDER (2 respondents); (d) FASTAN (9 respondents); (e) DIFFERENTIALS (2 respondents); and (f) NEWTON (1 respondent).

Subjects in the G section would have appreciated use of the *Master Grapher* software. G+ subjects had been provided this software throughout the semester. Otherwise, many subjects in both sections indicated that at least some of the investigator-written programs would have been beneficial to them for further investigation outside of class.

ES8: Have you ever taken Calculus before?

Table 28

Frequency of ES8 Responses by Treatment

TRTMNT	Response	
	Yes	No
G (n = 27)		
Count	11	16
Row %	40.7	59.3
G+ (n = 32)		
Count	14	18
Row %	43.8	56.3

Frequencies of responses by treatment to question ES8 are presented in Table 28. Of the 59 subjects who responded to question ES8, similar percentages of subjects in both treatments had previously taken calculus. Based on a χ^2 test of the data, no significant difference between treatment frequency distributions were detected, $\chi^2(1) = 0.054$, $p < .816$. Similar percentages of subjects in each group had had prior calculus experience.

Subjects who had had prior calculus experience were asked the following questions:

a) Why did you take calculus this semester?

b) How was this class similar to your former class?

c) How was it different?

d) Which did you prefer? Why?

Not all subjects who had taken calculus prior to enrollment in the current course responded to questions (a), (b), (c), and (d). For those subjects who did respond further, these responses were dichotomized into two groups for each treatment: (1) those subjects who preferred the current course (C), and (2) those subjects who preferred their previous course (P) or had no preference (B). The number of subjects in each category; C, P, or B; are presented in Table 29.

Table 29

Preference of Calculus Courses by Treatment

TRTMNT		Preference	
	Both (B)	Current (C)	Previous (P)
G ($n = 9$)			
Count	0	8	1
Row %	0.0	88.9	11.1
G+ ($n = 14$)			
Count	1	8	5
Row %	7.1	57.1	35.7

No clear patterns appear between subjects who preferred the current course and those who preferred their previous course, either in reasons for retaking the course or similarities observed.

Subjects in the G section who had previously taken calculus heavily preferred their current course to their prior calculus experience. Subjects in the G+ section did not indicate as strong a preference for the current course as had the G subjects. However, G+ subjects were more often in favor of the current course than their previous experience. Sections are not significantly different in their preference, however, based on a χ^2 test of the responses for this question, no significant difference between treatment frequency distributions were detected, $\chi^2(2) = 2.708$, $p < .258$. The results of the χ^2 test are questionable, however, since three of the cells contain less than the expected frequency for those cells.

Question (a), "Why did you take calculus this semester?", was asked to determine if subjects' prior experiences with calculus had been in high school or in college. Subjects gave several reasons for retaking calculus. Among these are: (a) the course was required for the subject's major or minor; (b) the subject was not satisfied with his/her previous understanding; (c) the subject placed into Calculus I on the WMU placement test, or had placed into precalculus and had completed this course in the Fall '87 semester; (d) the subject had dropped the course during the previous semester; and (e) the subject chose Calculus I to satisfy the WMU general education science requirement. From these responses, it was not always possible to determine where the previous instruction had taken place.

Subjects found a few similarities between the current course and their previous course. Two subjects mentioned that both courses were challenging. Two subjects mentioned that the assignments were similar. It is interesting that 13 subjects

recognized the basic content as being the same for both courses. Of more interest is that five subjects stated that no similarities existed between their previous and present courses.

Subjects observed a number of differences between their previous and current courses. Subjects who preferred the current course were likely to mention that the emphasis in the current course was much different from that of their previous course (9 C, 2 P respondents). Generally, subjects mentioned that emphasis in the previous course was on solving problems which most often required the memorization of many rules (8 C, 2 P respondents). The emphasis in the current course was on understanding the theorems, definitions, and subsequent rules; including how each related to subjects' existing knowledge and when each was appropriately used (8 C, 2 P respondents). Subjects who had taken the course before and preferred the current course appreciated this emphasis on understanding. Both groups – those who preferred the previous course and those who preferred the current course – mentioned the heavy use of graphic illustrations with equal frequency (3 C, 3 P respondents).

In answering the final question, ES8d, several reasons were cited for subject preference of the current or the previous course. Fourteen of the subjects who answered question ES8d stated that they had learned more in the current class; one of these noted a preference for the previous course because it was easier. Subjects stated that previous courses stressed memorizing results to solve problems. At least 11, of the 14 above, appreciated the reduced emphasis on manipulative skill

and the increased emphasis on conceptual understanding that was characteristic of the current course. Subjects mentioned that the current course stressed understanding. Among other things, when finding a numeric or algebraic solution to a problem, it was necessary to explain why such a solution made sense. While such a requirement frustrated at least 2 of the subjects who preferred their previous courses, subjects felt that they learned more, and were better prepared for future mathematics courses, as a result of this requirement in the current course Two subjects who preferred their previous course mentioned that the content of the previous course was easier to learn. Judging from the comments of subjects who preferred the current course, those few subjects who preferred their previous course because it was easier, preferred to memorize or mimic techniques rather than to understand more deeply.

One subject who indicated preference of the previous course ended her rationale with "I really don't know which I prefer". The subject who liked both prior and current courses equally indicated that both were interesting.

Summary of Investigation 1

Investigation 1 was undertaken to compare the Graphics and Graphics Plus treatment sections on cognitive and affective measures related to the use of a Cartesian coordinate graphic representation system. In Analysis 1, the performances of the G and G+ subjects on the cognitive measures GLEVEL2, GAPP, GSR, and GSN were compared. In Analysis 2, subject attitudes toward the use and usefulness of graphs were compared. Other descriptive data further investi-

gating subject attitudes and opinions toward the use and usefulness of graphs, particularly as they had been used in the present study, were summarized and reported. The conclusions of Analyses 1 and 2 and a summary of the related descriptive data follow.

Conclusions, Analysis 1

For Analysis I, pre-treatment differences were detected between G and G+ subjects only for the pre-attitude variable PREIIM, with subjects in the G section displaying a significantly ($p < .05$) higher level of interest in mathematics. In subsequent multivariate analysis of the graphic variables, GLEVEL2, GAPP, GSR, and GSN, using PRETEST, PREIIM, and PREITM as covariates; differences due to interaction of treatment by prior calculus experience, treatment main effect, and prior calculus main effect were not significant ($p < .05$). Hypotheses 1a, 2, and 3 were not rejected. The provision of the computer graphics software, *Master Grapher* (Waits and Demana, 1987a), and related supplemental assignments requiring its use, did not differentially affect subject performance on the graphic cognitive variables.

Conclusions, Analysis 2

Hypothesis 1b, there is no significant difference between treatments on subject attitudes toward the use and usefulness of the Cartesian coordinate graphic representation system in understanding calculus concepts, was not rejected. All χ^2 tests failed to show significant differences ($p < .05$) between frequency distributions by treatment. The provision of the graphics software, *Master Grapher*,

and related supplemental assignments requiring its use, did not differentially affect subject attitudes toward the use and usefulness of graphs. These results are not surprising since nearly half (48.5%) of the subjects in the G+ section admitted that they had rarely or never used the computer on assignments where its use was required or suggested.

Subject attitude toward the use and usefulness of graphs was most often positive. Subjects indicated that they used graphs more often in the current course than they had in previous courses, and that they would like to continue using graphs in future math courses. They indicated that graphs helped them think about functions. They agreed that computer graphs were useful in helping them understand calculus ideas. Many subjects indicated that they had to draw (or visualize) a graph to understand a problem. Differences between treatments were negligible.

Conclusions, Other Descriptive Data

In general, subject attitudes toward the use of graphs were very positive. Subjects reported that their personal use of graphs had increased over their previous use of graphs as a result of the development in the course. At the very least, for a few subjects, use of graphs remained at the same high level in which they had been used in previous courses.

Subjects appreciated the use of computer graphs in class as visual aids in understanding the concepts presented. They noted that graphs were particularly useful in modeling velocity and acceleration dynamically, and in showing the re-

lationships between various functions.

Subjects in the G section–the section that had not been provided computer software–would have appreciated the availability of the graphics software, *Master Grapher*, as well as many of the other investigator- written programs. Only about half of the G+ subjects made use of the software, *Master Grapher*, that was provided them. Subjects who use the available software indicated that it might have been beneficial to have had the investigator-written programs available for out-of-class use.

Subjects who had prior calculus experience compared their previous course with the current course. Subjects were most often in favor of the current course. Most admitted that developing the deep level of understanding of concepts that was expected in the current course made it more difficult than the technique- oriented course they had taken previously. However, they reported appreciating the level of understanding acquired in the current course and felt encouraged that they understood calculus much better through this approach. One subject stated, "This class was a stepping stone to Calculus II, not a wall."

Investigation 2

Hypotheses

The following hypotheses were investigated to compare treatment sections differing in the level of exposure to a graphically-developed, conceptual calculus course–varying from very little exposure, as in a traditional course, to very high exposure heavily dependent on the use of computer graphics displays:

1. There is no significant difference between treatments on: (a) subject performance on routine applied, routine symbolic, and nonroutine symbolic calculus exam and quiz questions; (b) subject performance on the departmental final exam and on subscales of questions further categorized as: (i) questions pertaining to the concepts of limit, continuity, and derivative; and (ii) questions pertaining to precalculus, antiderivative, and integral concepts; and (c) pre to post changes in subject interest in mathematics and interest in taking more courses in mathematics.

2. There is no significant difference between subjects with no prior calculus experience and subjects with prior calculus experience in: (a) subject performance on routine applied, routine symbolic, and nonroutine symbolic calculus exam and quiz questions; and (b) subject performance on the departmental final exam and on subscales of questions further categorized as: (i) questions pertaining to the concepts of limit, continuity, and derivative; and (ii) questions pertaining to precalculus, antiderivative, and integral concepts.

3. There is no significant interaction between treatment and prior calculus experience in: (a) subject performance on routine applied, routine symbolic, and nonroutine symbolic calculus exam and quiz questions; and (b) subject performance on the departmental final exam and on subscales of questions further categorized as: (i) questions pertaining to the concepts of limit, continuity, and derivative; and (ii) questions pertaining to precalculus, antiderivative, and integral concepts.

Variable Names and Descriptions

The abbreviations used for various variable names throughout the following discussions are given below:

Covariates:

PRETEST: The score on precalculus symbol manipulation and graphing abilities pretest consisting of eight symbol manipulation questions and eight graphic questions.

PREIIM: The score on the pre-attitude survey for the five statements pertaining to interest in mathematics.

PREITM: The score on the pre-attitude survey for the five statements pertaining to interest in taking more mathematics courses.

Cognitive Dependent Variables:

APP: The score on the five exam or quiz questions requiring application to a real-world model.

SR: The score on the 37 exam and quiz questions that were symbolic and routine.

SN: The score on the 16 exam and quiz questions that were symbolic and nonroutine.

FINAL: The score on the 30-question comprehensive departmental final exam.

FDERIV: The score on the 18 final exam questions pertaining to limits, continuity, or derivatives.

FANTI: The score on the 12 final exam questions pertaining to pre-calculus skills, antidifferentiation, or integration.

Affective Dependent Variables:

PREPNMn: The response on the pre-attitude survey for the statements ($n = 1, 2, 3, 4, 5$) concerning perceptions of the nature of mathematics.

POSTPNMn: The response on the post-attitude survey for the statements ($n = 1, 2, 3, 4, 5$) concerning perceptions of the nature of mathematics.

POSTIIM: The score on the post-attitude survey for the five statements pertaining to interest in mathematics.

POSTITM: The score on the post-attitude survey for the five statements pertaining to interest in taking more courses in mathematics.

DIFFPNMn: The difference score, POSTPNMn - PREPNMn, ($n = 1, 2, 3, 4, 5$) pertaining to changes in perceptions of the nature of mathematics.

DIFFIIM: The difference score, POSTIIM - PREIIM, for the five statements pertaining to interest in mathematics.

DIFFITM: The difference score, POSTITM - PREITM, for the five statements pertaining to interest in taking more courses in mathematics.

EVALn: The score on the evaluation survey for the statements (n = 16, 17, 18, 19, 20, 21, 22) concerning subject opinions about the course.

Factors:

TRTMNT: Treatment

G : Graphics (conceptual course with computer graphics in class).

G+: Graphics Plus (same course as G plus software and supplemental assignments).

S1: Standard 1 (graphic, conceptual course, no computer).

S2: Standard 2 (traditional course).

PRICALC: Prior Calculus experience

N: less than half a semester.

P: at least half a semester.

Three separate analyses were performed for Investigation 2:

1. Analysis 1: To test hypotheses 1a, 2a, and 3a, a multivariate analysis of covariance was performed on the cognitive dependent variables, APP, SR, and SN, using PRETEST, PREIIM, and PREITM as covariates.

2. Analysis 2: To test hypotheses 1b, 2b, and 3b, a univariate analysis of covariance was performed on FINAL, then a multivariate analysis of covariance was performed on the subscales FDERIV and FANTI. For both analyses, the variables PRETEST, PREIIM, and PREITM were used as covariates.

3. Analysis 3: To test hypothesis 1c, univariate analyses of variance were performed on the difference scores, DIFFIIM and DIFFITM.

Analysis 1

The Final Sample

Of the subjects participating in Investigation 2, as described in Chapter III, 30 of the G subjects, 35 of the G+ subjects, 25 of the S1 subjects, and 21 of the S2 subjects were retained in Analysis 1. Retained subjects for the analysis are defined to be subjects for whom data sets were: (a) complete for the APP subscale, (b) missing no more than four entries for the SR subscale, and (c) missing no more than two entries for the SN subscale. For these subjects, each missing entry was replaced with the subject's average score for the remaining responses in the particular subscale. These adjusted scores were used in the final analyses. Removed subjects are subjects whose data sets did not meet the above criteria.

For each of the G, G+, S1, and S2 treatments, comparisons were made between retained subjects and removed subjects on the covariates, PRETEST, PREIIM, PREITM, and on the final exam score, FINAL.

In the G treatment, only one subject was removed from the analysis. The descriptive statistics for the retained subjects and the subject's scores pertaining to the variables of interest were reported in Table 9. The removed subject's scores were lower than the corresponding means for retained subjects with the exception of PRETEST. The PRETEST scores were comparable.

In the G+ treatment, one subject was removed from this analysis. The descriptive statistics for the retained subjects and this subject's scores on PRETEST, PREIIM, PREITM, and FINAL are given in Table 30. The highest possible score for each variable is listed to provide a basis for comparison.

Table 30

G+ Treatment: Retained Subjects vs. Removed Subject, Investigation 2, Analysis 1

Variables	Highest Possible Score	Retained Subject ($n = 35$) M	SD	Removed Subject's ($n = 1$) Score
PRETEST	16	11.71	2.18	12
PREIIM	25	17.43	3.37	16
PREITM	25	17.97	3.93	13
FINAL	30	12.74	4.23	12

The removed subject's scores on PRETEST and FINAL are comparable to the means for the retained subjects for these variables. The removed subject's PREIIM and PREITM scores were somewhat lower than the retained subjects' corresponding mean scores.

In the S1 treatment, eight subjects were removed from Analysis 1. The

descriptive statistics for the 25 retained subjects and 8 removed subjects for the variables PRETEST, PREIIM, PREITM, and FINAL are given in Table 31. Means for PREIIM and PREITM are comparable for these two groups. The PRETEST mean for removed subjects was slightly lower than that for retained subjects. The FINAL mean for removed subjects is considerably lower than that for retained subjects.

Table 31

Descriptive Statistics for Retained vs. Removed S1 Subjects, Investigation 2, Analysis 1

	Retained Subjects ($n = 25$)		Removed Subjects ($n = 8$)	
Variables	M	SD	M	SD
PRETEST	10.24	2.54	8.50	3.42
PREIIM	18.16	3.37	16.88	3.09
PREITM	18.76	3.69	19.13	3.18
FINAL	14.24	5.21	10.50	2.39

To compare retained subjects with removed subjects in S1 on PRETEST, PREIIM, PREITM, and FINAL, the folded form of the F-statistic[1], F', was used to determine equality of variances for each variable ($p < .10$). Variances for PRETEST, PREIIM, and PREITM were equal between groups. FINAL variances

were unequal[2]. T-tests were conducted to determine equality of means for removed vs. retained S1 subjects. The results of the t-tests are presented in Table 32.

Nearly one-fourth of the S1 subjects were removed from Analysis 1 due to excessive missing data. Removed subjects did not differ significantly ($p < .05$) from the retained subjects on the covariate measures of PRETEST, PREIIM, and PREITM. However, differences in performance on the departmental final exam, FINAL, were significant ($p < .05$) with retained subjects outperforming removed subjects. Such information indicates that the retained subjects were probably not representative of the full sample of S1 subjects. In further analysis, interpretation of results obtained using the covariates of PRETEST, PREIIM, and PREITM must be made cautiously.

Table 32

S1 Treatment: Comparisons of Retained vs. Removed Subjects, Investigation 2, Analysis 1.

Variable	*t*	*df*	*p*
PRETEST	-1.551	31	0.131
PREIIM	-0.955	31	0.347
PREITM	0.251	31	0.804
FINAL	-2.786	26.6[a]	0.010*

* significant for $p < .05$
[a] Satterthwaite's approximation for the degrees of freedom (SAS, 1985, p. 797)

[2] Satterthwaite's approximation for the degrees of freedom was used to determine the t-statistic for a variable with unequal variances.

In the S2 treatment, seven subjects were removed from Analysis 1. The descriptive statistics for the 21 retained subjects and 7 removed subjects for the variables PRETEST, PREIIM, PREITM, and FINAL are given in Table 33. Means for PREIIM and PREITM are somewhat higher for the removed subjects than for the retained subjects. The PRETEST mean for removed subjects is slightly lower than that for retained subjects. The FINAL mean for removed subjects is very low compared to the mean for retained subjects.

Table 33

Descriptive Statistics for Retained vs. Removed S2 Subjects, Investigation 2, Analysis 1

	Retained Subjects ($n = 21$)		Removed Subjects ($n = 7$)	
Variables	*M*	*SD*	*M*	*SD*
PRETEST	11.86	1.93	10.14	3.24
PREIIM	18.04	3.09	19.29	2.63
PREITM	18.48	3.87	20.43	2.57
FINAL	13.76	4.69	9.29	3.40

To compare retained subjects with removed subjects in S2 on PRETEST, PREIIM, PREITM, and FINAL, the folded form of the F-statistic[1], F', was used

to determine equality of variances for each variable ($p < .10$). Variances for PREIIM, PREITM, and FINAL were equal between groups. PRETEST variances were unequal[2]. T-tests were conducted to determine equality of means for removed vs. retained S2 subjects. The results of the t-tests are presented in Table 34.

Table 34

S2 Treatment: Comparisons of Retained vs. Removed Subjects, Investigation 2, Analysis 1

Variable	*t*	*df*	*p*
PRETEST	-1.325	7.5[a]	0.224
PREIIM	0.959	26	0.347
PREITM	1.239	26	0.226
FINAL	-2.318	26	0.029*

* significant for $p < .05$
[a] Satterthwaite's approximation for the degrees of freedom (SAS, 1985, p. 797)

One-fourth of S2 subjects were removed from Analysis 1 due to excessive missing data. Removed subjects did not differ significantly ($p < .05$) from retained subjects on any of the premeasures. However, differences in performance on the departmental final, FINAL, were significant ($p < .05$) with retained subjects outperforming removed subjects. Such information indicates that the retained subjects were probably not representative of the full sample of S2 subjects. In further analysis, interpretation of results obtained using the covariates of PRETEST, PREIIM, and PREITM must be made cautiously.

For Analysis 1, the S1 and S2 sections were represented by only three-fourths of their respective possible samples. In the case of the S1 section, the retained subjects represented 71% of the S1 subjects who completed Calculus I and 63% of those who were initially enrolled in that section. The retained S2 subjects represented 66% of the S2 subjects who completed the course and 46% of those S2 subjects initially enrolled. The retention patterns were much different in the G and the G+ sections. In both the G and the G+ sections 97% of the subjects who completed the course were retained in the analyses. These groups represented 81% and 88% respectively of those G and G+ subjects who initially enrolled in the course.

In both of the S1 and S2 sections, retained and removed subjects' performances on the pre-treatment measures were comparable. For the post-treatment measure, FINAL, retained S1 and S2 subjects performed at a significantly higher rate than removed S1 and S2 subjects respectively. These results indicate the possibility that the cognitive dependent measures for the retained S1 and S2 subjects reflect better results than would be expected and thus bias this analysis in favor of the S1 and S2 treatments over the G and G+ treatments.

Profile of Subjects. To further describe subjects remaining in the final sample for Analysis 1, data pertaining to sex, class, and college are displayed in Figures 7, 8, and 9 respectively.

The highest percentage of males were in the S2 section. The G, G+, and S1 sections had similar male/female populations. The highest percentage of fresh-

men, class level 1, were in the G section, with other sections being comparable in class breakdown. The highest percentage of engineering students were in the S2 section. The most even distribution of college interests were represented in the S1 section. The G and G+ sections had similar College of Engineering enrollments. The G and S1 sections had the largest percentages of Arts and Sciences majors.

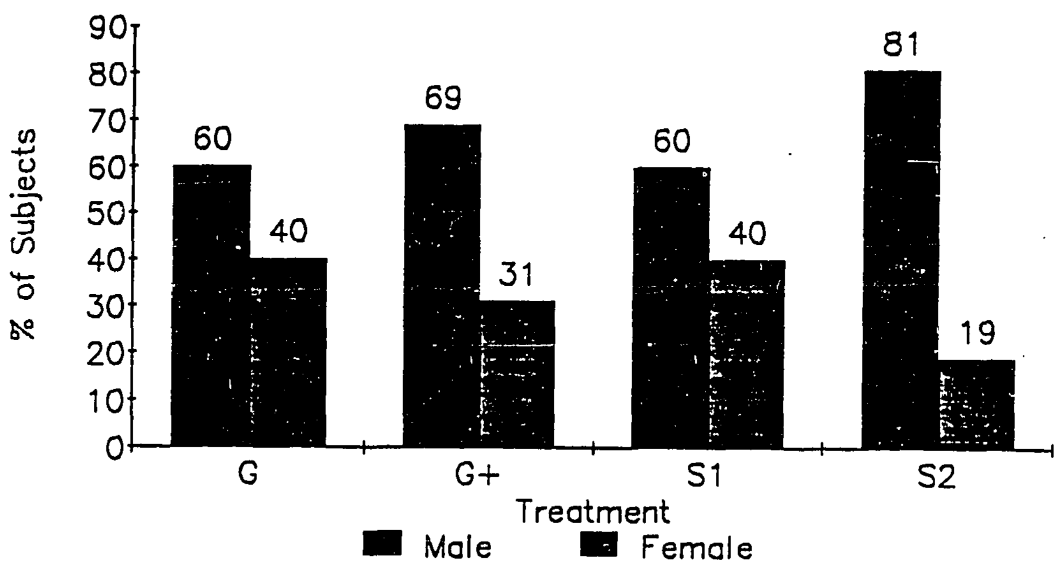


Figure 7. Males versus females, Investigation 2, Analysis 1.

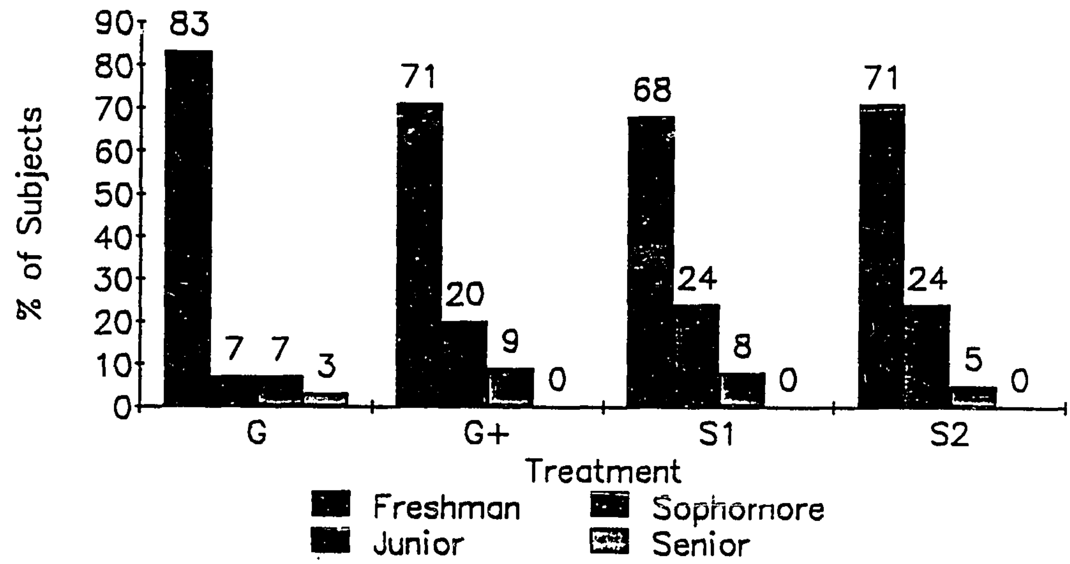


Figure 8. Class level, Investigation 2, Analysis 1.

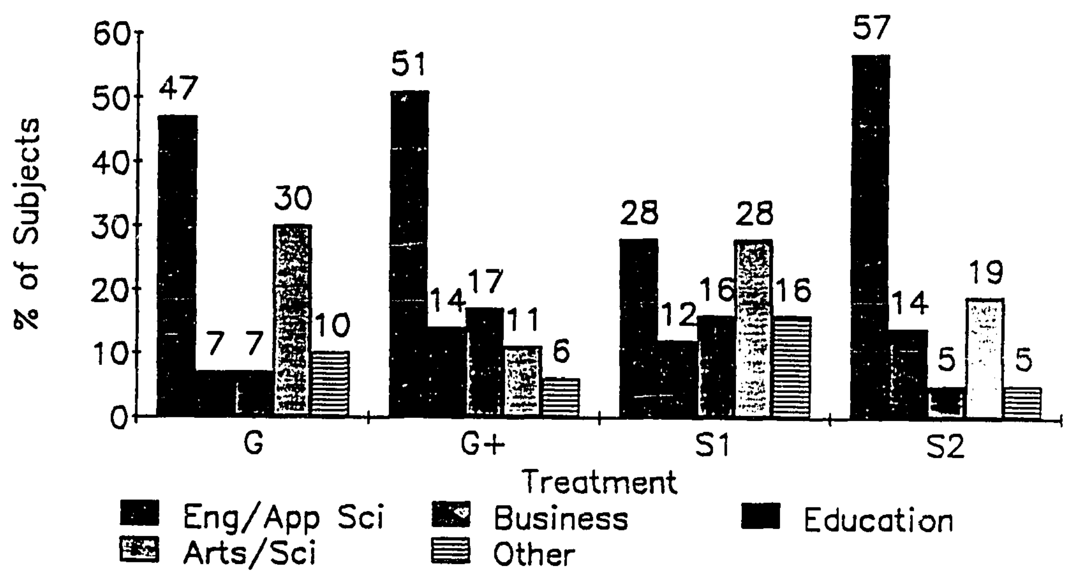


Figure 9. College enrollment, Investigation 2, Analysis 1.

Pre-Investigation Differences. Although random assignment of subjects to treatments was not possible, sections were assumed to be equal in mathematical ability and attitude prior to treatment. To test this assumption, PRETEST, PREIIM, and PREITM scores were compared for subjects in the G, G+, S1, and S2 sections. Treatment means and standard deviations for these variables are listed in Table 35.

Table 35

Descriptive Statistics for Pre-Treatment Measures,
Investigation 2, Analysis 1

Variable	TRTMNT			
	G (n = 30)	G+ (n = 35)	S1 (n = 25)	S2 (n = 21)
PRETEST				
M	11.77	11.71	10.24	11.86
SD	2.36	2.18	2.54	1.93
PREIIM				
M	19.07	17.43	18.16	18.04
SD	2.36	3.37	3.37	3.09
PREITM				
M	18.37	17.97	18.76	18.48
SD	2.95	3.93	3.69	3.87

The PRETEST mean for the S1 treatment is somewhat lower than that for the other three sections, suggesting a slight difference in initial mathematical ability. The PREIIM mean was highest for subjects in the G section, suggesting that these subjects might have been more interested in mathematics initially than subjects in the other three sections. The PREITM mean was lowest for subjects in the G+ section, suggesting that these subjects initially might have been least interested in taking more mathematics courses than subjects in the other three sections. These means suggest the possibility of pre-treatment bias.

To test the hypothesis that sections did not differ significantly in pre-treatment ability and attitude, one-way analyses of variance were performed for each of the variables PRETEST, PREIIM, and PREITM. A summary of these analyses appears in Table 36.

Table 36

Analysis of Variance Results for PRETEST, PREIIM, and PREITM, Investigation 2, Analysis 1

Variable	*F*	*df*	*p*
PRETEST	2.939	3,107	0.037*
PREIIM	1.540	3,107	0.209
PREITM	.243	3,107	0.866

$* \, p < .05$

The analysis of variance supports the hypothesis of equality of treatment

means on the pre-attitude measures, PREIIM and PREITM. The noticeable difference in group means was not significant. A significant pre- treatment bias was detected among sections concerning subject initial precalculus symbolic abilities and graphic competencies. Multiple comparisons, using the Bonferroni procedure with a 95% family confidence coefficient, indicated that the G+ section performed significantly better than the S1 section on PRETEST. No other significant differences were detected.

Design

For Investigation 2, Analysis 1, a 2×4 unbalanced fixed-effects factorial design was used with three dependent variables. The factors were treatment, G, G+, S1, and S2, and prior calculus experience, N and P. The three dependent variables were the subscales of exam and quiz questions, APP, SR, and SN. The design matrix indicating factors and corresponding cell sizes is given in Table 37.

Table 37

The Experimental Design with Final Cell Sizes,
Investigation 2, Analysis 1

PRICALC	TRTMNT			
	G	G+	S1	S2
N	17	18	16	11
P	13	17	9	10

A 2×4 multivariate analysis of covariance was conducted for the subscales

APP, SR, and SN. Although treatments did not differ significantly with respect to PREIIM, PREITM, these variables were used together with PRETEST as covariates to account for some of the variability in the model. The analysis was conducted using the SAS (1986) MANOVA program.

The raw cell means and standard deviations and the cell means and standard errors, adjusted for the covariates, for each of the subscales APP, SR, and SN are presented in Tables 38, 39, and 40 respectively.

Table 38

Raw and Adjusted Cell Means for APP

PRICALC	TRTMNT							
	G		G+		S1		S2	
	Raw	Adj.	Raw	Adj.	Raw	Adj.	Raw	Adj.
N								
M	11.82	10.70	9.11	8.78	14.06	15.15	11.91	12.09
SD[a]	4.72	1.24	4.96	1.18	3.86	1.27	5.89	1.51
P								
M	10.69	11.11	11.76	11.73	11.67	12.69	14.90	14.06
SD[a]	5.01	1.39	6.08	1.20	7.05	1.67	4.70	1.58

[a] Standard errors are given for adjusted means.

Table 39

Raw and Adjusted Cell Means for SR

PRICALC	TRTMNT							
	G		G+		S1		S2	
	Raw	Adj.	Raw	Adj.	Raw	Adj.	Raw	Adj.
N								
M	96.94	92.25	85.89	85.37	102.06	106.74	94.09	94.47
SD[a]	18.53	4.26	22.86	4.04	13.77	4.35	21.30	5.18
P								
M	99.01	99.84	100.81	100.75	103.79	107.28	99.10	95.94
SD[a]	17.64	4.76	15.97	4.12	20.65	5.73	15.20	5.43

[a] Standard errors are given for adjusted means.

Table 40

Raw and Adjusted Cell Means for SN

PRICALC	TRTMNT							
	G		G+		S1		S2	
	Raw	Adj.	Raw	Adj.	Raw	Adj.	Raw	Adj.
N								
M	34.86	32.23	31.71	31.24	31.81	34.30	26.09	26.63
SD[a]	10.48	2.30	9.93	2.18	9.50	2.35	12.16	2.80
P								
M	34.86	35.47	33.84	33.86	31.79	33.82	28.44	26.51
SD[a]	11.13	2.57	9.09	2.23	6.57	3.10	9.58	2.94

[a] Standard errors are given for adjusted means.

Trends are not immediately evident from a perusal of adjusted and raw cell means. The adjusted means for the S1 section are high for APP and SR. Raw means for the S1 section are high only for SR. The raw and adjusted means for the G+ x N cell are low for APP and SR. For the S2 section, the raw cell means are relatively high for the applied questions, and both raw and adjusted means are low for the symbolic nonroutine questions.

To provide more information concerning trends for treatment and prior calculus experience, the raw and adjusted marginal means for each of the main effects are provided in Tables 41 and 42.

Table 41

Raw and Adjusted Treatment Means for APP, SR, and SN

Variable	TRTMNT							
	G ($n = 30$)		G+ ($n = 35$)		S1 ($n = 25$)		S2 ($n = 21$)	
	Raw	Adj.	Raw	Adj.	Raw	Adj.	Raw	Adj.
APP								
M	11.33	10.91	10.40	10.25	13.20	13.92	13.33	13.08
SD[a]	4.79	0.93	5.61	0.84	5.22	1.06	5.44	1.08
SR								
M	97.83	96.05	93.13	93.06	102.69	107.01	96.48	95.20
SD[a]	17.87	3.19	20.94	2.89	16.17	3.64	18.36	3.72
SN								
M	34.86	33.85	32.74	32.55	31.81	34.06	27.21	26.57
SD[a]	10.57	1.73	9.45	1.56	8.41	1.97	10.80	2.01

[a] Standard errors are given for adjusted means.

Table 42

Raw and Adjusted PRICALC Means for APP, SR, and SN

Variable	PRICALC			
	N ($n = 62$)		P ($n = 49$)	
	Raw	Adj.	Raw	Adj.
APP				
M	11.63	11.68	12.10	12.40
SD[a]	5.05	0.64	5.77	0.73
SR				
M	94.55	94.71	100.53	100.95
SD[a]	19.88	2.20	16.74	2.51
SN				
M	31.60	31.10	32.63	32.42
SD[a]	10.54	1.19	9.41	1.36

[a] Standard errors are given for adjusted PRICALC means.

Adjusted treatment means for the S1 section are consistently higher than those for the other three sections. This suggests a possible difference in performance due to treatment when covariates are used. However, since the covariate measures for S1 and S2 subjects are suspect, analyses using adjusted means must

be conducted cautiously. Considering raw means, the S1 section is high for SR, the S1 and S2 sections are high for APP, and the G and G+ sections are high for SN.

Both raw and adjusted PRICALC means are consistently higher for subjects who have had prior calculus experience. This suggests a possible difference in performance due to prior calculus experience.

A summary of the multivariate analysis of covariance is presented in Table 43. The multivariate analysis did not show a significant interaction between treatment and prior calculus experience. Hypothesis 3a, there is no significant interaction, was not rejected; $F\ (9,300) = 0.84,\ p < .576$.

Table 43

Multivariate Analysis of Covariance for APP, SR, and SN

Source	*F*	*df*	*p*
TRTMNT × PRICALC	0.84[a]	9,300	0.576
TRTMNT	3.15[a]	9,300	0.001*
PRICALC	1.30	3,98	0.280

[a] *F* approximates Pillai's Trace statistic which is both powerful and robust.

$* p < .05$

The multivariate analysis did not show a significant main effect for prior calculus experience, $F(3,98) = 1.30$, $p < .280$. Hypothesis 2a, there is no significant prior calculus main effect, was not rejected. While marginal means for subjects with prior calculus experience were slightly higher than for those subjects with no prior calculus experience, the difference is not significant.

The multivariate analysis indicated a significant main effect for treatment as was suggested by the marginal means. Three univariate analyses of covariance were performed to determine the source of treatment differences detected in the multivariate analysis. A summary of these analyses is presented in Table 44. All three univariate tests indicate significant differences between the treatment sections on the subscales APP, SR, and SN.

Table 44

Univariate Analyses of Covariance to Determine Treatment Effects for APP, SR, and SN

Variable	*F*	df	p
APP	3.21	3,100	0.026*
SR	3.17	3,100	0.028*
SN	3.22	3,100	0.026*

*$p < .05$

To examine the nature of the observed differences, all pairwise comparisons were estimated by means of the Bonferroni procedure, using a family confidence coefficient of 95%. Individual unprotected pairwise comparisons of adjusted treatment means for APP, SR, and SN are displayed in Tables 45, 46, and 47 respectively.

Table 45

Pairwise Treatment Comparisons for APP

Treatment	G+	S1	S2
G	0.607	0.038	0.132
G+		0.009	0.042
S1			0.584

Table 46

Pairwise Treatment Comparisons for SR

Treatment	G+	S1	S2
G	0.494	0.027	0.863
G+		0.004*	0.650
S1			0.027

*$p < .008$

Table 47

Pairwise Treatment Comparisons for SN

Treatment	G+	S1	S2
G	0.582	0.936	0.007*
G+		0.552	0.021
S1			0.010

$^*p < .008$

The Bonferroni multiple comparison method yields conservative estimates of significance between treatments. For the six pairwise comparison, significant differences by variable are detected where the p-values given in Tables 45, 46, and 47 are less than $\frac{.05}{6} = .0083$. No significant protected differences were detected between treatments for the APP subscale. A significant difference was detected for the SR subscale between the G+ and S1 treatment sections, in favor of the S1 section. A significant difference was detected for the SN subscale between the G and S2 treatment sections, in favor of the G section. No other significant pairwise protected differences were detected for the subscales APP, SR, and SN.

A near significant difference was detected for APP between the G+ and S1 sections, in favor of the S1 section. Part of the difference between the sections might be attributed to the placement of the questions comprising the APP subscale on assessments. The APP subscale contains only five questions. All of

these questions were placed at the end of two exams for the G and G+ sections. Four of the APP questions appeared on S1 quizzes. The remaining APP question appeared in the center of an exam for this section.

As previously discussed, there is reason to question the validity of the use of covariates for the S1 and S2 sections. For this reason, univariate analyses of variance, without covariates, were conducted for the variables APP, SR, and SN. No significant differences were detected for interaction or PRICALC main effect. A significant difference ($p < .05$) in subject performance due to treatment was observed only for the SN subscale, $F(3,100) = 2.96$, $p < .034$ (see Table 48). No significant treatment main effect was observed for the APP and SR subscales.

Table 48

Univariate Analyses of Variance to Determine
Treatment Effects for APP, SR, and SN

Variable	*F*	*df*	*p*
APP	2.37	3,100	0.075
SR	1.57	3,100	0.201
SN	2.96	3,100	0.036*

*$p < .05$

To examine the nature of the differences between treatments, without the use of covariates, pairwise comparisons were estimated by means of the Bonferroni procedure using a 95% family confidence coefficient. Significant differences were

detected between the G and S2 treatments for SN, in favor of the G treatment. These findings are consistent with those obtained in the analyses with covariates.

The retained S2 subjects were not representative of the full S2 sample. They comprised 75% of the full S2 sample for Investigation 2. They performed significantly better on the FINAL assessment than had removed S2 subjects, yet pre-measures for these groups were not significantly different. The retained G subjects comprise 97% of the full G sample. Yet, the G subjects performed significantly better than the S2 subjects on the symbolic nonroutine subscale of exam and quiz questions.

The retained S1 subjects were not representative of the full S1 sample. Retained S1 subjects performed significantly better on the FINAL assessment than had removed S1 subjects, yet pre-measures for these groups were not significantly different. The retained S1 subjects also comprised only 76% of the full S1 sample for Investigation 2. Retained subjects for the G+ section comprised 97% of the full G+ sample. The significant difference for the SR questions detected in favor of the S1 section between S1 and G+, is a comparison between subjects from a select group of S1 students and a group of G+ subjects representing a broader range of abilities. This difference was not significant when the analysis was conducted without covariates. The significant difference detected between the G+ and S1 treatments for SR is therefore open to question.

Analysis 2

Analysis 2 was conducted to test hypotheses 1b, 2b, and 3b. To test each

hypothesis, a univariate analysis of covariance was performed on scores of the departmental final exam FINAL, and a multivariate analysis of covariance was performed on the subscales into which FINAL was partitioned, FDERIV and FANTI.

The Final Sample

All of the subjects ($N = 128$) participating in Investigation 2 were retained in Analysis 2. Sample sizes are given in Table 49.

Table 49

Sample Sizes for Investigation 2, Analysis 2.

Treatment	*n*
G	31
G+	36
S1	33
S2	28

Profile of Subjects. Sex, class, and college data were available for all subjects involved in the study. These data appear in Figures 10, 11, and 12 respectively. They are given as percentages so that comparisons can be made between sections of unequal size.

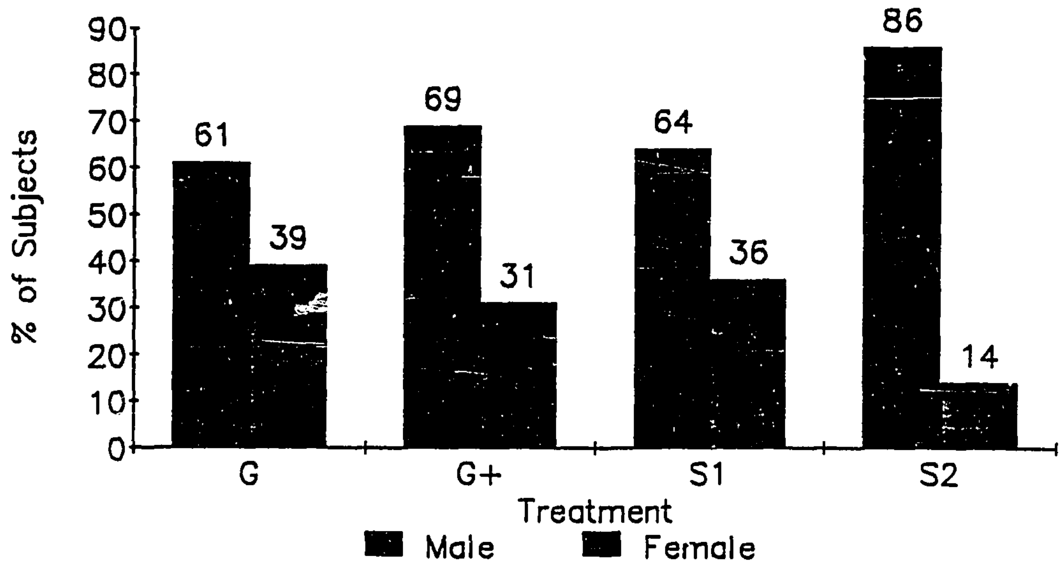


Figure 10. Males versus females, Investigation 2, Analysis 2.

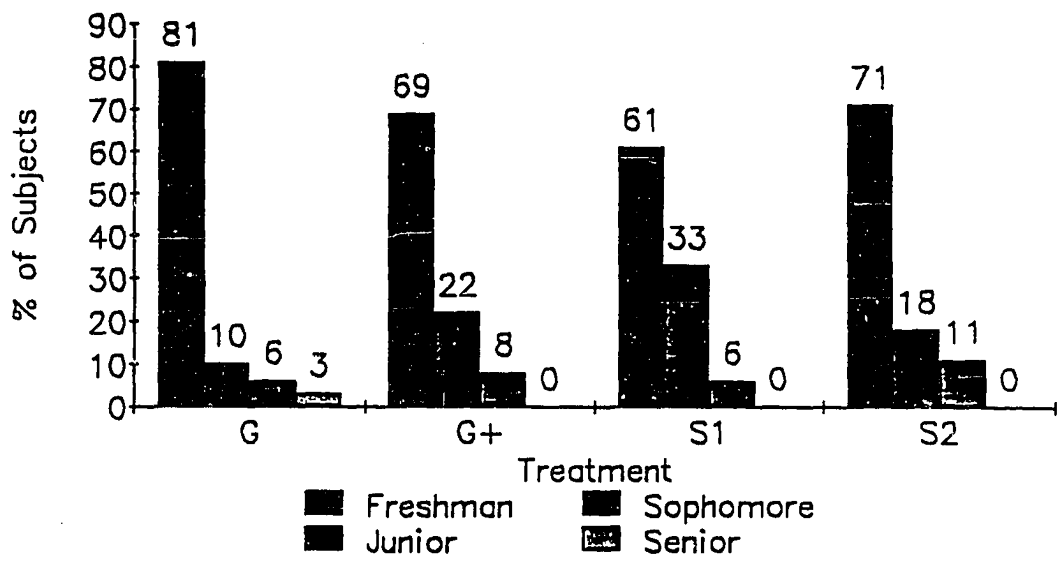


Figure 11. Class level, Investigation 2, Analysis 2.

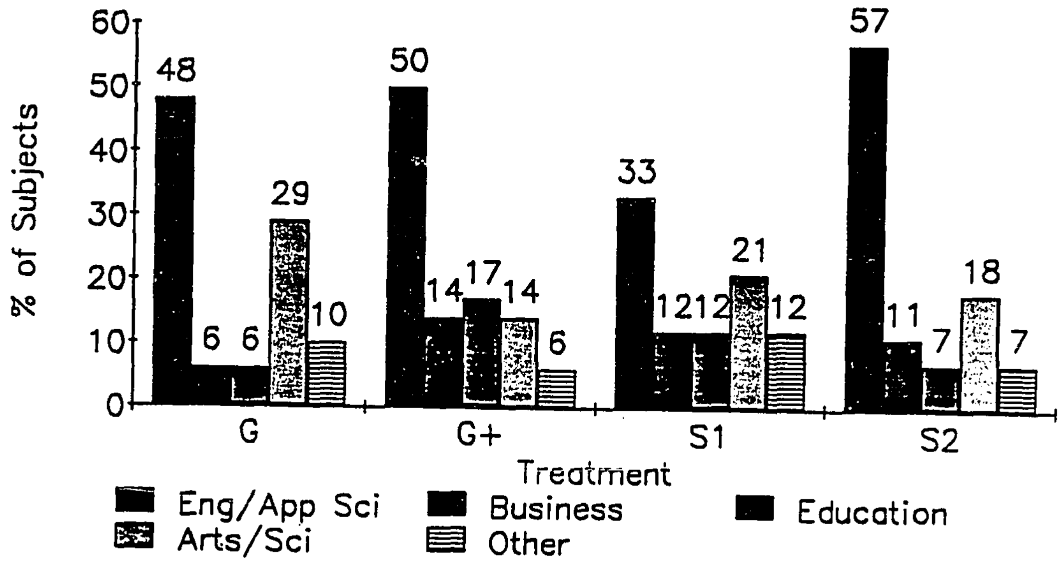


Figure 12. College enrollment, Investigation 2, Analysis 2.

The S2 section contained the highest percentage of males. The remaining three sections had similar male/female populations. The largest percentage of freshmen, class level 1, were in the G section. The smallest percentage of freshmen were in the S1 section. The class level data suggest that previous mathematics exposure for S1 subjects might have been less recent than for subjects in the other three sections. The G+ and S2 sections had similar class level breakdowns. The highest percentage of engineering students were in the S2 section. The most balanced range of college interests were represented in the S1 section. The G and G+ sections had similar College of Engineering enrollments. The G section had a larger percentage of Arts and Sciences majors than any of the other sections.

Pre-Investigation Differences. Although random assignment of subjects to treatment sections was not possible, sections were assumed to be equal in mathematical ability and attitude prior to treatment. To test this assumption, PRETEST, PREIIM, and PREITM scores were compared for subjects in the G, G+, S1, and S2 sections. Sample means and standard deviations are presented in Table 50.

Table 50

Descriptive Statistics for Pre-Treatment Measures, Investigation 2, Analysis 2

Variable	TRTMNT			
	G ($n = 31$)	G+ ($n = 36$)	S1 ($n = 33$)	S2 ($n = 28$)
PRETEST				
M	11.77	11.72	9.82	11.43
SD	2.32	2.15	2.82	2.38
PREIIM				
M	18.97	17.39	17.85	18.35
SD	2.39	3.33	3.31	2.98
PREITM				
M	18.23	17.83	18.85	18.96
SD	3.01	3.96	3.53	3.65

The PRETEST mean for the S1 section is lower than that of the other three sections, suggesting a difference in initial mathematical ability or in the recency

of the S1 subjects previous mathematics exposure. The pre- attitude means, for PREIIM and PREITM, are somewhat lower for the G+ section than for the other three sections, suggesting that these subjects initially might have been less interested in mathematics and in taking more mathematics courses than subjects in the other three sections. These means suggest the possibility of pre-treatment bias.

To test the hypothesis that sections do not differ significantly in pre-treatment ability and attitude, one-way analyses of variance were performed for each of the variables PRETEST, PREIIM, and PREITM. A summary of these analyses appears in Table 51.

Table 51

Analysis of Variance Results for PRETEST, PREIIM, and PREITM, Investigation 2, Analysis 2

Variable	*F*	*df*	*p*
PRETEST	4.780	3,124	.004*
PREIIM	1.630	3,124	.186
PREITM	0.730	3,124	.536

$^{*}p < .05$

The analysis of variance supports the hypothesis of equality of treatments on the pre-attitude measures PREIIM and PREITM. The noticeable difference in group means on these attitude scales was not significant. A significant pre-

treatment bias was detected among sections concerning subject initial precalculus symbolic and graphic competencies. Multiple comparisons using the Bonferroni procedure with a 95% family confidence coefficient detected significant PRETEST differences between the G and S1 treatments and between the G+ and S1 treatments with the S1 section performing significantly lower than both the G and G+ section.

Design

For Investigation 2, Analysis 2, a 2×4 unbalanced fixed effects factorial design was used. The factors were treatment, G, G+, S1, and S2, and prior calculus experience, N and P. The design matrix indicating factors and corresponding cell sizes is given in Table 52.

Table 52

The Experimental Design with Final Cell Sizes,
Investigation 2, Analysis 2

PRICALC	TRTMNT			
	G	G+	S1	S2
N	18	19	19	14
P	13	17	14	14

A 2×4 univariate analysis of covariance was conducted for the variable, FINAL. Although treatments did not differ significantly with respect to PREIIM

and PREITM, these variables were used as covariates together with PRETEST to account for some of the variability in the model. The analysis was conducted using the SAS (1986) MANOVA program. Raw cell means and standard deviations and cell means and standard errors adjusted for the covariates, are presented in Table 53.

Table 53

Raw and Adjusted Cell Means for FINAL

PRICALC	TRTMNT							
	G		G+		S1		S2	
	Raw	Adj.	Raw	Adj.	Raw	Adj.	Raw	Adj.
N								
M	11.78	10.49	12.42	11.71	13.32	14.69	12.57	13.15
SD[a]	2.82	0.98	4.46	0.93	3.80	0.97	5.05	1.09
P								
M	13.23	13.54	13.06	12.77	13.36	14.40	12.71	11.93
SD[a]	4.97	1.12	3.93	0.97	6.31	1.09	4.66	1.08

[a] Standard errors are given for adjusted cell means.

Raw cell means do not seem to be significantly different, ranging from 11.78 to 13.36. All raw cell means for prior calculus experience are slightly lower than the corresponding raw cell mean for no prior calculus experience. Differences due to treatment are not evident.

Adjusted means for the S1 section are higher than those for the other three sections, suggesting the possibility of a significant main effect for treatment. For the graphic sections, G and G+, subjects with prior calculus experience had higher adjusted means than subjects with no prior calculus. The opposite was true for the standard sections, S1 and S2. This suggests the possibility of an interaction effect. It does not seem likely that a main effect for prior calculus experience exists.

Profile curves by PRICALC and by TRTMNT for adjusted means are presented in Figures 13 and 14 respectively. The obvious departure from parallelism for the treatment curves suggests an interaction effect.

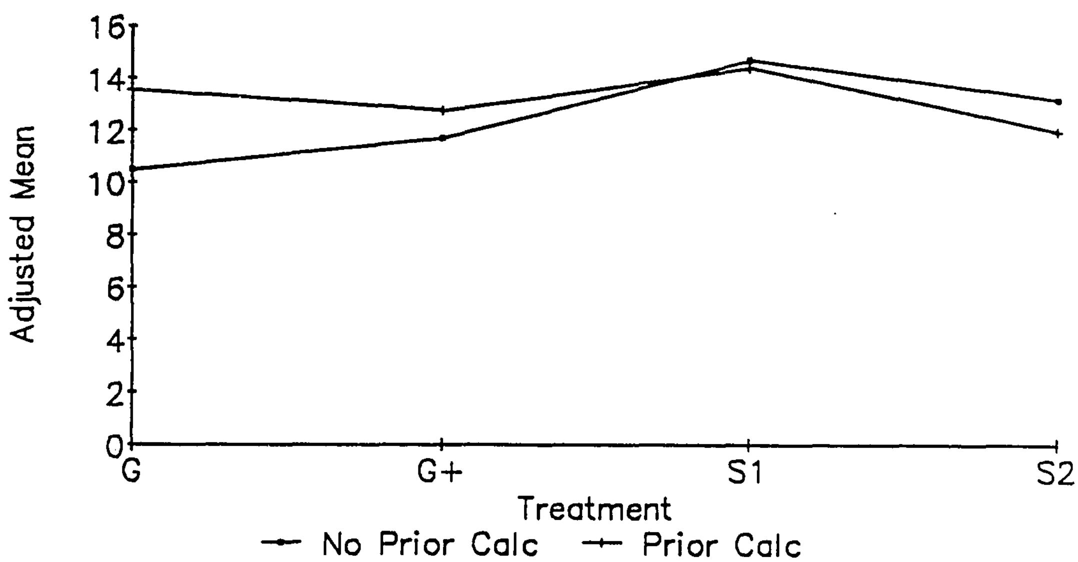


Figure 13. PRICALC profile curves, Investigation 2, Analysis 2.

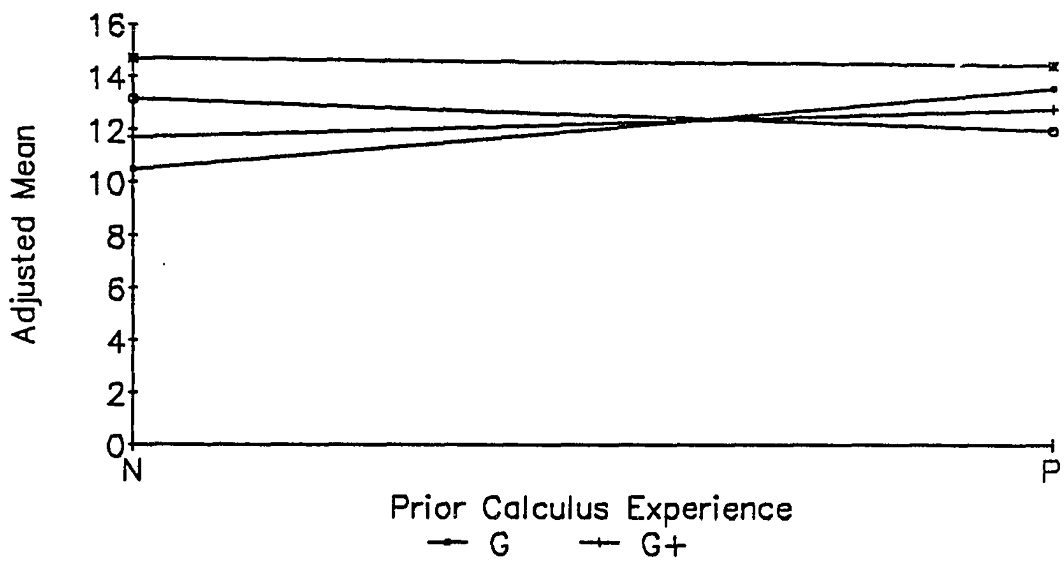


Figure 14. Treatment profile curves, Investigation 2, Analysis 2.

The results of the analysis of covariance conducted for FINAL are presented in Table 54. The analysis of covariance did not show a significant interaction between treatment and prior calculus experience. Hypothesis 3b, there is no significant interaction, was not rejected, $F(3,117) = 1.48, p < .224$.

As expected the analysis of covariance did not indicate a significant main effect for prior calculus experience. The analysis of covariance also did not indicate a significant ($p < .05$) main effect for treatment. Hypotheses 2b and 1b, no main effects for TRTMNT and PRICALC, were not rejected for FINAL.

Table 54

Analysis of Covariance for FINAL

Source	*F*	*df*	*p*
PRICALC × TRTMNT	1.48	3,117	0.224
TRTMNT	2.38	3,117	0.074
PRICALC	0.81	1,117	0.370

The departmental final exam was comprehensive, containing questions from material not covered in the present study. Subject performance on questions pertaining to limit, continuity, and derivative were of particular interest for this investigation. The questions found on the final exam were subsequently partitioned into two subscales: (a) FDERIV – questions pertaining to the concepts of limit, continuity, and derivative; and (b) FANTI – questions pertaining to pre-calculus concepts, antidifferentiation, and integration. A 2×4 multivariate analysis of covariance was conducted for these subscales, FDERIV and FANTI, with covariates PRETEST, PREIIM, and PREITM. Raw cell means and standard deviations, and cell means and standard errors adjusted for the covariates, for each of the subscales, FDERIV and FANTI, are presented in Table 55 and 56 respectively.

Table 55

Raw and Adjusted Cell Means for FDERIV

PRICALC	TRTMNT							
	G		G+		S1		S2	
	Raw	Adj.	Raw	Adj.	Raw	Adj.	Raw	Adj.
N								
M	7.22	6.47	8.11	7.59	9.16	9.92	8.50	8.88
SD[a]	2.26	0.61	2.40	0.58	2.83	0.60	3.13	0.68
P								
M	8.54	8.83	8.47	8.27	8.64	9.34	8.79	8.32
SD[a]	3.18	0.70	2.40	0.61	3.69	0.68	2.91	0.67

Note: This subscale contained 18 questions
[a] Standard errors are given for adjusted means.

Table 56

Adjusted Cell Means for FANTI

PRICALC	TRTMNT							
	G		G+		S1		S2	
	Raw	Adj.	Raw	Adj.	Raw	Adj.	Raw	Adj.
N								
M	4.56	4.02	4.32	4.13	4.16	4.77	4.07	4.27
SD[a]	1.54	0.54	2.58	0.51	1.92	0.53	2.50	0.60
P								
M	4.69	4.71	4.59	4.50	4.71	5.06	3.93	3.61
SD[a]	2.18	0.62	2.18	0.54	3.00	0.60	2.62	0.60

Note: This subscale contained 12 questions.
[a] Standard errors are given for adjusted means.

Not surprisingly, trends similar to those observed with FINAL are evident in this analysis. Adjusted means for S1 subjects are consistently high. An interaction of P and N levels of prior calculus, appears to be occurring between graphic and standard sections for adjusted means. No main effect for prior calculus experience is evident.

Trends are not as evident when the raw cell means are observed. The S1 means are only slightly higher than the S2 means for FDERIV. The G means are

somewhat higher than all other sections for FANTI.

A summary of the multivariate analysis of covariance is presented in Table 57. The multivariate analysis did not show a significant interaction between treatment and prior calculus experience. Hypothesis 3b, there is no significant interaction for FINAL, FDERIV, and FANTI, was not rejected.

Table 57

Multivariate Analysis of Covariance for FDERIV and FANTI

Source	*F*	*df*	p
TRTMNT × PRICALC	1.24[a]	6,234	0.287
TRTMNT	2.17[a]	6,234	0.046*
PRICALC	0.55	2,116	0.577

[a] F-statistic approximates Pillai's Trace Statistic for the given degrees of freedom.
$^{*}p < .05$

The multivariate analysis did not show a significant main effect prior calculus experience. Hypothesis 2b, there is no significant prior calculus experience effect, was not rejected for FINAL, FDERIV, and FANTI.

The multivariate analysis for FDERIV and FANTI did indicate a significant treatment main effect that was not found to be significant using the univariate analysis for FINAL. To determine the source of the treatment effect, two univariate

analyses of covariance were performed. A summary of these analyses is presented in Table 58.

Table 58

Univariate Analyses of Covariance to Determine Treatment Effects for FDERIV and FANTI

Variable	*F*	*df*	*p*
FDERIV	3.55	3,117	0.017*
FANTI	0.94	3,117	0.426

*$p < .05$

The univariate analysis for FANTI did not show a significant treatment effect, however, the univariate analysis for FDERIV did indicate a significant treatment effect ($p < .05$). Hypothesis 1b, there is no significant treatment effect, was not rejected for FINAL or its subscale FANTI. This hypothesis was rejected for FDERIV ($p < .05$).

To determine the nature of differences in individual treatment means, adjusted for covariates, Bonferroni multiple comparison procedures were used with a family confidence coefficient of 95%. Pairwise comparisons for adjusted treatment means for FDERIV are presented in Table 59.

Table 59

Pairwise Treatment Comparisons for FDERIV

Treatment	G+	S1	S2
G	0.662	0.004*	0.154
G+		0.008*	0.290
S1			0.124

$^{*}p < .0083$

Significant differences are observed between treatments for which the given p-values are less than $\frac{.05}{6} = .0083$. For FDERIV, observed differences occur between the G and S1 sections, and between the G+ and S1 sections, in favor of the S1 section.

PRETEST results are suspect for subjects participating in the S1 section. Class and college data indicate that S1 subjects are not representative of the full sample population participating in the present study. They tend to have had more experience in college since the lowest percentage of freshmen are in this group. Although data on past mathematics enrollments was not available, it is likely, judging from subject performance on the FINAL, that subjects in the S1 section had not had recent exposure to a mathematics course prior to the study. This suggests that the low pretest scores of the S1 subjects do not indicate lower entering mathematics abilities, but instead indicate that longer time periods had passed since S1 subjects had last worked in mathematics.

For the above reasons, univariate analyses of variance without covariates were conducted for FINAL, FDERIV, and FANTI. No significant interaction ($p < .05$), PRICALC main effect, or TRTMNT main effect were observed for any of these variables, and in particular, for FDERIV.

Analysis 3

Analysis 3 was conducted to test hypothesis 1c, there is no significant change in attitude between treatments concerning subject interest in mathematics and interest in taking more courses in mathematics. To test this hypothesis, an analysis of variance was performed on the difference scores, DIFFIIM and DIFFITM. DIFFIIM (POSTIIM - PREIIM) measures changes in subject levels of interest in mathematics. DIFFITM (POSTITM - PREITM) measures changes in subject levels of interest in taking more courses in mathematics.

The Final Sample

Of the subjects participating in Investigation 2; all of the G subjects (n = 31), 35 of the G+ subjects, all of the S1 subjects (n = 33), and 25 of the S2 subjects were retained in Analysis 3. Retained subjects for this analysis are defined to be subjects who: (a) responded to both the pre- and post-attitude surveys, and (b) were missing no more than 1 entry for the PREIIM, POSTIIM, PREITM, or POSTITM scales. For these subjects, each missing entry was replaced with the subject's average score for the remaining responses in the particular scale. Removed subjects either did not complete the post-attitude survey or did not respond to two or more items on the scales concerning interest in mathematics

and interest in taking more courses in mathematics.

As with the two previous analyses, the S2 section again displays results from a more select group of students. A wider range of abilities and attitudes are represented in the G, G+, and S1 sections.

Design

For Investigation 2, Analysis 3, a one factor fixed-effects design was used. The factor used was treatment, G, G+, S1, and S2. Univariate one-way analyses of variance were conducted for each of the difference scores, DIFFIIM and DIFFITM. The analyses were conducted using the SPSS-X (Release 3.0 for Vax/VMS) ONEWAY program.

Treatment means and standard deviations for PREIIM, POSTIIM, PREITM, and POSTITM are presented in Table 60. Treatment means, standard deviations, minimums, and maximums for DIFFIIM and DIFFITM are presented in Tables 61 and 62 respectively. Subject responses for each of the items comprising the subscales ranged from 1 to 5, with a response of 5 indicating the most positive interest in mathematics or in taking more mathematics courses. The range for each score PREIIM, POSTIIM, PREITM, and POSTITM is 5 to 25. The resulting possible range for DIFFIIM and DIFFITM scores is -20 to 20.

Table 60

Treatment Means for PREIIM, POSTIIM, PREITM, and POSTITM, Investigation 2, Analysis 3

Variable	TRTMNT							
	G ($n = 31$)		G+ ($n = 35$)		S1 ($n = 33$)		S2 ($N = 25$)	
	PRE	POST	PRE	POST	PRE	POST	PRE	POST
IIM								
M	18.97	17.65	17.31	16.60	17.85	17.73	18.28	18.60
SD	2.39	3.07	3.35	3.87	3.31	2.99	3.03	2.94
ITM								
M	18.23	17.97	17.91	17.29	18.85	17.45	18.72	18.52
SD	3.01	3.36	3.99	4.79	3.53	3.63	3.71	3.61

The highest and lowest initial levels of interest in mathematics were exhibited by the G and G+ treatment sections respectively. Average initial responses for the G section ($M = 3.79$) indicate that subjects generally agreed that mathematics is a subject that they found enjoyable and interesting. Initial average responses for the G+ section ($M = 3.46$) indicate that subjects were undecided or neutral in their interest in mathematics. The lowest final level of interest in mathematics were also exhibited by the G+ section ($M = 16.60$). Subjects'

final average responses $(M = 3.32)$ indicated that they were still undecided or neutral concerning their level of interest in mathematics.

For interest in taking more mathematics courses, the highest and lowest initial levels of interest were exhibited by the S1 $(M = 18.85)$ and G+ $(M = 17.91)$ sections respectively. Initial average responses for the S1 section $(M = 3.77)$ indicate that these subjects would have been interested in pursuing the study of mathematics further. Subjects in the G+ section also were interested in pursuing their studies in mathematics initially $(M = 2.58)$ but to a slightly lesser degree.

Final levels of interest in taking more mathematics courses were slightly lower for all sections, with S1 subjects exhibiting the largest drop in such interest and G+ subjects exhibiting the least final interest. Still, final average responses indicate even G+ subjects $(M = 3.46)$ are at least neutral or undecided in their interest in taking more mathematics courses. Overall, initial and final responses, as indicated, were generally positive toward interest in mathematics and interest in taking more mathematics courses for all subjects participating in the study.

Table 61

Descriptive Statistics for DIFFIIM

TRTMNT	*n*	*M*	*SD*	Minimum	Maximum
G	31	-1.32	3.09	- 8	6
G+	35	-0.71	3.22	- 7	11
S1	33	-0.11	3.64	-10	7
S2	25	0.29	2.30	- 3	5

Table 62

Descriptive Statistics for DIFFITM

TRTMNT	*n*	*M*	*SD*	Minimum	Maximum
G	31	-0.26	3.28	- 7	6
G+	35	-0.63	4.81	-10	19
S2	33	-1.39	3.11	- 8	4
S2	25	-0.20	2.00	- 3	5

Treatment means for DIFFIIM and DIFFITM are negative in all but one case, indicating a very slight decrease over the semester in both subject interest in mathematics and in subject interest in taking more mathematics courses. In all cases, the means have absolute values less than 1.4, compared to a possible range of -20 to 20 for the difference scores. The largest negative means for DIFFIIM occurred for the G section, indicating that subject interest in mathematics decreased most for this section. The largest negative mean for DIFFITM occurred for the S1 section, indicating that subject interest in taking more mathematics courses decreased most for this section. In both cases, the changes were negligible. The observed differences between treatments are small, suggesting that these differences are also not significant.

To determine changes in attitude between treatments concerning subject level of interest in mathematics and interest in taking more courses in mathematics, analyses of variance for DIFFIIM and DIFFITM by treatment were conducted.

The result of the analysis of variance conducted for DIFFIIM did not show a significant difference between treatments, $F(3,120) = 1.46$, $p < .23$. The results of the analysis of variance conducted for DIFFITM also did not show a significant difference between treatments; $F(3,120) = 0.074$, $p < .53$. Hypothesis 1c, there is no significant difference between treatments on subject interest in mathematics and in subject interest in taking more mathematics courses, was not rejected.

To determine significant pre to post changes, for individual treatments, 95%

Least Significant Difference (LSD) confidence intervals were determined for DIFFIIM and DIFFITM. These are presented in Tables 63 and 64 respectively.

Table 63

LSD Confidence Intervals for DIFFIIM

TRTMNT	95% C.I. for Mean
G	-2.457 to -0.189
G+	-1.821 to 0.393
S1	-1.405 to 1.177
S2	-0.661 to 1.241

Table 64

LSD Confidence Intervals for DIFFITM

TRTMNT	95% C.I. for Mean
G	-1.456 to 0.944
G+	-2.283 to 1.025
S1	-2.497 to -0.291
S2	-1.026 to 0.626

The only significant pre to post change concerning interest in mathematics occurred within the G section. Subject scores decreased from an initial mean of 18.97 to a mean of 17.65. For the five questions on the DIFFIIM subscale, such a change denotes a slight decrease in interest while still indicating a positive interest in mathematics.

The only significant pre to post change concerning interest in taking more mathematics courses occurred within the S1 section. Subject scores decreased from an initial mean of 18.85 to 17.45. Again, the change denotes only a slight decrease in interest. The level of interest is still positive toward taking more mathematics courses.

As indicated by the profile data given in Analysis 2, the G section contained the largest number of freshmen among the four participating sections. The post responses for the G section are similar to the post response for the other three sections. Perhaps the enthusiasm of the freshmen was tempered with realism over the semester.

The S1 section contained the least number of freshmen (see profile data in Analysis 2). While subjects indicated an interest in taking more courses in mathematics at the beginning of the semester, their slightly less enthusiastic post response might more accurately reflect the realism of their program requirements.

Other Descriptive Data

Perceptions of the Nature of Mathematics

Students responded to attitude survey items concerning their perceptions of the nature of mathematics. Difference scores DIFFPNMn ($n = 1,2,3,4$) were examined to determine pre to post changes in subject perceptions.

The Final Sample. Retained subjects are those subjects who responded to both of the items PREPNMn and POSTPNMn for a given n ($n = 1,2,3,4,5$). Sample sizes for each item are summarized in Table 65. Retained subjects are identical for the first four items. Three of the S1 subjects and one of the S2 subjects chose not to respond to POSTPNM5.

Table 65

Sample Sizes for Comparisons of PNM Items

TRTMNT	n	
	1,2,3,and 4	5
G	31	31
G+	35	35
S1	33	30
S2	25	24

Procedure. Treatments G, G+, S1, and S2, were compared. For each of the five items concerning subject perceptions of the nature of mathematics on the pre- and post-attitude survey, the frequencies of the difference scores,

$$\text{DIFFPNM}n = \text{POSTPNM}n - \text{PREPNM}n,$$

were tabulated and examined. The statements for each item are given in the discussion that follows. Subject responses to items ranged from 1 to 5, strongly disagree to strongly agree. Pre and post means and standard deviations are presented for each item.

PNM1: There is only one way to solve most mathematics problems.

Means and standard deviations for PREPNM1 and POSTPNM1 by treatment appear in Table 66. Average responses, both pre and post, indicate that subjects perceive that there is more than one way to solve most mathematics problems. POST means are lower than pre means for the G, G+, and S1 sections, with G subjects indicating that they now strongly disagree with statement PNM1. The S2 subjects still disagreed with the statement but slightly less strongly at the end of the semester than they had at the beginning. An examination of difference scores, DIFFPNM1, indicated that most subjects did not change their perception.

Table 66

Descriptive Statistics for PREPNM1 and POSTPNM1 by Treatment

TRTMNT	n	PREPNM1		POSTPNM1	
		M	SD	M	SD
G	31	1.52	0.81	1.19	0.40
G+	35	1.57	1.07	1.51	0.70
S1	33	1.82	1.07	1.67	0.78
S2	25	1.56	0.51	1.64	0.57
Overall	124	1.62	0.92	1.50	0.66

PNM2: Mathematics helps to develop the mind and teaches a person to think.

Average pre and post responses to this item are presented in Table 67. The data indicate that subjects in all sections tended to agree with the statement. Only subjects in section G agreed more strongly at the end of the semester. Subjects in the S2 section agreed less strongly late in the semester than they had at the beginning of the semester.

An examination of difference scores, DIFFNM2, indicated that subject changes in perception were similar across treatments G+, S1, and S2. Over half of the subjects in each section did not change their initial perception. The perceptions of 21% to 32% of the subjects in each of these sections became more negative over the semester; fewer subjects perceptions increased. Overall, sub-

jects in each section agreed pre to post that mathematics helps to develop the mind and teaches a person to think.

Table 67

Descriptive Statistics for PREPNM2 and POSTPNM2 by Treatment

TRTMNT	*n*	PREPNM2		POSTPNM2	
		M	*SD*	*M*	*SD*
G	31	4.32	0.65	4.48	0.63
G+	35	4.06	0.84	4.00	0.84
S1	33	4.18	1.10	4.18	0.95
S2	25	4.44	0.51	4.20	0.50
Overall	124	4.23	0.83	4.21	0.78

PNM3: Mathematics involves mostly memorizing.

Average pre and post responses to this item are presented in Table 68. The data indicate that only in the S1 treatment did subject agreement with this statement increase during the study. In the remaining three sections, there was a slight decrease in subject agreement during the study. An examination of the difference scores, DIFFPNM3, indicated that subject perceptions concerning the amount of memorization involved in mathematics did not change with respect to treatment.

Table 68

Descriptive Statistics for PREPNM3 and POSTPNM3 by Treatment

TRTMNT	*n*	PREPNM3		POSTPNM3	
		M	*SD*	*M*	*SD*
G	31	2.29	0.97	2.13	1.23
G+	35	2.46	0.89	2.31	0.96
S1	33	2.52	1.03	2.58	0.83
S2	25	2.44	0.92	2.36	0.91
Overall	124	2.43	0.95	2.35	1.00

PNM4: Mathematics involves creativity.

Average pre and post responses to this item are presented in Table 69. This data indicates that subjects in all treatments tended toward slight agreement with this item both before and after this study, with S2 subjects showing the greatest increase over the semester. An examination of the difference scores, DIFFPNM4 indicated that subject perceptions concerning the amount of creativity involved in mathematics did not change with respect to treatment.

Table 69

Descriptive Statistics for PREPNM4 and POSTPNM4 by Treatment

TRTMNT	*n*	PREPNM4		POSTPNM4	
		M	*SD*	*M*	*SD*
G	31	3.74	1.00	3.61	1.20
G+	35	3.51	0.78	3.40	0.85
S1	33	3.55	0.90	3.64	0.74
S2	25	3.44	0.87	3.80	0.71
Overall	124	3.56	0.89	3.60	0.90

PNM5: Most problems in mathematics can be solved by following a rule.

Average pre and post means for this item are presented in Table 70. The data indicate that, overall, subjects agree with this statement. An examination of difference scores, DIFFPNM5, indicated that no S1 subject showed stronger agreement with this statement from pre to post, and that S2 subjects were as likely to change their opinion to express stronger agreement as stronger disagreement with the statement.

Table 70

Descriptive Statistics for PREPNM5 and POSTPNM5 by Treatment

TRTMNT	*n*	PREPNM5		POSTPNM5	
		M	*SD*	*M*	*SD*
G	31	3.97	0.71	3.81	0.95
G+	35	3.80	0.83	3.94	0.59
S1	30	4.23	0.57	3.97	0.61
S2	24	3.75	0.74	3.71	0.91
Overall	120	3.94	0.74	3.87	0.77

Retention Rates

Additional data was gathered for the study: (a) enrollment patterns describing retention rates for all subjects enrolled in Calculus I at WMU during the Winter 1988 semester; and (b) departmental student evaluation survey data for the G, G+, S1, and S2 sections which solicited subject opinions concerning various aspects of the course.

Enrollment data is presented in Table 71. Retention rates were very high for the G, G+ and S1 sections. Retention rates for the S2 section and the three sections not participating in the study were considerably lower, although exceeding

those generally reported for large universities (Petersen, 1986). The course presentation for the sections with high retention rates was conceptual. Emphasis was placed on understanding concepts, which were often developed graphically. For at least the G and G+ sections, a corresponding deemphasis was placed on skill development. For the S2 section and the other sections that did not participate in the study, instruction was traditional. Emphasis in these sections was on skill building.

Table 71

WMU Enrollment Patterns, Calculus I, Winter 1988

Group	Initially Enrolled	Completed Course	Retention Rate
G	37	31	83.8%
G+	40	36	90.0%
S1	40	35	87.5 %
S2	46	32	69.6 %
Students Not in Study	140	105	75.0%

Evaluation Survey Data

In addition to the data gathered for each of the G, G+, S1, and S2 subjects which were analyzed in Investigation 2, data gathered from subject responses to

each of seven items on the WMU student evaluation survey, EVALn ($n = 16, 17, 18, 19, 20, 21, 22$), were compared by treatment. To make comparisons, one-way analyses of variance were conducted for each of the items.

The Final Sample. All subjects responded to evaluation survey data under conditions of anonymity. Subject responses were not identifiable. Therefore, some subjects, who were not included in any of the other analyses but who were enrolled in the course at the time the evaluation survey was administered, are included in the present sample. Sample sizes for the following analyses are presented in Table 72.

Table 72

Sample Sizes for Evaluation Survey Analyses

TRTMNT	n
G	30
G+	35
S1	35
S2	28

As with the analyses of Investigation 2, a considerable amount of self-selection had taken place in the S2 section. While the subjects of the G, G+, and S1 sections represented over 80% of the subjects for these sections who were initially enrolled

in the course, S2 subjects represented only 61% of those initially enrolled.

Design. A one-factor design was used with factor levels G, G+, S1, and S2. Treatment means were determined for each of the seven items on the evaluation survey, then analyses of variance were conducted for each item. The statements for each item are given below. Subject responses for items EVAL16 through EVAL20 varied from 1 to 5 as indicated in Table 73.

Table 73

Response Interpretation for EVAL16 through EVAL20

poor	below average	average	above average	excellent
1	2	3	4	5
disagree	disagree somewhat	neutral	agree somewhat	agree strongly

EVAL16: Content and/or processes developed in this course are useful.

Treatment means are similar for EVAL16 (see Table 74). Subjects in all treatments "agree somewhat" that content and/or processes developed in this course are useful. Based on an analysis of variance, no significant differences were detected between treatments, $F(3,124) = 0.157$, $p < .925$. Subject attitudes toward the usefulness of course content and/or processes did not differ by treatment.

Table 74

Treatment Means for EVAL16

TRTMNT	*n*	*M*	*SD*
G	30	3.77	0.90
G+	35	3.86	0.91
S1	35	3.80	0.87
S2	28	3.71	0.66

EVAL17: Assignments seem carefully selected and contribute significantly to student learning.

Table 75

Treatment Means for EVAL 17

TRTMNT	*n*	*M*	*SD*
G	30	3.40	0.97
G+	35	3.66	1.00
S1	35	3.80	0.83
S2	28	4.04	0.64

Treatment means are quite different for EVAL 17 (see Table 75). The mean responses for subjects in the G+, S1, and S2 sections indicate that subjects "somewhat agree" that assignments were carefully selected and contributed significantly to student learning. Subjects in the G section are only in "average" agreement with this statement. The difference between the G and S2 sections is particularly large. An analysis of variance detected a significant difference between treatments for EVAL17, $F(3,124) = 2.690$, $p < .049$. Multiple comparisons, using the Tukey procedure with a family confidence coefficient of 95%, indicate that the G and S2 treatment means are unequal. S2 subject responses indicate that assignments were perceived as consistent with the course goals. Responses for the G treatment indicate that subjects were neutral or undecided as to whether or not assignments contributed significantly to student learning.

EVAL18: Course text and materials contribute to student learning.

Table 76

Treatment Means for EVAL18

TRTMNT	*n*	*M*	*SD*
G	30	3.40	1.13
G+	35	3.83	1.04
S1	35	3.94	0.87
S2	28	3.93	0.66

Treatment means for the G+, S1, and S2 sections are similar for EVAL18 (see Table 76). The mean for the G treatment is somewhat lower. Based on an analysis of variance, no significant differences ($p < .05$) were detected between treatments for EVAL18, $F(3, 124) = 2.202$, $p < .091$. Observed differences in treatment means are not significant. However, subjects in the G+, S1, and S2 sections agreed more strongly that the course text and materials contributed to student learning than did subjects in the G section.

EVAL19: Examinations cover the course material well.

Table 77

Treatment Means for EVAL19

TRTMNT	*n*	*M*	*SD*
G	30	3.40	1.40
G+	35	3.34	1.30
S1	35	3.69	0.90
S2	28	3.71	0.81

Treatment means for EVAL19 are similar (see Table 77). The graphic sections, G and G+ agreed less strongly that examinations cover the course material well than did the standard sections, S1 and S2. These differences are small.

An analysis of variance conducted for the data did not detect significant differences between treatments for EVAL19, $F(3,124) = 0.911$, $p < .438$. Observed differences between treatment means are not significant.

EVAL20: General evaluation of the course.

Table 78

Treatment Means for EVAL20

TRTMNT	n	M	SD
G	30	3.63	1.07
G+	35	3.54	1.01
S1	35	3.71	0.93
S2	28	3.79	0.69

Treatment means for EVAL20 are similar (see Table 78). Mean responses for all treatments indicate that subject general evaluations of the course were above average. Subjects in the G+ section were in slightly less agreement with the general evaluation than were subjects in the other sections but this difference is negligible.

An analysis of variance conducted for EVAL20 failed to detect significant differences between treatments, $F(3,124) = 0.395$, $p < .757$. Mean responses

indicate that subjects generally rate the course as above average. Slight differences in treatment means are not significant.

EVAL21: I gained _____ understanding of the concepts and principles in this course. 1) no; 2) minimal; 3) adequate; 4) good; 5) excellent.

Table 79

Treatment Means for EVAL21

TRTMNT	*n*	*M*	*SD*
G	30	3.37	0.69
G+	35	3.37	1.00
S1	35	3.66	0.84
S2	28	3.61	0.69

Treatment means are somewhat lower for the G and G+ sections than for the S1 and S2 sections for EVAL21 (see Table 79). Mean responses for the G and G+ sections indicate that subjects in these sections viewed their understanding of concepts and principles in the course as adequate. Mean responses for the S1 and S2 sections inaicate that these subjects viewed their understanding as good.

An analysis of variance conducted for EVAL21 failed to detect significant differences between treatments, $F(3,124) = 1.135$, $p < .338$. Observed

differences in subject self-evaluation of levels of understanding were not significant between treatments.

EVAL22: On the average, approximately how many hours do you devote to this course outside of class for each hour in class: 1) one or less; 2) two; 3) three; 4) four; 5) five or more.

Table 80

Treatment Means for EVAL22

TRTMNT	*n*	*M*	*SD*
G	30	2.00	1.02
G+	35	2.09	1.31
S1	35	1.94	1.03
S2	28	2.04	1.04

Treatment means are similar for all sections for EVAL22 (see Table 80). Subjects in each section devoted approximately two hours to the course outside of class for each hour in class.

An analysis of variance conducted for EVAL22 responses failed to detect significant differences between treatments, $F(3,124) = 0.101$, $p < .959$. No significant differences exist between sections concerning hours devoted to the course outside of class.

Summary of Investigation 2

Investigation 2 was undertaken to compare treatment sections differing in the level of exposure to a graphically-developed, conceptual calculus course on cognitive and affective measures. Instruction in the treatment sections varied from instruction in which the use of graphs was infrequent to instruction in which concept development was heavily dependent on the use of computer graphic displays.

In Analysis 1, the performances of G, G+, S1, and S2 subjects on the subscales comprised of exam and quiz questions, APP, SR, and SN, were compared. In Analysis 2, the performances of G, G+, S1, and S2 subjects on the cognitive measure, FINAL, and its subscales, FDERIV and FANTI, were compared. In Analysis 3, subject pre to post changes in attitude toward interest in mathematics and interest in taking more courses in mathematics were compared. Descriptive data concerning subject pre to post changes in perceptions of the nature of mathematics; enrollment and retention rates; and subject attitudes and opinions toward content, processes, assignments, text, materials, examinations, personal understanding and time commitment were also compared.

Retained subjects for each analysis differed somewhat. A summary of the retained subjects for each analysis is given in Table 81. Initial enrollments are given as a basis for comparison. Full samples for each of the G, G+, S1, and S2 sections represent 83.8%, 90%, 82.5%, and 60.9% respectively, of the initial enrollments for these sections.

Table 81

Subjects Retained in Investigation 2 by Analysis

Analysis	TRTMNT			
	G	G+	S1	S2
1	30	35	25	21
2	31	36	33	28
3	31	35	33	25
Perceptions of Nature of Math	31	35	33[a]	25[a]
Evaluation Surveys	30	35	35[b]	28[b]
Full Sample	31	36	33	28
Initial Enrollment	37	40	40	46

[a] Only 30 S1 subjects and 24 S2 subjects remained for comparisons of DIFFPNM5.
[b] Subjects responded to Student Evaluation Surveys under conditions of anonymity. Some subjects who were not included in the study for reasons given in Chapter III are represented here.

In the analyses for Investigation 2, S2 subjects represented a much smaller percentage of subjects who initially enrolled in the course as compared to at least the G and G+ sections. The S1 section was not fully represented in Analysis 1.

Retention rates for the sections in which conceptual understanding was emphasized, G, G+ and S1, were considerably higher than for sections in which the presentation was more traditional – S2 and three other WMU Calculus I sections which did not participate in the study.

The conclusions of each analysis and related descriptive data follow.

Conclusions, Analysis 1

Hypothesis 3a, there is no significant interaction between treatment and prior calculus experience on subject performance on exam and quiz questions categorized as applied routine, symbolic routine, and symbolic nonroutine, was not rejected at the .05 level of significance.

Hypothesis 2a, there is no significant difference between subjects with no prior calculus experience on performance on applied routine, symbolic routine, and symbolic nonroutine exam and quiz questions, was not rejected at the .05 level of significance.

Hypothesis 1a, there is no significant difference between treatments on subject performance on applied routine, symbolic routine, and symbolic nonroutine exam and quiz questions, was rejected at the .05 level for the symbolic routine and symbolic nonroutine subscales. Significant differences were detected: (a) between G+ and S1 subjects in favor of the S1 treatment for symbolic routine questions, and (b) between G and S2 subjects in favor of the G treatment for symbolic nonroutine questions.

Suspecting irregularities in the pre-treatment measures for the S1 and S2

sections, univariate analyses of variance were conducted for applied routine, symbolic routine, and symbolic nonroutine subscales. No significant interaction or prior calculus experience main effects were detected. A treatment main effect was detected for the symbolic nonroutine variable only. Multiple comparisons, performed to determine the nature of the treatment differences, were consistent with the findings of the analysis of covariance for symbolic nonroutine questions.

Considering the differences between treatments, the S1 and S2 sections representing more selective samples than the G and G+ sections, it is surprising that more differences were not found in favor of the S1 and S2 sections. The G section, representing a broad range of abilities, performed better on symbolic nonroutine questions than the S2 section who represented a smaller range of abilities. While the differences were not significant, the G and G+ sections also performed better than the S1 section on the symbolic nonroutine questions. These differences might have been more pronounced had the S1 sample been complete.

Conclusions, Analysis 2

Hypothesis 3b, there is no significant interaction between treatment and prior calculus experience in subject performance on the departmental final exam or either of its subscales FDERIV or FANTI, was not rejected at the .05 level of significance.

Hypothesis 2b, there is no significant difference between subjects with prior calculus experience and subjects with no prior calculus experience in subject performance on the departmental final exam or either of its subscales FDERIV or

FANTI, was not rejected at the .05 level of significance.

Hypothesis 1b, there is no significant difference between treatments in subject performance on the departmental final exam or either of its subscales FDERIV or FANTI, was not rejected for the departmental final and its subscale FANTI, but was rejected for the subscale, FDERIV, questions pertaining to the concepts of limit, continuity, and derivative.

Questioning the validity of PRETEST results for the S1 section, univariate analyses of variance were conducted for FINAL, FDERIV, and FANTI. No significant $(p < .,05)$ interaction or main effect differences were detected for any of these variables.

Conclusions, Analysis 3

Hypothesis 1c, there is no significant difference between treatments in pre to post changes in subject interest in mathematics and interest in taking more mathematics, was not rejected.

Individual within-treatment differences from pre to post were detected only for: (a) the G section for DIFFIIM, and (b) the S1 section for DIFFITM. These differences were only slightly negative, showing decreased interest from pre to post. In both cases, post responses reflected a view that was more consistent with the other sections.

In all cases, pre and post, the levels of interest were positive, both toward mathematics and toward taking more mathematics courses.

Conclusions, Other Descriptive Data

Perceptions of the Nature of Mathematics. There were no differences between treatments in pre to post changes in subject perceptions of the nature of mathematics for DIFFPNMn ($n = 1, 2, 3, 4$). Subject perceptions of the nature of mathematics were generally positive.

Between treatments, changes in subject perceptions of the nature of mathematics were observed for the perception that most mathematic problems can be solved by following a rule. Subjects in the G+ section disagreed with the statement less often than expected, S1 subjects agreed with the statement less often than expected, and S2 subject responses changed more often than expected.

Within treatments, pre to post changes in subject perceptions indicated that: (a) G subjects disagreed more strongly that there is only one way to solve most mathematics problems, and (b) S1 subjects disagreed more strongly that most problems in mathematics can be solved by following a rule. S2 subjects tended to disagree more strongly that mathematics helps to develop the mind and teaches a person to think, and they tended to agree more strongly that mathematics involves creativity.

Evaluation Survey Data. While over 80% of the subjects initially enrolled in the G, G+, and S1 sections responded to the evaluation survey, only 61% of the S2 subjects did so. Responses for the S2 section represented a more select group than did responses for the other sections. However, the only significant difference ($p < .05$) between treatment sections concerned the selection of

appropriate assignments. Subjects in the G section perceived assignments as being adequately chosen to contribute to student learning. Subjects in this section were presented a course that was often very different from that presented in the course text. They were given no supplemental assignments to more adequately fit the course they received. Of the three instructors participating in the present study, the S2 instructor presented a course most consistent with that presented in the text. Assignments for all sections were chosen from text and were consistent with its presentation. G + subjects, but not G subjects, received supplemental assignments along with the textbook assignments.

It is not surprising that a difference in treatment means would occur between sections G and S2 on appropriateness of selected assignments. It is somewhat surprising that significant differences did not occur in other areas, especially those related to the course materials and examinations.

CHAPTER V

SUMMARY, CONCLUSIONS, IMPLICATIONS, AND RECOMMENDATIONS

The present study was undertaken to investigate the extent to which use of the Cartesian coordinate graphical representation system, developed and enhanced through the use of computer graphics software, aids student understanding of the concepts in a first course in calculus. The study suggests possible directions for reform in the content and teaching of calculus and directions for future research.

An experimental, graphically-developed, technology-enhanced, conceptual calculus course served as the vehicle with which to carry out the present study. The conceptual development of the course was based primarily on the cognitive science/information processing theory as presented by Davis (1984), the psychology of learning mathematics as presented by Skemp (1987), and the theory of developing competence with mathematical symbol systems as presented by Hiebert (1987).

Two concurrent investigations were undertaken to compare subject performance and attitudes related to: (a) student understanding and use of graphs – Investigation 1, and (b) student understanding of calculus concepts especially as developed in a graphic environment – Investigation 2.

Four instructional treatments were used in the study: (1) Graphics (G) – exposure to a graphically-developed, conceptual calculus course heavily dependent

on the use of computer graphics for in-class exposition; (2) Graphics Plus (G+) – exposure to the same course as the G treatment plus the provision of use of computer graphics software and supplemental assignments for further exploration or reinforcement of course content; (3) Standard 1 (S1) – exposure to a graphically-developed, conceptual calculus course where the computer was not used; and (4) Standard 2 (S2) – exposure to a non-graphical calculus course where skill development was emphasized. Subjects in each treatment section were partitioned into two subgroups with respect to each subject's prior calculus experience: (1) No prior calculus experience – either no previous experience with calculus, or experience only with the part of the course preceding the use of rules for finding derivatives; and (2) Prior calculus experience – experience with at least one-half of a semester of calculus including the rules for finding derivatives.

For Investigation 1, data was collected for the G and G+ treatments concerning student: (a) facility with a graphic representation, including the use of such a representation in understanding calculus concepts; and (b) attitudes toward the use and usefulness of hand-drawn and computer-generated graphs.

For Investigation 2, data was collected for the G, G+, S1 and S2 treatments concerning student: (a) performance on applied, routine symbolic, and nonroutine symbolic questions; and (b) attitudes toward mathematics and the course.

Summary of Results

Investigation 1

In Investigation I, no significant differences were detected due to treatment, prior calculus experience, or the interaction of treatment by prior calculus experience for the graphic cognitive measures. In each case, performances on cognitive measures were in favor of the subjects who had been provided supplemental assignments and computer software.

Subject reactions toward the use and usefulness of graphs were positive and often enthusiastic. Subjects reported that their ability to use graphs had increased and that their use of graphs in the present course was more frequent than it had been in previous courses. No significant differences were detected between the graphics treatments on affective measures.

Subjects agreed overwhelmingly that the use of graphs in general, and computer graphs in particular, helped them understand functions and calculus ideas. Subjects indicated that in previous courses, graphs were the end result of an exercise. In the present course, subject responses included indications that graphs: (a) allowed the mind to receive a lot of information at once, (b) were easier to understand than algebraic or calculus symbols used alone, (c) generated with the computer helped students develop efficient methods of graphing functions by hand, (d) reduced the amount of memorizing necessary by helping students organize their thoughts and reason through problems, and (e) were useful tools in understanding problems and determining the reasonableness of solutions.

Although subjects indicated strong support for the use of graphs in and out of class, they did not express strong support for drawing graphs to solve problems. As indicated in Appendix A, subject responses to questions in class, and on exams and quizzes, were often stated in graphic terms. The investigator observed subjects occasionally drawing graphs in the air. Subjects indicated that they did not always need to draw graphs. Rather, they had developed enough facility with graphs to efficiently mentally estimate with them.

About half of the G+ subjects indicated that they either rarely or never used the computer software provided to them. Under these conditions, it is not surprising that the G and G+ subjects did not differ significantly on any of the graphic cognitive or affective measures.

Investigation 2

In Investigation 2, no significant differences were detected for prior calculus experience, or for interaction of treatment by prior calculus experience for any of the cognitive variables. Treatment main effects were detected for symbolic routine questions, symbolic nonroutine questions, and final exam questions pertaining to the concepts of limit, continuity, and derivative when the pre-measures of precalculus competence, interest in mathematics, and interest in taking more courses in mathematics were used as covariates. Significant differences for symbolic routine questions were detected between the G+ and S1 treatment sections, in favor of the S1 section. Significant differences for symbolic nonroutine questions were detected between the G and S2 treatments in favor of the G section. Significant

differences for the final exam questions related to limits, continuity, and derivatives were detected between the G and S1 treatments and between the G+ and S1 treatments, both in favor of the S1 section.

As explained in Chapter IV, the validity of the results of the pretest of precalculus symbol manipulation and graphing competencies were questionable for at least the S1 section. In subsequent univariate analyses without covariates, only one significant treatment effect was detected. The G subjects performed significantly better than the S2 subjects on symbolic nonroutine questions.

In comparing treatment effects on course exam and quiz measures, over 29% of the subjects in each of the standard sections were removed from the analysis because of missing data. These subjects' cognitive abilities, as measured by their performance on the departmental final exam, were much lower than the abilities of the subjects retained for the analysis. Subjects in both of the standard sections also were not representative of the full sample for comparisons made on the final exam measures. In contrast, only 3% of the subjects from the graphics sections were removed from either analysis. Overall, this could have had the effect of biasing the results in favor of the standard sections. Considering these facts it is surprising that more differences were not detected in favor of the standard sections. Differences in ability to solve nonroutine calculus problems were found in favor of the graphics sections – these differences were significant between G and S2 subjects with G subjects outperforming S2 subjects. This indicates that a computer- graphically developed, conceptual calculus course could potentially

improve student understanding of calculus concepts. Further research is needed before conclusive statements can be made. Also, while no definitive statement can be made concerning the differences between treatments in performance on routine questions, results suggest that if student facility with manipulative technique is affected, the effect is negligible. Further study is warranted in this regard, as well.

In analyses pertaining to affective measures, no significant differences were detected between treatments for pre to post changes in subject interest in mathematics, or in interest in taking more courses in mathematics. Within treatments, slight significant negative pre to post changes in: (a) interest in mathematics were detected for the G subjects, and (b) interest in taking more courses in mathematics were detected for the S1 subjects. In both cases, interests remained "above average", and post responses were comparable to the responses of subjects in the other sections.

Subject perceptions of the nature of mathematics were positive for all items. Treatments varied for subject responses to the statement, "Most problems in mathematics can be solved by following a rule." S1 subjects perceptions changed to disagree more strongly with the perception, and G+ subjects perceptions changed to agree more strongly with the statement.

Subject assessment of course content, processes, materials, exams, and time commitment indicated that they perceived these as being of "above average" appropriateness. No significant differences were detected between treatments on these items. The only significant treatment effect detected concerned subject as-

sessment of the appropriateness of assignments between the G and S2 sections. Assignments given these sections were taken directly from the course text. The S2 section considered assignments to be "good", the G section considered them to be "adequate". This difference was to be expected since the instruction given the S2 subjects closely paralleled the text while the course presentation for the G subjects often diverged from the text in major ways.

Retention rates – percentage of students completing the course – were considerably higher for the sections in which the emphasis was on conceptual understanding than for the more traditional section where skills were emphasized. In all of the analyses, subjects in the sections in which conceptual understanding was emphasized performed favorably when compared to subjects in the traditional section. It is not known if the difference in retention is due to the conceptual approach or other factors such as student/instructor interaction.

Subjects in the G and G+ sections performed better than traditionally presented subjects on symbolic nonroutine problems, indicating that these subjects were better able to transfer the understanding they had developed. Since these subjects were probably as capable in performing routine manipulative techniques and since much higher percentages of these students successfully completed the course, these results speak very positively in favor of the graphically-developed, conceptual approach to calculus over the more traditional skill-dominated approach.

Conclusions

The results of the data analysis for the study support several conclusions. The generalizability of these conclusions is influenced by the characteristics of the sample, the experimental setting, and the measurement variables.

1. Student ability to solve nonroutine calculus problems can be enhanced through a graphically-developed, conceptual presentation of calculus.

2. Student acquisition of routine manipulative skills in calculus is probably not adversely affected by the reduced emphasis on skill development in a graphically-developed conceptual calculus course.

3. The retention rate for students participating in a graphically-developed conceptual calculus course is much higher than in more traditional courses which emphasize skill development.

4. Early introduction of Cartesian graphs allows the use of this representation system in developing calculus concepts while still keeping within the usual time-frame for the course.

5. The use of the computer in developing a graphic representation system can promote student development of: (a) efficient methods of drawing graphs by hand, and (b) mental estimation skills with graphs.

6. The use of graphs in problem solving aids students in understanding the problem, in planning and organizing the solution process, and in checking the reasonableness of results obtained.

7. The provision of computer software and supplemental assignments support-

ing in-class graphical, conceptual development encourages student facility with a graphic representation, especially as such is used as a referent for the development of understanding in calculus.

Implications for the Use of Graphs in Teaching Calculus

In the present study, graphs were used heavily to develop the concepts of calculus. The results indicate that use of graphs for conceptual development has potential for increasing student understanding and interest in calculus.

As developed in this study, several advantages arise from the use of graphs:

1. Computer generated graphs create a dynamic display of the relationships between variables and provide a link between concrete experiences and abstract symbols which describe these variables.

2. Graphs create a medium through which students can develop efficient mental estimation abilities.

3. Graphs reduce the amount of material that students need to commit to memory and often provide meaning to that which must be memorized.

4. Students have more confidence in the results they obtain symbolically when they can verify these results graphically.

In addition the study suggests that it is possible to greatly reduce the time spent in class on developing and reviewing manipulative skills and still maintain acceptable levels of student competence with these skills. Perhaps the increased time spent on conceptual development and problem solving enhances student ability to transfer learned knowledge to new situations.

Use of the Computer in Developing Competence With Graphs

Competence with written mathematical symbols develops as students construct connections between individual symbols and familiar referents. For graphs, the referent is location. The use of computer graphs allowed the quick generation of many graphs from which properties could be generalized.

Experiences in this the study suggest that, upon entry into calculus, student competencies with graphs include little more than plotting points and sketching curves by connecting such points. Computer graphs allowed the basic polynomial, trigonometric, greatest integer, and absolute value functions to be displayed and allowed the discussion of general properties such as domain, range, extrema, intervals of increase or decrease, periodicity, and symmetry. Relationships between new functions obtained through operations on familiar functions were displayed and generalized. Students developed competence with the graphic symbol system as presented by Hiebert (1987), for the second process – developing symbol manipulation procedures – and the third process – elaborating procedures for symbols – through the use of computer graphic displays (see Appendix B).

The fourth process – routinizing the procedures for manipulating symbols – was conducted on an individual basis by students, either by graphing functions by hand, or by graphing functions with the computer software provided. Subjects who had been provided the computer graphics software performed slightly better than subjects who had not been provided the software on graphic exam and quiz questions, though the difference was not significant. Such results

suggest that the fourth process might be engaged successfully through the use of technological tools.

The fifth process of developing competence with written mathematical systems – using the symbols and results as referents for building more abstract symbol systems – was also engaged successfully, through the use of computer graphs as exhibited by graphics subject performance on symbolic nonroutine questions.

Hiebert's (1987) theory of developing competence with written mathematical symbols addresses the development of such competencies by school-age children for the mathematical symbol systems of elementary mathematics. The present study supports the extension of the theory to college-age students and to more advanced written symbol systems.

Graphs as a Referent for Building More Advanced Symbol Systems

In the present study, graphs were used as a referent for developing the symbols of calculus. Calculus involves the study of how quantities change with respect to each other. The quantities of distance and time were displayed dynamically. Students observed a computer spider moving vertically while a graph of his position was sketched with respect to time. Through this medium, meaning was built for the concepts, and consequently the symbols, of first and second derivatives, by actually observing the spider's velocity and acceleration with respect to the properties of the related graph.

For the symbols of calculus, graphs served as a referent as properties of graphs were paralleled with the familiar experiential referent of distance vs. time dis-

cussed above. The first process of developing competence with written mathematical symbols – connecting individual symbols with referents – was engaged completely through the use of computer graphs. The second process – developing symbol manipulation procedures – was also engaged graphically. Meaning for basic calculus procedures was developed through observing the actions on graphs and translating these actions into the symbols of calculus. The third and fourth processes were engaged using the symbols and rules that had been developed graphically, without using the graphic referent necessarily, but with the possibility of translating results back to the referent if such was desired or necessary.

As with the development of competence for the graphic symbol system, the theory for such development proposed by Hiebert (1987) is supported by the development of competence with the symbols of calculus.

Recommendations for Further Study

The experiences and results of the present study suggest the following directions for further research and development:

1. Written responses were useful in ascertaining student levels of understanding of calculus concepts. However, some studies (Skemp, 1987) suggest that verbal techniques, such as reading a problem out loud, cue student memory. A similar study in which students are interviewed could provide more complete information concerning student conceptual understanding.

2. Attitude seemed to play a role in student acceptance of, and participation in, the instructional approach to teaching calculus used in the graphics sections.

Hesitation displayed by some students might have been due to sources such as: (a) apprehension of computer use, (b) unwillingness to do extra work involving supplemental assignments, or (c) disinterest in mathematics. The role of attitude in willingness to learn mathematics, especially when the computer is used as a tool, should be further explored.

3. Current literature on suggestions for revision of the calculus sequence give few suggestions as to which calculus techniques should be mastered by students. With full use of the computer as a tool in mathematics, the decision concerning which skills are fundamental to the course should receive attention in future research.

4. In the present study, computers were used to display graphs. Students in one section were provided graphics software and supplemental assignments to provide further experiences with graphs. Approximately half of the students in this section rarely or never used the computer. A similar study using the computer in classroom presentation, but requiring the use of graphics calculators by students might encourage more students to engage in the treatment. Such a follow-up study would provide valuable information regarding optimal use of the technology.

5. A similar study which allows student use of the investigator-written programs outside of class might provide valuable information, especially if supplemental assignments are written to optimize use of the programs, and if these assignments are subsequently collected and graded.

6. Interaction between students and instructor was very high for the graphics

sections. It is not known if this high level of participation caused the noticeable differences between groups, or if the differences were due to the use of a graphic, conceptual presentation of concepts. Future research might be conducted to determine the relative efficacy of a conceptual/lecture versus a conceptual/interactive approach to calculus.

7. In the S1 section, where concepts were presented graphically without the computer, the first 15 to 20 minutes of each class period were devoted to review of previous assignments. Assignments were generally skill-oriented. Students in this section outperformed students in all other sections on symbolic routine questions, though this difference was of questionable significance. However, students in this section completed the course with a stronger perception that mathematics involves mostly memorizing. Such results indicate that students might have perceived the course as emphasizing skills rather than concepts. The graphics sections received very few in-class review sessions, yet students performed favorably on routine questions as compared to the standard sections in which these reviews were commonplace. Future research could focus on determining the need for such reviews including a study of the effect on student perceptions of the general emphasis of the course with or without such reviews.

8. The Focused Holistic Scoring Point Scale first presented by Malone et al. (1980) and Charles et al. (1987), which was modified for use in the present study to rate student levels of understanding on exams and quizzes, proved to be reliable and valid for the purposes of the study. Future research extending the use of the

scale to other mathematics courses is warranted.

9. The present study was conducted with a relatively small number of students. Replication of the study using a larger number of sections where the classroom could be used as the unit of analysis, rather than the subject as was done in this study, would extend the generalizability of the results.

10. Finally, a replication of this study with the following modifications seems warranted: (a) at least some of the problems on each of the supplemental problem sets should be collected and graded; (b) computer use or at least computer output should be required on exams; (c) prior calculus experience should be further defined to include three categories: (i) no prior calculus experience, (ii) prior calculus experience in high school, and (iii) prior calculus experience in college; (d) when more than one instructor participates in the study, consistent grading policies that do not allow students to drop their lowest exam or quiz score, and subsequently does not allow them to miss an exam or quiz completely, should be adopted; and (e) data on recency of student previous experience with mathematics should be gathered and used as covariates for pre- measures of ability.

Such modifications will: (a) encourage students to engage in the treatment, (b) shed light on the effect of the treatment on special populations, (c) prevent excessive instances of missing data, and (d) help determine causes of pre-treatment biases.

Recommendations for the Teaching of Calculus

A graphically-developed, conceptually-presented first course in calculus has

potential for improving student understanding of graphs and of calculus concepts without hindering their acquisition of manipulative skill. The experimental course presented in Appendix A provides a first approximation of such a course. Reevaluation and modification should be inherent in this course as in any other. The following suggestions are made as the result of the first step in the revaluation process:

1. Students should be encouraged to obtain graphics calculators for daily personal use.

2. The investigator-written programs provided in Appendix G should be revised to make them user-friendly and then should be made available to students for use outside of class. Supplemental assignments should be written to optimize student use of these programs.

3. Supplemental assignments and course materials, consistent with the course presentation, should be provided to all students. Assignments included in Appendix F should be revised for use with graphics calculators.

4. At least some of the problems on each assignment should be graded to encourage students to complete them.

5. Computer software and/or graphics calculators use should be required on exams and quizzes if possible.

6. To enhance student note-taking, hard copies of selected computer graphs and demonstrations should be provided.

7. The Focused Holistic Scoring Point Scale is a viable grading alternative.

APPENDIX A

DAILY LOG OF LESSONS

The daily plans presented to the Graphics (G) and Graphics Plus (G+) treatments, are outlined in detail for lessons that vary significantly from those described in the "Instructor's Manual" for "Calculus" by J.F. Hurley (1987). The ideas expressed within are an aggregate of ideas from various sources (Douglas, 1986; Epp, 1986; Janvier,1975 ; Lucas, 1987; Sawyer, 1961; Tall, 1985, 1986a, 1986b, 1986c; Tall & Schwarzenberger, 1978; Waits & Demana, 1987b, 1987c, 1988) and the investigator, assimilated into a coherent whole. Suggestions and/or ideas from a specific source are noted within. Assignments and computer programs referred to in this log can be found in Appendices F and G respectively.

Day 1

The Course Information Sheet was discussed. It contained information such as: instructor's name, office, phone, office hours, the conceptual orientation of the course, the use of computer graphics, attendance and grading policies, grading scale and tentative exam dates.

Section 1.1: Absolute Values and Inequalities was covered.

Subjects were informed that they would be quizzed over the material in sections 1.1, 1.2, and 1.3 on Day 3.

Day 2

Class began with a discussion of "What is calculus?":

> Calculus is the study of how quantities change with respect to each other. For example, the following quantities could be studied as they change with respect to each other: (1) the relative movement of the planets; (2) the amount of medication in the bloodstream over time, depending on a person's height, weight, metabolism, etc.; (3) the net profit of a company's product line where profit depends on the number of units produced, expenses for materials, labor, energy costs, machinery, etc. The primary concern in Calculus I is the relationship between two quantities as they change with respect to each other.
>
> A very important concept in calculus is that of function. Functions can be found in various forms: (1) as a graph on an oscilloscope; (2) as a list of data in a table – from a newspaper, gathered in a lab, etc; (3) in equation form, e.g.
>
> $$f(x) = ax^2 + bx + c; \quad \text{and}$$
>
> (4) as a set of ordered pairs, e.g.
>
> $$\{(1,3),\ (2,7),\ (-1,3)\} \quad \text{(Douglas, 1986, p. ix).}$$

The preceding introduction was followed by a discussion of selected topics in Sections 1.2: Coordinates, Lines and Circles, and 1.3: Functions and Graphs, in

a slightly different order than given in the text. The topics were covered in the following order:

1. definition of function (subjects were asked to suggest familiar functions and then to give examples of non-functions);

2. domain (find the domains of $f(x) = \frac{1}{x}$ and $g(x) = \sqrt{x^2 + x - 6}$).

3. range (find the domain and range of $f(x) = x^2$);

4. functional notation;

5. the Cartesian coordinate plane where each point on the plane describes a unique location denoted by an ordered pair. The first coordinate represents the point's distance from the origin in the horizontal direction and the second coordinate represents its distance from the origin in the vertical direction;

6. the graph of a function;

7. determining graphically when the graph of a relation is a function;

8. undirected and directed distance;

9. equations and slopes of lines;

10. determining the equation in an applied situation if the relationship between the two quantities is known to be directly proportional;

11. symbolic operations on functions and determining the corresponding domains;

12. composite functions as a way of analyzing complex relationships in stages; and

13. domains of composite functions.

Topics left for subject independent study were: in Section 1.2, parallel and perpendicular lines, distance formula, and circles; and in Section 1.3, parabolas, completing the square, polynomials, and odd and even functions. Each of these topics was covered in later lessons.

Subjects were reminded that they would be taking a quiz over Sections 1.1, 1.2, and 1.3 on Day 3.

Day 3

Subjects responded to the Mathematics Attitude Scale followed by the Pretest. Subjects in the Graphics (G), Graphics Plus (G+), and Standard 1 (S1) sections were told that some of the questions on the pretest would be graded for the first quiz, but they were not told which questions would be graded. Due to a misunderstanding in carrying out testing instructions, subjects in the Standard 2 (S2) Section were told which questions were to be graded for their quiz and they were asked to circle the answers to those particular questions in the test booklet.

G+ subjects were given a diskette containing the software *Master Grapher* (Waits & Demana, 1987a) with a description of the software and its utilities. It was suggested that they try the software before the next class meeting to ensure the disk functioned properly. G+ and G subjects were given Assignment 1 to be completed before Day 4.

Day 4

Assignment 1 was collected.

The lesson began with a discussion of the calculus concern with instantaneous rates of change versus the precalculus concern with average rates of change. These ideas were presented through the following scenario:

> When traveling by automobile, it is often of interest to determine one's average velocity given the distance traveled and the time it takes to make the journey. A police officer's (John Law) interest in the traveler's velocity is much narrower. The officer's major concern is with one's velocity at the specific time his vehicle is traveling over the stretch of road being patrolled – the concern is with instantaneous velocity rather than average velocity.
>
> A precalculus problem is reflected in the traveler's concern with average velocity. A calculus problem is reflected in John Law's concern with the traveler's instantaneous velocity.

Various quantities were listed that change with respect to each other. Two quantities that change with respect to each other–position vs. time–were considered in more depth. The rate of change of position vs. time was investigated through the following situation inspired by Sawyer (1961):

> As a spider climbs a wall, photographs of his progress are taken at uniform time intervals. The photos are cut into narrow vertical strips, retaining just that part of each containing the image of the spider. The

strips are placed next to each other in the order that they were taken, and the progress of the spider is observed (see Figures A-1 and A-2).

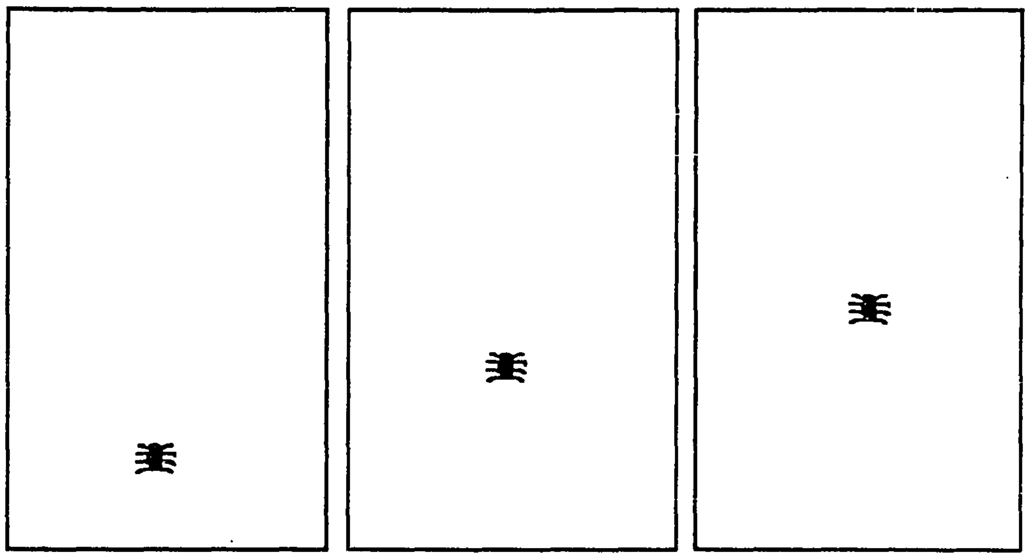


Figure A-1. The spider's progress over a few subintervals.

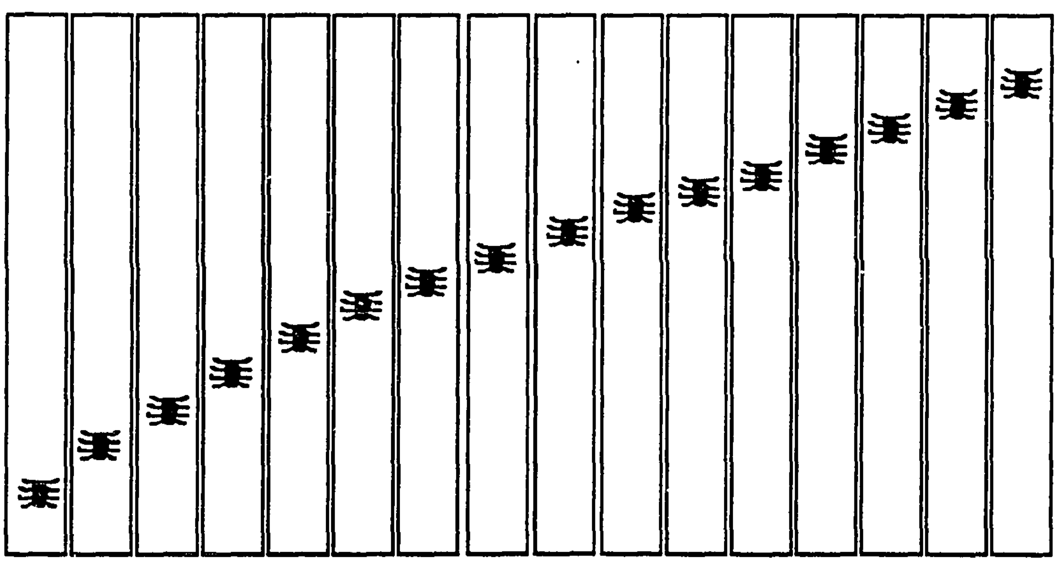


Figure A-2. The spider's progress over several uniform subintervals.

The spider's movement was simulated on the overhead projector. A 9 inch narrow vertical slit was cut through the center of an $8\frac{1}{2}$" $\times$ 11" solid piece of cardboard. Three overhead transparencies with graphs the width of the slit in the cardboard were pulled under the slit one at a time. This allowed subjects first to watch "the spider climb the wall" observing his rate as he climbed and then observing the shape of the graph that represented the spider's progress. The graphs of $f(x) = c$, a constant (horizontal line), $f(x) = ax$ (a diagonal line), and $f(x) = a^x$, $a > 1$ (a curve with a gentle then steep slope, no axes) were pulled under the cardboard with the rate of change for each discussed.

The spider's movement was further simulated by the investigator-written program, SPIDER. In SPIDER, an oval (the spider) "climbs" the right side of the computer screen while the graph modeling his movement is drawn to the left. The vertical distance of the spider from the base of the wall is observed, on the right as the spider climbs, and on the left with respect to time.

The following functions were observed over the indicated intervals and the spider's rates of change, velocity, at various times were discussed:

function	x − interval	y − interval
2	[0, 4]	[0, 4]
$2x$	[0, 3]	[0, 6]
$\sin(x)$	[−3.2, 3.2]	[−2, 2]
$x + \sin(x)$	[−6.3, 6.3]	[−7, 7]

Subjects were asked to suggest quantities other than distance vs. time which could change with respect to each other at similar rates. A discussion of speed as the absolute value of velocity followed, noting that velocity is of particular interest

in calculus.

An application of the rate of change of the population of Manhattan Island over time from the 1790-1970 census data was given (Usiskin, 1979, pp. 408-409). Subjects suggested that the greatest positive and negative rates of change are most easily found when the data points are graphed and a connecting curve is drawn.

Through the above activities, subjects observed that graphs are very useful for representing two quantities and for determining how these quantities change with respect to each other.

Having motivated the usefulness of graphs, subjects' skills in efficient methods of graphing functions were developed. As a preview of what was to come, the graphs of $h(x) = f(x) \diamondsuit g(x)$ were plotted using the investigator-written program, PLOT, with $\diamondsuit$ replaced by the successive operations of $+$ (addition), $-$ (subtraction), $*$ (multiplication), and $/$ (division). PLOT draws the graphs of $f(x) = x$, $g(x) = \sin x$, and $h(x) = f(x) \diamondsuit g(x)$ on the same Cartesian coordinate plane, allowing subjects to observe relationships between, f, g, and $f \diamondsuit g$. When plotting $f * g$ and f/g, $-f(x)$ was drawn instead of $g(x)$. This allowed the subjects to observe that $y = x \sin x$ oscillates between $y = x$ and $y = -x$ while $x/\sin x$ lies "outside" of the graphs of $y = x$ and $y = -x$; ie, $-|x| \leq x \sin x \leq |x|$ and $|x| \leq |x/sin\ x|$. Subjects were challenged to determine why the graphs of $h(x) = f(x) \diamondsuit g(x)$ behaved as they did given $f(x)$ and $g(x)$ using operation $\diamondsuit$. Subjects in the G+ section were given the assignment

Getting Acquainted with *Master Grapher* and Assignment 2. Getting Acquainted with *Master Grapher* introduces subjects to many of the utilities of the software, *Master Grapher* (Waits & Demana, 1987a), through problem settings. Assignment 2 extends the work begun in Getting Acquainted with *Master Grapher*. In these assignments, subjects use the Transform option to observe the effect on the graph of $g(x) = af(bx + c) + d$ when one of the parameters a, b, c, or d is changed (see Appendix F).

Day 5

The pretest was returned and the graded problems - numbers 1, 2, 3, 4, 5, 7, 8, 9, 11, and 13 - were reviewed. Problems 10 and 15 were also reviewed.

Subjects were given a copy of the basic graphs to be used in the day's discussion to increase class involvement and to help them take notes (see Figure A-3).

On Day 4, graphs had been shown to be useful in modeling the rate of change of two quantities with respect to each other. For graphs to be used efficiently, it is necessary that subjectss recognize the graphs of certain basic functions. Familiarity with these basic functions both symbolically and graphically, makes information available about many related functions.

Lecture Notes 1
Math 122 - Calculus I

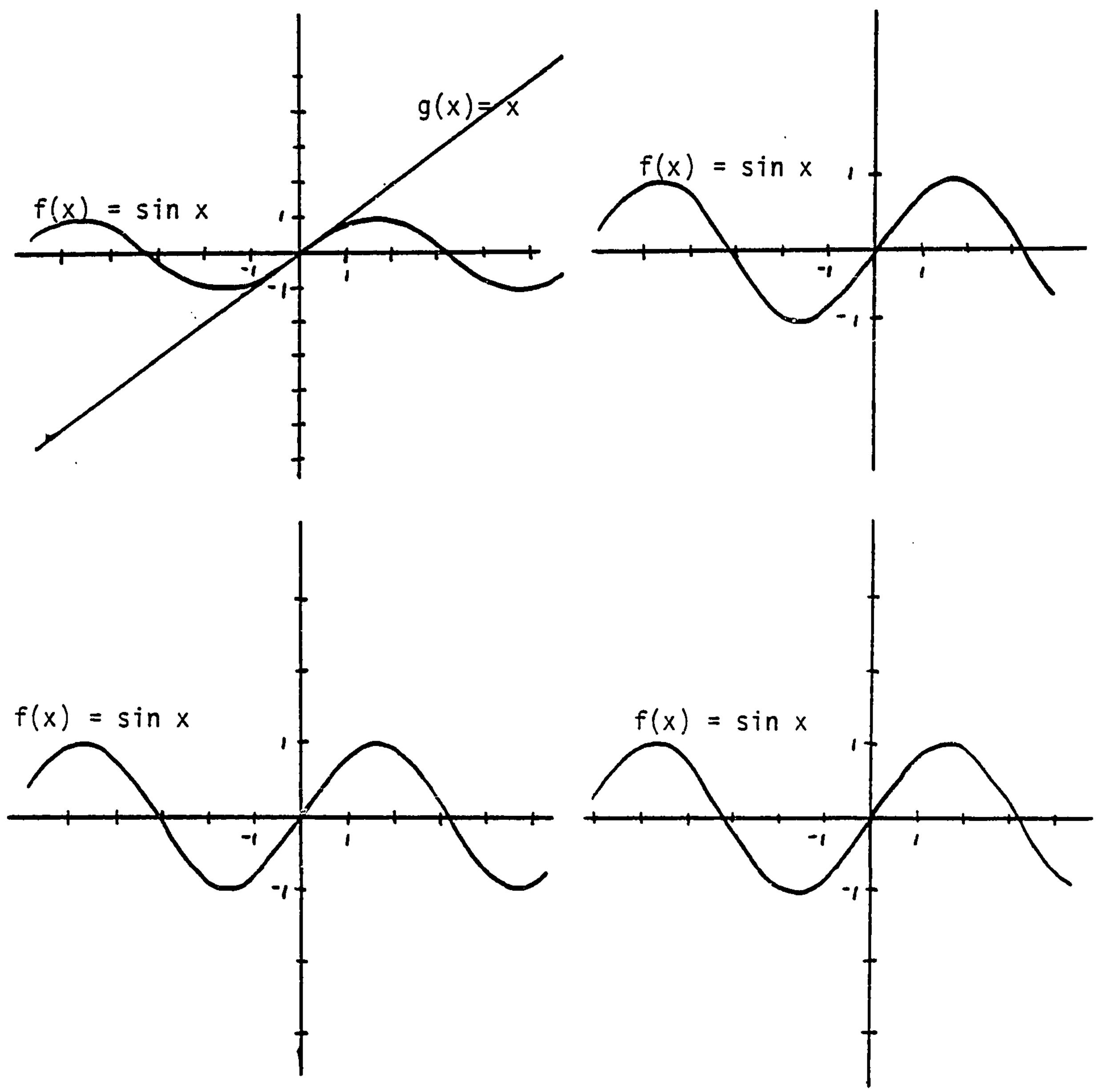


Figure A-3. Lecture Notes, Day 5

Subjects become familiar with: (1) domain and range; (2) end behavior, behaviors as $|x|$ grows large; (3) maximum(s), minimum(s); and (4) rates of change for $y = f(x)$ with respect to x for any x in the domain.

To develop subject familiarity with functions, the "cast of characters", as suggested in the Tulane Conference Syllabus (Douglas, 1986, pp. ix-xi), was introduced (see Table A-1).

Table A-1

Cast of Characters

Type of function	Function
polynomial	x^n where $n \in Z^+$
other algebraic	roots, x^q where $q \in \Re$,
	rational functions,
	absolute value
step function	greatest integer
trigonometric	sin, cos, tan

New functions were obtained from those in Table A-1 by:

1. adding constants, $g(x) = f(x) + d$;

2. multiplying by scalars, $g(x) = a \cdot f(x)$;

3. operations on functions, $h(x) = f(x) \diamondsuit g(x)$ where $\diamondsuit$ is replaced by $+, -, *, /$;

4. composition of functions, $g(x) = f(x + c)$, $g(x) = f(bx)$; and

5. inversion, if $f(x)$ is one-to-one and $y = f(x)$ then $x = f(y)$.

Each of parts (1) through (4) above were investigated first through use of the overhead projector. The software, *Master Grapher*, was used to further investigate parts (1), (2), and (4). Part (3) was investigated using the program PLOT. Part (5) was addressed during a later session.

The referent of points in the Cartesian coordinate plane as unique locations was reviewed as follows:

> For the point $P(a,b)$, the horizontal distance of P from the origin $(0,0)$ is a and the vertical distance of P from the origin is b. If point $P(a,b)$ is on the graph of $f(x)$, then the coordinates (a,b) can be written $(a, f(a))$ where $b = f(a)$ is the vertical distance of P from the origin.

Operation on functions were described graphically. The result $h(x)$ of adding two functions $f(x)$ and $g(x)$ at $x = c$ was described as the result of adding the respective directed vertical distances $f(c)$ and $g(c)$ to obtain $h(c)$ for various choices of c. Subjects were asked to determine the graph of $h(x) = f(x) + g(x)$ where $f(x) = x$ and $g(x) = \sin x$ as given in the lecture notes handout. The investigator served as recorder using an overhead transparency of the handout while the subjects described the location of several points along the curve of $h(x)$. The resulting graph was checked against the output of the program PLOT.

To investigate $g(x) = f(x) + d$, the Transform $F(x)$ option on *Master Grapher* was used. This option allows the user to choose replacements for a, b, c, and d in the equation

$$g(x) = a \cdot f(bx + c) + d.$$

Using the session handout, subjects were asked to determine the graph of $g(x) = \sin x + 2$ from the graph of $f(x) = \sin x$. Subjects suggested the position of $g(x)$ for various choices of x. These were plotted and the resulting curve was drawn on an overhead transparency which displayed the graph of $f(x) = \sin x$. Subjects observed that to obtain the graph of $g(x)$, one need only "slide" the graph of $f(x)$ up 2 units. To reinforce this concept, the graphs of $g(x) = f(x) + 3$ and $g(x) = f(x) - 2$ were plotted using *Master Grapher* and three different replacements of $f(x)$: $f(x) = x$, $f(x) = x^2$, and $f(x) = |x|$. For each new function $f(x)$, the screen was cleared, $f(x)$ was plotted, and the graphs of $g(x) = f(x) + 3$ and $g(x) = f(x) - 2$ were overlayed on the same screen.

A similar process was used to investigate $g(x) = a \cdot f(x)$. The graph of $g(x) = 2 \cdot \sin x$ was found on the overhead while subjects did the same at their seats. This was followed by the computer investigation, generating many cases to observe the relationship between $g(x)$ and $f(x)$. Each of the graphs of $f(x) = x$, $f(x) = x^2$, $f(x) = |x|$, and $f(x) = [x]$ were investigated, letting the parameter a take on the values 1, 2, and $\frac{1}{4}$. When graphing $f(x) = [x]$,

subjects were made aware of the necessity of questioning the reasonableness of computer-drawn graphs. The default plotting mode of *Master Grapher*, as is the case for most graphing utilities, draws graphs by connecting consecutively plotted points with line segments. For $f(x) = [x]$, this plotting mode produces a stair-step effect where vertical segments are drawn at each integer. Changing the plotting mode to point-plotting only gives the accurate graph.

The investigation of the effect of the parameter c on the graph of $g(x) = f(x + c)$ began symbolically by letting $g(x) = f(x + 2)$ where $f(x) = x^2$. For $c = 2$, $f(0) = f(-2 + 2) = g(-2) = 0$. The discussion continued graphically noting that $f(0) = g(-2)$. One more point was obtained for each function f and g, then their graphs were drawn. Subjects observe that the graph of $f(x + 2) = g(x)$ could be found by "moving" $f(x)$ 2 units to the left. Subjects continued the investigation by graphing $g(x) = \sin(x + 1)$ on their handout. The computer was used to generate the cases $g(x) = f(x + c)$ for $f(x) = x^2$ and $f(x) = \frac{1}{x}$ for $c = -2$ and $c = 4$.

To investigate $g(x) = f(bx)$, the computer was used with $f(x) = \sin x$. Subjects observed that b shortens the period if $b > 1$, and lengthens the period if $b < 1$. The discussion ended with the observation that $k \cdot f(x) \neq f(kx)$ for $k \neq 1$ and $f(x) \neq x$. The functions $g(x) = k \cdot f(x)$ and $h(x) = f(kx)$ were overlayed together with $f(x)$ for $f(x) = x^2$ and $f(x) = \sin x$ where $k = 2$.

Day 6

Previous discussions had focused on the individual effects of the parameters a, b, c, and d in $g(x) = a \cdot f(bx + c) + d$. To investigate the effect of changing more than one parameter at a time, an efficient method of graphing parabolas was presented. Using the example $g(x) = x^2 - 2x + 7$, subjects observed that an equivalent parameterized expression could be obtained by completing the square, ie.

$$G(x) = x^2 - 2x + 7 = (x-1)^2 + 6.$$

If $f(x) = x^2$ then $g(x) = f(x-1) + 6$. This form of the function indicates that the shape of the graph of g is exactly that of f. It also indicates that the position of g can be found by translating f 6 units up and 1 unit to the right resulting in a parabola with vertex $(1, 6)$.

The same process was applied to $g(x) = 2x^2 + x - 5$, obtaining the parameterized form $g(x) = 2 \cdot f(x + \frac{1}{4}) - 5\frac{1}{8}$ where $f(x) = x^2$. This time g was determined to be narrower than f with vertex $(-\frac{1}{4}, -5\frac{1}{8})$ which translates the graph of $f(x)$ down $5\frac{1}{8}$ units and to the left $\frac{1}{4}$ unit. The discussion of parameters was concluded by graphing $y = 2 \cdot \sin(x - \frac{\pi}{2}) + 3$ given the graph of $y = \sin x$.

Subjects were informed that another aid in efficient graphing exploits the symmetry of a graph. The discussion of symmetry began with determining the point P in the coordinate plane symmetric to a given point (a, b) where P was symmetric to (a, b) with respect to: (a) the x-axis; (b) the y-axis;

(c) the origin in $(0,0)$; and (d) the line $y = x$. An overhead transparency displaying a coordinate grid with a point labelled (a, b) was used in the discussion. Subjects suggested possible locations of P for a given line or point of symmetry. To determine symmetry of functions, a curve was drawn on a transparency of a coordinate grid; another was traced on a clear transparency. The second copy was flipped or turned appropriately depending on the symmetry being discussed. Definitions of even and odd functions were given with subjects suggesting examples of even, then odd functions.

The ideas of parameterized functions were extended to circles. Subjects observed that the circle is not a function but can be described using two functions, upper and lower semicircles for example. The discussion began using the unit circle in standard position, $x^2 + y^2 = 1$. Subjects observed that the upper semi-circle of radius 1 and center $(3,0)$ is given by the equation $g(x) = f(x-3)$ where $f(x) = \sqrt{1-x^2}$, and the lower semicircle of radius 2 with center $(3,5)$ is given by $g(x) = f(x-3) + 5$ where $f(x) = -\sqrt{1 - x^2}$. This result was compared with the equation of a circle with radius r and center (h,k) familiar to subjects from precalculus courses.

The session ended with an introduction of the concept of limit. The police officer's interest in our speed at a particular moment in time was briefly discussed. The police officer uses a radar detector which essentially reads an initial and final position over a small time interval. Average velocity is determined by

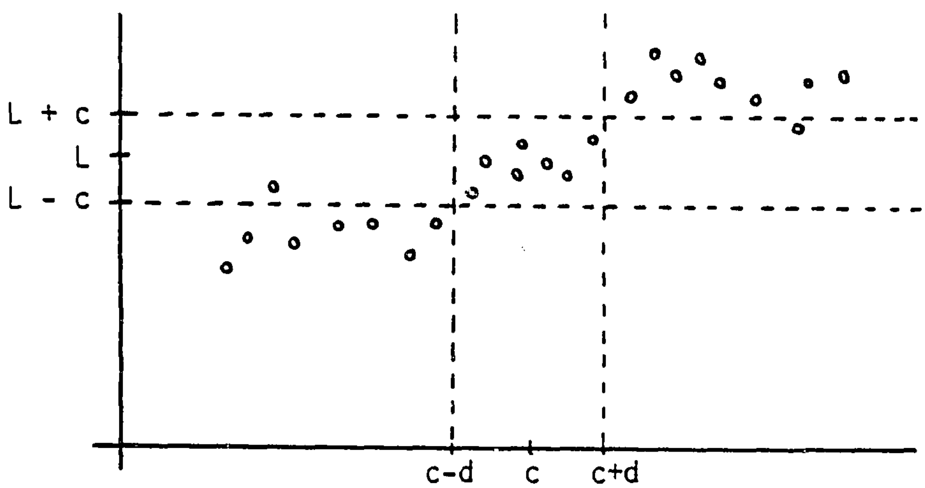
$$\text{average velocity} = \frac{\text{final position - initial position}}{\text{final time - initial time}}.$$

The process of looking at smaller and smaller time intervals was described as a limiting process.

Day 7

Subjects were given Quiz 1 during the first 20 minutes of class.

Subjects were reminded of the introduction to the idea of limit discussed previously. The discussion was continued with the use of the following scenario given to the subjects in the form of lecture notes. A graph was drawn to further illustrate quality control ideas as they related to limits (see Figure A-4).



o indicates the piston ring size at setting x.

Figure A-4. Graph representing the scenario given in lecture notes.

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We define limit informally as follows:

The limit of f(x) as x approaches c is L, denoted

$$\lim_{x \to c} f(x) = L$$

means that f(x) can be made arbitrarily close to L by making x sufficiently close but not equal to c.

To illustrate the idea of limit, consider the manufacture of piston rings. In order to work properly, piston rings must be manufactured precisely. However, no machine will make a perfect part every time. The best the manufacturer can hope for is that the piston rings are made within a very small acceptable tolerance level of the ideal size. We call the ideal size of the piston ring L, the manufactured size of the piston ring f(x) (where x is the particular setting of the machine at the time this piston ring is made), and we accept the piston ring if it is within a specified tolerance, say e (for error), of L. The piston ring of size f(x) is no larger that L + e and no smaller than L - e. This L is the limiting value referred to in our informal definition.

We now consider the machine which produces the piston rings. Because of power surges, wear and tear on the machinery, length of time between routine maintenance, etc., the machine does not operate at exactly the same levels at all times. The machine operator determines from the operation manual that when the machine controls are set at level c, the machine ideally produces the piston ring of size L. However, the above mentioned situations influence the machine's operation causing it to operate near setting c, but perhaps not actually at c. A gauge on the machine reads the actual operation level x of the machine. We determine that acceptable parts are produced when the machine's actual operation level lies within a tolerance level d either above or below the setting c.

The mathematical concept of limit requires that we be able to attain any level of accuracy e for the size of the piston ring, no matter how small. This means that for every part produced, the actual part size f(x) can be made as close to the perfect size L as we wish when the machine operates sufficiently closely to the machine setting c.

Assignment:

Using the illustration and informal definition above (and referring to our earlier discussion in class), construct an original example that illustrates the concept of limit in its mathematical sense. Do not use a manufacturing example such as that given above. Be creative, yet mathematically correct. Explain carefully!

NOTE: Lecture notes varied in that the Assignment section appeared only on notes given to students in the GF treatment.

Each of the values given in the scenario was labelled on the graph and was described as follows:

L the ideal piston ring size;

e the allowable error; the piston ring is acceptable if its size is within the range $(L-e,\ L+e)$;

c the ideal machine setting that the machine manual suggests will produce the piston ring of size L; and

d the amount of allowable play in the machine's operation level; in Figure A-4, that the piston rings produced when the machine level x is operating within d units of c, $(c-d,\ c+d)$, will be acceptable.

A discussion of the mathematical concept of limit as it relates to the limitations of accuracy of measuring tools, manufacturing tools, etc. followed. Although the theoretical limit cannot be achieved in a world of limited accuracy, the idea of tolerance is important and leads to the understanding of the concept of limit. It was suggested that subjects write their own scenario to illustrate the idea of limit. G+ subjects were asked to turn in their scenario.

To motivate the usefulness of limit subjects were told that, among other things, limits are useful in investigating the behavior of functions near a given point. An informal definition of limit was given (adapted from Douglas, 1986, p. ix):

DEFINITION.

$\lim_{x \to c} f(x) = L$ *means that* $f(x)$ *can be made arbitrarily close to* L *by making* x *sufficiently close, but not necessarily equal, to* c.

The behavior of the function $f(x) = \frac{\sqrt{x}-1}{x-1}$ at $c = 1$ was investigated. Subjects were asked to determine the value of $f(1)$. They were also asked what answer their calculators would give for $f(1)$. Subjects suggested that $f(1)$ cannot be found with the calculator by simply plugging $x = 1$ into the given expression since this would produce a zero in the denominator; division by zero causes an error message to be displayed. A discussion of how the computer might handle such a situation followed. Subjects suggested that the computer would operate similarly. When asked to suggest other alternatives for determining the behavior of the function $f(x)$ near $c = 1$ they suggested two alternatives: graphing $f(x)$, and looking at a table of values of $f(x)$ for x close to 1 but not equal to 1. Subjects did not suggest simplifying the function algebraically. The function had been chosen so that such a possibility would not be an obvious choice.

The investigation continued through the use of the investigator-written program LIMITS. LIMITS produces a table of values for h, $f(x+h)$, and $f(x-h)$ where the user determines $f(x)$ and x, and h is halved iteratively varying from 0.5 to 0.001. Subjects observed that the values of $f(x+h)$ and $f(x-h)$ approached $\frac{1}{2}$.

The investigation was continued through the use of *Master Grapher*. The

function $f(x)$ was graphed on the default screen $[-10,\ 10]$ by $[-10,\ 10]$. Subjects observed that there did not appear to be a "hole" in the graph at $x = 1$. The zoom-in option was used repeatedly about the intersection of the graph of $f(x)$ with the vertical line $x = 1$, illustrating the limiting process. Each time, subjects observed that the graph of $f(x)$ appeared to be solid. The limit of $f(x)$ as x approaches 1, as closely as machine precision will allow (8 or 9 decimal places), is $\frac{1}{2}$. The graph of $f(x)$ appears to contain the point $(1,\ \frac{1}{2})$ even though the computer could not be using f to calculate that point, as an error would be produced if this were the case. The function $f(x) = \frac{\sqrt{x} - 1}{x - 1}$ was then factored and shown to be algebraically equivalent to $g(x) = \frac{1}{\sqrt{x} + 1}$ for all values of $x \neq 1$. The intuitive groundwork had been laid for the following theorem:

THEOREM. *If $f(x) = g(x)$ for all x in an open interval containing c (except possibly at c itself), and if $\lim_{x \to c} g(x) = L$, then $\lim_{x \to c} f(x) = L$.*

Since $\lim_{x \to 1} \frac{1}{\sqrt{x} + 1} = \frac{1}{2}$, by the above theorem, $\lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1} = \frac{1}{2}$.

It is often possible to find the limit of a function by algebraic manipulation at a point where it is undefined. The function $f(x) = \frac{\sin x}{x}$ was considered since it cannot be algebraically manipulated to obtain an equivalent function that is defined for $c = 0$. The process described above was repeated for $f(x) = \frac{\sin x}{x}$. The limit at $c = 0$ was determined to be 1, both through a table of values (LIMITS) and graphically (*Master Grapher*).

Day 8

Quiz 2 was returned and discussed. Questions over the previous day's assignments were answered. In particular, the following exercise was of interest to both sections:

$$\text{Find } \lim_{x \to 2} \frac{1}{x-2}.$$

The function $f(x) = \frac{1}{x}$ was graphed and subjects recalled that $g(x) = \frac{1}{x-2} = f(x-2)$ translates $f(x)$ 2 units to the right.

The above problem was used to introduce the notation of left- and right-hand limits. The numerical and graphical behavior of $g(x)$ as x approached 2 from the left $(x \to 2^-)$ and as x approached 2 from the right $(x \to 2^+)$ were discussed. Subjects agreed that $g(x)$ did not approach one particular value L for every x in any small open interval containing 2. It was concluded that $\lim_{x \to 2} g(x)$ does not exist. The behavior of $g(x) = \frac{1}{x-2}$ as x approaches 2 from the right, is better described using the notation of infinite limits. For the example, to denote that the function grows positively without bound:

$$\lim_{x \to 2^+} g(x) = +\infty.$$

Similarly,

$$\lim_{x \to 2^-} g(x) = -\infty.$$

The previous day's session was reviewed briefly. The intuitive definition of limit was rewritten on the board and subjects were asked to recall that the functions $f(x) = \frac{\sqrt{x}-1}{x-1}$ and $f(x) = \frac{\sin x}{x}$ each have limits as x approaches 1 and 0 respectively. The function $g(x) = \frac{1}{x-2}$ was given as an example of a function that does not have a limit as x approaches 2. Subjects were asked to suggest other functions that had a limit at $x = 0$ and functions whose limit did not exist at $x = 0$.

Several limit properties were developed intuitively by observing the graph of a function $f(x)$, using the zoom-in option, and looking at a table of values of $f(x)$ for values of x near c. This process permitted subjects to discover the following limit properties graphically and to interpret their meanings symbolically:

1) $\lim_{x \to c} k = k$ where k is a constant,

2) $\lim_{x \to c} x = c$, and

3) $\lim_{x \to c} k \cdot f(x) = k \cdot \lim_{x \to c} f(x)$.

Using the investigator-written program CLIMITS and the software *Master Grapher*, the limits of sums, differences, products, and quotients of functions were also investigated. CLIMITS allows the user to investigate, through a table of values,

$$\lim_{x \to c}(f(x) \diamond g(x))$$

where the user chooses the x-interval of interest and the operation $\diamond$ from among $+$, $-$, $*, /$, and $\wedge$. The functions f and g are given as:

$$f(x) = \frac{\sin x}{x}, \quad g(x) = \frac{1}{x+1}.$$

The functions f and g may be redefined.

The functions $f(x)$ and $g(x)$ were combined using the operations $+$, $-$, $*$, and $/$ in [-0.175, 0.175] then on [-1.175, -0.825] with step size of 0.025. The required functions, entered into the *Master Grapher* function index prior to class, were displayed on small intervals containing $x = 0$ then $x = 1$ through the use of the zoom-in option. In each case, subjects observed that:

$$\lim_{x \to c}(f(x) \Diamond g(x)) = \lim_{x \to c} f(x) \Diamond \lim_{x \to c} g(x)$$

when both of these limits exist. Subjects observed that

$$\lim_{x \to 0}(f(x) \Diamond g(x)) = \lim_{x \to 0} f(x) \Diamond \lim_{x \to 0} g(x)$$

existed for the given functions for the operations of addition, subtraction, and multiplication. However, $\lim_{x \to 0} f(x)/g(x)$ does not exist. Subjects observed that the limit

$$\lim_{x \to c}(f(x) \Diamond g(x))$$

exists provided $\lim_{x \to c} g(x) \neq 0$ when the operation $\Diamond$ is division. On the other hand, if one of the individual function limits did not exist, then $\lim_{x \to c}(f(x) \Diamond g(x))$ might or might not exist depending on f and g. (Subjects were often required

to critically analyze the hypotheses and conclusions of theorems given an intuitive base from which to proceed.)

The following limits were considered graphically and symbolically:

(a) $\lim_{x \to -1} \frac{|x+1|}{x+1}$,

(b) $\lim_{x \to 3} \frac{x^2-9}{x-3}$, and

(c) $\lim_{x \to \frac{1}{2}} \frac{2x^2+3}{x-4}$.

The existence of limits of rational functions $f(x)$ with zero denominators at $x = c$ were discussed. Subjects were asked to suggest when $\lim_{x \to c} f(x)$ exists or does not exist. It was suggested that the limit exists as x approaches c for functions such as:

$$f(x) = \frac{x^2-4}{x+2}.$$

The graphs of such functions appeared to have no "holes". They could be algebraically manipulated to obtain equivalent functions for $x \neq c$ but for which no discontinuity existed at $x = c$. Functions such as

$$g(x) = \frac{x^2-4}{x+1}$$

whose discontinuity could not be "factored out" were suggested as examples of functions where $\lim_{x \to c} g(x)$ did not exist. The graph of such a function had an asymptote at $x = c$. In this case, $\lim_{x \to -1^+} g(x) = -\infty$ and $\lim_{x \to -1^-} g(x) =$

$+\infty$. The function $g(x)$ grew positively or negatively without bound on either side of the discontinuity.

The rational functions discussed above differed from the absolute value function $h(x) = \frac{|x+1|}{x+1}$ in appearance. The function $h(x)$ had no asymptote, each side of the function approached a particular value. For $h(x)$,

$$\lim_{x \to -1^-} h(x) = -1 \text{ and}$$

$$\lim_{x \to -1^+} h(x) = +1.$$

In this case both left- and right-hand limits exist. However, since these one-sided limits are not equal, $\lim_{x \to ^-1} h(x)$ does not exist.

Day 9

The previous work with limit properties was reviewed briefly. Using these properties, the generalized sum and product rules were developed using Mathematical Induction and the Σ and π notation. Subjects concluded that limits of polynomial functions exist for all real numbers $x = c$ and that limits of rational functions $f(x) = p(x)/q(x)$ exist wherever $q(x) \neq 0$. When $q(x) = 0$, it was concluded that further investigation was necessary to determine the existence of a limit. Limit properties of roots and composite functions were determined as discussed in Hurley's "Instructor's Manual".

The Sandwich Theorem was developed graphically using the investigator-written program PLOT3 and the functions

$$h(x) = f(x) * g(x) = x^2 \sin\frac{1}{x}.$$

The development was begun with a discussion of the behavior of $g(x) = \sin\frac{1}{x}$ as x approaches 0. The values of $\pm\frac{2}{\pi}, \pm\frac{2}{3\pi}, \pm\frac{1}{2\pi}$ were suggested as replacements for x with subjects observing that the function oscillates more rapidly between $y = 1$ and $y = -1$ as $|x|$ becomes small. Using PLOT3 with $f(x) = 1$ and $g(x) = \sin\frac{1}{x}$, the graphs $y = f(x)$ and $y = -f(x)$ were plotted followed by the graph of $h(x) = f(x) * g(x) = 1 \cdot \sin\frac{1}{x} = \sin\frac{1}{x}$ on the intervals $[-0.25, 0.25]$ for x and $[-1.2, 1.2]$ for y. Subjects observed the rapid oscillation of $g(x)$ near $x = 0$, as they suggested from the preceding numeric work, noting the limited precision of the graph. (A brief discussion of the program's plotting mode and the resolution on the screen took place at this point. Subjects saw the limitations of "discrete machines" as they were introduced to the possibilities of this powerful tool, little by little!) The function $f(x) = 1$ was changed to $f(x) = x^2$. Subjects were asked to suggest how the function $h(x) = f(x) * g(x) = x^2 \sin\frac{1}{x}$, might appear. The graphs $y = f(x) = x^2$, and $y = -f(x)$, followed by $h(x) = x^2 \sin\frac{1}{x}$, were plotted on the viewing screen $[-0.1, 0.1]$ by $[-0.01, 0.01]$. Subjects observed that $h(x)$ oscillates between $y = x^2$ and $y = -x^2$ and were asked to suggest reasons for the graphs' behavior near $x = 0$. The following theorem was displayed on the overhead:

SANDWICH THEOREM. *Suppose that for all* $x \neq c$ *in an open interval* I *containing* c,

$$f(x) \leq h(x) \leq g(x).$$

If $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = L$ *then* $\lim_{x \to c} h(x) = L$.

Observing the appearance of the graph of $h(x)$ relative to the graphs of $y = x^2$ and $y = -x^2$, subjects suggested that $f(x) = -x^2$, $g(x) = x^2$ and $h(x) = x^2 \sin \frac{1}{x}$, in the theorem above, since $-x^2 \leq x^2 \sin \frac{1}{x} \leq x^2$. Subjects observed that $\lim_{x \to 0} -x^2 = \lim_{x \to 0} x^2 = 0$. The hypotheses of the theorem were satisfied, so the conclusion must follow. The symbolic argument below was also considered:

For $x \neq 0$,

$$-1 \leq \sin \frac{1}{x} \leq 1$$

$$\text{so } -x^2 \leq x^2 \sin \tfrac{1}{x} \leq x^2$$

$$\text{since } x^2 \geq 0 \text{ for all } x.$$

$$\text{Then } \lim_{x \to 0}(-x^2) \leq \lim_{x \to 0}(x^2 \sin \frac{1}{x}) \leq \lim_{x \to 0}(x^2).$$

Since

$$\lim_{x \to 0} \pm x^2 = 0, \quad 0 \leq \lim_{x \to 0} (x^2 \sin \frac{1}{x}) \leq 0.$$

In words, since 0 is both smaller and larger than $\lim_{x \to 0} x^2 \sin \frac{1}{x}$, then

$$\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0.$$

The following theorem was discussed and illustrated graphically:

THEOREM. *Let I be an open interval containing c. Suppose that*

$$\lim_{x \to c} f(x) = L.$$

(a) If $f(x) \geq 0$ *for all* $x \neq c$ *in* I, *then* $L \geq 0$.

(b) If $f(x) \leq 0$ *for all* $x \neq c$ *in* I, *then* $L \leq 0$.

Day 10

Motivated by questions asked of the investigator during office hours, the session began with a discussion of

$$\lim_{x \to c} f(x) \cdot g(x) = \lim_{x \to} f(x) \cdot \lim_{x \to c} g(x).$$

Subjects wanted to determine $\lim_{x \to 0} x \cdot \cos \frac{1}{x}$ (for any similar problems) by stating the following:

$$\begin{aligned} \lim_{x \to 0} x \cdot \cos \frac{1}{x} &= \lim_{x \to 0} x \cdot \lim_{x \to 0} \cos(\frac{1}{x}) \\ &= 0 \cdot \lim_{x \to 0} \cos \frac{1}{x} \\ &= 0. \end{aligned}$$

The property of limits of products was reviewed, noting carefully that the hypotheses of the theorem require that for $\lim_{x \to 0} f(x) \cdot g(x)$ to exist, each of the limits $\lim_{x \to c} f(x)$ <u>and</u> $\lim_{x \to c} g(x)$ must exist. The difficulty experienced above is most likely due to the "rule" memorized as "0 times anything is 0". Subjects

were reminded that in order to multiply zero by another number, they must first have the other number!

Assignment 3 was given to the G+ section with the instruction to have it completed by Day 12.

The function $f(x) = \sqrt{x}$ was graphed and its limit as x approaches 0 was discussed. Subjects suggested that a limit of 0 was approached from the right, but observed that the function was not defined for values of $x < 0$. The definition of limit was again given with subjects determining that $\lim_{x \to 0} f(x)$ could not exist, since the requirement that the function be defined in an open interval containing 0 was not met. A similar observation was made for the function $g(x) = \sqrt{4 - x^2}$ as x approaches -2 and 2. This discussion, as well as earlier discussions of limits of such functions as $y = \frac{1}{x}$ and $y = \frac{|x|}{x}$, motivated the need for a definition of one-sided limits.

The following intuitive definitions of one-sided limits were given. It is analogous to the definition of limit given earlier.

DEFINITION. *Suppose that f is defined on an open interval (a, b). The one-sided limits of f are defined by*

(a) $\lim_{x \to a^+} f(x) = R$ means that as x approaches a from the right (positive side), $f(x)$ becomes and stays arbitrarily close to R.

(b) $\lim_{x \to b^-} f(x) = L$ means that as x approaches b from the left (negative side), $f(x)$ becomes and stays arbitrarily close to L.

By the definitions given above, subjects determined that $\lim_{x \to 0^+} \sqrt{x} = 0$

and $\lim_{x \to 0^-} \sqrt{x}$ did not exist. Similarly, $\lim_{x \to 2^-} \sqrt{4 - x^2} = 0$ (analogously to properties of two-sided limits) but $\lim_{x \to 2^+} \sqrt{4 - x^2}$ did not exist.

Subjects were introduced to schizophrenic functions–functions that have multiple formulas–using the example given in the text concerning the heat content of water (Hurley, 1987, pp. 46-47). They were asked to determine the one-sided limits of the following function at $x = 0$:

$$f(x) = \begin{cases} x^2 - 3x + 1 & \text{for } x \leq 0 \\ 1 + 5x - x^2 & \text{for } x > 0. \end{cases}$$

The graph of $f(x)$ was sketched on the overhead and both one-sided limits determined to be 1. Subjects suggested that $\lim_{x \to 0} f(x) = 1$ since the function approaches 1 as x approaches 0 from both sides.

The above discussion motivated the necessity of the following theorem to determine if a limit exits:

THEOREM. *Let* f *be defined on an open interval I containing* c, *except possibly at* c, *then*

$$\lim_{x \to c} f(x) \text{ exists if and only if}$$

$$\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = L,$$

$$\text{in which case, } \lim_{x \to c} f(x) = L.$$

The following functions and their corresponding one- and two-sided limits as x approaches c (if these exist) were considered both graphically and symbolically;

1) $f_1(x) = [x]$, $c = 2$

2)

$$f_2(x) = \begin{cases} x^2 - x + 1 & \text{for } x \leq 1 \\ 2x - 1 & \text{for } x \geq 1, \quad c = 1 \end{cases}$$

3) $f_3(x) = \frac{1}{x^2}, \ c = 0$

4) $f_4(x) = \frac{1}{x}, \ c = 0.$

Subjects observed that graphically, the limit of $f(x)$ exists as x approaches c if the graph of $f(x)$ is unbroken or if the only break is a hole in the graph such that if the hole were filled by a single point, the graph would no longer be broken.

Intuitive definitions were given for infinite limits. These had been discussed earlier as a means of being more descriptive of the behavior of a graph at a point. The definition were given as follows:

DEFINITION. *Suppose that* f *is defined on some open interval having left endpoint* a. *Then*

(a) $\lim_{x \to a^+} f(x) = +\infty$ *means that* $f(x)$ *is arbitrarily large and positive whenever* $x > a$ *is sufficiently close to* a.

(b) $\lim_{x \to a^+} f(x) = -\infty$ *means that* $f(x)$ *is arbitrarily large and negative whenever* $x > a$ *is sufficiently close to* a.

Subjects were again reminded that for a function to have a limit as $x \to c$ it must approach a <u>real</u> number. The symbols $+\infty$ and $-\infty$ are <u>not</u> real numbers, but denote $f(x)$ growing without bound positively or negatively respectively.

Day 11

Class began with questions from subjects over recent assignments. Subjects completed Quiz 2.

The concept of continuity was presented as: A function is continuous at c if, as x gets closer to c, $f(x)$ gets closer to $f(c)$. Continuity was defined as follows:

DEFINITION. *The function* $f(x)$ *is continuous at* $x = c$ *if*

$$\lim_{x \to c} f(x) = f(c).$$

This definition requires that :

1. $f(x)$ is defined at $x = c$,
2. $\lim_{x \to c} f(x)$ exists, and
3. $f(c) = \lim_{x \to c} f(x)$.

If any one of these conditions is violated, f is not continuous at c.

To critically analyze the requirements of the definition of continuity, the following investigation was pursued with subjects suggesting functions both symbolically and graphically:

1. Subjects were asked to determine functions that were not defined at $x = 4$. Suggested functions included $f_1(x) = \frac{1}{x-4}$, $f_2(x) = \frac{x^2-16}{x-4}$, $f_3(x) = \sqrt{x-5}$, and $f_4(x) = \frac{1}{4-x}$. Each of these was graphed with subjects observing that f_2 had a point missing at $x = 4$, f_1 and f_4 had an asymptote at $x = 4$, and f_3 was undefined for $x < 5$.

2. Subjects were asked to suggest functions $f(x)$ that are defined at $x = {}^-3$ but for which $\lim_{x \to {}^-3} f(x)$ does not exist. Suggestions included

$$f(x) = \begin{cases} \frac{|x+3|}{x+3} & \text{for } x \neq -3 \\ 1 & \text{for } x = {}^-3 \end{cases}$$

and $f(x) = [x]$. These functions were graphed and subjects observed that in each case the graph jumped at $x = {}^-3$.

3. Subjects were asked to suggest functions $f(x)$ that are defined at $x = \frac{1}{2}$ for which $\lim_{x \to \frac{1}{2}} f(x) = L$ exists, but $L \neq f(c)$.

Only one suggestion was made:

$$f(x) = \begin{cases} \frac{x(x - \frac{1}{2})}{x - \frac{1}{2}} & \text{for } x \neq \frac{1}{2} \\ 1 & \text{for } x = \frac{1}{2} \end{cases}$$

Graphically, subjects observed that functions that are defined at $x = c$ and for which the limit exists at c but $\lim_{x \to c} f(x) \neq f(c)$ have a hole in them. Such functions can be made continuous by redefining $f(c)$ to be $\lim_{x \to c} f(x)$.

In each case above, subjects noted that discontinuous function were easily determined by showing their graphs had a hole or some other type of gap – either an asymptote or a jump where left and right hand limits were defined but were not equal. They also noted that graphs of continuous functions could be drawn in one continuous stroke, without lifting the pencil from the page.

Day 12

Quiz 2 was returned to subjects. In the process of going over problems 2 and 3 of this quiz, limits were reviewed. Subjects were reminded that for a given func-

tion, graphs, tables of values, and symbolic work must give the same information. If this is not the case, then the work with one of these representations is incorrect. Graphs provide another way of deciding if symbol work is reasonable.

Subjects continued to have difficulty in understanding limits. In particular they had trouble recognizing situations where a limit might fail to exist. The graphs in Figure A-5 were used with 8 transparent overlays. Each successive overlay had a paper rectangle of size $2(\delta_i)$ by $2(\epsilon_i)$, centered on it, where $\delta_1 > \delta_2 > \cdots > \delta_8$ and $\epsilon_1 > \epsilon_2 > \cdots > \epsilon_8$. The transparencies were removed successively to display larger parts of the graph of $f(x)$ near $x = c$ and a hypothetical $y = L$.

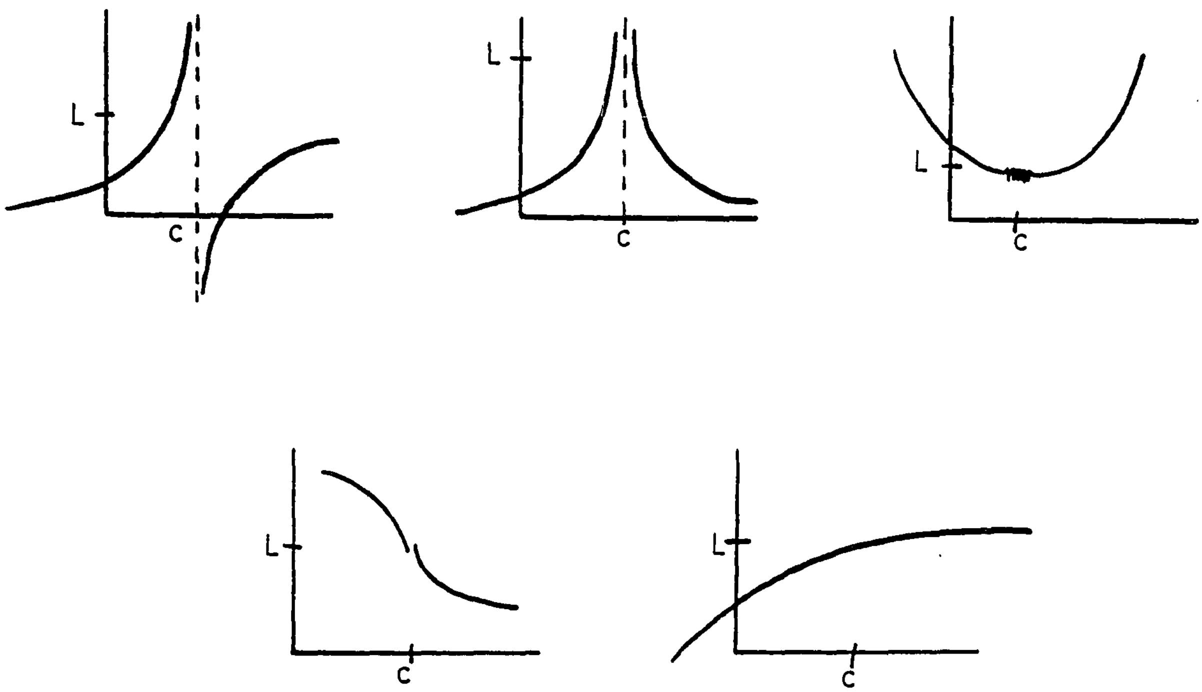


Figure A-5. Graphs illustrating various behaviors of functions at a point $x = c$.

The following definition of continuity on an interval was given:

DEFINITION. *The function f is continuous on an interval I if f is continuous at every point c in I. If f is not continuous at $x = c$, it is said to be discontinuous.*

Subjects were asked to determine whether or not certain functions were continuous. If a function was not continuous, subjects were asked to determine what part(s) of the definition of continuity was (were) not satisfied for that function. The functions used in the discussion were:

1. polynomials,
2. rational expressions,
3. roots,
4. $f(x) = \frac{|x|}{x}$,
5. $f(x) = \frac{\sin x}{x}$,
6. $f(x) = \frac{1}{x-1}$, and
7. $f(x) = \frac{x^2-4}{x-2}$.

Each of the functions (4) through (7) was graphed.

The definitions of removable and essential discontinuities were given symbolically (as in "Calculus" by Hurley, 1987) and interpreted graphically. The graph of a function with a removable discontinuity was described as one that "looks" continuous when drawn on the computer. Only a single point of an otherwise continuous graph is missing. To make this type of function continuous at its point of discontinuity c, it is necessary to define $f(c)$ to be $\lim_{x \to c} f(x)$.

The graph of a function with an essential discontinuity is, in Hurley's words, "beyond repair" –the discontinuity cannot be fixed. There is a visible gap in the graph at $x = c$ and $\lim_{x \to c} f(x)$ does not exist.

The functions used earlier in the lesson were revisited and discontinuities for functions (2) through (7) were classified according to the symbolic definitions and their corresponding graphic interpretations.

Using the definition of continuity and the results from the discussion of removable and essential discontinuities, subjects were asked to list functions that are continuous on every open interval for which they are defined. Suggestions included polynomials, rational functions, and roots.

Day 13

The discussion began with a quick review of the definitions of continuity and removable and essential discontinuities. Questions over homework problems were answered.

Subjects were given that f and g are continuous functions at $x = c$ and were asked to recall the properties of sums, difference, products, and quotients of functions. Then the properties of the limits of these combined functions as x approaches c were reviewed. Subjects suggested that, analogous to the properties of limits, if f and g are continuous at $x = c$ then so are $f + g$, $f - g$, and $f \cdot g$, If $g(c) \neq c$, then $\frac{1}{g}$ and f/g are also continuous. The proof of continuity at $x = c$ for $f + g$, given continuous f and g, was presented. Subjects were asked to try proving a few other cases on their own.

The continuity of $f(x) = \sqrt[3]{x}$ was discussed. It was determined that it might be helpful to define continuity on closed intervals for functions such as these. Definitions were given for the continuity of a function on a closed interval and for continuity of composite functions. Continuity at $x = 0$ was discussed for $f(x) = \sqrt{x}$. Continuity at $x = -1$ was discussed for

$$g(x) = \sqrt{\frac{x^2 - 1}{(x+1)^2}}.$$

Two results for continuous functions that were discussed but not proved were the Intermediate Value Theorem and the Extreme Value Theorem. In each case, the theorem was stated and explained graphically. Subjects were asked why the hypothesis "$f(x)$ is continuous on a closed interval $[a, b]$" is necessary for each example. More pointedly, subjects needed to determine a function that was not continuous on $[0, 1]$, but was continuous on $[0, 1)$; for which the theorem under discussion failed. The suggestion of $f(x) = [x]$ was given. Subjects determined that only integer values between $f(0) = 0$ and $f(1) = 1$ were taken on so the Intermediate Value Theorem failed. The Extreme Value Theorem failed for $f(x) = \frac{1}{x-1}$ since no minimum value for $f(x)$ is achieved on $[0, 1)$. Subjects were also asked to determine a function that was discontinuous on some interval $[a, b]$ for which the above theorems failed.

The Bisection Method for determining roots to an equation was discussed graphically using an overhead transparency of the graph of $f(x) = x^3 - 2x^2 - 1$ on the rectangle $[-5, 5]$ by $[-5, 5]$. (This was done after the discussion on the Intermediate Value Theorem and before the discussion of the Extreme Value

Theorem). The importance of the Intermediate Value Theorem in determining a root was discussed. Beginning with $[a, b] = [2, 3]$, the algorithm for the Bisection Method was used, successively determining new subintervals upon which to apply the Intermediate Value Theorem. As each successive subinterval was determined algebraically, it was illustrated graphically, determining the root to within one decimal place of accuracy. Subjects suggested that the Bisection Method could always be used to find a root to a function that was continuous on a closed interval containing the root, but that this process might be very slow.

Subjects were reminded that the machinery of limits and continuity had been built to lay a foundation for the concepts of calculus, the study of how quantities change with respect to each other. The ideas that follow would use the ideas of limits and continuity.

The discussion of the concept of derivative began with the following scenario that had been discussed earlier. It was reviewed and discussed in more depth here.

> On a long vacation, it might be of interest to know what a person's average speed was as she traveled from city A to city B. To find the average velocity, the traveler determines the distance traveled and divides the quantity by the time it took to travel that distance.

Suppose the traveler's position at various times was graphed. The graph might resemble that in Figure A-6 (without the secant line through $(a, f(a))$ and $(b, f(b)))$:

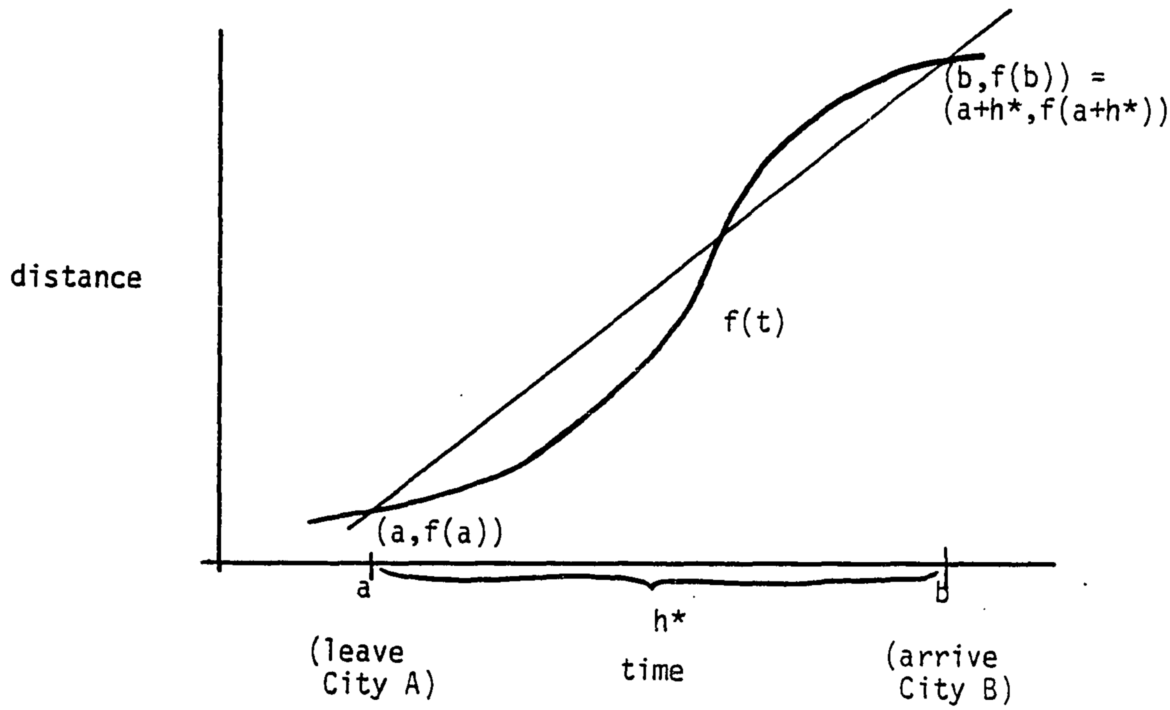


Figure A-6. Graph of a traveler's journey over time.

The average velocity would be:

$$\text{Average velocity} = \frac{\Delta \text{ distance}}{\Delta \text{ time}} = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{f(a + h^*) - f(a)}{(a + h^*) - a} = \frac{f(a + h^*) - f(a)}{h^*}$$

Graphically, this quantity is exactly the slope of the secant line containing

the points $(a, f(a))$ and $(b, f(b)) = (a+h^*, f(a+h^*))$. (The secant line was added to the graph in Figure A-6 at this time.)

The same situation can be considered from the police officer's point of view. He/she is interested in the traveler's velocity at a particular point in time c, ie. the police officer is interested in the traveler's instantaneous velocity at c. The police officer determines average velocity just as the traveler might, but on a very short time interval (with the aid of a radar detector)

$$\text{Average velocity} = \frac{\Delta \text{ distance}}{\Delta \text{ time}} = \frac{f(c+h)-f(c)}{h}.$$

Ideally, the police officer would like the traveler's velocity at c, as $h \to 0$. This interest is in

$$\lim_{h \to 0} \frac{f(c+h)-f(c)}{h}.$$

As the time interval decreases in length, the calculated average velocity becomes more accurate. The traveler doesn't have time to apply the brakes!

Day 14

The first 30 minutes of class were used as a review session for the first exam (to be given on Day 15). Subjects were particularly concerned with the difference between essential and removable discontinuities which also brought in a discussion of infinite limits. The following functions were discussed graphically and

symbolically with respect to continuity and infinite limits: $f(x) = \frac{x^2-4}{x-2}$ and $g(x) = \frac{x^2-4}{x-1}$.

During the remaining 20 minutes of class the discussion of derivatives begun during the previous day was reviewed and continued.

Earlier in the semester, it had been observed through the use of the investi - gator-written program SPIDER, that graphs model rate of change. The SPIDER was again observed as he "climbed the wall" following the distance function $f(x) = x^5 - 3x^2$ on the viewing rectangle $[-1, 2]$ by $[-4, 8]$. Subjects were asked to notice the shape of the graph relative to the spider's rate of change – velocity. They observed that when the spider was moving rapidly, his position vs. time graph was quite steep. When his rate of change was slow, the graph was less steep, and when he stopped moving, the graph appeared level. Speed vs. velocity were discussed noting that velocity is directional and speed is the absolute value of velocity.

It was determined previously that the expression for average velocity over the interval $[x,\ x+h]$ is the same expression as that of the slope of the secant line of $f(x)$ containing the points $(x, f(x))$ and $(x+h,\ f(x+h))$. The policeperson is interested in instantaneous velocity – the average velocity as the difference between starting and ending times goes to zero $((x+h)-x = h \to 0)$. The software *Master Grapher* (Waits & Demana, 1987a) was used to demonstrate how closely the police officer's estimate of the traveler's velocity matches the traveler's actual speed. Graphically, since average velocity is

the slope of the secant containing the endpoints of a time interval, if the graph of a function over a very small time interval appears to be straight, then the secant line containing the endpoints of the interval must closely approximate the function on that interval. This implies that the average rate of change of the function over the given interval is approximated very closely by the slope of the secant line containing the endpoints of the interval.

The following functions were used to illustrate the concept of derivative discussed above:

1. $f_1(x) = \sqrt{4-x^2}$ near $c = 1$,

2. $f_2(x) = \sin\left(\frac{1}{x}\right)$ slightly away from $c = 0$, and

3. $f_3(x) = \frac{|\cos 25x|}{25} + x$ for $c \neq \frac{k\pi}{50}$ where k is an odd integer.

The zoom-in option of *Master Grapher* was used to magnify function f_1 near $c = 1$. After using the zoom-in option three or four times, the graph of $f_1(x)$ appeared to be straight. The experiment was performed slightly differently for function f_2. Setting the number of points plotted to 500 and the viewing window to $[^-0.1,\ 0.1]$ by $[^-1.2,\ 1.2]$, the graph of $f(x) = \sin\left(\frac{1}{x}\right)$ was displayed and its behavior discussed, especially for x near 0. Subjects suggested that the graph was too crooked near 0 to straighten it out. Subjects were hesitant when asked if it were possible to locate an interval over which $f(x)$ could be made straight near $c = 0$ but not containing 0. Keeping the y-interval at $[^-1.2,\ 1.2]$, the x-interval was changed successively to $[0.001,\ 0.002]$ then to $[0.001,\ 0.001001]$. In the first x interval, the graph of $f(x)$ was still oscillating

wildly. In the second x-interval, the graph of $f(x)$ was a gentle, monotonically decreasing curve. Using the zoom-in option of *Master Grapher* on a small part of this curve elicits a straight line segment.

In each case above, subjects were asked to suggest why it is of interest to determine, that for small x intervals, the graphs of each function can be made locally straight. They were reminded of the expression for average rate of change – velocity – and its graphical interpretation.

Asked if they thought all graphs of continuous functions $f(x)$ could be made straight for every value of x, subjects seemed confident that the answer was yes. Function f_3 was displayed and the graph shown to be locally straight for a small interval containing $c = 1$. Asked what would happen if the graph were magnified around one of the visible "corners", subjects suggested that the graph could be made straight. The zoom-in option was used successively around a corner of $f(x) = \frac{|\cos 25x|}{25} + x$. Each time, the corner became more defined. Subjects were surprised that this graph could not be made locally straight for certain values of x. They were challenged to determine why this was the case. Asked what such a corner in the graph meant in terms of a distance vs. time function, subjects suggested that a stopped vehicle suddenly started traveling at a constant rapid rate of speed. There was no gradual acceleration. They suggested that it was unlikely that such a function would model a distance vs. time situation. Several subjects remained after class to experiment with f_3 on the computer, zooming-in several times on more than one of the corners. Each time the results were the

same.

Day 15

Exam 1 was given. Graphics calculators were not allowed for this first exam. Subjects were asked to read the directions and were reminded to show all of their work and to explain their solutions. Subjects were required to stay in the room for the full 50-minute class period.

Day 16

The session of Day 14 was reviewed briefly:

> Graphs have been shown to model rate of change. The steepness of the graph determines how y changes with respect to x. In an application of distance vs. time, the expression for average velocity v on the time interval from an initial time c to a final time $c + h$ was found to be represented symbolically by the expression
>
> $$v = \frac{f(c+h) - f(c)}{(c+h) - c}.$$
>
> The expression is also that of the slope of the secant line containing the points $(c, f(c))$ and $(c+h, f(c+h))$. Having graphed a few functions, it was observed that in each case they could be made locally straight over small intervals. It was concluded that, on small intervals, the slope of the secant is a good approximation for the rate of change over this interval.

It was also observed that not all functions are locally straight and that for such functions it was not possible to achieve an accurate estimate for the rate of change. With respect to the application of distance vs. time, as the traveler's speed is observed over shorter time intervals, a more accurate estimate of this velocity can be determined.

Before giving a definition for the limiting process used to determine instantaneous rate of change, one last observation was made. The investigator-written program SECANT was used, displaying the graph of $f(x) = 2 \sin x$ on the viewing rectangle $[^{-}6, 6]$ by $[^{-}4, 4]$ with $c = 1.5708 \approx \frac{\pi}{2}$ and the initial choice for $(c + h) = 4$. Subjects were asked to observe the appearance of successive secant lines containing $(c, f(c))$ and $(c + h, f(c + h))$ as the distance between these points was reduced by a factor of $\frac{1}{2}$. The value of the slope appeared in the upper left corner of the screen. Subjects suggested that for a very small h, the secant line appeared to touch $f(x)$ in only one point. Subjects had seen tangents to circles in previous courses. Now they had been introduced to tangents to more general curves.

The following definition was given:

DEFINITION. *If* $y = f(x)$, *then the derivative of* y *with respect to* x *at* $x = c$ *is*

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

if this limit exists. When the limit does exist, it is called the instantaneous rate

of change of y *with respect to* x *at* $x = c$, *or graphically, the slope of the tangent to the graph of* $y = f(x)$ *at* $x = c$.

Derivative notation was introduced. The following application (Hurley, 1987, p. 30) was given and discussed:

> A skydiver experiences free fall before opening his parachute. The number of feet fallen after t seconds is closely approximated by the function $y = 16t^2 = f(t)$. Find the skydiver's velocity 4 seconds after beginning his fall.

This problem was solved using the ideas of average rate of change developed earlier, then discussed in terms of the language and notation of the definition of the derivative. After finding the solution, subjects were asked to determine the skydiver's velocity 2 seconds after beginning his fall, and then at c seconds after beginning his fall. Subjects agreed that derivatives of functions were also functions. Derivative functions could be used to determine the instantaneous rate of change at any given point in time.

A second problem was given:

> Find the tangent to the curve of $f(x) = x^2 + 1$ at the point $x = 2$.

This time, $f'(c)$ was found first using the definition of derivative, recalling that the slope of the tangent is the limit of the slope of the secant over the x-interval $[x, x+h]$ as the width of this interval goes to 0.

Following the exercise above, subjects were asked when a function would fail to be differentiable. In other words, when would

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

fail to exist?

Subjects suggested that the limit would fail to exist when h wouldn't "factor out," when the function was undefined at a point $x = c$, when the function was discontinuous at c but defined there, or when a corner appeared on the graph. In this last case, subjects mentioned that left and right slopes would be different so that left and right limits would not be equal and the limit would not exist.

The following theorem was stated and proved:

THEOREM. *If* f *is differentiable at* $x = c$, *then* f *is continuous at* $x = c$.

Class concluded with a discussion of a hierarchy of properties of functions (Hurley, 1987):

1. Having a limit at a point $x = c$,
2. Being continuous at a point $x = c$, and
3. Being differentiable at a point $x = c$.

Subjects were encouraged to respond to the following:

1. Find a function that has a limit at $x = 2$ but is not continuous there.
2. Find a function that is continuous at $x = -1$ but is not differentiable there.

3. Find a function that is differentiable at $x = 0$. Is it also continuous?

Responses included, respectively:

1. $f(x) = \frac{x^2-4}{x-2}$,
2. $f(x) = |x + 1|$, and
3. $f(x) = x^2$, yes.

Day 17

During this class period subjects learned to differentiate graphically. Given a function $f(x)$ and noticing certain "landmarks" of f, subjects were asked to determine the graph of $f'(x)$. These ideas were then used to begin developing rules for efficient methods of finding derivatives symbolically (see Janvier, 1975).

Subjects were asked to recall that the steepness of a graph models how two quantities change with respect to each other. The steepness of a graph at a given point $x = c$ is found by determining $f'(c)$, if it exists.

The program SPIDER was used with $f(x) = x^5 - 3x^3$ on the viewing rectangle $[^-1, 2]$ by $[^-5, 5]$. Subjects were asked to determine the appearance of the graph as the spider climbed up, down or stopped.

To graphically estimate $f'(x)$ given the graph of $f(x)$, subjects observed the following landmarks of f to help sketch f' :

1. Values of x where the slope of the tangent to the curves is 0.
2. Intervals over which the slopes of the tangents to $f(x)$ are positive.
3. Intervals over which the slopes of the tangents to $f(x)$ are negative.

Lecture notes were given to aid subject participation in the exercise and to provide a hard copy of the discussion for their notes. Included in the notes was the definition of derivative, questions to aid in their understanding of the appearance of $f'(x)$ and sample graphs which were differentiated by the investigator on the overhead while subjects worked at their seats.

Differentiating the graph of $f(x)$ given on page 2 of the lecture notes, subjects suggested approximate values for the slopes of each of the points indicated. At the very least, subjects noticed when the slopes were positive, negative on zero. They observed that in each case, a corresponding point was plotted on the coordinate plane provided. For each point $(x, f(x))$, if the slope of the tangent to the curve at $(x, f(x))$ was negative, the corresponding point $(x, f'(x))$ was plotted below the x-axis. If the slope of the tangent to the curve at $(x, f(x))$ was positive, then $(x, f'(x))$ was plotted above the x-axis, and if the slope of the tangent to the curve at $(x, f(x))$ was zero then $(x, f'(x))$ was plotted on the x-axis. Subjects observed that this cubic-looking function appeared to have a somewhat quadratic-looking derivative function.

Lecture Notes
Math 122 - Calculus I

Recall: If $y = f(x)$, then the derivative of y with respect to x at $x = c$ is :

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

if the limit exists. When the limit does exist, it is called the instantaneous rate of change of y with respect to x at x = c. In graphic terms, $f'(c)$ is the slope of the tangent line to the graph of $y = f(x)$ at $P(c,f(c))$.

In this assignment we will graphically estimate $f'(x)$ given the graph of $f(x)$. We will notice certain "landmarks" of $f(x)$ to help us sketch $f'(x)$. Some helpful "landmarks" are:

i) Values of x where the slope of the tangent is 0.

If $f'(c) = 0$ then what point is on the graph of $f'(x)$? ______________________

ii) Intervals where the slope of $f(x)$ is negative.

If $f'(x) < 0$ on an interval, where does the graph of $f'(x)$ lie with respect to the x-axis? ________
____

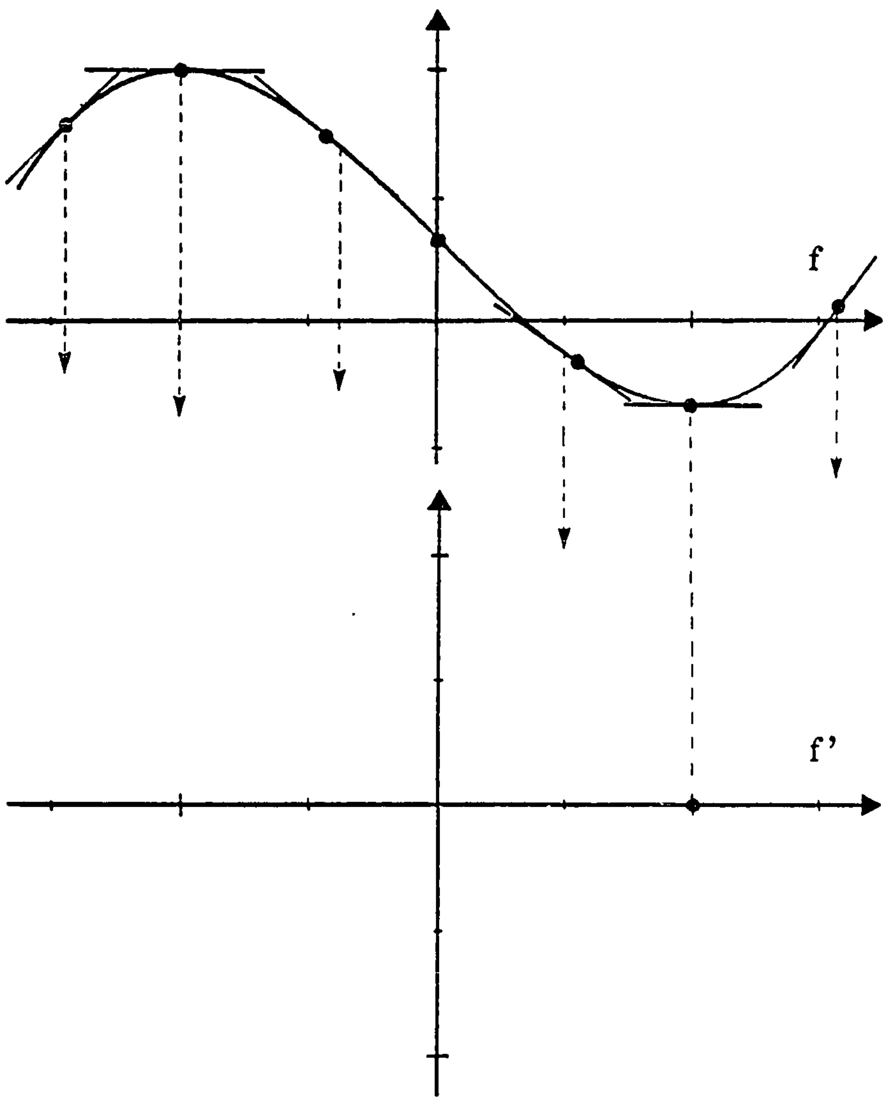
iii) Intervals where the slope of $f(x)$ is positive.

If the slope of $f(x)$ is positive on an interval, then $f'(x)$ _____ 0. Where does the graph of $f'(x)$ lie with respect to the x-axis for x in such intervals? ______________________

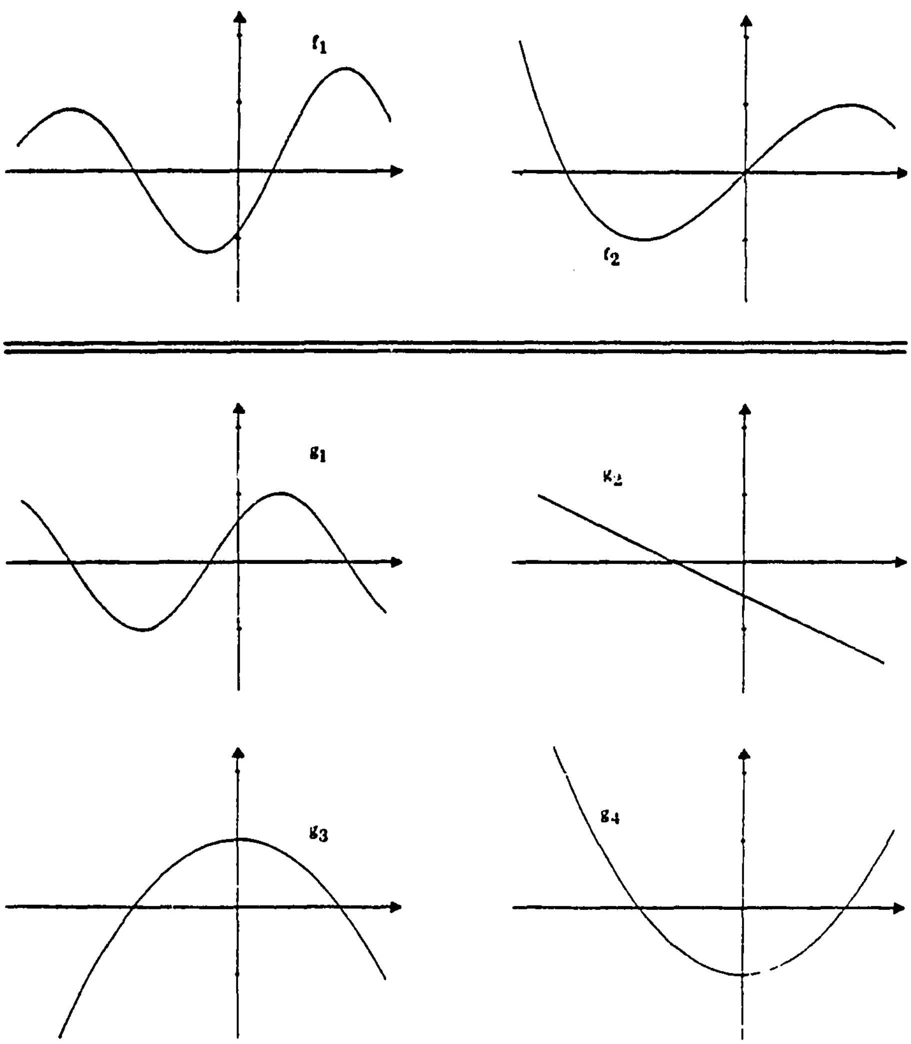
Taken from:

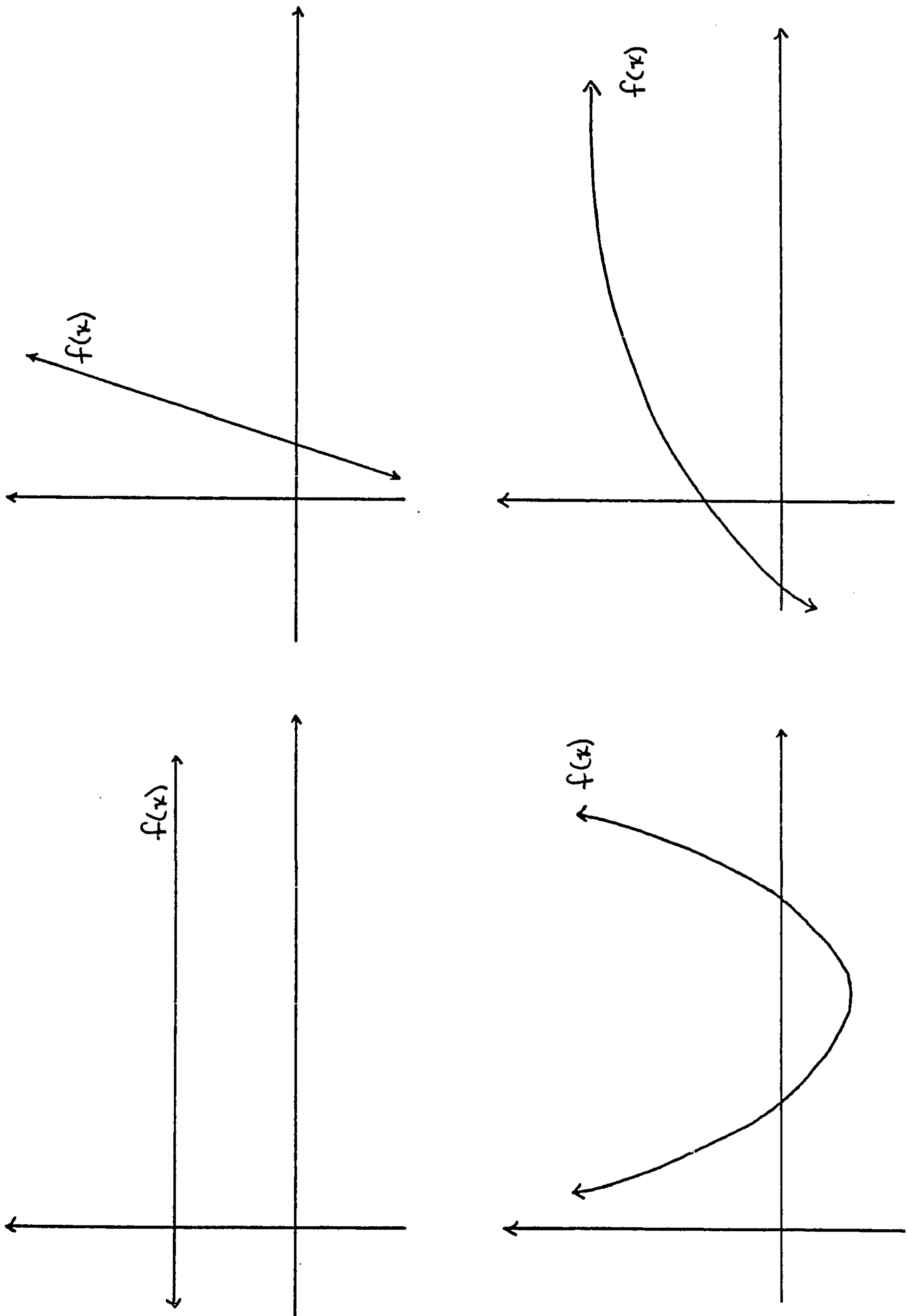
Janvier, C. (1975). Calcul differentiel et integral, module 1, la dérivée. Universite du Quebec, Tele-Universite, Permama X, pp. 2-17 - 2-25.

We will now practice sketching the graphs of the derivatives of the functions given graphically below.



On the first part of this page we have the graphs of functions f_1 and f_2. Among the functions g_1 through g_4, find the graph of the derivative of each function f.





4

More practice for finding derivatives graphically was provided by the graphs shown on page 3 of the lecture notes.

Using the skills developed in the opening exercises, subjects were asked to differentiate the first three graphs given on page 4 of the lecture notes. While the investigator worked on the overhead, subjects suggested slopes to various points along each curve determining the following:

(1) If $f(x) = c$, where c is a constant, then $f'(x) = 0$;

(2) If $f(x) = mx + b$ is a line then the slope of the tangent for every choice of x is the same as the slope of $f(x)$ so $f'(x) = m$;

(3) If $f(x)$ is a parabola, then it appears that $f'(x)$ is linear.

The investigator-written program FASTAN was used to allow the computer to more accurately perform the graphic differentiation of the user-chosen function $f(x)$ on the viewing rectangle $[^-6, 6]$ by $[^-5, 5]$. FASTAN draws successive tangents along the curve of f then plots the values of the slopes of each tangent where f' is estimated by

$$g(x) = \frac{f(x + .01) - f(x)}{.01}.$$

Using FASTAN, subjects observed that the derivative of $f(x) = x^2$ is linear. Moreover, they observed that $D_x(x^2) = 2x$. The graph of $f(x) = x^3$ was plotted along with its derivative with subjects observing that $D_x(x^3)$ was parabolic, but narrower than x^2, so the coefficient of the x^2 term in $D_x(x^3)$

must be greater than 1. Subjects also observed that $D_x(x^4)$ was cubic. A table listing each of the functions that had been differentiated was built (see Table A-2).

Table A-2

Polynomials and Their Derivatives

$f(x)$	$f'(x)$
$C = cx^\circ$	0
$mx + b = mx^1 + b$	m
x^2	$2x^1$
x^3	parabola, ax^2
x^4	cubic, bx^3
$\dots$	
x^n	

Subjects guessed that the derivative of x^n would be of degree $n - 1$. Each of $D_x(c)$, $D_x(mx + b)$, $D_x(x^2)$, and $D_x(x^3)$ were derived symbolically. As with many concepts developed throughout the course, subject intuition was developed through applications and/or graphs. This intuition then supported the symbolic work. The Power Rule for derivatives was stated.

The following application was stated and discussed:

If the function $s(t) = t^2 - 4t + 6$ gives the position of an object after t seconds, what is its velocity at time $t = 1$; at time $t = 3$?

Day 18

The rules for differentiation developed on Day 17 were reviewed. Using the Transform $F(x)$ option of *Master Grapher*, the graphs of $f(x) = x$, $f(x) = x^2$, and $f(x) = \sin x$ were viewed separately, each time overlaying the graph of $c \cdot f(x)$ for $c = 2$, $c = \frac{1}{2}$, $c = {}^-1$. Subjects were asked how the slopes of tangents to $y = c \cdot f(x)$ compared to slopes of tangents to $f(x)$. The rule, $D_x(c \cdot f(x)) = c \cdot D_x(f(x))$ was derived symbolically following the intuitive development.

The graphs of $f(x)$ and $g(x)$ were drawn on the overhead (see Figure A-7). The investigator began drawing the graph of $h(x) = f(x) + g(x)$ using the graphs of f and g as had been done earlier in the course. Subjects were asked how the slope of the tangent to $h(x)$ at $x = c$ would correspond to the slopes of the tangents to $f(x)$ and $g(x)$ at $x = c$.

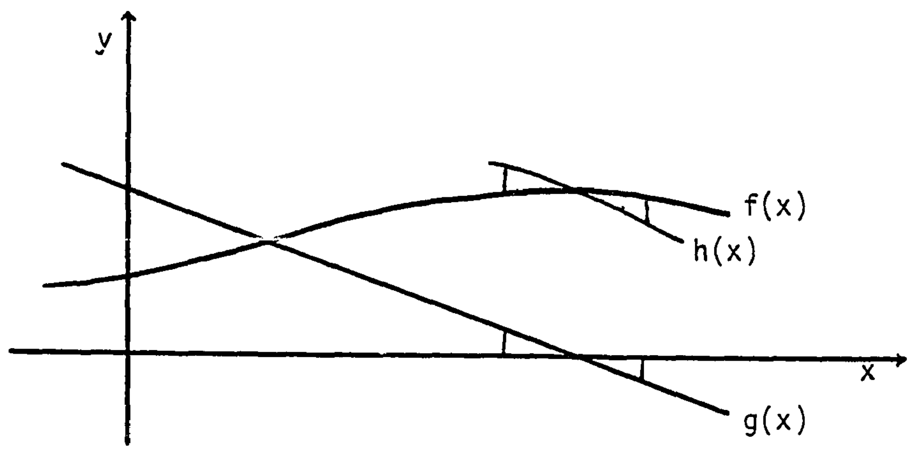


Figure A-7. Illustration of adding $f(x)$ and $g(x)$ graphically.

Using *Master Grapher*, the graphs of $f(x) = x$ and $g(x) = \frac{1}{2}x$ were displayed on the viewing rectangle [0, 5] by [0, 5]. The graph of $h(x) = x + \frac{1}{2}x = \frac{3}{2}x$ was overlayed. Subjects suggested that the slope of the graphs of f and g determined the slope of h. The slopes of tangents to h were the results of adding the slopes of the tangents of f with the slopes of the tangents of g. Symbolically, $h'(x) = f'(x) + g'(x)$. The rule was derived symbolically from the definition of the derivative. When asked to conjecture a rule for finding the derivative of the difference of two functions $h(x) = f(x) - g(x)$, subjects suggested that similarly to the rule for the sum of two functions, the derivative would be $h'(x) = f'(x) - g'(x)$. Subjects suggested that $D_x((f+g+h)(x)) = (f' + g' + h')(x)$. This was also derived symbolically. The summation notation

was reviewed, and the rule

$$D_x(\sum_{i=1}^{n} f_i(x)) = \sum_{i=1}^{n} D_x(f_i(x))$$

was derived using mathematical induction.

The special case of $g(x) = d$, a constant, was illustrated on the overhead and discussed. The result of adding a constant to a function $f(x)$ had been previously discussed. Subjects suggested that the shape of the graph of $h(x) = f(x) + d$ was unchanged, that its position was different vertically from f (for $d \neq 0$), and that the slopes of tangents to $h(x)$ must be the same as slopes of tangents to $f(x)$ at any given x. Symbolically:

$$h'(x) = D_x(h(x)) = D_x(f(x) + d) = D_x(f(x)) + D_x(d)$$

$$= f'(x) + 0 = f'(x).$$

Using the power rule, the generalized sum and difference rules, the derivative of a constant, and the rule for differentiating a function multiplied by a constant, subjects were asked to differentiate the polynomial $f(x) = -3x^3 - 2x^2 + 4x - 5$. (Subjects suggested that they'd appreciate being tested on finding derivatives of polynomial. The investigator related an experience during which she had "taught" her fourth grade daughter to differentiate polynomials without the child knowing anything about functions, variables, exponents or derivatives. College subjects would be asked questions that required deeper understanding!)

Subjects were asked to suggest rules for finding the derivative of the product of two functions. Generalizing the derivative rule for sums of functions, subjects

suggested that $h'(x) = f'(x) \cdot g'(x)$. Using $f(x) = x^3$ and $g(x) = x^5$, subjects discovered that $(f \cdot g)'(x) \neq f'(x) \cdot g'(x)$. The product rule was stated and proved symbolically.

Derivative rules for the products of 3 and 4 functions were derived using the Product Role.

The generalization:

$$D_x(\prod_{i=1}^{n} f(x)) = D_x[f(x)^n] = n\, f(x)^{n-1} \cdot f'(x)$$

was derived using the product rules for 2, 3, and 4 functions where each function was equal to $f(x)$.

Examples which used the product rule and the generalized power rule were given and discussed.

Day 19

After 15 minutes of review of homework assignments, subjects took 15 minutes to respond to Quiz 3.

During the remaining 20 minutes of class, the reciprocal rule was stated. The reciprocal rule was proved using the definition of derivative. The generalized power rule for derivatives was extended to functions raised to a negative exponent. The reciprocal rule was shown to be consistent with such a rule. The derivative rule of $f(x) = x^{\frac{m}{n}}$ for $m, n \in Z$, $n \neq 0$ was given without proof. The quotient rule was derived using the product and reciprocal rules. Examples

of uses of the reciprocal and quotient rules were demonstrated.

Day 20

Homework exercises requiring the use of rules for differentiation were reviewed during the first 25 minutes of class. The example of $PV = nRT$ from the text was discussed giving subjects alternate notation for the product rule (Hurley, 1987, pp. 75-76).

Subjects were presented the problem:

> Find the slope of the tangent line to the graph of $y = \sqrt{x}$ at the point $(c, \sqrt{c})$ where $c > 0$, and then at the point $(0, 0)$.

Using the definition of derivative, $y' = \frac{1}{2\sqrt{c}}$. Subjects recognized that y' exists for $c > 0$ but that the slope of the tangent line at the point $(0, 0)$ is undefined. Looking at the graph of $y = \sqrt{x}$, subjects suggested that as c approaches 0, the slope of the tangent gets very steep and that a tangent to $y = \sqrt{x}$ at $x = 0$ would be vertical so its slope is undefined. The definition of derivative was extended to include the derivative at an endpoint. It was observed that even by this definition, the derivative of $y = \sqrt{x}$ does not exist at $x = 0$.

Subjects were asked to determine the derivative of $f(x) = |x|$ at $x = 0$ where f is defined on the intervals:

1. $[0, \infty)$,
2. $(-\infty, 0]$, and
3. $(-a, a)$ for $a \neq 0$.

The graph of $f(x)$ was displayed. Subjects suggested that for the respective situations, (1) $f'(x) = 1$, (2) $f'(x) = -1$; (3) $f'(0)$ does not exist since left and right limits are unequal.

Day 21

Exam 1 and Quiz 3 were returned. The problems on Quiz 3 and the first three pages of Exam 1 were reviewed.

The trigonometry review of Section 2.3 was covered including radian measure, right triangle trigonometry, the interpretation of right triangle trigonometry on the unit circle, and the periodicity of the trigonometric functions. The presentation varied from that suggested in the Instructor's Manual in that the review of periodicity was first done graphically on the overhead and then with the *Master Grapher* option Transform $F(x)$. Subjects suggested that $y = \cos x$ could be obtained from $y = \sin x$ by shifting $y = \sin x$ to the left $\frac{\pi}{2}$ units; so $\cos x = \sin(x + \frac{\pi}{2})$. Also presented graphically were the relationships $\cos x = \cos(-x)$ and $-\sin x = \sin(-x)$, noticing that $y = \cos x$ is an even function and $y = \sin x$ is an odd function. The relationships between $f(x)$ and $f(-x)$ for the functions $f(x) = \cos x$ and $f(x) = \sin x$ were discussed first through the use of the unit circle. The *Master Grapher* option Transform $f(x)$ was used to compare the graphs of $f(x)$ and $f(-x)$.

Day 22

As with other functions encountered in the course, the existence of

$\lim_{x \to c} f(x)$ and the continuity of $f(x)$ for the trigonometric functions were discussed. Continuity of sine and cosine functions was discussed as follows (see Hurley, 1987, pp. 86-90; Renz, 1986, p. 106):

> Consider any two points A and B on the unit circle (see Figure A-8). The length of arc OA is a and the length of arc OB is b. The length of arc AB is then $|b - a|$. Draw a line parallel to the x-axis through point A and another line parallel to the y-axis through point B. Call the point of intersection of these two lines C with coordinates $(\cos b, \sin a)$. The length of segment AC is $|\cos b - \cos a|$. The length of segment BC is $|\sin b - \sin a|$. By the Pythagorean Theorem, the length of segment AB is $\sqrt{(\cos b - \cos a)^2 + (\sin b - \sin a)^2}$. The length of segment AB is less than or equal to the length of arc AB so
>
> $$\sqrt{(\cos b - \cos a)^2 + (\sin b - \sin a)^2} \leq |b - a|$$
>
> which implies that
>
> $$|\cos b - \cos a| \leq |b - a| \text{ and}$$
>
> $$|\sin b - \sin a| \leq |b - a| .$$

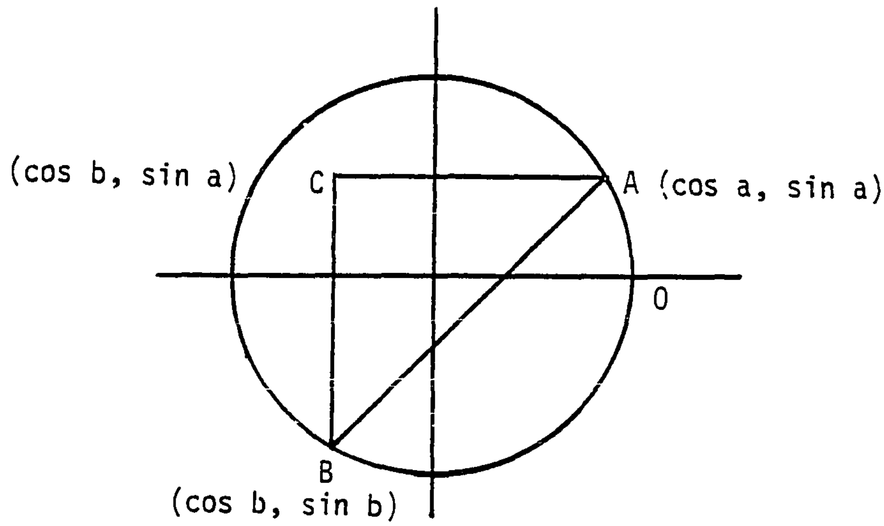


Figure A-8. Geometric illustration of the proof of continuity of sine and cosine.

The distance between any two points on the unit circle $(\cos x, \sin x)$, $(\cos c, \sin c)$ is measured by the length of the arc between them $|x - c|$. To determine that $f(x) = \sin x$ is a continuous function, it is necessary to show that as the distance $|x - c|$ decreases to 0 then the vertical distance between the points above, $|\sin x - \sin c|$, also decreases to 0. From the previous work,

$$|\sin x - \sin c| \leq |x - c|$$

for any two points $(\cos x, \sin x)$ and $(\cos c, \sin c)$ on the unit

circle. Then $\lim_{x \to c} |\sin x - \sin c| \leq \lim_{x \to c} |x - c|$ which implies $\lim_{x \to c} |\sin x - \sin c| = 0$ so $\lim_{x \to c} \sin x = \sin c$. By the definition of continuity $f(x) = \sin x$ is continuous. Analogously, $f(x) = \cos x$ is continuous.

Since all of the other trigonometric functions – tangent, cotangent, secant, and cosecant – are defined using sine and cosine functions; by the reciprocal and quotient properties of continuity, these functions are also continuous wherever they are defined.

To be able to symbolically determine the derivatives of sine and cosine, the following special limits are necessary:

$$\lim_{x \to 0} \frac{\sin x}{x} \text{ and } \lim_{x \to 0} \frac{1 - \cos x}{x}.$$

Subjects had discovered $\lim_{x \to 0} \frac{\sin x}{x} = 1$ by looking at successive graph of $f(x) = \frac{\sin x}{x}$ on successively smaller intervals containing $x = 0$. Subjects noticed that successive graphs appeared to be continuous at $x = 0$ and that $f(x)$ appeared to take on the value of 1. This discussion had taken place during the conceptual development of limit.

Subjects were reminded of their earlier work with $f(x) = \frac{\sin x}{x}$. They were then shown an overhead transparency with the unit circle containing acute angle $\angle POR$ whose center was at the origin O (see Figure A-9). The points P (cos t, sin t), Q (cos t, 0), and R (1, 0) were labelled. Subjects were reminded that the length of arc PR was t radians and that the length of

segment PQ was $\sin t$. The meaning of $\lim_{t \to 0} \frac{\sin t}{t}$ was interpreted graphically as a comparison between the lengths of segment $PQ = \sin t$ and arc $PR = t$ as t becomes very small. Overlays of successively smaller values of t were shown, each time comparing the corresponding lengths t and $\sin t$. As t approached 0, subjects observed that $\sin t$ and t were both becoming very close to the same length. For all practical purposes, as t approaches 0, $\frac{\sin t}{t}$ behaves just as $\frac{t}{t}$. In other words, $\lim_{t \to 0} \frac{\sin t}{t} = \lim_{t \to 0} \frac{t}{t} = 1$.

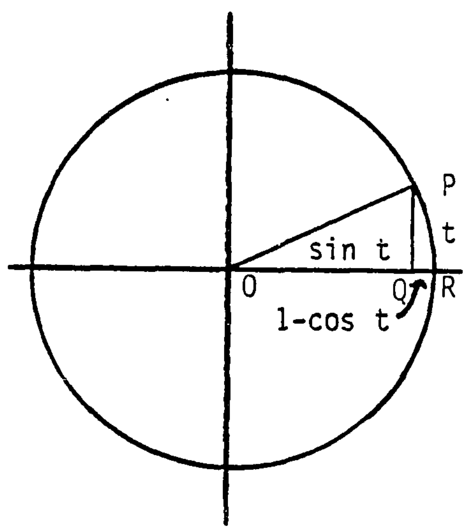


Figure A-9. Comparisons of lengths t, $\sin t$, and $1 - \cos t$.

Subjects were asked to notice that segment OQ had length $\cos t$ so segment QR had length $1 - \cos t$. The overlays used earlier were used again to aid subject visualization of the ratio $\frac{1 - \cos t}{t}$ as t approaches 0. Subjects recognized that the length $1 - \cos t$ approaches 0 much more

rapidly than the length t so $\lim_{t \to 0} \frac{1 - \cos t}{t}$ appeared to be 0. The graph of $f(x) = \frac{1 - \cos x}{x}$ was drawn with *Master Grapher*, noticing that $f(0)$ appeared to be 0.

Having discussed continuity of sine and cosine functions, and having laid the foundation for symbolically determining their derivatives, subjects were asked to determine the derivatives of sine and cosine graphically. The functions were shown on the overhead with a discussion of their shapes. Subjects mentioned that these functions had the same shapes. Subjects were asked, if they knew the derivative of the sine function, could they find the derivative of the cosine function? They suggested that the derivative graphs would have the same shape since cosine and sine have the same shape, so to find the derivative of the cosine, one could move the derivative of the sine to the left $\frac{\pi}{2}$ units.

Looking at the graph of $f(x) = \sin x$, subjects were asked to suggest values of the slopes of tangents to $f(x)$ for x equal to $\pm 2\pi$, $\pm \frac{3\pi}{2}$, $\pm \pi$, $\pm \frac{\pi}{2}$, and 0. The investigator then drew the graph of $f'(x)$ connecting these points to resemble $g(x) = \cos x$. When asked what the derivative of $g(x) = \cos x$ would be, subjects suggested that $g'(x) = \cos(x + \frac{\pi}{2})$ since $\cos x = f'(x) = \sin(x + \frac{\pi}{2})$. They then determined that
$g'(x) = \cos(x + \frac{\pi}{2}) = \sin(x + \frac{\pi}{2} + \frac{\pi}{2}) = -\sin x$.

The program, FASTAN, was used to reinforce subject intuition. Subjects observed that the derivative function of $f(x) = \sin x$ being sketched was as they expected: $f'(x) = \cos x$. Using the definition of derivative and the special

limits found earlier in the class period, subjects were led through the symbolic proof.

The graph of $f(x) = \cos x$ was also viewed using FASTAN. Subjects observed that they were correct in their assumption that the derivative of $f(x)$ is $f'(x) = -\sin x$.

Day 23

Subjects were given 15 minutes to complete Quiz 4.

During office hours, subjects asked many questions about the relative lengths of $\sin t$, $1 - \cos t$, and t. Subjects clearly were not convinced that $\sin t$ and t become very close to the same length as t approaches 0. Assuming that other subjects were as equally unsure as the conscientious ones who regularly came to office hours, it was decided that the above comparison required another round.

Using *Master Grapher* in conic graphing mode, the unit circle $x^2 + y^2 - 1 = 0$ was displayed. The graphs of the lines $y = .5x$, $y = .1x$, and $y = .05x$ had been entered into the function index before class to use during the demonstration. Changing the graphing speed to plot 250 points, the unit circle was redrawn on the viewing rectangle $[-1.2, 1.2]$ by $[-1.2, 1.2]$. The line $y = .5x$ was overlayed and a vertical line drawn from the intersection of the unit circle with this line to the x-axis (see Figure A-9). Subjects were reminded of the graphical interpretation of the values $\sin t$, $1 - \cos t$ and t, noting that in this case $t \approx 0.46$ rad .

The function $y = .1x$ was overlayed on the graph in Figure A-9 with another vertical line drawn through the intersection of $y = .1x$ and the unit circle. The lengths $\sin t$, $1 - \cos t$, and t were compared for $t \approx 0.10$ rad. Subjects noticed that $\sin .10$ and $.10$ rad were much closer in length than $\sin .46$ and $.46$ rad. They also noted that $1 - \cos t$ in each case was much smaller than t.

The speed was changed to 350. Using the zoom-in option of *Master Grapher*, a rectangle was drawn around the segments and arc whose lengths were approximately $\sin .10$, $1 - \cos .10$ and $.46$, respectively, and the enclosed region magnified. The line $y = .1x$ was redrawn, as well as the vertical line through its intersection point with the unit circle. Subjects observed that, within this window, $\sin .10$ and $.10$ were only slightly different in length. They also observed that $1 - \cos .10$ was much smaller as compared to $.10$ than the corresponding comparison for $t = .46$.

The graph of $y = .05x$ was overlayed with a vertical line drawn through its intersection point with the unit circle. The observations comparing lengths of approximately $.05$, $\sin .05$ and $1 - \cos .05$ were made. Subjects observed once more that $\sin t$ and t were closer in length than before and that $1 - \cos t$ appeared to be negligible. Subjects seemed convinced as a result of the investigation above that

$$\lim_{t \to 0} \frac{\sin t}{t} = 1 \quad \text{and}$$
$$\lim_{t \to 0} \frac{1 - \cos t}{t} = 0.$$

Subjects viewed the computer generated graphs as more reliable and believable than those drawn by the investigator and displayed on the overhead.

In the time that remained, subjects were introduced to tangent approximations and differentials through the following problem and the use of computer graphs:

> Problem: Find the equation of the tangent line to the graph of $f(x) = \sqrt{x} + 1$ through the point $(1, 2)$.

The functions $y = \sqrt{x}$ and $y = \frac{1}{2}(x - 1) + 2$ had been entered into the function index of *Master Grapher* prior to class time. Subjects were asked to suggest the steps toward finding the requested tangent line. The function $f(x)$ and its tangent line through $(1, 2)$ were graphed. Using the zoom-in option, the graphs were magnified a few times in a region containing the point $(1, 2)$. Subjects observed that the tangent line and $f(x)$ coincided for values of x near 1. It was concluded that the line

$$\begin{aligned} T(x) &= \frac{1}{2}(x - 1) + 2 \\ &= f'(1)(x - 1) + f(1) \end{aligned}$$

could be used to estimate function values for values of x near 1, ie.

$$f(1.01) \approx T(1.01) = f'(1)(1.01 - 1) + f(1).$$

Subjects used calculators to compare the results found by using the tangent line approximation for $f(x)$ with the value of $f(1.01)$. The values $T(1.01)$ and $f(1.01)$ were found to agree to four decimal places.

Subjects were informed that the process is especially important in applications. If an initial data point and the related rate of change for two quantities are known, a function relating the two quantities can be estimated using tangent approximations. For example, Kepler knew the relative positions of the planets at various points in time and the rates at which they seemed to be revolving about the sun. It can be imagined that by using tangent approximations, one could describe the orbits of the planets using just this information.

Day 24

The previous day's work with tangent approximation was reviewed. The remaining work with tangent approximations and differentials was similar to that described in the Instructor's Manual in Section 3.5 of *Calculus* by Hurley (1987). The following examples were discussed:

1. Estimate $\sqrt{9.006}$ using differentials.

2. Suppose a room is measured to be 100 ft by 100 ft with a possible error of measurement of $\pm\, 0.01$ ft. Determine the greatest possible error in determining the area of the room. What is the relative error?

Tangent approximation were used to develop the chain rule, as suggested in Hurley (1987) Section 2.6. Using the example of the food chain of fish in a lake, the growth of the population of fish was shown to be dependent on the change

in volume of algae in the lake which in turn was dependent on time. The fish population, dependent on both algae and time, could be written as a function of time. The rate of change of the fish population was determined to be dependent on the change in time.

The chain rule was stated using both composite function and Leibniz notation. The following examples were completed in class:

1. Find $D_x(\sin 3x)$.
2. Find $D_x(\sqrt{\sin x})$

The generalized power rule was shown to be a special case of the chain rule.

Day 25

Exercises and definition of differentials were reviewed. Work begun on the chain rule on Day 24 was reviewed with the chain rule being derived again (see Hurley, section 2.6) and the definition stated. The following examples were discussed:

1. Let $y = \sqrt{x^2 + 3}$, find $\frac{dy}{dx}$.
2. Let $y = \sin^3 x$, find $\frac{dy}{dx}$.

The chain rule was extended to composites of three or more functions. The following example was completed and discussed:

$$\text{Let } y = \sin^3 x^5. \text{ Find } \frac{dy}{dx}.$$

Example 7 (Hurley, 1987, p. 110) was used. The problem discusses the food chain of a population of large fish in a lake.

Day 26

This class period was used to review for Exam 2 to be given on Day 27.

Day 27

Subjects were given Exam 2. The exam was monitored by the investigator's advisor. All subjects were required to remain in the classroom for the entire testing period. Graphing calculators were not allowed on this exam.

Day 28

Inverse functions were defined and discussed graphically. Subjects were told that the inverse of a function is obtained by changing the roles of x and y. Graphically, subjects were shown that this meant reflecting the graph of a function across the line $y = x$. Using grid paper with the axes drawn on it, subjects were shown the graph of $f(x) = x^2$, for $x \in [0, \infty)$. They suggested points that were on $f(x)$, then suggested corresponding points that must be on $f^{-1}(x)$. Solving $y = x^2$ for x, subjects concluded that the graph of $f^{-1}(x)$ containing the points they had found, was the graph of $f^{-1}(x) = \sqrt{x}$.

Subjects were shown various graphs of the functions on the overhead. Using separate individual transparencies of each graph, subjects were also shown the inverse relations. As a result of this discussion, subjects suggested that a function has an inverse function if the function is always increasing or always decreasing on an interval. They suggested that a function that has an inverse must pass a

horizontal line test – any horizontal line drawn through the graph of a function $f(x)$ intersects $f(x)$ in, at most, one point.

The definitions of increasing and decreasing functions were stated with subjects suggesting the wording "less than" or "greater than" in the definitions while looking at the graph of an increasing function, then a decreasing function. The definitions of monotonic and one-to-one functions were also stated.

Returning to the graphs of $f(x) = x^2$ and $f^{-1}(x) = \sqrt{x}$, subjects were asked how the slope of $f(x)$ at $x = 2$ related to the slope of $f^{-1}(x)$ at $x = 4$. The investigation was completed symbolically, finding the derivatives of f and f^{-1}, and evaluating them at $x = 2$ and $x = 4$ respectively. Subjects were asked to consider the graphs and to suggest why the symbolic work turned out as it did. They suggested that since the roles of x and y were exchanged from f to f^{-1} that the rise-over-the-run for f would translate to the run-over-the-rise when determining the slope of f^{-1}, *ie.* $\frac{\Delta y}{\Delta x}$ for f becomes $\frac{\Delta x}{\Delta y}$ with respect to f when determining the slope of f^{-1}.

Subjects were given the function $f(x) = \sin x$ on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. The graph was drawn. Since the function $f^{-1}(x) = \sin^{-1} x$ had not been discussed in the course, subjects were asked to suggest how they might find the slope of f^{-1} at $x = \frac{1}{2}$. Subjects suggested finding the slope of $f(x)$ where $f(x) = \frac{1}{2}$, then taking the reciprocal.

The observation was made that the inverse function rule for differentiation allows differentiation of functions such as $y = x^{\frac{m}{n}}$ and $y = [f(x)]^{\frac{m}{n}}$. This

was accepted without proof.

Day 29

The session began with a review of homework problems assigned for Section 2.7: Differentiation of Inverse Functions and Roots.

Section 2.8: Implicit Differentiation was discussed similarly to that described in the text and the Instructor's Manual. The examples $x^2 + y^2 = c$ were discussed where subjects suggested appropriate and inappropriate replacements for c. A tangent approximation example for implicit functions was discussed symbolically where $x^2y^2 + x^2y + x^3y = 3$ at $x = .98$ and at $x = .96$.

The following scenario was used to illustrate the usefulness of tangent approximations and differentials in application:

> Given a table of data gathered in a lab, it is usually possible through statistical methods to determine a relationship between the quantities being observed. Very often, it's useful to predict a trend for a particular value near a given data point. This can be done using successive tangent line approximations when $f(x, y)$ or $f'(x, y)$ is known, ie. when a relationship between the quantities is known and/or a related rate of change is known.

A graphical description of this process was given, drawing $y = f(x)$ and the successive tangents for 2 values of x near $x = c$ (see Figure A-10).

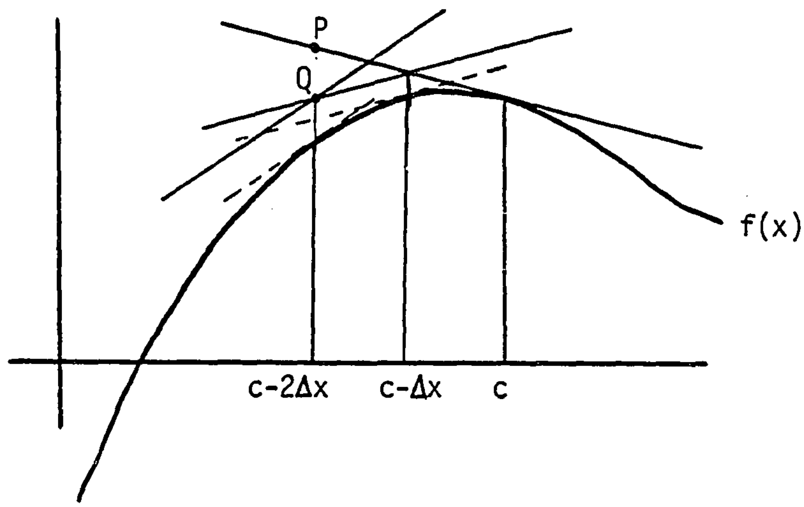


Figure A-10. Successive tangent approximations of $f(x)$ starting with $x = c$.

The rate of change of y with respect to x is used to determine successive approximations, noting that the tangent line approximation at $c - 2\Delta x$, labelled Q, is a much better approximation than that found by using the tangent line through $(c, f(c))$, labelled P.

The investigator-written program DIFFERENTIALS was used to illustrate the use of tangent approximations when an initial data point and an expression for the rate of change of two quantities is known. DIFFERENTIALS used the tangent approximation for successive points to estimate and sketch the graph of $f(x)$ when an initial data point $(x_0, y_0) = (3, 2)$ and a rate of change relationship

$f'(x) = \frac{x^2}{3}$ was known. The resulting graph contained tick marks showing the data points being estimated. These were joined by line segments. The graph of $f(x) = \frac{x^3}{9} - 1$ that had been estimated by the tangent-approximated graph above was overlayed. Subjects observed that the approximated graph varied only slightly from the graph of $f(x)$. It was therefore shown that, not only can tangent approximations be used to obtain good estimates of values of $f(x)$ for x near x_0, but also, can be used successively to obtain good estimates of $f(x)$ for values of x away from x_0.

Day 30

Class began with a review of homework problems from Section 2.8: Implicit Differentiation.

Subjects were asked to consider successive derivatives of the function $f(x) = 3x^2 + 2x + 9$. The notation for higher-order derivatives was introduced during the example. Subjects suggested that all polynomials would eventually have a derivative $f^{(n)}(x) = 0$ for some n. They also suggested that $f^{(n)}(x)$ would not necessarily be zero for $f(x)$ not a polynomial. Examples of functions not having 0 as an nth derivative included

$f(x) = \sin x$ and $f(x) = \sqrt[3]{x}$.

Subjects were asked to recall the definition of the first derivative. They were asked to suggest a definition for the second derivative. They did so appropriately. Implicit second derivatives were discussed using the example:

Given $x^2 - y^2 = 9$, find y'' in terms of x and y.

Subjects were asked to recall the graphic meaning of the first derivative. They were also asked to recall the meaning of the first derivative $f'(x)$ if the function $f(x)$ modelled distance vs. time. Subjects were told that the second derivative also had special interpretations, both graphically and with respect to the application of distance vs. time.

To illustrate these meanings simultaneously the investigator-written program SPIDER was used to illustrate graphically the position versus time function of the spider as it "climbed the wall". Subjects were asked to observe the spider's velocity with respect to the shape of the graph. They were reminded that the first derivative $f'(x)$ being positive or negative meant that the function $f(x)$ was increasing or decreasing respectively. They were asked to determine where the velocity of the spider was increasing. They suggested that the velocity was increasing when the graph was getting steeper. Subjects were then reminded that velocity is directional, i.e., not the same as speed. Acceleration described the rate of change of velocity.

Subjects were shown several graphs of increasing functions (see Figure A-11) on the overhead. They were asked to describe the spider's movement as it is modelled by each graph.

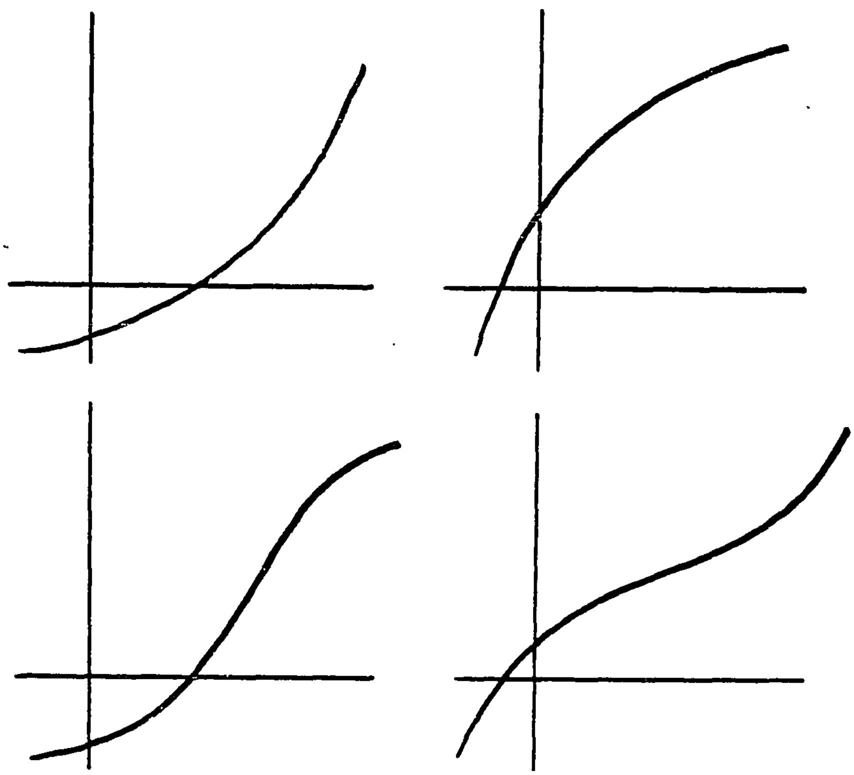


Figure A-11. Graphs of increasing functions.

The concavity of each graph was discussed. Subjects observed that when a graph is concave up, the spider's velocity is increasing so the second derivative is positive, $f''(x) > 0$. When a graph is concave down, the spider would slow down so velocity is decreasing and the second derivative is negative, $f''(x) < 0$.

The discussion of change in velocity for graphs of decreasing functions (see Figure A-12) proved to be more difficult to comprehend. Subjects appeared to have very strong primitive notions of speed. They have used the term velocity to mean speed. This seemed to be the first time in their experiences that velocity depended on the direction of travel. Subjects described the movement of

the spider relative to speed – the absolute value of velocity. The distinction was made that as the spider moves faster in the negative direction, the graph that models his movement becomes steeper. Describing the movement in terms of velocity, subjects were told that the velocity becomes more negative so acceleration is negative, ie. the second derivative is negative. Subjects seemed to be having difficulty understanding the idea of negative acceleration, especially where it meant the spider was moving faster. It was suggested that they copy down the graphs of the decreasing functions and describe the spider's motion in terms of the shape of the graph. It was also suggested that the subjects determine whether the second derivative was positive or negative by looking at the slopes of tangents to each graph. They were reminded that when the slopes are decreasing, the first derivative is decreasing which implies that the second derivative is negative.

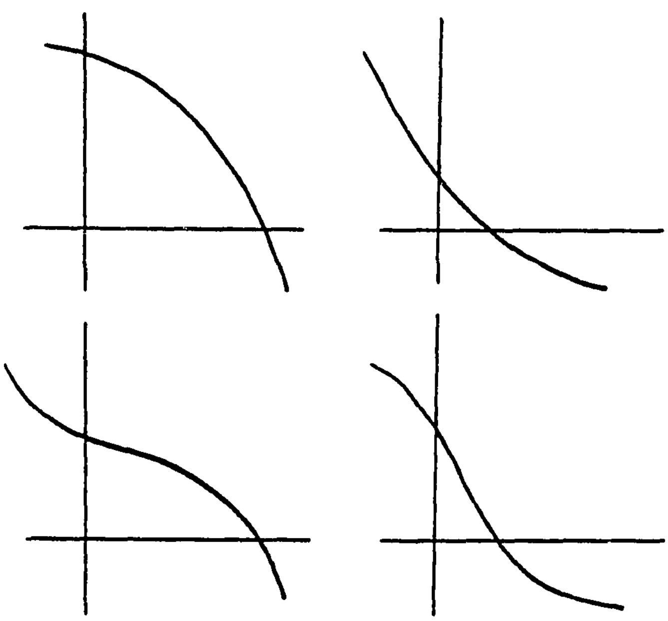


Figure A-12. Graphs of decreasing functions.

Day 31

Subjects were given Quiz 5 in the first 15 minutes of class.

A substitute instructor taught the class. He covered Section 3.1: Related Rates as prescribed in the instructor's manual. Section 2.10: The Mean Value Theorem for Derivatives was delayed for presentation until the investigator's return because of its graphical nature. Section 3.1 was not dependent on the sections immediately preceding or following it. Section 3.2: Extreme Values of a Function over a Closed Interval seemed to follow Section 2.10 well. It was determined that covering these sections out of textbook-defined order would not be detrimental to the subjects.

Day 32

The Mean Value Theorem for Derivatives was introduced with the scenario presented in the text, determining the average speed of a motorist on the Kansas turnpike knowing the amount of time it took to travel the 137 mile stretch of road from Topeka to Wichita. The Mean Value Theorem was explained through the application as guaranteeing that, at some time during the motorist's travel, he had to have been traveling at the same rate of speed as his average speed for the time interval of travel.

The Mean Value Theorem was then interpreted graphically, drawing a graph of a function that could model the motorist's position function. The average velocity over the time interval $[a, b]$ was determined to be the slope of the secant

containing the points $(a,\ f(a))$ and $(b,\ f(b))$. Subjects were asked if there was some point in time c when the slope of a tangent to the curve was the same as the slope of the secant through the endpoints. It was determined that these two lines would be parallel and that such a line existed. Subjects recalled that the slope of a tangent at $x = c$ gave the instantaneous velocity at $x = c$.

Before stating and proving the Mean Value Theorem, the Extreme Value Theorem was written on the board with subjects being asked to explain its meaning. They recalled that the Extreme Value Theorem guaranteed the existence of both a maximum and a minimum value for a continuous function $f(x)$ over a closed interval $[a,\ b]$. It was then shown that if f is differentiable on $(a,\ b)$, the slope of a tangent to f at an extremum occurring in $(a,\ b)$ is 0.

Rolle's Theorem was stated, illustrated, described graphically and proved. Using the graphs of various differentiable functions, the Mean Value Theorem, of which Rolle's Theorem is a special case, was illustrated. The points $(a,\ f(a))$, and $(b,\ f(b))$ were marked on each graph. Subjects were asked to determine if the graphs were continuous or not on $[a,\ b]$. They were also asked to determine if the functions graphed were differentiable on $(a,\ b)$. An overhead transparency with several parallel lines on it was overlayed. Subjects were asked to determine if there was a value $x = c$ in $(a,\ b)$ for which the slope of the tangent at c was the same as the slope of the secant containing $(a,\ f(a))$ and $(b,\ f(b))$. In each case, subjects suggested that at least one such c existed. They also described its position.

The Mean Value Theorem was stated and proved with subjects being asked to participate in determining various steps in the proof. The following questions were asked with subjects suggesting functions for which the Mean Value Theorem failed:

1. Why is it necessary that f be continuous on $[a, b]$? Is it possible to find a function f that is continuous on $(1, 3]$ and differentiable on $(1, 3)$ for which the Mean Value Theorem fails?

2. Why is it necessary that f be differentiable on (a, b)? Is it possible to find a function f that is continuous on $[-4, -1]$ but not differentiable for at least one value of $x \in (-4, -1)$ for which the Mean Value Theorem fails?

Subjects suggested answers to the above questions graphically. For the first question, a subject suggested that a function with an asymptote at $x = 1$ like $y = \frac{1}{x-1}$, would fail to have a maximum value at $x = 1$ so no average value for the interval could be found. Another subject suggested that a step function like the greatest integer function that is defined differently at 1 than at values close to but greater than 1 would be continuous on $(1, 3]$ but not continuous on $[1, 3]$ (see Figure A-13). The Mean Value Theorem would fail since

$$f'(c) = 0 \text{ for all } c \in (1, 3] \text{ but } \frac{f(3) - f(1)}{3 - 1} \neq 0.$$

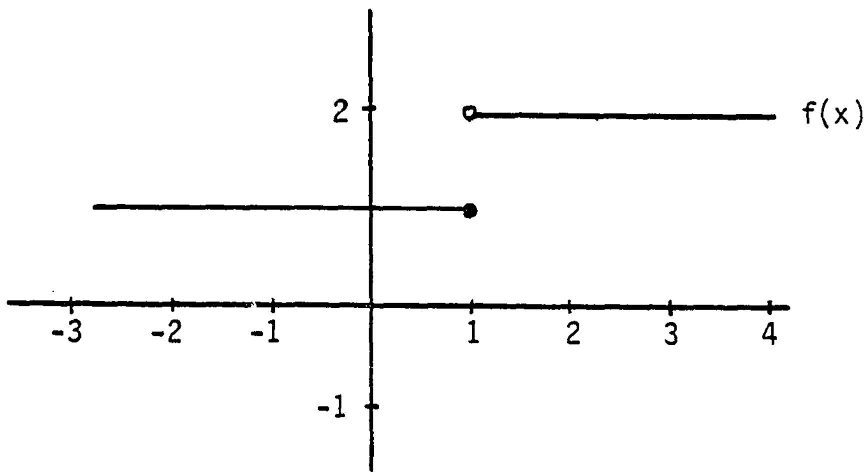


Figure A-13. Function for which the Mean Value Theorem fails at $x = 1$.

For question (2), subjects suggested that $y = |x|$ moved to the left 2 units, $f(x) = |x + 2|$ would have a corner in it so it would not be differentiable at $x = {}^{-}2$. They recognized that the slope of any tangent for $c \in (-4, -1)$ would be 1 or -1 but that

$\frac{f(-1) - f(-4)}{-1 - (-4)} = \frac{1 - 2}{3} = \frac{-1}{3} \neq \pm 1$ indicating that the Mean Value Theorem fails.

Day 33

Assigned Problems on related rates were discussed. The discussion of the Mean Value Theorem was continued by investigating the following problems both graphically and symbolically:

1. Let $m(x) = \frac{1}{x}$ on $[1, 4]$. Find $c \in (1, 4)$ whose existence is guaranteed by the Mean Value Theorem.

2. Let $\ell(x) = |x|$ on $[-1, 1]$. Show that there is no point $c \in (-1, 1)$

such that the Mean Value Theorem holds for the function $\ell(x)$ on the interval $(-1,\ 1)$. Which hypotheses of the Mean Value Theorem fail to hold?

3. Repeat problem (2) for $m(x) = \frac{1}{x}$ on $[0,\ 4]$.

Section 3.2: Extreme Values of a Function Over a Closed Interval was discussed briefly, defining absolute and local maximum and minimum, extreme value versus extreme point and critical points. Subjects recognized that this section seemed to be a review of ideas that had been discussed earlier, the only difference being the introduction of more "proper" terminology.

Subjects were asked to describe the characteristics of the graph of a function at its maximum and/or minimum points. They suggested that a maximum or a minimum could occur at an endpoint or at a place where the slope of a tangent to the graph was zero. Shown the graph of $y = |x|$ subjects also suggested that a maximum or a minimum could occur at a point where the derivative did not exist.

It was stated, that in real life applications, there are many occasions where the interest is to optimize a quantity. For example, car manufacturers are interested in maximizing the internal space of the vehicle being designed, or minimizing the amount of fuel necessary to operate the vehicle. Subjects are interested in maximal grade point averages and minimal tuition.

Subjects were asked how they could determine if a point was a local or absolute maximum or minimum. They suggested that one should find and compare the function values for endpoint and critical points. They suggested that these values

would determine if a point was an extreme point. Given the function $y = x^3$, subjects determined that an $f'(0) = 0$ but that an extreme point does not exist at $x = 0$. Asked to be more specific on how to determine extreme points, subjects suggested that the values of the derivatives for x on both sides of a critical point could be checked to see if the function changed direction. Determining maximums and minimums from a graph was elementary. Subjects seemed to be thinking graphically when stating rules for using the first derivative to determine maximums and minimums.

Day 34

Assigned problems on related rates were discussed. The Mean Value Theorem was reviewed in the process of discussing assigned exercises. Subjects responded to Quiz 6.

Day 35

Section 3.3: Increasing and Decreasing Functions/The First Derivative Test and Section 3.4: Concavity/The Second Derivative Test were discussed with subjects recognizing the material from earlier graphical work. The statement of the First Derivative Test was given. Subjects were asked not to memorize it, but to use what they knew about the signs of the derivative for increasing and decreasing functions to determine maximums and minimums.

The term "inflection point" was defined and discussed graphically. Subjects were asked to describe the concavity of a differentiable function at a maximum

and a minimum point. They concluded with the content of the Second Derivative Test. The Second Derivative Test was stated. Again it was suggested that subjects not memorize the rule, but reason from their knowledge of the second derivative and its graphical interpretation.

Subjects were asked if a maximum or a minimum must occur when the first derivative is zero. They were given the examples $f(x) = x^2$, $f(x) = x^3$, and $f(x) = -x^4$ and asked to discuss the shape of $f(x)$ at $x = 0$. In each case subjects determined that $f'(0) = 0$. They also determined that $f''(0) = 0$ so the function was not concave up or concave down at $x = 0$. Subjects were reminded that concavity is a property of intervals, not of points and that when the first derivative is 0 at $x = c$, the second derivative does not give enough information to determine concavity at c.

Subjects had a week long break from classes between Days 35 and 36.

Day 36

Exam 2, with an answer key, and Quizzes 5 and 6 were returned to subjects. Problems on Quizzes 5 and 6 were reviewed with a discussion of the first derivative and the information it gives about the function $f(x)$. Subjects were asked where $f'(x) = 0$ for a quadratic function $f(x)$. They determined that the vertex of a parabola could easily be found by setting the first derivative equal to 0.

The rest of the class period was devoted to determining as much information as possible about the function $f(x) = x^4 - 4x^3 + 4x^2 - 3$ using precalculus ideas, and the first and second derivatives. Using precalculus ideas subjects

determined that since the leading coefficient of $f(x)$ was positive, $f(x)$ must open upward. They suggested possible shapes and positions of $f(x)$ graphically using the fact that it is a fourth degree polynomial. They suggested that the function was not factorable so that it would be necessary to use the Intermediate Value Theorem to determine if the function crossed the x-axis. Subjects noticed that $f(0) = -3$ so the function, which opens upward and is positive when $|x|$ is large, must have at least two roots. They agreed that the information above was about all they could determine efficiently about $f(x)$ without using the derivatives f' and f''.

Subjects found f' and set it equal to 0 to find the critical points at 0, 1, and 2. They determined $f''(x)$ and evaluated it for x equal to 0, 1, and 2 to determine concavity and maximums and minimums. To find inflection points, f'' was set equal to zero. The coordinates of the inflection points and critical points were graphed, the concavity of f on all intervals noted and also graphed.

It was suggested that such a function for $x \leq 0$ might model a profit function of a company t years after its founding. Subjects were asked to interpret the graph of $f(x)$ in terms of this application during the first year, second year, and subsequent years. Subjects were asked to suggest other possibilities for applications that might be modelled by this graph.

It was suggested that subjects read through Section 3.5: Applications Involving Extreme Values. They were asked to complete a few "simple" problems from the section. Discussion of Section 3.5 would begin on Day 37.

Day 37

The class period was spent discussing Section 3.5: Applications Involving Extreme Values. It was suggested that subjects write down all of the information given in the problem, draw and label a picture if possible, decide what information is being asked for in the problem, and carefully define any variables being used. Once an equation had been found, subjects were asked to determine a suitable domain for the problem situation. Several problems were set up using the above guidelines. The first three were solved completely and the solutions properly labelled with the appropriate units. Each problem was reread and subjects were asked to determine if the solutions were sensible.

Day 38

Section 3.6: Limits Involving Infinity was discussed. The following example described in the text (Hurley, 1987, p. 183) was used to motivate the use of limits at infinity:

> Publishing companies put out new editions of textbooks every so often since they get no revenue from the sale of used texts. If used texts flood the market, they will sell no new texts. Suppose the sale of calculus texts can be modelled by the equation:
>
> $$s(t) = \frac{20,000}{1 + \frac{1}{4}(t - 1)^2} \text{ for time } t.$$
>
> What happens as time gets large? How many texts are sold?

The function $s(t)$ was graphed on the computer using *Master Grapher*. The graph was not visible on the default screen $[-10, 10]$ by $[-10, 10]$. Subjects suggested the window $[0, 15]$ by $[0, 20,000]$. On this viewing rectangle, subjects observed that as time passed, the function was asymptotic to the x-axis. They correctly interpreted the situation as meaning that few if any, textbooks would be sold after 15 years, the function $s(t)$ was rewritten as

$$s(t) = \frac{20,00}{0.25\, t^2 - 0.5\, t + 1.25}.$$

Discussing the limit of $s(t)$ as t approaches infinity, the subjects were asked to consider the relative size of each term in the denominator as t gets very large. They correctly suggested that the constant term was minimal. They also suggested that $.5t$ would be small as compared to $.25t^2$. Since the numerator is constant, and the denominator growing without bound, the limit as t approaches infinity is 0.

The discussion of relative size of terms was continued with the discussion of $f(x) = \frac{x^3 - 10x^2 + x + 50}{x - 2}$.

This function is suggested by Waits and Demana (1987b) for discussing the behavior of rational functions. The graph of $f(x)$ was displayed on the default screen which showed three almost-vertical lines. The graph was then viewed on the rectangle $[-20, 20]$ by $[-200, 200]$ with speed set at 200 points. On this window, the asymptote at $x = 2$ appears. A brief discussion of left and right limits at $x = 2$ occurred. The zoom factor was set at $x = 10$

and $y = 100$ then the zoom-out option was used to look at $f(x)$ on the viewing rectangle $[-200, 200]$ by $[-20,000, 20,000]$. Subjects asked why the asymptote no longer appeared. They were reminded that only 200 points were being plotted along the x-axis and that the width of the x-interval was 400 units. Subjects recognized that there could be at most one point plotted on the x-interval $[1, 3]$. Subjects were asked if the function displayed resembled any function with which they were familiar. They suggested that $f(x)$ resembled $y = x^2$. The graph of $y = x^2$ was overlayed. Subjects observed that the shapes of the graphs were very close but that there was still some space between the graphs, at least near the origin, zooming-out one last time, subjects viewed $f(x)$ and $y = x^2$ on the rectangle $[-2\,000, 2\,000]$ by $[-2\,000\,000, 2\,000\,000]$. This time the graphs coincided for all but a few points. The relative magnitude of the terms in the numerator and then the terms in the denominator were discussed Subjects suggested that $f(x)$ behaves like $y = x^2$ for large absolute values of x since the dominating term in the numerator is x^3 and the dominating term in the denominator is x. The following conclusion was made:

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{x^3}{x} = \lim_{x \to \infty} x^2 = +\infty.$$

Similarly, $\lim_{x \to -\infty} f(x) = +\infty.$

Although not sensible for the application, subjects were asked to determine

$$\lim_{x \to -\infty} \frac{20,000}{0.25t^2 - 0.5t + 1.25}.$$

They noted readily that this limit is also 0.

In both of the previously discussed examples, $\lim_{x \to -\infty} f(x) = \lim_{x \to +\infty} f(x)$. To show that this is not always the case, $g(x) = \frac{2x + 4}{\sqrt{x^2 + 4}}$ was used. Similar to the development above, the relative sizes of each term were discussed. Subjects observed that $2x$ is the dominating term in the numerator and $\sqrt{x^2}$ is the dominating term in the denominator. Subjects were asked to determine $\lim_{x \to \infty} \frac{2x+4}{\sqrt{x^2+4}}$. They suggested the following:

$$\lim_{x \to \infty} \frac{2x + 4}{\sqrt{x^2 + 4}} = \lim_{x \to \infty} \frac{2x}{\sqrt{x^2}} = \lim_{x \to \infty} \frac{2x}{x} = 2$$

They were then asked to determine $\lim_{x \to -\infty} g(x)$. Several subjects suggested that these were the same. Subjects were cautioned to consider the case of $x = -2$. They observed that $\sqrt{(-2)^2} = \sqrt{4} = 2 = -(-2) = -x$. The following conclusion was made:

$$\lim_{x \to -\infty} g(x) = \lim_{x \to -\infty} \frac{2x}{\sqrt{x^2}} = \lim_{x \to -\infty} \frac{2x}{-x} = -2.$$

The definitions of limits at positive and negative infinity were given. Subjects were reminded of the definition of asymptote and specifically given a definition of horizontal asymptote. Subjects were also reminded that in order for a limit to exist, the function must approach a real number. The symbols $+\infty$ and $-\infty$ do not denote real numbers.

A four-hour review session was held during the afternoon of Day 38. Nearly all of the 70 subjects enrolled in the investigator's two sections were in attendance

during some part of those four hours.

Day 39

Subjects responded to Exam 3. They were allowed to use graphing calculators except on problem 4. They were reminded to show all of their work and to explain their solutions. Their attention was called to problem 5 in which they were given the graph of f', not f, and they were being asked to answer questions about the function f using information from f'.

Subjects took the full 50 minutes to complete the exam.

Day 40

Section 3.8: Newton's Method was the topic of discussion. Subjects were reminded that a critical step in optimization is determining where $f'(x) = 0$. Given the function $f(x) = \frac{x^6}{6} - x^2 - x$, where f' is a fifth degree equation, it was determined that it is not always an easy matter to solve $f'(x) = 0$.

Subjects were reminded that the tangent to the curve of $f(x)$ at $x = c$ approximates the curve very well for choices of x near c. The function $g(x) = f'(x) = x^5 - 2x - 1$ was graphed with *Master Grapher.* The graph shows a root of $g(x)$ in the interval $[-1, 0]$. Subjects were asked to determine the equation of the tangent to $g(x)$ at $x = 0$. This function $T(x) = -2x - 1$ was overlayed using the Transform $F(x)$ option on *Master Grapher* for the function $f(x) = x$ in the function index. The tangent line appeared to coincide with $g(x)$ on the x-interval $[-0.6, 0.6]$. The graph of $g(x)$ was redrawn on

the viewing rectangle $[-1, 2]$ by $[-2.5, 1]$. The graph of $T(x) = -2x - 1$ was again overlayed. Zooming-in on the intersections of $g(x)$ and $T(x)$ with the x-axis and redrawing $T(x)$, subjects observed that $T(x) = 0$ could be used to approximate the root of $g(x)$. The tangent line for values of x near the root of g was shown to approximate $g(x)$ fairly well. It was suggested that, by using the root of $T(x)$ which is easily found since T is linear, the process could be continued. The equation of the tangent line to $g(x)$ at $x = -\frac{1}{2}$, the root of $T(x)$, was found to be $T_2(x) = -1.6875\,(x + 0.5) - 0.03125$. $T_2(x)$ was overlayed and the observation made that $T_2(x) = 0$ gave a much closer approximation of the root of $g(x)$ than had $T(x)$. $T_2(x)$ appeared to coincide with $g(x)$ on the viewing rectangle $[-0.57035, -0.41959]$ by $[-0.06407, 0.09422]$.

Newton's Method for finding roots was derived symbolically, as explained in the text.

Using the investigator-written program NEWTON, Newton's method was illustrated graphically for $f(x) = (x - 2)^3 + 2$ on the viewing rectangles; (a) $[-10, 10]$ by $[-10, 10]$, (b) $[-1, 2]$ by $[-25, 2]$, and (c) $[0, 1]$ by $[-6, 1]$. The search for a root was begun at $x = -0.5$. NEWTON displays Newton's Method graphically, and with a table of values, showing successive approximations for the root of a function. A tangent line to the graph of a function $f(x)$ is drawn, its root found and displayed numerically. A vertical line segment is drawn from the root of the tangent to $f(x)$ and the process is repeated until the root

is found accurate to 6 decimal places.

For the above examples, Newton's Method found the root of $f(x)$ accurate to 6 decimal places in 4 or 5 iterations.

Subjects were reminded of the Bisection Method for finding roots of functions. Subjects suggested that the Bisection Method would always be successful in determining a root to a given level of accuracy. When asked if Newton's Method would always be successful in locating a root, subjects seemed to think it would. Asked to look at the iterative formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

subjects suggested that Newton's method would fail if the first derivative was 0 at $x = x_n$. It was suggested that Newton's Method could be written in such a way that division by zero would not occur. Subjects were asked to consider what was occurring graphically. They then suggested that Newton's Method would fail if the first derivative was zero since the tangent line to $f(x)$ would be parallel to the x-axis.

To illustrate a case in which Newton's Method fails, the investigator-written program NEWTON2 was used. NEWTON2 is identical to NEWTON with the exception that NEWTON2 illustrates Newton's Method for $f(x) = x^4 - 3x^3 + 2x + 1$. Setting the viewing rectangle for $[1, 2]$ by $[-3, 1]$ and beginning the search for a root at $x = 1$, subjects observed that a root is estimated to be $x = 1.26914$ after 3 iterations. Changing the viewing window to $[-2, 2]$

by $[-5, 5]$ and beginning the search for a root at $x = 0$, subjects observed that Newton's Method failed to find a root. Successive iterations cycled back and forth between values near $x = 0$ and values near $x = -0.5$ (see Figure A-14).

Subjects asked how it was possible to determine if a function had a root without looking at its graph. Other subjects suggested that the Intermediate Value Theorem could be used to locate an interval containing a root.

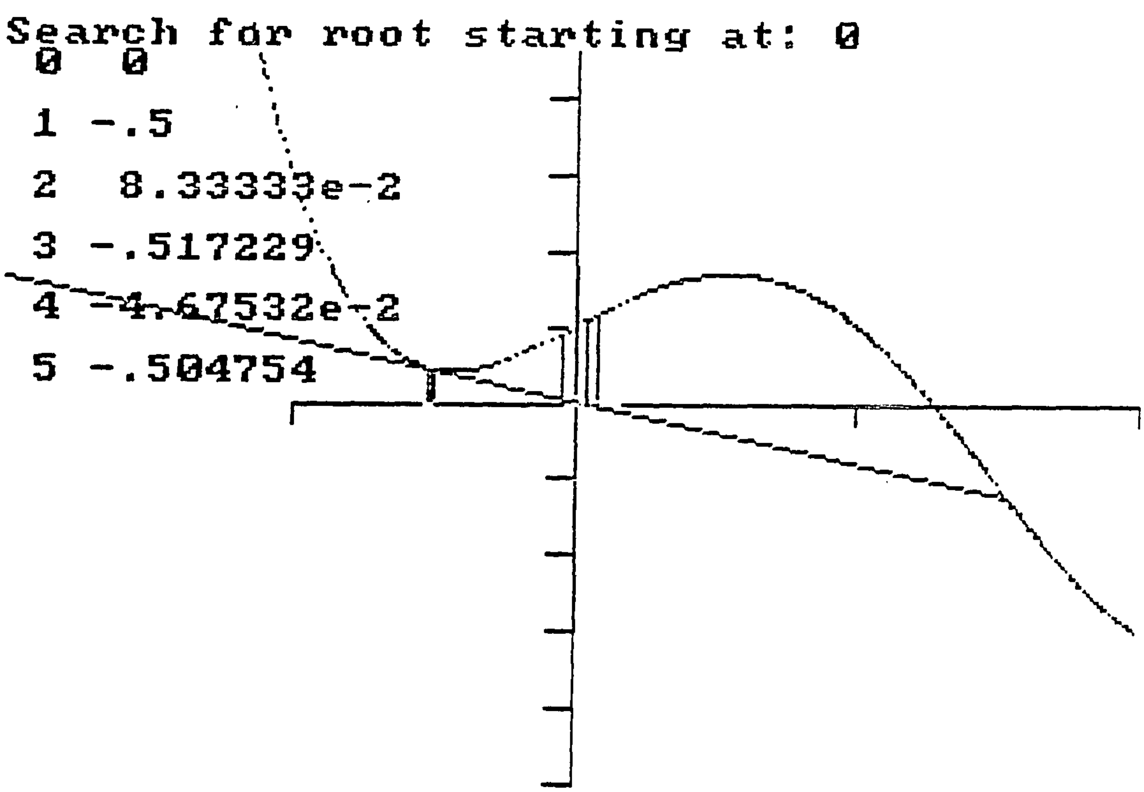


Figure A-14. Computer illustration of NEWTON2 beginning search for a root at $x = 0$.

APPENDIX B

SYMBOL REPRESENTATION SYSTEM LEVELS OF COMPETENCY

Five major processes in developing competence with written mathematical symbols are proposed by Hiebert:

> "1. connecting individual symbols with referents; 2. developing symbol manipulation procedures; 3a. elaborating procedures for symbols; 3b. routinizing the procedures for manipulating symbols; and 4. using the symbols and rules as referents for building more abstract symbol systems. The third and fourth processes are numbered together because they operate concurrently. (Hiebert, 1987)

In the present study, Hiebert's theory was used to develop competency with both the first-level Cartesian coordinate graphic representation system and the second-level symbol system of calculus. Graphic and calculus examples are provided for each of the identified processes.

Connecting Individual Symbols with Referents

Written mathematical symbols represent "quantities" or operations on quantities. The process of connecting individual symbols with referents involves developing familiarity with the form of written symbols, and establishing connections between symbols and meaningful referents. Meaningful referents can be familiar

experiential referents and/or alternate representations with which students are already familiar.

For the Cartesian coordinate graphic representation symbol system, the referent is location. Each point, labelled (x, y), in the plane represents a unique location x horizontal units and y vertical units, as indicated on horizontal and vertical number lines - axes, from a standard position–the origin $(0,0)$–at the intersection of the axes. At this level, students create meaning for combined alphanumeric and graphic symbols such as ordered pairs, axes, scales of the axes, and curves sketched by plotting and connecting points. Operation symbols for graphs include the basic operation symbols of arithmetic with which students are already familiar. In the connecting process, operations on graphs are carried out point-by- point, operating as in arithmetic, on the vertical distances of each point from the x-axis. For example, when adding two functions $y = f(x)$ and $y = g(x)$ represented graphically, the vertical distances from the origin are added for points $(x, f(x))$ and $(x, g(x))$. The graph of $h(x) = f(x) + g(x)$ contains the point $(x, f(x) + g(x))$ where $f(x) + g(x)$ is the vertical distance of $y = h(x)$ from the origin.

For calculus, the quantities being represented are functions. In the present study, students created meaning for the alphanumeric symbols representing functions through the use of the Cartesian coordinate graphic representation system and through familiar experiential referents relating distance and time. These referents were used in tandem, creating meaning for calculus symbols through qualities

of the graph which represented a distance vs. time situation.

Meaning for the operation symbol, $\lim_{x \to c} f(x)$, was created through the distance vs. time referent, discussing average velocity from the perspective of a traveler on a long journey as compared to average velocity from the perspective of a police officer observing the traveler's velocity over a small time interval. Meaning for the operation symbols $\frac{d}{dx}$ and D_x were developed through the referent used for limits which was modelled graphically and translated algebraically. As the time interval decreased, students observed that the process of finding a derivative was the process of determining the slope of a tangent line to the graph of the function, or equivalently, of determining the instantaneous velocity of the traveler. Rules for performing the operations of finding a limit or finding a derivative were not discussed. Meaning was created for each of these operations without discussion of specific rules to perform them.

Developing Symbol Manipulation Procedures

The rules for operating on symbols are developed through observing actions on their referents and translating these actions symbolically. For the graphic representation system, rules for operations on graphs are developed point-by-point using the location referent. For example, the rule for adding a constant to a function represented graphically is developed by observing that adding a constant c to each of the y coordinates – vertical distances of points on a curve from the x-axis–has the effect of translating the graph c units vertically.

For calculus, rules are developed using the graphic referent. Basic rules for

finding derivatives of functions are developed by observing the result obtained from plotting the values of slopes of tangents of a given function. For example, the rule for finding the derivative of the sine function is developed by sketching tangents to the curve of the sine function at its maximum and minimum points and at points where it crosses the x-axis. The values of each of these slopes are plotted, points are connected with a smooth curve, and the resulting graph is observed to be the cosine function.

Elaborating Procedures for Symbols

Once symbols and basic procedures for operating with symbols have been established, elaborating symbol procedures to more complex procedures can proceed, usually without the referent in mind. The referent is always available for recall whenever it is needed to check the validity or consistency of the result of an elaborate procedure.

For graphs, once the rules for the basic transformations with graphs have been developed using the location referent, the rules are combined to determine graphs of more complicated functions. For example, if $f(x) = \sin x$ then the graph of $f(x) = 2 \sin x + 5 = 2 \cdot f(x) + 5$ can be obtained from the graph of $f(x)$ by stretching the graph of $f(x)$ twice as far away from the x-axis and translating the resulting graph 5 units vertically in the positive direction. While the location referent is available to be recalled if difficulties arise or to check the result, the procedures developed in the second level of competency are used to create more complex procedures.

For calculus, procedures for finding derivatives of constant, linear, quadratic, cubic, sine and cosine functions were developed by finding slopes of tangents of these functions and observing the results. The slope of the tangent was translated symbolically as the limit of slopes of secants:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{(x+h) - x}.$$

The rules above were developed graphically and verified symbolically. The procedure for finding derivatives of sums of functions was also developed using the graphic referent for simple functions. The product and quotient rules for finding derivatives were developed using the symbolic techniques illustrated after graphic development of each of the aforementioned rules. Although not developed through a graphic referent, results of these techniques were illustrated graphically for simple functions and found to be sensible.

Routinizing the Procedures for Manipulating Symbols

A symbol system is used more efficiently if procedures necessary for its use are routinized. The graphic representation system is used more efficiently if students recall procedures for producing graphs, visually or by hand, without having to recall the location referent. Graphs that must be produced point-by-point are often not useful since the time required to produce them prohibits their use in further investigations. In calculus, the graphs of basic functions $f(x)$ such as first-, second-, and third-degree polynomials, trigonometric functions, absolute

value functions, and the greatest integer function and transformations of these, $g(x) = a \cdot f(bx + c) + d$, were often used as test cases to verify procedures and to develop the underlying meaning of definitions and theorems. Decisions about the correctness of results produced symbolically in solving problems can also be made using graphs produced either mentally or on paper.

For calculus, the first and second derivative rules were "discovered" and used long before they were presented in the course text. Students developed meaning for increasing and decreasing functions through observing a computer graphic spider climbing the right side of the screen as a graph of his position with respect to time was drawn to the left. When the meaning for the derivative was developed, students observed that the derivative was positive (negative) when the function was increasing (decreasing) since corresponding slopes of tangents were positive (negative). The derivative was determined to be zero when the spider stopped momentarily which corresponded to the graph of his position changing from increasing to decreasing. The slope of the graph at this moment was observed to be zero. In later work with optimization problems, students found derivatives and set them equal to zero to determine critical points. They tested for maximums and minimums symbolically, not as the result of a procedure that had been presented, but as a result of a procedure they had developed through use of a graphic referent. They no longer relied on the referent, and were locating optimal points for functions that would have been difficult to graph.

Building More Abstract Symbol Systems

The final process in developing competence with written mathematical systems requires using the symbols and rules as referents for building more abstract symbol systems. Essentially, the process of developing competency for written mathematical symbols cycles back to Process 1. Once competence is developed with a symbol system, it can then be used as a referent for building second-level symbol systems.

In the present study, this level was achieved for the graphic representation system, as has been illustrated by the calculus examples given for each of the prior levels in this discussion. Use of the calculus symbols and rules as referents for building more abstract symbol systems is beyond the scope of the present study.

APPENDIX C

FOCUSED HOLISTIC SCORING POINT SCALE

This scale was devised to objectively evaluate student competence and understanding in calculus.

Construction of the Scale

To construct the Focused Holistic Scoring Point Scale (FHSP Scale), the investigator combined two existing scales which were devised to evaluate student progress in problem solving (Malone, Douglas, Kissane, & Mortlock, 1980, p. 209; Charles, Lester, & O'Daffer, 1987, p. 35.) The resulting scale was modified for clarification and adaptation to a graphic and/or calculus environment through incorporation of: (a) statements adapted from the calculus error classification scheme devised by Geuther (1986, pp. 116-118), and (b) statements and phrases the investigator found to be necessary during piloting of the scale. The resulting scale was determined to be appropriate for rating student responses to routine as well as nonroutine graphic and/or calculus questions.

In the FHSP Scale, passages printed using: (a) ALL CAPITAL LETTERS were taken verbatim from Malone et al. (1980); (b) **Bold-face letters** were taken verbatim from Charles et al. (1987); and *italic letters* were adapted from Geuther's (1986) error classification scheme. The remaining words and passages were the investigator's own.

The Scale

Subjects' responses to questions on exams and quizzes were rated according to the criteria given below:

0 POINTS – NONCOMMENCEMENT

THE STUDENT IS UNABLE TO BEGIN THE PROBLEM OR HANDS IN WORK THAT IS MEANINGLESS. **These papers have one of the following characteristics: (a) They are blank. (b) The data in the problem may be simply recopied, but nothing is done with the data or there is work but no apparent understanding of the problem. (c) There is an incorrect answer and no other work is shown.**

1 POINT – APPROACH

THE STUDENT APPROACHES THE PROBLEM WITH MEANINGFUL WORK, INDICATING SOME UNDERSTANDING OF THE PROBLEM, BUT AN EARLY IMPASSE IS REACHED. **These papers have one of the following characteristics: (a) There is a start toward finding the solution beyond just copying data that reflects some understanding, but the approach used would not have led to a correct solution. (b) An inappropriate strategy is started but not carried out, and there is no evidence that the student turned to another strategy. It appears that the student tried one approach that did not work and then gave up. (c) The student tried to reach a subgoal but never did.** (d) *There are few and often inappropriate connections made between multiple representations*

and/or subprocedures in a given problem, little evidence of an overall plan is given. (e) A memorized rule is implemented incorrectly and a substrategy is performed incorrectly. (f) Two or more of the errors listed in the 2 point scale are performed.

2 POINTS – SUBSTANCE

SUFFICIENT DETAIL DEMONSTRATES THAT THE STUDENT HAS PROCEEDED TOWARD A RATIONAL SOLUTION, BUT MAJOR ERRORS OR MISINTERPRETATIONS OBSTRUCT THE CORRECT SOLUTION PROCESS. **These papers have one of the following characteristics: (a) The student used an inappropriate strategy and got an incorrect answer, but the work showed some understanding of the problem. (b) An appropriate strategy was used, but (i) it was not carried out far enough to reach a solution (eg., there were only two entries in an organized list), or (ii) it was implemented incorrectly and thus led to no answer or to an incorrect answer. (c) The student successfully reached a subgoal, but went no further. (d) The correct answer is shown, but (i) the work is not understandable, or (ii) no work is shown.** (e) *The student tried to apply a memorized rule but forgot more than a small part of it. (f) The student did not pay attention to the hypotheses of an algorithm and applied an algorithm in an inappropriate situation. (g) The student generalized inappropriately from a previous content area or previous example; the student invented a new algorithm.* (h) The appropriate algorithm is applied, some piece of the algorithm is not reconstructed correctly, but not enough evidence is given

to determine student's approach beyond choosing the algorithm. (i) In problems that require a graph to be drawn, the graph is drawn correctly, but no explanation is given.

3 POINTS – RESULT

THE PROBLEM IS VERY NEARLY SOLVED; MINOR ERRORS PRODUCED AN INVALID SOLUTION. **These papers have one of the following characteristics: (a) The student has implemented a solution strategy that could have led to the correct solution, but misunderstood part of the problem, ignored** a minor **condition of the problem, or carried out a substrategy incorrectly. (b) Appropriate solution strategies were properly applied, but (i) the student answered the problem incorrectly for no apparent reason, or (ii) the correct numerical part of the answer was given and the answer was not labeled or was labeled incorrectly,** or there was a minor misuse of notation; **(iii) no answer is given. (c) The correct answer is given, and there is some evidence that appropriate solution strategies were selected, however, either the implementation of the strategies is not completely clear,** or minor errors occurred in implementation of strategies. (d) *The appropriate algorithm is applied, but some piece of the algorithm is not reconstructed correctly. Student knows which rule to apply, but carries out some part of the rule incorrectly.*

4 POINTS – COMPLETION

AN APPROPRIATE METHOD IS APPLIED TO YIELD A

VALID SOLUTION, with perhaps only minor errors in arithmetic or algebra occurring. **These papers have the following characteristics: (a) The student made an error in carrying out an appropriate solution strategy. However, this error does not reflect misunderstanding of either the problem or how to implement the strategy, but rather it seems to be a copying or computational** (arithmetic or algebraic) **error. (b) Appropriate strategies were selected and implemented. The correct answer was given in terms of the data in the problem.**

Reliabilities

To determine the reliability of the FHSP Scale in measuring subject competence and understanding in calculus, the written work of 14 subjects chosen at random from those participating in the G and G+ sections was used. Data for these subjects included: (a) all of the questions contained in the four graphics subscales - GLEVEL2, GAPP, GSR, and GSN - analyzed in Investigation 1; and (b) all of the questions contained in the three exam and quiz subscales - APP, SR, and SN - analyzed in Investigation 2. A subset of this data was used to estimate inter-grader and intra-grader reliabilities.

The investigator initially graded 39 exam and quiz questions, at least two questions from each of the seven subscales used in the study, for each of the 14 subjects in the sample. Three months later, the investigator regraded the same 39 questions. Independently of the investigator, a professor in the WMU Department of Mathematics and Statistics also used the FHSP Scale to grade these questions.

Table C-1 lists for each subscale: (a) the number of questions graded, (b) inter-grader reliabilities, (c) intra-grader reliabilities, and (d) the significance levels of each of the reliabilities in (b) and (c). With one exception, the Spearman correlation coefficients (S_ρ) used to measure grader reliability were above 0.75. All correlations were found to be significant at the 0.01 level or below.

Table C-1

Inter-grader and Intra-grader Reliabilities
for the FHSP Scale by Subscale

Subscale	Number of Questions Graded	Inter-Grader Reliability		Intra-Grader Reliability	
		S_ρ	ρ	S_ρ	ρ
Investigation 1					
GLEVEL2	5	.7590	.002	.9218	.000
GAPP	3	.6640	.010	.8387	.000
GSR	8	.7956	.001	.9424	.000
GSN	5	.8653	.000	.7802	.001
Investigation 2					
APP	2	.9492	.000	.9513	.000
SR	11	.9150	.000	.9202	.000
SN	5	.7953	.001	.8583	.000

Validity

To further determine the usefulness and validity of the FHSP Scale, both Pearson (R) and Spearman (S_ρ) correlation coefficients were obtained for the variables CUMUL and FGRADE, for the full sample ($N = 113$) and by treatment. CUMUL is the cumulative score for each subject for exam and quiz questions used in the study, CUMUL = APP + SR + SN, determined by the investigator using the FHSP Scale. FGRADE is the final grade earned by the subject as determined by the subject's respective instructor. The S1 and S2 instructors graded exam and quiz questions independently of the investigator. They did not use the FHSP Scale in determining the FGRADE for their students. The CUMUL by FGRADE correlation coefficients are presented in Table C-2.

Table C-2

Correlation of CUMUL with FGRADE

TRTMNT	n	R	S_ρ
G, G+, S1, S2	113	.9031	.8903
G	30	.8915	.8803
G+	35	.9008	.8844
S1	26	.9109	.9007
S2	22	.9193	.9379

Note: All p-values were less than .0005.

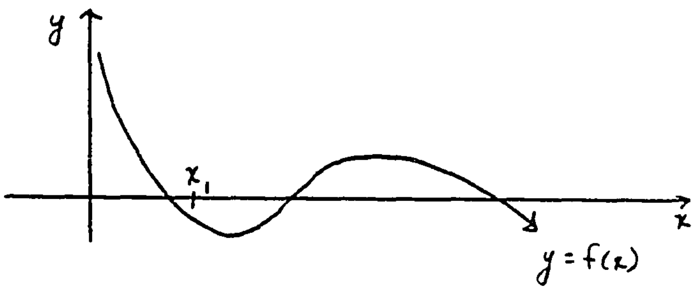
All of the Pearson and Spearman correlation coefficients were above .88. All correlations of CUMUL with FGRADE were significant at the .0005 level or below.

An Illustration

To illustrate the use of the FHSP scale, 5 subjects' responses to one particular problem from the study are given. Figure C-1 presents the problem, the individual student responses, related investigator-assigned scores, and the rationale for assigning each score as based on the FHSP Scale.

Problem:

Suppose you are using Newton's method to find a root of $f(x) = 0$ for the function f shown below. If x_1 indicates where your first approximation is, lable where (approximately) your second approximation x_2 and your third approximation x_3 will be. Your completed illustration should indicate how you located x_2 and x_3.



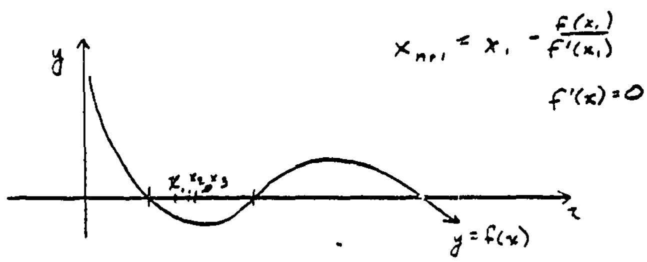
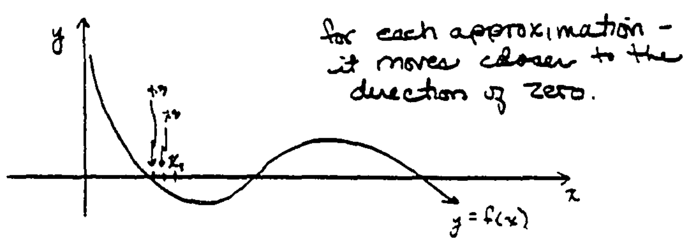
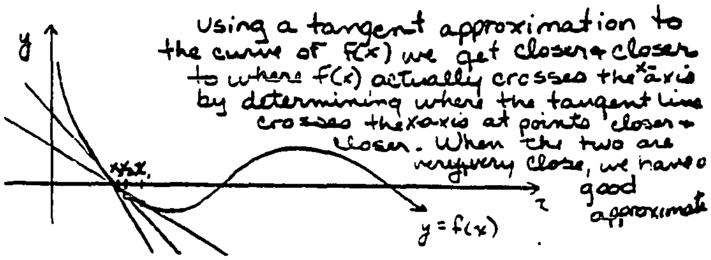
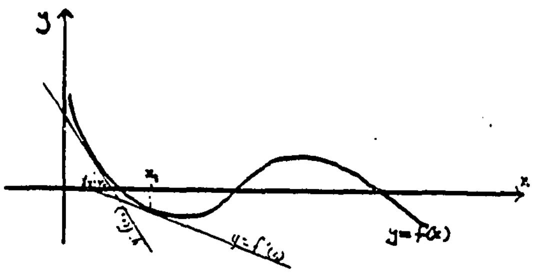
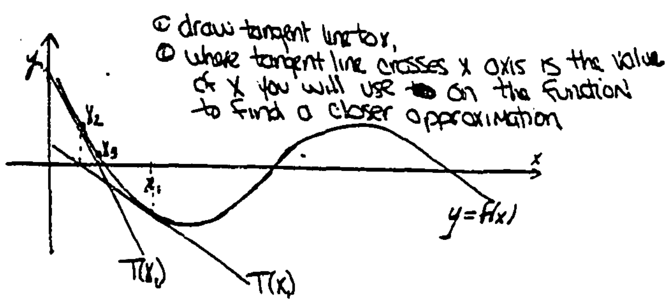
Student Response	Score	FHSP Scale Criteria
	0	0b
	1	1a

Figure C-1. Sample Implementation of the FHSP Scale.

Figure C-1. (continued)

Student Response	Score	FHSP Scale Criteria
	2	2a
	3	3bii, 3c
	4	4b

APPENDIX D

COGNITIVE MEASURES

The instruments used to assess subject academic progress in the course are contained in this appendix. These are:

1. The pretest administered to all subjects in sections G, G+, S1, and S2;

2. Exam and quiz subscales: (a) subscales consisting of exam and quiz questions given to subjects in sections G and G+ used in Investigation 1; (b) subscales consisting of the exam and quiz questions given to all subjects in sections G, G+, S1, and S2 used in Investigation 2; and (c) cross-reference for exam and quiz questions.

3. The departmental final exam administered to all subjects in sections G, G+, S1, and S2.

The Pretest

The pretest is a multiple-choice assessment of subject precalculus manipulative skill and graphing ability. The pretest was used to assess pre-experimental differences among treatment sections and as a covariate in further analyses.

This precalculus test consists of 20 multiple choice items. These items were either investigator-written or adapted from questions found in various sources to cover a wide spectrum of precalculus symbolic and graphic abilities. The sources were: (a) the Cooperative Mathematics Tests (ETS), Algebra II, Algebra III, and

Calculus – questions 5, 9, and 11; b. Heid's dissertation – question 14 (1984, p. xix); (c) Janvier's dissertation (1978, p. A2.4) – questions 6 and 16; (d) Pedersen and Ross (1985) – questions 2, 12, 18, and 20; (e) Patterson's dissertation (1983, p. 17) – question 17; (f) WMU's Mathematics Placement Exam – questions 7, 15, and 19; and (g) investigator-written questions – numbers 1, 3, 4, 8, 10, and 13.

The pilot pretest of 30 questions was administered during the Fall 1987 semester, and an item analysis was performed. Inappropriate or ambiguous questions were removed or rewritten. The pretest was then examined for content validity by a group of five faculty members in the Department of Mathematics and Statistics at WMU.

An item-analysis was performed on the pretest given to all subjects ($N = 128$) in the study. This item analysis indicated that questions 9, 12, 16 and 17 had very low discrimination indices. Questions 9 and 16 proved to have very low difficulty indices and questions 12 and 17 proved to have very high difficulty indices. These questions were subsequently removed from the instrument. The reliability of the resulting precalculus test was calculated as .56 using the Kuder-Richardson Formula 20.

Exam and Quiz Subscales

Each subscale is a free response assessment of subject understanding of calculus and/or a graphic representation system.

The exam and quiz questions that comprise each of the subscales came from a variety of sources. These include: (a) questions written by the S1 and S2 instruc-

tors and the investigator; (b) the investigator's exams from previous semesters, including those used in the pilot study; (c) the S1 instructor's previous exams and quizzes; (d) WMU departmental final exams; (e) exams and quizzes published in Heid's dissertation (1984); (f) exams from former Advanced Placement Calculus AB and BC exams (Steen, 1988); (g) Janvier's dissertation (1978); (h) Pedersen's and Ross's article (1985); and (i) Cooperative Testing Services Calculus Exam.

A bank of questions compiled by the investigator from the above sources was shared with the S1 and S2 instructors. The three instructors met weekly to discuss, among other things, potential exam and quiz questions. Only questions that were judged as appropriate, by all three instructors for all four sections were used as common questions for the study. Instructors wrote their own exams and quizzes and inserted the common questions within these. Each question used in the study was photocopied and then graded by the investigator using the Focused Holistic Scoring Point Scale given in Appendix C.

Investigation 1: Exam and Quiz Subscales

The following exam and quiz questions were used to assess subject levels of understanding of a Cartesian coordinate graphical representation system, including the use of such a representation system in understanding calculus concepts.

Each of these exam and quiz questions required the use of a graphic representation. They were used in comparisons between the G and G+ sections. Such questions were categorized into the two components of symbol systems identified below (and elaborated in Appendix B):

1. Level 2: The development of symbol manipulation procedures for graphs (GLEVEL2), and

2. Level 4: The use of symbols and rules as referents for building more abstract symbol systems.

Level 4 questions were further categorized as Applied (GAPP), symbolic routine (GSR), and symbolic nonroutine (GSN).

Each item is labeled to indicate on which exam or quiz it appeared. The numbers, $Ei.j.$ or $Qi.j.$, indicate that the problem is question j on Exam i ($Ei.j.$) or Quiz i ($Qi.j.$).

Level 2: The Development of Symbol Manipulation Procedures for Graphs (GLEVEL2)

Questions included in this subscale are graphic and assess subject understanding of rules for manipulating graphic symbols.

Q1.1.

For the graphs of $f(x)$ and $g(x)$ given below, if

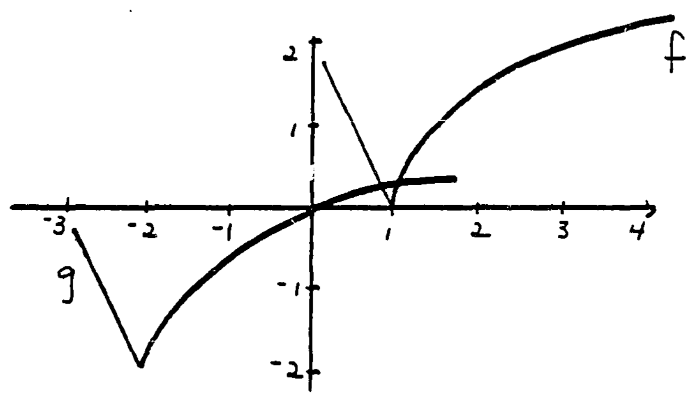
$$g(x) = a \cdot f(x+c) + d$$

what are $a, c,$ and d?

a = ______

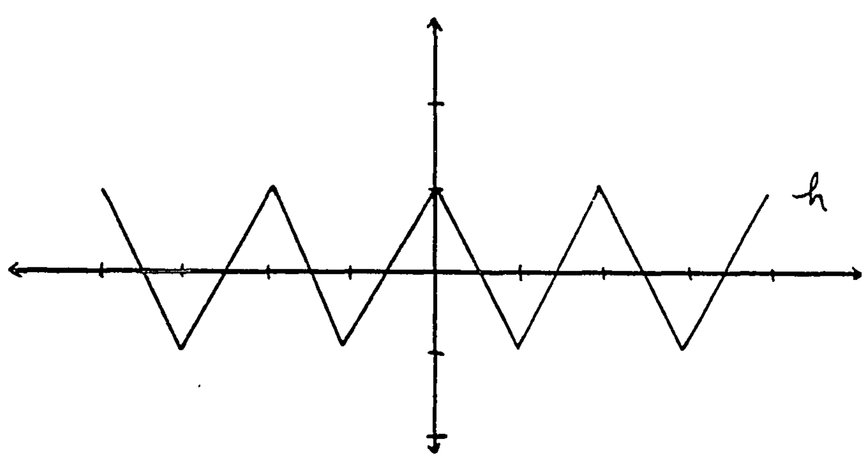
d = ______

c = ______



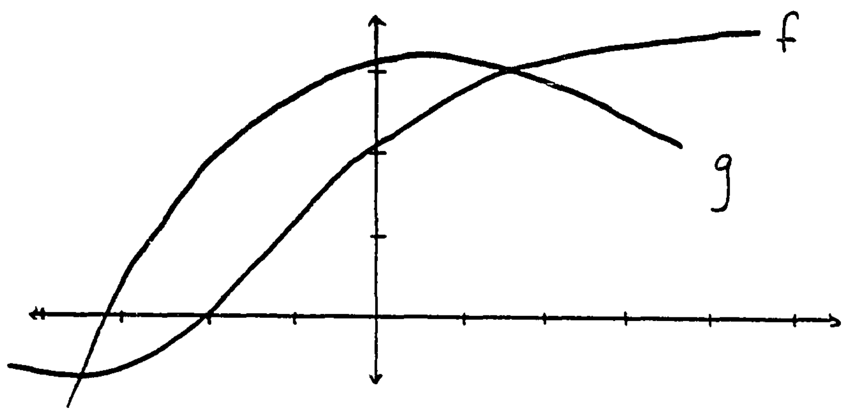
Q1.3.

The function $h(x)$ graphed below is periodic. Sketch $2 \cdot h(x)$ on the same set of axes.



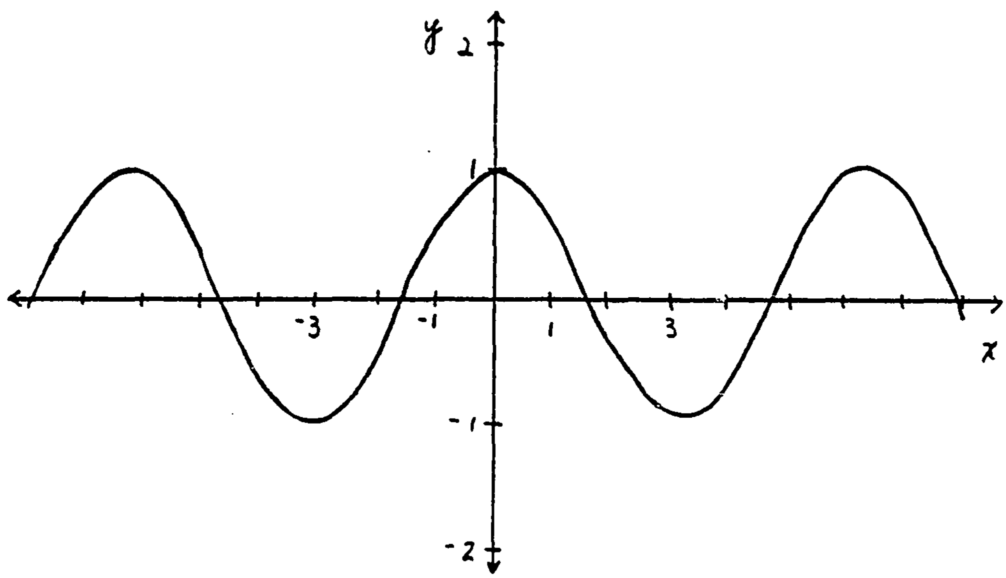
Q1.4.

Let $f(x)$ and $g(x)$ be the functions graphed below. Assume $h(x) = f(x) - g(x)$. Locate the point(s) $(x,\ h(x))$ on the coordinate plane where $h(x) = 0$. Explain how you found these points.



E1.2.

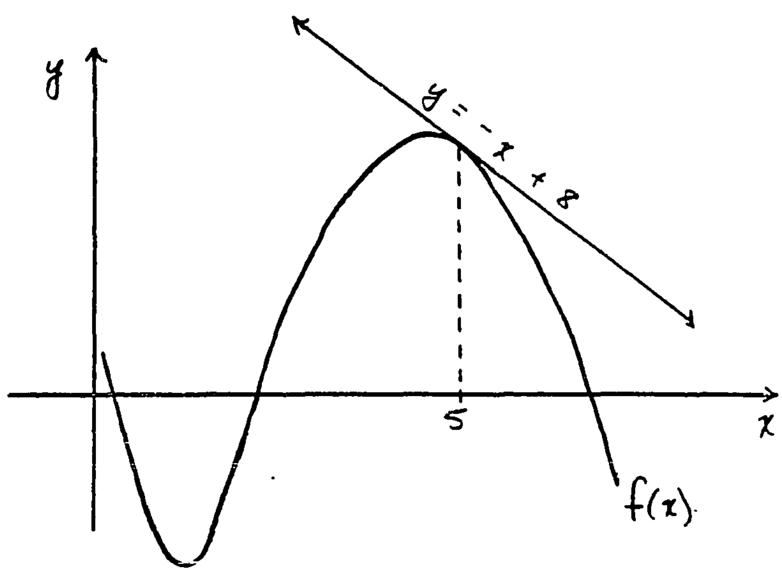
The graph of $f(x) = cos\ x$ is given below. Sketch the graph of $sec\,x = 1 / \cos x$ on the same coordinate system. Sketch vertical asymptotes to aid in your drawing.



Q3.1.

Consider the curve $y = f(x)$ in the accompanying sketch. The line $y = -x + 8$ is tangent to $f(x)$ at $x = 5$.

a. Find $f(5)$



Level 4: The Use of Symbols and Rules as Referents for Building More Abstract Symbol Systems

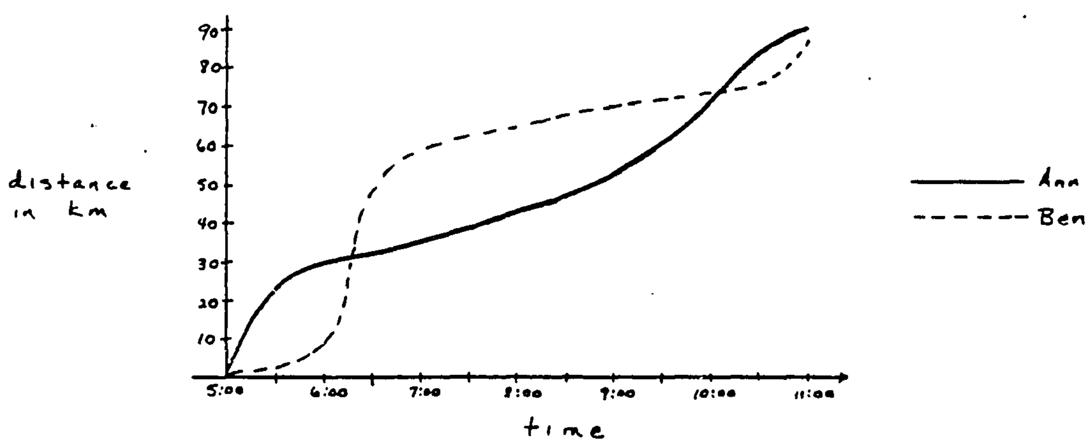
Level 4 questions are graphic and assess subject use of a graphic representation system in understanding the concepts of calculus.

Applied (GAPP)

Questions included in this subscale are graphic. They each required application to a real-world model.

E1.3.

The following graph describes the distances Ann and Ben traveled.



a. During which time interval(s) had Ann traveled further than Ben? Explain.

b. At what time did Ann travel her fastest? Explain.

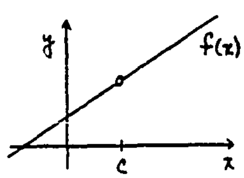
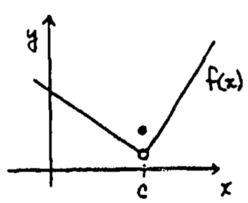
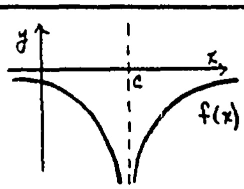
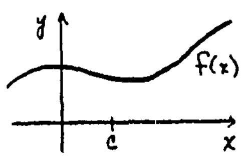
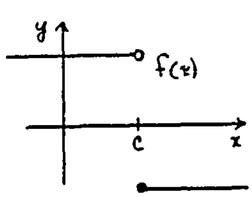
c. Who traveled faster between 8:00 and 9:00, Ann, Ben or neither? Explain.

Symbolic Routine (GSR)

Questions included in this subscale are graphic. They are similar to questions that appeared in the course presentation and/or on assignments. They did not require application to a real-world model.

E1.7.

Complete the chart for each function graphed below.

	f is defined at c (Yes or No)	$\lim\limits_{x \to c} f(x)$ exists (Yes or No)	f is continuous at c (Yes or No and briefly EXPLAIN using the definition of continuity.)
			
			
			
			
			

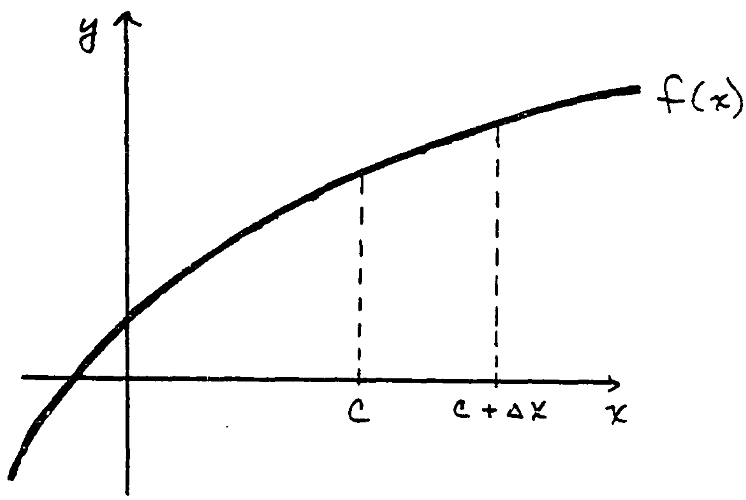
E2.6.

In the figure below, draw the tangent to the curve of $y = f(x)$ at $x = c$.

Indicate Δy and dy.

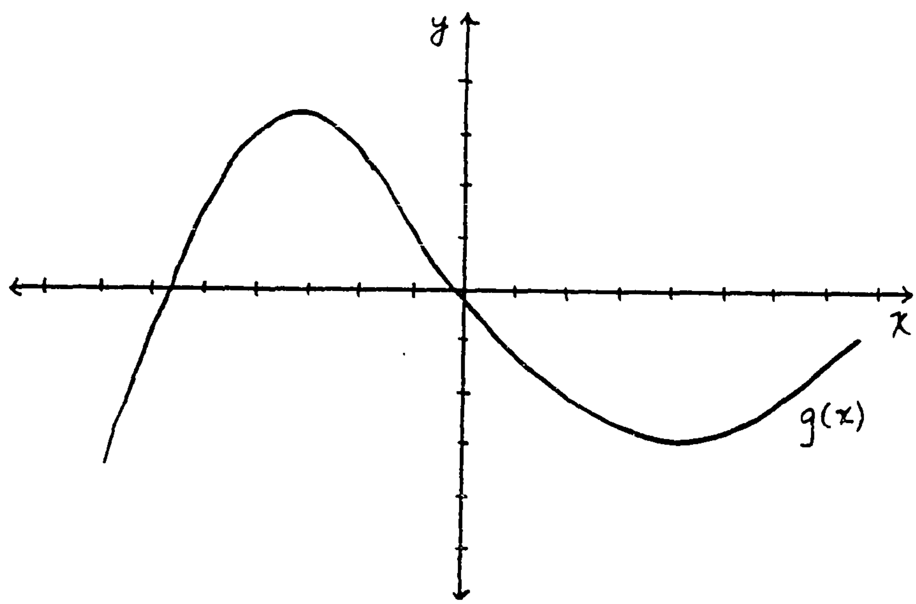
i. dy

ii. dy



E2.9.

Given $g(x)$ as graphed below, sketch $g'(x)$ on the same set of axes.

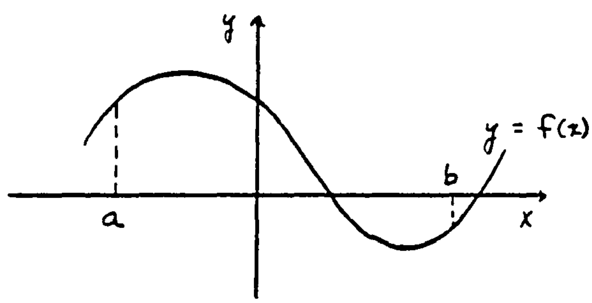


E3.2.

Complete the following statement: The mean value theorem implies that if $f'(x)$ exists for $a \leq x \leq b$, then there exists at least one number c between a and b such that ____________________________.

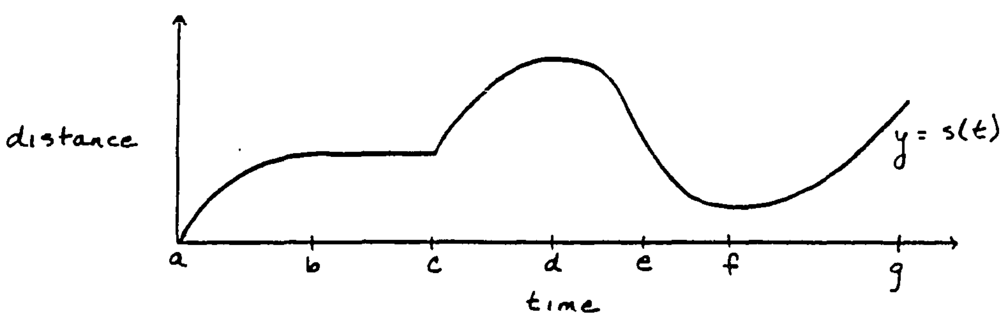
(Note: This question is not part of the analyses for G vs. G+ sections. It was included here to make sense of E32b which is part of the analysis.)

b. How many such numbers c are there for the function shown below? Mark on the x-axis approximately where each such number c is. EXPLAIN.



E3.3.

The following is the graph of a function of distance vs. time. Estimate the following times and/or intervals of time. EXPLAIN your answers.



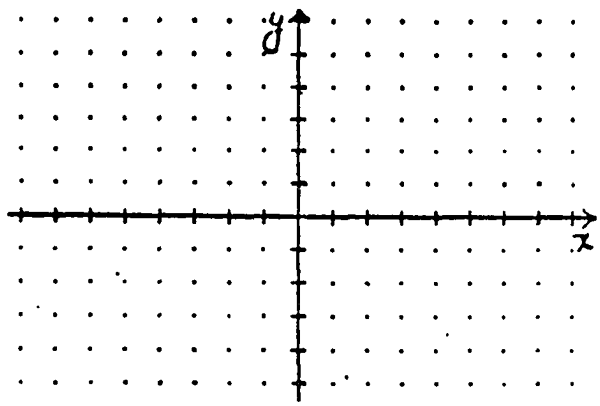
e. At what times are there points of inflection for the graph of $s(t)$?

E3.4.

Answer the following questions for the function

$$y = x^3 - 6x^2 + 9x - 3.$$

d. Sketch the graph using the information obtained in a, b, and c.

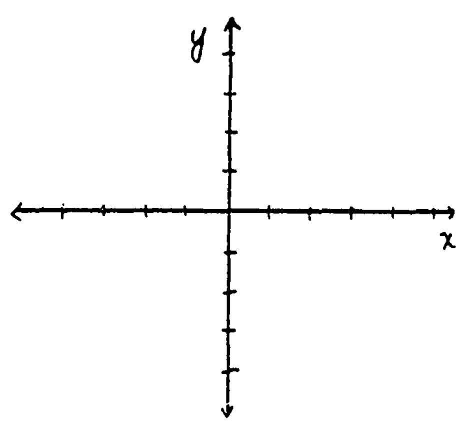


Q7.2.

Determine all horizontal and vertical asymptotes of

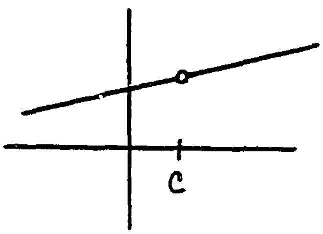
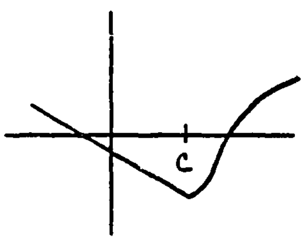
$$f(x) = \frac{3(x - 2)}{x^2 - 4}.$$

g. Sketch f on the axes below.



E4.1.

Complete the chart for each function given below. In each square, state true or false and briefly explain.

$f(x)$	$\lim\limits_{x \to c}$ $x \to c$ exists	$f(x)$ is continuous at $x = c$	$f'(c)$ exists
			
			

Symbolic Nonroutine (GSN)

Questions included in this subscale are graphic. They are nonroutine in that subjects attempting such problems possessed neither a known answer nor a previously established (routine) procedure for finding one. They did not required application to a real-world model.

Q2.2.

Give an example of a function $f(x)$ that is defined on the interval $(1, 3)$ except at $x = 2$ for which

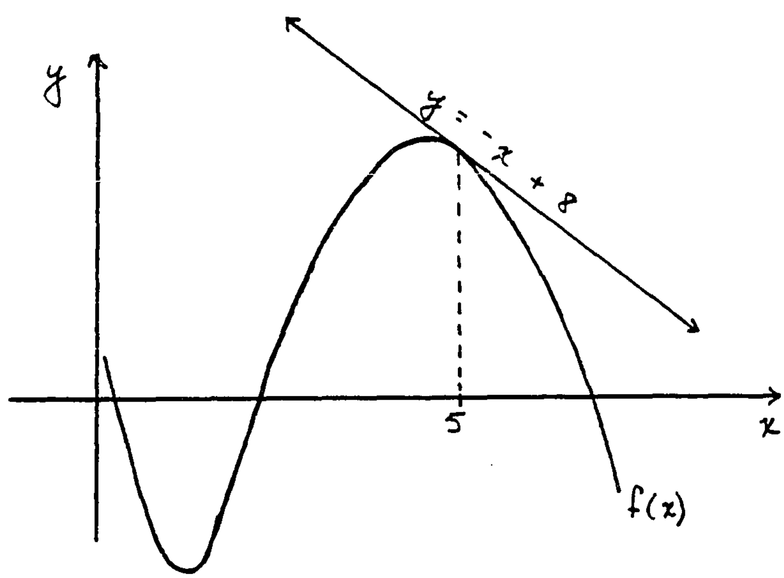
$$\lim_{x \to 2} f(x) \text{ exists.}$$

Explain your choice. (Your function may be given in graphical form, however, you must explain how it satisfies the above conditions.)

Q3.1.

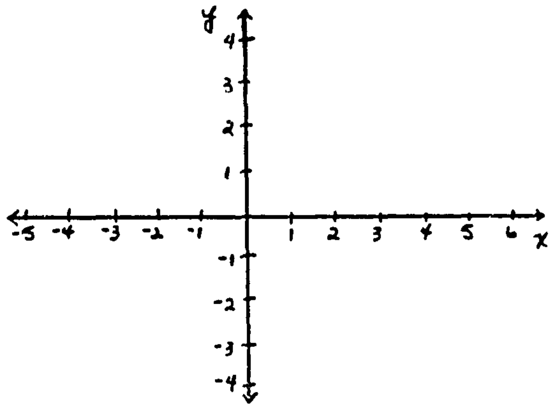
Consider the curve $y = f(x)$ in the accompanying sketch. The line $y = -x + 8$ is tangent to $f(x)$ at $x = 5$.

b. Find $f'(5)$



E2.8.

Sketch the graph of a function which is continuous for all x but is not differentiable at $x = -1$. Explain why the function whose graph you sketched fails to be differentiable at $x = -1$.



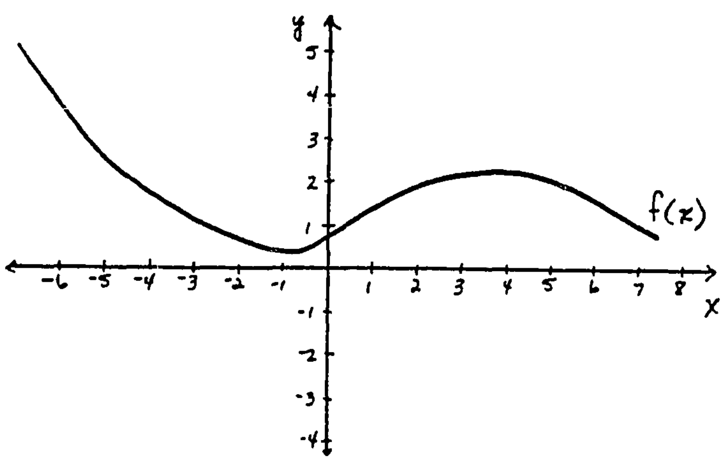
E2.10.

Sketched below is the graph of a function f. Suppose another function g has the following properties:

$a.$ $g(-1) = -2$ and

$b.$ $g'(x) = f'(x)$ for all x.

Sketch the graph of g using the same axes.

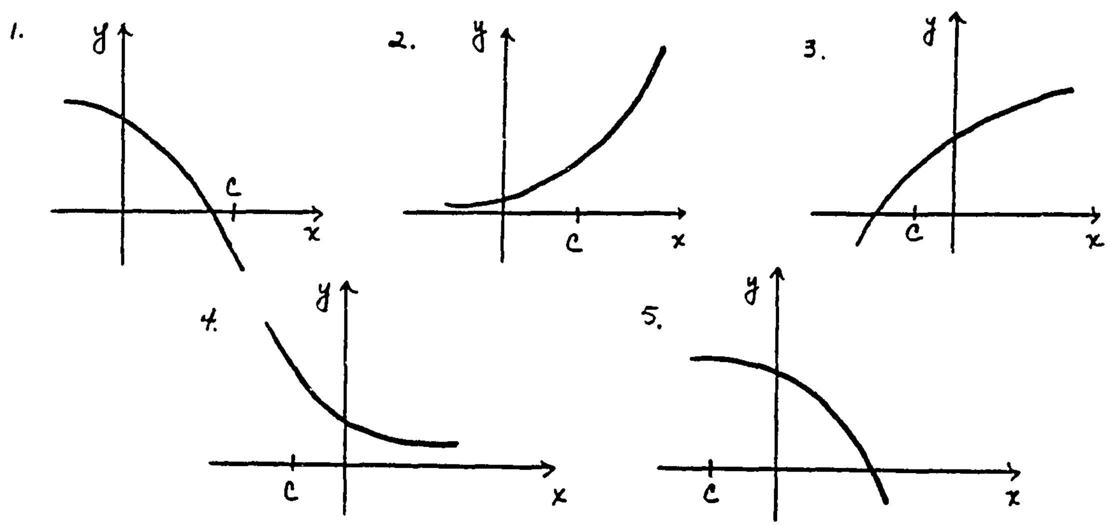


Q6.3.

Indicate which of the graphs below correspond to a function f that has the following properties:

$$f(c) > 0, \quad f'(c) < 0, \quad f''(c) > 0 \quad \text{at the point } x = c.$$

Briefly, support your answer.

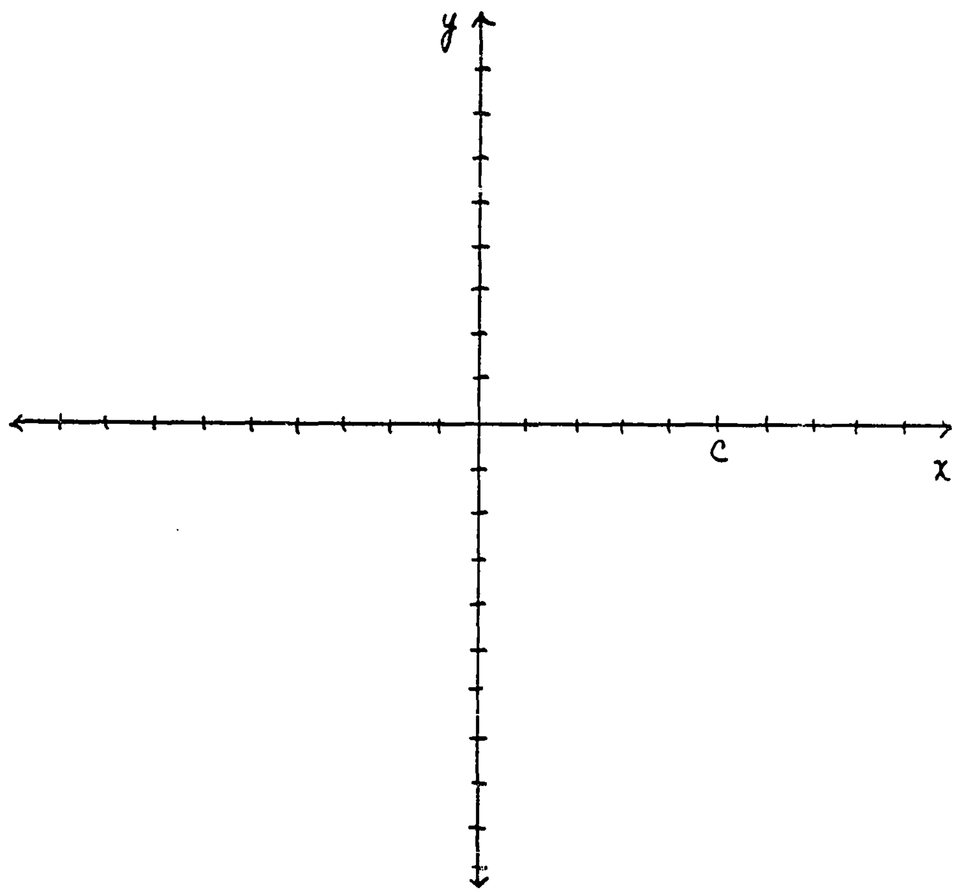


E3.1.

On the axes below, sketch the graph of a function $f(x)$ with the properties:

i) $f(c) < 0$ ii) $f'(c) < 0$ iii) $f''(c) > 0$

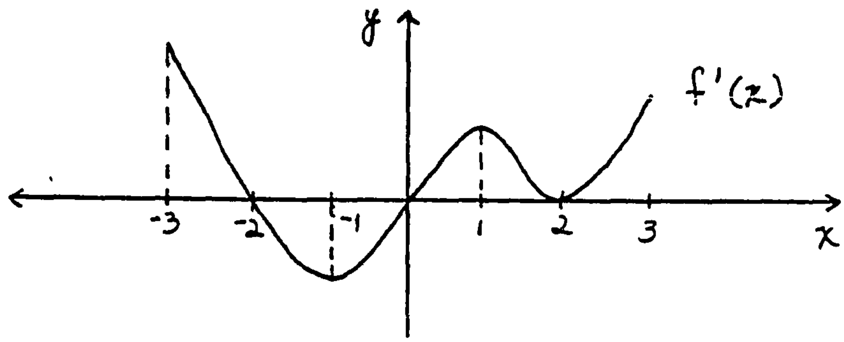
EXPLAIN!



E3.5.

The figure below shows the graph of f', the derivative of a function f.

The domain of f is the set of all $x \in [-3, 3]$.



NOTE: This is the graph of the <u>derivative</u> of f, not the graph of f.

a. For what values of x, $-3 < x < 3$, does f have a local maximum? a local minimum? EXPLAIN.

i. maximum

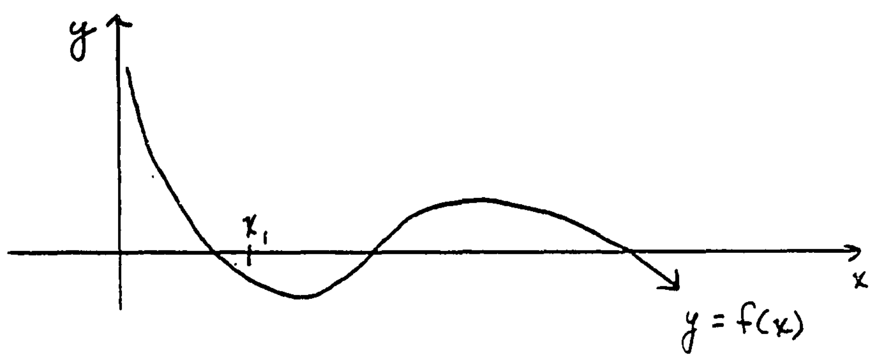
ii. minimum

b. For what values of x is the graph of f concave up? EXPLAIN.

c. For what values of x does f have a point of inflection? EXPLAIN.

Q8.1.

Suppose you are using Newton's method to find a root of $f(x) = 0$ for the function f shown below. If x_1 indicates where your first approximation is, label where (approximately) your second approximation x_2 and your third approximation x_3 will be. Your completed illustration should indicate how you located x_2 and x_3.



Investigation 2: Exam and Quiz Subscales

Following are the collection of the exam and quiz questions given to all subjects in sections G, G+, S1, and S2. These questions were used to assess subject understanding of calculus. They were categorized as applied (APP), symbolic routine (SR), and symbolic nonroutine (SN).

The numbering $Ei.j.$ or $Qi.j.$ correspond to the question j on Exam i ($Ei.j.$) or on Quiz i ($Qi.j.$) as given to the subjects in the G and G+ sections. A cross-reference is provided indicating for the S1 and S2 sections, the day on which the question was given to subjects in these sections. The corresponding exam or quiz question numbers are also indicated. Certain questions were used in both Investigation 1 and Investigation 2.

Applied (APP)

Questions included in this subscale require application to a real-world model.

E2.7.

Suppose a rectangle has dimensions of x inches by $2x$ inches. Then the area of the rectangle is given by

$$A = 2x^2 \text{ in}^2$$

a. Use differentials to approximate the change in area if x changes from 10 inches to 10.01 inches.

b. What is the actual change in area given by the conditions of (b)?

E3.6.

The body of a snowman is in the shape of a sphere and is melting at a rate of $2 \text{ ft}^3/\text{hr}$. How fast is the radius changing when the body is 3 feet in diameter (assuming the body stays spherical)?

a. Determine an equation relating volume and radius. Determine an equation relating their rates of change.

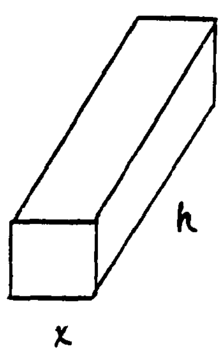
b. Substitute appropriate values into related rates equation. Interpret the solution in terms of the application.

E3.7.

Postal requirements specify that parcels must have length plus girth of at most 84 inches. Find the dimensions of the square-ended rectangular package of greatest volume that is mailable. Note: The length plus girth is $h + 4x$.

a. Determine an equation to be maximized and interpret the solution in terms of the application.

b. Symbolic Routine - technique in finding solution.



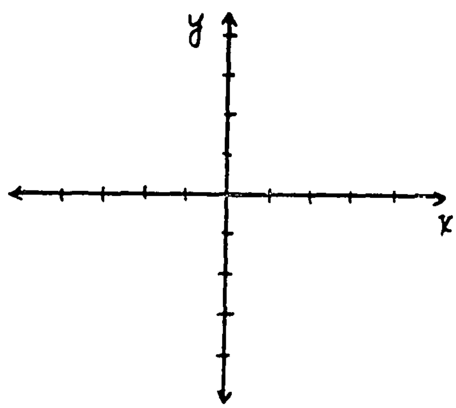
Symbolic Routine (SR)

Questions included in this subscale are similar to questions that appeared in the course presentation and/or on assignments. They did not require application to a real-world model.

Q2.3.

g. Sketch the graph of

$$f(x) = \begin{cases} x^2 - 3 & \text{if } x < -1 \\ \frac{x^2 - 1}{x + 1} & \text{if } x > -1 \end{cases}$$



Find the following limits if they exist. Explain your answers.

a. $\lim_{x \to -1^+} f(x)$

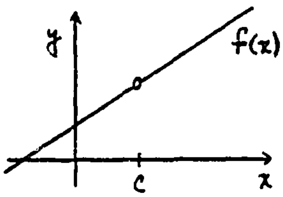
b. $\lim_{x \to -1^-} f(x)$

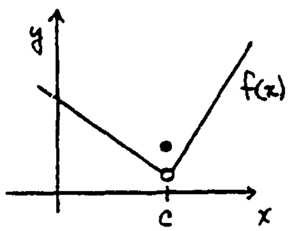
c. $\lim_{x \to -1} f(x)$

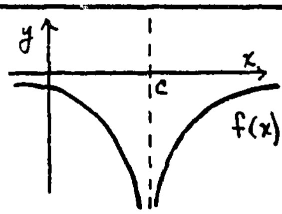
E1.7.

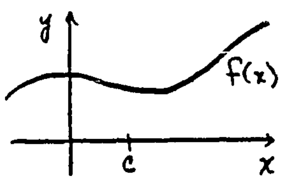
Complete the chart for each function graphed below.

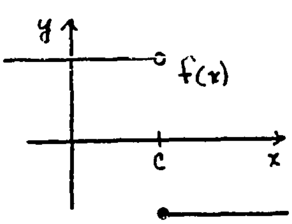
	f is defined at c (Yes or No)	$\lim_{x \to c} f(x)$ exists (Yes or No)	f is continuous at c (Yes or No and briefly EXPLAIN using the definition of continuity.)











E1.8.

Suppose $f(x) = (x^2 - 16)/(x - 4)$ for $x \neq 4$. How should $f(4)$ be defined so that f is continuous at $x = 4$?

E2.1.

Let $f(x) = \sqrt[3]{x^2}$.

a. Write $f'(x)$ without negative exponents.

b. What is the domain of $f'(x)$? Explain.

E2.2.

What is the slope of the tangent line to the graph of

$$f(x) = \frac{\cos(x)}{\sin(x) + 1}$$

at $x = 0$?

E2.5.

Determine the limit:

$$\lim_{x \to 0} \frac{1 - \cos(5x)}{3x}.$$

Q5.2.

Let $x^2 + y^2 = 9$. Using implicit differentiation:

a. Find an expression for y'.

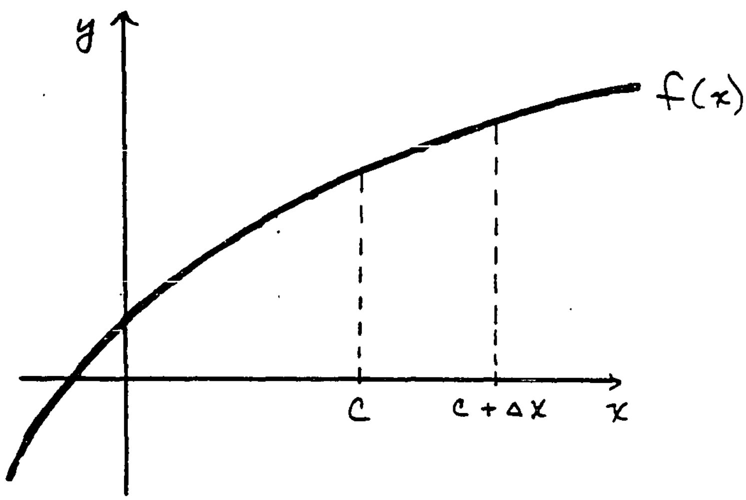
b. Find an expression for y''. Use the expression for y' to write y'' in terms of x and y.

E2.6.

In the figure below, draw the tangent to the curve of $y = f(x)$ at $x = c$. Indicate Δy and dy.

i. dy

ii. δy

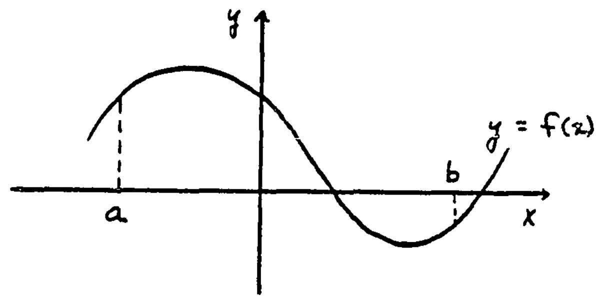


E3.2.

a. Complete the following statement: The mean value theorem implies that if $f'(x)$ exists for $a \leq x \leq b$, then there exists at least one number c between a and b such that ____________________.

(This question is categorized as Symbolic Nonroutine but is included here to help make sense of E32b which follows.)

b. How many such numbers c are there for the function shown below? Mark on the x-axis approximately where each such number c is. EXPLAIN.



c. Illustrate the mean value theorem for the function $f(x) = x^3 - x - 2$, and the interval $[-1, 2]$ by finding a number c in $(-1, 2)$ that satisfies the theorem.

Q6.1.

What is the minimum value of the function $f(x) = x^2 - 5x + 4$ on the interval $[1, 2]$.

E3.4.

Answer the following questions for the function

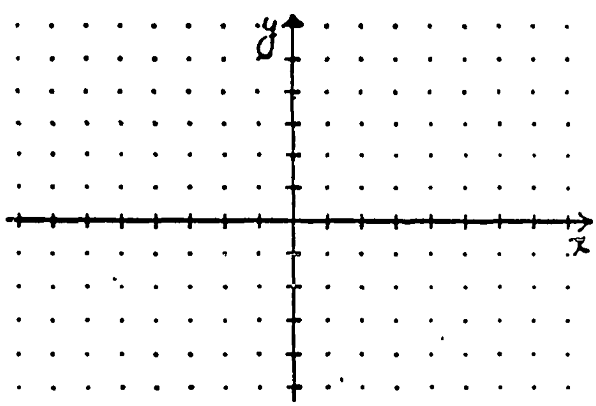
$$y = x^3 - 6x^2 + 9x - 3.$$

a. What are all the coordinates of all local minima and all local maxima (if any)? Justify your answer.

b. What are the coordinates of all points of inflection (if any)? Justify your answer.

c. On what intervals is the function concave up? On what intervals is the function concave down? EXPLAIN.

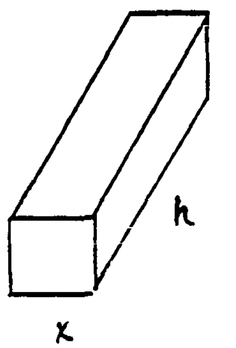
d. Sketch the graph using the information obtained in a, b, and c.



E3.7.

b. Postal requirements specify that parcels must have length plus girth of at most 84 inches. Find the dimensions of the square-ended rectangular package of greatest volume that is mailable. Note: The length plus girth is $h + 4x$.

(NOTE: This question was graded in two parts. E3.7.b. was an assessment of students technical facility after the problem was set up. Set up and interpretation were graded as E3.7.a.)



Q7.1.

Determine whether or not each of the following is true or false. Explain your choice.

If $f(x) = x/\sqrt{x^2 + 3}$, then

a. $y = 1$ is a horizontal asymptote for $f(x)$.

b. $\lim_{x \to -\infty} f(x) = 1$.

Q7.2.

Determine all horizontal and vertical asymptotes of

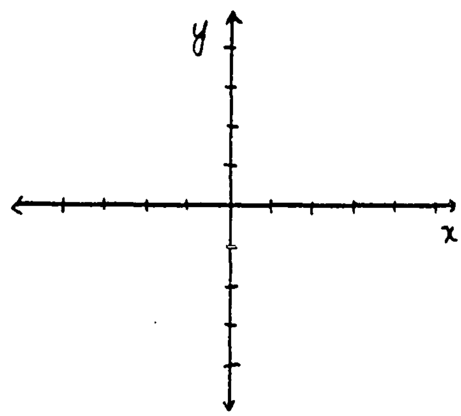
$$f(x) = \frac{3(x - 2)}{x^2 - 4}.$$

Sketch f on the axes below.

h. horizontal asymptotes

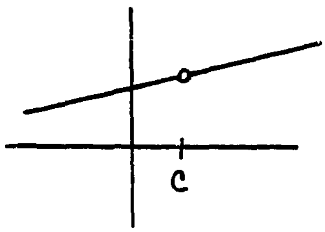
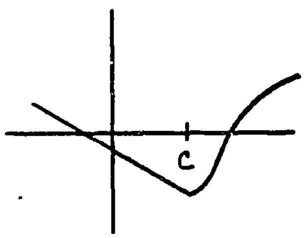
i. vertical asymptotes

g. graph



E4.1.

Complete the chart for each function given below. In each square, state true or false and <u>briefly</u> explain.

$f(x)$	$\lim\limits_{x \to c}$ $x \to c$ exists	$f(x)$ is continuous at $x = c$	$f'(c)$ exists
			
	a.	b.	c.
$f(x) = \|x - 2\|$ where $c = 2$			
$f(x) = 3\sqrt{x}$ where $c = 0$	d.	e.	f.

Symbolic Nonroutine (SN)

Questions included in this subscale are nonroutine in that subjects attempting such problems possessed neither a known answer nor a previously established (routine) procedure for finding one. These questions did not require application to a real-world model.

Q2.2.

Give an example of a function $f(x)$ that is defined on the interval $(1,3)$ except at $x = 2$ for which

$$\lim_{x \to 2} f(x) \text{ exists.}$$

Explain your choice. (Your function may be given in graphical form, however, you must explain how it satisfies the above conditions.)

E1.6.

Let

$$f(x) = \begin{cases} bx^2 + 1 & \text{if } x < -2, \\ x & \text{if } x \geq {}^-2 \end{cases}$$

Find all possible values of b such that $\lim_{x \to {}^-2} f(x)$ exists.

E1.9.

Find a closed interval of length 1 in which the equation

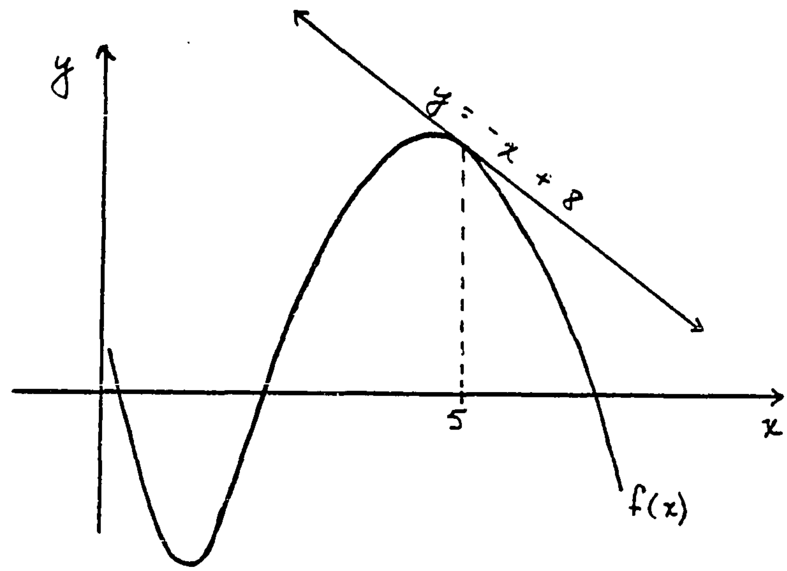
$$x^3 + 4x^2 - 10 = 0$$

has a positive real number solution. Explain.

Q3.1.

Consider the curve $y = f(x)$ in the accompanying sketch. The line $y = -x + 8$ is tangent to $f(x)$ at $x = 5$.

a. Find $f(5)$



b. Find $f'(5)$

Q6.2.

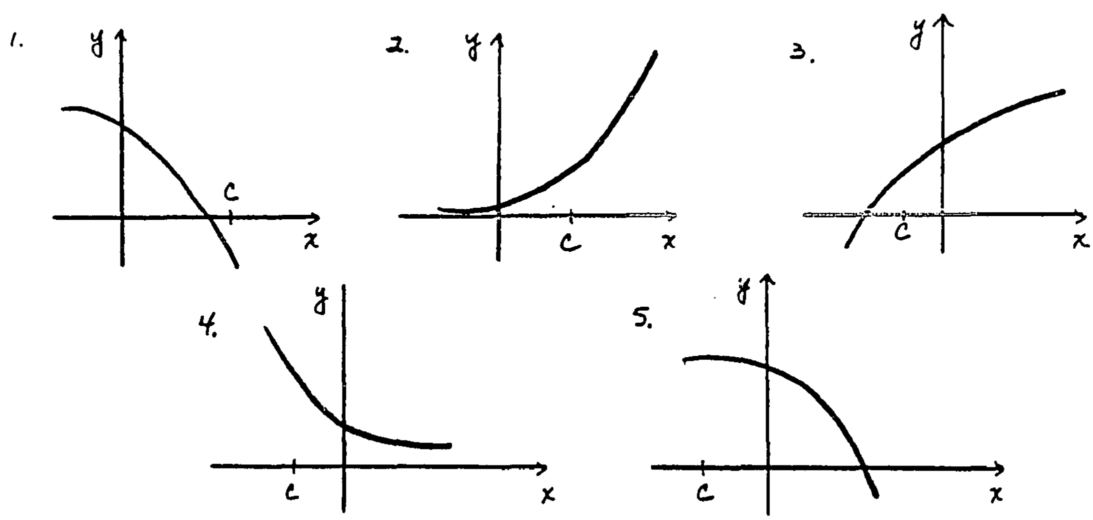
What are <u>all</u> the values of x for which the function $f(x) = x^2 - 4x + 3$ is increasing?

Q6.3.

Indicate which of the graphs below correspond to a function f that has the following properties:

$$f(c) > 0, \quad f'(c) < 0, \quad f''(c) > 0 \quad \text{at the point } x = c.$$

Briefly support your answer.

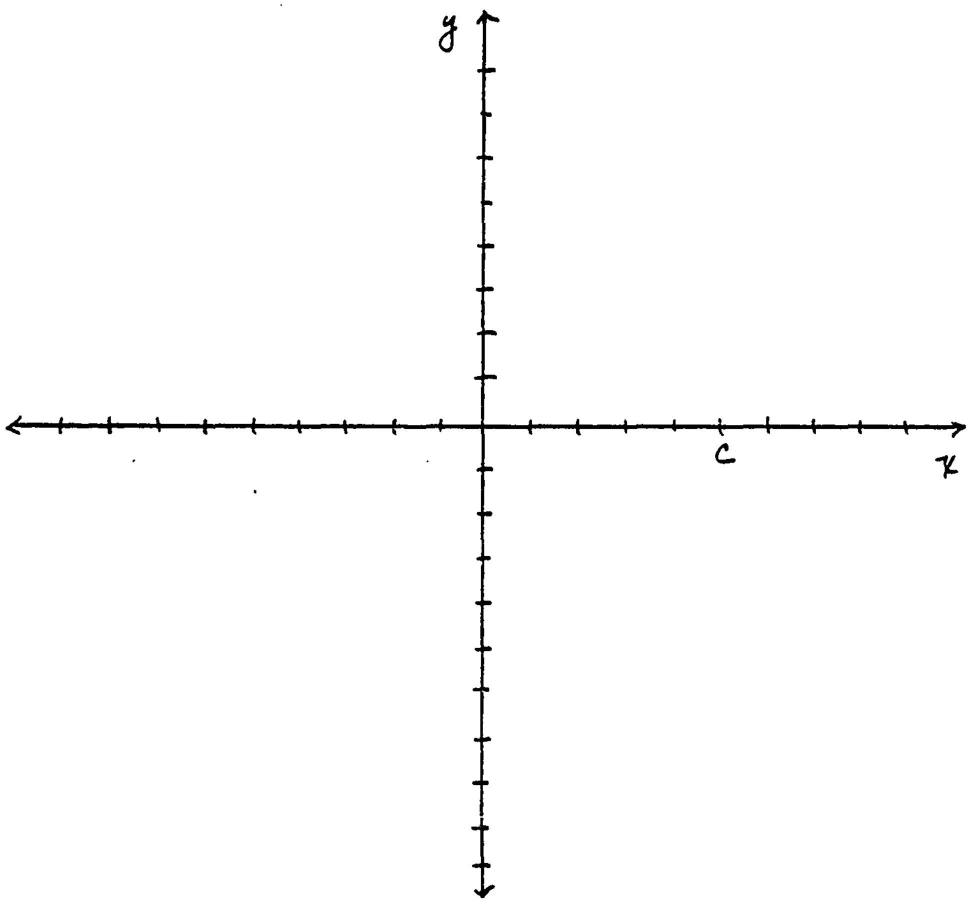


E3.1.

On the axes below, sketch the graph of a function $f(x)$ with the properties:

i) $f(c) < 0$ ii) $f'(c) < 0$ iii) $f''(c) > 0$

EXPLAIN!

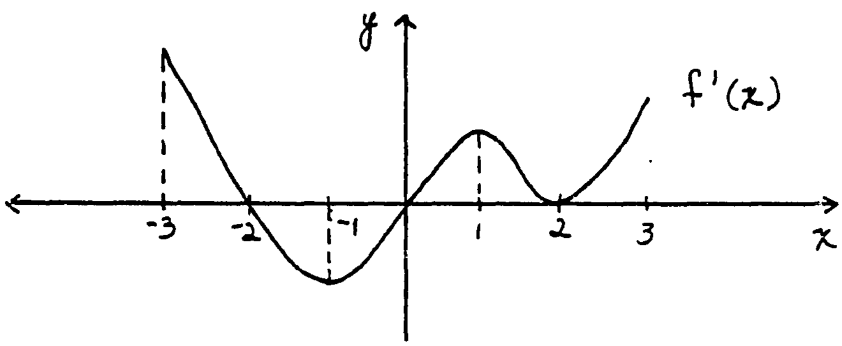


E3.2.

a. Complete the following statement: The mean value theorem implies that if $f'(x)$ exists for $a \leq x \leq b$, then there exists at least one number c between a and b such that ________________________.

E3.5.

The figure below shows the graph of f', the derivative of a function f. The domain of f is the set of all $x \in [-3, 3]$.



NOTE: This is the graph of the <u>derivative</u> of f, not the graph of f.

a. For what values of x, $-3 < x < 3$, does f have a local maximum? a local minimum? EXPLAIN.

i) maximum

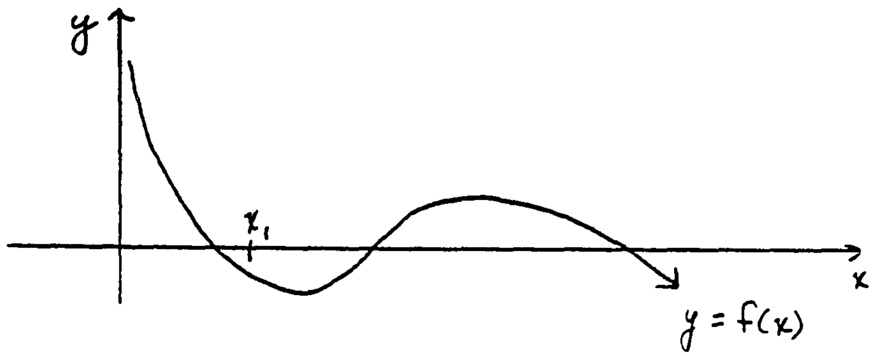
ii) minimum

b. For what values of x is the graph of f concave up? EXPLAIN.

c. For what values of x does f have a point of inflection? EXPLAIN.

Q8.1.

Suppose you are using Newton's method to find a root of $f(x) = 0$ for the function f shown below. If x_1 indicates where your first approximation is, label where (approximately) your second approximation x_2 and your third approximation x_3 will be. Your completed illustration should indicate how you located x_2 and x_3.



Q8.2.

If Newton's method is applied to $f(x) = 3x + 4$, and x_1 is chosen to be 15, what will x_7 be? (Hint: Think graphically!) Explain.

E4.3.

Let $f''(x) > 0$ for all x in the interval $(2, 7)$. Is $f'(4) < f'(5)$? Explain why or why not.

Cross-reference of Exams and Quizzes

Following is a cross-reference of the exam and quiz questions given to subjects in all sections participating in the study. Classes met four days each week. All sections met on the same days during the semester.

For each question given to the subjects in the G and G+ sections, the day and question number is indicated. The corresponding days and question numbers are given for subjects in the S1 and S2 sections.

Cross-Reference: Exam and Quiz Questions

Graphics and Graphics Plus		Standard 1		Standard 2	
Day	Question	Day	Question	Day	Question
11	Q22	15	E19	15	E15
11	Q23a,b,c	11	Q32a,b,c	15	E16a,b,c
15	E16	15	E1,10	15	E19
15	E17a,b,c,d,e	11	Q31a,b,c,d,e	15	E18a,b,c,d,e
15	E18	15	E17	11	Q31
15	E19	15	E18	15	E12
19	Q31a,b	15	E16a,b	15	E1, 10a,b
27	E21a,b	27	E21a,b	27	E23 a,b
27	E22	27	E22	27	E21
27	E25	27	E25	15	E11
27	E26	27	E26	27	E25
27	E27a,b	23	Q52a,b	27	E29a,b
31	Q52a,b	27	E27a,b	27	E24 a,b
35	Q61	35	Q71	31	Q61
35	Q62	35	Q72	31	Q62
35	Q63	35	Q73	31	Q63
39	E31	39	E32	39	E31
39	E32a,b	39	E31a,b	27	E27a,b
39	E32c	31	Q61	27	E22
39	E34a,b,c,d	39	E36a,b,c,d	39	E33a,b,c,d
39	E35 a,i,ii,b,c	39	E356a,b,c	39	E36a,b,c
39	E36	31	Q62	27	E28
39	E37	39	E34	35	Q81
43	Q71 a,b	43	Q82a,b	39	E34a,b
43	Q72	43	Q81	39	E35
46	Q81	46	Q91	43	Q91
46	Q82	46	Q92	43	Q91
53	E41a-f	53	E41a-f	53	E4,11a-f
53	E43	53	E43	53	E4,10

Departmental Final Exam
Calculus I - Winter '88
Form A

This exam is a multiple-choice assessment of subject academic progress in the course. One of three parallel forms of the exam was given to all students completing Calculus I during the Winter 1988 semester.

Questions over the material covered in the study (FDERIV) are numbered 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 16, 17, 24, 25, and 30. The remaining questions were over antidifferentiation, integration, or precalculus ideas (FANTI).

MATH 122 FINAL EXAM
FORM A

Name ______________________________

Instructions:

1. Write your instructor's name, course, and section in the space provided at the bottom of your answer sheet.
2. Turn answer sheet sideways and print/code in your name, date, FORM, and social security number.
3. In addition to the work space provided under each problem, you may tear off the extra sheets at the end of this packet. Be sure to put your answers on the answer sheet during the time of the test.
4. Do not ask your instructor to interpret any questions, except for English definitions of non-mathematical words.
5. When finished, hand in the Exam, the answer sheet, and all scrap paper.

---

1. Find the equation of the line through $(2, -1)$ perpendicular to $2x - 3y = 8$.

   1) $2x + 3y - 1 = 0$ 2) $4x + 8y = 0$

   3) $3x + 2y - 4 = 0$ 4) $3x + 2y - 8 = 0$ 5) $2x - 3y - 7 = 0$

---

2. $\lim\limits_{x \to 2} \dfrac{\frac{1}{x+1} - \frac{1}{3}}{x - 2} =$

   1) $\frac{1}{3}$ 2) $\frac{-1}{3}$ 3) $\frac{1}{9}$ 4) $\frac{-1}{9}$ 5) does not exist

3. Let

$$f(x) = \begin{cases} \frac{x^2 - 9}{x - 3}, & x < 2, \\ x^2 - x + 3, & x \geq 2. \end{cases}$$

This function is

1) not continuous at $x = 3$ and not continuous at $x = -3$;

2) not continuous at $x = 3$ and continuous at $x = 2$;

3) not continuous at $x = 2$ and continuous at $x = -3$;

4) continuous at $x = 2$ and continuous at $x = 3$;

5) none of the previous.

---

4. Find $\lim_{x \to -1^+} \frac{|x + 1|}{x + 1}$, if it exists.

1) 1 2) -1 3) 0 4) $+\infty$ 5) does not exist

---

5. At what point on the graph of $y = 3x^2 + 2x + 1$ does the tangent to the curve have slope 4?

1) $(1, 6)$ 2) $(\frac{2}{3}, \frac{11}{3})$ 3) $(\frac{1}{3}, 2)$ 4) $(2, 17)$ 5) $(\frac{1}{3}, 4)$

---

6. What is the slope of the graph of $f(x) = |3x^3|$ at $x = -1$?

1) -9 2) -3 3) 3 4) -1 5) 9

7. An individual's pulse rate during exercise (in beats per minutes) is given by

$$P(t) = \frac{t^2}{4} - t + 71$$

where t represents the number of minutes of exercise with $0 \le t \le 10$. Find the instantaneous rate of change of $P(t)$ 4 minutes after beginning to exercise.

1) 1 2) 2 3) 70 4) 71 5) 86

8. Evaluate the following limit if it exists.

$$\lim_{x \to 0} \frac{2x + \sin x}{x}$$

1) 0 2) 2 3) 3 4) $-\infty$ 5) $+\infty$

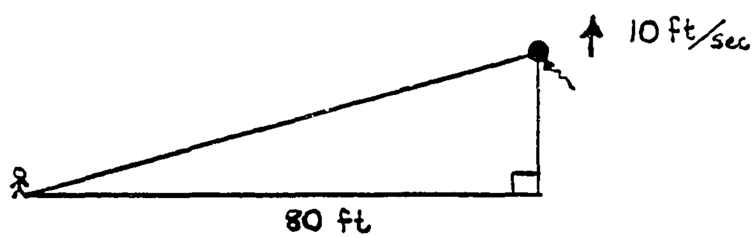
9. Let $h(x) = f(g(x))$. Determine $h'(2)$ if $g(2) = 4$, $f'(2) = 11$, $f'(7) = 5$, $g'(2) = 7$, and $f'(4) = 3$.

1) 20 2) 44 3) 3 4) 21 5) 5

10. Given the function $f(x) = \sqrt{x}$, for what value c in the interval $(1, 4)$ is the instantaneous rate of change of $f(x)$ with respect to x at c equal to the average rate of change over the interval $[1, 4]$?

1) $\frac{1}{3}$ 2) $\frac{3}{2}$ 3) 0 4) $\frac{9}{4}$ 5) $\frac{1}{9}$

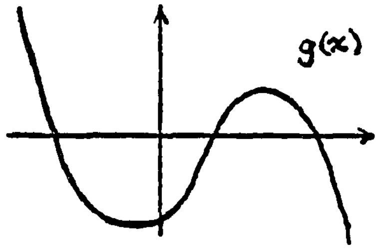
11. A small balloon is released at a point 80 feet away from an observer, who is on level ground. If the balloon goes straight up at a rate of 10 feet per second, how fast is the distance from the observer to the balloon increasing when the balloon is 60 feet high?

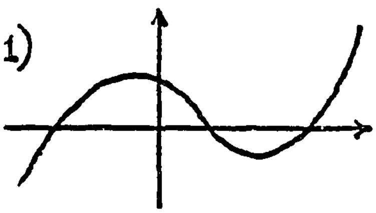


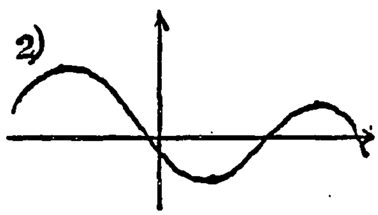
1) 7.5 ft/sec 2) 38 ft/sec

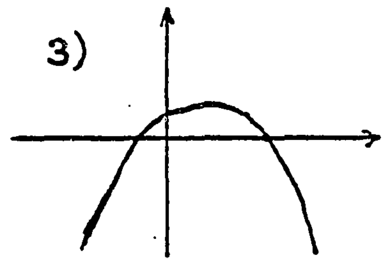
3) 16.67 ft/sec 4) 6 ft/sec 5) 40 ft/sec

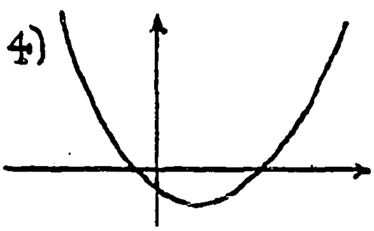
12. Which of the graphs below could be the graph of the derivative of $g(x)$?

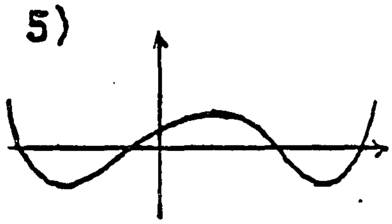


1)


2)


3)


4)


5)


13. Determine the number of inflection points for the curve $y = 3x^5 - 5x^4$.

1) 1 2) 2 3) 3 4) 4 5) 0

14. For what values of x is the graph $y = x^3 - 3x + 2$ concave upward?

1) $x > 0$ 2) $x > 1$ and $x < -1$

3) $-1 > x < 1$ 4) $x < 0$ 5) All values of x

15. An apple orchard containing 200 trees per acre yields an average of 100 apples per tree. For every additional 20 trees planted per acre, the yield per tree drops by 5 applies. Find an expression for the total apple yield per acre, A, in terms of t, the total number of trees per acre.

1) $A = 20,000 - 20(t - 5)$

2) $A = [100 - 5(t - 200)]t$

3) $A = [100 - (t - 200)/4](200)$

4) $A = [100 - (t - 200)/4]t$

5) $A = [200 - 5(t - 100)]t$

---

16. Which one of the following functions has $y = 0$ as an asymptote?

1) $f(x) = \frac{x+1}{x}$

2) $f(x) = \frac{x^2+x}{x}$

3) $f(x) = x$

4) $f(x) = \frac{x}{x-1}$

5) $f(x) = \frac{1}{x-1}$

---

17. If the maximum value of $f(x)$ is 2 and if $f'(x) = 2 - 2x$, then $f(x) =$

1) $-x^2 + 2x - 2$ 2) $-x^2 + 2x - 1$ 3) $-x^2 + 2x$

4) $-x^2 + 2x + 1$ 5) $-x^2 + 2x + 2$

---

18. On the surface of the moon, the acceleration due to gravity is approximately -5 ft/sec^2. An object is thrown straight up from moon level with initial velocity of 20 ft/sec. The intervals below represent distance in feet. Find the interval containing the maximum height of the object above moon level.

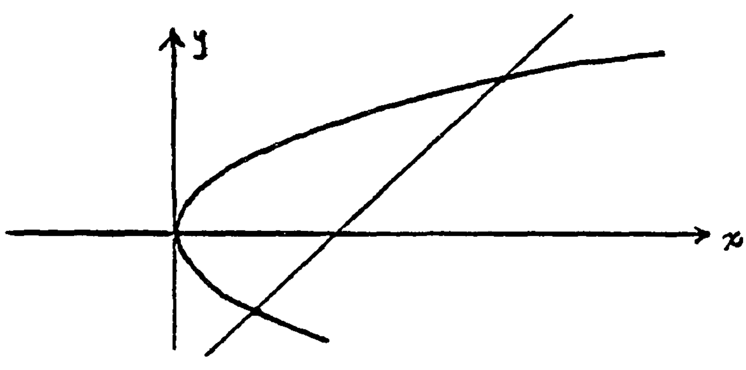
1) $[0, 30)$ 2) $[20, 60)$ 3) $[60, 90)$ 4) $[90, 120)$ 5) $[120, +\infty)$

19. Evaluate

$$\int_0^{\frac{\pi}{6}} \sin 2x \, dx$$

1) $\frac{1}{2}$ 2) $\frac{1}{4}$ 3) $-\frac{1}{4}$ 4) $\frac{2-\sqrt{3}}{4}$ 5) $\frac{\sqrt{3}}{2}$

20. Find the area of the region bounded by the graphs of $x = y^2$ and $y = x - 2$.



1) $\frac{9}{2}$ 2) $\frac{19}{6}$ 3) $\frac{15}{2}$ 4) $\frac{3}{2}$ 5) $\frac{7}{3}$

21. Find an integral representing the region bounded by $y = x^2$, $y = 0$, and $x = 2$ revolved about the y-axis.

1) $\pi \int_0^4 (4 - y)dy$ 2) $\pi \int_0^2 (2 - \sqrt{y})^2 dy$ 3) $\pi \int_0^2 x^4 dx$

4) $2\pi \int_0^4 (2 - \sqrt{y}) y\, dy$ 5) $\pi \int_0^4 y\, dy$

22. Which integral will give the arc length of the graph of $y = \sin 2x$ over the interval $[0, \pi]$?

1) $\int_0^\pi \sqrt{1 - 4 \cos^2 2x}\; dx$

2) $\int_0^\pi \sqrt{1 + \cos^2 2x}\; dx$

3) $\int_0^\pi \sqrt{1 + 4 \cos^2 2x}\; dx$

4) $\int_0^\pi \sqrt{1 - \cos^2 2x}\; dx$

5) $\int_0^\pi \sqrt{\sin^2 2x + \cos^2 2x}\; dx$

23. Given that $\int_1^5 f(x)dx = A$ and $\int_1^5 (f(x))^2 dx = B$ find $\int_1^5 (f(x) + 3)^2\, dx$

1) $B + 6A + 36$ 2) $A^2 + 6A + 9$ 3) $\frac{4}{3}(A + 3)^3$

4) $B + 24A + 36$ 5) $4(A + 3)^2$

24. The absolute maximum of the function $f(x) = 2\cos x + x$ on the interval $[0, \frac{\pi}{2}]$ is

1) $1 + \frac{\pi}{3}$ 2) 2 3) $\sqrt{3} + \frac{\pi}{6}$ 4) $\sqrt{2} + \frac{\pi}{4}$ 5) $\frac{\pi}{2}$

---

25. The asymptotes of $f(x) = \frac{2x^2 - 2}{x^2 - 4}$ are

1) vertical: $x = 2$, horizontal: $y = 2$

2) vertical: $x = 2$ horizontal: $y = 1$

3) vertical: $x = 2$ horizontal: $y = 1, y = -1$

4) vertical: $x = 2, x = -2$ horizontal: $y = 2$

5) vertical: $x = 2, x = -2$ horizontal: $y = 1, y = -1$.

---

26. $\int x(x^2 - 1)^5 dx =$

1) $\frac{(x^2 - 1)^6}{12} + C$ 2) $\frac{x^2(x^2 - 1)^6}{12} + C$ 3) $10x(x^2 - 1)^4 + C$

4) $\frac{(x^2 - 1)^6}{6} + C$ 5) $\frac{x^{12}}{12} - \frac{x^2}{2} + C$

---

27. $\int_0^{\frac{\pi}{2}} \cos x\,(1 + \sin x)^2 dx$

1) $\frac{8}{3}$ 2) $\frac{7}{3}$ 3) $\frac{4}{3}$ 4) 7 5) 8

---

28. $\int_{-2}^{1} |x|dx$

1) $-\frac{3}{2}$ 2) -1 3) 3 4) $\frac{5}{2}$ 5) $\frac{3}{2}$

---

29. If $F(x) = \int_0^x \cos 3t \, dt$, then $F'(x) =$

1) $-3 \sin 3x$ 2) $3 \cos 3x$ 3) $\sin 3x$ 4) $\cos 3x$ 5) $\frac{1}{3} \cos 3x$

---

30. Find the derivative of

$$g(t) = \frac{5t + 2}{t^2 + 1}$$

1) $\frac{5}{2t}$ 2) $\frac{5}{t^2 + 1}$ 3) $\frac{5t^2 + 4t - 5}{t^2 + 1}$

4) $\frac{5t^2 + 4t - 5}{(t^2 + 1)^2}$ 5) $\frac{5 - 4t - 5t^2}{(t^2 + 1)^2}$

APPENDIX E

ATTITUDE AND OPINION SURVEYS

Copies of the attitude assessments and opinion surveys administered during the study are contained in this appendix. Following are copies of:

1. The Mathematics Attitude Survey,

2. The Evaluation Survey, and

3. The Exit Survey.

Mathematics Attitude Survey

The attitude survey was administered to all subjects in sections G, G+, S1, and S2 on Day 3 and again on Day 49 of the Winter 1988 semester. It was composed of 15 items designed to assess student attitudes on: (a) interest in mathematics–questions 1, 7, 8, 11, 14, (b) interest in taking more mathematics–questions 2, 3, 5, 10, 12, and (c) perceptions of the nature of mathematics – questions 4, 6, 9, 13, 15.

Items for the survey were either taken directly from, or adapted from, items found in two sources. Items numbered 1, 2, 3, 5, 6,7, 8, 11, and 12 were selected from a "Scale of Attitudes toward Mathematics" (Aiken, 1979). Item 10, "I would enjoy taking additional mathematics classes," was weakened from the Aiken version that read, "I plan to take as much mathematics as I can during my education." Since college students are bound by the requirements of their pro-

grams, the former statement was believed to be more realistic for use with the participants in the present study.

Items 4, 9, and 14 were taken from an attitude survey published in Heid's dissertation (1984). Items 13 and 15 were adapted from Heid's survey. The corresponding statements in Heid's survey were the negatives of each of these.

All items are Likert-type with 5 possible responses: (1) strongly disagree, (2) disagree, (3) undecided, (4) agree, and (5) strongly agree. For statistical analysis, responses to items concerning interest in mathematics and interest in taking more courses in mathematics were changed so that a score of 5 indicated the most positive response toward mathematics.

Mathematics Attitude Scale

On the answer sheet, enter your name, the date, and your Social Security Number.

Each of the statements on this opinionaire expresses a feeling which a particular person has toward mathematics. You are to express, on a five-point scale, the extent of agreement between the feeling expressed in each statement and your own personal feeling. The following statements are to be rated on a scale 1-5 with the interpretation:

Strongly Disagree	Disagree	Undecided	Agree	Strongly Agree
1	2	3	4	5

1. Mathematics is not a very interesting subject.
2. I want to develop my mathematical skills and study this subject more.
3. I don't want to take any more mathematics than I have to.
4. There is only one way to solve most mathematics problems.
5. I am interested in acquiring further knowledge of mathematics.
6. Mathematics helps to develop the mind and teaches a person to think.
7. Mathematics is enjoyable and stimulating to me.
8. I am not motivated to work very hard on mathematics lessons.
9. Mathematics involves mostly memorizing.
10. I would enjoy taking additional mathematics classes.
11. I like trying to solve new problems in mathematics.
12. I am not willing to take more than the required amount of mathematics.
13. Mathematics involves creativity.
14. I like to explore mathematical ideas.
15. Most problems in mathematics can be solved by following a rule.

Evaluation Survey

Items on this survey were administered to all subjects in sections G, G+, S1 and S2 on Day 49 of the semester in which the study took place. It was administered under conditions of anonymity.

Evaluation Survey

The information on the response sheet will be tabulated for departmental use and returned to the instructor only after the final grades have been submitted. Rate the following statements on a scale of 1 - 5 with the following interpretation for Part A. Use black pencil only.

poor	below average	average	above average	excellent
1	2	3	4	5
disagree	disagree somewhat	neutral	agree somewhat	agree strongly

PART A: The Course

16. Content and/or processes developed in this course are useful.

17. Assignments seem carefully selected and contribute significantly to student learning.

18. Course text and materials contribute to student learning.

19. Examinations cover the course material well.

20. General evaluation of the course.

PART B: The Student

21. I gained ________________ understanding of the concepts and principles in this course.
Please code the number beside the word which completes the sentence in your case.
1) no; 2) minimal; 3) adequate;
4) good; 5) excellent

22. On the average, approximately how many hours do you devote to this course outside of class for each hour in class:
1) one or less; 2) two; 3) three;
4) four; 5) five or more

Exit Survey

The Exit Survey was given to subjects in the G and G+ sections. It served to assess their post-treatment attitudes toward the usefulness of hand-drawn and computer-generated graphs and to solicit their opinions of the computer graphics demonstrations used throughout the course.

Two separate forms of the survey were given. The survey included here was given to subjects in the G+ section. The G+ Exit Survey also solicited information concerning student use of the available software on assignments and subsequent coursework. Items 1 and 2 were not included on the Exit Survey for subjects in the G section.

Name________________________

Exit Survey
Math 122 - Calculus I

Please answer the following questions as completely and honestly as possible. The results will not be tabulated until final grades are turned in. Please return this survey as soon as possible, but before April 15, 1988. I realize time is short with your upcoming exams, but I hope you will take the time to answer the survey. Thanks for your help!

1. How often did you use the computer on assignments where its use was required/suggested?
   a) very often b) often c) some of the time
   d) rarely e) never

2. How often did you use the computer for the course when its use was not required/suggested?
   a) very often b) often c) some of the time
   d) rarely e) never

3. How often do you find yourself using graphs when solving calculus problems?
   a) almost always b) most of the time
   c) about half of the time d) some of the time
   e) almost never

4. Rate the statements below on a scale of 1 - 5 with the following interpretation:

Strongly Disagree	Disagree	Undecided	Agree	Strongly Agree
1	2	3	4	5

_____a) The computer graphs were useful in helping me understand calculus ideas.
_____b) I would like to continue using graphs in other math classes.
_____c) I find graphs useful in helping me think about functions.
_____d) I find I have to draw a graph to understand the problem.
_____e) I find that I use graphs to solve math problems more often now than I did in previous courses.

Please feel free to add extra pages to answer the following if you need more space:

5. Do you think there is any difference in the frequency with which you tend to use graphs now from work you have done in mathematics previous to this course?

Has the graphics approach to this course caused you to change your view of the usefulnes of graphs in any way? Please state specifically how your use of graphs and your view of their usefulness has changed over this semester. Give examples if possible.

6. Are there any concepts that were developed with the computer or any programs that were demonstrated on the computer that you found particularly interesting or useful?

If yes, what were they? Describe your reaction/reason.

7. There were a variety of programs that were instructor written which were not made available to students outside of class. Do you think it might have been helpful to have been able to use any of these programs on your own?

If yes, please discuss specifically which program(s) you might have found useful outside of class and why.

8. Have you ever taken calculus before?__________________________

If yes, please answer the following:

a) Why did you take calculus this semester?

b) How was this class similar to your former class?

c) How was it different?

d) Which did you prefer? Why?

APPENDIX F

ASSIGNMENTS

This appendix contains:

1. Textbook assignments given to all subjects in the G, G+, S1 and S2 sections.

2. Investigator-written Assignment 1 given to subjects in the G and G+ sections only, and

3. Investigator-written assignments given to the subjects in the G+ section only.

Textbook Assignments for G, G+, S1, and S2 Sections

All of the textbook assignments from *Calculus* by Hurley (1987) are listed by section and exercise number. These assignments were given to all subjects participating in the study. No other assignments or supplemental review problems were given to subjects in the S1 and S2 sections. Only one other assignment was given to subjects in the G section.

Subjects in the G+ section were given additional assignments which are fully described and presented in this Appendix.

Textbook Assignments

Chapter 1: Functions and Limits

Section	Exercises
1.1	1, 5, 9, 13, 17, 18, 23, 25a, 26, 28, 35
1.2	1b, 3b, 5b, 7b, 8b, 10, 11a, 13, 14, 15a, 16a, 18a, 31, 36
1.3	3, 4, 7, 17, 25, 27, 29, 31, 35, 39, 41, 47, 49
1.4	1, 3, 5, 7, 9, 11, 13, 15, 17, 33, 35, 36, 38
	graph 33, 35, 36
1.5	3, 7, 11, 17, 21, 22, 23, 25, 33, 34, 36, 42,
	graph 21, 22
1.6	1-11 odd, 14, 15, 17, 23, 30
1.7	4, 5, 7, 11, 12, 13, 16, 19, 21, 23, 31, 44-47
1.8	all exercises except 47, 48, 49

Chapter 2: Derivatives

Section	Exercises
2.1	1, 3b, 6a, 7, 8b, 9, 10, 11, 15a, 17d, 19d, 21, 28, 31, 37
2.2	1-25 odd, 33, 41
2.3	1a, 1c, 1e, 5e, 6f, 11c, 11f, 13, 17, 18, 19-33 odd, 39
2.4	1, 3, 5, 11, 13, 15, 19, 23, 26
2.5	1, 3, 5, 9, 13, 16, 17, 19, 20 26
2.6	1-23 odd, 27, 29, 31, 35, 47

Chapter 2: Derivatives (continued)

Section	Exercises
2.7	1, 4, 11, 13, 15, 17, 19
2.8	1-17 odd, 21, 23, 31, 37, 41
2.9	1, 3, 11, 15, 17, 19, 23, 25, 31, 33, 35, 43, 49
2.10	2, 3, 5, 7, 9, 11, 15, 17, 19 21, 24, 27
2.11	all but 63

Chapter 3: Applications of Differentiation

Section	Exercises
3.1	1, 3, 7, 9, 11, 13, 15, 17, 19, 32
3.2	1, 2, 3, 5, 7, 10, 11, 13, 25, 26, 27, 31
	interpret graphically 25, 26, 27
3.3	1-21 odd, 31, 35, 42
3.4	1-11 odd, 27, 31, 37
3.5	1, 3, 4, 5, 6, 7, 9, 11, 15, 16, 17, 20, 25, 29, 33, 37, 39, 43
3.6	1-11 odd, 12, 15, 23, 25, 35, 37
3.7	3, 7, 15, 21, 23, 31, 47
3.8	3
3.9	1-11 odd, 15, 49, $\int \sin x \, dx$, $\int \cos x \, dx$
3.10	1, 3, 5, 7, 19, 27

Assignment 1 for G and G+ Sections Only

Assignment 1 was given to subjects in the G and G+ sections only. It was designed to encourage subjects to begin thinking about the relative usefulness of the graphical, tabular, and symbolic representations in displaying mathematical information.

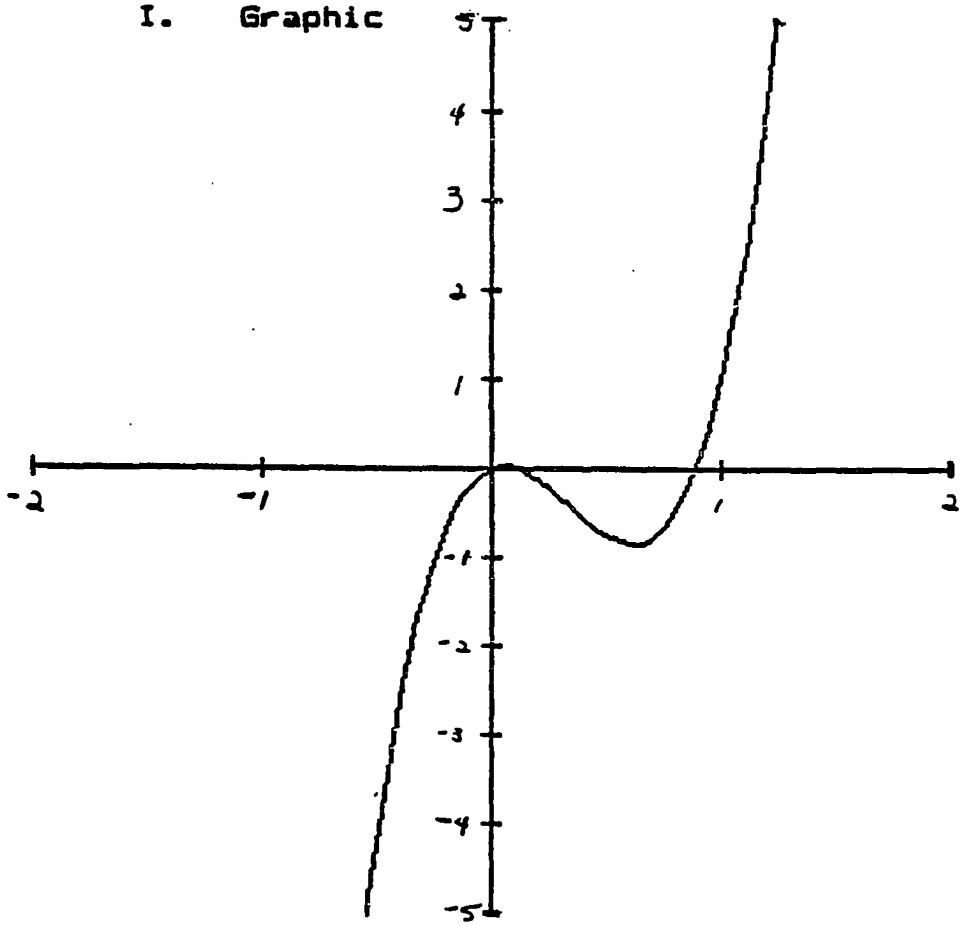
Name________________________
Winter '88

Math 122
Assignment 1

1. You have used graphs in previous mathematics courses. Based on these experiences, what does the word "graph" mean to you?

2. Below are three representations of the same function.

I. Graphic



II. Tabular

x	f(x)
-5	-1505
-4	- 804
-3	- 363
-2	- 122
-1	- 21
0	0
1	1
2	42
3	183
4	484
5	1005

III. Symbolic

$$f(x) = 10x^3 - 10x^2 + x$$

a) Which of these representations do you prefer? Why?

b) What information about the function can be drawn from representation I?

c) What information about the function can be drawn from representation II?

d) What information about the function can be drawn from representation III?

e) Compare and contrast the information given in each representation:

i) What kinds of information does each representation give?

ii) What information do all three representations give?

iii) Does one representation give information that another does not? What information specifically?

Create a chart like the one below to help you answer the questions above.

Representation	Graph	Table	Equation
rate of change			
zeros/roots			
inflection points			
intercepts			
etc.			

Assignments for the G+ Section Only

This section contains copies of the investigator-written supplemental assignments given to subjects in the G+ section only. They are

1. **Master Grapher.** These information sheets describe the *Master Grapher* software (Waits & Demana, 1987a), subject responsibilities concerning the software, and the available utilities.

2. **Getting Acquainted with** *Master Grapher.* This assignment, adapted from Waits and Demana (1987b) acquaints subjects with the use of various *Master Grapher* utilities.

3. **Assignment 2.** This assignment is designed to aid subjects in developing efficient methods of graphing functions.

4. **Lecture Notes for January 15, 1988.** These notes describe a real world example to aid subject understanding of the limit concept and contains an assignment which asks subjects to construct an original example illustrating the concept of limit. This was the only assignment subjects were required to turn in.

5. **Assignment 3.** This assignment, adapted from Epp (1986), alerts subjects to the relationship between the values a and $f(a)$ and the point $(a, f(a))$ on the graph of $f(x)$. Subjects are also given the opportunity to begin experimentation with derivatives by looking at suggested functions on small intervals.

6. **Assignment 4.** This assignment, adapted from Janvier (1975), gives subjects practices with graphical differentiation.

Master Grapher
Release Version 3.00

Master Grapher is a very fast, machine language function, conic, polar, and parametric grapher with many enhancements. These include: variable viewing window, zoom in, zoom out, point or line segment plot modes, user determined graph speed/accuracy setting, function overlay capability, function transformation capabilities, etc.

Master Grapher requires an IBM or compatible microcomputer with at least 256K of memory and a color or enhanced graphics board. (The software will not run on IBMs with certain older monochrome graphics boards.) The Zenith microcomputers in Maybee Hall meet these hardware requirements.

Master Grapher is copyrighted and may not be reproduced or copied by any means without the permission of the developers, Bert Waits or Franklin Demana. The copies provided for this course are the property of Western Michigan University Department of Mathematics and Statistics and must be returned to the instructor upon completion of the semester or upon withdrawal from the course. If the software is not returned by April 14, 1988, the student will be assessed a $25 fine during finals week. The fine could cancel the student's registration for the following semester. It will be removed when the software is returned. Those students who drop the course must return the software within a week of the day they drop or be assessed a $25 fine (which will be removed once the software is returned.

The developers are continually revising and improving the graphing software and appreciate being informed of difficulties encountered and of suggestions for improvement. Please inform the instructor of difficulties/suggestions so that these may be forwarded to the developers.

Some Features of Master Grapher

Master Grapher allows the user to graph various types of equations. The Main Menu contains the following options:

1) Function Grapher
2) Conic Grapher
3) Parametric Grapher
4) Polar Grapher
5) Help
6) Quit

In Math 122, we will use Function Grapher most often. You are encouraged to experiment with each of the utilities. To run Master Grapher, insert a DOS 2.1 disk into drive A (the disk drive on the left or top in a two disk drive machine) and turn on the machine. If the machine is on, press the CTRL, ALT, and DEL keys simultaneously to boot the system disk. Put Master Grapher in drive B. Type b: and press the RETURN key, then type master and again press the RETURN key. You should now see a title screen or a menu with several choices on it, indicating that Master Grapher is ready for use.

To obtain a printout of any part of a session with "Master Grapher", before typing B: in the directions above, type graphics then press RETURN. Continue with the directions above. During the computer session, press the PRINT SCREEN function key to obtain a printout of the computer screen when desired.

By typing 1 followed by 6, we obtain a screen with a graph displayed on the right and the function menu on the left. Below is a description of the options, each of which is accessed by typing the letter indicated.

Option	Description
A) Quit	Returns control to DOS saving current index functions. To boot Master Grapher from system, type master and press return.
B) Previous Menu	Returns control to previous menu.
C) Change View	Contains the following features: 1) Zoom in (point): Magnifies viewing rectangle around user-chosen (x,y). 2) Zoom in: Using arrow keys, user determines new viewing rectangle. 3) Zoom out (point): Obtains larger viewing rectangle around (x,y). 4) Zoom out: Obtains larger viewing rectangle determined by zoom factor.

Option	Description
	5) Set Zoom Factor: Allows user to define factor for magnification or reduction of viewing window. 6) Set Window: Allows user to determine the viewing rectangle by choosing L(eft), R(ight), B(ottom), and T(op) coordinates of the rectangle. 7) Default Window: Graphs function in [-10,10] by [-10,10] viewing rectangle. 8) Cancel: Returns control to function menu.
D) Redraw Graph	Allows user to redraw the graph(s) in current, default, or previous viewing window.
E) Clear Screen	Clears the graphics screen. Using L, O, or P, user may graph functions on graphics screen without axes after using this option.
F) Draw Line	Allows moving or fixed vertical or horizontal lines to be drawn on the graph.
G) Draw Grid	Draws lattice points on the graph.
H) Read Position	Allows user to move an arrow indicator with cursor keys to a desired position on the graphics screen and determine the x and y coordinates of the chosen position.
I) Change Speed	Allows user to change the speed with which the graph is drawn by determining the number of points to be plotted (from 1 to 1716). The larger the number, the more accurate the plot, and the slower the plotting speed. Default is 100.
J) Change Plot Mode	Changes the plot mode from plotting by points to plotting by segments and vice versa.

Option	Description
K) C/D/R Function	Change/Display/Remove Functions: Lists eight functions available for graphing. User may change the display status of the function by typing the index number of the function. Functions may be changed by typing 0 and the index of the function to be changed; functions must be entered in BASIC syntax (see attached). Function does not change if it is entered incorrectly. Typing 9 returns control to the function menu.
L) Overlay Function	Allows the user to display a function in the index over those currently displayed on the graphics screen.
M) Rotate Equation	Graphs an index function rotated about the origin. User chooses the counterclockwise angle of rotation.
N) Trans. Equation	Graphs a translated index function. User chooses horizontal and vertical translation vectors.
O) Inverse Equation	Given y = f(x) from the index, this option graphs the inverse relation, y = $f^{-1}(x)$.
P) Transform F(x)	Allows user to graph y = a f(bx + c) + d where f(x) is an index function. User chooses a, b, c, and d.

Note: The error in using a point (x, y) in the viewing rectangle [L, R] by [B, T] to approximate any point (a, b) in the viewing rectangle is at most R − L for x and T − B for y. Better error bounds are possible by overlaying a lattice in a viewing rectangle or by using scale marks appearing in the viewing rectangle.

Equations must be entered using the BASIC programming language. This allows each expression to be written on one line. While using the Function Grapher utility, all expressions must be written with x as the independent variable.

A list of operations and their corresponding notation in BASIC is given below.

Operation	BASIC notation
Addition	+
Subtraction	-
Multiplication	*
Division	/
Raising to a power	^

The computer evaluates expressions using these operations following the usual algebraic rules for order of operations.

Notes:
a) Only "(" and ")" may be used as grouping symbols.
b) Evaluation of expressions involving built-in functions such as SIN and SQR is done first of all, ie. sin(x)^2 means "the square of the sine of x."
c) Multiplication is not implied. The symbol "*" must be used to indicate multiplication (3x must be entered as 3 * x.)

Each BASIC function has an abbreviation and an argument. The argument is enclosed in parentheses and may be any BASIC expression. Some common functions are listed below.

Function	Abbreviation
Absolute value of x	abs(x)
Square root of x	sqr(x)
Greatest integer less than or equal to x	int(x)
Sine of x in radians	sin(x)
Cosine of x in radians	cos(x)
Tangent of x in radians	tan(x)
Natural logarithm of x	log(x)
Exponential function, e^x	exp(x)
Sign of x; -1, 0, or 1 as x is negative, zero or positive	sgn(x)
Nth root of x where n is an integer	root(x,n)
x to the nth power where n is an integer	power(x,n)

The numeric output given by the program is often written in E notation, the computer equivalent of scientific notation. Examples are given below:

Number	Scientific Notation	E Notation
3,260,415,000	3.260451×10^6	3.260451E+06
.000004233	4.233×10^{-6}	4.233E-06

Getting Acquainted with Master Grapher

Insert DOS 2.1 (or higher) in drive A and Master Grapher in drive B. Turn on the computer and wait for date, time prompts. You may press return twice to obtain A> prompt. Type b: and press return then type master and press return. You will see the title screen for Master Grapher. Press return to get the main menu. We will use the Function Grapher utility, obtained by typing 1. The menu appearing on the screen gives the current status of the graphing utility. Type 6 to obtain the graphing screen and function menu.

1. Enter the function f(x) = (x^3-10*x^2+x+50)/(x-2) by typing the letter K to obtain the C/D/R option. Type 0 and 1 to enter f(x) as function #1. It is given in BASIC syntax above. If you enter the function incorrectly, type 0 then 1 and begin again.

   To graph function #1, the word "Displayed" should appear before the function. If "Not displayed" appears, type 1 to change the display status. If any other functions are listed as "Displayed", type their function numbers to change their status so that only function #1 is displayed. Type 9 to return control to the function menu. Any function whose status is "Displayed" will now be graphed on th default screen, [-10,10] by [-10,10].

2. The current displayed function is listed in the lower left corner of the screen. Can you determine what happens to the function values as |x| gets large from the default viewing rectangle?

3. Press C then 6 to enter the new viewing rectangle [-20,20] by [-200,200]. Press return after entering the coordinates for each of L, R, B, and T (in that order). Do you now "see" the complete graph? Explain.

4. Press J then D followed by 1. Explain the difference(s) observed.

5. Explore the speed option (Key I). Note that plotting speed and plot resolution are inversely proportional. Type I and choose a speed, then press D followed by 1 to redraw f(x) on the current window.

6. Press D then 2 to obtain the default plot. Press G. What do you observe?

7. Press C then 5. Set the zoom factors to x = 10 (horizontal direction) and y = 100 (vertical direction). Press 4 to zoom out. Do you see why these two zoom factors were chosen?

8. Press K to enter f(x) = x^2 into the function index as function #2. Press 9 to return to function menu. Redraw the function on the last viewing rectangle by typing D followed by 4. Overlay function #2 by typing L then 2 then pressing return. What do you observe?

9. Zoom out by default factors (10 in both directions). Again overlay function #2. What do you observe?

10. What can you conclude about the end behavior of function #1?

11. Change the viewing rectangle of f(x) to [-20,20] by [-100,100]. How many zeros does f have? Estimate the values of the x-intercepts (zeros) by pressing C followed by 2 to zoom-in on each of these several times, then use the H option to read the position of each zero. Note that the x and y coordinates of the zero are listed in the lower right corner of the screen.

12. Enter the following functions into the function index as indicated:

Function number	Function
3	x
4	x^2
5	x^0.5
6	sin(x)

Change the display status so that only function #3 is displayed.

13. We will now investigate g(x) = a * f(bx+c) + d for the functions f(x) entered in (12) above. Type P followed by 3. Enter 1, 1, 0, 0 for A, B, C, D respectively, pressing return after each entry. What do you observe? Why?

14. Using each of the functions above, experiment with the parameter A as listed below as well as other replacements you choose. Keep a record of your observations.

A	B	C	D	g(x) = ?	observation
2	1	0	0	2 f(x) = 2x	
.5	1	0	0		
-3	1	0	0		

Experiment with other values of A, predict the appearance of f(x) before displaying the graph. Press D to redraw on default view to experiment with each new function.

Name________________________

Assignment 2

1. You should recognize the graphs of the following functions. Plot each one with Master Grapher and print out the graph. Write each function in standard notation.

Function type	f(x)
Polynomial:	x, x^2, x^3, x^4
Algebraic:	root(x,2), root(x,3), power (x,2), power(root(x,2),3)
Trigonometric:	sin(x), cos(x), tan(x)
Transcendental:	abs(x), int(x)

2. Enter the functions give below into the function index as indicated.

Function Number	Function
1	1/x
2	x^2
3	x^4 - 5*x^2
4	x^3 - 3*x
5	x^(1/3)
6	root(x,3)
7	cos(x)

We will investigate the affect of changing parameters b, c, and d where g(x) = a * f(bx + c) + d using these functions.

Complete the charts below for each of functions 1, 2, and 7. Carefully observe each graph drawn. (Redraw f(x) after each section; sections are separated by blank lines. You might find it helpful to print out the graphs for each section and compare results.)

a	b	c	d	g(x)	position of g(x) wrt f(x)
1	2	0	0		
1	.5	0	0		
1	-1	0	0		

1	1	2	0		
1	1	.5	0		
1	1	-1	0		

1	1	0	2		
1	1	0	.5		
1	1	0	-1		

a	b	c	d	g(x)	position of g(x) wrt f(x)
1	2	0	0		
1	.5	0	0		
1	-1	0	0		
1	1	2	0		
1	1	.5	0		
1	1	-1	0		
1	1	0	2		
1	1	0	.5		
1	1	0	-1		

a	b	c	d	g(x)	position of g(x) wrt f(x)
1	2	0	0		
1	.5	0	0		
1	-1	0	0		
1	1	2	0		
1	1	.5	0		
1	1	-1	0		
1	1	0	2		
1	1	0	.5		
1	1	0	-1		

a) From our earlier investigations (which you might want to repeat here for function #2 above), given f(x) = x^2, how do the graphs of k*f(x) and f(kx) differ? Why?

b) For f(x) = cos(x), how are the shape and position of the graph of f(bx) similar to (different from) that of the graph of f(x)?

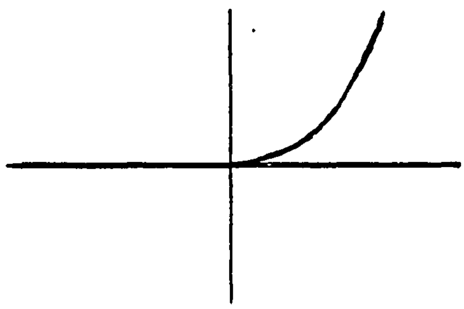
c) For any function f(x), how are the shape and position of the graph of f(x + c) similar to (different from) that of the graph of f(x)?

d) - For any function f(x), how are the shape and position of the graph of f(x) + d similar to (different from) that of the graph of f(x)?

3. An even function is a function that is symmetric to the y-axis. Symbolically, if f is an even function then f(x) = f(-x). Investigate this graphically by using the P option on the function menu for each of the functions in the index with parameters a = 1, b = -1, c = 0, and d = 0.

a) Which of the functions in the index are even?

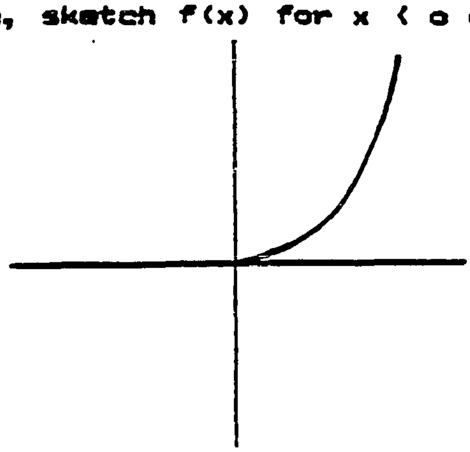
b) The graph below shows the values of f(x) for x > 0. If f(x) is an even function, sketch f(x) for x < o.



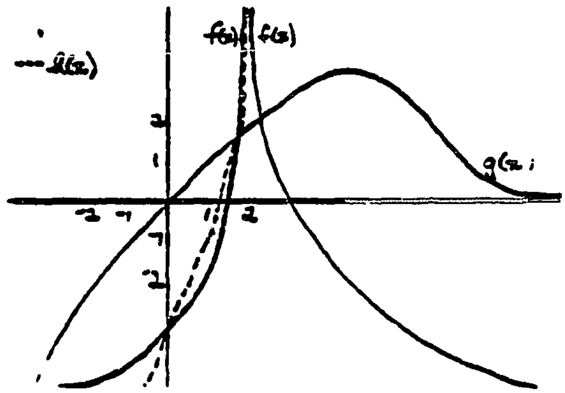
4. An odd function is a function that is symmetric with respect to the origin, (0,0). Geometrically, if f(x) is an odd function, rotating it 180 about the origin returns the original graph. Investigate this graphically for each of the functions in the index using the M option on the function menu.

a) Which of the functions in the index are odd?

b) If the function f(x) (shown only for positive values of f) given in the graph of problem 3b is an odd function, sketch f(x) for x < o on the axes below.



5. The function $h(x) = f(x) + g(x)$ is partially drawn on the axes below. Complete the sketch of $h(x)$.



a) $f(x)$ is not defined (does not exist) for $x = 2$. What is $h(2)$? Explain.

b) As x becomes large positively, $g(x)$ approaches 0. What does $h(x)$ approach? Why?

Lecture Notes, January 15, 1988
Math 122 - Calculus I

We define limit informally as follows:

> The limit of f(x) as x approaches c is L, denoted
> $$\lim_{x \to c} f(x) = L$$
> means that f(x) can be made arbitrarily close to L by making x sufficiently close but not equal to c.

To illustrate the idea of limit, consider the manufacture of piston rings. In order to work properly, piston rings must be manufactured precisely. However, no machine will make a perfect part every time. The best the manufacturer can hope for is that the piston rings are made within a very small acceptable tolerance level of the ideal size. We call the ideal size of the piston ring L, the manufactured size of the piston ring f(x) (where x is the particular setting of the machine at the time this piston ring is made), and we accept the piston ring if it is within a specified tolerance, say e (for error), of L. The piston ring of size f(x) is no larger that L + e and no smaller than L - e. This L is the limiting value referred to in our informal definition.

We now consider the machine which produces the piston rings. Because of power surges, wear and tear on the machinery, length of time between routine maintenance, etc., the machine does not operate at exactly the same levels at all times. The machine operator determines from the operation manual that when the machine controls are set at level c, the machine ideally produces the piston ring of size L. However, the above mentioned situations influence the machine's operation causing it to operate near setting c, but perhaps not actually at c. A gauge on the machine reads the actual operation level x of the machine. We determine that acceptable parts are produced when the machine's actual operation level lies within a tolerance level d either above or below the setting c.

The mathematical concept of limit requires that we be able to attain any level of accuracy e for the size of the piston ring, no matter how small. This means that for every part produced, the actual part size f(x) can be made as close to the perfect size L as we wish when the machine operates sufficiently closely to the machine setting c.

Assignment:

> Using the illustration and informal definition above (and referring to our earlier discussion in class), construct an original example that illustrates the concept of limit in its mathematical sense. Do not use a manufacturing example such as that given above. Be creative, yet mathematically correct. Explain carefully!

Assignment 3
Math 122 - Calculus I

1. Let f and g be functions defined for all real numbers.
   a) Suppose $f(4) = 9$. What point must lie on the graph of f?

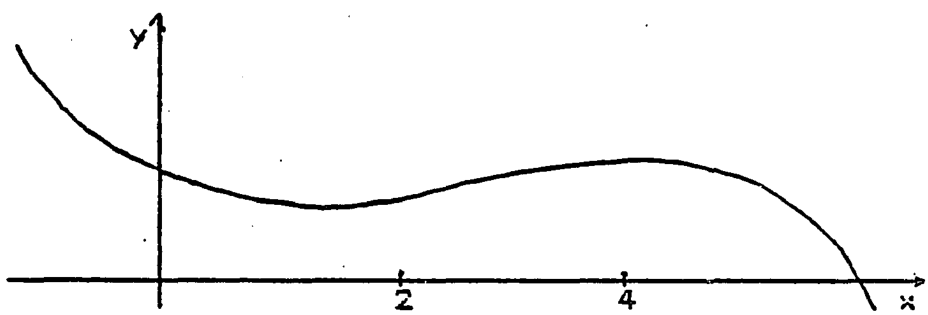
   b) Suppose the point $(-1,2)$ lies on the graph of f. What can be inferred about f?

   c) Suppose that the point $(3, g(3))$ lies on the graph of f. What can be inferred about the relation between f and g?

   d) Suppose the graphs of f and g have a point in common, say (w,z). What can be inferred about the relation between $f(w)$ and $g(w)$?

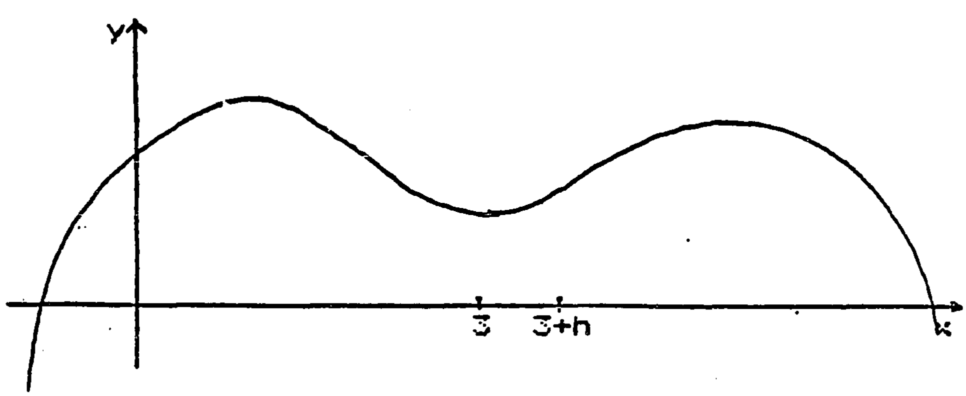
2. Let f be the function whose graph is given below.
   a) Label the points $(2,f(2))$ and $(4,f(4))$ on the graph and draw the secant line through these two points.

   b) Find an expression for the slope of the secant line through the points $(2,f(2))$ and $(4,f(4))$.



3. Let f be the function whose graph is indicated below and suppose h is a (small) positive number.
   a) Label the points $(3,f(3))$ and $(3+h,f(3+h))$ and draw the secant line through these points.

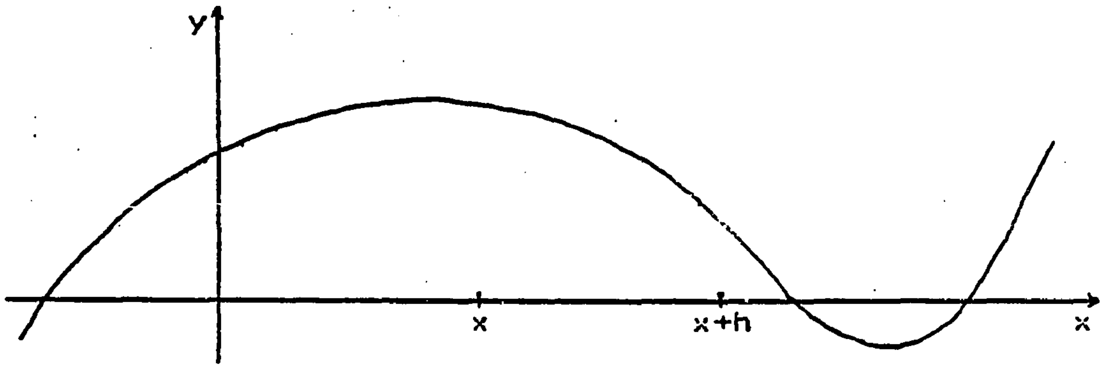
   b) Find an expression for the slope of the secant line through the points $(3,f(3))$ and $(3+h,f(3+h))$.



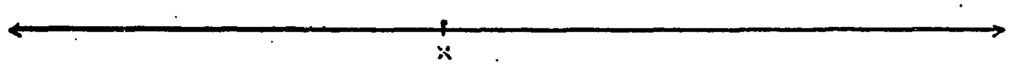
4. Let f be the function whose graph is indicated below. Suppose that x is a number and h is a positive number and f takes values at x and x+h.

   a) Label the points (x,f(x)) and (x+h,f(x+h)) on the graph of f and draw the secant line through these points.

   b) Find an expression for the slope of the secant line through the points (x,f(x)) and (x+h,f(x+h)).



5. Let x be the number represented by the labeled point on the number line below and suppose h is a negative number. Indicate and label a reasonable choice of point to represent the number x+h.



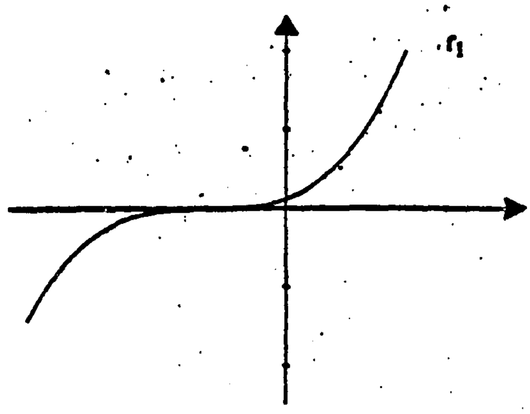
6. Using Master Grapher, enter the following functions into the function index.

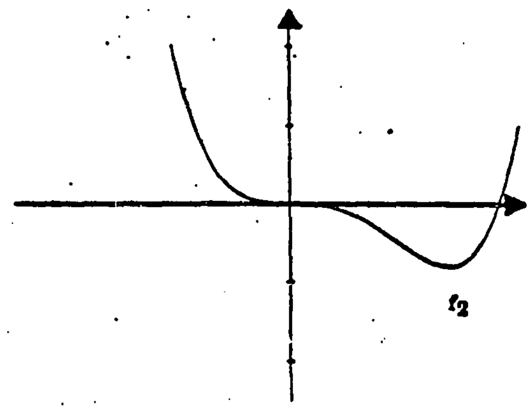
function number	function f(x)
1	sqr(4 - x^2)
2	sin(1/x)
3	x^5 - 5*x^3
4	abs(cos(25x))/25 + x

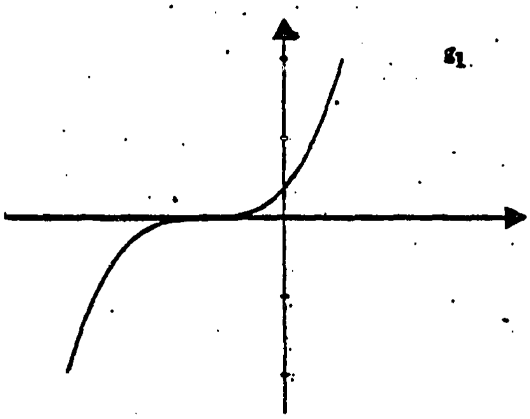
   a) Investigate the behavior of each of the functions around the point (1,f(1)) by using the F option to draw a vertical line through x=1, and zooming in on the intersection of f(x) and x=1 several times. (You will have to redraw x=1 for each new viewing rectangle.) Describe the graphs of each function above when the x interval shown on the screen is no wider than .02.

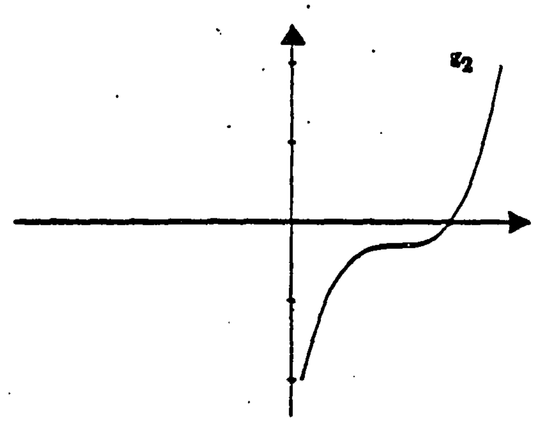
   b) Is it possible to find a value x = c for any of the functions above such that whenever the graph is drawn on an interval including c (no matter how small the interval is) that the graph does not appear as a straight line? If so, which functions produce such graphs and for what value(s) of c?

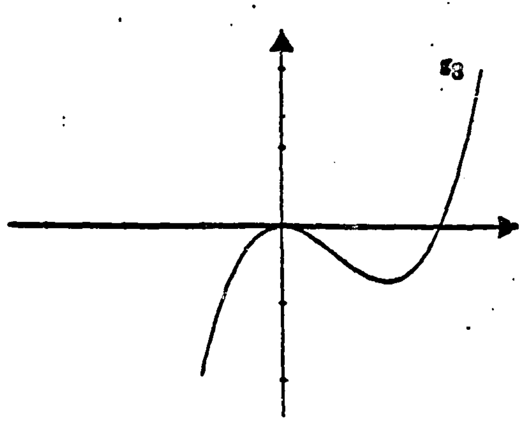
On the first part of this page we have the graphs of functions f_1 and f_2. Among the functions g_1 through g_4, find the graph of the derivative of each function f.

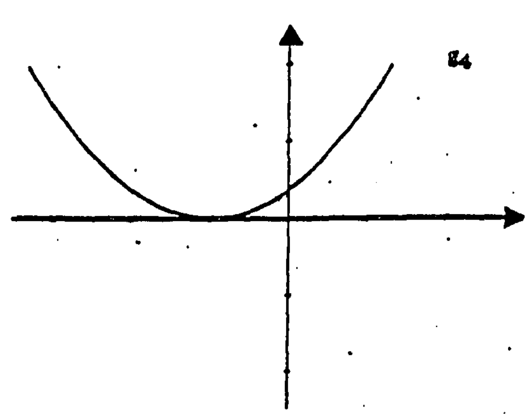




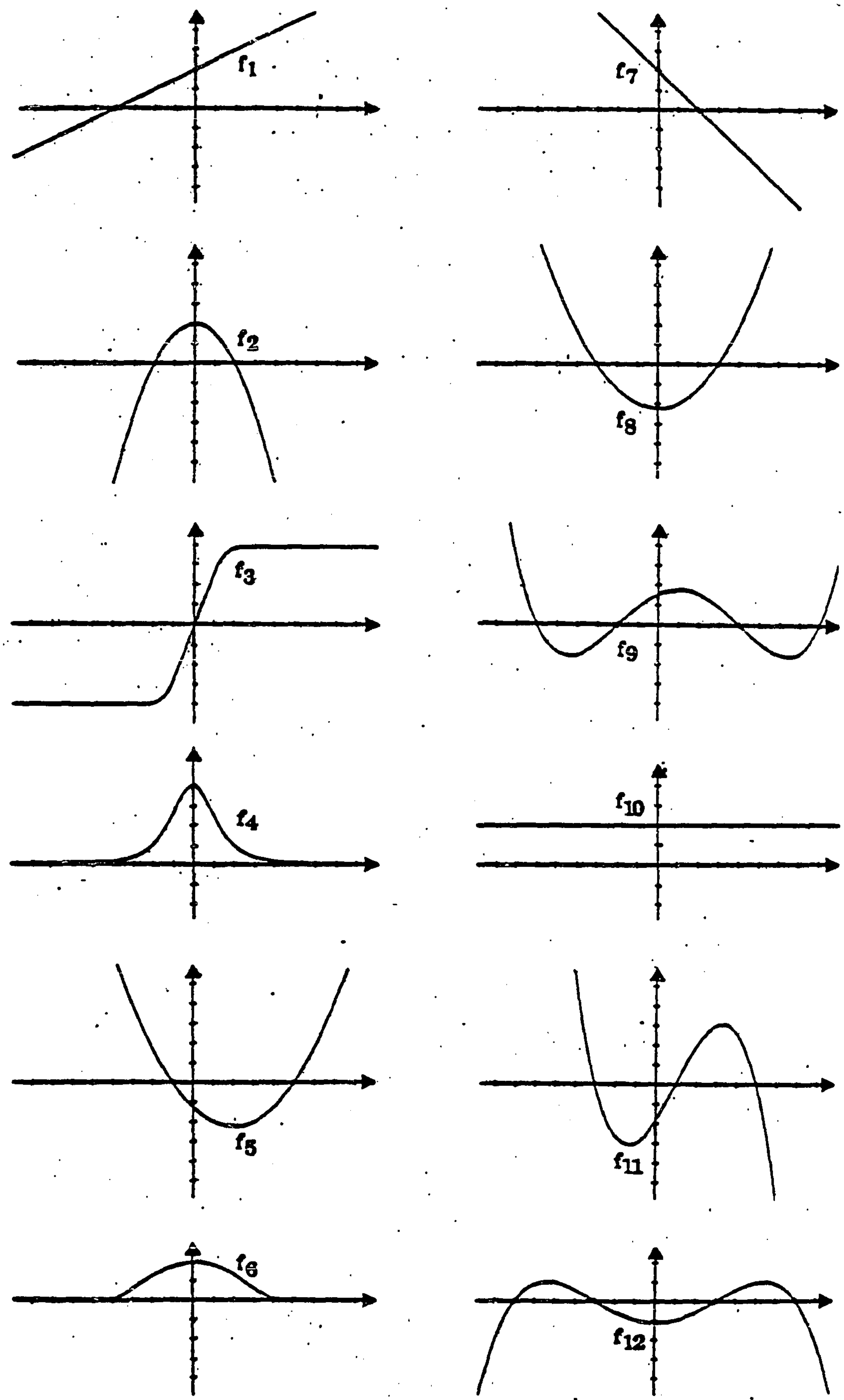


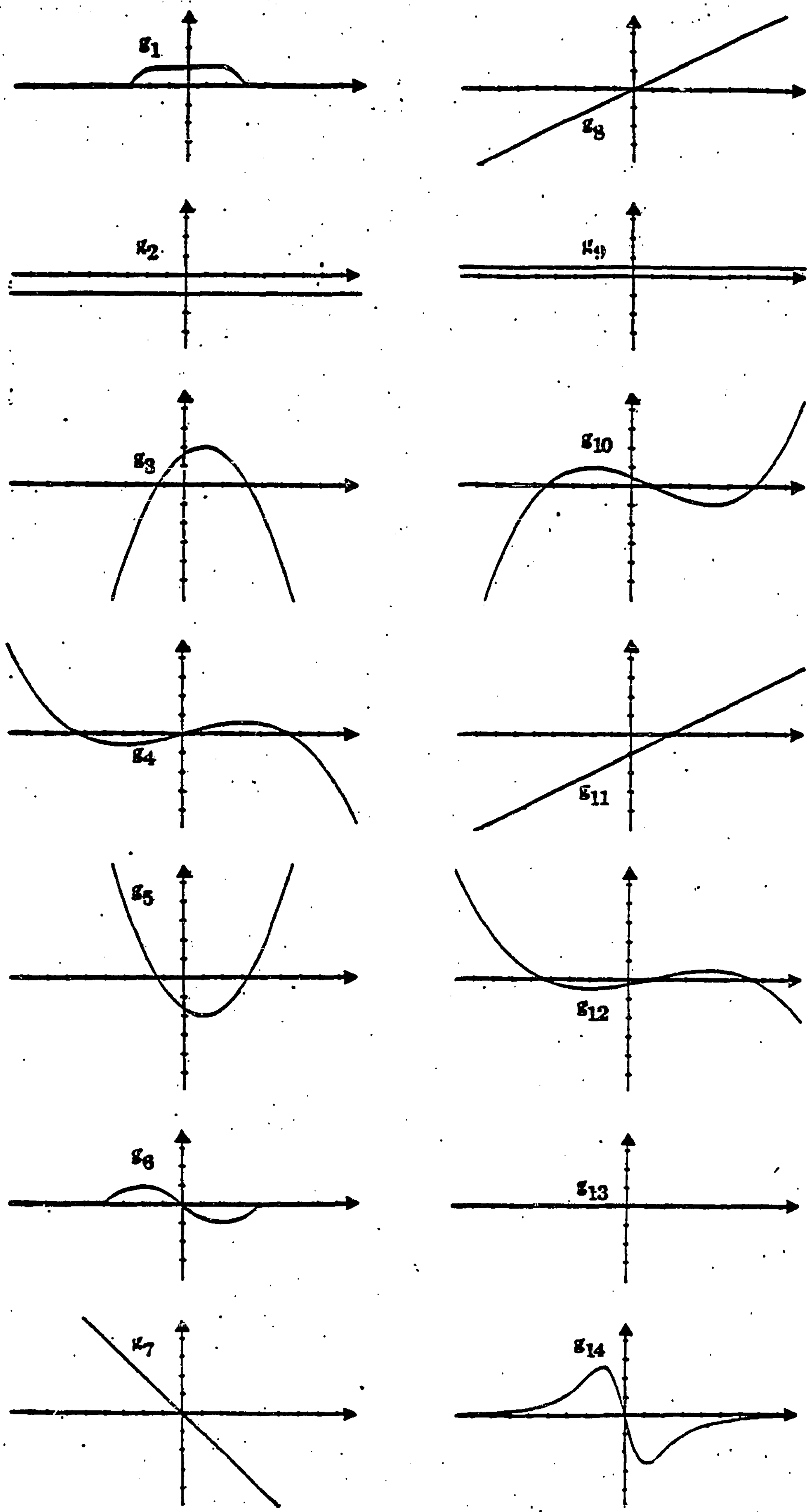






Now consider the graphs f_i and g_j on the following pages. For each graph f_i there is a corresponding graph g_j such that $f_i{}' = g_j$. Determine this correspondence.




g_1
g_2
g_3
g_4
g_5
g_6
g_7
g_8
g_9
g_{10}
g_{11}
g_{12}
g_{13}
g_{14}

APPENDIX G

COMPUTER DEMONSTRATION PROGRAMS

This appendix contains the investigator-written computer programs that were used for in-class demonstration in the development of various calculus concepts.

Programs written in the BASIC language for the Apple IIe are:

1. LIMITS and
2. CLIMITS.

Programs written in True BASIC for the IBM compatible computer in order of their use in class are:

1. SPIDER
2. SPIDERT
3. PLOT
4. PLOT2
5. PLOT3
6. SECANT
7. FASTAN
8. DIFFERENTIALS and
9. NEWTON.

Programs and their uses are described in detail in Appendix A.

LIMITS

```
100   HOME
110   PRINT "THIS PROGRAM WILL PRINT A TABLE": PRINT
120   PRINT "OF VALUES FOR H, F(X+H), AND F(X-H).": PRINT
130   PRINT
140   PRINT "ENTER F(X) BY TYPING: ": PRINT
150   PRINT
160   PRINT "300 DEF FN F(X) = < YOUR FUNCTION >": PRINT
170   PRINT
180   PRINT "ONCE YOUR HAVE DEFINED THE FUNCTION": PRINT
190   PRINT "TYPE 'RUN 300' AND FOLLOW INSTRUCTIONS.": PRINT
200   STOP
300   DEF FN F(X) = (SQR(X) - 1) / (X-1)
310   DEF FN R(X) = INT(10000 * X + .5) / 10000
320   HOME: LIST 300: PRINT
330   H = .5
340   INPUT "LET X = "; X
350   HOME: PRINT "X = "; X
360   LIST 300: PRINT "H"; TAB(9); "F(X+H)"; TAB(24); "F(X-H)"
370   PRINT " "; TAB(9); "----------"; TAB(24); "----------"
380   PRINT: PRINT
390   FOR I = 1 TO 10
410   PRINT FN R(H); TAB(9); FN R(FN F(X+H)); TAB(24); FN R(FN F(X-H))
420   H = H/2
430   NEXT I
440   END
```

CLIMITS

```
100  HOME: PRINT "THIS PROGRAM WILL PRINT A TABLE": PRINT
110  PRINT "OF VALUES FOR X, F(X), G(X), AND": PRINT
120  PRINT "F(X) AND G(X) COMBINED BY AN": PRINT
130  PRINT "OPERATION OF THE USER'S CHOICE.": PRINT
140  PRINT "ENTER F(X) AND G(X) BY TYPING": PRINT
150  PRINT "300 DEF FN F(X) = < YOUR FUNCTION >": PRINT
160  PRINT "310 DEF FN G(X) = < YOUR FUNCTION >": PRINT: PRINT
170  PRINT "ONCE YOU HAVE DEFINED THE FUNCTIONS": PRINT
180  PRINT "TYPE 'RUN 300' AND FOLLOW INSTRUCTIONS.": PRINT
200  STOP
300  DEF FN F(X) = SIN(X) / X
310  DEF G(X) = 1 / (X+1)
320  DEF FN R(X) = INT(10000 * X + .5) / 10000
330  HOME: LIST 300,310
340  INPUT "CHOOSE AN OPERATION (+,-,*,/,^)"; P$
350  IF P$ = "+" THEN P = 1: GOT TO 410
360  IF P$ = "-" THEN P = 2: GOT TO 410
370  IF P$ = "*" THEN P = 3: GOT TO 410
380  IF P$ = "/" THEN P = 4: GOT TO 410
390  IF P$ = "^" THEN P = 5: GOT TO 410
400  GOTO 340
410  INPUT "BEGIN X AT: "; B
420  INPUT "END X AT: "; E
430  INPUT "STEP X BY: "; S
440  GOSUB 700
450  FOR XN = B TO E STEP S
460  LET X = FN R(XN)
470  PRINT X; TAB(9); FN R(FN F(X)); TAB(20); FN R(FN G(X)); TAB(31);
480  IF FN G(X) = 0 AND P$ = "/" THEN PRINT "UNDEFINED": GOTO 600
490  ON P GOTO 500, 510, 520, 530, 540
500  PRINT FN R(FN F(X) + FN G(X)): GOTO 550
510  PRINT FN R(FN F(X) - FN G(X)): GOTO 550
520  PRINT FN R(FN F(X) * FN G(X)): GOTO 550
530  PRINT FN R(FN F(X) / FN G(X)): GOTO 550
540  PRINT FN R(FN F(X) ^ FN G(X))
550  C = C + 1: IF C < 15 THEN GOTO 600
560  PRINT: INPUT "TO CONTINUE PRESS RETURN"; C$
570  GOSUB 700
600  NEXT XN
610  INPUT "TRY AGAIN? (Y/N) "; A$
620  IF A$ <> "Y" THEN GOTO 670
630  INPUT "NEW FUNCTIONS? (Y/N) "; B$
640  IF B$ = "Y" THEN HOME: GOTO 140
650  IF B$ = "N" THEN HOME: LIST 300, 310: GOTO 340
660  GOTO 630
670  END
700  HOME: LIST 300, 310: PRINT
710  PRINT "X"; TAB(9); "F(X)"; TAB(20); "G(X)"; TAB(31); "F(X)"; P$; "G(X)"
720  C = 0
730  RETURN
```

```
! Program SPIDER draws the graph of y = f(x) and a
! "spider" who's height is f(x).

def f(x) = x^5-5*x^2
plot 0,0
print "Enter coordinates for viewing rectangle"
input prompt " a, b, c, d  ":a,b,c,d
set window a, b, c, d
let m = (b-a)/15
let n = (d-c)/10
set color 2
box circle x,x+m,y,y+n
box keep x, x+m,y,y+n in ball$
clear
for x = a to b step abs((a-b)/500)
    set color 1
    plot x, f(x)
    set color 2
    box show ball$ at b-m-1/30,f(x)
next x
end
```

```
! Program SPIDERT draws the graph of y = f(x) and a
! "spider" who's height is f(x).

def f(x) =x^5-5*x^2
let a=-1
let b=2.3
let c=-6
let d=8
set window a, b, c, d
let m = (b-a)/15
let n = (d-c)/10
set color 2
box circle x,x+m,y,y+n
box keep x, x+m,y,y+n in ball$
clear
for x = a to b step abs((a-b)/500)
    set color 1
    plot x, f(x)
    set color 2
    box show ball$ at b-m-1/30,f(x)
next x
end
```

```
! Program PLOT graphs f(x), g(x) and (f+g)(x)
! (can be changed to plot any two functions combined
! with any operation +,-,*,/,^) on [-12,12]x[-10,10].

def f(x) = x
def g(x) = sin(x)
def h(x) = f(x) + g(x)
set window -12,12,-10,10
plot -12,0;12,0     ! x-axis
plot 0,-10;0,10     ! y-axis
for i = -12 to 12
    plot i,-.3; i,0
next i
for j = -10 to 10
    plot -.1,j;0,j
next j
plot text, at -.5, 8.5  : "9"
plot text, at -1.25,-9  : "-9"
plot text, at 9, -1   : " 10"
plot text, at -11,-1 : "-10"
for x = -10 to 10 step .13
    set color 1
    plot x, f(x)
    set color 2
    plot x, g(x)
!    plot x, -f(x)
next x
input prompt"Continue? ":a$
set color 3
for x = -12 to 12 step .15
    plot x,h(x);
next x
plot
end
```

```
! Program PLOT2 draws f(x) and -f(x) then f(-x) to
! investigate whether a function is even or odd.

def f(x) = cos(x)
set window -12,12,-10,10
plot -12,0;12,0     ! x-axis
plot 0,-10;0,10     ! y-axis
for i = -12 to 12
    plot i,-.3; i,0
next i
for j = -10 to 10
    plot -.1,j;0,j
next j
for x = -10 to 10 step .2
    set color 1
    plot x, f(x)
 !   set color 3
 !   plot x, -f(x)
next x
set color 2
for x = -12 to 12 step .05
    plot x,f(-x)
next x
end
```

```
! Program PLOT3 graphs f(x), -f(x), and f*g where f*g
! oscillates between  f and -f.

def f(x) = x^2
def g(x) = sin(1/x)
def h(x) = f(x) * g(x)
plot 0,0
print "Enter coordinates for viewing rectangle"
input prompt " a, b, c, d  ":a,b,c,d
set window a, b, c, d
plot a,0;b,0  ! x-axis
plot 0,c;0,d  ! y-axis
set color 3
for x = a to b step abs((a-b)/200)
    plot x, f(x)
!    plot x, g(x)
    plot x, -f(x)
next x
set color 2
for x = a to b step abs((a-b)/200)
    plot x,h(x);
next x
end
```

```
! Program SECANT draws f(x) and secants between
! user-chosen points c and c+h where c+h is redefined
! iteratively to be the midpoint of [c, c+h].
! The last secant drawn looks like a tangent.  The slope
! of each secant is also displayed.

input prompt "What is the fixed point, c?  ":t
input prompt "What is c+h?  ":c
def f(x) = 2*sin(x)
let m=(f(t)-f(c))/(t-c)
def h(x)=m*(x-t)+f(t)
set window -6,6,-5,5
plot -6,0;6,0     ! x-axis
plot 0,-5;0,5     ! y-axis
for i = -6 to 6  !tick marks x-axis
    plot i,-.2; i,0
next i
for j = -5 to 5  !tick marks y-axis
    plot -.1,j;0,j
next j
for x = -6 to 6 step .04
    set color 3
    plot x, f(x)
next x
do while abs(t-c) > .001
       set color 2
       if t < c then
       let a = t - 1.5
       let b = c + 1.5
       else
       let a = t + 1.5
       let b = c - 1.5
       end if
       plot a,h(a);b,h(b)
       for i = 1 to 1500
           let w = w + 1
       next i
       set color 0
       if a < b then
       for q = a to b step .06
       plot q, h(q)
       next q
       else
       for q = b to a step .06
       plot q, h(q)
       next q
       end if
       set color 2
       let c = (t+c)/2
       let m = (f(t)-f(c))/(t-c)
       plot text, at -5,3.5 : using$("######.#####",m)
```

```
! SECANT continued

loop
let a = t - 1.5
let b = c + 1.5
set color 3
plot a, h(a); b, h(b)
end
```

```
! Program FASTAN plots f(x) , tangents to the curve, and
! the values of  the slopes of these tangents, erasing
! the tangents after each one is drawn...  To be used to
! discover the derivative rules -- power rule for
! polynomials and sin and cos.

def f(x) =x^2
def g(x) = (f(x+.01)-f(x))/.01 !approximate f'
def h(t) = g(x)*(t-x)+f(x)
set window -6,6,-5,5
plot -6,0;6,0    ! x-axis
plot 0,-5;0,5    ! y-axis
for i = -6 to 6  !tick marks x-axis
    plot i,-.2; i,0
next i
for j = -5 to 5  !tick marks y-axis
    plot -.1,j;0,j
next j
set color 3
for x = -6 to 6 step .08
    plot x, f(x);
next x
plot
for x = -6 to 6 step .15
       if f(x)>=-5 and f(x)<=5 then
       set color 2
       let a = x - 1.5
       let b = x + 1.5
       plot a,h(a);b,h(b)
       set color 3
       for t =a to b step .07
           plot t,f(t)
       next t
       set color 1
       plot x, g(x)
       set color 0
       plot a,h(a);b,h(b)
       end if
next x
set color 3
for x = -6 to 6 step .1
    plot x, f(x);
next x
plot
set color 1
for x = -6 to 6 step .1
    plot x, g(x);
next x
end
```

```
! Program DIFFERENTIALS estimates f(x) using the tangent
! approximation method drawing the segments between
! successive approximations.  DIFFERENTIALS then draws
! f(x) to show how good the estimate is.  Note:  This
! program could be improved by drawing the tangent lines
! longer than they are in alternating colors so that the
! intersections of these segments will mark the
! successive approximations.

set window 2.8,3.8,1,5
dim x(100)
dim g(100)
dim t(100)
plot 2.80,1;2.80,5
plot 2.8,1.0;3.8,1.0
for i=1 to 5
    plot -.02,i;0,i
next i
for j = 28 to 38
    plot j/10,-.1;j/10,0
next j
for n = 1 to 9
    let x(n) = 3 + (n-1)/10
    let g(n) = x(n)^2/3
next n
let t(1) = 2
for n = 2 to 9
    let t(n) = t(n-1) + g(n-1) * (x(n) - x(n-1))
next n
set color 1
for n = 1 to 9
    plot x(n), t(n);
    plot text, at x(n),t(n) : "|"
next n
plot x(9),t(9)
input prompt "Continue? " :q$
set color 2
def f(x) = x^3/9 - 1
for j = 28 to 38
    plot j/10, ((j/10)^3)/9-1;
next j
end
```

```
! Newton's method displayed graphically and w/table

def f(x) =(x-2)^3+2
def f1(x)=3*(x-2)^2         !first derivative of f
print "Enter coordinates for viewing rectangle"
input prompt " a, b, c, d  ":a,b,c,d
set window a, b, c, d
set color 6
plot a,0;b,0  ! x-axis
plot 0,c;0,d  ! y-axis
for i = int(a) to b       !tick marks
    plot i,-.3;i,0
next i
for j = int(c) to d
    plot -.1,j;0,j
next j
for x = a to b step abs((a-b)/500)
    plot x, f(x)
next x
input prompt "Search for root starting at: ":w
let count = 0
let e = 1000
def t(x) = f(w) + f1(w)*(x-w)
do while abs(e)>.0000001
   set color 6
   print count;w
   let v = w - f(w)/f1(w)
   set color 4
   if v<w then
      plot v-1.5,T(v-1.5);w+1.5,T(w+1.5)
   else
      plot v+1.5,T(v+1.5);w-1.5,T(w-1.5)
   end if
   for k = 1 to 2000
       let s = s*2
   next k
   set color 5
   plot v,0;v,f(v)
   set color 0
   input prompt "continue? ":q
   if v<w then
      plot v-1.5,T(v-1.5);w+1.5,T(w+1.5)
   else
      plot v+1.5,T(v+1.5);w-1.5,T(w-1.5)
   end if
   set color 6
   if v<w then
      for x = v - 1.5 to w + 1.5 step .03
          plot x, f(x)
      next x
   else
```

```
! NEWTON continued

        for x = w - 1.5 to v + 1.5 step .03
            plot x, f(x)
        next x
    end if
    let e = abs(w-v)
    let w = v
    let count = count + 1
loop
end
```

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