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Generalized Distance in Graphs

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GENERALIZED DISTANCE IN GRAPHS

by

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A Dissertation
Submitted to the
Faculty of The Graduate College
in partial fulfillment of the
requirements for the
Degree of Doctor of Philosophy
Department of Mathematics and Statistics

Western Michigan University
Kalamazoo, Michigan
August 1988

GENERALIZED DISTANCE IN GRAPHS

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In this dissertation, variations of distance in graphs are investigated. First, for a connected graph G, a subset S of $V(G)$ and vertices u, v of G, the S-distance $d_S(u, v)$ from u to v is defined as the length of a shortest $u - v$ walk in G that contains every vertex of S. Extensions of eccentricity, radius, diameter, center and periphery are studied. Several sufficient conditions are given for a connected graph G to contain a subset S of $V(G)$ and a vertex v such that $d_S(v, v) = max\{d_S(u, w) | u, w \in V(G)\}$.

Second, for a connected graph G, the distance $d(v)$ of a vertex $v \in V(G)$ is defined as the sum of the distances from v to all vertices of G. The margin of G is the subgraph induced by the vertices of G having the maximum distance d. Several results concerning the margin of a graph are obtained. In particular, bounds on d are given and every graph is shown to be isomorphic to the margin of some graph.

Third, an extension of antipodal graphs is given for digraphs; namely, for a digraph D, the antipodal digraph $A(D)$ is the digraph with $V(A(D)) = V(D)$ and with the arc (u, v) if and only if $d_D(u, v) = diam\ D$.

If D is not strongly connected, then $diam\ D = \infty$. A characterization of antipodal digraphs is given, and graphs G and digraphs D for which $A(G) \cong G$ and $A(D) \cong D$ are studied. For a graph G, an antipodal sequence is formed, the antipodal period of G is defined, and it is shown, for each positive integer k, that there exists a graph having antipodal period k. This result also holds for digraphs.

Finally, a subgraph distance is studied. In particular, for a connected graph G of order p, an integer n such that $1 \leq n \leq p$ and two subgraphs F and H of G with $p(F) = p(H) = n$, if π is a one-to-one correspondence from the set $V(F)$, say $\{v_1, v_2, \cdots, v_n\}$ to the set $V(H)$, then

$$d(F, H) = \min_{\pi}(\sum_{i=1}^{n} d(v_i,\ \pi(v_i))).$$

The n-subgraph diameter $diam_n(G)$ of G is defined and sharp bounds for it are given. Also properties of the sequence $diam_1(G)$, $diam_2(G)$, $\cdots$, $diam_{p-1}(G)$ are investigated.

Order Number 8901472

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Western Michigan University, 1988

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For my family and friends,
who knew that I would make it.

ACKNOWLEDGEMENTS

I wish to thank Professor Gary Chartrand for his advice and guidance, and, especially, for his patience and encouragement during the writing of this dissertation. My thanks also go to many others in the Department of Mathematics and Statistics of Western Michigan University who have stimulated my interest in mathematics and have given their time and friendship freely. I would like to thank Professor A.J. Boals, Professor C.R. Hirsch, Professor S.F. Kapoor, and Professor O.R. Oellermann for serving on my committee. Finally, my thanks go to Margo Johnson for typing this manuscript.

Garry L. Johns

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CHAPTER I

PRELIMINARIES

1.1 Introduction

Historically, graphs have been used as models for studying the structure and relationships in many real-world situations. One relationship that has received considerable attention is the distance between vertices of a graph. For a connected graph G and a pair u, v of vertices of G, we define the distance $d_G(u,v)$ (or simply $d(u,v)$) as the length of a shortest $u-v$ path in G. With the aid of an algorithm developed by Dijkstra [7], computing distances is straightforward.

Many applications require altering the definition of distance; however, one concept has always been of interest, namely, the idea of "centrality". Intuitively, in a graph G, the "central" vertices are those vertices of G whose distance to all other vertices of G is "small." This idea of "centrality" has led to the study of graphical structures such as centers, centroids, k-centra and medians (see [6]).

Recently, applications have arisen where we consider the opposite extreme. That is, we consider "boundary" vertices in a graph G whose distance to all other vertices of G is "large." Many of the results in this dissertation deal with these "boundary" vertices.

The primary purpose of this dissertation is to study generalizations of distance in graphs. In Chapters II and V, we introduce, for a connected graph G, new definitions for the distance between vertices of G and the distance between subgraphs of G, respectively. In Chapters III and IV, we expand on known results; namely, the distance of a vertex and antipodal graphs.

In Chapter II, we define, for a connected graph G, a subset S of $V(G)$ and vertices u, v of G, the S-distance $d_S(u,v)$ from u to v as the length of a shortest $u-v$ walk in G that contains every vertex of S. We also consider the S-eccentricity $e_S(v)$ of a vertex v, the S-radius of a graph and the S-diameter $diam_S(G)$ of a graph G. We then define the S-center of a graph and show, for every graph G and each nonnegative integer $n \neq 1$, that there exists a graph H and a subset S of $V(H)$ with $|S| = n$ such that the S-center of H is isomorphic to G. Similarly, we define the S-periphery of a graph and, for a graph G and a positive integer n, we show that there exists a graph H and a subset S of $V(H)$ with $|S| = n$ such that the S-periphery of H is isomorphic to G. An unusual property of S-distance is that, for a connected graph G and a subset S of $V(G)$, there often exists a vertex v such that $d_S(v,v) = diam_S(G)$. Several sufficient conditions are given for a graph G that insure the occurrence of this property. Several results for S-distance in trees are presented and the chapter is concluded by studying the n-eccentricity $e_n(v)$ for a vertex v where $e_n(v) = \max_{|S|=n} e_S(v)$.

be a pairing from $V(F)$ to $V(H)$ such that $d(F, H) = d_{\pi_1}(F, H)$ and $\pi_1(u_i) = u_i$ for a maximum number n of vertices in $\{u_1, u_2, \ldots, u_k\}$. Without loss of generality, suppose that $\pi_1(u_1) \neq u_1$. Now, let $\pi_1(u_1) = v$ and $u \in V(F)$ satisfy $\pi(u) = u_1$. Then $d(u, v) \leq d(u, u_1) + d(u_1, v)$. Thus, we can define the pairing π from $V(F)$ to $V(H)$ so that $\pi(u_1) = u_1$, the image $\pi(u) = v$ and, for $w \neq u_1, u$, the image $\pi(w) = \pi_1(w)$. Therefore, $\pi(u_i) = u_i$ for $m + 1$ vertices in $\{u_1, u_2, \ldots, u_k\}$ and, since $d_\pi(F, H) \leq d_{\pi_1}(F, H)$, it follows that $d_\pi(F, H) = d(F, H)$, which contradicts the choice of π_1. •

An immediate consequence of Theorem 5.1 is the following.

For a connected graph G of order p, if F and H are subgraphs of order n, with $\lceil \frac{p}{2} \rceil \leq n \leq p$, then $V(F)$ and $V(H)$ have vertices in common, say $u_1, u_2, \ldots, u_k$, where $k \leq n$. Let $F' = F - \{u_1, u_2, \ldots, u_k\}$ and let $H' = H - \{u_1, u_2, \ldots, u_k\}$. Then by Theorem 5.1,

$$d(F,H) = \sum_{i=1}^{k} d(u_i,u_i) + d(F',H') = d(F',H'). \tag{5.1}$$

For a subgraph F of a connected graph G with $p(F) = n$, we define the <u>subgraph eccentricity</u> $e(F)$ of F as $e(F) = \max_{p(H)=n} d(F,H)$. We define the n- <u>radius</u> $rad_n\, G$ of G as $rad_n\, G = \min_{p(F)=n} e(F)$ and the n-<u>diameter</u> $diam_n\, G$ of G as $diam_n\, G = \max_{p(F)=n} e(F)$. For a connected graph G, we define the <u>diameter sequence</u> as the sequence

is the smallest positive integer such that there exists a nonnegative integer ℓ with $A^{\ell}(G) \cong A^{\ell+k}(G)$. We define the antipodal period for digraphs in the same way, and show that for each positive integer k there exists a graph and a digraph having antipodal period k.

In Chapter V, we define a subgraph distance. In particular, for a connected graph G of order p, an integer n such that $1 \leq n \leq p$ and two subgraphs F and H of G with $p(F) = p(H) = n$, we define a pairing π from the set $V(F)$, say $\{v_1, v_2, \ldots, v_n\}$, to the set $V(H)$ that associates a vertex of $V(F)$ with one of $V(H)$. Now the subgraph distance between F and H is

$$d(F, H) = \min_{\pi}(\sum_{i=1}^{n} d(v_i, \pi(v_i))).$$

For a graph G and an integer n such that $1 \leq n \leq p$, we define the n-subgraph radius and the n-subgraph diameter $diam_n(G)$. We present sharp bounds for $diam_n(G)$ and study the subgraph diameter sequence $diam_1(G), diam_2(G), \ldots, diam_{p-1}(G)$.

1.2 Definitions and Notations

In this section we present some basic definitions and notation that will be used throughout the dissertation. Additional more specialized definitions will be given later as required. Definitions of all other terms will be consistent with [4].

For a connected graph G, we define the distance $d(u,v)$ between two vertices u and v as the length of a shortest $u - v$ path in G. The eccentricity $e(v)$ of a vertex v of a connected graph G is the number $max_{u \in V(G)} d(u,v)$. The radius $rad\, G$ is defined as $min_{v \in V(G)}\, e(v)$ while the diameter $diam\, G$ is $max_{v \in V(G)}\, e(v)$. A vertex is a central vertex if $e(v) = rad\, G$ and the center $C(G)$ of G is the subgraph induced by the set of central vertices. Similarly, a vertex v is a peripheral vertex if $e(v) = diamG$ and the periphery $Per\, G$ of G is the subgraph induced by the set of peripheral vertices.

For a strongly connected digraph D, the distance $d(u,v)$ between two vertices u and v is the length of a shortest $u - v$ directed path in D. Now, the above definitions extend naturally to arbitrary digraphs. It is well-known, for a connected graph G, that

$$rad\, G \leq diam\, G \leq 2\, rad\, G.$$

Similarly, for a strongly connected digraph D, we have $rad\, D \leq diam\, D$; however, $diam\, D$ is not bounded above by $2\, rad\, D$. A second well-known result is that for any graph G, there exists a graph H such that $C(H) \cong G$ and $p(H) \leq p(G) + 4$. We state a similar result of O. R. Oellermann [13] for the periphery of a graph.

PROPERTY 1.1: A graph G of order $p \geq 1$ is the periphery of a connected graph H if and only if $\Delta(G) \leq p - 2$ or $G \cong K_p$.

We will write $F \subset G$ if the graph F is a subgraph of the graph G and $F \prec G$ if F is an induced subgraph of G. As a matter of convenience for the reader, the symbol $\bullet$ is used to designate the end of a proof.

CHAPTER II

S-DISTANCE

2.1 Introduction

When determining the delivery route for a distributor, one tries to find a shortest route that begins and ends at the dispatching center and passes through each of the required stops.

In this chapter we discuss a variation of distance which models the delivery route just described. For a connected graph G, a subset S of $V(G)$ and $u, v \in V(G)$, we define a $u - v$ S-walk as a $u - v$ walk in G that contains every vertex of S. The S- distance $d_S(u, v)$ from u to v, is the length of a shortest $u - v$ S- walk. These ideas are illustrated in the graph G in Figure 2.1. We will indicate the vertices of S, here and in the sequel, by shading them. Hence, $S = \{v, x\}$ and a shortest $u - v$ S- walk is $W : u, w, x, w, v$. The length of W is 4, so $d_S(u, v) = 4$. Similarly, $d_S(u, x) = 4$, $d_S(u, w) = 5$ and $d_S(u, u) = 6$. It is interesting to note that the S-distance from u to itself is greater than the S-distance from u to any other vertex of G. This is not coincidental as we will show later.

Note that the S-distance in a connected graph G is a generalization of distance because $d_\emptyset(u, v) = d(u, v)$ for all $u, v \in V(G)$.

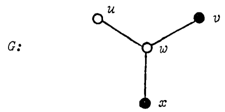


Figure 2.1

The following observations will be used throughout this chapter. For $u, v, w \in V(G)$ and $S \subseteq V(G)$:

(a) $d_S(u,v) \geq max\{|S|-1,\ 0\}$;

(b) $d_S(u,v) = d_S(v,u)$; [Symmetry]

(c) if $S \subseteq T \subseteq V(G)$, then $d_S(u,v) \leq d_T(u,v)$;

(d)

$$d_S(u,v) \leq \begin{cases} d(u,w) + d_S(w,v) \\ d_S(u,w) + d(w,v); \end{cases}$$

(e)

$$\begin{aligned} d_S(u,v) &\leq d(u,w) + d_S(w,v) && \text{[Triangle Inequality]} \\ &= d_\emptyset(u,w) + d_S(w,v) \\ &\leq d_S(u,w) + d_S(w,v); \text{ and} \end{aligned}$$

(f) $d_S(u,v) = 0$ if and only if $u = v$ and $S \subseteq \{u\}$.

In general, the S-distance is not a metric (not even a pseudometric) because (f) states that $d_S(u,u) \not\equiv 0$.

For each vertex v of a connected graph G with $S \subseteq V(G)$, we define the S- eccentricity of v as

$$e_S(v) = max\{d_S(v,u)|u \in V(G)\}.$$

Recall that $d(v,v) \equiv 0$; however, as we showed for the graph in Figure 2.1, $e_S(u) = d_S(u,u) > 0$.

A useful property that relates the S-eccentricities of adjacent vertices in a graph is given next.

THEOREM 2.1. *If G is a connected graph, $S \subseteq V(G)$ and $uv \in E(G)$, then* $|e_S(u) - e_S(v)| \leq 1$.

PROOF: Without loss of generality, we may assume that $e_S(v) \leq e_S(u)$. Then there is a vertex $w \in V(G)$ such that $e_S(u) = d_S(u,w)$, where possibly $w = u$. Since

$$\begin{aligned} d_S(u,w) &\leq d(u,v) + d_S(v,w) \\ &\leq 1 + e_S(v), \end{aligned}$$

we have

$$e_S(u) \leq 1 + e_S(v) \text{ or } e_S(u) - e_S(v) \leq 1.$$

Therefore $|e_S(u) - e_S(v)| \leq 1$. •

We will now concentrate on two particular values of the S- eccentricity. For a connected graph G and $S \subseteq V(G)$, the S-radius of G is $rad_S(G) = min\{e_S(v)|v \in V(G)\}$ and the S-diameter of G is $diam_S(G) = max\{e_S(v)|v \in V(G)\}$. Note that if $S = \emptyset$, these definitions give the conventional radius and diameter of G.

Two results that hold for distance are also true for the S- distance.

THEOREM 2.2. *For a connected graph G, a subset S of $V(G)$ and an integer c such that $rad_S(G) < c < diam_S(G)$, there is a vertex v of G such that $e_S(v) = c$.*

PROOF: Let u and v be vertices of G that satisfy $e_S(u) = rad_S(G)$ and $e_S(v) = diam_S(G)$. Let P be a $u-v$ path in G and let w_1 be the last vertex of P with $e_S(w_1) < c$. If w_2 is the next vertex on P, then $e_S(w_2) \geq c$ and $|e_S(w_1) - e_S(w_2)| \leq 1$ by Theorem 2.1. Therefore, $e_S(w_2) = c$. •

The second result gives a familiar bound for $diam_S(G)$.

THEOREM 2.3. *For a connected graph G and $S \subseteq V(G)$, the S-radius $rad_S(G) \leq diam_S(G) \leq 2\, rad_S(G)$.*

PROOF: The first inequality follows from the definitions. For the second inequality, let u and v be vertices of G (possibly $u = v$) such that $d_S(u,v) = diam_S(G)$ and let $w \in V(G)$ such that $e_S(w) = rad_S(G)$.

Then $diam_S(G) = d_S(u,v) \leq d_S(u,w) + d_S(w,v) \leq 2e_S(w) = 2rad_S(G)$. •

We now consider the sharpness of these bounds. For the lower bound, if $|S| = 0$, it is known that $rad(C_p) = diam(C_p) = \lfloor \frac{p}{2} \rfloor = rad_S(C_p) = diam_S(C_p)$ for any positive integer p where we define $C_1 \cong K_1$ and $C_2 \cong K_2$. If $|S| = 1$ and G is a connected graph such that $rad_S(G) = diam_S(G)$, then $G \cong K_1$. This will follow from Theorem 2.9. For each integer $n > 1$, the infinite class $\mathcal{G}_n$ of paths P_m, $m \geq n$, with $S \subseteq V(P_m)$ such that $|S| = n$ and both end-vertices of P_m in S, satisfies $rad_S(P_m) = diam_S(P_m)$. To see this let $P_m : v_1, v_2, \ldots, v_{m-1}, v_m$ be such a path. Now for $v_i \in V(P_m)$,

$$d_S(v_i, v_j) = \begin{cases} d(v_i,v_1) + d(v_1,v_m) + d(v_m,v_j) & \text{if } i \leq j \\ d(v_i,v_m) + d(v_m,v_1) + d(v_1,v_j) & \text{if } i > j. \end{cases}$$

In either case, $d_S(v_i,v_j) = 2(m-1) - d(v_i,v_j)$ and $e_S(v_i) = d_S(v_i,v_i) = 2(m-1)$. Therefore, $rad_S(P_m) = diam_S(P_m) = 2(m-1)$.

A more general class of graphs than $\mathcal{G}_n$ that satisfies the lower bound for $n > 1$ is the class of all trees whose end-vertices are in S. This will be proved in Corollary 2.4A.

The upper bound is also sharp for each nonnegative integer n. For $|S| = n = 0$, consider the class of stars $K(1,m)$, $m \geq 2$. In this case, $diam_S(K(1,m)) = 2$ and $rad_S(K(1,m)) = 1$. For $|S| = n > 0$, we

first define an n- longated star $K_n(1,m)$, $m \geq 2$, formed by subdividing $n - 1$ times each edge of $K(1,m)$. For example, $K_1(1,m) \cong K(1,m)$ and $K_3(1,4)$ is shown in Figure 2.2.

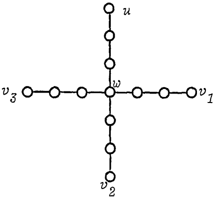


Figure 2.2

Let u be an end-vertex of $K_n(1,m)$ and w the vertex of degree m. For $m \geq 2$, we define $G_m \cong K_n(1,m)$ with S as the set of n vertices on the $u - w$ path in $K_n(1,m)$ other than w. If $v_1, v_2, \ldots, v_{m-1}$ are the end-vertices of G_m other than u, then for $1 \leq i \leq m - 1$, we have $d_S(u,v_i) = e_S(u) = 2n$ and $d_S(v_i,v_i) = e_S(v_i) = 4n$. If x is an interior vertex of G_m, then there is an end-vertex v_j such that x is on the $u - v_j$ path in G_m. Since $d_S(u,v_j) = e_S(v_j) - e_S(u) = 2n$, it follows that $e_S(u) < e_S(x) < e_S(v_j)$ by Theorem 2.1 and Theorem 2.2. Therefore, $diam_S(G_m) = e_S(v_i) = 2e_S(u) = 2rad_S(G_m)$, for $1 \leq i \leq m - 1$. Hence, for $n > 0$, the class $\mathcal{G}_n = \{G_m | m \geq 2\}$ is an infinite class of graphs that verifies the sharpness of the upper bound

in Theorem 2.3.

By focusing our attention on trees, even more can be said about S-distance.

THEOREM 2.4. *Let T be a tree with a nonempty subset S of $V(T)$ such that every end-vertex of T is in S. Then for every pair u, v of vertices, $d_S(u,v) = 2q(T) - d(u,v)$.*

PROOF: We first show that $d_S(u,v) \geq 2q(T) - d(u,v)$. Let W be a shortest $u-v$ S-walk and let P be the unique $u-v$ path in T. Suppose that e is an edge of P. Since W is a $u-v$ walk, it contains every edge in P; so e appears at least once in W.

Now suppose f is an edge of T that is not in P. Then there exists an end-vertex w of T such that f is on the $u-w$ path P_u in T. Since $w \in S$ and every edge in P_u appears in W, it follows that f occurs at least once in W. If f is also on the $v-w$ path P_v in T, then f occurs at least twice in W. If f is not on P_v, then P_u is a $u-w$ path containing f and P followed by P_v is a $u-w$ walk (containing a $u-w$ path) not including f. Hence, T would contain a cycle, which is impossible. Therefore $d_S(u,v) \geq d(u,v) + 2[q(T) - d(u,v)] = 2q(T) - d(u,v)$.

To show that $d_S(u,v) \leq 2q(T) - d(u,v)$, let M be the multigraph formed by duplicating each edge of T that is not on the $u-v$ path P. Now the degree of every vertex is even except for u and v. Hence M contains an eulerian $u-v$ trail of length $2q(T) - d(u,v)$ and the associated

$u-v$ walk W in T is an S-walk of length $2q(T) - d(u,v)$. Since $d_S(u,v) \leq 2q(T) - d(u,v)$, we have the desired result. •

An important corollary follows immediately.

COROLLARY 2.4A. *Let T be a tree with a nonempty subset S of $V(T)$ such that every end-vertex of T is in S. Then for every vertex v, the S-eccentricity $e_S(v) = d_S(v,v) = 2q(T)$.*

Theorem 2.4 can be extended to all trees with the help of the next lemma. To do this we first define for a tree T and a nonempty subset S of $V(T)$, the tree T_S generated by S as the smallest subtree of T containing every vertex of S. It is useful to note that every end-vertex of T_S is a vertex of S and for each vertex u in T there exists a unique vertex u' of T_S satisfying $d(u,u') = min\{d(u,w)|w \in V(T_S)\}$.

LEMMA 2.5. *Let T be a tree with a nonempty subset S of $V(T)$. For every pair u,v of vertices in T_S, the S-distance $d_S(u,v) = 2q(T_S) - d(u,v)$.*

PROOF: Let W be a shortest $u-v$ S-walk in T. Suppose that e is an edge in W that is not in $E(T_S)$. Then at least one of the vertices, say w, incident with e is not in $V(T_S)$; otherwise, T would contain a cycle. Let x be the last vertex in T_S that precedes w on W and let y be the first vertex in T_S that follows w on W. Now $x = y$. For if not, then T contains an $x-y$ path in T_S and an $x-y$ path passing through w that is not in T_S. Hence T contains a cycle. Now a shorter $u-v$ S-walk

W' can be formed from W by removing the edges between x and w and those between w and y. However, this contradicts the minimality of W. Therefore, every edge of W is in $E(T_S)$. Since Theorem 2.4 applies to T_S, it follows that $d_S(u,v) = 2q(T_S) - d(u,v)$. •

We are now ready to give a formula for $d_S(u,v)$ for every pair u,v of vertices in a tree T with a nonempty subset S of $V(T)$.

THEOREM 2.6. *Let* T *be a tree with a nonempty subset* S *of* $V(T)$. *For every pair* u,v *of vertices, define vertices* u', v' *of* T_S *such that* $d(u,u') = min\{d(u,w)|w \in V(T_S)\}$ *and* $d(v,v') = min\{d(v,w)|w \in V(T_S)\}$. *Then* $d_S(u,v) = d(u,u') + d(v,v') + 2q(T_S) - d(u',v')$.

PROOF: Let W be a shortest $u-v$ S-walk in T. Let w be the first vertex of T_S on W and let x be the last vertex of T_S on W. Then

$$d_S(u,v) = d(u,w) + d_S(w,x) + d(x,v).$$

Suppose $w \neq u'$. Then there exists a $w-u'$ path in T_S and a $w-u'$ path passing through u that is not in T_S. Thus, T would have a cycle; a contradiction. Therefore, $w = u'$ and by a similar argument, $x = v'$. Finally, by Lemma 2.5 we have $d_S(w,x) = 2q(T_S) - d(w,x)$, so $d_S(u,v) = d(u,u') + d(v,v') + 2q(T_S) - d(u',v')$. •

We can now give two results on the S-diameter for trees.

THEOREM 2.7. *Let T be a tree with a nonempty subset S of $V(T)$. If vertices u, v satisfy $d_S(u,v) = diam_S(T)$, where possibly $u = v$, then $e_S(u) = d_S(u,u) = diam_S(T)$ and $e_S(v) = d_S(v,v) = diam_S(T)$.*

PROOF: By Theorem 2.6, there exist vertices u' and v' in T_S such that $d(u,u') = min\{d(u,w)|w \in T_S\}$ and $d(v,v') = min\{d(v,w)|w \in T_S\}$ and such that

$$d_S(u,u) = 2d(u,u') + 2q(T_S) \text{ and}$$

$$d_S(v,v) = 2d(v,v') + 2q(T_S).$$

Suppose that $d_S(u,u) < d_S(u,v)$. Then, because $d_S(u,v) = d(u,u') + 2q(T_S) - d(u',v') + d(v',v)$, we have $d(u,u') < d(v,v') - d(u',v')$ and by the triangle inequality $d(u,v') < d(v,v')$. This implies that $d(u,u') + d(u',v') < d(v,v')$, and by the triangle inequality, $d(u,v') < d(v,v')$.

Similarly, if $d_S(v,v) < d_S(u,v)$ then $d(v,u') < d(u,u')$. Thus, suppose that $d_S(u,u) < d_S(u,v)$ and $d_S(v,v) < d_S(u,v)$. Then $d(u,u') < d(u,v')$ and $d(v,v') < d(v,u')$, so that

$$d(u,v') < d(v,v') < d(v,u') < d(u,u') < d(u,v'),$$

which is a contradiction. Therefore, either $e_S(u) = d_S(u,u) = diam_S(T)$ or $e_S(v) = d_S(v,v) = diam_S(T)$.

Without loss of generality, suppose that $d_S(u,v) = e_S(u) = diam_S(T)$. We show next that $e_S(v) = diam_S(T)$ also. Assume, to the contrary, that

$d_S(v,v) < d_S(u,u) = diam_S(T)$. Then, since $d_S(v,v) = 2d(v,v') + 2q(T_S) < d_S(u,u) = 2d(u,u') + 2q(T_S)$, we have $d(v,v') < d(u,u')$. However, since $d_S(u,u) = diam_S(T) = d_S(u,v)$ we have, by Theorem 2.6 and the above remark, $2d(u,u') + 2q(T_S) = d(u,u') + 2q(T_S) - d(u',v') + d(v',v)$. This implies that $d(u,u') = d(v,v') - d(v',u') \leq d(v,v')$, giving a contradiction. Therefore, $e_S(u) = d_S(u,u) = diam_S(T)$ and $e_S(v) = d_S(v,v) = diam_S(T)$. •

We can now state an immediate corollary.

COROLLARY 2.7A. *For a tree T and a nonempty subset S of $V(T)$, there exists a vertex v of T such that $e_S(v) = d_S(v,v) = diam_S(T)$.*

The next result follows easily.

COROLLARY 2.7B. *For a tree T and a nonempty subset S of $V(T)$, the S-diameter of T is always even.*

PROOF: Let u be a vertex of T such that $d_S(u,u) = diam_S(T)$ and let u' be the vertex in T_S such that $d(u,u') = min\{d(u,w)|w \in T_S\}$. Then

$$d_S(u,u) = 2d(u,u') + 2q(T_S)$$

and $diam_S(T)$ is even. •

2.2 The S-Center and S-Periphery of a Graph

Having defined $rad_S(G)$ and $diam_S(G)$ as an extension of $rad(G)$ and $diam(G)$ for a connected graph G and $S \subseteq V(G)$, it is natural to define extensions of $C(G)$ and $Per(G)$. For a connected graph G with $S \subseteq V(G)$, the S-center of G is $C_S(G) = \langle\{v \in V(G)|e_S(v) = rad_S(G)\}\rangle$ and the S-periphery of G is $Per_S(G) = \langle\{v \in V(G)|e_S(v) = diam_S(G)\}\rangle$.

We first investigate those graphs that are isomorphic to the S-center of some connected graph.

THEOREM 2.8. *For a nonempty graph G and a nonnegative integer $n\ (\neq 1)$, there is a connected graph H and subset of S of $V(H)$ for which $|S| = n$ and $C_S(H) \cong G$.*

PROOF: For $n = 0$, let $F = G + \overline{K_2}$ and label the vertices of F that are not in $V(G)$ as u_1 and u_2. Add two vertices v_1 and v_2 to F such that v_i is adjacent to u_i for $i = 1, 2$ and denote this graph by H (see Figure 2.3(a)). Note, since $S = \emptyset$, it follows that $e_S(v_i) = e(v_i) = d(v_i, v_j) = 4$ where $i \neq j$ and $e_S(u_i) = e(u_i) = d(u_i, v_j) = 3$ where $i \neq j$. For $w \in V(G)$, we have $e_S(w) = e(w) = d(w, v_1) = 2$. Therefore, $G = C_S(H)$ when $|S| = 0$.

For $n \geq 2$, let $F = G + \overline{K_n}$ and label the vertices of F that are not in $V(G)$ as $u_1, u_2, \dots, u_n$. Add n vertices $v_1, v_2, \dots, v_n$

to F such that v_i is adjacent to u_i for $1 \leq i \leq n$, and call this graph H_n (see Figure 2.3(b)). Let $S = \{u_1, u_2, \ldots, u_n\}$. Then for $w \in V(G)$, we have $e_S(w) = d_S(w,w) = 2n$. If $1 \leq i \leq n$, then $e_S(v_i) = d_S(v_i, v_i) = 2n + 2$ and by Theorem 2.1, $e_S(u_i) = 2n + 1$ for $1 \leq i \leq n$. Thus $G = C_S(H_n)$ when $|S| \geq 2$ and this completes the proof. •

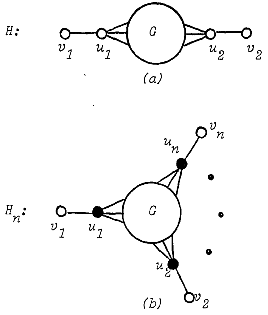


Figure 2.3

We can generalize the graphs in Figure 2.3 by replacing each vertex v_i with a copy of K_{m_i} where m_i is a positive integer for $1 \leq i \leq n$. The S-eccentricities of the vertices of K_{m_i} are the same as $e_S(v_i)$ and the values of $e_S(u_i)$, $1 \leq i \leq n$, and $e_S(w)$, $w \in V(G)$, remain the same.

For each $n \geq 2$, we have constructed a class of graphs H such that $|S| = n$ and $S \cap V(C_S(H)) = \emptyset$. We have also shown earlier a class of graphs H where $rad_S(H) = diam_S(H)$. For these graphs $S \subseteq V(C_S(H))$. For the class of graphs H that we formed having $diam_S(H) = 2rad_S(H)$, we have $V(C_S(H)) \subseteq S$. In fact, if $H \cong K_{n+m}$, $n \geq 2$, $m \geq 1$, with $V(H) = \{v_1, v_2, \ldots, v_{n+m}\}$ and $S = \{v_1, v_2, \ldots, v_n\}$, then $S = V(C_S(H))$ with $rad_S(H) = n$ and $diam_S(H) = n + 1$. Therefore, there appears to be no consistent relationship between S and $V(C_S(H))$ when $|S| > 1$. So far we have ignored the case when $|S| = 1$. It is interesting to observe that for $|S| = 1$, a definite relationship holds between S and $V(C_S(H))$ as the next result shows.

THEOREM 2.9. *If G is a connected graph and $S \subseteq V(G)$ with $|S| = 1$, then $S = V(C_S(G))$.*

PROOF: Suppose that $S = \{v\}$ and $v \notin V(C_S(G))$. Then there is a vertex $w \in V(G)$ such that $d_S(v,w) > rad_S(G)$. Let $u \in V(C_S(G))$. Then $d_S(u,w) = d(u,v) + d(v,w) = d(u,v) + d_S(v,w) > rad_S(G)$, contradicting the fact that $u \in V(C_S(G))$. Therefore, $S \subseteq C_S(G)$.

Now, let $u \in V(G) - S$ where $S = \{v\}$. Since $S \subseteq C_S(G)$, there is a vertex $w \in V(G)$ such that $d_S(v,w) = rad_S(G)$. Then, $d_S(u,w) = d(u,v) + d(v,w) \geq 1 + d(v,w) > rad_S(G)$ and $u \notin V(C_S(G))$. Hence, $V(C_S(G)) \cap [V(G) - S] = \emptyset$ and so $V(C_S(G)) \subseteq S$. Therefore $S = V(C_S(G))$ when $|S| = 1$. •

We now turn to the S-periphery of a graph. Recall from Chapter I, that not every graph G is isomorphic to the periphery of a connected graph H. In our present context, this result can be restated as follows.

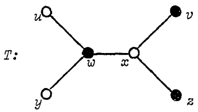
THEOREM 2.10. *A graph G of order $p \geq 1$ is the $\emptyset$-periphery of a connected graph H if and only if $\Delta(G) \leq p - 2$ or $G \cong K_p$.*

The next result shows that these restrictions do not apply to $Per_S(H)$ for nonempty subsets S of $V(H)$.

THEOREM 2.11. *For a graph G and a positive integer n, let $H = G + K_n$. If S is the set of n vertices of H not in $V(G)$, then $Per_S(H) = G$.*

PROOF: Suppose $u \in S$. If $v \in S$ and $w \in V(G)$, then $d_S(u,v) = n - 1$ and $d_S(u,w) = n$. Thus, $e_S(u) = n$ for $u \in S$. On the other hand, suppose $u \in V(G)$. If $v \in S$ and $w \in V(G)$, then $d_S(u,v) = n$ and $d_S(u,w) = n + 1$. Therefore, $e_S(u) = n + 1$ for $u \in V(G)$ and $Per_S(H) = G$. •

Recall, for a tree T and a nonempty subset S of $V(T)$, that T_S denotes the tree generated by S. We conclude this section with two results that relate $C_S(T)$ and $Per_S(T)$ to T_S. In Corollary 2.4A, we saw that it is possible for $T_{V(C_S(T))}$ and T_S to be the same; however, this is not always the case. For example, in the tree T in Figure 2.4, if $S = \{v, w, z\}$, then $T_S = \langle\{v, w, x, z\}\rangle$ and $T_{V(C_S(T))} = \langle\{v, x, z\}\rangle$. This leads to our next result.



$$e_S(v) = e_S(x) = e_S(z) = 6$$
$$e_S(w) = 7$$
$$e_S(u) = e_S(y) = 8$$

Figure 2.4

THEOREM 2.12. *Let T be a tree and let S be a nonempty subset of $V(T)$. If $V = V(C_S(T))$, then $T_V \subset T_S$.*

PROOF: Since every subtree of a given tree T' is an induced subgraph of T', it follows that $T_V \not\subset T_S$ only if there exists an end-vertex v of T_V that is not in $V(T_S)$. Since every end-vertex of T_V is in $V = V(C_S(T))$, suppose that $v \in V$ and $v \notin V(T_S)$. Let v' be the vertex of T_S such that $d(v,v') = min\{d(v,w)|w \in V(T_S)\}$. For every vertex u of T, the S-distance $d_S(v,u) = d(v,v') + d_S(v',u)$. Thus, $e_S(v) > e_S(v') \geq rad_S(T)$ and so, $v \notin V(C_S(T))$ which is a contradiction. Therefore, $T_V \subset T_S$. •

Finally, for a tree T and a nonempty subset S of $V(T)$, we can use T_S to classify those vertices of T that are in $Per_S(T)$.

THEOREM 2.13. *Let T be a tree and let S be a nonempty subset of $V(T)$ that does not include every end-vertex of T. A vertex v of*

T is in $Per_S(T)$ if and only if v is an end-vertex of T such that $d(v, T_S) = max\{d(w, T_S)|w \in V(T)\}$.

PROOF: First, let v be a vertex of $Per_S(T)$. Then there exists a vertex v' in $V(T_S)$ such that $d(v, v') = min\{d(v, w)|w \in V(T_S)\}$ and by Theorem 2.7, the S-distance $d_S(v, v) = 2d(v, v') + 2q(T_S) = diam_S(T)$. Suppose that v is not an end-vertex of T. Then there exists a vertex u adjacent to v with $d(u, v') = d(v, v') + 1$. However, then $d_S(u, u) = 2d(u, v') + 2q(T_S) = 2 + 2d(v, v') + 2q(T_S) > diam_S(T)$, which is impossible. Now, suppose that $d(v, T_S) < d(w, T_S)$. Then $diam_S(T) = d_S(v, v) = 2d(v, T_S) + 2q(T_S) < 2d(w, T_S) + 2q(T_S) = d_S(w, w) \leq diam_S(T)$, but this is an contradiction. Therefore, if $v \in V(Per_S(T))$ then v is an end-vertex of T such that $d(v, T_S) = max\{d(w, T_S)|w \in V(T)\}$.

For the converse, let v be an end-vertex of T such that $d(v, T_S) = max\{d(w, T_S)|w \in V(T)\}$ and let u be a vertex in $Per_S(T)$. We know that $diam_S(T) = 2d(u, T_S) + 2q(T_S) \leq 2d(v, T_S) + 2q(T_S) = d_S(v, v) \leq diam_S(T)$. Therefore, $v \in V(Per_S(T))$. •

2.3 When Does $e_S(v) = d_S(v, v) = diam_S(G)$?

As we have seen throughout this chapter, it is possible for a vertex v in a connected graph G with $S \subseteq V(G)$ to satisfy $e_S(v) = d_S(v, v)$. In fact, as we now show, it is often the case that a vertex v exists such that $e_S(v) = d_S(v, v) = diam_S(G)$.

We begin with a useful concept. Let G be a connected graph with

$S \subseteq V(G)$. For vertices u, v of G, an ordering $\mathcal{O}$ of a $u - v$ S-walk W is a sequence of $|S| + 2$ (not necessarily distinct) vertices with u as the initial vertex, v as the terminal vertex and, in addition, listing each vertex of S once as an intermediate vertex such that if $w_1, w_2 \in S$ and w_1 appears before w_2 in $\mathcal{O}$, then w_1 appears before w_2 at least once in W. As an example, let G be the graph in Figure 2.5 and let $S = \{u, w, y, z\}$. For the $u - v$ S-walk $W : u, w, x, y, z, y, x, v$, the possible orderings are $\mathcal{O}_1 : u, u, w, y, z, v$ and $\mathcal{O}_2 : u, u, w, z, y, v$.

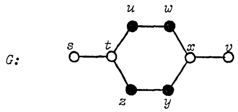


Figure 2.5

We are interested in orderings for two reasons. First, suppose that G is a connected graph with $S \subseteq V(G)$ and containing vertices u and v. If $\mathcal{O}$ is an ordering for a $u - v$ S-walk W, then by reversing the order of $\mathcal{O}$ we obtain an ordering for the $v - u$ S-walk formed by traversing W in reverse. Second, given an ordering for a shortest $u - v$ S-walk, there is a formula for finding $d_S(u, v)$ as the next lemma shows.

LEMMA 2.1. *Let G be a connected graph with $S \subseteq V(G)$ and $|S| = n \geq 1$. If, for vertices u and v, a shortest $u - v$ S-walk has an*

ordering $\mathcal{O}: u, w_1, w_2, \ldots, w_n, v$ *where* $w_i \in S$, $1 \leq i \leq n$, *then*

$$d_S(u,v) = d(u,w_1) + \sum_{i=1}^{n-1} d(w_i, w_{i+1}) + d(w_n, v).$$

PROOF: We will prove this by induction on $|S| = n$. For $n = 1$, let $S = \{w_1\}$. If W is a shortest $u - v$ S-walk, then an ordering $\mathcal{O}$ of W must be $\mathcal{O}: u, w_1, v$. Since $d_S(u,v) = d(u,w_1) + d(w_1,v)$, the result holds. Now for $n \geq 2$, assume that the formula holds for all sets S' such that $1 \leq |S'| < n$. Let S be a subset of $V(G)$ with $|S| = n$. Let W be a shortest $u - v$ S-walk and let $\mathcal{O}: u, w_1, w_2, \ldots, w_n, v$ be an ordering of W. Since W is a shortest $u - v$ S-walk and every vertex of S other than w_1 appears (possibly reappears) after w_1, the length of the $u - w_1$ subwalk in W is $d(u, w_1)$. Now if $S' = S - \{w_1\}$, then the $w_1 - v$ subwalk W' in W is a shortest $w_1 - v$ S'-walk and $\mathcal{O}': w_1, w_2, \ldots, w_n, v$ is an order for W'. Hence, by the inductive hypothesis, the length of W' is $\sum_{i=1}^{n-1} d(w_i, w_{i+1}) + d(w_n, v)$ and therefore

$$d_S(u,v) = d(u,w_1) + \sum_{i=1}^{n-1} d(w_i, w_{i+1}) + d(w_n, v). \qquad \bullet$$

We can now give sufficient conditions for a graph G with $S \subseteq V(G)$ to have a vertex v such that $e_S(v) = d_S(v,v) = diam_S(G)$.

THEOREM 2.14. *Let* G *be a connected graph with* $S \subseteq V(G)$ *such that* $1 \leq |S| \leq 2$. *Then there exists a vertex* v_0 *of* G *satisfying*

$e_S(v_0) = d_S(v_0, v_0) = diam_S(G)$.

PROOF: We consider two cases.

CASE 1: Suppose that $|S| = 1$. Let $S = \{w\}$ and suppose u and v are vertices of G such that $diam_S(G) = d_S(u,v) = d(u,w) + d(w,v)$. Without loss of generality, let $d(u,w) \leq d(w,v)$. Then $diam_S(G) \leq 2d(w,v) = d_S(v,v) \leq e_S(v) \leq diam_S(G)$ and the result holds for $v_0 = v$.

CASE 2: Suppose that $|S| = 2$. Assume there is no vertex v_0 such that $e_S(v_0) = d_S(v_0, v_0) = diam_S(G)$. Let u and v be vertices of G such that $d_S(u,v) = diam_S(G)$ and let W be a shortest $u - v$ S-walk in G. Label the vertices of S as w_1 and w_2 so that an ordering of W is $\mathcal{O}:\ u,\ w_1,\ w_2,\ v$. Assume that $d(u,w_2) \geq d(w_2,v)$. Then

$$\begin{aligned} d_S(u,u) &= min\{d_S(u,w_2) + d(w_2,u),\ d(u,w_2) + d_S(w_2,u)\} \\ &= d_S(u,w_2) + d(w_2,u) \\ &\geq d_S(u,w_2) + d(w_2,v) = diam_S(G). \end{aligned}$$

Therefore, $e_S(u) = d_S(u,u) = diam_S(G)$, which is a contradiction. Hence, $d(u,w_2) < d(w_2,v)$. Similarly, $d(v,w_1) < d(u,w_1)$ for if $d(v,w_1) \geq d(u,w_1)$, then $e_S(v) = d_S(v,v) = diam_S(G)$, again a contradiction. However, by the ordering of $\mathcal{O}$ of W,

$$\begin{aligned} d_S(u,v) &= d(u,w_1) + d(w_1,w_2) + d(w_2,v) \\ &> d(v,w_1) + d(w_1,w_2) + d(w_2,u) \\ &= d_S(v,u). \end{aligned}$$

But this contradicts the symmetry of S-distance so G must contain a vertex v_0 such that $e_S(v_0) = d_S(v_0,v_0) = diam_S(G)$. •

The same result holds for a connected graph G if $S \subseteq V(G)$ and $|S| = 3$.

THEOREM 2.15. *Let G be a connected graph with $S \subseteq V(G)$ such that $|S| = 3$. Then there exists a vertex v_0 of G satisfying $e_S(v_0) = d_S(v_0,v_0) = diam_S(G)$.*

PROOF: Suppose that there is no vertex v_0 such that $e_S(v_0) = d_S(v_0,v_0) = diam_S(G)$ and let u and v be vertices of G such that $d_S(u,v) = diam_S(G)$. Let W be a shortest $u - v$ S-walk in G and label the vertices of S as w_1, w_2, w_3 such that an ordering of W is $\mathcal{O}:\ u,\ w_1,\ w_2,\ w_3,\ v$. By our assumption, $d_S(u,u) < diam_S(G)$ and $d_S(v,v) < diam_S(G)$. Now, let W_u be a shortest $u - u$ S-walk in G and let W_v be a shortest $v - v$ S-walk in G. There are six possible orderings for W_u and for W_v. We list them below using a superscript R if the vertices of S in the ordering are reversed in a previous ordering.

U_1: u, w_1, w_2, w_3, u	V_1: v, w_1, w_2, w_3, v
U_1^R: u, w_3, w_2, w_1, u	V_1^R: v, w_3, w_2, w_1, v
U_2: u, w_1, w_3, w_2, u	V_2: v, w_1, w_3, w_2, v
U_2^R: u, w_2, w_3, w_1, u	V_2^R: v, w_2, w_3, w_1, v
U_3: u, w_2, w_1, w_3, u	V_3: v, w_2, w_1, w_3, v
U_3^R: u, w_3, w_1, w_2, u	V_3^R: v, w_3, w_1, w_2, v

We consider four cases.

CASE 1: Suppose that U_1 is an ordering for W_u. Since $d_S(u,u) = d_S(u,w_3) + d(w_3,u) < d_S(u,v) = d_S(u,w_3) + d(w_3,v)$, we have $d(w_3,u) < d(w_3,v)$. Thus, neither V_1^R nor V_3^R is an ordering of W_v; for if so, then $diam_S(G) = d_S(u,v) \leq d(u,w_3) + d_S(w_3,v) < d(v,w_3) + d_S(w_3,v) = d_S(v,v)$, which contradicts the maximality of $diam_S(G)$. Similarly, no ordering of W_v can be V_1 or V_3 because then $diam_S(G) = d_S(u,v) = d_S(v,u) \leq d_S(v,w_3) + d(w_3,u) < d_S(v,w_3) + d(w_3,v) = d_S(v,v)$, which is again impossible. Finally, since there exists no shortest $v - v$ S-walk with ordering V_1, no ordering of W_v can be V_2 or V_2^R; for otherwise, $d_S(v,v) = d(v,w_1) + d(w_1,w_3) + d(w_3,w_2) + d(w_2,v) < d(v,w_1) + d(w_1,w_2) + d(w_2,w_3) + d(w_3,v)$ and $d(w_1,w_3) + d(w_2,v) < d(w_1,w_2) + d(w_3,v)$. However, then $d_S(u,v) \leq d(u,w_1) + d(w_1,w_3) + d(w_3,w_2) + d(w_2,v) < d(u,w_1) + d(w_1,w_2) + d(w_2,w_3) + d(w_3,v) = d_S(u,v)$, which is a contradiction. Therefore, since $d_S(u,u) < diam_S(G)$

and $d_S(v,v) < diam_S(G)$ it follows that U_1 is not an ordering of W_u.

CASE 2: <u>Suppose that U_2 is an ordering for W_u.</u> Since $d_S(u,u) = d(u,w_1) + d(w_1,w_3) + d(w_3,w_2) + d(w_2,u) < d_S(u,v) = d(u,w_1) + d(w_1,w_2) + d(w_2,w_3) + d(w_3,v)$, we have $d(w_1,w_3) + d(w_2,u) < d(w_1,w_2) + d(w_3,v)$. Thus, neither V_1 nor V_1^R is an ordering of W_v; for if so, then $diam_S(G) = d_S(u,v) = d_S(v,u) \leq d(v,w_1) + d(w_1,w_3) + d(w_3,w_2) + d(w_2,u) < d(v,w_1) + d(w_1,w_2) + d(w_2,w_3) + d(w_3,v) = d_S(v,v)$, which contradicts the maximality of $diam_S(G)$. Similarly, no ordering of W_v can be V_3 or V_3^R because then $diam_S(G) = d_S(u,v) \leq d(u,w_2) + d(w_2,w_3) + d(w_3,w_1) + d(w_1,v) < d(v,w_1) + d(w_1,w_2) + d(w_2,w_3) + d(w_3,v) = d_S(v,v)$, which is again impossible. Finally, since there exists no shortest $v - v$ S-walk with ordering V_1, no ordering of W_v can be V_2 or V_2^R; for otherwise, $d_S(v,v) = d(v,w_1) + d(w_1,w_3) + d(w_3,w_2) + d(w_2,v) < d(v,w_1) + d(w_1,w_2) + d(w_2,w_3) + d(w_3,v)$ and $d(w_1,w_3) + d(w_2,v) < d(w_1,w_2) + d(w_3,v)$. However, then $d_S(u,v) \leq d(u,w_1) + d(w_1,w_3) + d(w_3,w_2) + d(w_2,v) < d(u,w_1) + d(w_1,w_2) + d(w_2,w_3) + d(w_3,v) = d_S(u,v)$, which is a contradiction. Therefore, since $d_S(u,u) < diam_S(G)$ and $d_S(v,v) < diam_S(G)$ it follows that U_2 is not an ordering of W_u.

CASE 3: <u>Suppose that U_3 is an ordering for W_u.</u> If U_1 is an ordering for a shortest $u - u$ S-walk W_u', then we reach a contradiction by applying Case 1 to W_u'. If there exists no shortest $u - u$ S-walk with ordering

U_1, then $d_S(u,u) = d(u,w_2) + d(w_2,w_1) + d(w_1,w_3) + d(w_3,u) < d(u,w_1) + d(w_1,w_2) + d(w_2,w_3) + d(w_3,u)$ and $d(u,w_2) + d(w_1,w_3) < d(u,w_1) + d(w_2,w_3)$. However, then $d_S(u,v) \leq d(u,w_2) + d(w_2,w_1) + d(w_1,w_3) + d(w_3,v) < d(u,w_1) + d(w_1,w_2) + d(w_2,w_3) + d(w_3,v) = d_S(u,v)$, which is a contradiction. Therefore, since $d_S(u,u) < diam_S(G)$ and $d_S(v,v) < diam_S(G)$ it follows that U_3 is not an ordering of W_u.

CASE 4: Suppose that one of U_1^R, U_2^R or U_3^R is an ordering for W_u. By traversing W_u backwards we obtain a $u - u$ S-walk W_u^R with U_1, U_2 or U_3 as an ordering. Now, if we apply the arguments of cases 1, 2, or 3 to W_u^R we reach a contradiction. Therefore, since $d_S(u,u) < diam_S(G)$ and $d_S(v,v) < diam_S(G)$, none of U_1^R, U_2^R nor U_3^R is an ordering of W_u.

Since W_u must have one of the six orderings either $d_S(u,u) = diam_S(G)$ or $d_S(v,v) = diam_S(G)$. •

Theorem 2.14 and Theorem 2.15 can be combined to give the following result.

COROLLARY 2.15A. *Let G be a connected graph with a subset S of $V(G)$ such that $1 \leq |S| \leq 3$. Then there exists a vertex v of G satisfying $e_S(v) = d_S(v,v) = diam_S(G)$.*

Corollary 2.15A does not hold if $|S| = 4$. For example, the graph G in Figure 2.5 has $S = \{u,w,y,z\}$, yet for every vertex v_0 of G, the S-distance $d_S(v_0,v_0) < e_S(v_0)$. This will be proved as part of our next

result.

THEOREM 2.16. *Let* $n \geq 4$ *be an integer. If* p *is an integer with*

$$p \geq \begin{cases} 8 & if\ 4 \leq n \leq 6 \\ n & if\ n \geq 7, \end{cases}$$

then there is a graph G *of order* p *and a subset* S *of* $V(G)$ *with* $|S| = n$ *such that no vertex* v *of* G *satisfies* $e_S(v) = d_S(v, v)$.

PROOF: We will prove this in two cases.

CASE 1: Suppose that $4 \leq n \leq 6$. Consider the graph G of order $p \geq 8$ in Figure 2.6. If $S = \{u_2, u_3, u_5, u_6\}$, then $d_S(v_i, v_i) = 8$ for $0 \leq i \leq p - 7$ and $d_S(u_i, u_i) = 6$ for $1 \leq i \leq 6$. However, for each vertex v of G, we can give a shortest S-walk beginning at v, that induces an S-distance exceeding $d_S(v, v)$.

$$\begin{aligned}
d_S(v_0, v_1) &= 9 \text{ by } W_{v_0} : v_0, u_1, u_2, u_3, u_4, u_5, u_6, u_5, u_4, v_1 \\
d_S(v_i, v_0) &= 9 \text{ by } W_{v_i} : v_i, u_4, u_5, u_6, u_1, u_2, u_3, u_2, u_1, v_0 \ (1 \leq i \leq p-7) \\
d_S(u_1, v_1) &= 8 \text{ by } W_{u_1} : u_1, u_2, u_3, u_4, u_5, u_6, u_5, u_4, v_1 \\
d_S(u_2, v_1) &= 7 \text{ by } W_{u_2} : u_2, u_1, u_6, u_5, u_4, u_3, u_4, v_1 \\
d_S(u_3, v_0) &= 7 \text{ by } W_{u_3} : u_3, u_4, u_5, u_6, u_1, u_2, u_1, v_0 \\
d_S(u_4, v_0) &= 8 \text{ by } W_{u_4} : u_4, u_5, u_6, u_1, u_2, u_3, u_2, u_1, v_0 \\
d_S(u_5, v_0) &= 7 \text{ by } W_{u_5} : u_5, u_4, u_3, u_2, u_1, u_6, u_1, v_0 \\
d_S(u_6, v_1) &= 7 \text{ by } W_{u_6} : u_6, u_1, u_2, u_3, u_4, u_5, u_4, v_1
\end{aligned}$$

Hence, for $|S| = 4$, no vertex of G satisfies $e_S(v) = d_S(v,v)$. In fact, if $S = \{u_2, u_3, u_4, u_5, u_6\}$ or $S = \{u_1, u_2, u_3, u_4, u_5, u_6\}$, then for each vertex v of G, the S-distance $d_S(v,v)$ remains the same as when $|S| = 4$. Also, since each vertex of this new set S is in each of the eight walks listed above, these walks are still shortest S-walks. Therefore, for $5 \leq |S| \leq 6$, no vertex v of G satisfies $e_S(v) = d_S(v,v)$.

CASE 2: Suppose that $n \geq 7$. We construct the graph G of order $p \geq n$ from $C_n : u_1, u_2, \ldots, u_n$ by joining $p - n$ vertices $v_1, v_2, \ldots, v_{p-n}$ to vertices u_1 and u_2. Let $S = \{u_1, u_2, \ldots, u_n\}$. Then, for $1 \leq i \leq n$, the S-distance $d_S(u_i, u_i) = n$ and if $d(u_i, u_j) = 3$, then $d_S(u_i, u_j) = n + 1$. For $1 \leq i \leq p - n$, the S-distance $d_S(v_i, v_i) = n + 1$ and $d_S(v_i, u_4) = d(v_i, u_1) + d_S(u_1, u_4) = 1 + (n+1) = n + 2$. Thus, no vertex v of G satisfies $e_S(v) = d_S(v,v)$. •

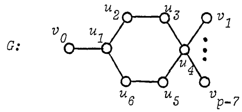


Figure 2.6

By this result, Corollary 2.15A is sharp; because, for each $n \geq 4$, we

constructed an infinite class of graphs G with nonempty subsets S of $V(G)$, such that for every vertex v of G, the S-distance $d_S(v,v) < e_S(v) \leq diam_S(G)$. A natural question now arises. For $n \geq 4$, does Theorem 2.16 give a graph G of minimum order and a nonempty subset S of $V(G)$ with $|S| = n$ such that no vertex v of G satisfies $d_S(v,v) = e_S(v)$? For $n \geq 7$, the question has an affirmative answer because we constructed a graph G of order n. For $n = 6$, such is not the case. If $G \cong C_6$ and $S = V(G)$, then for every vertex v of G, the S-distance $d_S(v,v) = 6$ and $e_S(v) = 7$. Hence, for $n = 6$, there is a graph G of order 6 and a set S of $V(G)$ such that no vertex v of G satisfies $e_S(v) = d_S(v,v)$. However, no such graph G of order 7 is known for $n = 6$. For $4 \leq n \leq 5$, it is not known if $p = 8$ is the minimum order. However, by our next result, the order p is at least 6.

THEOREM 2.17. *For every connected graph G with $p(G) \leq 5$ and nonempty subset S of $V(G)$, there exists a vertex v of G such that $e_S(v) = d_S(v,v) = diam_S(G)$.*

PROOF: If $0 < |S| \leq 3$ or $p(G) \leq 3$, then by Corollary 2.15A, there exists a vertex v of G such that $e_S(v) = d_S(v,v) = diam_S(G)$. Also, if G is a tree, then the result holds by Corollary 2.7B. Thus, we can assume that $|S| > 3$, the order of G is either 4 or 5, and G contains a cycle. If $p(G) = 4$, then $S = V(G)$ and the only graphs of order 4 containing cycles are shown in Figure 2.7(a). Note that every vertex of these

graphs satisfies $e_S(v) = d_S(v,v) = diam_S(G)$.

For $p(G) = 5$, we consider two cases.

Case 1: Suppose that $|S| = 5$. If G contains a 5-cycle, then for each vertex v of G, we have $d_S(v,v) = 5$ and for a vertex $u \neq v$, the S-distance from v to u is either 4 or 5. Therefore, every vertex v satisfies $e_S(v) = d_S(v,v) = diam_S(G)$. Similarly, if the largest cycle in G is a 3-cycle or a 4-cycle, then the result also holds for every vertex v of G. All such graphs are shown in Figure 2.7. For those in Figure 2.7 (b), we have $d_S(v,v) = 7$ and for those graphs in Figure 2.7(c), the S-distance $d_S(v,v) = 6$ for all $v \in V(G)$.

Case 2: Suppose that $|S| = 4$. Let v be the vertex of G that is not in S. Since $S \subseteq V(G)$, we know for each pair u, w of vertices of G, that $d_S(u,w) \leq d_{V(G)}(u,w)$ and so $diam_S(G) \leq diam_{V(G)}(G)$. However, a shortest $v-v$ S-walk contains every vertex of G; so $d_S(v,v) = d_{V(G)}(v,v) = diam_{V(G)}(G) \leq diam_S(G)$. Therefore $e_S(v) = d_S(v,v) = diam_S(G)$. •

This result is sharp; for, if $p \geq 7$, then we have a counterexample by Theorem 2.16 and if $p = 6$, then $G \cong C_6$ with $S = V(G)$ is a counterexample as discussed previously.

We conclude this section and our discussion of sharpness with a weaker version of Theorem 2.16.

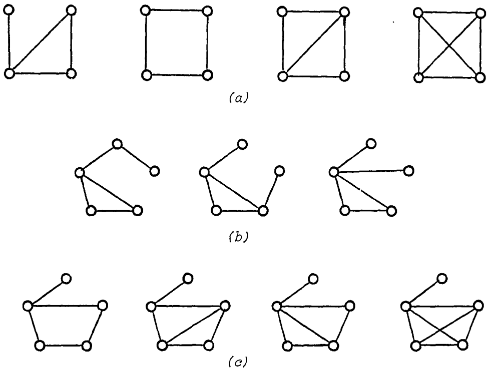


Figure 2.7

THEOREM 2.18. *For positive integers* $n \geq 4$ *and* $p \geq 6$, *there exists a graph* G *of order* p *and a nonempty subset* S *of* $V(G)$ *with* $|S| = n$ *such that no vertex* v *of* G *satisfies* $d_S(v,v) = diam_S(G)$.

PROOF: Suppose that $p = 6$. Let $G \cong C_6$ with $V(G) = \{v_1, v_2, \ldots, v_6\}$ and $E(G) = \{v_i v_{i+1} | 1 \leq i \leq 5\} \cup \{v_6 v_1\}$. If $n = 4$, let $S = \{v_2, v_3, v_5, v_6\}$. If $n = 5$, let $S = \{v_2, v_3, \ldots, v_6\}$. If $n = 6$, let $S = V(G)$. In each case, for $1 \leq i \leq 6$, the S-distance $d_S(v_i, v_i) = 6$;

but, $d_S(v_1, v_4) = 7 = diam_S(G)$.

Next, suppose that $p = 7$ and let G be the graph in Figure 2.8. If $n = 4$, let $S = \{v_2, v_3, v_5, v_6\}$. If $n = 5$, let $S = \{v_2, v_3, \ldots, v_6\}$. For either set S, the S- distance $d_S(v_i, v_i) = 6$ for $1 \le i \le 7$, while $d_S(v_1, v_4) = 7 = diam_S(G)$.

For $n = 6$ and $p = 7$, let $G \cong C_7$ with $V(G) = \{v_1, v_2, \ldots, v_7\}$ and $E(G) = \{v_i v_{i+1} | 1 \le i \le 6\} \cup \{v_7 v_1\}$. If $S = \{v_2, v_3, \ldots, v_7\}$, then $d_S(v_i, v_i) = 7$ for $1 \le i \le 7$; however, $d_S(v_1, v_4) = 8 = diam_S(G)$.

For all other choices of n and p, we can let G and S be the graph and set constructed in the proof of Theorem 2.16. •

G:

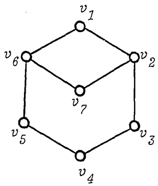


Figure 2.8

2.4 The n-Eccentricity of a Vertex

Throughout this chapter, we have always considered a connected graph G and a specified subset S of $V(G)$ when defining the S-eccentricity $e_S(v)$ for a vertex v of G. We now extend this idea by considering, for a nonnegative integer $n \le p(G)$, all subsets S of $V(G)$ with $|S| = n$. In particular, for a graph G, a vertex v of G, and a nonnegative integer $n \le p(G)$, the n- eccentricity $e_n(v)$ of the vertex v is

$$e_n(v) = max\{e_S(v) | S \subseteq V(G) \text{ and } |S| = n\}.$$

We define the n-radius $rad_n(G)$ of G and the n-diameter $diam_n(G)$ of G by

$$rad_n(G) = \min_{v \in V(G)} e_n(v) \text{ and}$$

$$diam_n(G) = \max_{v \in V(G)} e_n(v).$$

A useful property that relates the n-eccentricities of adjacent vertices in a graph is given next.

THEOREM 2.19. *Let G be a connected graph and let n be a nonnegative integer with $n \le p(G)$. If $uv \in E(G)$, then $|e_n(u) - e_n(v)| \le 1$.*

PROOF: Without loss of generality, we may assume that $e_n(v) \le e_n(u)$. Let S be a subset of $V(G)$ with $|S| = n$ such that $e_n(u) = e_S(u)$. By Theorem 2.1, we have $e_S(u) - 1 \le e_S(v) \le e_S(u) + 1$. However,

$e_S(v) \leq e_n(v) \leq e_n(u) = e_S(u)$; so $e_n(u) - 1 \leq e_n(v) \leq e_n(u) + 1$. Therefore, $|e_n(u) - e_n(v)| \leq 1$. •

We now present two results relating $rad_n(G)$ and $diam_n(G)$.

THEOREM 2.20. *For a connected graph G, a nonnegative integer $n \leq p(G)$ and an integer c such that $rad_n(G) < c < diam_n(G)$, there is a vertex v of G such that $e_n(v) = c$.*

PROOF: Let u and v be vertices of G that satisfy $e_n(u) = rad_n(G)$ and $e_n(v) = diam_n(G)$. Let P be a $u - v$ path in G and let w_1 be the last vertex of P with $e_n(w_1) < c$. If w_2 is the next vertex on P, then $e_n(w_2) \geq c$ and $|e_n(w_1) - e_n(w_2)| \leq 1$, by Theorem 2.1. Therefore, $e_n(w_2) = c$. •

The second result gives a familiar bound for $diam_n(G)$.

THEOREM 2.21. *For a connected graph G and a nonnegative integer $n \leq p(G)$,*

$$rad_n(G) \leq diam_n(G) \leq 2rad_n(G).$$

PROOF: The first inequality follows from the definitions. For the second inequality, let u and v be vertices of G such that $e_n(u) = diam_n(G)$ and $e_n(v) = rad_n(G)$. Also, let S be a subset of $V(G)$ such that $|S| = n$ and $e_n(u) = e_S(u)$. Then $e_S(v) \leq e_n(v)$. Now $diam_n(G) = e_S(u) \leq diam_S(G) \leq 2rad_S(G)$, by Theorem 2.3. Therefore, $diam_n(G) \leq 2e_S(v) \leq 2e_n(v) = 2rad_n(G)$. •

Although it is not known if the upper bound in Theorem 2.21 can be improved for graphs in general, it can be improved for trees. We begin with some notation. For a tree T, a vertex v of T and a subset S of $V(T)$, recall that $T_{S \cup \{v\}}$ is the tree generated by $S \cup \{v\}$. For a positive integer $n \leq p(T)$, let

$$q_n(v) = max\{q(T_{S \cup \{v\}}) | S \subseteq V(T) \text{ and } |S| = n\}.$$

In [5] it was shown that if $q_n^* = max\{q_n(v) | v \in V(T)\}$ and $q_n = min\{q_n(v) | v \in V(T)\}$, then

$$q_n^* \leq \left(\frac{n+1}{n}\right) \cdot q_n. \tag{2.1}$$

COROLLARY 2.21A. *For a tree* T *and a positive integer* $n \leq p(T)$,

$$rad_n(T) \leq diam_n(T) \leq \left(\frac{n+1}{n}\right) rad_n(T).$$

PROOF: The lower bound is immediate. For the upper bound, note that there exists a vertex v of T and a subset S of $V(T)$ with $|S| = n$ such that $diam_n(T) = e_n(v) = e_S(v)$. Suppose that $e_S(v) < diam_S(T)$. Then there exists a vertex $u(\neq v)$ in T such that $diam_S(T) = e_S(u) \leq e_n(u) \leq diam_n(T)$. However, this implies that $diam_n(T) < diam_n(T)$, which is impossible. Therefore, $diam_n(T) = e_S(v) = diam_S(T)$. By Corollary

2.7A, we have $diam_S(T) = d_S(v,v) = 2d(v,T_S) + 2q(T_S) = 2q(T_{S\cup\{v\}})$, and therefore, by the definition of q_n^*,

$$diam_n(T) \leq 2q_n^*. \tag{2.2}$$

Also,

$$\begin{aligned} q_n &= min\{max\{q(T_{S\cup\{u\}})|S \subset V(T) \text{ and } |S| = n\}|u \in V(T)\} \\ &= min\{max\{\frac{1}{2}d_S(u,u)|S \subseteq V(T) \text{ and } |S| = n\}|u \in V(T)\} \\ &\leq min\{max\{\frac{1}{2}e_S(u)|S \subseteq V(T) \text{ and } |S| = n\}|u \in V(T)\} \\ &= \frac{1}{2}\, rad_n(T). \end{aligned} \tag{2.3}$$

By combining (2.1), (2.2) and (2.3) we have

$$diam_n(T) \leq 2q_n^* \leq 2\left(\frac{n+1}{n}\right) q_n \leq \left(\frac{n+1}{n}\right) rad_n(T). \quad \bullet$$

CHAPTER III

THE DISTANCE OF A VERTEX

3.1 Introduction

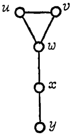
For a city planner, the best location for a service facility normally does not depend on the distance to one other point in the city but on its distance to many, if not all, other points. For example, ideally, a post office should be built in a location so that the sum of the distances from that point to all other points is minimized, while a toxic waste site should be located so that the sum of the distances from that point to all other points is maximized. These ideas can be expressed using a connected graph G by defining the distance $d(v)$ of a vertex v by

$$d(v) = \sum_{u \in V(G)} d(v,u).$$

A vertex of minimum distance is called a median vertex and the distance of a median vertex is denoted by $med(G)$. The median $M(G)$ of G is the subgraph of G induced by its median vertices. Similarly, we refer to a vertex v of maximum distance as a marginal vertex and let $d(v) = mar(G)$. The margin $\mathcal{M}(G)$ of G is the subgraph induced by the marginal vertices of G. These ideas are illustrated in Figure 3.1.

One can easily find the distance of a vertex v in a graph G by first applying Dijkstra's algorithm to G to find $d(v,u)$ for each $u \in V(G)$ and then summing those distances to calculate $d(v)$. Since the order of complexity of Dijkstra's algorithm is $O(p^2)$ and the extra steps to find $d(v)$ require only $p - 1$ additions, the order of complexity remains $O(p^2)$.

G:



$d(u) = d(v) = 7$	$med(G) = 5$
$d(w) = 5$	$M(G) = \langle\{w\}\rangle$
$d(x) = 6$	$mar(G) = 9$
$d(y) = 9$	$M(G) = \langle\{y\}\rangle$

Figure 3.1

We consider a property of distances in bipartite graphs. It is known for a connected bipartite graph G that there is exactly one way to partition $V(G)$ into two partite sets.

THEOREM 3.1. *Let G be a connected bipartite graph with partite sets V_1 and V_2. If $v \in V(G) - V_i$, for $i = 1,2$, then $d(v)$ and $|V_i|$ have the same parity.*

PROOF: Without loss of generality, let $v \in V_1$. Now

$$d(v) = \sum_{u \in V_1} d(v,u) + \sum_{u \in V_2} d(v,u).$$

For each $u \in V_1$, the distance $d(v,u)$ is even. Thus, $\sum_{u \in V_1} d(v,u)$ is even. For $u \in V_2$, the distance $d(v,u)$ is odd. Now if $|V_2|$ is even, then $\sum_{u \in V_2} d(v,u)$ is an even sum of odd distances and therefore is even. Thus, $d(v)$ is even and has the same parity as $|V_2|$.

On the other hand, if $|V_2|$ is odd, then $\sum_{u \in V_2} d(v,u)$ is odd, $d(v)$ is odd and again $d(v)$ and $|V_2|$ have the same parity. •

Two corollaries follow immediately.

COROLLARY 3.1A. *A connected graph G of odd order is bipartite if and only if for each pair u, v of adjacent vertices, the distances $d(u)$ and $d(v)$ have opposite parity.*

PROOF: Suppose G is a connected bipartite graph of odd order with partite sets V_1 and V_2, and let u and v be adjacent vertices of G.

Without loss of generality, let $u \in V_1$. Thus, $v \in V_2$. By Theorem 3.1, $d(u)$ and $|V_2|$ have the same parity, as do $d(v)$ and $|V_1|$. However, since the order of G is odd, $|V_1|$ and $|V_2|$ have opposite parity and so $d(u)$ and $d(v)$ have opposite parity.

For the converse, let G be a connected graph such that for each pair u, v of adjacent vertices, $d(u)$ and $d(v)$ have opposite parity. Suppose that G contains an odd cycle $C:\ u_1, u_2, \ldots, u_{k-1}, u_k, u_1$. Note for all vertices u_i of C with i even that $d(u_i)$ will have the same parity,

as will $d(u_i)$ for i odd. However, u_k and u_1 are adjacent and $d(u_k)$ has the same parity as $d(u_1)$, a contradiction. Thus, G is bipartite. •

COROLLARY 3.1B. *Let G be a connected graph of even order. If G is bipartite with partite sets V_1 and V_2, then the distances of all vertices of G have the same parity as $|V_1|$ (and $|V_2|$).*

PROOF: Suppose $v \in V(G) - V_1$. Then, by Theorem 3.1, $d(v)$ and $|V_1|$ have the same parity. For $v \in V(G) - V_2$, the numbers $d(v)$ and $|V_2|$ have the same parity. However, since G has even order, $|V_1|$ and $|V_2|$ have the same parity and this completes the proof. •

Note that the converse of Corollary 3.1B is not true, for if all vertices of a connected graph G have the same parity, G need not be bipartite. For example, consider K_p or $K(p_1, p_2, p_3)$ with p_1, p_2 and p_3 of the same parity.

In [14] Slater showed that for any graph G there exists a graph H with $M(H) = G$. We will now present an analogous result by using the following observation. If v is a vertex of a graph G of order p and diameter 2, then

$$\begin{aligned} d(v) &= deg(v) + 2[p - 1 - deg(v)] \\ &= 2(p - 1) - deg(v) \end{aligned} \tag{3.1}$$

Thus, a vertex of maximum degree will be a median vertex of G and one of minimum degree will be a marginal vertex of G.

THEOREM 3.2. *For a given graph G of order p, there exists a graph H with $\mathcal{M}(H) = G$.*

PROOF: We consider three cases.

Case 1: Suppose that $G \cong K_p$. Then $H \cong K_p$ has the desired property.

Case 2: Suppose that G is r- regular with $r \leq p - 2$. Let $F \cong K_1$ and set $H = G + F$, where u is the vertex of H that is not in G. Then $d_H(u) = p$. Now *diam* $H = 2$, so by (3.1) if v is a vertex of G, then

$$\begin{aligned} d_H(v) &= 2p - deg_H(v) \\ &= 2p - (r+1) \\ &\geq 2p - 1 - (p-2) \\ &= p + 1. \end{aligned}$$

So $d_H(v) > d_H(u)$ and $M(H) = F$ and $\mathcal{M}(H) = G$.

Case 3: Suppose that G is not regular. Let

$$V_i = \{v \in V(G) | deg(v) = i\}$$

for $\delta(G) \leq i \leq \Delta(G)$. Also, let $n = \Delta(G)$ and let $x \in V(G)$ be a vertex of maximum degree. We will construct H in two steps. First, define G_1 by

$$V(G_1) = V(G) \cup \{w_1, w_2, \ldots, w_{n+2}\} \text{ and}$$

$$E(G_1) = \cup_{i=\delta(G)}^{n} \{vw_j | v \in V_i, 1 \leq j \leq n - i\} \cup$$

$$E(G) \cup \{w_i w_j | 1 \leq i < j \leq n + 2\}.$$

Note that $deg_{G_1}(v) = n$ for every $v \in V(G)$. Also, each vertex w_i is adjacent to the $n + 1$ vertices w_j, $j \neq i$, and to at most $p - 1$ vertices of $G - x$. Thus,

$$n + 1 \leq deg_{G_1}(w_i) \leq (n+1) + (p-1) = n + p.$$

Since $xw_1 \notin E(G_1)$, we have $diam\ G_1 \geq 2$.

Finally, let $F \cong K_1$ and set $H = G_1 + F$ with u being the vertex of H that is not in G_1. Then $d_H(u) = p + n + 2$ and $n + 2 \leq d_H(w_i) \leq p + n + 1$; while for $v \in V(G)$, we have $d_H(v) = n + 1$. Since $diam\ H = 2$, it follows that $M(H) = F$ and $\mathcal{M}(H) = G$, by the observation preceding the theorem. •

In the proof of Theorem 3.2, for each graph G of order p, other than K_p, the graph H that was constructed to satisfy $\mathcal{M}(H) = G$ had diameter 2 and contained a single median vertex with maximum degree $p(H) - 1$. Such a graph H can also be constructed with $\mathcal{M}(H) \cong K_p$.

Define $H \cong (K_p \cup K_{p+1}) + K_1$. Note that $diam\ H = 2$. Let

$$V(H) = \{v_1, v_2, \ldots, v_p, w_1, w_2, \ldots, w_{p+1}, u\}$$

such that $\langle\{v_1, v_2, \ldots, v_p\}\rangle \cong K_p$, $\langle \{w_1, w_2, \ldots, w_{p+1}\}\rangle \cong K_{p+1}$, and u is adjacent to every other vertex of H. Then,

$$deg(u) = p + (p+1) = 2p+1 = p(H) - 1,$$

$$deg(w_i) = p + 1, \text{ and}$$

$$deg(v_i) = (p-1) + 1 = p.$$

Therefore, $\mathcal{M}(H) \cong K_p$ and $M(H) \cong K_1$.

Hence for every graph G of order p, there exists a graph H with $diam\ H = 2$ such that $\mathcal{M}(H) = G$, and $M(H) \cong K_1$, and for $v \in M(H)$, the degree $deg_H\ v = p(H) - 1$. We can now prove a more general corollary.

COROLLARY 3.2A. *For a given graph G of order p and a positive integer k, there exists a graph H_k such that $diam\ H_k = 2$ with $\mathcal{M}(H_k) = G$ and $M(H_k) \cong K_k$ and $deg\ v = p(H_k) - 1$ for $v \in M(H_k)$.*

PROOF: Let G be a graph of order p. For $k = 1$, let $H_1 \cong (K_p \cup K_{p+1}) + K_1$. The result follows from the preceding discussion. For $k \geq 2$, we proceed by induction. Suppose there exists a graph H_{k-1} such that $diam\ (H_{k-1}) = 2$ with $\mathcal{M}(H_{k-1}) \cong G$ and $M(H_{k-1}) \cong K_{k-1}$,

and for $v \in M(H_{k-1})$, the degree $deg\ v = p(H_{k-1}) - 1$. Define $H_k \cong H_{k-1} + K_1$, and let u be the vertex of H_k that is not in H_{k-1}. Now $diam\ (H_k) = 2$ since $diam H = 2$, and $deg\ u = p(H_{k-1}) = p(H_k) - 1$.

For $v \in V(H_{k-1})$ we have $deg_{H_k} v = 1 + deg_{H_{k-1}} v$. That is, $deg_{H_k}(v) = \delta(H_k)$ if and only if $v \in V(\mathcal{M}(H_{k-1}))$ and $deg_{H_k}(v) = p(H_k) - 1$ if and only if $v \in V(M(H_{k-1}))$ or if $v = u$. Therefore, $\mathcal{M}(H_k) \cong G$ and $M(H_k) \cong M(H_{k-1}) + K_1 \cong K_k$ and $deg\ v = p(H_k) - 1$ for $v \in M(H_k)$. •

In the proof of Theorem 3.2, for a regular graph G, the graph H was constructed by adding at most one vertex to G. This is best possible because $H \cong K(1,n)$ is a graph of minimum order having $\mathcal{M}(H) \cong \overline{K_n}$. On the other hand, if G is non- regular, then H had $\Delta(G) + 3$ more vertices than G. It is unclear if $\Delta(G) + 3$ vertices are ever required but the next two results show that it is not always needed.

THEOREM 3.3. *Let G be a non-regular connected graph and let $V_\delta = \{v \in V(G) | deg\ v = \delta(G)\}$. If*

a) $|V_\delta| > \delta(G) + 1$,

b) $diam\ G \leq 2$, and

c) every vertex of maximum degree in G is adjacent to a vertex of minimum degree in G,

then there is a graph H such that $\mathcal{M}(H) \cong G$ and $p(H) =$

$p(G) + \Delta(G) - \delta(G)$.

PROOF: To construct H, let $n = \Delta(G) - \delta(G)$ and label the vertices of K_n by $v_1, v_2, \ldots, v_n$. Now let $H \cong G \cup K_n$ with the added edges

$$\{u\ v_i | u \in V(G) \text{ and } deg_G\, u + i \leq \Delta(G)\}.$$

For $u \in V(G)$ and $m = \Delta(G) - deg_G u$, the vertex u is adjacent to $v_1, v_2, \ldots, v_m$ so $deg_H u = deg_G u + \Delta(G) - deg_G u = \Delta(G)$. Each vertex v_i is adjacent to the other $n - 1$ vertices $v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n$ and to every vertex in V_δ; so $deg_H\, v_i \geq n - 1 + |V_\delta| > \Delta(G) - \delta(G) - 1 + \delta(G) + 1 = \Delta(G)$. Since H is not regular, $diam\ H \geq 2$. However, if $diam\ H = 2$, then $\mathcal{M}(H) \cong G$ and we are done.

Let w_1 and w_2 be vertices of H. If w_1 and w_2 are vertices of G, then $d_H(w_1, w_2) \leq 2$. If w_1 and w_2 are in $\{v_1, v_2, \ldots, v_n\}$, then $d_H(w_1, w_2) = 1$. Now, suppose that $w_1 \in V(G)$ and $w_2 = v_i$ for some $i = 1, 2, \ldots, n$. If $deg_G\, w_1 < \Delta(G)$, then w_1 is adjacent to v_1, and v_1 is either equal to or adjacent to w_2; so $d_H(w_1, w_2) \leq 2$. If $deg\ w_1 = \Delta(G)$, then by (c), the vertex w_1 is adjacent to a vertex of V_δ that is adjacent to every vertex v_i. Thus, $d_H(w_1, w_2) = 2$ and therefore $diam H = 2$. •

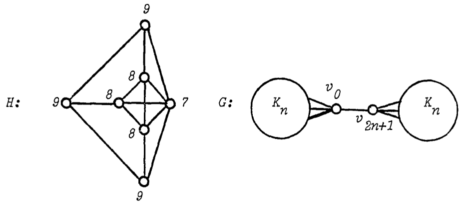


Figure 3.2

Before giving a related corollary, we consider the sharpness of Theorem 3.3.

First, if either condition (a) or (b) is removed, then the theorem is not true. For example, P_3 satisfies conditions (b) and (c) but not (a), and no graph with $p(P_3) + \Delta(P_3) - \delta(P_3) = 4$ vertices contains P_3 as its margin. In fact, the graph H in Figure 3.2 is a graph of minimum order that satisfies $\mathcal{M}(H) \cong P_3$. The number next to each vertex is the distance of that vertex.

Now consider the graph G in Figure 3.2. The graph G satisfies conditions (a) and (c); however, $diam G = 3$. Note that $d_G(v_0) = d_G(v_{2n+1}) = 3n + 1$ and if the vertices of one copy of K_n are labeled $v_1, v_2, \ldots, v_n$ and the vertices of the second copy are labeled $v_{n+1}, \ldots, v_{2n}$.

then $d_G(v_1) = d_G(v_2) = \cdots = d_G(v_{2n}) = 4n+2$. Since $\Delta(G) - \delta(G) = 1$, the theorem requires a graph H such that $p(H) = p(G) + 1$. Thus, let H be a graph with $V(H) = V(G) \cup \{u\}$. We must add edges incident to u to guarantee that $d_H(v_i) = d_H(v_j)$ for $0 \le i, j \le 2n+1$ and $d_H(v_i) > d_H(u)$ for all i. If u is adjacent to v_0, then u must be adjacent to v_{2n+1}; for otherwise, $d_H(v_0) < d_H(v_{2n+1})$. Similarly, if u is adjacent to one of the vertices $v_1, v_2, \ldots, v_{2n}$, then it is adjacent to all of them; for if u is adjacent to v_i and not to v_j, $1 \le i, j \le 2n$, then $d_H(v_i) < d_H(v_j)$. Therefore, three possibilities exist for H. First, suppose that u is adjacent only to v_0 and v_{2n+1}. Then $d_H(v_0) = 3n+2$ and $d_H(v_1) = 4n+4$. These are equal only when $n = -2$; so $G \neq \mathcal{M}(H)$. Second, suppose u is adjacent to only $v_1, v_2, \ldots, v_{2n}$. Then $d_H(v_0) = 3n + 3$ and $d_H(v_1) = 4n+3$. These are only equal when $n = 0$; so again $G \neq \mathcal{M}(H)$. Third, suppose u is adjacent to every vertex of G. Then $d_H(v_0) = 3n + 2$ and $d_H(v_1) = 4n+3$. Hence equality holds if $n = -1$; so $G \neq \mathcal{M}(H)$. Therefore, if H is a graph such that $\mathcal{M}(H) = G$, then $p(H) > p(G) + \Delta(G) - \delta(G)$.

It is not known if there are graphs that satisfy conditions (a) and (b), but fail to meet condition (c) and the conclusion of Theorem 3.3. Hence, the sharpness of this result is uncertain.

Conditions (b) and (c) are very restrictive on our choice of the graph G. Hence, our next result is a powerful generalization of Theorem 3.3.

COROLLARY 3.3A. *Let* G *be a non-regular graph with* $V_\delta = \{v \in V(G) | deg\ v = \delta(G)\}$. *If* $|V_\delta| > \delta(G) + 1$, *then there is a graph* H_1 *such that* $\mathcal{M}(H_1) = G$ *and* $p(H_1) = p(G) + \Delta(G) - \delta(G) + 1$.

PROOF: Let H be the graph constructed in the proof of Theorem 3.3. Define $H_1 = H + K_1$ and let w be the vertex of H_1 not in $V(H)$. Then, $diam\ H_1 \leq 2$. If $u \in V(G)$, then $deg_{H_1} u = \Delta(G) + 1$. If $u \in V(H) - V(G)$, then $deg_{H_1} u > \Delta(G) + 1$ and if $u = w$, then $deg_{H_1} u = p(H) = p(G) + \Delta(G) - \delta(G) > \Delta(G) + 1$. Thus, since H_1 is not regular and $diam\ H_1 = 2$, it follows that $\mathcal{M}(H_1) = G$. Further, $p(H_1) = p(H) + 1 = p(G) + \Delta(G) - \delta(G) + 1$. •

This corollary is sharp. For each positive integer n, consider the graph G in Figure 3.2. The graph H_1, constructed from G in Corollary 3.3A has order $p(G) + 2$ and $\mathcal{M}(H_1) = G$, and we have shown that no graph H exists that has order $p(G) + 1$ and satisfies $\mathcal{M}(H) = G$.

Our previous results have shown ways to add vertices to a graph G to create a graph H satisfying $\mathcal{M}(H) = G$. The number of additional vertices was always related to the degrees of the vertices of G. Is it possible that a constant k exists such that for each graph G we can construct a graph H satisfying $\mathcal{M}(H) = G$ and $p(H) \leq p(G) + k$? The answer is no, as our next result shows.

THEOREM 3.4. *For each positive integer k, there exists a graph G such that no graph H satisfying $p(H) \leq p(G) + k$ contains G as its margin.*

PROOF: Let $n = \dfrac{k^3 + 3k + 10}{2}$ and let G be a copy of K_n with one additional vertex v adjacent to one vertex of K_n. Also, let u be a vertex of K_n not adjacent to v. Now $d_G(u) = 1(n-1) + 2 = n + 1$ and $d_G(v) = 1 + 2(n-1) = 2n - 1$. If we add a set S of $m(\leq k)$ vertices to G as well as edges that join pairs of vertices of S or exactly one vertex of G and one of S to produce a connected graph H, then $d_H(v)$ will be a minimum if v is adjacent to the m vertices of S. Hence, $d_H(v) \geq d_G(v) + m = 2n - 1 + m = k^3 + 3k + 9 + m$. Similarly, $d_H(u)$ will be a maximum if the m vertices of S induce a path with one end- vertex adjacent to v and no edge that joins a vertex of G and one of S. In this case,

$$\begin{aligned}
d_H(u) &\leq d_G(u) + [3 + 4 + \cdots + (m+2)] = n - 2 + \binom{m+3}{2} \\
&= \frac{(k^3 + m^2) + (3k + 3m) + 16 + 2m}{2} - 2 \\
&\leq \frac{2k^3 + 6k + 16 + 2m}{2} - 2 = k^3 + 3k + 6 + m.
\end{aligned}$$

Thus, $d_H(v) > d_H(u)$ and so $\mathcal{M}(H) \neq G$. $\bullet$

3.2 Distance in Trees

By focusing our attention on trees, some properties of the distance of a vertex can be seen that will be useful in proving later results. A very early

result, due to Jordan [11], states that for every tree T, either $M(T) \cong K_1$ or $M(T) \cong K_2$. By Corollary 3.1A, this can be improved for trees of odd order.

COROLLARY 3.1C. *For every tree T of odd order, the median $M(T) \cong K_1$.*

PROOF: Let T be a tree of odd order. If $M(T) \cong K_2$, then two adjacent vertices have the same distance, contradicting the fact that adjacent vertices have distances of opposite parity.

Similar to Jordan's result which characterizes the medians of trees, we can also characterize the margins of trees. Specifically, a tree $T \cong K_2$ if and only if $\mathcal{M}(T) \cong K_2$; otherwise, $\mathcal{M}(T) \cong \overline{K_n}$ for some $n \geq 1$. This follows as a corollary to the next result.

THEOREM 3.5. *For a connected graph G of order $p \geq 3$ with a cut-vertex v, there is a vertex w adjacent to v such that $d(w) > d(v)$.*

PROOF: Let G be a connected graph of order $p \geq 3$, and let v be a cut-vertex of G. Suppose $deg\ v = k$, where the neighbors of v are $u_1, u_2, \ldots, u_k$.

Now let $H = G - v$ and for each $i = 1, 2, \ldots, k$, define the set

$$V_i' = \{u \in V(H) | d_G(u_i, u) \leq d_G(u_j, u),\ j \neq i\}.$$

We can partition $V(H)$ into k sets $V_1, V_2, \ldots, V_k$ by letting

$$\begin{aligned}
V_1 &= V_1', \\
V_2 &= V_2' - V_1, \\
V_3 &= V_3' - (V_1 \cup V_2), \\
&\vdots \\
V_k &= V_k' - (\bigcup_{i=1}^{k-1} V_i).
\end{aligned}$$

If we let $d_H(u_i) = \sum_{u \in V_i} d_G(u_i, u)$ for each i, then

$$d(v) = \sum_{i=1}^{k} d_H(u_i) + \sum_{i=1}^{p-1} 1 = \sum_{i=1}^{k} d_H(u_i) + p - 1$$

Since H has at least two components and the order of H is $p - 1$, it follows that H contains a component C such that $p(C) \leq \frac{p-1}{2}$. Thus, $p(H) - p(C) \geq \frac{p-1}{2}$. Without loss of generality, suppose that u_1 is a vertex in C. Then

$$\begin{aligned}
d(u_1) &\geq \sum_{i=1}^{k} d_H(u_i) + 2\left(\frac{p-1}{2}\right) + 1 \\
&= \sum_{i=1}^{k} d_H(u_i) + p \\
&= d(v) + 1.
\end{aligned}$$

Hence, if we let $w = u_1$, then $d(w) > d(v)$. •

A special case of this theorem follows for trees because a vertex of a tree that is not an end-vertex is a cut-vertex.

Corollary 3.5A. *For a tree* T *of order* $p \geq 3$ *and* $v \in V(T)$ *that is not an end-vertex, there exists a vertex* w *adjacent to* v *such that* $d(w) > d(v)$.

We can now give a result that was proved differently in [15] by Zelinka.

Corollary 3.5B. *Every marginal vertex of a tree is an end-vertex.*

Proof: Suppose v is a marginal vertex of a tree T and v is not an end-vertex. Then there exists a vertex w with $d(w) > d(v)$, contradicting the maximality of $d(v)$. •

Our characterization of the margins of trees is an immediate consequence of this corollary.

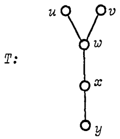
Corollary 3.5C. *Let* T *be a tree. Then* $T \cong K_2$ *if and only if* $\mathcal{M}(T) \cong K_2$; *otherwise,* $\mathcal{M}(T) \cong \overline{K_n}$, *for some* $n \geq 1$.

Since the converse of Corollary 3.5B is not true, as can be seen by the tree T of Figure 3.3, two natural questions now arise. First, which end-vertices of a tree are marginal vertices? Second, for $n \geq 1$, which trees T satisfy $\mathcal{M}(T) \cong \overline{K_n}$? At present, neither question has been answered.

A related result in [15] is that for a tree T of order $p \geq 3$, no end-vertex of T can be a median vertex of T. This is immediate from the fact that if v is an end-vertex of T and u is the vertex adjacent to v, then

$$d(u) = d(u,v) + \sum\{d(v,w) - 1 | w \in V(T),\ w \neq v,u\} = d(v) - (p-2).$$

That is, $d(u) < d(v)$ since $p - 2 > 0$.



$d(u) = d(v) = 8$

$d(y) = 9$

$\mathcal{M}(T) = \langle\{y\}\rangle$

Figure 3.3

We close this section with three lemmas that we will use to find bounds for *med(G)*.

The first lemma was proved in [14].

LEMMA 3.6. *For each positive integer* n *and the path* $P_n : v_1, v_2, \ldots, v_n$, *the median* $med(P_n) = d(v_{\lceil \frac{n}{2} \rceil}) = \lceil \frac{n^2-1}{4} \rceil$.

LEMMA 3.7. *Let* n *be a nonnegative integer. If* T *is a tree of order* p *with adjacent vertices* u *and* v *such that* u *is in a component* C *of* $T - v$ *with* $|V(C)| \geq \frac{p+n}{2}$, *then* $d(u) \leq d(v) - n$.

PROOF: Let $V = V(T) - (V(C) \cup \{v\})$. Since $|V(C)| \geq \frac{p+n}{2}$ it follows that $|V| \leq \frac{p-n}{2} - 1$. Now,

$$\begin{aligned} d(u) &= \sum_{w \in V(C)} d(u,w) + \sum_{w \in V} d(u,w) + d(u,v) \\ &= \sum_{w \in V(C)} (d(v,w) - 1) + \sum_{w \in V} (d(v,w) + 1) + 1 \\ &= d(v) - |V(C)| + |V| + 1 \\ &\leq d(v) - (\frac{p+n}{2}) + (\frac{p-n}{2} - 1) + 1. \end{aligned}$$

So $d(u) \leq d(v) - n$.

We next present an immediate consequence of Lemma 3.7.

LEMMA 3.8. *Let* n *be a nonnegative integer and let* T *be a tree of order* p *with vertices* u *and* v *such that* $d(u,v) = m$. *If* u *is in a component* C_v *of* $T - v$ *with* $|V(C_v)| \geq \frac{p+n}{2}$ *and there exists a component* C_u *of* $T - u$ *with* $|V(C_u)| \geq \frac{p+n}{2} - 1$ *and not containing* v, *then* $d(u) \leq d(v) - mn$.

PROOF: Let $P : v = w_0, w_1, \ldots, w_m = u$ be the $v - u$ path in T of length m. Note that for each $i = 1, 2, \ldots, m - 1$, there exists a component C_i of $T - w_i$ with $|V(C_i)| \geq \frac{p+n}{2}$ and $\{w_{i+1}, w_{i+2}, \ldots, w_m = u\} \subset V(C_i)$. By repeated application of Lemma 3.7, we have the following inequalities:

$$d(w_1) \leq d(w_0) - n$$

$$d(w_2) \leq d(w_1) - n \leq d(w_0) - 2n$$

$$\vdots$$

$$d(w_m) \leq d(w_{m-1}) - n \leq d(w_0) - mn.$$

Therefore, $d(u) \leq d(v) - mn$. •

3.3 Bounds for $mar(G)$ and $med(G)$

For a connected graph G of order p, it is useful to know the values that $mar(G)$ and $med(G)$ can attain. This was studied in [8]. We now extend some of those results.

THEOREM 3.9. *For a connected graph G of order p,*

$$p - 1 \leq mar(G) \leq \binom{p}{2}$$

where the lower bound and upper bound are attained only for K_p and P_p, respectively.

PROOF: First, since every vertex of K_p is adjacent to every other vertex, $mar(K_p) = p - 1$. If $G \not\cong K_p$, then there exist nonadjacent vertices u and v of G. Thus

$$\begin{aligned} d(u) &= \sum_{w \in V(G)} d(u,w) \\ &= \sum \{d(u,w) | w \in V(G), w \neq u, v\} + d(u,v) \\ &\geq 1(p-2) + 2, \end{aligned}$$

and it follows that $mar(G) \geq d(u) \geq p$. Hence, the lower bound holds and is attained only for K_p.

Now consider the upper bound. Let v_1 and v_2 be the end-vertices of P_p. By symmetry, $d(v_1) = d(v_2)$ and by Corollary 3.5B, the margin $mar(P_p) = d(v_1)$. Since

$$d(v_1) = \sum_{i=1}^{p-1} i = \binom{p}{2},$$

we have $mar(P_p) = \binom{p}{2}$. Thus the upper bound is attained for $mar(P_p)$.

To verify that $\binom{p}{2}$ is an upper bound and is attained only for P_p, let $G \ncong P_p$ be a connected graph of order p and let v be a marginal vertex of G. Let P be a longest subpath of G having v as an end-vertex and let k denote the length of P. Hence, $p - k \geq 2$, the distance $d(v,u) \leq k$ for every $u \in V(G)$ and

$$\begin{aligned} d(v) &= \sum_{u \in P} d(v,u) + \sum_{u \in P} d(v,u) \\ &\leq [1 + 2 + \cdots + k] + k[(p-1) - k]. \end{aligned}$$

Now

$$\binom{p}{2} = [1+2+\cdots+k] + [(k+1)+(k+2)+\cdots+(p-1)]$$
$$= [1+2+\cdots+k] + k[(p-1)-k] + [1+2+\cdots+(p-1)-k]$$

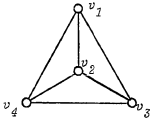
and so $\binom{p}{2} \geq d(v) + \binom{p-k}{2}$.

Since $\binom{p-k}{2} \geq \binom{2}{2} = 1 > 0$, we have $mar(G) < \binom{p}{2}$ and this completes the proof. •

For $p = 1$ or $p = 2$, we have $p - 1 = \binom{p}{2}$. Further, for $p = 3$ we have $p - 1 = 2$ and $\binom{p}{2} = 3$. However, for $p \geq 4$, there exist integers k such that $p - 1 < k < \binom{p}{2}$. A natural question to ask is: For which of these values of k does there exist a graph G of order p such that $mar(G) = k$?

In Figure 3.4, we show a sequence of labeled graphs G_0, G_1, G_2, G_3, each of order 4, with $d(v_1) = mar(G_n) = (p-1) + n$.

G_1:



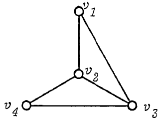
$d(v_1) = 3 = p - 1$

$d(v_2) = 3$

$d(v_3) = 3$

$d(v_4) = 3$

G_2:



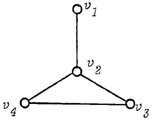
$d(v_1) = 4$

$d(v_2) = 3$

$d(v_3) = 3$

$d(v_4) = 4$

G_3:



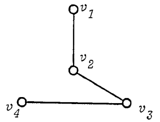
$d(v_1) = 5$

$d(v_2) = 3$

$d(v_3) = 4$

$d(v_4) = 4$

G_4:



$d(v_1) = 6 = \binom{p}{2}$

$d(v_2) = 4$

$d(v_3) = 4$

$d(v_4) = 6$

Figure 3.4

Note that each graph was obtained by removing an edge from the preceding graph. That this will always be true follows from Theorem 3.10 which was proved in [8].

THEOREM 3.10. *For fixed integers* $p \geq 4$ *and* k *such that* $p - 1 \leq k \leq \binom{p}{2}$, *there exists a graph* G *with order* p *and* $mar(G) = k$.

Similar results concerning bounds, sharpness and intermediate values also hold for $med(G)$. We first present a useful theorem about trees.

THEOREM 3.11. *Let* T *be a tree of order* p. *If* $T \not\cong P_p$, *then*

$$med(T) < med(P_p) = \lceil \frac{p^2 - 1}{4} \rceil.$$

PROOF: First, recall from Lemma 3.6 that $med(P_p) = \lceil \frac{p^2-1}{4} \rceil$. Now let T be a tree of order p such that $T \not\cong P_p$, let $v \in M(T)$ and let $r = e(v)$. Observe that $r \leq \frac{p}{2}$; for if not, then there exist $u, w \in V(T)$ such that $d(v, u) = r \geq \frac{p+1}{2}$, the edge $wv \in E(T)$ and w is on the $v - u$ path in T. Thus, the component of $T - v$ containing u has at least $\frac{p+1}{2}$ vertices and by Lemma 3.7, we have $d(w) < d(v)$ which implies that $v \notin M(T)$, a contradiction.

To complete the proof, we consider two cases.

CASE 1: <u>Suppose that</u> $r \leq \frac{p-2}{2}$. Then there exists $u \in V(T)$ such that $d(u,v) = r$, a $v - u$ path P of length r and a component C of $T - v$ containing u. Let $s = max\{d(v,x) | x \notin V(C)\}$, let $w \in V(T) - V(C)$

with $d(v,\, w) = s$, and let P' be the $v - w$ path in T. Then $s \leq r$ and

$$\begin{aligned} d(v) &= \sum_{x \in V(P)-\{v\}} d(v,x) + \sum_{x \in V(C)-[V(P)-\{v\}]} d(v,x) + \\ &+ \sum_{x \in V(P')} d(v,x) + \sum_{x \in V(T)-[V(P') \cup V(C)]} d(v,x) \\ &\leq [1+2+\cdots+r] + r[|V(C)|-r] + [1+2\cdots+s] + \\ &+ s[p-(s+1)-|V(C)|]. \end{aligned}$$

However,

$$\begin{aligned} s[p-(s+1)-|V(C)|] &\leq [(s+1)+(s+2)+\cdots+r] \\ &\quad + s[p-(s+1)-|V(C)| - (r-s)] \\ &\leq [(s+1)+(s+2)+\cdots+r] \\ &\quad + r[p-1-|V(C)|-r]; \end{aligned}$$

so

$$\begin{aligned} d(v) &\leq [1+2+\cdots+r] + r[|V(C)|-r] + [1+2+\cdots+r] \\ &\quad + r[p-1-|V(C)|-r] \\ &= 2\binom{r+1}{2} + r[p-1-2r] \\ &= rp - r^2. \end{aligned} \tag{3.2}$$

Since $r \leq \frac{p-2}{2}$, equation (3.2)becomes $d(v) \leq \frac{p^2-4}{4} < \frac{p^2-1}{4}$ and so $med(T) < med(P_p) = \lceil\frac{p^2-1}{4}\rceil$.

CASE 2: <u>Suppose that</u> p <u>is even and</u> $r = \frac{p}{2}$ <u>or</u> p <u>is odd and</u> $r = \frac{p-1}{2}$. Then there exists $u \in V(T)$ such that $d(v,u) = r$, a $v-u$ path P in T, and a component C of $T - v$ containing u. If C contains vertices other than those of P, then $p(C) \geq \frac{p+1}{2}$ and by Lemma 3.7, we have $v \in M(T)$, a contradiction.

Therefore the $p - r$ vertices of T that are not in C form a subtree T_v of T and

$$d(v) = \sum d(v,x) + \sum d(v,x) = \binom{r+1}{2} + d_{T_v}(v).$$

$$d(v) = \sum_{x \in V(C)} d(v,x) + \sum_{x \in V(T_v)} d(v,x) = \binom{r+1}{2} + d_{T_v}(v).$$

Since $\binom{r+1}{2}$ is fixed, $d(v)$ will be maximized by making $d_{T_v}(v)$ as large as possible. By Theorem 3.9, $d_{T_v}(v)$ will be a maximum only when T_v is a path and v is an end-vertex of T_v. Then $d_{T_v}(v) = \binom{p-r}{2}$. However, this forces $T \cong P_p$, which contradicts the choice of T. Hence,

$$d_{T_v}(v) \leq \binom{p-r}{2} - 1$$

and

$$d(v) \leq \binom{r+1}{2} + \binom{p-4}{2} - 1. \tag{3.3}$$

If p is even and $r = \frac{p}{2}$, then by (3.3),

$$d(v) \leq \frac{p^2}{4} - 1 < \frac{p^2}{4} = med(P_p).$$

On the other hand, if p is odd and $r = \frac{p-1}{2}$, then (3.3) implies that

$$d(v) \leq \frac{p^2 - 1}{4} - 1 < \frac{p^2 - 1}{4} = med(P_p),$$

and this completes the proof. ●

If G is a connected graph of order p and T is a spanning tree of G, then for $u, v \in V(G)$,

$$d_G(u,v) \leq d_T(u,v)$$

because T has potentially fewer edges than G. Hence, for $v \in V(G)$,

$$d_G(v) \leq d_T(v)$$

and

$$med(G) \leq med(T). \tag{3.4}$$

If, in addition T is a distance-preserving tree from $v \in V(G)$, then for each $u \in V(G)$

$$\begin{aligned} d_T(v,u) &= d_G(v,u), \\ d_T(v) &= d_G(v), \end{aligned} \tag{3.5}$$

and

$$deg_T(v) = deg_G(v). \tag{3.6}$$

We can now give bounds for $med(G)$ when $p(G)$ is fixed.

COROLLARY 3.11A. *For a connected graph G of order p,*

$$p - 1 \leq med(G) \leq \lceil \frac{p^2-1}{4} \rceil .$$

PROOF: For the first inequality, let G be a connected graph of order p and let $v \in M(G)$. Then

$$med(G) = d(v) = \sum_{u \in V(G)-\{v\}} d(v,u) \geq \sum_{u \in V(G)-\{v\}} 1 = p - 1.$$

For the second inequality, let G be a connected graph of order p and let T be a spanning tree of G. By (3.4) we have $med(G) \leq med(T)$, and by Theorem 3.8, we have $med(T) \leq \lceil \frac{p^2-1}{4} \rceil$. Hence, $med(G) \leq \lceil \frac{p^2-1}{4} \rceil$, as desired. •

Our next corollary shows that these bounds are sharp and also characterizes all graphs G of order p such that $med(G) = p - 1$ or $med(G) = \lceil \frac{p^2-1}{4} \rceil$.

COROLLARY 3.11B. *Let G be a connected graph of order p. Then*

a) $med(G) = p - 1$ if and only if $\Delta(G) = p - 1$, and

b) $med(G) = \lceil \frac{p^2-1}{4} \rceil$ if and only if $G \cong P_p$ or $G \cong C_p$.

PROOF: For (a), suppose that $med(G) = p - 1$. Then there exists $u \in M(G)$ with $d(u) = p - 1$. That is, u is adjacent to every other vertex of G and so $\Delta(G) = p - 1$. Conversely, assume $\Delta(G) = p - 1$ and let $u \in V(G)$ with $deg(u) = \Delta(G)$. Then $d(u) = p - 1$ and since $med(G) \geq p - 1$, by Corollary 3.11A, the median $med(G) = p - 1$.

For (b), if $G \cong P_p$, then $med(G) = \lceil \frac{p^2-1}{4} \rceil$, by Lemma 3.6. Thus, assume $G \cong C_p$ and let $v \in M(C_p)$. Let T be a distance-preserving tree from v. Then by (3.4) and (3.5) we have $med(C_p) = med(T)$. However, $T \cong P_p$ so $med(C_p) = med(P_p) = \lceil \frac{p^2-1}{4} \rceil$. For the converse, suppose that G is connected and of order p but that $G \ncong P_p$ and $G \ncong C_p$. Then G has a vertex u with $deg(u) \geq 3$. Let T be a distance-preserving tree from u. By (3.6) we have $deg_T(u) = deg_G(u) \geq 3$ and so $T \ncong P_p$. Thus, by Theorem 3.11, the median $med(T) < \lceil \frac{p^2-1}{4} \rceil$, and so $med(G) < \lceil \frac{p^2-1}{4} \rceil$, and this completes the proof. •

It is also interesting to note that for a fixed positive integer p, there is a tree of order p whose median value is n for each n satisfying $p - 1 < n < \lceil \frac{p^2-1}{4} \rceil$.

THEOREM 3.12. *For positive integers* p *and* n *such that* $p-1 < n < \lceil \frac{p^2-1}{4} \rceil$, *there exists a tree* T *of order* p *with* $med(T) = n$.

PROOF: For $p \leq 4$, there is no integer n such that $p-1 < n < \lceil \frac{p^2-1}{4} \rceil$. Therefore, we can assume that $p \geq 5$. First, for $n = \lceil \frac{p^2-1}{4} \rceil - 1$, let $V(P_p) = \{u_1, u_2, \ldots, u_p\}$ with $E(P_p) = \{u_i u_{i+1} | i = 1, 2, \ldots, p-1\}$ and let $T_n = P_p - u_1u_2 + u_1u_3$. Recall that $med(P_p) = \lceil \frac{p^2-1}{4} \rceil$ and $u_{\lceil \frac{p}{2} \rceil} \in V(M(P_p))$, by Lemma 3.6, and no end-vertex of T_n is in $M(T_n)$; so

$$V(M(T_n)) \subseteq \{u_3, u_4, \ldots, u_{\lceil \frac{p}{2} \rceil}, \ldots, u_{p-1}\}.$$

For $3 \leq i \leq p-1$, the distance $d_{T_n}(u_i, u_1) = d_{P_p}(u_i, u_1) - 1$ and for $u_j \neq u_i$ where $j \geq 2$, we have $d_{T_n}(u_i, u_j) = d_{P_p}(u_i, u_j)$. Therefore, $d_{T_n}(u_i) = d_{P_p}(u_i) - 1$ for $3 \leq i \leq p-1$, and so $med(T_n) = med(P_p) - 1 = n$.

We proceed by induction on the decreasing value of n.

Assume, for each integer m such that

$$p-1 < n < m \leq \lceil \frac{p^2-1}{4} \rceil,$$

that there exists a tree T_m with $med(T_m) = m$. Let $\ell = diam(T_{n+1})$. If $\ell = 1$, then $T_{n+1} \cong K_p$, contradicting the fact that T_{n+1} is a tree. If

$\ell = 2$, then $T_{n+1} \cong K(1, p-1)$ and $med(T_{n+1}) = n + 1 = p - 1$, contradicting $p - 1 < n$. Therefore, $\ell \geq 3$.

Now, consider a longest path $P: u_1, u_2, \ldots, u_{\ell+1}$ in T_{n+1} labelled so that $deg(u_2) \leq deg(u_\ell)$. Then u_2 is adjacent to u_3, u_1 and $k = deg(u_2) - 2$ other end-vertices, say $w_1, w_2, \ldots, w_k$. Otherwise, T_{n+1} contains a cycle or a longer path than P. Also, no vertex, other than possibly u_3, is adjacent to both u_2 and u_ℓ so

$$deg\ u_2 + deg\ u_\ell \leq (p-2) + 1$$

and since $deg\ u_2 \leq deg\ u_\ell$,

$$deg\ u_2 \leq \frac{p-1}{2}.$$

Now, u_3 is in a component C of $T_{n+1} - u_2$ with $|V(C)| = p - deg\, u_2 \geq p - (\frac{p-1}{2}) = \frac{p+1}{2}$. Thus, by Lemma 3.7, we have $d_{T_{n+1}}(u_3) < d_{T_{n+1}}(u_2)$; so $u_2 \notin V(M(T_{n+1}))$ and none of $u_1, w_1, w_2, \ldots, w_k$ are in $V(M(T_{n+1}))$ because they are end-vertices of T_{n+1}.

Now construct the desired tree $T_n = T_{n+1} - u_1u_2 + u_1u_3$. For any vertex $v \notin \{u_1, u_2, w_1, w_2, \ldots, w_k\}$ of T_n,

$$\begin{aligned} d_{T_n}(v) &= d_{T_{n+1}}(v) - d_{T_{n+1}}(v, u_1) + d_{T_n}(v, u_1) \\ &= d_{T_{n+1}}(v) - 1. \end{aligned}$$

Thus, $med(T_n) = med(T_{n+1}) - 1 = (n+1) - 1 = n$. •

Recall for a connected graph G that $rad(G) \leq diam(G) \leq 2\,rad(G)$. One might naturally ask if such an inequality holds for $med(G)$ and $mar(G)$. Clearly, $med(G) \leq mar(G)$. However, as the next result shows, no such upper bound exists.

THEOREM 3.13. *For each positive integer* n, *there exists a tree* T *such that* $mar(T) > n[med(T)]$.

PROOF: Let n be a positive integer. We construct T from P_{n+2} as follows. Let u and v be the end-vertices of P_{n+2} and let w be the vertex adjacent to u. Add $k = \frac{n^3-n}{2}$ vertices to P_{n+2} and join them to w. Now,

$$\begin{aligned} mar(T) \geq d(v) &= \binom{n+2}{2} + k(n+1) \\ &= \frac{n^4 + n^3 + 2n}{2} + 1, \end{aligned}$$

and

$$\begin{aligned} n[med(T)] \leq n[d(w)] &= n[\binom{n+1}{2} + k + 1] \\ &= \frac{n^4 + n^3 + 2n}{2}. \end{aligned}$$

Thus, $mar(T) \geq d(v) > n[d(w)] \geq n[med(T)]$. •

Even more can be said if we analyze the tree T described in the proof of Theorem 3.13. First, we already have shown, that

$$d(v) = \binom{n+2}{2} + \left(\frac{n^3-n}{2}\right)(n+1)$$

and

$$d(w) = \frac{n^3 + n^2 + 2}{2}.$$

It is in fact true that these values equal $mar(T)$ and $med(T)$, respectively. To prove this, we need only consider an end-vertex adjacent to w, say u, and an interior vertex, say x, on the $w - v$ path.

For the vertex u we have $med(T) < d(u)$ since u is an end-vertex of the tree. Also, $n+2$ vertices form the $u - v$ path in T while the remaining $\frac{n^3-n}{2}$ vertices are a distance 2 from u. Thus,

$$\begin{aligned} d(u) &= \binom{n+2}{2} + \left(\frac{n^3-n}{2}\right) \cdot 2 \\ &\leq \binom{n+2}{2} + \left(\frac{n^3-n}{2}\right)(n+1) \\ &= d(v). \end{aligned}$$

Therefore, $med(T) < d(u) \leq d(v)$. For the vertex x, the distance $d(x) < mar(T)$ because x is not an end-vertex of T. Also, if we let $\ell = d(x,w)$, then $n - \ell = d(x,v)$. Further, $n - \ell$ vertices of $T - x$ lie on an $x - v$ path in T, with ℓ vertices of $T - x$ lie on an $x - w$ path and the remaining $\frac{n^3-n}{2} + 1$ end-vertices adjacent to w are a distance $\ell + 1$ from x. Thus,

$$d(x) = \binom{n-\ell+1}{2} + \binom{\ell+1}{2} + \left(\frac{n^3-n}{2} + 1\right)(\ell + 1)$$

$$= \frac{n^3+n^2+2}{2} + \ell^2 + \frac{\ell}{2}(n^3 - 3n + 2)$$

$$= d(w) + \ell^2 + \frac{\ell}{2}(n^3 - 3n + 2).$$

If $n = 1$, then $n^3 - 3n + 2 = 0$ and $d(x) > d(w)$. If $n \geq 2$, then $n^3 - 3n + 2 \geq 4n - 3n + 2 = n + 2 > 0$ and again $d(x) > d(w)$. Therefore,

$$d(w) < d(x) < mar(T)$$

and so

$$d(w) = med(T) \text{ and } d(v) = mar(T).$$

The importance of this statement is that for T,

$$n[med(T)] < mar(T) \leq (n+1)[med(T)]$$

and since $diam(T) = n+1$, we have

$$[diam(T) - 1] \cdot [med(T)] < mar(T) \leq [diam(T)] \cdot [med(T)]. \qquad (3.7)$$

In fact, the next corollary shows that for any connected graph G, the product $diam(G) \cdot med(G)$ is always an upper bound for $mar(G)$.

COROLLARY 3.13A. *For a connected graph* G,

$$mar(G) \leq diam(G) \cdot med(G).$$

PROOF: Let p be the order of G. Then $mar(G) \leq (p-1) \cdot diam(G)$ and $med(G) \geq (p-1) \cdot 1$.

Thus,

$$\frac{mar(G)}{med(G)} \leq \frac{(p-1)diam(G)}{(p-1) \cdot 1} = diam(G),$$

and so $mar(G) \leq diam(G) \cdot med(G)$. •

This bound is sharp in the sense that, by (3.7), there exists an infinite class $\mathcal{T}$ of trees such that for $T \in \mathcal{T}$, the margin $mar(T) \leq diam(T) \cdot med(T)$ and $mar(T) > [diam(T) - 1] \cdot med(T)$.

3.4 Margin and Periphery

We have established for a tree, that every marginal vertex is an end-vertex. It is also true that every peripheral vertex of a tree is an end-vertex. One might ask if any relationship exists between marginal and peripheral vertices.

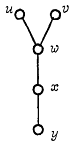
Note that if $T_1 \cong K(1,n)$, then every end-vertex is in both $Per(T_1)$ and $\mathcal{M}(T_1)$. Hence, $Per(T_1) \cong \mathcal{M}(T_1)$. However, if T_2 is as shown in Figure 3.5, then $Per(T_2) = \langle\{u,v,y\}\rangle$, and $\mathcal{M}(T_2) = \langle\{y\}\rangle$; so $V(\mathcal{M}(T_2)) \subset V(Per(T_2))$.

If G_1 is as in Figure 3.6(a), then

$$Per(G_1) = \langle\{w,z\}\rangle \text{ and } \mathcal{M}(G_1) = \langle\{s,w,z\}\rangle;$$

so $V(Per(G_1)) \subset V(\mathcal{M}(G_1))$. By adding the vertex α to G_1 to create G_2 as shown in Figure 3.6(b), we see that neither $V(Per(G_2)) \subseteq V(\mathcal{M}(G_2))$ nor $V(\mathcal{M}(G_2)) \subseteq V(Per(G_2))$. Note that $Per(G_2) = \langle\{w,z\}\rangle$ and $\mathcal{M}(G_2) = \langle\{w,\alpha\}\rangle$.

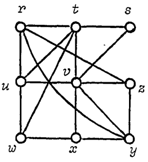
T_2:



vertex	ecc.	distance
u	3	8
v	3	8
w	2	5
x	2	6
y	3	9

Figure 3.5

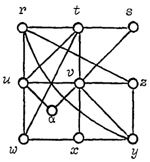
G_1:



(a)

vertex	ecc.	distance
r	2	12
s	2	14
t	2	11
u	2	12
v	2	10
w	3	14
x	2	13
y	2	12
z	3	14

G_2:



(b)

vertex	ecc.	distance
r	2	13
s	2	16
t	2	13
u	2	14
v	2	11
w	3	17
x	2	15
y	2	14
z	3	16
α	2	17

Figure 3.6

A similar result relating the center of a graph G and the median of G was proved by Hendry [9] and extended by Holbert [10]. A part of the result was that for $k \geq 1$ there is a graph G such that

$$d_G(C(G), M(G)) = k.$$

That is, the center and median of a graph can be arbitrarily far apart. We conclude this chapter with two results that show that the periphery and margin of a graph can also be arbitrarily far apart.

THEOREM 3.14. *For each integer* $k \geq 1$, *there is a tree* T_k *such that* $d_{T_k}(Per(T_k), \mathcal{M}(T_k)) \geq k$.

PROOF: For $k \geq 5$, we construct T_k as follows. Begin with the path P of order $n = 2k - 5$ and label the vertices $w_1, w_2, \ldots, w_n$ so that $w_i\, w_{i+1} \in E(P)$ for $i = 1, 2, \ldots, n-1$. Now join $m = \frac{k^2-5k+10}{2}$ vertices to w_1 and a second set of m vertices to w_n. Finally, let v_0 be a new vertex adjacent to w_{k-2} and let v be a vertex adjacent only to v_0.

A sketch of T_k is in Figure 3.7

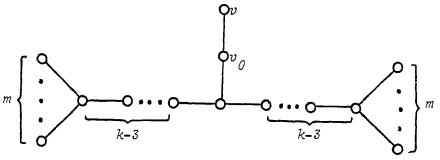


Figure 3.7

All end-vertices of T_k other than v are similar and thus have the same eccentricity and distance. Let u be one of these end-vertices. We need only calculate the eccentricity and distance for u and v to find the periphery

and margin for T_k.

Now $e(v) = k$ while $e(u) = 2k - 4$. Since $k \geq 5$, $e(u) > e(v)$ and v is the only end- vertex not in $Per(T_k)$.

Also,

$$\begin{aligned} d(v) &= 1 + 2 + 2[3+4+\cdots+(k-1)] + 2mk \\ &= k^3 - 4k^2 + 9k - 3, \end{aligned}$$

while

$$\begin{aligned} d(u) &= [1+2+\cdots+n] + 2(m-1) + m(n+1) + (k-1) + k \\ &= k^3 - 4k^2 + 8k - 3. \end{aligned}$$

Since $d(v) - d(u) = k$, we have $\mathcal{M}(T_k) = \langle\{v\}\rangle$, and so $d_{T_k}(Per(T_k), \mathcal{M}(T_k)) = d(u,v) = k$. If $k = 1,2,3$ or 4, let $T_k = T_5$ and the inequality follows. $\bullet$

This result can be extended to arbitrary graphs.

COROLLARY 3.14A. *For each integer $k \geq 1$, there is a graph G_k containing cycles such that $d_{G_k}(Per(G_k), \mathcal{M}(G_k)) \geq k$.*

PROOF: For $k \geq 5$, we construct G_k from the tree T_k in Theorem 3.14. Join the m end-vertices $u_1, u_2, \ldots, u_m$ adjacent to w_1, and join the m end-vertices $u_{m+1}, u_{m+2}, \ldots, u_{2m}$ adjacent to w_n to form two copies of K_m. This is the graph G_k. For $x \in V(G_k)$, the

eccentricity $e_{G_k}(x) = e_{T_k}(x)$. Therefore, $V(Per(G_k)) = V(Per(T_k)) = \{u_1, u_2, \ldots, u_{2m}\}$. Similarly, for $x \in V(G_k) - \{u_1, u_2, \ldots, u_{2m}\}$, the distance $d_{G_k}(x) = d_{T_k}(x)$ and for u_i $(1 \le i \le 2m)$ the distance $d_{G_k}(u_i) < d_{T_k}(u_i)$. Thus, $\mathcal{M}(G_k) = \mathcal{M}(T_k) = \langle\{v\}\rangle$. Finally, since $d_{G_k}(u_i, v) = k$ for $1 \le i \le 2m$, we have $d_{G_k}(Per(G_k), \mathcal{M}(G_k)) = k$. If $k = 1, 2, 3$ or 4, let $G_k \cong G_5$ and the inequality follows.

CHAPTER IV

ANTIPODAL GRAPHS AND DIGRAPHS

4.1 Introduction

Recall, for a pair u, v of vertices in a strong digraph D, that the distance $d(u,v)$ is the length of a shortest directed $u - v$ path. We can extend this definition to all digraphs D by defining $d(u,v) = \infty$ if there is no directed $u - v$ path in D. Similarly, if G is a disconnected graph with vertices u, v in different components, then we can define $d(u,v) = \infty$. Hence, for a digraph D that is not strongly connected or a graph G that is disconnected, we can define the diameter $diam(D)$ or $diam(G)$ to be ∞.

For a digraph D, the antipodal digraph $A(D)$ of D is the digraph with $V(A(D)) = V(D)$ and

$$E(A(D)) = \{(u,v) | u, v \in V(D) \text{ and } d_D(u,v) = diam(D)\}.$$

Our first result gives a useful property of antipodal digraphs. The proof is straightforward, so we omit it

LEMMA 4.1. *If D is a symmetric digraph, then $A(D)$ is also symmetric.*

For $1 \leq p(D) \leq 3$, it is easy to check that the converse of Lemma 4.1 is true. However, for $p(D) \geq 4$, there exist asymmetric digraphs with

symmetric antipodal digraphs. Figure 4.1(a) shows an asymmetric digraph D_1 of order $p \geq 4$ with $diam(D_1) = \infty$ and the corresponding symmetric antipodal digraph $A(D_1)$. Figure 4.1(b) shows an asymmetric strong digraph D_2 of order 4 with finite diameter $(diam(D_2) = 3)$ and its symmetric antipodal digraph $A(D_2)$.

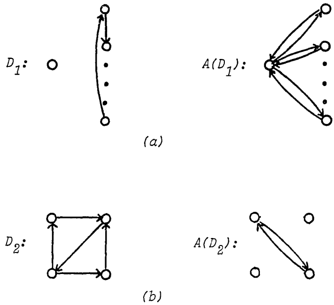


Figure 4.1

When a digraph D contains each of the arcs (u, v) and (v, u), it is customary to represent this symmetric pair of arcs by the single edge uv. This convention induces a one-to-one correspondence φ from the set of

symmetric digraphs to the set of graphs. For example, in Figure 4.1, we have $\varphi(A(D_1)) = K(3, p-3)$ and $\varphi(A(D_2)) = K_2 \cup \overline{K_2}$. Therefore, by Lemma 4.1, it is natural to define, for a graph G, the antipodal graph $A(G)$ of G as the graph with $V(A(G)) = V(G)$ and

$$E(A(G)) = \{uv | u, v \in V(G) \text{ and } d_G(u,v) = diam(G)\}.$$

Antipodal graphs have been studied in [1], [2] and [3].

4.2 Which Digraphs Are Antipodal Digraphs?

In [2] and [3] a characterization of antipodal graphs was given which we now state.

THEOREM 4.2. *A graph G is an antipodal graph if and only if it is the antipodal graph of its complement.*

Using other results in [2], we can give a second form of Theorem 4.2.

THEOREM 4.2$'$. *A graph G is an antipodal graph if and only if*

(1) $diam(\overline{G}) = 2$ or

(2) $\overline{G}$ is disconnected and the components of $\overline{G}$ are complete graphs.

In this section, we will generalize several of the results in [2] and give a characterization of antipodal digraphs. This will lead to a proof of the characterization of antipodal graphs in Theorem 4.2, which is simpler than the proof given in [2].

LEMMA 4.3. *For a digraph* D *of order* p, *the antipodal digraph* $A(D) = D$ *if and only if* $D \cong K_p^*$.

PROOF: First, suppose that $A(D) = D$. If $(u,v) \in E(D)$ then $(u,v) \in E(A(D))$. Therefore, $d_D(u,v) = 1 = diam(D)$. Since K_p^* is the only digraph of diameter 1, we have $D \cong K_p^*$. For the converse, if $D \cong K_p^*$, then $diam(D) = 1$ and for every pair u,v of vertices in D, the distance $d_D(u,v) = 1$. Hence, $A(D) \cong K_p^*$ and $A(D) = D$. •

Since $\varphi(K_p^*) = K_p$, the next result, from [2], follows immediately.

COROLLARY 4.3A. *For a graph* G *of order* p, *the antipodal graph* $A(G) = G$ *if and only if* $G \cong K_p$.

Suppose that $D \ncong K_p^*$. Then $diam(D) \geq 2$ and if (u,v) is an arc of D, then (u,v) will not be an arc of $A(D)$. Similarly, if (u,v) is an arc of $A(D)$, then it is not an arc of D. Thus, $A(D)$ is always an subdigraph of $\overline{D}$. This is our next result.

LEMMA 4.4. *If* D *is a digraph of order* p *that is not complete symmetric, then* $A(D) \subset \overline{D}$.

As a special case, we have the following result from [2].

COROLLARY 4.4A. *If* G *is a non-complete graph of order* p, *then* $A(G) \subset \overline{G}$.

We can now present a result that is useful in our characterization of antipodal digraphs.

THEOREM 4.5. *For a digraph* D, *the antipodal digraph* $A(D) = \overline{D}$ *if and only if either (1)* $diam(D) = 2$ *or (2)* D *is not strongly connected and for every pair* u, v *of vertices of* D, *the distance* $d_D(u,v) = 1$ *or* $d_D(u,v) = \infty$.

PROOF: First, suppose that $diam(D) = 2$. If $(u,v) \in E(D)$, then $d_D(u,v) = 1$; so $(u,v) \notin E(A(D))$. If $(u,v) \notin E(D)$, then $d_D(u,v) = 2$ and $(u,v) \in E(A(D))$. Therefore, $A(D) = \overline{D}$. Now, suppose that D is not strongly connected and for every pair u, v of vertices of D, the distance $d_D(u,v) = 1$ or $d_D(u,v) = \infty$. If $d_D(u,v) = \infty$ for every pair u, v of vertices, then $D \cong \overline{K_p^*}$, for some positive integer p, and $A(D) = A(\overline{K_p^*}) \cong K_p^* \cong \overline{D}$. If, on the other hand, $diam(D) = \infty$ and $(u,v) \in E(D)$, then $(u,v) \notin E(A(D))$. If $(u,v) \notin E(D)$, then $d_D(u,v) = \infty$ and so $(u,v) \in E(A(D))$. Hence, $A(D) = \overline{D}$.

For the converse, suppose that $A(D) = \overline{D}$. Assume that the diameter is finite and not equal to 2. If $diam\ D = 1$, then $D \cong K_p^*$. However, then $A(K_p^*) = \overline{K_p^*}$, which contradicts Lemma 4.3. Thus, assume that $2 < diam\ D < \infty$. Let u and v be vertices of D such that $d_D(u,v) = 2$. Note that $(u,v) \notin E(D)$ and $(u,v) \notin E(A(D))$; so $A(D) \neq \overline{D}$. Now, assume that $diam\ D = \infty$ and there exist vertices u and v such that $1 < d_D(u,v) < \infty$. Then $(u,v) \notin E(D)$ and $(u,v) \notin E(A(D))$ and again, $A(D) \neq D$. •

If D is a symmetric digraph of diameter 2, then $\varphi(D)$ is a graph of

diameter 2. On the other hand, if D is symmetric but not strongly connected and for every pair u, v of vertices of D, the distance $d_D(u,v) = 1$ or $d_D(u,v) = \infty$, then $\varphi(D)$ is a disconnected graph where each component is complete. This gives our next result, from [2].

Corollary 4.5A. *For a graph G, the antipodal graph $A(G) = \overline{G}$ if and only if (1) G is of diameter 2 or (2) G is disconnected and the components of G are complete graphs.*

We can now give a characterization of antipodal digraphs.

Theorem 4.6. *A digraph D is an antipodal digraph if and only if D is the antipodal digraph of its complement.*

Proof: First, if D is the antipodal digraph of its complement, then D is an antipodal digraph. For the converse, suppose that D is an antipodal digraph and let H be a digraph such that $A(H) = D$. We consider three cases based on $diam\ H$.

Case 1: Suppose that $diam\ H = 1$. Then $H \cong K_p^*$, for some positive integer p and $A(H) = A(K_p^*) \cong \overline{K_p^*} \cong D$. Since $\overline{D} \cong K_p^*$ and $A(K_p^*) = \overline{K_p^*}$, it follows that $D = A(\overline{D})$, as desired.

Case 2: Suppose that $1 < diam\ H < \infty$. Since the diameter of H is finite, H is strongly connected and for every pair u, v of vertices of H, the distance $d_H(u,v) \leq diam\ H$. Define H' as the digraph formed by adding the arc (u,v) to $E(H)$ if $1 < d_H(u,v) < diam\ H$. Note that

$d_{H'}(u,v) = 1$ when $d_H(u,v) < diam(H)$ and $d_{H'}(u,v) = 2$ when $d_H(u,v) = diam(H)$. Thus, $D = A(H) = A(H')$. Since $diam(H') = 2$, we have $A(H') = \overline{H'}$ by Theorem 4.5. Therefore, $D = \overline{H'}$ and $\overline{D} = H'$ which gives $D = A(\overline{D})$, as desired.

CASE 3: <u>Suppose that</u> $diam(H) = \infty$. Define H' as the digraph formed by adding the arc (u,v) to $E(H)$ if $1 < d_H(u,v) < diam(H)$. Now, if $d_H(u,v) < \infty$, then $d_{H'}(u,v) = 1$ and if $d_H(u,v) = \infty$, then $d_{H'}(u,v) = \infty$ also. Thus, $D = A(H) = A(H')$. Since H' is not strongly connected and for every pair u,v of vertices of H' the distance $d_{H'}(u,v) = 1$ or $d_{H'}(u,v) = \infty$, we have $A(H') = \overline{H'}$ by Theorem 4.5. Therefore, $D = \overline{H'}$ and $\overline{D} = H'$ which gives $D = A(\overline{D})$, as desired. •

This characterization can be restated, with the aid of Theorem 4.5, as follows.

THEOREM 4.6$'$. *A digraph D is an antipodal digraph if and only if (1) $diam(\overline{D}) = 2$ or (2) $\overline{D}$ is not strongly connected and for every pair u,v of vertices of D, the distance $d_{\overline{D}}(u,v) = 1$ or $d_{\overline{D}}(u,v) = \infty$.*

With the correspondence φ between symmetric digraphs and graphs, the characterizations of antipodal graphs in Theorem 4.2 and Theorem 4.2$'$ follow immediately.

4.3 Self-Antipodal Digraphs and Graphs

In the previous section, we proved, for a digraph D of order p, that the antipodal digraph $A(D)$ is identical to D if and only if $D \cong K_p^*$. Similarly, for a graph G of order p, the antipodal graph $A(G)$ is identical to G if and only if $G \cong K_p$. A more interesting question can also be asked. When is $A(D)$ isomorphic to D or when is $A(G)$ isomorphic to G? If $A(D) \cong D$, then we will call D a self-antipodal digraph and if $A(G) \cong G$, we will call G a self-antipodal graph. Self-antipodal graphs were studied in [1].

Although no characterization is known for self-antipodal digraphs, we can show that certain types of digraphs are self-antipodal while others are not. First, the complete symmetric digraphs are self-antipodal; for if $D \cong K_p^*$ for some positive integer p, then $A(D)$ is identical to D and therefore $A(D)$ is also isomorphic to D. Since this is the only type of digraph D such that $A(D) = D$, no other self-antipodal digraph will be identical to its antipodal digraph. For example, given a positive integer p, the directed cycle C_p', where $V(C_p') = \{v_1, v_2, \ldots, v_p\}$ and $E(C_p') = \{(v_i, v_{i+1}) | 1 \leq i \leq p - 1\} \cup \{(v_p, v_1)\}$, is self-antipodal but $A(C_p') \neq C_p'$ because $E(A(C_p')) = \{(v_{i+1}, v_i) | 1 \leq i \leq p - 1\} \cup \{(v_1, v_p)\}$.

If D is a disconnected digraph, then D is not self-antipodal because $A(D)$ is strongly connected and *diam* $A(D) \leq 2$. To see this, let u and v be vertices of D. If u and v are in different components

of D, then $d_D(u,v) = \infty = diam(D)$. Thus $(u,v) \in E(A(D))$ and $d_{A(D)}(u,v) = 1$. If, on the other hand, u and v are in the same component of D, then there exists a vertex w in a second component of D. Now, $d_D(u,w) = d_D(w,v) = \infty$, so $(u,w) \in E(A(D))$ and $(w,v) \in E(A(D))$ and $d_{A(D)}(u,v) \leq 2$. Therefore $A(D)$ is strongly connected and $diam(A(D)) \leq 2$.

A second type of digraph D that is not self-antipodal is if D is strongly connected and the eccentricity of some vertex v of D is less than the diameter of D. Then $od_{A(D)}(v) = 0$ and $A(D)$ is not strong.

We combine these two observations in the next result.

LEMMA 4.7. *If D is a self-antipodal digraph, then D is weakly connected. If, in addition, D is strongly connected, then D is self-centered.*

For each positive integer p, the complete symmetric digraph K_p^* and the directed cycles C_p' $(p \geq 3)$ are strongly connected self-antipodal digraphs. The self-antipodal digraph D in Figure 4.2 is an example of minimum order that is weakly connected but not unilaterally connected. For a class of self-antipodal digraphs D that are unilaterally connected but not strongly connected, let $p > 1$ be an integer and let $D \cong T_p$, the transitive tournament of order p. Note that if $(u,v) \in E(T_p)$, then $d_{T_p}(v,u) = \infty$. Thus, $od_{A(T_p)}(v) = id_{T_p}(v)$ and since the sequence of indegrees of the vertices of T_p is $0, 1, 2, \ldots, p-1$ we have the same sequence as the score sequence for $A(T_p)$. Because T_p is the only tournament

with score sequence $0,\ 1,\ \ldots,\ p\ -\ 1,$ it follows that $A(T_p)\ \cong\ T_p$.

D:

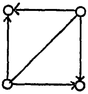


Figure 4.2

For each transitive tournament T_p, the antipodal digraph $A(T_p)\ =\ \overline{T_p}$ and since T_p is self-complementary, we have $A(T_p)\ \cong\ T_p$. This is a special case of the next result, which follows immediately form Theorem 4.5.

THEOREM 4.8. *If* D *is a self-complementary digraph of order* $p\ \geq\ 2$, *then* $A(D)\ \cong\ D$ *if and only if (1)* $diam(D)\ =\ 2$ *or (2)* D *is not strongly connected and for every pair* u, v *of vertices of* D, *the distance* $d_D(u,v)\ =\ 1$ *or* $d_D(u,v)\ =\ \infty$.

We now present one final result on self-antipodal digraphs.

THEOREM 4.9. *If* D *is a non-complete self-antipodal digraph of order* $p\ \geq$ 3, *then* $p\ \leq\ q(D)\ \leq\ p(p-1)/2$.

PROOF: By Lemma 4.7, the digraph D must be weakly connected. The minimum number of arcs in a weakly connected digraph is $p\ -\ 1$, so

$p - 1 \leq q(D)$. Suppose that $q(D) = p - 1$. Then D can contain no directed cycles and hence D is not strongly connected. If D is unilaterally connected, then D contains a directed walk that passes through each vertex of D. This can only be done with $p - 1$ arcs if D is a directed path P'. However, since $A(P')$ is isomorphic to a transitive tournament, D is not self- antipodal. Finally, if D is weakly connected, but not unilaterally connected, then there exist two vertices u and v in D such that no $u - v$ directed path and no $v - u$ directed path exist in D. Therefore, $d_D(u,v) = d_D(v,u) = \infty$ and the arcs (u,v) and (v,u) are both in $A(D)$. Since D contains no directed cycles and $A(D)$ contains a directed 2-cycle, $A(D) \ncong D$. Therefore, $q(D) \geq p$.

For the upper bound, we know since $D \ncong K_p^*$ that $A(D) \subset \overline{D}$. Now $D = A(D) \subset \overline{D}$ implies that $q(D) \leq q(\overline{D})$. Therefore $q(D) \leq \frac{1}{2}q(K_p^*) = \frac{p(p-1)}{2}$. •

The bounds in Theorem 4.9 are sharp. The lower bound is sharp for the class of directed cycles and the upper bound is sharp for the class of transitive tournaments. It is not true, however, for given positive integers p and c with $p < c < p(p-1)/2$, that there is a self-antipodal digraph D having $q(D) = c$. For instance, there is no self-antipodal digraph with four vertices and five arcs.

We now turn to self-antipodal graphs. If G is a self-antipodal graph and φ is the natural one-to-one correspondence from the set of symmetric

digraphs to the set of graphs, then $\varphi^{-1}(G)$ is a self-antipodal digraph. By Lemma 4.7, the digraph $\varphi^{-1}(G)$ is weakly connected; so G is connected. Also, since $\varphi^{-1}(G)$ is symmetric, it is strongly connected and thus $\varphi^{-1}(G)$ and G are self-centered. Therefore, we have the following result.

THEOREM 4.10. *If G is a self-antipodal graph, then G is connected and self- centered.*

Now, suppose that G is a self-antipodal self-complementary graph of order $p \geq 2$. By Theorem 4.10, the graph G is connected, and so Corollary 4.5A implies that $A(G) = \overline{G} \cong G$ if and only if G has diameter 2. We state this as the next result.

COROLLARY 4.10A. *Let G be a self-complementary graph of order $p \geq 2$. Then G is a self-antipodal graph if and only if diam $G = 2$.*

This result is related to a result in [1], which we state.

THEOREM 4.11. *If G is a self-antipodal graph, then G is*

(1) a bipartite self-complementary graph of diameter 2, or

(2) G is nonbipartite.

An important class of self-antipodal graphs is the class of odd cycles. To see this, we will show that $A(C_{2d+1})$, $d \geq 1$, is 2-regular and connected, which characterizes cycles. First, let $V(C_{2d+1}) = \{v_1, v_2, \ldots, v_{2d+1}\}$ and $E(C_{2d+1}) = \{v_i v_{i+1} | 1 \leq i \leq 2d\} \cup \{v_{2d+1}\, v_1\}$. Now, *diam* $(C_{2d+1}) = d$ and for each vertex v_i, $1 \leq i \leq 2d+1$, there exist exactly two vertices v_j

and v_k with $j \equiv (i+d)\, mod(2d+1)$ and $k \equiv (i+d+1)\, mod(2d+1)$ such that $d(v_i,\ v_j) = d(v_i,\ v_k) = d$. Thus, v_i is adjacent to only v_j and v_k in $A(C_{2d+1})$, and so $A(C_{2d+1})$ is 2-regular. Next, if u and v are distinct vertices in C_{2d+1}, then, without loss of generality, we can let $u = v_{2d+1}$ and $v = v_n$ with $1 \leq n \leq d$. A $u - v$ path in $A(C_{2d+1})$ is $P: u = v_{2d+1},\ v_{d+1},\ v_1,\ v_{d+2},\ v_2,\ \ldots\ , v_{d+n},\ v_n$, so $A(C_{2d+1})$ is connected and it follows that $A(C_{2d+1}) \cong C_{2d+1}$.

Two useful observations are now easy to see. First, in showing $A(C_{2d+1})$ to be connected, we found, for vertices u and v with $d_{C_{2d+1}}(u,v) = n$, a $u - v$ path P in $A(C_{2d+1})$ of length $2n$. Since $A(C_{2d+1})$ is a cycle of order $p = 2d + 1$, we have $d_{A(C_{2d+1})}(u,v) = min\{2n,\ p-2n\}$. Thus, we have the following result.

LEMMA 4.12. *Let p be an odd integer and let u and v be vertices of C_p. If $n = d_{C_p}(u,v)$, then there exists a $u - v$ path in $A(C_p)$ of length $2n$ and $d_{A(C_p)}(u,v) = min\{2n,\ p-2n\}$.*

A second observation is that for each integer $d \geq 3$, there exists a graph G of order $2d + 2$ and size $2d + 3$ that is self-antipodal. Let $V(G) = \{v_0,\ v_1,\ \ldots\ ,\ v_{2d+1}\}$ and let

$$E(G) = \{v_i\, v_{i+1} | 1 \leq i \leq 2d\} \cup \{v_{2d+1}\, v_1,\ v_{2d+1}\, v_0,\ v_0\, v_2\}.$$

Vertices v_0 and v_1 are similar and $G - v_0 \cong C_{2d+1}$. For vertices u and v in $G - v_0$, the distance $d_{G-v_0}(u,v) = d_G(u,v)$. Thus,

$A(G-v_0) \prec A(G)$. Now $A(G-v_0) \cong C_{2d+1}$ with v_1 adjacent to v_{d+1} and v_{d+2}. Since v_0 and v_1 are similar in G, the vertex v_0 is also adjacent only to v_{d+1} and v_{d+2} in $A(G)$ and hence, $A(G) \cong G$. We will henceforth denote this graph by $G(2d+2)$.

We now present one last necessary condition for self-antipodal graphs.

THEOREM 4.13. *Let G be a non-complete self-antipodal graph of order $p \geq 5$. Then*

(1) $p \leq q(G) \leq \lfloor p(p-1)/4 \rfloor$, if p is odd, and

(2) $p + 1 \leq q(G) \leq \lfloor p(p-1)/4 \rfloor$, if p is even.

PROOF: We first show that $\lfloor p(p-1)/4 \rfloor$ is an upper bound. The antipodal graph $A(G) \subset \overline{G}$ implies that $G \subset \overline{G}$; so $q(G) \leq \lfloor \frac{1}{2}q(K_p) \rfloor = \lfloor p(p-1)/4 \rfloor$.

Now consider the lower bounds. By Theorem 4.10, we know that G is connected. Hence, $p - 1 \leq q(G)$. However, if $q(G) = p - 1$, then G is a tree and G is not self- centered. This is because if u is an end-vertex of G and v is the vertex adjacent to u, then $e(u) = e(v) + 1$. Thus, $p \leq q(G)$. Now suppose that p is even and there exists a self-antipodal graph G of order p. Then G is connected and contains one cycle. Remove one cycle edge e from G to form the tree $G - e$ and let V be the set of end-vertices of $G - e$. Suppose that there exists a vertex $u \in V$ such that u is not incident with e in G. Then, if v is the vertex adjacent to u in G, then $e(u) = e(v) + 1$. Hence, G is not self-centered and

therefore, not self- antipodal. On the other hand, suppose that every vertex of V is incident with the edge e. Then $|V| = 2$ and $G \cong C_p$. However, for p even, $A(C_p) \cong (\frac{p}{2})K_2$ and again G is not self-antipodal.

Therefore, if $p \geq 5$ is odd, then $p \leq q(G) \leq \lfloor p(p-1)/4 \rfloor$ and if $p \geq 5$ is even, then $p + 1 \leq q(G) \leq \lfloor p(p-1)/4 \rfloor$. •

The bounds in Theorem 4.13 are sharp. First, for each odd integer $p \geq 5$ the cycle C_p is self-antipodal and $q(C_p) = p$. For each even integer $p \geq 8$, the graph $G(p)$ is self-antipodal and $q(G(p)) = p + 1$. Second, for each positive integer $p \geq 5$ such that $p \equiv 0 (mod\, 4)$ or $p \equiv 1 (mod\, 4)$, there exists a self-complementary graph G of diameter 2. By Corollary 4.10A, the graph G is self-antipodal. Hence $q(G) = q(\overline{G}) = \frac{1}{2}q(K_p) = p(p-1)/4$.

Two questions still remain. Do there exist self-antipodal graphs of order $p \equiv 2(mod\, 4)$ or $p \equiv 3(mod\, 4)$ and size $\lfloor p(p-1)/4 \rfloor$? Also, for given positive integers $p \geq 5$ and c such that $p \leq c \leq \lfloor p(p-1)/4 \rfloor$, does there exist a self- antipodal graph G with $c = q(G)$?

4.4 The Antipodal Period of Graphs and Digraphs

For a digraph D of order p and an integer $n \geq 1$, the nth <u>iterated antipodal digraph</u> $A^n(D)$ of D is defined to be $A(A^{n-1}(D))$, where $A^1(D)$ denotes $A(D)$ and $A^0(D)$ denotes D. Since there exist only finitely many digraphs of order p, there exist nonnegative integers n_1 and n_2 such that $n_1 < n_2$ and $A^{n_1}(D) \cong A^{n_2}(D)$. In fact, $A^{n_1+i}(D) \cong A^{n_2+i}(D)$ for all $i \geq 0$. Hence, a sequence of isomorphic copies

of the digraphs $A^{n_1}(D)$, $A^{n_1+1}(D)$, ... , $A^{n_2-1}(D)$ repeats itself continually. Thus, it is natural to define, for a digraph D, the antipodal period k of D as the smallest positive integer for which there exists a nonnegative integer ℓ such that $A^{\ell}(D) \cong A^{\ell+k}(D)$. For instance, for $p \geq 2$, the transitive tournament T_p is a self-antipodal digraph; so T_p has antipodal period 1.

For a graph G, we define the nth iterated antipodal graph $A^n(G)$ of G similarly. If G is a graph and k is the smallest positive integer for which there exists a nonnegative integer ℓ such that $A^{\ell}(G) \cong A^{\ell+k}(G)$, then G has antipodal period n.

We now consider the following question. For which integers $n \geq 1$ is there a digraph or graph with antipodal period n? We will show, through the next results, that there exists a digraph and a graph with antipodal period n for every $n \geq 1$. Since we have already discussed self- antipodal digraphs and graph, we may assume that $n \geq 2$. The next result gives a class of graphs with antipodal period 2.

THEOREM 4.14. *Let* $n \geq 2$ *be an integer and let* $p_1, p_2, \ldots, p_n$ *be a sequence of positive integers with* $p_1 > 1$. *If* $G \cong \bigcup_{i=1}^{n} K_{p_i}$, *then* G *has antipodal period* 2.

PROOF: Since $\overline{G} \cong K(p_1, p_2, \ldots, p_n)$ has diameter 2, the graph G is an antipodal graph, by Theorem 4.2′. In fact, $A(G) = \overline{G} \not\cong G$, by Corollary 4.5A. Also, $A^2(G) = A(\overline{G}) = \overline{G} \cong G$, so G has period 2. •

For the graph G in Theorem 4.14, the digraph $\varphi^{-1}(G)$ is a symmetric digraph with antipodal period 2; however, our next result gives a second class of digraphs with antipodal period 2.

THEOREM 4.15. *For each positive integer k, there exists an asymmetric digraph of order $p = 12k + 3$ that has antipodal period 2.*

PROOF: Let k be a positive integer. We form a digraph D with antipodal period 2 as follows. Let

$$V(D) = \{u_1, u_2, \ldots, u_{2k+1}, v_1, v_2, \ldots, v_{4k+1}, w_1, w_2, \ldots, w_{6k+1}\}$$

and let

$$\begin{aligned} E(D) = \{(u_i, u_j) | 1 \leq i \leq 2k+1,\ 1 \leq \ell \leq k \\ \text{and } j \equiv (i+\ell) mod(2k+1)\} \cup \\ \{(v_i, v_j) | 1 \leq i \leq 4k+1,\ 1 \leq \ell \leq 2k \\ \text{and } j \equiv (i+\ell) mod(4k+1)\} \cup \\ \{(w_i, w_j) | 1 \leq i \leq 6k+1,\ 1 \leq \ell \leq 3k \\ \text{and } j \equiv (i+\ell) mod(6k+1)\} \cup \\ \{(u_i, v_j) | 1 \leq i \leq 2k+1,\ 1 \leq j \leq 4k+1\} \cup \\ \{(v_i, w_j) | 1 \leq i \leq 4k+1,\ 1 \leq j \leq 6k+1\} \cup \\ \{(w_i, u_j) | 1 \leq i \leq 6k+1,\ 1 \leq j \leq 2k+1\}. \end{aligned}$$

The diameter of D is 2. To see this, let x and y be vertices of D such that x is not adjacent to y. First, suppose that $x = u_i$ and $y = u_j$. If $i < j$, then the length ℓ of the $u_i - u_j$ directed path P_1 : $u_i, u_{i+1}, \ldots, u_j$ in D satisfies $k+1 \leq \ell \leq 2k$. If $i > j$, then the length ℓ of the $u_i - u_j$ directed path P_2 : $u_i, u_{i+1}, \ldots, u_{2k+1}, u_1, \ldots, u_j$ in D also satisfies $k + 1 \leq \ell \leq 2k$. Hence, in either case, for $m \equiv (i+k) mod(2k+1)$, the arcs (u_i, u_m) and (u_m, u_j) are in $E(D)$; so $d_D(x,y) = 2$. Second, suppose that $x = v_i$ and $y = v_j$. If $i < j$, then the length ℓ of the $v_i - v_j$ directed path P_1 : $v_i, v_{i+1}, \ldots, v_j$ in D satisfies $2k + 1 \leq \ell \leq 4k$. If $i > j$, then the length ℓ of the $v_i - v_j$ directed path $P_2 : v_i, v_{i+1}, \ldots, v_{4k+1}, v_1, \ldots, v_j$ in D also satisfies $2k + 1 \leq \ell \leq 4k$. Thus, in both cases, for $m \equiv (i+2k) mod(4k+1)$, the arcs (v_i, v_m) and (v_m, v_j) are in $E(D)$; so $d_D(x,y) = 2$. Third, suppose that $x = w_i$ and $y = w_j$. For either $i < j$ or $i > j$, there exists a $w_i - w_j$ directed path of length ℓ such that $3k + 1 \leq \ell \leq 6k$. Thus, for $m \equiv (i+3k) mod(6k+1)$, the arcs (w_i, w_m) and (w_m, w_j) are in $E(D)$; so $d_D(x,y) = 2$. Finally, if $x = u_i$ and $y = w_j$ or $x = v_i$ and $y = u_j$ or $x = w_i$ and $y = v_j$, then $d(u_i, w_j) = 2$ by passing through the vertex v_1, the distance $d(v_i, u_j) = 2$ by passing through the vertex w_1 and $d(w_i, v_j) = 2$ by passing through vertex u_1. Thus $diam(D) = 2$ and by Theorem 4.5, we have $A(D) \cong \overline{D}$.

To show that $D \ncong \overline{D}$, consider the score of each vertex. For the vertex u_i $(1 \le i \le 2k+1)$, the scores are $s_D(u_i) = 5k+1$ and $s_{\overline{D}}(u_i) = 7k + 1$. For the vertex v_i $(1 \le i \le 4k+1)$, we have $s_D(v_i) = 8k+1$ and for the vertex w_i $(1 \le i \le 6k+1)$, the score is $s_D(w_i) = 5k + 1$. Since there exist vertices in $\overline{D}$ with a score of $7k + 1$ and no such vertices in D, it follows that $D \ncong \overline{D}$.

Now, $diam\ (\overline{D}) = 2$ by a similar argument to the one used to show that $diam\ (D) = 2$; so $A^2(D) = A(\overline{D}) = \overline{D} = D$ by Theorem 4.5 and, hence, D is an asymmetric digraph of order $p = 12k + 3$ with antipodal period 2. •

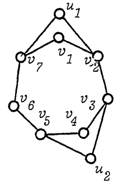
We now show a general construction that will give a graph with antipodal period n for every $n \ge 3$. We begin with some notation. Let $p \ge 7$ be an odd integer and let n be an integer such that $2 \le n \le \lfloor \frac{p-2}{2} \rfloor$. We define the graph $G(n,p)$ as follows. Let $V(G(n,p)) = \{v_1, v_2, \ldots, v_{p-2}, u_1, u_2\}$ and let

$$E(G(n,p)) = \{v_i v_{i+1} | 1 \le i \le p-3\} \cup \{v_{p-2}v_1, v_{p-2}u_1, v_2u_1, v_nu_2, v_{n+2}u_2\}.$$

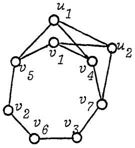
Then the order of $G(n,p)$ is p and $d(u_1, u_2) = n$. For instance, Figure 4.3 shows the graph $G(3,9)$, the antipodal graph $A(G(3,9))$, which we will call $G(1,9)$ and $A^2(G(3,9)) \cong G(2,9)$. Note that, in each graph $G(n,p)$, the vertices u_1 and v_1 are similar and the vertices u_2 and v_{n+1} are

similar.

$G(3,9)$:



$A(G(3,9))$:



$A^2(G(3,9))$:

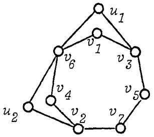


Figure 4.3

This sequence is always achieved when we begin with $G \cong G(\lfloor \frac{p-2}{2} \rfloor, p)$. That is, $A(G) \cong G(1,p)$ and $A^2(G) \cong G(2,p)$. To see this, note that for vertices u and v in $G - u_2$, we have $d_{G-u_2}(u,v)$. Therefore, $A(G-u_2) \prec A(G)$. Since $G - u_2 \cong G(p-1)$, we have shown earlier

that $A(G - u_2) \cong G - u_2$. Now, G contains exactly three vertices whose distance from u_2 is $diam(G)$, namely, u_1, v_1 and v_{p-2}. Thus $A(G(\lfloor \frac{p-2}{2} \rfloor, p)) \cong A(G(\lfloor \frac{p-2}{2} \rfloor, p) - u_2) + u_2u_1 + u_2v_1 + u_2v_{p-2} \cong G(1,p)$. By repeating this process, we have $A(G(1,p) - u_2) \prec A(G(1,p))$ and $A(G(1,p) - u_2) \cong G(1,p) - u_2$. There exist exactly two vertices, say w and x, in $G(1,p)$ whose distance from u_2 is $diam(G(1,p))$. Since u_1 and u_2 are adjacent vertices on a $(p-2)$-cycle in $G(1,p)$, one of the two vertices, say w, satisfies $d(u_1,w) = d(u_2,w) = diam(G(1,p))$. Therefore, $d_{A(G(1,p))}(u,v) = 2$ and $A^2(G(\lfloor \frac{p-2}{2} \rfloor, p)) \cong G(2,p)$.

Now, we present our last result.

THEOREM 4.16. *Let* $n \geq 3$ *be an integer and let* $p = 2^n + 3$. *Then the graph* $G(2,p)$ *has antipodal period* n.

PROOF: In $G(2,p)$ the vertex u_1 is similar to v_1 and the vertex u_2 is similar to v_3. Therefore, the vertices u_1 and u_2 play the same role as v_1 and v_3, respectively, on the cycle $C: v_1, v_2, \ldots, v_{p-2}, v_1$. Thus, by Lemma 4.12, the distance $d_{A(G(2,p))}(u_1,u_2) = min\{4,\ p-4\} = 4$ and $A(G(2,p)) \cong G(4,p)$. Since the diameter of $A^k(G(2,p))$ is always $\lfloor \frac{p-2}{2} \rfloor = 2^{n-1}$, the kth iterated antipodal graph of $G(2,p)$ will be of the form $A^k(G(2,p)) \cong G(2^{k+1},p)$ until $k = n-1$. Hence, $A^{n-2}(G(2,p)) \cong G(2^{n-1},p)$ and, by the discussion preceding this theorem,

$$A^n(G(2,p)) \cong A^2(A^{n-2}(G(2,p))) \cong A^2(G(2^{n-1},p))$$
$$\cong A^2(G(\lfloor \frac{p-2}{2} \rfloor, p)) \cong G(2,p).$$

Since this is the first time $G(2,p)$ has reappeared, $G(2,p)$ has antipodal period n. •

Certainly, the digraph $\varphi^{-1}(G(2,2^n+3))$, for $n \geq 3$, has antipodal period n by Theorem .16; however, it is also symmetric. It is not known whether there exist non-symmetric digraphs with antipodal period $n \geq 3$.

CHAPTER V

SUBGRAPH DISTANCE

5.1 Introduction

The Mayor's office is investigating the connection between crime and unemployment in their city. To do this, they have divided the city into 50 districts and have located the ten districts with the highest crime rates and the ten districts with the highest rates of unemployment. If these two sets of districts are "close" and overlap, then there may be reason to suspect a strong relation between crime and unemployment. On the other hand, if the two sets of districts are far apart then the problems may not be related. How can they determine how far apart these two sets are?

Our last generalization of distance can be used to answer this question. For a connected graph G of order p and an integer n such that $1 \leq n \leq p$, let F and H be induced subgraphs of G of order n. We define a pairing π from the set $V(F)$, say $\{v_1, v_2, \ldots, v_n\}$, to the set $V(H)$ as a one-to-one correspondence that associates a vertex of $V(F)$ with one of $V(H)$. The distance induced by π between F and H is defined as

$$d_\pi(F, H) = \sum_{i=1}^{n} d(v_i, \pi(v_i))$$

and the subgraph distance between F and H is

$$d(F, H) = \min_{\pi} d_\pi(F, H).$$

First we show that the subgraph distance is, in fact, a metric. For a connected graph G of order p and an integer n such that $1 \leq n \leq p$, let F, H and J be subgraphs of G of order n. First, $d(F, H) \geq 0$ and $d(F, H) = 0$ if and only if $F = H$. Second, $d(F, H) = d(H, F)$. Third, the triangle inequality holds. To see this, let $V(F) = \{v_1, v_2, \ldots, v_n\}$ and let π_1 and π_2 be pairings such that $d(F, H) = d_{\pi_1}(F, H)$ and $d(H, J) = d_{\pi_2}(H, J)$. Then

$$\begin{aligned} d(F, J) \leq d_{\pi_2(\pi_1)}(F, J) &= \sum_{i=1}^{n} d(v_i, \pi_2(\pi_1(v_i))) \\ &\leq \sum_{i=1}^{n} [d(v_i, \pi_1(v_i)) + d(\pi_1(v_i), \pi_2(\pi_1(v_i)))] \\ &= d_{\pi_1}(F, H) + d_{\pi_2}(H, J) \\ &= d(F, H) + d(H, J). \end{aligned}$$

We now present a useful result.

THEOREM 5.1. *For a connected graph* G, *let* F *and* H *be subgraphs of* G *with* $p(F) = p(H)$. *If* $\{u_1, u_2, \ldots, u_k\} \subseteq V(F) \cap V(H)$, *then there exists a pairing* π *from* $V(F)$ *to* $V(H)$ *such that* $d(F, H) = d_\pi(F, H)$ *and* $\pi(u_i) = u_i$ *for* $i = 1, 2, \ldots, k$.

PROOF: Suppose that no pairing π from $V(F)$ to $V(H)$ exists such that $d(F, H) = d_\pi(F, H)$ and $\pi(u_i) = u_i$ for $i = 1, 2 \ldots, k$. Thus, let π_1

be a pairing from $V(F)$ to $V(H)$ such that $d(F, H) = d_{\pi_1}(F, H)$ and $\pi_1(u_i) = u_i$ for a maximum number n of vertices in $\{u_1, u_2, \ldots, u_k\}$. Without loss of generality, suppose that $\pi_1(u_1) \neq u_1$. Now, let $\pi_1(u_1) = v$ and $u \in V(F)$ satisfy $\pi(u) = u_1$. Then $d(u, v) \leq d(u, u_1) + d(u_1, v)$. Thus, we can define the pairing π from $V(F)$ to $V(H)$ so that $\pi(u_1) = u_1$, the image $\pi(u) = v$ and, for $w \neq u_1, u$, the image $\pi(w) = \pi_1(w)$. Therefore, $\pi(u_i) = u_i$ for $m + 1$ vertices in $\{u_1, u_2, \ldots, u_k\}$ and, since $d_\pi(F, H) \leq d_{\pi_1}(F, H)$, it follows that $d_\pi(F, H) = d(F, H)$, which contradicts the choice of π_1. •

An immediate consequence of Theorem 5.1 is the following.

For a connected graph G of order p, if F and H are subgraphs of order n, with $\lceil \frac{p}{2} \rceil \leq n \leq p$, then $V(F)$ and $V(H)$ have vertices in common, say $u_1, u_2, \ldots, u_k$, where $k \leq n$. Let $F' = F - \{u_1, u_2, \ldots, u_k\}$ and let $H' = H - \{u_1, u_2, \ldots, u_k\}$. Then by Theorem 5.1,

$$d(F,H) = \sum_{i=1}^{k} d(u_i,u_i) + d(F',H') = d(F',H'). \qquad (5.1)$$

For a subgraph F of a connected graph G with $p(F) = n$, we define the subgraph eccentricity $e(F)$ of F as $e(F) = \max_{p(H)=n} d(F,H)$. We define the n- radius $rad_n\, G$ of G as $rad_n\, G = \min_{p(F)=n} e(F)$ and the n-diameter $diam_n\, G$ of G as $diam_n\, G = \max_{p(F)=n} e(F)$. For a connected graph G, we define the diameter sequence as the sequence

$diam_1(G)$, $diam_2(G)$, $\ldots$, $diam_{p-1}(G)$. We now give two results on the diameter sequence of a graph. The first result describes the first "half" of the sequence.

COROLLARY 5.1A. *Let* G *be a connected graph of order* p. *If* n *is an integer such that* $1 \leq n \leq \lfloor \frac{p}{2} \rfloor - 1$, *then* $diam_n(G) \leq diam_{n+1}(G)$.

PROOF: Let F and H be subgraphs of G of order n such that $d(F,H) = diam_n(G)$. Since $|V(F) \cup V(H)| < p$, there exists a vertex v of G such that $v \notin V(F) \cup V(H)$. Now, define the subgraphs $F' = \langle V(F) \cup \{v\} \rangle$ and $H' = \langle V(H) \cup \{v\} \rangle$ of order $n + 1$, and let π be a pairing from the set $V(F')$ to $V(H')$ such that $\pi(v) = v$. Then $d(F',H') = d(F,H)$ by (5.1), and since $d(F',H') \leq diam_{n+1}(G)$, it follows that $diam_n(G) \leq diam_{n+1}(G)$. •

The next result describes the second "half" of the diameter sequence of a graph.

COROLLARY 5.1B. *Let* G *be a connected graph of order* p. *If* n *is an integer such that* $1 \leq n \leq p - 1$, *then* $diam_n(G) = diam_{p-n}(G)$.

PROOF: Without loss of generality, we may assume that $1 \leq n \leq \frac{p}{2}$. If p is even and $n = \frac{p}{2}$, then $p - n = \frac{p}{2}$ and $diam_n(G) = diam_{p-n}(G)$. Thus, we may assume that $1 \leq n < \frac{p}{2}$. Let F and H be subgraphs of G of order n such that $d(F,H) = diam_n(G)$. There exist $k(\geq p - 2n)$ vertices $u_1, u_2, \ldots ,u_k$ of G such that $u_i \notin V(F) \cup V(H)$ for

$i = 1, 2, \ldots, k$. Thus, let $S = \{u_1, u_2, \ldots, u_{p-2n}\}$ and define the subgraphs $F' = \langle V(F) \cup S \rangle$ and $H' = \langle V(H) \cup S \rangle$ of order $p - n$. Then, by (5.1), we have $d(F', H') = d(F, H) = diam_n(G)$, and it follows that $diam_n(G) \leq diam_{p-n}(G)$.

Now, let F_1 and F_2 be subgraphs of G of order $p - n$ such that $d(F_1, H_1) = diam_{p-n}(G)$. Since $p - n > \frac{p}{2}$, the set $V(F) \cap V(H)$ is nonempty. In fact, there exist $k(\geq p - 2n)$ vertices $v_1, v_2, \ldots, v_k$ in $V(F) \cap V(H)$. Let $S_1 = \{v_1, v_2, \ldots v_{p-2n}\}$. For subgraphs $F_1' = F_1 - S_1$ and $H_1' = H_1 - S_1$, we have $d(F_1, H_1) = d(F_1', H_1') \leq diam_n(G)$, and so $diam_{p-n}(G) \leq diam_n(G)$. Therefore, $diam_n(G) = diam_{p-n}(G)$, as desired. $\bullet$

As an example of a connected graph and its diameter sequence, consider $G \cong K_p$. If $p = 2k$ and n is an integer such that $1 \leq n \leq k$, then every subgraph F of G with order n has $e(F) = n$. Thus, the diameter sequence for K_{2k} is $\mathcal{S}_1 :\ 1, 2, 3, \ldots, k-1, k, k-1, \ldots, 2, 1$. Similarly, for $G \cong K_{2k+1}$, the diameter sequence is $\mathcal{S}_2 :\ 1, 2, 3, \ldots, k-1, k, k, k-1, \ldots, 2, 1$. Since complete graphs are the only graphs G such that $diam\, G = 1$, it follows that $\mathcal{S}_1$ and $\mathcal{S}_2$ are diameter sequences only for K_{2k} and K_{2k+1}, respectively, and that no other sequence beginning with 1 is a diameter sequence.

For the graph K_p and an integer n with $1 \leq n \leq \lfloor \frac{p}{2} \rfloor - 1$, we have $diam_n(K_p) < diam_{n+1}(K_p)$. For Corollary 5.1A to be sharp, there

must exist graphs G of order p and an integer n with $1 \leq n \leq \lfloor \frac{p}{2} \rfloor - 1$ such that $diam_n G = diam_{n+1} G$. Such a graph is $G \cong K(2k, 2k)$ for every positive integer k. To see this, let V_1 and V_2 be the partite sets of G, and let F be a subgraph of G with $p(F) = k$ and $V(F) \subseteq V_1$. If $H = \langle V_1 \rangle - V(F)$, then $d(F, H) = 2k$. Since $diam\ G = 2$ and $diam_n(G_0) = n(diam\ G_0)$ for any connected graph G_0, it follows that $diam_k G = 2k$. Now, for every subgraph F of G with $p(F) = 2k$, if $H = \langle V(G) - V(F) \rangle$, then $d(F, H) = 2k$. This is because there exists a pairing π from $V(F)$ to $V(H)$ such that for each vertex $v \in V_1 \cap V(F)$ there is a vertex $u \in V_2 \cap V(H)$ and for $v \in V_2 \cap V(F)$ there exists a vertex $u \in V_1 \cap V(H)$ such that $\pi(v) = u$ and $d(v, \pi(v)) = 1$. To see that $diam_{2k} G = 2k$, suppose that H' is a subgraph of G of order $2k$ with $\ell(\geq 1)$ vertices in $V(F) \cap V(H')$. Then there exists a pairing π from $V(F)$ to $V(H')$ such that for $v \in V(F)$, we have $d(v, \pi(v)) = 0$ for ℓ vertices, $d(v, \pi(v)) = 2$ for at most ℓ vertices and $d(v, \pi(v)) = 1$ for the remaining vertices of F. Thus, $d(F, H') \leq 2k$, and so $diam_{2k} G = 2k$. Therefore, if $G \cong K(2k, 2k)$, then

$$diam_k G = diam_{k+1} G = \cdots = diam_{2k} G = 2k.$$

Next, we develop a characterization for the diameter sequence of a caterpillar. A <u>caterpillar</u> is a tree of order $p \geq 3$ whose subgraph obtained by removing its end-vertices is a path. Let G be a caterpillar of odd order p

with vertices $v_1,\ v_2 \in V(G)$ satisfying $d(v_1,\ v_2) = diam\ G$. There exists a vertex v on the $v_1 - v_2$ path in G such that the graph $G - v$ consists of a component G_1 containing v_1, a component G_2 containing v_2 and $\ell = \ell_1 + \ell_2$ isolated vertices $(\ell_1,\ \ell_2 \geq 0)$ so that $|V(G_1)| + \ell_1 = |V(G_2)| + \ell_2$.

Define the set V_1 as $V(G_1)$ together with ℓ_1 isolated vertices of $G - v$ and the set V_2 as $V(G_2)$ and the remaining ℓ_2 isolated vertices of $G - v$. Now, if $m_1 = d(v_1,\ v)$ and $m_2 = d(v_2,\ v)$, then partition the vertices of V_1 into sets $A_i = \{u | u \in V_1$ and $d(u,\ v) = i\}$ for $i = 1,\ 2,\ \ldots,\ m_1$, and the vertices of V_2 into sets $B_i = \{w | w \in V_2$ and $d(w,\ v) = i\}$ for $i = 1,\ 2,\ \ldots,\ m_2$. Note that if $u \in A_i$ and $w \in B_j$ for some i and j, then $d(u,\ w) = i + j$. For convenience, label the vertices of V_1 as $u_1,\ u_2,\ \ldots,\ u_{\lfloor \frac{p}{2} \rfloor}$ such that if $u_{j_1} \in A_{i_1}$ and $u_{j_2} \in A_{i_2}$ for $1 \leq i_1,\ i_2 \leq m_1$, then $j_1 < j_2$ whenever $i_1 > i_2$. Similarly, we can label the vertices of V_2 as $w_1,\ w_2,\ \ldots,\ w_{\lfloor \frac{p}{2} \rfloor}$ such that if $w_{j_1} \in B_{i_1}$ and $w_{j_2} \in B_{i_2}$ for $1 \leq i_1,\ i_2 \leq m_2$, then $j_1 < j_2$ whenever $i_1 > i_2$.

Analogously, suppose that G is a caterpillar of even order p, with vertices v_1 and v_2 of $V(G)$ such that $d(v_1,\ v_2) = diam\ G$. Then there exists a vertex v on the $v_1 - v_2$ path in G such that the graph $G - v$ consists of a component G_1 containing v_1, a component G_2 containing v_2 and $\ell = \ell_1 + \ell_2$ isolated vertices $(\ell_1,\ \ell_2 \geq 0)$ so that

$|(|V(G_1)| + \ell_1) - (|V(G_2)| + \ell_2)| = 1$. Without loss of generality, suppose that $|V(G_1) + \ell_1| < |V(G_2) + \ell_2|$. Define the set V_1 as $V(G_1) \cup \{v\}$ with ℓ_1 isolated vertices of $G - v$ and define the set V_2 as $V(G_2)$ with the remaining ℓ_2 isolated vertices of $G - v$. We define the sets A_i and B_i as for p odd and label the vertices of V_1 as $u_1, u_2, \ldots, u_{\frac{p}{2}}$ and the vertices of V_2 as $w_1, w_2, \ldots, w_{\frac{p}{2}}$, as before.

Several useful observations can now be made. First, for an integer n such that $1 \le n \le \lfloor \frac{p}{2} \rfloor$, we have

$$\begin{aligned} diam_n\, G &= d(\langle\{u_1, u_2, \ldots, u_n\}\rangle, \langle\{w_1, w_2, \ldots, w_n\}\rangle) \\ &= \sum_{i=1}^{n} d(u_i, w_i). \end{aligned}$$

Second, for each integer n such that $2 \le n \le \lfloor \frac{p}{2} \rfloor$, if we let $k_1 = diam\, G$ and $k_n = diam_n\, G - diam_{n-1} G$, then $k_n = diam(G - \{u_i, w_i | 1 \le i \le n-1\})$. Hence, $2 \le k_1 \le p - 1$ and $k_{n-1} - 2 \le k_n \le k_{n-1}$. The inequalities $2 \le k_1 \le p - 1$ follow from the usual bounds on the diameter of a noncomplete graph. To see that the inequality $k_n \le k_{n-1}$ holds, let $d(u_{n-1}, w_{n-1}) = k_{n-1}$ and $d(u_n, w_n) = k_n$. If $k_n > k_{n-1}$, then $d(\langle\{u_1, u_2, \ldots, u_{n-2}, u_n\}\rangle, \langle\{w_1, w_2, \ldots, w_{n-2}, w_n\}\rangle) > \sum_{i=1}^{n-1} d(u_i, w_i) = diam_{n-1}\, G$, which is a contradiction. Thus, $k_n \le k_{n-1}$. For the inequality $k_{n-1} - 2 \le k_n$, let u be the vertex adjacent to u_{n-1} on the $u_{n-1} - w_{n-1}$ path in G, and let w be the vertex adjacent to w_{n-1}. Then $d(u, w) = k_{n-1} - 2$ and

$diam_n\, G \ \geq \ diam_{n-1}\, G \ + \ k_{n-1} \ - \ 2$. That is, $k_n \ \geq \ k_{n-1} \ - \ 2$ and the inequality holds. Finally, if p is odd, then $k_{\lfloor \frac{p}{2} \rfloor} \ = \ d(u_{\lfloor \frac{p}{2} \rfloor}, w_{\lfloor \frac{p}{2} \rfloor}) \ = \ 2$ and if p is even, then $k_{\lfloor \frac{p}{2} \rfloor} \ = \ d(u_{\lfloor \frac{p}{2} \rfloor}, w_{\lfloor \frac{p}{2} \rfloor}) \ = \ 1$ and $k_n \ > \ 1$ for $n \ \neq \ \frac{p}{2}$ since the $u_n \ - \ w_n$ path contains v.

Thus, the diameter sequence of a caterpillar G of order p satisfies the following:

a) $diam_n\, G \ \leq \ diam_{n+1}\, G$ for $1 \ \leq \ n \ \leq \ \lfloor \frac{p}{2} \rfloor \ - 1$,

b) $diam_n\, G \ = \ diam_{p-n} G$,

c) $2 \ \leq \ diam\, G \ \leq \ p - \ 1$,

d) if $k_1 \ = \ diam\, G$ and $k_n \ = \ diam_n\, G \ - \ diam_{n-1}\, G$ for $2 \ \leq \ n \ \leq \ \lfloor \frac{p}{2} \rfloor$, then $k_{n-1} \ - \ 2 \ \leq \ k_n \ \leq \ k_{n-1}$,

e) if p is odd, then $k_{\lfloor \frac{p}{2} \rfloor} \ = \ 2$, and

f) if p is even, then $k_{\frac{p}{2}} \ = \ 1$ and $k_n \ > \ 1$ for $n \ \neq \ \frac{p}{2}$.

As we will now show, any sequence $a_1, \ a_2, \ \ldots \ , a_{p-1}$ of positive integers satisfying properties (a) - (f) is, in fact, the diameter sequence of some caterpillar of order p. That is, we show there exists a caterpillar G of order $p \ \geq \ 3$ such that $diam_n\, G \ = \ a_n$ for each $n\, (1 \ \leq \ n \ \leq \ p - \ 1)$.

We construct a caterpillar G recursively as follows. Let P be a path of length a_1. Label the end-vertices of P as u_1 and w_1 and let u be the vertex of P adjacent to u_1 and let w be the vertex of P adjacent to w_1. If we let $k_1 \ = \ a_1$ and $k_n \ = \ a_n \ - \ a_{n-1}$ for $2 \ \leq \ n \ \leq \ \lfloor \frac{p}{2} \rfloor$, then by property (d), there exist three possibilities for k_2. If $k_2 \ = \ k_1$, then

add a new vertex u_2 and join it to u, and add a new vertex w_2 and join it to w. If $k_2 = k_1 - 1$, then add a new vertex w_2 and join it to w, let $u = u_2$ and label the vertex on the $u_2 - w_2$ path adjacent to u_2 as u. If $k_2 = k_1 - 2$, then let $u = u_2$ and $w = w_2$, and label the vertex on the $u_2 - w_2$ path adjacent to u_2 as u and the vertex adjacent to w_2 as w.

In general, for $2 \leq n \leq \lfloor \frac{p}{2} \rfloor$, suppose that vertices u_{n-1} and w_{n-1} have been defined such that $d(u_{n-1}, w_{n-1}) = k_{n-1}$ and the vertex adjacent to u_{n-1} on the $u_{n-1} - w_{n-1}$ path is u and the vertex on P adjacent to w_{n-1} is w. Three possibilities exist for k_n. If $k_n = k_{n-1}$, then add a new vertex u_n and join it to u and add a new vertex w_n and join it to w. If $k_n = k_{n-1} - 1$, then add a new vertex w_n and join it to w, let $u = u_n$ and label the vertex on the $u_n - w_n$ path adjacent to u_n as u. If $k_n = k_{n-1} - 2$, then let $u = u_n$ and $w = w_n$ and label the vertex on the $u_n - w_n$ path adjacent to u_n as u and the vertex adjacent to w_n as w. Note that in each case, the resulting graph G is a caterpillar and $a_n = diam_n\, G$ for each $n\,(1 \leq n \leq p - 1)$.

Thus, we have proved the following result

THEOREM 5.2. *A sequence* $S: a_1, a_2, \ldots, a_{p-1}$ $(p \geq 3)$ *of positive integers is the diameter sequence for a caterpillar of order* p *if and only if* S *satisfies the following:*

a) $a_n \leq a_{n+1}$ *for* $1 \leq n \leq \lfloor \frac{p}{2} \rfloor - 1$,

b) $a_n = a_{p-n}$,

c) $2 \le a_1 \le p - 1$,

d) if $k_1 = a_1$ *and* $k_n = a_n - a_{n-1}$ *for* $2 \le n \le \lfloor \frac{p}{2} \rfloor$, *then* $k_{n-1} - 2 \le k_n \le k_{n-1}$,

e) if p *is odd, then* $k_{\lfloor \frac{p}{2} \rfloor} = 2$, *and*

f) if p *is even, then* $k_{\frac{p}{2}} = 1$ *and* $k_n > 1$ *for* $n \ne \frac{p}{2}$.

A characterization for the diameter sequence of a tree remains unsolved.

5.2 Computing Subgraph Distances

For a connected graph G of order p, an integer n such that $1 \le n \le p$ and subgraphs F and H of G of order n, how can we efficiently compute $d(F, H)$? One could use the definition. However, this approach would require us to determine all pairings π from $V(F)$ to $V(H)$ and then to compute $d_\pi(F, H)$ for each π and finally to find $min_\pi\, d_\pi(F, H)$. But this procedure is not efficient since there exist $n!$ pairings from $V(F)$ to $V(H)$.

Fortunately, there is an efficient way of determining a pairing π from $V(F)$ to $V(H)$ for which $d(F, H) = d_\pi(F, H)$. To accomplish this, we begin by constructing a complete weighted bipartite graph $G_0 \cong K(n, n)$ of order $2n$, where the vertices of one partite set of G_0 correspond to and are labeled the same as the vertices of F and the vertices of the second partite set of G_0 correspond to and are labeled the same as the vertices of H. The

weight of an edge in G_0 is defined as the distance between the corresponding vertices in G. This can be done by applying Dijkstra's algorithm to each vertex of F. Thus, the complexity of finding G_0 is $\mathcal{O}(np^2)$.

We next determine a matching of G_0 whose weight is as small as possible. This can be done as follows. Let m be the maximum weight among the edges of G_0 and set $m' = m + 1$. Define $G_0' \cong K(n, n)$ to be the complete weighted bipartite graph of order $2n$ obtained by replacing the weight $w_{G_0}(e)$ of every edge e of G_0 by $m' - w_{G_0}(e)$. In [12], an algorithm by Kuhn for finding a maximum weight matching in a weighted bipartite graph is given. We may apply this algorithm to G_0' to obtain a maximum weight matching M for G_0'. Then M is a minimum weight matching for G_0 and the pairing π from $V(F)$ to $V(H)$ induced by M gives $d_\pi(F, H) = d(F, H)$.

In determining a minimum weight matching of G_0, the only step with a significant complexity is the one involving Kuhn's algorithm. Its complexity is $\mathcal{O}(n^3)$, so the complexity of determining $d(F, H)$ is $\mathcal{O}(np^2)$.

This method is good for finding the subgraph distance between two specified subgraphs of a connected graph with equal orders; however, it is cumbersome to use when determining the n-radius or n-diameter of a graph. In the next section, we discuss a graphical structure that is useful for proving theoretical results.

5.3 The n-Subgraph Graph

Let G be a connected graph of order p and let F and H be subgraphs of order n with $1 \leq n \leq p - 1$. Then $d(F, H) = 1$ if and only if there exist adjacent vertices $u \in V(F)$ and $v \in V(H)$ such that $V(F - u) = V(H - v)$. We define the n-subgraph graph of G (or simply the n-graph of G) as that graph G_n with $V(G_n) = \{v_i | S_i$ is a set of n vertices in G and $1 \leq i \leq \binom{p}{n}\}$ and $E(G_n) = \{v_i v_j | d(\langle S_i \rangle, \langle S_j \rangle) = 1, 1 \leq i, j \leq \binom{p}{n}\}$. For example, for every connected graph G, we have $G_1 \cong G$ and for the graph G in Figure 5.1(a), the graph G_2 is in Figure 5.1(b). The vertices of G_2 correspond $(v_i \leftrightarrow S_i)$ to the six subsets $S_1 = \{u_1, u_2\}$, $S_2 = \{u_1, u_3\}$, $S_3 = \{u_1, u_4\}$, $S_4 = \{u_2, u_3\}$, S_5 $\{u_2, u_4\}$, $S_6 = \{u_3, u_4\}$ of $V(G)$.

For a connected graph G of order p and an integer n such that $1 \leq n \leq p$, the graph G_n is connected. Otherwise, suppose that F and H are subgraphs of G of order n such that if $V(F) = S_1$ and $V(H) = S_2$, then G_n contains no path from v_1 to v_2 and $|V(F) \cap V(H)|$ is a maximum. If G contains a path $P: u, u_1, u_2, \ldots, v$ from a vertex $u \in V(H) - V(F)$ to a vertex $v \in V(F) - V(H)$ such that no internal vertex of P is in $V(F) \cup V(H)$, then there exists a subgraph H' of G with $V(H') = V(H - u) \cup \{v\}$ and if $V(H) = S_\alpha$ and $v(H - u) \cup \{u_i\} = S_{2+i}$, there exists a path P_1 in G_n from v_2

to v_α, namely,

$$P_1 : v_2, v_3, v_4, \dots , v_\alpha.$$

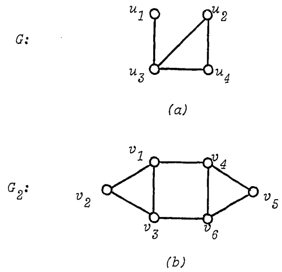


Figure 5.1

Since $| V(F) \cap V(H')| > |V(F) \cap V(H)|$, there must be a path P' in G_n from v_1 to v_α. However, paths P' and P_1 assure us of a path from v_1 to v_2, which is a contradiction. Since no such $u - v$ path P exists in G, there exist vertices $u' \in V(H) - V(F)$ and $v' \in V(F) - V(H)$ and a $u' - v'$ path P'' in G such that if w is an internal vertex of P'' and $w \in V(F) \cup V(H)$, then $w \in V(F) \cap V(H)$. If H'' is a subgraph of G with $V(H'') = V(H - u') \cup \{v'\}$, then if $V(H'') = S_\beta$, path P'' induces a path P''' in G_n from v_2 to v_3.

However, since $|V(F) \cap V(H'')| > |V(F) \cap V(H)|$, we again have a contradiction. Therefore, G_n is connected.

The importance of the n-graph is seen in the next result.

THEOREM 5.3. *Let G be a connected graph of order p and let n be an integer such that $1 \leq n \leq p$. If F and H are subgraphs of G of order n with $V(F) = S_\alpha$ and $V(H) = S_\beta$, then*

$$d_G(F,H) = d_{G_n}(v_\alpha, v_\beta).$$

PROOF: First, we show that $d_G(F,H) \leq d_{G_n}(v_\alpha, v_\beta)$. Let $P: v_\alpha = v_0, v_1, \dots, v_k = v_\beta$ be a shortest path from the vertex v_α to v_β in G_n. Let π_i $(1 \leq i \leq k)$ be a pairing from $V(F_{i-1})$ to $V(F_i)$ such that $d_{\pi_i}(F_{i-1}, F_i) = d(F_{i-1}, F_i)$ and let $\pi = \pi_k \circ \pi_{k-1} \circ \cdots \circ \pi_1$. Then π is a pairing from $V(F)$ to $V(H)$ and by the triangle inequality,

$$d_\pi(F,H) \leq \sum_{i=1}^{k} d_{\pi_i}(V(F_{i-1}), V(F_i)) = k = d_{G_n}(v_\alpha, v_\beta).$$

Since $d(F,H) \leq d_\pi(F,H)$, we have $d(F,H) \leq d_{G_n}(v_\alpha, v_\beta)$.

To show that $d_{G_n}(v_\alpha, v_\beta) \leq d(F,H)$, let π be a pairing from $V(F)$ to $V(H)$ such that $d_\pi(F,H) = d(F,H)$ and $\pi(v) = v$ if $v \in V(F) \cap V(H)$. Let $V(F) = \{v_1, v_2, \dots, v_n\}$ and define the subgraphs F_i such that $F_0 = F$ and $F_n = H$ and, for $1 \leq i \leq n-1$, let $S_i = \{\pi(v_1), \pi(v_2), \dots, \pi(v_i), v_{i+1}, \dots, v_n\}$ and $F_i = \langle S_i \rangle$.

Then $d_{G_n}(v_{i-1},\, v_i) = d_G(v_i,\, \pi(v_i))$ for $1 \leq i \leq n$. Thus, by the triangle inequality,

$$\begin{aligned} d_{G_n}(v_\alpha,\, v_\beta) &\leq \sum_{i=1}^{n} d_{G_n}(v_{i-1},\, v_i) \\ &= \sum_{i=1}^{n} d_G(v_i,\, \pi(v_i)) \\ &= d(F, H) \end{aligned}$$

and the result follows. $\bullet$

From Theorem 5.3, we see, for a connected graph G of order p, an integer n such that $1 \leq n \leq p$ and a subgraph F of order n, that if $V(F) = S_1$, then $e(F) = e_{G_n}(v_1)$. Also, $diam_n(G) = diam(G_n)$ and $rad_n(G) = rad(G_n)$.

We now state three immediate results, without proof.

COROLLARY 5.3A. *Let G be a connected graph of order p and n an integer such that $1 \leq n \leq p$. If F and H are subgraphs of G of order n and $d(F,H) = 1$, then $|e(F) - e(H)| \leq 1$.*

COROLLARY 5.3B. *Let G be a connected graph of order p and n an integer such that $1 \leq n \leq p$. For each integer c such that $rad_nG \leq c \leq diam_nG$, there exists a subgraph F of G of order n such that $e(F) = c$.*

COROLLARY 5.3C. *Let G be a connected graph of order p and n an integer such that $1 \leq n \leq p$. Then*

$$rad_n G \leq diam_n G \leq 2rad_n G.$$

The bounds in Corollary 5.3C are sharp for every positive integer n. For instance, we have already shown for $G \cong K_m$ with $m \geq 2n$, that $rad_n\, G = diam_n\, G$. For the upper bound, let $F \cong K_m$ and $H \cong \overline{K}_m$ where $m \geq n$. For $G = F + 2H$, we have $e(F) = n = rad_n G$ and $e(H) = 2n = diam_n G$. Thus $diam_n G = 2rad_n\, G$.

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