6-1988

Graph and Directed Graph Augmentation Problems

Zhuguo Mo
Western Michigan University

GRAPH AND DIRECTED GRAPH AUGMENTATION PROBLEMS

by

Zhuguo Mo

A Dissertation
Submitted to the
Faculty of The Graduate College
in partial fulfillment of the
requirements for the
Degree of Doctor of Philosophy
Department of Mathematics & Statistics

Western Michigan University
Kalamazoo, Michigan
June 1988

Order Number 8818435

Graph and directed graph augmentation problems

Mo, Zhuguo, Ph.D.

Western Michigan University, 1988

U·M·I
300 N. Zeeb Rd.
Ann Arbor, MI 48106

For my parents,

whose love and encouragement keep me smiling,

and for Sue-Firn,

who understands.

ACKNOWLEDGEMENTS

I wish to thank Professor Kenneth Williams, my dissertation advisor, for his guidance and encouragement. My thanks also go to many others in the Department of Mathematics and Statistics and the Department of Computer Science of Western Michigan University who have stimulated my interest in mathematics and computer science and have given their time and friendship freely. Finally, I would like to thank Professor Abdol Esfahanian of Michigan State University, Professor Alfred Boals, Professor Gary Chartrand, and Professor John Petro of Western Michigan University for their many helpful suggestions in the preparation of this manuscript.

Zhuguo Mo

TABLE OF CONTENTS

CHAPTER

ACKNOWLEDGEMENTS ii

I. PRELIMINARIES 1

Introduction 1

Basic Definitions and Notation 4

II. RADIUS r DOMINATION AND AUGMENTATIONS 10

Radius r Domination 11

m-Centrix Radius r Augmentation 26

III. K DISJOINT MATCHINGS AND AUGMENTATIONS . . . 36

r-Factorability 37

K Disjoint Maximum Matchings. 39

K Disjoint 1-Factors Augmentation for Forests 48

IV. (M, N, R_1, R_2)-TRANSITIVE AUGMENTATIONS 58

Results on Tournaments 59

Some Results on Graphs and Digraphs 62

$(M, 1, R_1, =)$-Transitive Augmentations 68

V. (r, s)-DOMINATION AND AUGMENTATIONS 75

Bounds on (r, s)-Domination Number 76

Computational Complexities 92

VI. ON BLOCK-COMPLETE GRAPHS 102

VII. SUMMARY 112

BIBLIOGRAPHY 114

CHAPTER I

PRELIMINARIES

Introduction

Let $G^* = (V, E^*)$ be a complete weighted graph with non-negative real-valued cost function c on the edges of G^*, G be a spanning subgraph of G^*, and P be a property defined for graphs. One may ask the following questions: "Is there a graph H such that $G \subseteq H \subseteq G^*$ and H has property P?" If the answer is yes, "Is there an efficient algorithm for finding such a graph H with a minimum-cost set of edges?" The problem of finding a spanning supergraph H of G with minimum-cost set of edges so that H has property P is called a (weighted) graph augmentation problem with respect to G^* and P. For unweighted graphs, the problem of finding a minimum spanning supergraph H of G having the desired property P is called an (unweighted) graph augmentation problem (with respect to P). Digraph augmentation problems can be defined similarly. Alternatively, graph (digraph) augmentation problems are sometimes called graph (digraph) completion problems. This dissertation presents solutions for unweighted augmentation problems, and it provides some general techniques for solving augmentation problems.

Graph and digraph augmentation problems have been studied by many people. In 1973, Eswaran [15] solved the problem of

adding a minimum-cost set of arcs to a digraph D_0 so that there is a cycle which contains all the arcs of D_0. In the same year, Goodman and Hedetniemi [24] gave an $O(|V|^2)$ time algorithm to find the minimum number of edges which must be added to a tree so that the resulting graph has a hamiltonian cycle. Later, they improved the algorithm to $O(|V|)$. In 1976, Eswaran and Tarjan [16] proposed a theoretical framework for studying graph (digraph) augmentation problems, gave well-known examples of such problems, and analyzed in detail the strong connectivity, bridge connectivity and biconnectivity augmentation problems. Since then a number of results about augmentation problems have been obtained. In 1977, Boesch, Suffel, and Tindell [7] characterized those graphs which span eulerian graphs and gave exact formulas for the number of edges which must be added to such graphs in order to obtain eulerian graphs. Further results about the three connectivity augmentation problems considered by Eswaran and Tarjan [16] were obtained by others. In 1977, Rosenthal and Goldner [46] presented an $O(|V|+|E|)$ time algorithm which, given a graph G, finds a smallest biconnectivity augmentation of G. Frederickson and Ja'Ja' [18] showed that the three augmentation problems considered in [16] are NP-complete in the restricted case of the graph being initially connected. They also obtained fast approximation algorithms which have constant worst-case performance ratio. Triangulation augmentation has been studied extensively and has not yet been solved (see [41][44][45]). In recent years, many augmentation problems

have been investigated, among them are K-edge-connected digraph augmentation for trees [35], mixed graph augmentation [26], interval graph augmentation [42], (M, N)-transitive augmentation [4][27][52], and (M, N, R_1, R_2)- transitive augmentation [5] etc.

Note that if the complete graphs K_n have property P, then for any graph G there exists a minimum spanning supergraph H of G having property P. In this dissertation, we focus on the properties of the complete graphs (digraphs). Given an arbitrary graph G = (V, E) and a graphical property P, let <u>Aug(G, P)</u> be the minimum number of edges to be added to G so that the resulting graph has property P. Aug(G, P) is called the <u>augmentation number of G with respect to the property P</u>. We will write Aug(G) for Aug(G, P) if no confusion arises. By a <u>minimum augmentation of a graph G with respect to a property P</u> sometimes is meant a minimum set of edges to be added to G so that the resulting graph has property P and sometimes is meant a graph obtained from G by adding a minimum set of edges with respect to the property P depending on the context. Many definitions and notation defined for graphs in this dissertation can be modified in a natural way to give corresponding definitions and notation for digraphs. In the rest of the dissertation, we deal with only those properties P for which Aug(G, P) are well defined.

Given an arbitrary graph G = (V, E) and a graphical property P, if the problem of determining whether G has property P is NP-hard, then the problem of finding Aug(G, P) is NP-hard since

G has property P if and only if Aug(G, P) = 0. In the case that the problem of determining whether an arbitrary graph G has property P is NP-complete (or NP-hard), we may provide heuristic approximation algorithms for the corresponding augmentation problem. We may also restrict the domain of the graphs to some classes of graphs, say trees, bipartite graphs, planar graphs, triangulated graphs, block-complete graphs etc.

Basic Definitions and Notation

For terminologies and notation, we follow Behzad, Chartrand, and Lesniak-Foster [3] unless defined here. A _graph_ G is a finite, nonempty set V together with a set E of two element subsets of (distinct) elements of V. Each element of V is referred to as a _vertex_ and V itself as the _vertex set_ of G; the members of the edge set E are called _edges_. As usual, $|S|$ denotes the cardinality of a set S. For a graph G, $|V|$ and $|E|$ are referred to as the _order_ and _size_ of G, respectively. The maximum degree and minimum degree of G are denoted by $\Delta(G)$ and $\delta(G)$ respectively. There is only one graph of order one (up to isomorphism) which is referred to as the _trivial graph_. A nontrivial graph then has order at least 2. The edge $e = \{u, v\}$ is said to _join_ the vertices u and v. If $e = \{u, v\}$ is an edge of a graph G, then u and v are _adjacent vertices_ while u and e are _incident_ as are v and e. Furthermore, if e_1 and e_2 are distinct edges of G incident with a common vertex, then e_1 and e_2 are _ad-_

jacent edges. Sometimes, an edge $e = \{u, v\}$ is denoted by uv or vu.

A graph is complete if every two of its vertices are adjacent. A complete graph of order n is denoted by K_n. An empty graph is a graph which has no edges. A graph $G = (V, E)$ is bipartite, if it is possible to partition V into subsets V_1 and V_2 (called partite sets) such that every element of E joins a vertex of V_1 to a vertex of V_2. A complete bipartite graph G is a bipartite graph with the added property that if $u \in V_1$ and $v \in V_2$ then $uv \in E(G)$. If $|V_1| = n_1$ and $|V_2| = n_2$, then this graph is denoted by $K(n_1, n_2)$ or K_{n_1, n_2}.

Let u and v be (not necessary distinct) vertices of a graph G. By a u-v walk of G is meant a finite, alternating sequence of vertices and edges of G, beginning with u and ending with v, such that every edge is immediately preceded and succeeded by the two vertices with which it is incident. A u-v walk is closed or open depending on whether $u = v$ or $u \neq v$. A u-v path is a u-v walk in which no vertex is repeated; a single vertex u forms the trivial u-u path. A hamiltonian path of G is a path which contains all vertices of G. In a graph G, a closed walk v_1, v_2, $\ldots$, v_n, v_1 ($n \geq 3$) whose n vertices v_i are distinct is called a cycle of G. A cycle edge is an edge that lies on a cycle. An acyclic graph has no cycles. A hamiltonian cycle of G is a cycle which contains all vertices of G. A graph G is said to be hamiltonian if it has a hamiltonian cycle.

Two distinct vertices or edges in a graph G are <u>independent</u> if they are not adjacent in G. A set of pairwise independent edges of G is called a <u>matching</u> in G, while a matching of maximum cardinality is a <u>maximum matching</u> in G. Let M be a specified matching in a graph G. An edge e of G that is not in M is called a <u>weak edge</u> (with respect to M). A <u>weak vertex</u> (with respect to M) is a vertex of G incident only with weak edges. In a graph G, a nonempty set U_1 of V(G), is said to be <u>matched</u> to a subset of U_2 of V(G) disjoint from U_1 if there exists a matching M in G such that each edge of M is incident with a vertex of U_1 and a vertex of U_2 and every vertex of U_1 is incident with an edge of M, as is every vertex of U_2. Let U be a nonempty set of vertices of a graph G and let its <u>neighborhood</u> N(U) denote the set of all vertices of G adjacent with at least one element of U. Then the set U is said to be <u>nondeficient</u> (relative to G) if $|N(S)| \geq |S|$ for every nonempty subset S of U. Otherwise, U is said to be <u>deficient</u>.

A <u>factor</u> of a graph G is a (possibly empty) spanning subgraph of G. If G is expressed as the edge sum of factors of G, then this sum is called a <u>factorization</u> of G. An r-regular factor of a graph G is referred to as an <u>r-factor</u> of G. If there exists a factorization of G such that each factor is an r-factor (for a fixed r), then G is <u>r-factorable</u>.

The set of positive integers is denoted by I^+. Note that an unweighted graph is a weighted graph with all the weights of the edges being one. For a weighted graph G = (V, E) with

weight function $c(e) \in I^+$, $e \in E$, the distance $d_G(u, v)$ between two vertices u and v is the minimum of the weighted lengths of the u-v paths of G if there exists a u-v path in G, otherwise the distance $d_G(u, v)$ is defined to be infinity. The eccentricity $e(v)$ of a vertex v of a weighted or unweighted connected graph G is the number max $d(u, v)$, where the max is taken over all the vertices $u \in V(G)$. The radius, rad G, is defined as $\min_{v \in V} e(v)$ while the diameter, diam G, is $\max_{u, v \in V} d(u, v)$. If two graphs G_1 and G_2 are isomorphic, we write $G_1 = G_2$.

A tree is an acyclic connected graph and a forest is an acyclic graph. A rooted tree, T, is a finite set of one or more vertices such that

(1) there is a specially designated vertex called the root of T;

(2) the remaining vertices are partitioned into $n \geq 0$ disjoint sets $T_1, T_2, \ldots, T_n$ where each of these sets is a rooted tree with the root of each of $T_1, T_2, \ldots, T_n$ adjacent to the root of T.

$T_1, T_2, \ldots, T_n$ are called the subtrees of the root. The roots of subtrees of a vertex v are called the children of v and v is the parent of the children.

In a rooted tree, the level of a vertex is defined by initially letting the root be at level one. If a vertex is at level p, then its children are at level p + 1. The height or depth of a rooted tree is defined to be the maximum level of any vertex in the rooted tree.

An assignment of colors to the edges of a nonempty graph G so that adjacent edges are colored differently is an edge coloring of G (an n-edge coloring if n colors are used). The graph G is n-edge colorable if there exists an m-edge coloring of G for some $m \leq n$. The minimum n for which a graph G is n-edge colorable is its edge chromatic number (or chromatic index).

A directed graph or digraph D is a finite nonempty set V (of vertices) together with a set A (disjoint from V) of ordered pairs of distinct elements of V. The elements of A are called arcs of D. Let u and v be two distinct vertices of a digraph D. A u-v path of D means a finite alternating sequence

$$u = u_0, a_1, u_1, a_2, \ldots, u_{n-1}, a_n, u_n = v$$

of distinct vertices and arcs such that $a_i = (u_{i-1}, u_i)$ for $1 \leq i \leq n$. A vertex v is said to be reachable from a vertex u in a digraph D if D contains a u-v path. For a (weighted) digraph $D = (V, A)$ with weight function $c(e) \in I^+$, $e \in A$, the distance $d_D(u, v)$ is defined to be the minimum of the (weighted) lengths of the u-v paths of D if there exists a u-v path in D, otherwise the distance $d_D(u, v)$ is defined to be infinity. The eccentricity $e(v)$ of a vertex v of D is defined as $e(v) = \max_{u \in V(D)} d(v, u)$. The radius rad D is defined by $\text{rad } D = \min_{v \in V(D)} e(v)$. If a given digraph D is weighted, the distance of two vertices of D means the weighted distance unless otherwise specified.

A digraph D is strongly connected or strong if for every two distinct vertices of D, each vertex is reachable from the other. A digraph D is called symmetric if whenever (u, v) is an

arc of D so too is (v, u). There is a natural one-to-one correspondence between the set of symmetric digraphs and the set of graphs. A digraph is <u>asymmetric</u> or <u>oriented</u> if whenever (u, v) is an arc of D, then (v, u) is not an arc of D. A digraph D is said to be <u>complete</u> if for every two distinct vertices u and v of D, both (u, v) and (v, u) are arcs of D. By the <u>underlying graph</u> of a digraph D is meant the graph G obtained from D by deleting all directions from the arcs of D and deleting an edge from a pair of multiple edges if multiple edges should be produced. A <u>tournament</u> is an asymmetric digraph whose underlying graph is complete. An <u>oriented tree</u> is an oriented digraph whose underlying graph is a tree. An <u>arborescence</u> is an acyclic digraph with one vertex, the root, having no entering arcs, and all other vertices having exactly one entering arc.

For graphs G = (V, E), a <u>linear time algorithm</u> is an algorithm having time complexity $O(|E| + |V|)$. Linear time algorithms for digraphs are defined similarly.

CHAPTER II

RADIUS r DOMINATION AND AUGMENTATIONS

Let V be the vertex set of a (weighted) graph or (weighted) digraph, v be a vertex in V, and U be a nonempty subset of V. Then the distance from the set U to the vertex v is defined as $d(U, v) = \min_{u \in U} d(u, v)$. Decision problems studied or referenced in this chapter are stated as follows:

Vertex Cover

Instance: Graph $G = (V, E)$, positive integer $K \leq |V|$.

Question: Is there a vertex cover of size K or less for G, i.e., a subset $V' \subseteq V$ with $|V'| \leq K$ such that for each edge $uv \in E$ at least one of u and v belongs to V'?

(Graph) Dominating Set of Radius r

Instance: Graph $G = (V, E)$, weight function $c(e) \in I^+$ for $e \in E$, positive integers r, $K \leq |V|$.

Question: Is there a dominating set of radius r with size K or less for G, i.e., a subset $V' \subseteq V$ with $|V'| \leq K$ such that for all $u \in V - V'$ there is a $v \in V'$ for which $d(v, u) \leq r$?

Digraph Dominating Set of Radius r

Instance: Digraph $D = (V, A)$, weight function $c(e) \in I^+$ for $e \in A$, positive integers $r, K \leq |V|$.

Question: Is there a dominating set of radius r with size K or less for D, i.e., a subset $V' \subseteq V$ with $|V'| \leq K$ such that for all

$u \in V - V'$ there is a $v \in V'$ for which $d(v, u) \le r$?

(Graph) m-Centrix Radius r Augmentation

Instance: Graph $G = (V, E)$, subset $C = \{c_1, c_2, \ldots, c_m\}$ of V, called an m-centrix, positive integers B and r.

Question: Is there a set E' of unordered pairs of vertices from V such that $|E'| \le B$ and for the graph $G' = (V, E \cup E')$, $d_{G'}(C, v) \le r$ for all $v \in V$?

Digraph m-Centrix Radius r Augmentation

Instance: Digraph $D = (V, A)$, subset $C = \{c_1, c_2, \ldots, c_m\}$ of V, called an m-centrix, positive integers B and r.

Question: Is there a set A' of ordered pairs of vertices from V such that $|A'| \le B$ and for the digraph $D' = (V, A \cup A')$, $d_{D'}(C, v) \le r$ for all $v \in V$?

Radius r Domination

For unweighted graphs, the Dominating Set of Radius 1 problem is referred to as the Dominating Set problem. Garey and Johnson [21][22] have shown that the Vertex Cover problem and the Dominating Set problem are NP-complete even when G is a (connected) planar graph of maximum degree at most 3 and of minimum degree 1. Kariv and Hakimi [36] have proved that (the optimization version of) the Dominating Set of Radius r problem is NP-hard which follows directly from the NP-completeness of the Dominating Set problem. We show that the Dominating Set of Radius r problem and the Digraph Dominating Set of Radius r problem are NP-complete for any fixed r. Both problems can be

solved in linear time if the graphs (digraphs) are restricted to trees (oriented trees).

Theorem 2. 1 For any fixed positive integer r, the Dominating Set of Radius r problem is NP-complete even in the case when the graph is an unweighted connected planar graph of maximum degree 3 and of minimum degree 1.

Proof: The Dominating Set of Radius r problem is in NP since a nondeterministic algorithm need only guess a subset V' of V and check in polynomial time that $|V'| \leq K$ and that for all $u \in V - V'$ there is a $v \in V'$ for which $d_G(v, u) \leq r$.

We transform the Vertex Cover problem to the Dominating Set of Radius r problem. Let $G_1 = (V_1, E_1)$ be an unweighted connected planar graph of maximum degree at most 3 and of minimum degree 1, and K be a positive integer in an instance of the Vertex Cover problem. We will construct a connected planar graph $G = (V, E)$ of maximum degree 3 and of minimum degree 1 such that there exists a subset $V' \subseteq V$ with $|V'| \leq |E_1| + K$ and for all $u \in V - V'$ there is a $v \in V'$ for which $d_G(v, u) \leq r$ if and only if there is a subset $V_0 \subseteq V_1$ with $|V_0| \leq K$ such that for each edge $\{u, v\} \in E_1$ at least one of u and v belongs to V_0.

Let G be the graph obtained from G_1 by replacing each edge x_iy_i by the graph H_i (see Figure 2. 1) which can be obtained from a path P of length $3r + 1$ with x_i and y_i as the end vertices by adding a u_i-v_i path of length $r + 1$, where u_i and v_i are vertices on P such that $d_P(x_i, u_i) = d_P(y_i, v_i) = r$. Clearly G is a connected

planar graph of maximum degree 3 and of minimum degree 1 and for $i \neq j$, $V(H_i) \cap V(H_j) = \{x_i, y_i\} \cap \{x_j, y_j\}$.

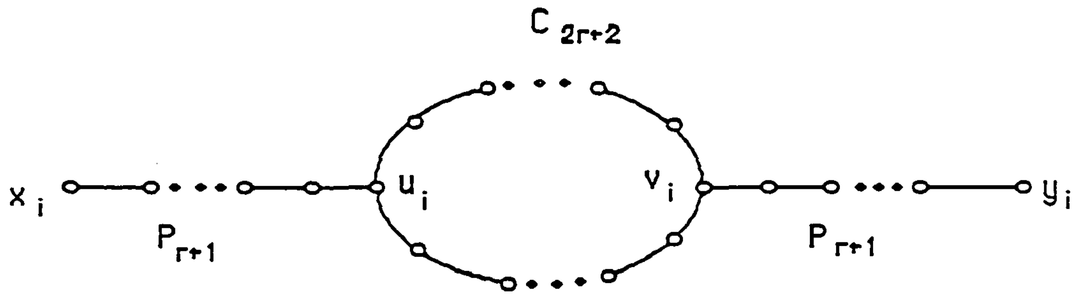


Figure 2. 1 Local replacement used in the proof of Theorem 2. 1

Suppose that there is a subset V_0 of the vertices of G_1 with $|V_0| \leq K$ such that each edge of G_1 is incident with a vertex of V_0. Then for each edge $x_iy_i \in E_1$, $1 \leq i \leq |E_1|$, either $x_i \in V_0$ or $y_i \in V_0$. Define vertex w_i to be v_i if $x_i \in V_0$, else to be u_i. Let $V' = V_0 \cup \{w_i \mid 1 \leq i \leq |E_1|\}$, then $|V'| = |V_0| + |E_1| \leq K + |E_1|$. For $1 \leq i \leq |E_1|$, either $\{x_i, v_i\} \subseteq V'$ or $\{y_i, u_i\} \subseteq V'$. It follows that for all $u \in V-V'$, there is a $v \in V'$ for which $d_G(v, u) \leq r$.

Conversely, suppose that there exists a subset $V' \subseteq V$ with $|V'| \leq |E_1| + K$ such that for all $u \in V-V'$, there is a $v \in V'$ for which $d_G(v, u) \leq r$. Without loss of generality, assume that $|V'|$ is minimum. We claim that G_1 has a vertex cover of size at most K, i.e., there is a subset $V_0 \subseteq V_1$ with $|V_0| \leq K$ such that each edge of G_1 is incident with a vertex of V_0.

Since $d_{H_i}(x_i, u_i) = d_{H_i}(y_i, v_i) = r$, each vertex in the cycle C_{2r+2} of length $2r + 2$ in H_i has to be dominated within distance r by some vertex in $V(H_i) \cap V'$. Note that at least two vertices are

needed to dominate the vertices in C_{2r+2} within distance r. Therefore $|V(H_i) \cap V'| \geq 2$, for all i, $1 \leq i \leq |E_1|$. The equality $d_G(x_i, y_i) = 3r + 1$ implies that $|(V(H_i) - \{x_i, y_i\}) \cap V'| \geq 1$, $1 \leq i \leq |E_1|$. Observe that $\{x_i, y_i, u_i\}$ is a dominating set of radius r for the graph H_i. By the minimality of $|V'|$, $|V(H_i) \cap V'| \leq 3$. Furthermore, in the case that $|V(H_i) \cap V'| = 3$, we may assume that $\{x_i, y_i, u_i\} \subseteq V'$; for otherwise, we could modify V' to have the desired property since $\{x_i, y_i, u_i\}$ is one of the triples of vertices in H_i which has best potential to cover more vertices in G.

Now we show that for each i, $1 \leq i \leq |E_1|$, such that $|V(H_i) \cap V'| = 2$, we may modify V' so that $V(H_i) \cap V' = \{x_i, v_i\}$ or $V(H_i) \cap V' = \{u_i, y_i\}$. Let $V(H_i) \cap V' = \{s_i, t_i\}$. We need to show that the set $\{s_i, t_i\}$ can be chosen in such a way that either $\{s_i, t_i\} = \{x_i, v_i\}$ or $\{s_i, t_i\} = \{u_i, y_i\}$. By the structure of the graph H_i, without loss of generality, suppose that s_i is on the x_i-u_i path while t_i is on the v_i-y_i path. (If necessary, V' could be modified in this way, the modified V' would be taken as V'.) Note that $|\{s_i, t_i\} \cap \{x_i, y_i\}| \leq 1$. If $|\{s_i, t_i\} \cap \{x_i, y_i\}| = 1$, for simplicity, suppose that $\{s_i, t_i\} \cap \{x_i, y_i\} = x_i$, then $\{s_i, t_i\} = \{x_i, v_i\}$. So we may assume that $\{s_i, t_i\} \cap \{x_i, y_i\} = \emptyset$. Note that x_i is an end-vertex in G if and only if x_i is an end-vertex in G_1. If x_i is an end-vertex in G_1, then we may choose the set $\{s_i, t_i\}$ to be $\{u_i, y_i\}$. Suppose that x_i is not an end-vertex in G_1. For each edge $x_j y_j$ ($j \neq i$) that is incident with x_i in G_1, we have $|V(H_j) \cap V'| = 2$. Without loss of generality, suppose that $x_j = x_i$. Let $V(H_j) \cap V'$

$= \{s_j, t_j\}$. Then either s_j or t_j is on the x_j-u_j path. Therefore $\{s_i, t_i\}$ could be replaced by $\{u_i, y_i\}$ if necessary.

Thus there exists a subset $V' \subseteq V$ with $|V'| \leq |E_1| + K$ such that for all $u \in V-V'$, there is a $v \in V'$ for which $d_G(v, u) \leq r$. Furthermore for all i, $1 \leq i \leq |E_1|$, $|(V(H_i) - \{x_i, y_i\}) \cap V'| = 1$ and either $\{x_i, v_i\} \subseteq V(H_i) \cap V'$ or $\{u_i, y_i\} \subseteq V(H_i) \cap V'$. Let $V_0 = V' \cap V(G_1) = V' \cap \{x_i, y_i \mid 1 \leq i \leq |E_1|\}$. Then $V_0 \subseteq V(G_1)$, $|V_0| = |V'| - |E_1| \leq K$, and each edge of G_1 is incident with a vertex of V_0, i.e., G_1 has a vertex cover V_0 of size at most K. □

Theorem 2.2 For any fixed positive integer r, the Digraph Dominating Set of Radius r problem is NP-complete even in the case when the digraph is unweighted, symmetric, and its underlying graph is connected planar of maximum degree 3 and of minimum degree 1.

Proof: It is easy to see that the Digraph Dominating Set of Radius r problem is in NP. Let $G = (V, E)$ be an unweighted connected planar graph of maximum degree 3 and of minimum degree 1 in an instance of the Dominating Set of Radius r problem and let $D = (V, A)$ be the symmetric digraph corresponding to G. Then D is an unweighted symmetric digraph whose underlying graph $G = (V, E)$ is a connected planar graph of maximum degree 3 and of minimum degree 1. We show that a subset V' of V is a dominating set of radius r for D if and only if V' is a dominating set of radius r for G. Let $V' \subseteq V$ be a dominating set of radius r for the digraph D. Then by definition for all $u \in V-V'$ there is a $v \in V'$ for which $d_D(v, u) \leq r$. Since G is the underlying graph of D, we

have $d_G(v, u) \le r$. Therefore V' is also a dominating set of radius r for G.

Conversely, suppose that V' is a dominating set of radius r for the graph G. Then for all $u \in V - V'$ there is a $v \in V'$ for which $d_G(v, u) \le r$. Let P be a shortest v-u path in G. Since G is the underlying graph of the symmetric digraph D, there is a directed v-u path P' in D having the same length as that of P in G. Therefore V' is also a dominating set of radius r for D. Thus a subset V' of V is a dominating set of radius r for D if and only if V' is a dominating set of radius r for G. The result follows from Theorem 2.1. □

Algorithm 2.1 Finds a minimum dominating set of radius r in a weighted oriented tree.

Procedure DOMINATE(T, ROOT, r, v, U, D, K)

INPUT:

T is a weighted oriented tree.

ROOT is the root of the tree T.

r is a positive integer, called a dominating radius.

v is a vertex of T.

OUTPUT:

U is a subset of V(T). At return, U is a minimum dominating set of radius r for T if v = ROOT. Otherwise, U is a subset of a minimum dominating set of radius r for the subtree of T rooted at v.

D is a boolean variable. Let T_v be the subtree of T rooted at v. If each vertex in $V(T_v)$ is dominated by U within

distance r, then D has value TRUE. Otherwise, D has value FALSE.

K is a nonnegative integer. If D has value TRUE, then K = $d(U, v)$. Otherwise there exists a vertex u in the subtree rooted at v such that $K = d(v, u) < r$ and u needs to be dominated by v or an ancestor (if any) of v.

VARIABLES:

COVERED is a boolean variable, if vertex v has been dominated by U within distance r, then COVERED has value TRUE; otherwise it has value FALSE.

Q is a temporary variable, its value is the maximum distance of v from a descendant which has not been dominated yet.

```
    /* In this procedure, an arc (u, v) is denoted by uv. */
begin
    if (v is an end-vertex) then
1)          D ← FALSE;  K ← 0
    else /* v is not an end-vertex */
            Let v_1, v_2, . . . , v_m be the children of v
2)          for i ← 1 to m loop
                    DOMINATE(T, ROOT, r, v_i, U, D_i, K_i)
3)          end loop
            /* Determine some children of v which need to be
            included in the dominating set U */
4)          for i ← 1 to m loop
```

```
            If (not D_i) and (v_i v ∈ A(T)) then
                U ← U ∪ {v_i};  D_i ← TRUE;  K_i ← 0
            end if
5)      end loop
6)      K ← r;  COVERED ← FALSE
        /* Determine whether the vertex v has been
        dominated by some of its descendants */
7)      for i ← 1 to m loop
            if D_i and ( v_i v ∈ A(T) ) and ( K_i + c(v_i v) ≤ r ) then
                COVERED ← TRUE;  K ← min {K, K_i + c(v_i v)}
            end if
8)      end loop
        if (not COVERED) then
9)          D ← FALSE; K ← 0
10)         for i ← 1 to m loop
                /* Determine some children of v which
                need to be included in the dominating
                set U*/
11)             if ( not D_i ) and ( K_i + c(vv_i) > r ) then
                /* not D_i implies vv_i is an arc of T */
                    U ← U ∪ {v_i}
12)             end if
                if ( not D_i ) and ( K_i + c(vv_i) ≤ r ) then
                    /* Update the maximum distance of
                    v to some vertex w in T_v such that w
                    has not been dominated yet*/
```

```
13)                K ← max {K, K_i + c(vv_i)}
               end if
           end loop
           if v = ROOT then
14)            U ← U ∪ {v};  D ← TRUE
           end if
       end if
       if COVERED then
15)        Q ← 0
           for i ← 1 to m loop
               /* determines some children of v which
               are required to be in the set U. */
16)            if ( not D_i ) and (c(vv_i) + K_i > r) then
                       U ← U ∪ {v_i};  D_i ← TRUE
17)            end if
               /* Updates the maximum distance of v
               from a descendant which has not been
               dominated yet. */
18)            if ( not D_i ) and ( K + c(vv_i) + K_i > r ) then
                   Q ← max {Q, c(vv_i) + K_i}
19)            end if
           end loop
20)        if Q > 0 then
               D ← FALSE;  K ← Q
           else
               D ← TRUE;
```

```
                end if
            end if
        end if
end Procedure DOMINATE
begin /* Algorithm 2.1 */
        U ← ∅
        Select a vertex v as the root of T
        ROOT ← v
        if |V(T)| = 1 then
                U ← {v}
        else
                DOMINATE(T,ROOT,r,v,U,D,K)
        end if
end Algorithm 2.1
```

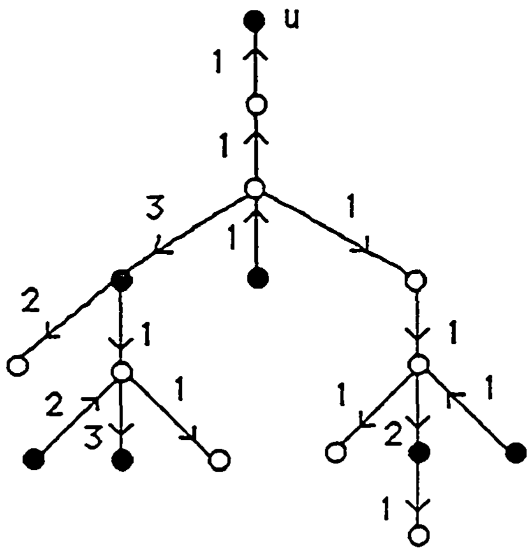


Figure 2.2

Example 2.1 Let T be the oriented tree in Figure 2.2, and let

$r = 2$. The vertex u is the root of T and the numbers in the figure are the weights of the arcs of T. If for each vertex v, Algorithm 2.1 calls procedure DOMINATE for the children of v in left to right order, then the minimum dominating set of radius r found is the set of solid vertices as indicated in Figure 2.2.

Note that the distance matrix of an oriented tree can be calculated in linear time.

Theorem 2.3 Algorithm 2.1 finds a minimum dominating set of radius r in a weighted oriented tree $T = (V, A)$ in $O(|V|)$ time.

Proof: If T is of order one, the result is clear. So we suppose that the oriented tree T is of order at least two. Algorithm 2.1 selects a vertex as the root of T and calls procedure DOMINATE. Procedure DOMINATE uses depth first search to find a set U of V. We will show that U is a minimum dominating set of radius r of T.

Note that procedure DOMINATE is recursive. We need only show that for each call DOMINATE(T, ROOT, r, v, U, D, K), an optimal decision is made, i.e., a required number of vertices is selected into the set U so that each vertex in the subtree rooted at v is either dominated by some vertex in the current set U within distance r or dominated by some ancestor of v, and the selected set of vertices has "best" potential of covering more vertices that are not in the subtree rooted at v within distance r. In the following "domination" means "domination of radius r", and T_v is the subtree of T with root v.

If v is an end-vertex, then by step 1, vertex v is not selected into U currently. Note that v might be selected into U later. So we may assume that v is not an end-vertex and let v_1, v_2, . . . , v_m be the children of v. The loop from step 2 to step 3 calls procedure DOMINATE for each child of v. By the loop from step 4 to step 5, for each child v_i of v, if v_i is not dominated yet and $v_iv \in A(T)$, then v_i is selected into U. This is necessary since no other vertices not in T_v can dominate v_i. Step 6 initializes K and COVERED. The loop from 7 to 8 determines whether the vertex v has been dominated by some of its descendants. At this point we consider two cases depending on whether the vertex v has been dominated.

Case 1: COVERED = FALSE.

Step 9 initializes D and resets K. Note that since D has value FALSE, K is the maximum distance of v to some vertex w in T_v such that w has not been dominated yet. Observe that at this point, if v_i is not dominated, then $(v, v_i) \in A(T)$. The if statement from step 11 to 12 determines some children of v which are required to be in the set U. Step 13 updates K. Step 14 selects v into U, since at this point v is the root of the tree T and v needs to be dominated.

Case 2: COVERED = TRUE.

Step 15 sets Q to 0. Steps 16 to 17 determine some children of v which are required to be in the set U. Steps 18 to 19 update Q. Steps 20 to 21 update D and K. □

Theorem 2.4 Algorithm 2.1 can be used to find a minimum dominating set of radius r for a weighted undirected tree.

Proof: Let T be an weighted undirected tree and r be a vertex of T. Orient the tree T so that T becomes an arborescence T'. Then use Algorithm 2.1 to find a minimum dominating set, say U, of radius r of T'. U is also a minimum dominating set of radius r of T. □

The cardinality of a smallest dominating set of radius r of an unweighted graph G (digraph D) is called the <u>r-domination number</u> of G (of D) and is denoted by $\gamma_r(G)$ ($\gamma_r(D)$).

Theorem 2.5 Let G be a connected graph. Then

$$\gamma_r(G) = \min \gamma_r(T),$$

where the minimum is taken over all spanning trees T of G.

Proof: Let G be a connected graph and T be a spanning tree of G. Then any dominating set of radius r of T is also a dominating set of radius r of G. Therefore,

$$\gamma_r(G) \leq \gamma_r(T).$$

It follows that,

$$\gamma_r(G) \leq \min \gamma_r(T),$$

where the minimum is taken over all spanning trees T of G.

Now we show the reverse inequality. If G is a tree, the theorem holds trivially. So we may assume that G is a connected non-acyclic graph. Let U be a minimum dominating set of radius r of G and C be a smallest cycle in G. Note that if we can show that U is a dominating set of radius r of G - e, for some cycle edge e, then $\gamma_r(G - e) \leq |U| = \gamma_r(G)$. By applying this

result a finite number of times, we will have $\gamma_r(T) \leq \gamma_r(G)$ for some spanning tree T of G, thus

$$\gamma_r(G) \geq \min \gamma_r(T),$$

where the minimum is taken over all spanning trees T of G.

Now we show that U is a dominating set of radius r of $G - e$, for some cycle edge e. Select a vertex x in V(C) such that $d_G(x, U) = \max\{d_G(v, U) \mid v \in V(C)\}$. Let y and z be the two vertices in V(C) which are adjacent to x. Without loss of generality, suppose that $d_G(z, U) \leq d_G(y, U)$. Let $e = xy$. Then e is a cycle edge. We will show that the edge e has the desired property.

Let $u \in U$ such that $d_G(y, u) = d_G(y, U) \leq r$ and let P be a y–u path of length $d_G(y, u)$ in G. By the way in which x was chosen, $d_G(y, U) \leq d_G(x, U)$. It follows that the edge $e = xy$ is not on the path P. Therefore $d_{G-e}(y, U) = d_G(y, U)$. Similarly, since $d_G(z, U) \leq d_G(y, U)$ and $d_G(z, U) \leq d_G(x, U)$, $d_{G-e}(z, U) = d_G(z, U)$. This implies that $d_{G-e}(x, U) = d_G(x, U)$. Therefore for any $v \in V(G)$, $d_{G-e}(v, U) = d_G(v, U) \leq r$. Thus U is a dominating set of radius r of $G - e$, where $e = xy$. □

Corollary 2.5 Let G be a connected graph of order n with at most K cycles, where K is a positive integer. Then the time complexity for computing the r-domination number $\gamma_r(G)$ is $O(n^{K+1})$.

Proof: Since G has at most K cycles, the number of spanning trees of G is bounded above by $O(n^K)$. For each spanning tree T of G, a minimum dominating set of radius r of T can be found in O(n) time, using Algorithm 2.1. It follows that $\gamma_r(T)$ can be

found in $O(n)$ time. By Theorem 2.5, $\gamma_r(G)$ can be found in $O(n^K) \cdot O(n) = O(n^{K+1})$ time. □

Theorem 2.6 Let D be a connected digraph. Then

$$\gamma_r(D) = \min \gamma_r(T),$$

where the minimum is taken over all spanning oriented trees T of D.

Proof: By a proof similar to Theorem 2.5, we have

$$\gamma_r(D) \leq \min \gamma_r(T),$$

where the minimum is taken over all spanning oriented trees T of D.

Now we show the reverse inequality. If D is an oriented tree, the theorem holds trivially. So we may assume that D contains a smallest subdigraph D_1 whose underlying graph is a cycle C. Then $V(D_1) = V(C)$. Let U be a minimum dominating set of radius r of D. Note that if we can show that U is a dominating set of radius r of $D - e$, for some arc e in D_1, then $\gamma_r(D - e) \leq |U| = \gamma_r(D)$. So by applying this result a finite number of times, we will have $\gamma_r(T) \leq \gamma_r(D)$ for some spanning oriented tree T of D, thus

$$\gamma_r(D) \geq \min \gamma_r(T),$$

where the minimum is taken over all spanning oriented trees T of D.

Now we show that U is a dominating set of radius r of $D - e$, for some arc e in D_1. Select a vertex x in $V(D_1)$ such that $d_D(U, x) = \max \{d_D(U, v) \mid v \in V(D_1)\}$. Let y and z be the two ver-

tices in $V(D_1)$ such that x is adjacent to y and z in the undirected cycle C. We consider two cases:

Case 1: $(x, y) \in A(D_1)$ or $(x, z) \in A(D_1)$

For simplicity, suppose that $e = (x, y) \in A(D_1)$. Since $d_D(U, y) \le d_D(U, x)$, $d_{D-e}(U, v) = d_D(U, v)$ for all $v \in V(D_1)$. It follows that U is a dominating set of radius r of $D - e$.

Case 2: $(y, x) \in A(D_1)$ and $(z, x) \in A(D_1)$.

Without loss of generality, we may assume that $d_D(U, z) \le d_D(U, y)$. Let $e = (y, x) \in A(D_1)$. Then $d_{D-e}(U, v) = d_D(U, v)$ for all $v \in V(D_1)$. Therefore U is a dominating set of radius r of $D - e$. □

A proof similar to Corollary 2.5 gives

Corollary 2.6 Let D be a connected digraph of order n whose underlying graph has at most K cycles, where K is a positive integer. Then the time complexity for computing the r-domination number $\gamma_r(D)$ is $O(n^{K+1})$. □

m-Centrix Radius r Augmentation

In this section, the m-Centrix Radius r Augmentation problem and the Digraph m-Centrix Radius r Augmentation problem are shown to be NP-complete for fixed m and r such that $r \ge 2$. Both problems can be solved in $O(m \cdot |V|)$ time for $r = 1$ and for trees (oriented trees).

Theorem 2.7 For $r = 1$, the problem of finding a minimum m-centrix radius r augmentation can be solved in $O(m \cdot |V|)$ time.

Proof: Let $G = (V, E)$ be a graph, m be a positive integer, $C \subseteq V$ be an m-centrix of G, and $r = 1$. The vertices in C are called central vertices. Note that a spanning supergraph G' of G has the property that $d_{G'}(C, v) \leq 1$ for all $v \in V(G')$ if and only if each vertex in $V(G') - C$ is adjacent to some vertex in C in the graph G'.

Let $C = \{c_1, c_2, \ldots, c_m\}$. For each central vertex c_i, $1 \leq i \leq m$, find the set V_i of vertices adjacent to c_i. Let $V' = V_1 \cup V_2 \cup \ldots \cup V_m$ and $V'' = V - (V' \cup C)$. Then V'' is the set of vertices in $V(G) - C$ which are not adjacent to vertices in C. Let $E' = \{c_1 v \mid v \in V''\}$. Then E' is a minimum set of edges to be added to G so that the resulting graph G' satisfies $d_{G'}(C, v) \leq 1$ for all $v \in V(G')$. Note that E' can be found in $O(m \cdot |V|)$ time. □

Corollary 2.7 For $r = 1$, the m-Centrix Radius r Augmentation problem can be solved in $O(m \cdot |V|)$ time. □

By an analogous proof, we have

Theorem 2.8 For $r = 1$, the problem of finding a minimum digraph m-centrix radius r augmentation can be solved in $O(m \cdot |V|)$ time. □

Corollary 2.8 For $r = 1$, the Digraph m-Centrix Radius r Augmentation problem can be solved in $O(m \cdot |V|)$ time. □

Lemma 2.1 Let $G = (V, E)$ be a graph and $C = \{c_1, c_2, \ldots, c_m\}$ be a subset of V, called an m-centrix. The vertices in C are called central vertices. Suppose that E' is a minimum set of edges such that for the graph $G' = (V, E \cup E')$, $d_{G'}(C, v) \leq r$ for all $v \in V$. Then there exists a set E'' of edges such that $|E''| = |E'|$, each edge of

E" joins a central vertex and a noncentral vertex, and for the graph $G" = (V, E \cup E")$, $d_{G"}(C, v) \leq r$ for all $v \in V$.

Proof: Let $G = (V, E)$ be a graph and $C = \{c_1, c_2, \ldots, c_m\} \subseteq V$ be a subset of vertices. Suppose that E' is a minimum set of edges such that for the graph $G' = (V, E \cup E')$, $d_{G'}(C, v) \leq r$ for all $v \in V$. Observe that edges joining pairs of central vertices (i.e., vertices in C) do not affect the distance $d(C, v)$ for any $v \in V$. Since E' is minimum, no edge in E' joins a pair of central vertices. Now, suppose that $e = uv \in E'$ is an edge joining two noncentral vertices. Without loss of generality, suppose that $d_{G'}(C, u) \leq d_{G'}(C, v)$. Let $E_1 = E' - uv + c_1v$ (see Figure 2.3), then for the graph $H = (V, E \cup E_1)$, $d_H(C, v) \leq d_{G'}(C, v) \leq r$ for all $v \in V$, $|E_1| = |E'|$, and the number of edges joining pairs of noncentral vertices in H is less than that of G'.

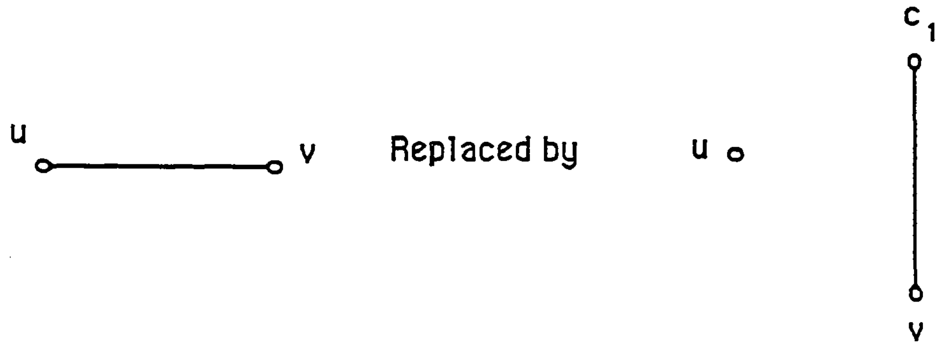


Figure 2.3 Edge replacement used in Lemma 2.1

If no edges in H join a pair of noncentral vertices, then we set $E" = E_1$. Otherwise, we replace edges joining pairs of noncentral vertices for H as we did for G'. After at most $O(|V|^2)$ steps a desired set E" will be obtained. □

Similarly, we have

Lemma 2.2 Let $D = (V, A)$ be a digraph and $C = \{c_1, c_2, \ldots, c_m\}$ be a subset of V, called an m-centrix. The vertices in C are called central vertices. Suppose that A' is a minimum subset of $V \times V - A$ such that for the digraph $D' = (V, A \cup A')$, $d_{D'}(C, v) \leq r$ for all $v \in V$. Then there exists a subset A'' of $V \times V - A$ such that $|A''| = |A'|$, each arc of A'' joins a central vertex and a non-central vertex, and for the digraph $D'' = (V, A \cup A'')$, $d_{D''}(C, v) \leq r$ for all $v \in V$.

Proof: Let $D = (V, A)$ be a digraph and $C = \{c_1, c_2, \ldots, c_m\} \subseteq V$ be a subset of vertices. Suppose that A' is a minimum subset of $V \times V - A$ such that for the digraph $D' = (V, A \cup A')$, $d_{D'}(C, v) \leq r$ for all $v \in V$. Observe that arcs joining pairs of central vertices do not affect the distance $d(C, v)$ for any $v \in V$. Since A' is minimum, no arc in A' joins a pair of central vertices.

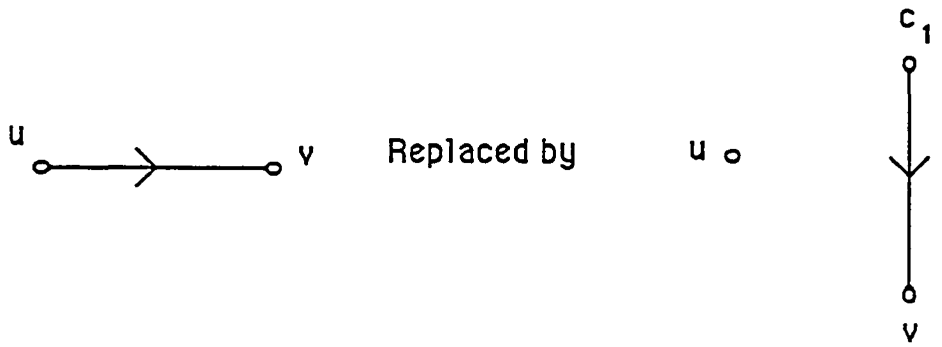


Figure 2.4 Arc replacement used in Lemma 2.2

Now, suppose that $e = (u, v) \in A'$, where u and v are non-central vertices. Let $A_1 = A' - (u, v) + (c_1, v)$ (Figure 2.4), then for the digraph $D_1 = (V, A \cup A_1)$, $d_{D_1}(C, w) \leq d_{D'}(C, w) \leq r$ for all $w \in V$, $|A_1| = |A'|$, and the number of arcs joining pairs of non-central vertices in D_1 is less than that of D'.

If no arcs in D_1 join a pair of noncentral vertices, then we set $A'' = A_1$. Otherwise, we replace arcs joining pairs of noncentral vertices for D_1 as we did for D'. After at most $O(|V|^2)$ steps a desired set A'' will be obtained. □

Theorem 2.9 For any fixed positive integers m and r such that $r \geq 2$. The m-Centrix Radius r Augmentation problem is NP-complete even in the case that the graph is connected planar of maximum degree 3.

Proof: It is easy to see that the m-Centrix Radius r Augmentation problem is in NP, since a nondeterministic algorithm need only guess a set of edges E' such that $E' \cap E = \emptyset$ and check in polynomial time that $|E'| \leq B$ and for the graph $G' = (V, E \cup E')$, $d(C, v) \leq r$ for all $v \in V$.

We will transform the Dominating Set of Radius r problem to the m-Centrix Radius r Augmentation problem. Let $H = (V_1, E_1)$ be a connected planar graph of maximum degree 3 and of minimum degree 1, r and K be two integers, $1 \leq r, K \leq |V_1|$ in an instance of the Dominating Set of Radius r problem. Set $r' = r + 1$. We will construct a connected planar graph $G = (V, E)$ of maximum degree 3 such that there exists an edge set E' with $|E'| \leq K$ and for $G' = (V, E \cup E')$, $d_{G'}(C, v) \leq r'$ for all $v \in V$ if and only if there is a subset $V_0 \subseteq V_1$ such that $|V_0| \leq K$ and $d_H(V_0, v) \leq r$ for all $v \in V_1$.

Let $V(G) = V_1 \cup V_2 \cup C$, where $V_2 = \{v_1, v_2, \ldots v_{r+1}\}$, $C = \{c_1, c_2, \ldots, c_m\}$, and V_1, V_2, and C are pairwise disjoint. There

exists a vertex, say v_{r+2}, of degree 1 in H. Let $E(G) = E_1 \cup E_2$, where $E_2 = \{c_i c_{i+1} \mid i = 1, 2, \ldots, m-1\} \cup \{c_m v_1\} \cup \{v_i v_{i+1} \mid 1 \leq i \leq r+1\}$.

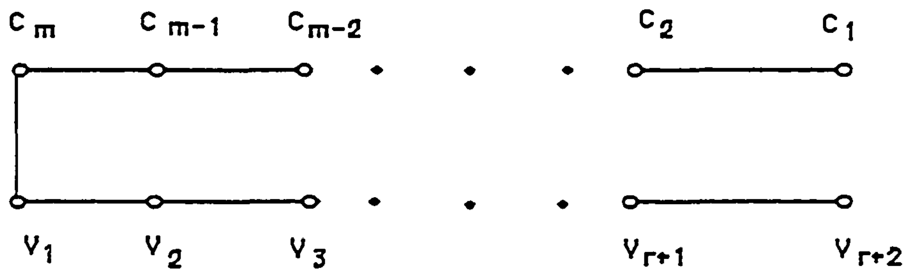


Figure 2.5 The graph induced by E_2

The graph G constructed above is clearly connected planar and of maximum degree 3. In graph G, only vertices in $C \cup V_2$ are within distance r + 1 from the m-centrix C. Suppose that H has a dominating set V_0 of radius r such that $V_0 \subseteq V_1$, $|V_0| \leq K$, and $d_H(V_0, v) \leq r$ for all $v \in V_1$. Let $E' = \{c_1 v \mid v \in V_0\}$, then $|E'| = |V_0| \leq K$ and for $G' = (V, E \cup E')$, $d_{G'}(C, v) \leq r' = r + 1$ for all $v \in V$.

On the other hand, suppose that there exists an edge set E' with $|E'| \leq K$ and for $G' = (V, E \cup E')$, $d_{G'}(C, v) \leq r' = r + 1$ for all $v \in V$. By Lemma 2.1, we may assume that each edge $e \in E'$ joins a central vertex $c_i \in C$ and a noncentral vertex. Let $V_0 = \{v \mid \exists\, e \in E'$ such that e is incident with v and $v \notin C\}$. Then clearly $|V_0| \leq |E'| \leq K$. Since for any $v \in V(H)$, $d(C, v) > r+1$ in G, and for any $v_0 \in V_0$, $d(C, v_0) = 1$ in G', we have $d_H(V_0, v) \leq r$ for all $v \in V_1$. This completes the proof. □

Theorem 2.10 For any fixed positive integers m and r, $r \geq 2$, the Digraph m-Centrix Radius r augmentation problem is NP-

complete even in the case that the digraph is symmetric and its underlying graph is planar of maximum degree 3.

Proof: In the proof of Theorem 2.9, replace each graph considered by the corresponding symmetric digraph. □

In the next algorithm, we use the fact that there is a linear time algorithm which finds the connected components of a graph [32].

Algorithm 2.2 m-Centrix Radius r augmentation for oriented trees.

INPUT: An oriented tree $T = (V, A)$, $|V| = n$, a subset of vertices $C = \{c_1, c_2, \ldots, c_m\} \subseteq V$, called an m-centrix, positive integer r. T is represented by its list structure, i.e., for each vertex, a list of vertices to which it is adjacent to is stored.

OUTPUT: A minimum set A′ of arcs such that for the digraph $D' = (V, A \cup A')$, $d_{D'}(C, v) \leq r$ for all $v \in V$.

VARIABLES

```
    d(v_i) is the minimum distance of vertex v_i to the m-centrix
          C after the algorithm
    W is the set of vertices in V such that A' = {c_1 w | w ∈ W}
begin
    /* initialize d(v_i) */
    for i ← 1 to n loop
          d(v_i) ← n
    end loop
    /* For each vertex v_i, determine the distance of v_i from
    the m-centrix C */
```

```
1)    for j ← 1 to m loop
            Let c_j be the root of the tree.
2)          Use Breadth First Search Method to determine
                  the distances d(c_j, v_i) in T for 1 ≤ i ≤ n.
            /* update d(v_i) */
            for i ← 1 to n loop
                  if (d(c_j, v_i) < d(v_i)) then
                        d(v_i) ← d(c_j, v_i)
                  end if
            end loop
      end loop
      /* Finds the set V_1 of vertices,  the distance from the
      m-centrix C to each vertex of V_1 is greater than r  */
      V_1 ← ∅
3)    for i ← 1 to n loop
            if (d(v_i) > r) then
                  V_1 ← V_1 ∪ {v_i}
            end if
4)    end loop
      if (r = 1) then  W ← V_1
      else
            W ← ∅
            Let F be the induced graph ⟨V_1⟩.
            /* F is represented by its list structure. */
5)          Find the connected components of F.
            Let T_1, T_2, . . . , T_s be the components of F.
```

6) for $i \leftarrow 1$ to s loop

7) Find a minimum dominating set W_i of

radius $r-1$ for T_i by Algorithm 2.1

8) $W \leftarrow W \cup W_i$

end loop

end if

9) $A' \leftarrow \{ (c_1, w) \mid w \in W \}$

end Algorithm 2.2

Theorem 2.11 Let $T = (V, A)$ be an oriented tree of order n, then Algorithm 2.2 finds a minimum m-centrix radius r augmentation of T in $O(m \cdot n)$ time.

Proof: In step 2, Algorithm 2.2 uses a Breadth First Search Method to determine the distances $d(v_i, c_j)$, for each fixed $c_j \in C$. Step 2 can be done in m times. Therefore the time required for step 2 is of order $O(m \cdot n)$. The loop from step 3 to step 4 finds the set V_1 of vertices each of which has distance at least $r+1$ from the m-centrix C. For $r = 1$, the result is trivial. So we asssume that $r \geq 2$. Let F be the induced graph $\langle V_1 \rangle$. Step 5 finds the connected components of F which can be done in $O(n)$ time [32]. Steps 7 to 8 use Algorithm 2.1 to find W which is a minimum dominating set of radius $r - 1$ of F. By Theorem 2.3, step 7 requires time $O(n)$. Therefore the time complexity of Algorithm 2.2 is $O(m \cdot n)$.

Let A' be the set of arcs returned from step 9, $D' = (V, A \cup A')$, and $v \in V$. If $v \notin V_1$, then $d_{D'}(C, v) \leq d_D(C, v) \leq r$. Otherwise let T_i be the component of F containing v, then

$d_{D'}(W_i, v) \le r - 1$ since W_i is a dominating set of radius $r - 1$ of T_i. From the way W was defined, c_1 must be adjacent to all the vertices in W_i. It follows that $d_{D'}(c_1, v) \le 1 + (r - 1) = r$. It only remains to show that $|A'|$ is minimum. Let $A"$ be a minimum set of arcs such that for all $v \in V$, $d_{D"}(C, v) \le r$, where $D"=(V, A \cup A")$. By Lemma 2.2, we may assume that each arc in $A"$ joins a central vertex $c \in C$ and a noncentral vertex. Let $V' = \{v \mid \exists e \in A", \exists c \in C$ such that $e = (c, v)\}$. Then V' is a dominating set of radius $r - 1$ of F. By the minimality of W, $|W| \le |V'|$. Therefore $|A'| = |W| \le |V'| \le |A"|$. Thus $|A'|$ is minimum. □

An algorithm for finding minimum m-centrix radius r augmentations for trees can be designed similarly.

Theorem 2.12 There is an $O(m \cdot n)$ time algorithm which finds a minimum m-centrix radius r augmentation for an arbitrary tree T of order n. □

CHAPTER III

K DISJOINT MATCHINGS AND AUGMENTATIONS

The following decision problems will be studied or referenced in this chapter. It can be shown easily that these problems are in NP.

Chromatic Index

Instance: Nonempty graph $G = (V, E)$, positive integer K such that $K \leq \Delta(G) + 1$.

Question: Does G have chromatic index K or less, i.e., Is there an assignment of k colors to the edges in E, with $k \leq K$, such that adjacent colors are colored differently?

r-Factorability

Instance: Graph $G = (V, E)$, positive integer $r \leq \Delta(G)$.

Question: Is G r-factorable, i.e., Is there a factorization of G such that each factor is an r-factor?

K Disjoint Maximum Matchings

Instance: Graph $G = (V, E)$, positive integer $K \leq \Delta(G)$.

Question: Does G contain K disjoint maximum matchings?

K Disjoint 1-Factors Augmentation

Instance: Graph $G = (V, E)$ of even order, positive integers B and K such that $B \leq |V|^2 - |V|$ and $K \leq \Delta(G)$.

Question: Is there a set E' of edges such that $|E'| \leq B$ and the graph $G' = (V, E \cup E')$ contains K disjoint 1-factors?

r-Factorability

By Vizing's theorem [51][3], the chromatic index of a nonempty graph G is either $\Delta(G)$ or $\Delta(G) + 1$, where $\Delta(G)$ is the maximum degree in G. It follows (see [3]) that for $r \geq 1$, an r-regular graph has chromatic index r if and only if it is 1-factorable. Holyer [30] has shown that the Chromatic Index problem is NP-complete even for cubic graphs. Thus we have

Theorem 3.1 The 1-Factorability problem is NP-complete even for cubic graphs.

Proof: Let G be a cubic graph. Then G is 1-factorable if and only if G has chromatic index 3. Thus, for cubic graphs, the NP-completeness of the 1-Factorability problem follows from that of the Chromatic Index problem. □

Corollary 3.1 The 1-Factorability problem is NP-complete even for cubic graphs having a 1-factor.

Proof: There is an $O(|V|^3)$ algorithm which finds a maximum matching in an arbitrary graph (e.g., Edmonds[13]). A 1-factor in a graph is a perfect matching. Thus there is an $O(|V|^3)$ time algorithm which determines whether an arbitrary graph has a 1-factor. If a graph G does not have a 1-factor, then G is not 1-factorable.

We transform the 1-Factorability problem for cubic graphs to 1-Factorability problem for cubic graphs having a 1-factor. Let $G = (V, E)$ be a cubic graph. Then G is of even order. Use an $O(|V|^3)$ time algorithm to determine whether G has a 1-factor.

If G has a 1-factor, G is mapped to G′ which is isomorphic to G (an identity map). Otherwise, G is mapped to G′ which is the Petersen graph (see Figure 3. 1). The Petersen graph is cubic and has a 1-factor but is not 1-factorable. Clearly , this is a polynomial transformation of order $O(|V|^3)$. G is 1-factorable if and only if G′ is 1-factorable. □

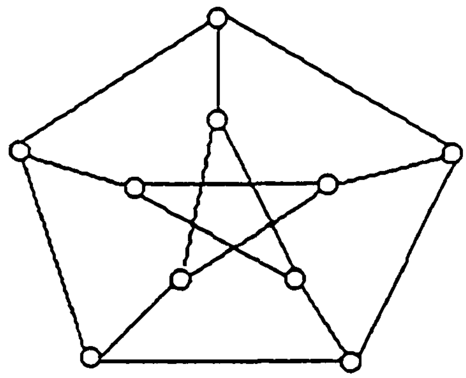


Figure 3. 1 The Petersen graph

By Theorem 3. 1, the r-Factorability problem is NP-complete for an arbitrary r. A natural question is "For a fixed value r, can the r-Factorability problem be solved in polynomial time?" For $r = 2$, the answer is affirmative.

The 2-Factorable graphs have been characterized by Petersen [43][3]: A nonempty graph G is 2-factorable if and only if G is 2n-regular for some $n \geq 1$.

Theorem 3.2 The 2-Factorability problem can be solved in $O(|V|^2)$ time.

Proof: Let $G = (V, E)$ be a graph which is represented by its adjacency matrix. The degree of the i-th vertex in G is the number of "one" entries in the i-th row of the adjacency matrix.

Thus the degrees of the vertices of G can be determined in $O(|V|^2)$ time. It follows that the problem of deciding whether a graph G is 2n-regular for some $n \geq 1$ can be solved in $O(|V|^2)$ time. By Petersen's characterization for 2-factorable graphs, the 2-Factorability problem can be solved in $O(|V|^2)$ time. □

K Disjoint Maximum Matchings

The majority of published results on matchings have focused on the problem of characterizing and finding maximum matchings in a graph. Notable among these are results of Edmonds [13], which gave rise to an $O(n^3)$ algorithm for finding a maximum weighted matching in an arbitrary graph G of order n, and Hopcroft and Karp [31], which include an $O(n^{5/2})$ algorithm for finding a maximum matching in a bipartite graph. We consider disjoint maximum matchings in graphs. Note that in a bipartite graph, the two partite sets might represent personnel and jobs. A maximum matching in such a bipartite graph represents a maximum assignment of people to jobs. The existence of K disjoint maximum matchings in such a graph would imply that on K successive days, maximum assignmemts could be scheduled in such a way that no person performs the same job twice, Cockayne, Hartnell, and Hedetniemi [10] have developed a linear algorithm for determining whether an arbitrary tree T contains two disjoint maximum matchings and finding them if they exist. A constructive characterization for a tree T containing K disjoint maximum matchings has been obtained by Slater (see [10]). We

show that the K Disjoint Maximum Matchings problem is NP-complete and present a linear algorithm for determining whether an arbitrary forest F contains K disjoint maximum matchings and finding them if they exist.

Theorem 3.3 The K Disjoint Maximum Matchings problem is NP-complete even for cubic graphs having a perfect matching and $K = 3$.

Proof: Clearly, the K Disjoint Maximum Matchings problem is in NP. Let $G = (V, E)$ be a cubic graph having a perfect matching (i.e., a 1-factor) and $K = 3$. Then G has K disjoint maximum matchings if and only if G is 1-factorable. By Corollary 3.1, the result follows. □

By a result due to König [37], the chromatic index of a bipartite graph is its maximum degree. Since a tree is a bipartite graph, the chromatic index of a tree T is $\Delta(T)$.

Algorithm 3.1 Colors the edges of a tree T using $\Delta(T)$ colors.

INPUT: A tree $T = (V, E)$ with maximum degree $K \geq 1$. The tree is represented in list structure, i.e., for each vertex, a list of vertices to which it is adjacent to is stored.

OUTPUT: K color sets $S_1, S_2, \ldots, S_K$, edges in the set S_i are colored with color i, $1 \leq i \leq K$.

Procedure COLOR(v, c_0)

INPUT

v is a vertex of T.

c_0 is the color that is used in coloring the edge uv, where u is the parent (if any) of v.

```
VARIABLES
    S is the set of colors used in coloring the edges
        adjacent with v in T .
    c(u_j) is the color that is used to color the edge vu_j,
        where u_j is a child of v, j ≥ 1.
begin /* Procedure COLOR */
    If (v has a child ) then
        S ← {c_0}
        Let u_1, u_2, ... , u_m be the children of v
        /* Color the edges vu_j, 1 ≤ j ≤ m */
        for j ← 1 to m loop
            Select a color i_0 in {1,2,...,K} - S
            c(u_j) ← i_0;  S_{i_0} ← S_{i_0} ∪ {vu_j};  S ← S ∪ {i_0}
        end loop
        /* recursive calls */
        for j ← 1 to m loop
            COLOR(u_j, c(u_j))
        end loop
    end if
end Procedure COLOR
begin /* Algorithm 3.1 */
    for i ← 1 to K loop
        S_i ← ∅
    end loop
    /* handle the first call to the procedure COLOR */
    c_0 ← -1
```

Select a vertex $r \in V$

Transform the tree T into a rooted tree having root r

COLOR(r, c_0)

end Algorithm 3.1

Theorem 3.4 Let $T=(V,E)$ be a tree with maximum degree $\Delta(T) \geq 1$. Then Algorithm 3.1 colors the edges of T using $\Delta(T)$ colors in linear time $O(|V|)$.

Proof: The proof proceeds by induction on the depth dep(T) of the rooted tree T with root r and maximum degree $\Delta(T) \geq 1$. If dep(T) = 2, then T is a star. The number of children of r is $\Delta(T)$, each S_i, $1 \leq i \leq K$, returns exactly one edge. So Theorem 3.4 is true for dep(T) = 2. Suppose that Theorem 3.4 holds for all rooted tree T' such that $2 \leq dep(T') < d$ and let T be a rooted tree of depth d (≥ 3) with root r. Let $u_1, u_2, \ldots, u_m$ be the children of r. Algorithm 3.1 colors the edges ru_j first, $1 \leq j \leq m$. Since $m \leq \Delta(T)$, the edges ru_j, $1 \leq j \leq m$, can be colored differently. The color of the edge ru_j is stored in $c(u_j)$. Let T_j be the subtree of T rooted at u_j, $1 \leq j \leq m$. By inductive hypotheses, for each j, $1 \leq j \leq m$, the edges of T_j can be colored with $\Delta(T_j)$ ($\leq \Delta(T)$) colors in linear time such that the color $c(u_j)$ is not used for those edges in T_j which are incident with u_j. It follows that the edges of T can be colored with $\Delta(T)$ colors in linear time $O(|V|)$. □

Corollary 3.4 Let $F=(V,E)$ be a forest with maximum degree $\Delta(F) \geq 1$. Then the edges of F can be colored using $\Delta(F)$ colors in linear time $O(|V|)$.

Given a forest F and a positive integer K such that $1 \le K \le \Delta(F)$, Algorithm 3.2 finds a maximum spanning subforest F_1 of F having a K-edge coloring. The algorithm first selects a vertex as the root of a component (which is a tree) of F. Then Algorithm 3.2 starts from the leaves going up to the root. Based on the number K of colors given, the algorithm determines the edges not to be colored at each vertex along the way from the leaves to the root. We consider deleting edges incident with a vertex v only after all the children (if any) of v have been considered. At the time when the edge which joins a vertex v and its parent and some other edges that are incident with v are considered to be deleted, deg(v) is the number of edges incident with v that have not been deleted yet. If $\deg(v) > K$, then the edge that joins v and the parent of v, together with $\deg(v) - K - 1$ additional (arbitrary) edges that are incident with v are deleted. The process continues until a forest F_1 satisfying $\Delta(F_1) \le K$ is obtained. At the end of Algorithm 3.2, F_1 is a maximum subforest of F such that $\Delta(F_1) \le K$.

Algorithm 3.2

INPUT: Forest $F = (V, E)$ with maximum degree $\Delta(F) \ge 1$, positive integer K such that $1 \le K \le \Delta(F)$.

OUTPUT: A maximum spanning subforest F_1 of F such that the edges of $F_1 = (V, E_1)$ can be colored using K colors.

Procedure DELETE_EDGES(T, v, K, E_0)

/* This procedure deletes a minimum set of edges from a tree

$T = (V, E)$ rooted at v such that the set of edges E_0 of the resulting forest $F_0 = (V(T), E_0)$ can be colored using K colors */

begin

    $E_0 \leftarrow E$; $d \leftarrow deg_{F_0}(v)$

1)   if $d > K$ then

2)   for each child u of v loop

            DELETE_EDGES(T, u, K, E_0)

3)   end loop

4)   Let w be the parent of v and let $v_1, v_2, \ldots, v_{d-K-1}$ be

            $d-K-1$ arbitrary children of v such that for each

            i, $1 \le i \le d - K - 1$, $vv_i \in E_0$.

5) $E_0 \leftarrow E_0 - \{wv \cup \{vv_i \mid 1 \le i \le d-K+1\}\}$

    end if

end Procedure DELETE_EDGES

begin /* Algorithm 3.2 */

    /* S is the set of vertices that have not been traversed */

    $S \leftarrow \emptyset$; $E_1 \leftarrow \emptyset$

    while $V - S \neq \emptyset$ loop

    begin

        Select a vertex $v \in V - S$ such that $deg_F\, v = 1$.

6)   Find the connected component of the forest F

            containing v, say $T_v = (V_v, E_v)$.

7)   DELETE_EDGES(T_v, v, K, E_0)

8) $E_1 \leftarrow E_1 \cup E_0$

        $S \leftarrow S \cup V_1$

    end loop

/* $F_1 = (V, E_1)$ */

for each component T of F_1 loop

Use Algorithm 3.1 to color the edges of T using colors {1, 2, ... , K}

end loop

end Algorithm 3.2

The following example illustrates Algorithm 3.2 for a tree.

Example 3.1 Let T be the tree in Figure 3.2 and K = 2. Suppose that v is the root of T. If Algorithm 3.2 is used to find a maximum spannning subforest F_1 of T such that F_1 has a K-edge coloring, then edge e_1 is deleted first, then e_2, then one of the edges e_3, e_4, and e_5 is deleted, finally e_6 is deleted.

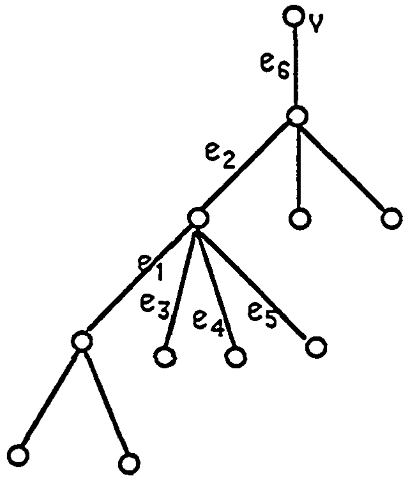


Figure 3.2

Theorem 3.5 Let F=(V, E) be a forest with maximum degree $\Delta(F) \geq 1$ and K be a positive integer such that $K \leq \Delta(F)$. Then Algorithm 3.2 finds a K-edge colorable maximum spanning subforest F_1 of F in linear time.

Proof: For each component of the forest F, Algorithm 3.2 calls procedure DELETE_EDGES to delete a set of edges from F (step 6 to step 7), returning the remaining edges (step 8). So it is sufficient to show that procedure DELETE_EDGES deletes a minimum set of edges for each component such that the resulting forest has a K-edge coloring. Note that procedure DELETE_EDGES is a recursive procedure (step 2 to step 3). Since all leaves of a tree have degree 1 and $K \geq 1$, by the condition in step 1, edges incident with leaves are not deleted.

Suppose that v is not a leaf. We consider the effect of calling procedure DELETE_EDGES(T, v, K, E_0). Let F_0 be the current forest, i.e., the remaining forest after all the calls to the procedure DELETE_EDGES for the children of v. Only edges in F_0 are considered at this point. If $deg_{F_0}(v) \leq K$, no edge incident with v needs to be deleted. If $d = deg_{F_0}(v) > K$, then at least $d - K - 1$ number of edges incident with v need to be deleted. The procedure DELETE_EDGES deletes the edge joining v with its parent first, followed by the $d - K$ other edges incident with v. Note that this is an optimal way of deleting the required number of edges so that the resulting forest is maximum and the edges incident with v can be colored with K colors in the resulting forest.

Algorithm 3.2 uses a depth first search method. It can be seen that the time required for finding F_1 is $O(|V|)$. □

For $K = 1$, Algorithm 3.2 finds a maximum matching of an arbitrary forest in linear time.

Let m be the size of a maximum matching of F and $K \le \Delta(F)$. Then F contains K disjoint maximum matchings if and only if $|E(F_1)| = K \cdot m$, where F_1 is the subforest of F found by Algorithm 3.2. Thus we have

Theorem 3.6 There is a linear time algorithm which determines whether a forest contains K disjoint maximum matchings and finds them if they exist. □

Corollary 3.6 There is a linear time algorithm which determines whether a forest contains K disjoint 1-factors and finds them if they exist. □

Lemma 3.1 Let M be a maximum matching of a graph $G = (V, E)$, $e \notin E$, and M' be a maximum matching of $G' = (V, E \cup \{e\})$. Then $|M'| \le |M| + 1$.

Proof: Note that $M' - e$ is a matching of G. Since M is a maximum matching of G, $|M' - e| \le |M|$. It follows that $|M'| \le |M| + 1$. □

Theorem 3.7 Let $G = (V, E)$ be a graph of even order $2n$, and let m be the cardinality of a maximum matching of G. Then the 1-factor augmentation number of G is $n - m$.

Proof: Let $Aug(G)$ be the 1-factor augmentation number of G. Since a 1-factor in a graph of order $2n$ contains n edges, by Lemma 3.1, $Aug(G) \ge n - m$. On the other hand, let M be a maximum matching of G, then $|M| = m$. Let V_1 be the set of weak vertices in G with respect to the matching M. Then $|V_1| = 2(n - m)$ and the induced subgraph $H = \langle V_1 \rangle$ is empty. Let E' be a 1-factor of the complement graph of H, then the graph $G' = (V, E \cup E')$ con-

tains a 1-factor and $|E'| = n - m$. Therefore $Aug(G) \le n - m$. Hence $Aug(G) = n - m$. □

K Disjoint 1-Factors Augmentation for Forests

Theorem 3.3 implies that the problem of deciding whether a graph has 3 disjoint 1-factors is NP-complete even for cubic graphs having a 1-factor. Therefore the K Disjoint 1-Factors Augmentation problem is NP-complete even for cubic graphs having a 1-factor. We prove that the K Disjoint 1-Factors Augmentation problem can be solved in polynomial time if the graphs are restricted to be forests and the positive integer K is small. Specifically, let F be a forest of even order 2n and K be a positive integer such that $K \le \max\{5, 2n-1\}$. Then we can determine Aug(F), the minimum number of edges to be added to F so that the resulting graph H contains K disjoint 1-factors. Furthermore, the K disjoint 1-factors in the graph H may be found in polynomial time. In order to present this result, we establish some other results first.

Theorem 3.8 and Theorem 3.9 provide sufficient conditions for a bipartite graph to contain a 1-factor.

Theorem 3.8 Let G be a bipartite graph with partite sets V_1 and V_2, n and K be positive integers such that $|V_1| = |V_2| = n$, $n \ge 2K$, and $\delta(G) \ge n - K$. Then G contains a 1-factor.

Proof: By a theorem of König [38] and Hall([28]), V_1 can be matched to a subset of V_2 if and only if V_1 is nondeficient.

Since $|V_1| = |V_2|$, V_1 cannot be matched to a proper subset of V_2. Thus G contains a 1-factor if and only if V_1 is nondeficient.

Suppose, to the contrary, that G does not contain a 1-factor. Then V_1 is deficient, i.e., there exists a nonempty subset U of V_1 such that $|N(U)| < |U|$, where N(U) is the set of all vertices of G adjacent with at least one element of U. It follows that $V_2 - N(U) \neq \emptyset$. Let $v \in V_2 - N(U)$. Since $\delta(G) \geq n-K$, $|U| \leq n - \deg(v) \leq n - (n - K) = K$. It follows that $|N(U)| < |U| \leq K$. On the other hand, $|N(U)| \geq \delta(G) \geq n - K \geq K$, since $n \geq 2K$. Thus we have a contradiction. Hence G contains a 1-factor. □

For any fixed positive integer K, the condition $n \geq 2K$ can not be improved upon. This is shown by the class of graphs $\{G_m \mid m \geq 1\}$, where $G_m = 2K_{m,m+1}$. Let $n = 2m+1$, $K = m+1$. Then G_m is a bipartite graph with partite sets of cardinality n, $\delta(G_m) = m = n-K$, $n = 2K-1$, and G_m does not contain a 1-factor.

Theorem 3.9 Let G be a bipartite graph with partite sets V_1 and V_2, n and K be positive integers such that $n \geq 2K$, $K \geq 2$, $|V_1| = |V_2| = n$, $u_1 \in V_1$, $u_2 \in V_2$, $\deg(v_i) \geq n-K$ for $v_i \neq u_1, u_2$, $\deg(u_1)$, $\deg(u_2) \geq n - K - 1$. Then G contains a 1-factor.

Proof: By a similar argument to that of Theorem 3.8, G contains a 1-factor if and only if V_1 is nondeficient. Suppose, to the contrary, that G does not contain a 1-factor. Then V_1 is deficient, i.e., there exists a nonempty subset U of V_1 such that $|N(U)| < |U|$. It follows that $V_2 - N(U) \neq \emptyset$.

Case 1: There exists $v_i \in V_2 - N(U)$ such that $\deg(v_i) \geq n-K$.

In this case, $|U| \leq n - \deg(v_i) \leq n - (n - K) = K$. It follows that $|N(U)| < |U| \leq K$. On the other hand if there exists $v_j \in U$ such that $\deg(v_j) \geq n - K$, then $|N(U)| \geq \deg(v_j) \geq n - K \geq K$, which is a contradiction. So we may assume that $U = \{u_1\}$ and $\deg(u_1) = n-K-1$. Thus $|N(U)| = |N(u_1)| = \deg(u_1) = n - K - 1 \geq 2K - K-1 = K - 1$, and $|U| \geq |N(U)| + 1 \geq K \geq 2$, contradicting that $|U| = |\{u_1\}| = 1$.

Case 2: $V_2 - N(U) = \{u_2\}$ and $\deg(u_2) = n - K - 1$.

Note that $|N(U)| = |V_2 - \{u_2\}| = n-1$ and $|N(U)| < |U| \leq |V_2| = n$. This implies that $|U| = n$ and $U = V_1$. Since $u_2 \notin N(V_1)$, $\deg(u_2) = 0$. But $\deg(u_2) = n - K - 1 \geq 2K - K - 1 = K-1 \geq 1$ by the assumption that $K \geq 2$. Again, we arrive at a contradiction.

Hence, G contains a 1-factor. □

Lemma 3.2 For $n \geq 2$, the graphs $K_{2n} - nK_2$ and $K_{2n} - C_{2n}$ are 1-factorable.

Proof: Let n be a positive integer such that $n \geq 2$. The complete graphs K_{2n} are 1-factorable (see [3]). Let $V(K_{2n}) = \{v_0, v_1, \ldots, v_{2n-1}\}$. Arrange the vertices $v_1, v_2 \ldots, v_{2n-1}$ in a regular $(2n-1)$-gon in clockwised order, and place v_0 in the center. Join every two vertices by a straight line segment. For $i = 1, 2, \ldots, 2n-1$, define the edge set of the factor F_i to be the edge v_0v_i together with all those edges perpendicular to v_0v_i. Then K_{2n} has a 1-factorization with 1-factors F_i, $1 \leq i \leq 2n-1$. It follows that the graph $K_{2n} - nK_2$ has a 1-factorization with

1-factors F_i, $1 \le i \le 2n-2$, and $K_{2n} - C_{2n}$ has a 1-factorization with 1-factors F_i, $1 \le i \le 2n-3$, since the edge sum of F_{2n-2} and F_{2n-3} is a hamiltonian cycle C_{2n}. □

Lemma 3.2 is illustrated in Figure 3.3 for $n = 3$.

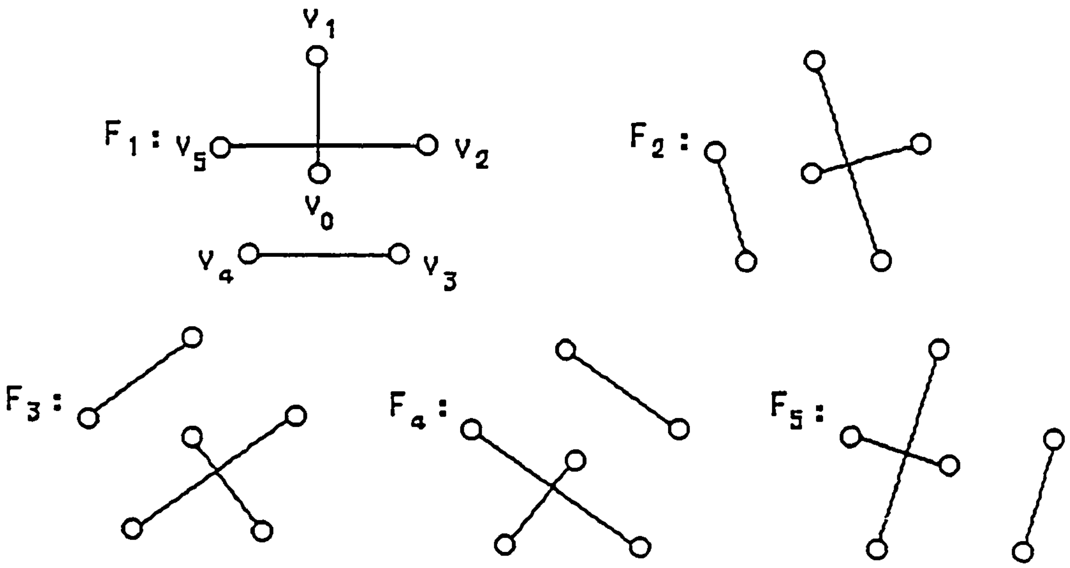


Figure 3.3 A 1-factorization of K_6.

Note that every 1-factorable graph is regular, so we may speak of the regularity of a 1-factorable graph.

Theorem 3.10 Let $F = (V, E)$ be a nonempty forest of even order $2m$ with maximum degree d such that $1 \le d \le 5$. Then F is a spanning subforest of some 1-factorable graph G of regularity d. Furthermore, there is an $O(m^{5/2})$ time algorithm which finds a set E' of edges such that $|E'| = d \cdot m - |E|$ and the graph $G = (V, E \cup E')$ is 1-factorable and of regularity d.

Proof: By Corollary 3.6, there is a linear time algorithm which determines whether a forest is 1-factorable. If F is 1-factorable, then $E' = \varnothing$. So we may assume that F is not

1-factorable in the following.

Case 1: $d = 1$.

Let V_1 be the set of isolated vertices in F. Since $|V|$ is even and each edge of F is incident with exactly two vertices, $|V_1|$ is even, say $V_1 = \{v_1, v_2, \ldots, v_{2k}\}$. Let $E' = \{v_i v_{i+k} \mid 1 \le i \le k\}$, then $|E'| = m - |E|$ and $G = (V, E \cup E')$ is a 1-factor.

Case 2: $d = 2$.

Let $G_1, G_2, \ldots, G_k$ be the components of F. Then each G_i, $1 \le i \le k$, is a path. Let u_i and v_i be the end vertices of G_i (if $G_i = K_1$, then $u_i = v_i$), and $E' = \{u_i v_{i+1} \mid 1 \le i \le k-1\} \cup \{v_k u_1\}$. Then $G = (V, E \cup E')$ is isomorphic to the cycle C_{2m} which is 1-factorable and of regularity 2.

Case 3: $d = 3, 4, 5$.

F is a forest of maximum degree d. By Corollary 3.4, the edges of F can be colored with d colors in linear time such that adjacent edges are colored differently. Thus the edges of F are partitioned into d sets, each set of edges is colored with one color. Let E_1 be such a set of edges with minimum cardinality, and n be an integer such that $|E_1| = m - n$, then $m - n \le \lfloor (2m-1)/d \rfloor$, i.e., $n \ge \lceil ((d-2) \cdot m + 1)/d \rceil$. Note that for $m \ge d$, $n \ge d-1$. We will use this fact for each of the following subcases. Let $F_1 = F - E_1$, then F_1 is a forest of order 2m with maximum degree $d-1$.

Subcase 3.1: $d = 3$.

In this case $m \ge 2$. If $m = 2$, then $F = K_{1,3}$ and F is a spanning subgraph of K_4 which is 1-factorable. $E' = E(K_4) - E(F)$.

So we may assume that $m \geq 3$. It then follows that $n \geq 2$. By Case 2, there is a linear time algorithm which finds a set E_2 of edges such that $|E_2| = 2m - |E(F_1)|$ and the graph $G_1 = (V, E(F_1) \cup E_2)$ is the hamiltonian cycle C_{2m} which is 1-factorable.

Let $H = (V, E(F) \cup E_2)) = (V, E(F_1) \cup E_1 \cup E_2)$, then the vertices of H are of degree 2 or degree 3 and the number of vertices of degree 2 is 2n. Let the vertices of degree 2 in H be $v_1, v_2, \ldots, v_{2n}$ in clockwise order along the hamiltonian cycle C_{2m}, $E_3 = \{v_1v_{n+1}, v_nv_{2n}\} \cup \{v_iv_{2n-i+1} \mid 2 \leq i \leq n-1\}$, and $E' = E_2 \cup E_3$. Then the graph $G = (V, E \cup E')$ is 1-factorable and of regularity 3.

Subcase 3.2: $d = 4$.

In this case $m \geq 3$. If $m = 3$, then $F = K_{1,4} \cup K_1$ or F is the tree of order 6 with maximum degree 4.

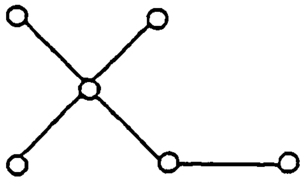


Figure 3.4 Tree of order 6 with maximum degree 4

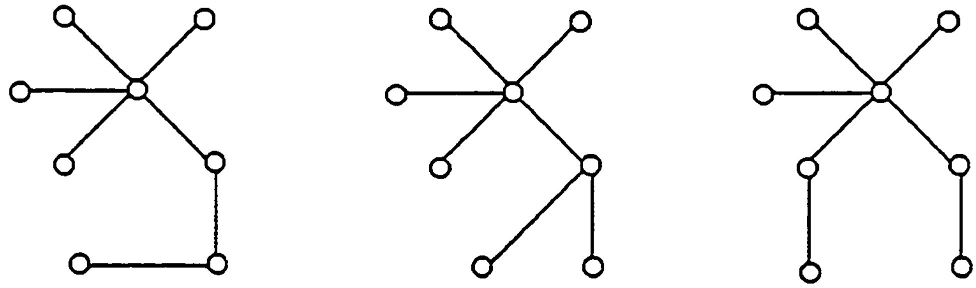
In either case F is a spanning subgraph of $K_6 - 3K_2$ which is 1-factorable and of regularity 4 by Lemma 3.2. If follows that $E(K_6 - 3K_2) - E(F)$ may be taken as the desired set E' of edges. So we may assume that $m \geq 4$. Then $n \geq 3$. By Subcase 3.1, there is a linear time algorithm which finds a set E_2 of edges such that $|E_2| = 3m - |E(F_1)|$ and the graph $G_1 = (V, E(F_1) \cup E_2)$ is 1-factorable,

hamiltonian, and of regularity 3.

Let $H = (V, E(F) \cup E_2)) = (V, E(F_1) \cup E_1 \cup E_2)$, then the vertices of H are of degree 3 or degree 4 and the number of vertices of degree 3 is 2n. Note that H contains a hamiltonian cycle C_{2m}, so there exists an edge e on C_{2m} such that e is adjacent to a vertex of degree 3 and a vertex of degree 4 in H. Without loss of generality, suppose that $e = uv_1$, $\deg_H u = 4$, $\deg_H v_1 = 3$, and v_1 is the next vertex of u along the hamiltonian cycle in clockwise direction. Let the vertices of degree 3 in H be $v_1, v_2, \ldots, v_{2n}$ in clockwise order along the hamiltonian cycle C_{2m} starting from the vertex v_1. Let $V_1=\{v_1, v_2, \ldots, v_n\}$, $V_2=\{v_{n+1}, v_{n+2}, \ldots, v_{2n}\}$, $E_3 = \{e \in E(H) \mid e = uv,\ u \in V_1, v \in V_2\}$, and $H_1 = K_{n,n} - E_3$. Note that v_nv_{n+1} may or may not be an edge in E_3, but in any case $E_3 - v_nv_{n+1}$ is a set of independent edges in $K_{n,n}$, so E_3 is contained in a 1-factor, say F_0, of $K_{n,n}$. By a theorem of König [37], every regular bipartite graph of degree $r \geq 1$ is 1-factorable. $K_{n,n} - F_0$ is regular bipartite of degree $n-1 \geq 2$ and therefore is 1-factorable. This implies that $K_{n,n} - F_0 - v_nv_{n+1}$ contains $n-2 \geq 1$ disjoint 1-factors. Thus H_1 contains a 1-factor. A 1-factor in a bipartite graph of order m can be found in $O(m^{5/2})$ time [31]. So a 1-factor of H_1 can be found in $O((2n)^{5/2}) = O(n^{5/2})$ time. The set of edges of a 1-factor in a graph is a perfect matching of the graph. Let E_4 be a perfect matching of H_1 and $E' = E_2 \cup E_4$, then the graph $G = (V, E \cup E')$ is 1-factorable and of regularity 4.

Subcase 3.3: $d = 5$.

In this case, $m \geq 3$. If $m = 3$, then $F = K_{1,5}$, F is a spanning subgraph of K_6 which is 1-factorable. $E' = E(K_6) - E(F)$. If $m = 4$, then there are three trees T_1, T_2, and T_3 of order 8 with maximum degree 5 (see Figure 3.5). It is easy to verify that these trees are spanning subgraphs of $K_8 - C_8$.



T_1 T_2 T_3

Figure 3.5 Trees of order 8 with maximum degree 5

Note that any forest of order 8 with maximum degree 5 is a subgraph of T_i, for some i, $1 \leq i \leq 3$. It follows that F is a spanning subgraph of $K_8 - C_8$ which is 1-factorable and of regularity 5 by Lemma 3.2. $E' = E(K_8 - C_8) - E(F)$. So we may assume that $m \geq 5$, thus $n \geq 4$. By Subcase 3.2, there is an $O(m^{5/2})$ time algorithm which finds a set E_2 of edges such that $|E_2| = 4m - |E(F_1)|$ and the graph $G_1 = (V, E(F_1) \cup E_2)$ is 1-factorable, hamiltonian and of regularity 4.

Let $H = (V, E(F) \cup E_2)) = (V, E(F_1) \cup E_1 \cup E_2)$, then the vertices of H are of degree 4 or degree 5 and the number of vertices of degree 4 is $2n$. Note that H contains a hamiltonian cycle C_{2m}, so there exists an edge e on C_{2m} such that e is adjacent to a ver-

tex of degree 4 and a vertex of degree 5 in H. Without loss of generality, suppose that $e=uv_1$, $\deg_H u=5$, $\deg_H v_1=4$, and v_1 is the next vertex of u along the hamiltonian cycle in clockwise direction. Let the vertices of degree 4 in H be $v_1, v_2, \ldots, v_{2n}$ in clockwise order along the hamiltonian cycle C_{2m} starting from the vertex v_1 and let $V_1 = \{v_1, v_2, \ldots, v_n\}$, $V_2 = \{v_{n+1}, v_{n+2}, \ldots, v_{2n}\}$, $E_3 = \{e \in E(H) \mid e = uv, u \in V_1, v \in V_2\}$, and $H_1 = K_{n,n}-E_3$. We show that H_1 satisfies the conditions of Theorem 3.9. Let $u_1 = v_n$, $u_2 = v_{n+1}$, and $K=2$, then $n \geq 4 = 2K$, $\deg_{H_1}(v_i) \geq n-K$ for $v_i \neq u_1, u_2$, and $\deg_{H_1}(u_1)$, $\deg_{H_1}(u_2) \geq n-K-1$. By Theorem 3.9, H_1 contains a 1-factor. A 1-factor of H_1 can be found in $O((2n)^{5/2}) = O(n^{5/2})$ time. Let E_4 be a perfect matching of H_1 and $E' = E_2 \cup E_4$, then the graph $G=(V, E \cup E')$ is 1-factorable and of regularity 5.

In case above, the desired set E′ can be found in $O(n^{5/2})$ time. □

Theorem 3.11 Let $F=(V,E)$ be a forest of even order 2n and K be a positive integer such that $K \leq \max\{5, 2n-1\}$. Then there is an $O(n^{5/2})$ time algorithm which finds a minimum set E′ of edges so that the graph $G'=(V, E \cup E')$ contains K disjoint 1-factors.

Proof: Use Algorithm 3.2 to find a maximum subforest F_1 of F such that F_1 is K-edge colorable. If the maximum degree d of F is less than K, let E_1 be a set of K − d edges such $E_1 \cap E(F_1) = \emptyset$ and the edges in E_1 are incident to some fixed vertex of F_1 having degree d. Otherwise, let $E_1 = \emptyset$. By Theorem 3.10, there is an $O(n^{5/2})$ time algorithm which finds a minimum set E_2 of edges so

that the graph $H=(V, E(F_1) \cup E_1 \cup E_2)$ is 1-factorable and of regularity K. Let $E'=E_1 \cup E_2$, then the graph $G'=(V, E(F) \cup E')$ contains K disjoint 1-factors and $|E'|$ is minimum. □

Based on Theorem 3.10, the following conjecture is made.

Conjecture 3.1 Every nonempty forest F of even order is a spanning subgraph of some 1-factorable graph of regularity $\Delta(F)$.

CHAPTER IV

(M, N, R_1, R_2)-TRANSITIVE AUGMENTATIONS

In a communications network it may be desirable to enforce some type of redundancy constraint on the paths that exist between vertices. This consideration leads to the following definition. Let M and N be two positive integers such that M > N, and let $R_1 \in \{=, \geq\}$, and $R_2 \in \{=, \leq\}$. A digraph D is (M, N, R_1, R_2)-transitive if for each u-v path P of length L in D such that L R_1 M there is a u-v path P' of length L' in D such that L' R_2 N and $V(P') \subset V(P)$. A digraph D is free (M, N, R_1, R_2)-transitive if for each u-v path P of length L in D such that L R_1 M there is a u-v path P' of length L' in D such that L' R_2 N. (M, N, R_1, R_2)-transitivity for graphs and free (M, N, R_1, R_2)-transitivity for graphs can be defined similarly.

(2, 1, =, =)-transitivity is the normal transitivity. (M, N, =, =)-transitivity is sometimes called (M, N)-transitivity. Boals and Williams [4][52] have developed some algorithms for finding minimum and minimal (M, N, =, =)-transitive augmentations (for digraphs). Gyárfás , Jacobson, and Kinch [27] have obtained several results about (M, N, =, =)-transitive tournaments. In this chapter, a number of characterization results for tournaments as well as for graphs and digraphs are derived. Then some efficient (M, N, R_1, R_2)-transitive augmentation algo-

rithms are presented. Finally heuristic algorithms are provided for free $(d+1, d, \geq, \leq)$-augmentation for graphs.

Results On Tournaments

Gyárfás et al. [27] have proved the following three theorems:

Theorem 4.1 A tournament T is $(M, 1, =, =)$-transitive if and only if it contains no cycles of length at least $M+1$. □

Theorem 4.2 If a tournament T is $(M, 1, =, =)$-transitive, then T is $(M, K, =, =)$-transitive for $K = 1, 2, \ldots, M-1$. □

Theorem 4.3 A tournament T is $(3, 1, =, =)$-transitive if and only if T is $(3, 2, =, =)$-transitive. □

By observation , we have

Lemma 4.1 Let G be a graph or digraph.

If G is $(M, N, \geq, =)$-transitive, then G is $(M, N, =, =)$-transitive.

If G is $(M, N, =, =)$-transitive, then G is $(M, N, =, \leq)$-transitive.

If G is $(M, N, \geq, =)$-transitive, then G is $(M, N, \geq, \leq)$-transitive.

If G is $(M, N, \geq, \leq)$-transitive, then G is $(M, N, =, \leq)$-transitive. □

Theorem 4.4 A tournament T is $(M, 1, \geq, =)$-transitive if and only if it contains no cycles of length at least $M+1$.

Proof: Let T be a $(M, 1, \geq, =)$-transitive tournament. We show that T contains no cycles of length at least $M+1$. Suppose, to

the contrary, that T contains a cycle C_L of length L such that $L \geq M+1$. Let (u, v) be an arc on C_L. Then $C_L - (u, v)$ is a v-u path of length $L-1 \geq M$. Since T is $(M, 1, \geq, =)$-transitive, $(v, u) \in A(T)$, contradicting that T is a tournament. Therefore T contains no cycles of length at least $M+1$.

On the other hand, suppose that tournament T contains no cycles of length at least $M+1$. Let P be a u-v path of length at least M in T, then $(v, u) \notin A(T)$; for otherwise T would contain a cycle of length at least $M+1$. Since T is a tournament, $(u, v) \in A(T)$. It follows that T is $(M, 1, \geq, =)$-transitive. □

By Theorem 4.1 and Theorem 4.4, we have

Theorem 4.5 Let T be a tournament. Then T is $(M, 1, \geq, =)$-transitive if and only if T is $(M, 1, =, =)$-transitive. □

By a result due to Camion [8][3], a tournament of order at least 3 is hamiltonian if and only if it is strong. This result is used in the following theorem.

Theorem 4.6 If a tournament T is $(M, 1, \geq, =)$-transitive, then T is $(M, K, \geq, =)$-transitive for $K = 1, 2, \ldots, M-1$.

Proof: Let T be a $(M, 1, \geq, =)$-transitive tournament, $P = v_0, v_1, \ldots, v_L$ be a path of length L, $L \geq M$, K be a positive integer, $1 \leq K \leq M-1$, n be the order of T. If $n \leq 2$, the result is trivial. Thus we assume that $n \geq 3$. Let T_1 be the tournament $\langle V(P) \rangle$ which is induced by the set of vertices in P. Then T_1 is not hamiltonian since T is $(M, 1, \geq, =)$-transitive and $L \geq M$. Therefore T_1 is not strong. Then $V(T_1)$ can be partitioned as $W_1 \cup W_2 \cup \ldots \cup W_k$, $K \geq 2$, such that W_i is a strong tournament for each i,

and if $u_i \in W_i$ and $u_j \in W_j$, where $i < j$, then $(u_i, u_j) \in A(T_1)$. Let $V_1 = W_1$ and $V_2 = W_2 \cup \ldots \cup W_K$, then $u_1u_2 \in A(T_1)$ for all $u_1 \in V_1$ and $u_2 \in V_2$. Let $p = |V_1|$. Since $v_0, v_1, \ldots, v_L$ is a hamiltonian path of T_1, $\{v_0, v_1, \ldots, v_{p-1}\} = V_1$ and $\{v_p, \ldots, v_L\} = V_2$ follow.

To see that T is $(M, K, \geq, =)$-transitive, we exhibit a suitable path of length K, P_{K+1}; from v_0 to v_L:

$$P_{K+1} = v_0, v_1, \ldots, v_{K-1}, v_L \quad \text{if } K \leq p,$$

$$= v_0, v_1, \ldots, v_{P-1}, v_q, v_{q+1} \ldots, v_L \quad \text{if } K > p,$$

where $q = P+L-K$. □

Theorem 4.7 A tournament T is $(3, 1, \geq, =)$-transitive if and only if T is $(3, 2, \geq, =)$-transitive.

Proof: Theorem 4.6 implies that if T is $(3, 1, \geq, =)$-transitive, then T is $(3, 2, \geq, =)$-transitive.

Let T be a $(3, 2, \geq, =)$-transitive tournament. Then T is $(3, 2, =, =)$-transitive by Lemma 4.1. By successively using Theorem 4.3 and Theorem 4.5, we have T is $(3, 1, =, =)$-transitive and $(3, 1, \geq, =)$-transitive. □

Theorem 4.8 A tournament T is $(3, 2, =, =)$-transitive if and only if T is $(3, 2, =, \leq)$-transitive.

Proof: Let T be a $(3, 2, =, =)$-transitive tournament, then T is $(3, 2, =, \leq)$-transitive by Lemma 4.1, hence it only remains to show that if T is a $(3, 2, =, \leq)$-transitive tournament, then T is $(3, 2, =, =)$-transitive. Let T be a $(3, 2, =, \leq)$-transitive tournament, we show that T is $(3, 2, =, =)$-transitive by showing that T is $(3, 1, =, =)$-transitive. Suppose, to the contrary, that T is not $(3, 1, =, =)$-transitive. By definition, it must be

the case that T contains a 4-cycle, say $v_0v_1v_2v_3$. Since T is $(3, 2, =, \le)$-transitive, either v_1v_3 or $v_0v_2 \in A(T)$. Without loss of generality, suppose $v_0v_2 \in A(T)$. By considering the path $v_2v_3v_0v_1$ it must be the case that $v_3v_1 \in A(T)$. Now consider the path $v_1v_2v_3v_0$; a contradiction results since there is no path P of length 1 or 2 in T such that $V(P) \subseteq \{v_0, v_1, v_2, v_3\}$. Thus T is $(3, 1, =, =)$-transitive. By Theorem 4.3, T is $(3, 2, =, =)$-transitive. □

Some Results on Graphs and Digraphs

Theorem 4.9 For any positive integer N, $(N+1, N, =, \le)$-transitivity is equivalent to $(N+1, N, \ge, \le)$-transitivity for digraphs.

Proof: Let D be an $(N+1, N, \ge, \le)$-transitive digraph. Since each path of length M is a path of length at least M, D is $(N+1, N, =, \le)$-transitive.

Now suppose that D is an $(N+1, N, =, \le)$-transitive digraph. Let P be a u-v path of length L in D such that $L \ge N+1$. We show that there is a u-v path P′ of length $\le N$ in D such that $V(P') \subset V(P)$ by induction on L. If $L = N+1$, the result is true since D is $(N+1, N, =, \le)$-transitive. Assume that the result is true for each u-v path P of length i satisfying $N+1 \le i < L$, i.e., there is a u-v path P′ of length $\le N$ in D such that $V(P') \subset V(P)$. Let P be a u-v path of length L in D, w be the vertex in P which is adjacent to v, and P_1 be the u-w subpath of P. The path P_1 is of length $L - 1 \ge N + 1$, by induction, there is a u-w path P_2 of length $\le N$ in D

such that $V(P_2) \subset V(P_1)$. Let P_3 be the path formed by concatenating P_2 and the arc wv, and let L_3 be the length of P_3. Then $V(P_3) \subset V(P)$. If $L_3 \leq N$, then $P' = P_3$ is a desired path. Otherwise $N + 1 \leq L_3 < L$. By the inductive hypothesis, there is a u-v path P′ of length $\leq N$ in D such that $V(P') \subset V(P_3) \subset V(P)$. Therefore the digraph D is $(N+1, N, \geq, \leq)$-transitive. This completes the proof. □

By a proof similar to that of Theorem 4.9, we have

Theorem 4.10 $(N+1, N, =, \leq)$-transitivity is equivalent to $(N+1, N, \geq, \leq)$-transitivity for graphs.

Williams and Boals [52] have shown that the intersection of two $(M, 1, =, =)$-transitive digraphs is $(M, 1, =, =)$-transitive and every digraph D has a unique minimum $(M, 1, =, =)$-transitive augmentation. Results similar to these are proved in Lemma 4.2 and Theorem 4.11.

Lemma 4.2 (1) The intersection of two $(M, 1, =, =)$-transitive graphs (digraphs) is $(M, 1, =, =)$-transitive.

(2) The intersection of two $(M, 1, \geq, =)$-transitive graphs (digraphs) is $(M, 1, \geq, =)$-transitive.

Proof: (1) Let G_1 and G_2 be two $(M, 1, =, =)$-transitive graphs. Suppose that P is a u-v path of length M in $G_1 \cap G_2$. Since G_1 and G_2 are $(M, 1, =, =)$-transitive, the edge uv must be in each, hence in the intersection. Therefore $G_1 \cap G_2$ is $(M, 1, =, =)$-transitive. Proof for the digraph case can be found in [52].

(2) can be proved similarly. □

Theorem 4.11 Let G be a graph or digraph. Then

(1) G has a unique minimum $(M, 1, =, =)$-transitive augmentation.

(2) G has a unique minimum $(M, 1, \geq, =)$-transitive augmentation.

Proof: (1) Suppose that G_1 and G_2 are $(M, 1, =, =)$-transitive augmentations of G. Then both G_1 and G_2 are $(M, 1, =, =)$-transitive. By Lemma 4.2, $G_1 \cap G_2$ is $(M, 1, =, =)$-transitive. Since $G \subseteq G_1$ and $G \subseteq G_2$, $G \subseteq G_1 \cap G_2$. Therefore $G_1 \cap G_2$ is an $(M, 1, =, =)$-transitive augmentation of G. By the minimality of G_1, $|E(G_1 \cap G_2)| = |E(G_1)|$. This implies that $E(G_1) \subseteq E(G_2)$. Similarly, $E(G_2) \subseteq E(G_1)$. Hence $G_1 = G_2$.

(2) can be proved similarly. □

Lemma 4.3 Let G be an $(M, 1, =, =)$-transitive graph, $M \geq 2$, and P be a path of length L in G such that $L \geq M + 1$. Then for each pair of vertices u and v in P such that $d_P(u, v) = 3$, u and v are adjacent in G.

Proof: Let $G = (V, E)$ be an $(M, 1, =, =)$-transitive graph, $M \geq 2$, and P be a path of length L in G such that $L \geq M + 1$. Note that for each pair of vertices u and v in P such that $d_P(u, v) = 3$, there exists a subpath P′ of P such that $d_{P'}(u, v) = 3$ and P′ is of length $M + 1$. So it is sufficient to pove this lemma for $L = M + 1$.

Let $P = v_0, v_1, \ldots, v_{M+1}$ be a path of length $M + 1$ in G. We need to show that $v_i v_{i+3} \in E$ for all i, $0 \leq i \leq M - 2$.

If $M = 2$, then $v_0 v_2 \in E$ since G is $(2, 1, =, =)$-transitive and v_0, v_1, v_2 is a path of length 2 in G. Now v_0, v_2, v_3 is a path of length 2 in G, therefore $v_0 v_3 \in E$.

If $M = 3$, then $v_0v_3 \in E(G)$ since G is $(3, 1, =, =)$-transitive and v_0, v_1, v_2, v_3 is a path of length 3 in G. Similarly, $v_1v_4 \in E(G)$.

So we assume that $M \geq 4$. Note that $v_0, v_1, \ldots, v_M$ is a path of length M in G. Since G is $(M, 1, =, =)$-transitive, $v_0v_M \in E$. Similarly, $v_1v_M \in E$. Now we show that $v_iv_{i+3} \in E$, $0 \leq i \leq M-2$, by presenting a v_i-v_{i+3} path of length M in G. All of the following paths are of length M.

v_0-v_3: $v_0, v_1, v_{M+1}, v_M, \ldots, v_4, v_3$.

v_1-v_4: $v_1, v_2, v_3, v_0, v_M, v_{M-1}, \ldots, v_4$.

v_2-v_5: $v_2, v_3, v_4, v_1, v_0, v_M, v_{M-1}, \ldots, v_5$.

v_3-v_6: $v_3, v_4, v_5, v_2, v_1, v_0, v_M, v_{M-1}, \ldots, v_6$.

. . . .

$v_{M-4}-v_{M-1}$: $v_{M-4}, v_{M-3}, v_{M-2}, v_{M-5}, v_{M-6}, \ldots, v_0, v_M, v_{M-1}$.

$v_{M-3}-v_M$: $v_{M-3}, v_{M-2}, v_{M-1}, v_{M-4}, v_{M-5}, \ldots, v_0, v_M$.

$v_{M-2}-v_{M+1}$: $v_{M-2}, v_{M-3}, \ldots, v_0, v_M, v_{M+1}$. □

Theorem 4.12 If G is an $(M, 1, =, =)$-transitive graph, $M \geq 2$, then G is $(M + 2K, 1, =, =)$-transitive for all nonnegative integers K.

Proof: The proof is by induction on K. For $K = 0$, the result holds trivially. Suppose that the result is true for $K < L$ and let G be an $(M, 1, =, =)$-transitive graph. We show that G is $(M+2L, 1, =, =)$-transitive. Let $P = v_0, v_1, \ldots, v_{M+2L}$ be a path of length $M + 2L$ in G. Since G is an $(M, 1, =, =)$-transitive graph, $M \geq 2$, $d_P(v_0, v_3) = 3$, and $M + 2L > M + 1$, by Lemma 4.3, $v_0v_3 \in E(G)$. Observe that $v_0, v_3, v_4, \ldots, v_{M+2L}$ is a path of

length $M + 2(L - 1)$ in G. By the inductive hypotheses, G is $(M + 2(L - 1), 1, =, =)$-transitive. Therefore $v_0v_{M+2L} \in E(G)$. Hence G is $(M + 2L, 1, =, =)$-transitive. By mathematical induction, the proof is completed. □

Theorem 4.13 A graph G is $(M, 1, \geq, =)$-transitive if and only if G is $(M, 1, =, =)$-transitive and $(M + 1, 1, =, =)$-transitive.

Proof: Clearly, $(M, 1, \geq, =)$-transitivity implies $(M, 1, =, =)$-transitivity and $(M + 1, 1, =, =)$-transitivity for graphs.

On the other hand, suppose that G is an $(M, 1, =, =)$-transitive and $(M + 1, 1, =, =)$-transitive graph. By Theorem 4.12, G is $(M + 2K, 1, =, =)$-transitive and $(M + 1 + 2K, 1, =, =)$-transitive for all nonnegative integers K. It follows that G is $(M, 1, \geq, =)$-transitive. □

Lemma 4.4 Let $G = (V, E)$ be a graph, $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{v_iv_{i+1} \mid 1 \leq i < n\} \cup \{v_i v_{i+3} \mid 1 \leq i \leq n - 3\}$, where n ($\geq 2$) is even. Then G contains a $v_{n-1}v_n$ hamiltonian path.

Proof: The proof proceeds by induction on n. For n = 2, 4, and 6, the paths P_2: v_1, v_2; P_4: v_3, v_2, v_1, v_4; and P_6: $v_5, v_4, v_1, v_2, v_3, v_6$ are v_{n-1}-v_n hamiltonian paths in G respectively. Suppose that the result holds for all even positive integers n such that n < k and let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_k\}$ and edge set $E = \{v_iv_{i+1} \mid 1 \leq i < k\} \cup \{v_i v_{i+3} \mid 1 \leq i \leq k - 3\}$, where k ($\geq 8$) is even. Let H be the subgraph of G induced by the set of vertices $\{v_1, v_2, \ldots, v_{k-6}\}$. Then $E(H) = \{v_iv_{i+1} \mid 1 \leq i < k - 6\} \cup \{v_iv_{i+3} \mid 1 \leq i \leq k - 9\}$. By the inductive hypothe-

ses, H contains a $v_{k-7}-v_{k-6}$ hamiltonian path P'. Note that the following two paths P_1: v_{k-1}, v_{k-2}, v_{k-5}, v_{k-4}, v_{k-7} and P_2: v_{k-6}, v_{k-3}, v_k are in G. The path P obtained by taking the path P_1 followed by P', then followed by P_2 is a $v_{k-1}-v_k$ hamiltonian path in G. □

Theorem 4.14 (2K, 1, =, =)-transitivity is equivalent to (2K, 1, ≥, =)-transitivity for graphs.

Proof: By Theorem 4.13, we need only show that (2K, 1, =, =)-transitivity implies (2K + 1, 1, =, =)-transitivity for graphs.

Suppose that G = (V, E) is a (2K, 1, =, =)-transitive graph and P: v_0, v_1, . . . , v_{2K+1} is a path of length 2K + 1 in G. We will show that $v_0v_{2K+1} \in E$. If K = 1, then P is a v_0-v_3 path of length 3, by Lemma 4.3, $v_0v_3 \in E$. Therefore G is (3, 1, =, =)-transitive. So we may assume that K ≥ 2. Since G is (2K, 1, =, =)-transitive, $v_0v_{2K} \in E$. By Lemma 4.3, $v_iv_{i+3} \in E$, $0 \le i \le 2K - 2$. Let $H = \langle v_1, v_2, \ldots, v_{2K-2}\rangle_G$, then $\{v_iv_{i+1} \mid 1 \le i < 2K-3\} \cup \{v_iv_{i+3} \mid 1 \le i \le 2K-5\} \subseteq E(H)$. By Lemma 4.4, H contains a $v_{2K-3}-v_{2K-2}$ hamiltonian path P'. Let $E' = \{v_0v_{2K}, v_{2K}v_{2K-3}, v_{2K-2}v_{2K+1}\} \cup E(P')$. Then $\langle E'\rangle_G$ induces a v_0-v_{2K+1} path of length 2K in G. Therefore $v_0v_{2K+1} \in E$. Thus G is (2K + 1, 1, =, =)-transitive. □

In general, an (M, 1, =, =)-transitive graph is not neces-

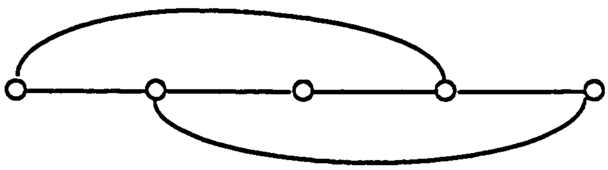


Figure 4.1

sarily $(M+1, 1, =, =)$-transitive. For example, the graph represented by Figure 4.1 is $(3, 1, =, =)$-transitive but is not $(4, 1, =, =)$-transitive.

$(M, 1, R_1, =)$-Transitive Augmentations

A number of results on the complexities of algorithms to compute certain minimum (M, N, R_1, R_2)-transitive augmentations for digraphs are known (e.g., see [4], [5], [52]). Algorithms for finding the minimum $(M, 1, =, =)$-transitive augmentations for digraphs have been developed by Williams and Boals [52]. Here we present algorithms for computing the minimum $(M, 1, R_1, =)$-transitive augmentations for graphs, where $R_1 \in \{=, \geq\}$.

Algorithm 4.1 Computes the minimum $(M, 1, =, =)$-transitive augmentation of a graph.

INPUT:

Graph $G = (V, E)$ of order p, positive integer $M > 1$.

OUTPUT:

The minimum $(M, 1, =, =)$-transitive augmentation $G' = (V, E')$ of G.

```
begin
1)   E' ← E;  DONE ← FALSE
2)   while not DONE loop
3)        ADD_EDGE ← FALSE
4)        for i ← 1 to p loop
5)             Find all paths of length M in G' with v_i as an
                    end-vertex.
```

for each path v_i–v of length M in G such that

$v_iv \notin E'$ loop

$E' \leftarrow E' \cup \{v_iv\}$

6) ADD_EDGE ← TRUE

end loop

7) end loop

8) DONE ← not ADD_EDGE

9) end loop

end Algorithm 4.1

Theorem 4.15 Algorithm 4.1 computes the minimum (M, 1, =, =)-transitive augmentation for any graph G in $O(p^{M+3})$ time, where p is the order of G.

Proof: Note that in step 8 DONE is TRUE if and only if no edges were added to G′ during a loop from step 4 to step 7. Thus G′ must be (M, 1, =, =)-transitive. G′ is the minimum (M, 1, =, =)-transitive augmentation of G since each edge in E(G′) - E(G) is required by the graph G essentially.

Now we show that the time complexity of Algorithm 4.1 is $O(p^{M+3})$. Since there are at most $O(p^2)$ number of edges in G, the loop from step 2 to step 9 can be done at most $O(p^2)$ times. Note that the loop from step 4 to step 7 can be done at most $O(p^2)\cdot O(p) = O(p^3)$ times. For a fixed vertex v_i, all paths of length M in G′ with v_i as an end-vertex can be found in $O(p^M)$ times. Therefore the time complexity of Algorithm 4.1 is $O(p^{M+3})$. □

Algorithm 4.2 Computes the minimum $(M, 1, \geq, =)$-transitive augmentation of a graph.

INPUT:

Graph $G = (V, E)$ of order p, positive integer $M > 1$.

OUTPUT:

The minimum $(M, 1, \geq, =)$-transitive augmentation $G' = (V, E')$ of G.

begin
1) Compute the minimum $(M, 1, =, =)$-transitive augmentation
        G' of G.
    If M is odd then
        DONE $\leftarrow$ FALSE
2)   while not DONE loop
3)   Compute the minimum $(M + 1, 1, =, =)$-
                transitive augmentation G_1 of G'.
4)   Compute the minimum $(M, 1, =, =)$-
                transitive augmentation G_2 of G_1.
5)   If $G_1 = G_2$ then
                DONE $\leftarrow$ TRUE
            end if
            $G' \leftarrow G_2$
6)   end loop
    end if
end Algorithm 4.2

Theorem 4.16 Algorithm 4.2 computes the minimum $(M, 1, \geq, =)$-transitive augmentation for any graph G in $O(p^{M+6})$ time,

where p is the order of G.

Proof: If M is even, by Theorem 4.14, Theorem 4.15, and step 1, the result is true. Suppose that M is odd and let G'' be the minimum $(M, 1, \geq, =)$-transitive augmentation of G, G' be the minimum $(M, 1, =, =)$-transitive augmentation of G, and G_1 be the minimum $(M + 1, 1, =, =)$-transitive augmentation of G'. Then G'' is an $(M, 1, =, =)$-transitive augmentation of G. It follows that $G' \subseteq G''$. Note that G'' is an $(M + 1, 1, =, =)$-transitive augmentation of G'. Therefore $G_1 \subseteq G''$. Similarly, it can be shown that, for any graph H obtained from G by taking arbitrary number of $(M, 1, =, =)$-transitive augmentations and $(M + 1, 1, =, =)$-transitive augmentations in any order, $H \subseteq G''$. Therefore, the graph G' returned from Algorithm 4.2 is a subgraph of G''.

Since the number of edges in the graph G is at most $O(p^2)$, the loop from step 2 to step 6 can be done at most $O(p^2)$ times. Therefore Algorithm 4.2 is guaranteed to halt. DONE has value TRUE if and only if Algorithm 4.2 finds an $(M, 1, =, =)$-transitive and $(M + 1, 1, =, =)$-transitive graph G'. By Theorem 4.13, G' is $(M, 1, \geq, =)$-transitive.

We may use Algorithm 4.1 to compute minimum $(K, 1, =, =)$-transitive augmentations, where $K = M, M + 1$. By Theorem 4.15, we have that the time complexity of Algorithm 4.2 is $O(p^2) \cdot O(p^{M+1+3}) = O(p^{M+6})$. □

Let G be a graph, M (> 1) be an odd positive integer, G'' be the minimum $(M, 1, \geq, =)$-transitive augmentation of G, G' be

the minimum $(M, 1, =, =)$-transitive augmentation of G, and G_1 be the minimum $(M + 1, 1, =, =)$-transitive augmentation of G′. In the proof of Theorem 4.16, we showed that $G_1 \subseteq G''$, it remains an open problem whether $G_1 = G''$.

For relational operators $R_1 \in \{=, \geq\}$ and $R_2 \in \{=, \leq\}$, and positive integers M and N such that $M > N$, we consider the following decision problem.

(M, N, R_1, R_2)-Transitive Augmentations (for Graphs)
Instance: Graph $G = (V, E)$, positive integer $K \leq |V|^2 - |V|$.
Question: Does there exist a set of edges E′ with $|E'| \leq K$ such that the graph $G' = (V, E \cup E')$ is (M, N, R_1, R_2)-transitive?

It follows from Theorem 4.15 and Theorem 4.16 that the $(M, 1, R_1, =)$-Transitive Augmentation problems can be solved in polynomial time. By similar proofs of corresponding results in [52], it can be shown that the $(M, N, =, R_2)$-Transitive Augmentation problems are in NP. Then, it follows from Theorem 4.10, that the $(N+1, N, \geq, \leq)$-Transitive Augmentation problem is also in NP. For $N \geq 2$ and $M \geq N+2$, we do not know whether the $(M, N, \geq, R_2)$-Transitive Augmentation problems are in NP. The following conjecture is made:

Conjecture 4.1 For $N \geq 2$, the $(M, N, \geq, R_2)$-Transitive Augmentation problems are NP-hard.

Note that a graph G is free $(d + 1, d, \geq, \leq)$-transitive if and only if the diameter of G is less than or equal to d. We consider the following minimization problem: Given a graph G and a positive integer d, find a minimum spanning supergraph G* of G hav-

ing diameter $\leq$ d. This problem is called diameter d augmentation problem. Note that a graph G has diameter 1 if and only if G is a complete graph of order at least 2. Therefore, for $d = 1$, the problem can be solved easily. For $d \geq 2$, no significant results are known except heuristic algorithms for solving the problem. Two heuristic algorithms are presented here, to be compaired.

Algorithm 4.3

INPUT

A graph G and a positive integer K.

OUTPUT

A supergraph G* of G such that diam(G*) $\leq$ K.

begin

G* ← G

while diam(G*) > K loop

Find two vertices u and v such that d(u, v) = diam (G*)

G* ← G*+ uv

end loop

end Algorithm 4.3

Algorithm 4.4

INPUT

A graph G and a positive integer K.

OUTPUT

A supergraph G* of G such that diam(G*) $\leq$ K.

begin

Find a vertex u in the center of G.

```
    G* ← G
    while rad(G*) > K/2 loop
        Find a vertex v such that d(u,v) = e_G*(u)
        G* ← G* + uv
    end loop
end Algorithm 4.4
```

Clearly, Algorithm 4.3 has time complexity $O(p^2)$ while Algorithm 4.4 has linear time complexity $O(p)$.

CHAPTER V

(r, s)-DOMINATION AND AUGMENTATIONS

Cockayne, Dawes, and Hedetniemi [9] initiated the study of total dominating sets in graphs in 1980. In this chapter, we generalize the concept of total dominating set to (r, s)-dominating set and obtain some results about the (r, s)-domination number of a graph. Some of these results are generalizations of those in [9]. In addition, results parallel to Chapter II are obtained.

A <u>total dominating set of a graph</u> $G = (V, E)$ is a subset U of V such that each vertex in V is adjacent to some vertex in U. Let $G = (V, E)$ be a graph, and r and s be two positive integers. A subset U of V is called an <u>(r, s)-dominating set of G</u> if for any $v \in V - U$ there exists $u \in U$ such that $d_G(u, v) \leq r$, and for any $u_1 \in U$ there exists $u_2 \in U$ ($u_2 \neq u_1$) such that $d_G(u_1, u_2) \leq s$. Similarly, let $D = (V, A)$ be a digraph, and r and s be two positive integers. A subset U of V is called an <u>(r, s)-dominating set of D</u> if for any $v \in V - U$ there exists $u \in U$ such that $d_D(u, v) \leq r$, and for any $u_1 \in U$ there exists $u_2 \in U$ ($u_2 \neq u_1$) such that $d_D(u_2, u_1) \leq s$. Clearly, an (r, s)-dominating set is a dominating set of radius r. Note that, if a digraph D has an (r, s)-dominating set, then no vertex of D has in-degree 0. Also a total dominating set is the same as a (1, 1)-dominating set for graphs.

Bounds on (r, s)-Domination Number

The cardinality of a smallest (r, s)-dominating set in a graph G is called the <u>(r, s)-domination number</u> and is denoted by $\gamma_{r,s}(G)$. We note that this parameter is only defined for graphs without isolated vertices and $\gamma_{r,s}(G) \geq 2$. In the case that $r = s = 1$, $\gamma_{r,s}(G)$ is the same as $\gamma_t(G)$ which is the total domination number.

Before presenting bounds on the (r, s)-domination number of a graph, an important equality is proved.

Theorem 5.1 Let G be a nontrivial connected graph, and r and s be two positive integers. Then

$$\gamma_{r,s}(G) = \min \gamma_{r,s}(T),$$

where the minimum is taken over all spanning trees T of G.

Proof: Let G be a nontrivial connected graph and T be a spanning tree of G. Then any (r, s)-dominating set of T is also an (r, s)-dominating set of G. Therefore

$$\gamma_{r,s}(G) \leq \gamma_{r,s}(T).$$

It follows that,

$$\gamma_{r,s}(G) \leq \min \gamma_{r,s}(T),$$

where the minimum is taken over all spanning trees T of G.

Now we show the reverse inequality. If G is a tree, the theorem holds trivially. So we may assume that G ia a connected non-acyclic graph. Let U be a minimum (r, s)-dominating set of G and C be a smallest cycle in G. If we can show that U is an (r, s)-dominating set of G − e for some cycle edge e, then

$\gamma_{r,s}(G - e) \leq |U| = \gamma_{r,s}(G)$. By applying this result a finite number of times, we have $\gamma_{r,s}(T) \leq \gamma_{r,s}(G)$ for some spanning tree T of G. Thus

$$\gamma_{r,s}(G) \geq \min \gamma_{r,s}(T),$$

where the minimum is taken over all spanning trees T of G.

Select two adjacent vertices x and y in V(C) such that $d_G(x, U) + d_G(y, U) = \max \{d_G(u, U) + d_G(v, U) \mid uv \in E(C)\}$. We will show that U is an (r, s)-dominating set of G - e, where e = xy.

Note that for any two adjacent vertices u and v in G, the difference of $d_G(u, U)$ and $d_G(v, U)$ is at most one. This implies that for t = x or y, $d_G(t, U) = \max \{d_G(v, U) \mid v \in V(C)\}$. Without loss of generality, suppose that $d_G(x, U) = \max \{d_G(v, U) \mid v \in V(C)\}$.

Let z be the vertex in V(C) such that $zx \in E(C)$ and $z \neq y$. By the way in which x and y were chosen, $d_G(z, U) \leq d_G(y, U)$. Since an (r, s)-dominating set is a dominating set of radius r, by the proof of Theorem 2.5, U is a dominating set of radius r of G - e. In addition $d_{G-e}(v, U) = d_G(v, U)$, for all vertices v in V(G). This equality will be used frequently in the rest of the proof.

Now it only remains to show that for any $u_1 \in U$, there exists $u_2 \in U$ $(u_2 \neq u_1)$ such that $d_{G-e}(u_1, u_2) \leq s$. Suppose, to the contrary, that there exists $u_1 \in U$ such that $d_{G-e}(u_1, U - u_1) > s$. Let x′ and y′ be vertices in U such that $d_{G-e}(x, x') = d_{G-e}(x, U)$ and $d_{G-e}(y, y') = d_{G-e}(y, U)$. Since U is an (r, s)-dominating set of G, there exists $u_2 \in U$ $(u_2 \neq u_1)$ for which $d_G(u_1, u_2) = d_G(u_1, U - u_1) \leq s$. Let P be a u_1–u_2 path of length $d_G(u_1, u_2)$ in G. Clearly, $e \in$

E(P). Observe that either the u_1-x subpath of P or the u_1-y subpath of P is in G - e. Thus we consider two cases:

Case 1: The u_1-y subpath P_1 of P is in G - e.

In this case, we may choose u_2 to be x'. For simplicity, we assume that $u_2 = x'$.

Let n and n_1 be the lengths of the paths P and P_1 respectively. Then $n = n_1 + 1 + d_{G-e}(x, U)$. If $u_1 \neq y'$, then

$$\begin{aligned} d_{G-e}(u_1, U - u_1) &\le d_{G-e}(u_1, y') \\ &\le d_{G-e}(u_1, y) + d_{G-e}(y, y') \\ &= d_{G-e}(u_1, y) + d_{G-e}(y, U) \\ &= n_1 + d_G(y, U) \\ &\le n_1 + d_G(x, U) \\ &= n_1 + d_{G-e}(x, U) \\ &< n \\ &\le s, \end{aligned}$$

which is a contradiction.

So we may assume that $u_1 = y'$(see Figure 5. 1). Let P_e be

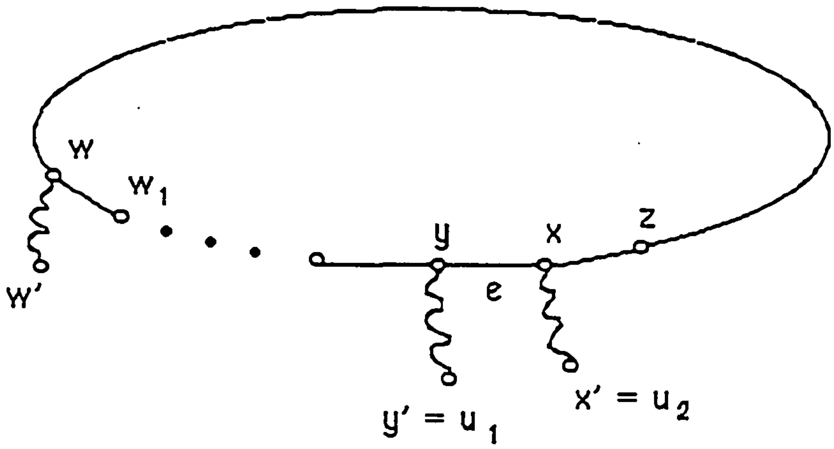


Figure 5. 1

the path obtained from C by removing the edge e and let w be the vertex in $V(P_e)$ such that $d_{G-e}(w, U) = d_{G-e}(w, w')$, for some $w' \in U$, $w \neq y$, $w' \neq y'$, and $d_P(w, y)$ is the smallest. The existence of the vertex w is provided by the fact that $x \in V(P_e)$ and $d_{G-e}(x, U) = d_{G-e}(x, x')$, where $x' \in U$ and $x' = u_2 \neq u_1 = y'$. Let w_1 be the vertex in $V(P_e)$ such that $d_{P_e}(w_1, y) = d_{P_e}(w, y) - 1$. Then w and w' are adjacent and $d_{G-e}(w_1, U) = d_{G-e}(w_1, y')$.

By the way in which x and y were chosen,

$$
\begin{aligned}
d_{G-e}(u_1, w') &= d_{G-e}(y', w') \\
&\leq d_{G-e}(w_1, y') + d_{G-e}(w, w') + 1 \\
&= d_{G-e}(w_1, U) + d_{G-e}(w, U) + 1 \\
&= d_G(w_1, U) + d_G(w, U) + 1 \\
&\leq d_G(x, U) + d_G(y, U) + 1 \\
&= d_{G-e}(x, U) + d_{G-e}(y, U) + 1 \\
&= d_{G-e}(x, U) + d_{G-e}(y, u_1) + 1 \\
&= d_{G-e}(x, U) + n_1 + 1 \\
&= n \\
&\leq s,
\end{aligned}
$$

which contradicts $d_{G-e}(u_1, U - u_1) > s$.

Case 2: The u_1–x subpath P_2 of P is in G − e.

The proof of this case is similar to Case 1. Without loss of generality, suppose that $u_2 = y'$. Let n and n_2 be the lengths of the paths P and P_2 respectively. Then $n = n_2 + 1 + d_{G-e}(y, U)$. If $u_1 \neq x'$, then

$$d_{G-e}(u_1, U - u_1) \leq d_{G-e}(u_1, x')$$

$$\le d_{G-e}(u_1, x) + d_{G-e}(x, x')$$
$$\le d_{G-e}(u_1, x) + d_{G-e}(x, U)$$
$$\le n_2 + d_{G-e}(y, U) + 1$$
$$= n$$
$$\le s,$$

which is a contradiction. So we may assume that $u_1 = x'$.

The rest of the proof is exactly the same as the second part of Case 1 where $u_1 = y'$, except we replace x, x', n_1 by y, y', n_2 respectively and vice versa.

A contradiction arises for Case 2.

Therefore in either case, a contradiction arises. Thus for any $u_1 \in U$, there exists $u_2 \in U$ ($u_2 \neq u_1$) such that $d_{G-e}(u_1, u_2) \le s$. In addition we have established that U is a dominating set of radius r of $G - e$. Therefore, U is an (r, s)-dominating set of $G - e$. This completes the proof. □

Lemma 5. 1 Let $G = (V, E)$ be a nontrivial connected graph, and r and s be two positive integers . If $\operatorname{rad} G \le r$, then $\gamma_{r,s}(G) = 2$.

Proof: Let v be a vertex in the center of G and u be a vertex adjacent to v. Since $\operatorname{rad} G \le r$, $\{u, v\}$ is an (r, s)-dominating set of G. Therefore $\gamma_{r,s}(G) = 2$. □

Lemma 5. 1 is useful when establishing certain upper bounds on $\gamma_{r,s}(G)$.

Lemma 5.2 Let G be a graph without isolated vertices, r_1, s_1, r_2, and s_2 be positive integers such that $r_1 \le r_2$ and $s_1 \le s_2$. Then

$$\gamma_{r_2,s_2}(G) \le \gamma_{r_1,s_1}(G).$$

Proof: Lemma 5.2 follows from the fact that an (r_1, s_1)-dominating set of G is also an (r_2, s_2)-dominating set of G, where r_1, s_1, r_2, and s_2 are positive integers such that $r_1 \le r_2$ and $s_1 \le s_2$.

□

Lemma 5.3 Let $G=(V,E)$ be a graph, and r and s be two positive integers such that $s \ge 2r+1$. A subset U of V is an (r, s)-dominating set of G if and only if U is an $(r, 2r+1)$-dominating set of G .

Proof: It is clear that an $(r, 2r+1)$-dominating set of G is an (r, s)-dominating set of G for $s \ge 2r+1$. Now suppose that U is an (r, s)-dominating set of a graph G, where $s \ge 2r+1$. Then U is a dominating set of radius r of G. For any vertex $u_1 \in U$, there exists $u_2 \in U$ such that $d_G(U-\{u_1\}, u_1) = d_G(u_2, u_1)$. Denote $d_G(u_2, u_1)$ by n. Let P be a u_2–u_1 path of length n in G and let $v \in V(P)$ such that $d_G(v, u_1) = \lfloor n/2 \rfloor$. If $d_G(u_2, u_1) > 2r+1$, then $d_G(U, v) > r$, contradicting that U is a dominating set of radius r of G. Therefore $d_G(u_2, u_1) \le 2r+1$. Thus U is an $(r, 2r+1)$-dominating set of G. □

By Lemma 5.3, for graphs, we need only consider (r, s)-dominating sets and (r, s)-domination numbers for $s \le 2r+1$.

The next algorithm will be used by Theorem 5.3.

Algorithm 5.1 SUBTREE_RS_DOMINATION(T, v, r, s, P, U, j)

/* This algorithm finds a minimum (r, s)-dominating set for some subtree of T, where s ≤ r + 1. */

INPUT

T is a tree with root v such that rad T > r .

r and s are positive integers such that $s \le r + 1$.

P is a longest path in T with end-vertices u and v.

x and y are the vertices on P such that $d(x, u) = r$ and
$d(y, u) = r + s$.

The x-y subpath of P is:

$x = v_0, v_1, \ldots, v_s = y$.

OUTPUT

j is the index of vertex v_j.

U is a minimum (r, s)-dominating set for the subtree of T
with root v_j.

begin

```
U ← {x}
For i = 1 to s loop
    For each child w ( ≠ v_{i-1} ) of v_i loop
        Let T_w be the subtree of T having root w
            and let w' be a vertex in T_w such that
            e(w) = d(w, w'), where e(w) is the
            eccentricity of w in T_w .
        if e(w) ≥ r then
            Let w" be a vertex in T_w such that
            d(w', w") = r and d(w", w) = e(v) - r.
            U ← U ∪ {w"}
        else
            if e(w) = r - 1 then
                U ← U ∪ {v_i}
            endif
```

```
            endif
        end loop
        Let T_{v_i} be the subtree of T having root v_i.
        if ( ∃ z ∈ V(T_{v_i}) such that d(U, z) > r ) or ( v_i ∈ U ) then
            U ← U ∪ {v_i};  j ← i
            exit loop
        endif
        if (i = s ) then
            if (|U| = 1) or (∃ z ∈ U such that d(U-{z}, z) > s)
                then U ← U ∪ {v_i}
            endif
            j ← i
        end if
    end loop
end Algorithm 5.1
```

Theorem 5.2 If T is a tree and r and s are two positive integers such that $s \le r + 1$, Algorithm 5.1 finds a minimum (r, s)-dominating set for some subtree of T.

Proof: Note that each vertex u in $U - \{v_j\}$ is required to be in U by an end-vertex descendant of u. If $v_j \in U$, then v_j is required to be in U to insure that U is an (r, s)-dominating set of the subtree T_{v_j} of T with root v_j. □

The (r, s)-domination number of a disconnected graph can be very large, for example, $\gamma_{r,s}(G) = |V(G)|$ for $G = mK_2$, $m \ge 1$. It is easy to see that mK_2 is the only graph with this property.

Cockayne et al. [9] have shown that for a connected graph of order $n \geq 3$,

$$\gamma_t(G) \leq 2n / 3.$$

Before presenting a generalization of this result, we define an r-star. An <u>r-star</u> is a graph which can be obtained from a set of disjoint paths of length r by identifying one end-vertex of each path to some fixed end-vertex of a path in the set. Thus each star is a 1-star.

Theorem 5.3 Let G be a connected graph of order $n \geq 2$, and r and s be two positive integers such that $s \leq r + 1$. Then

$$\gamma_{r,s}(G) \leq \max\{\, 2n / (r + s + 1),\; 2 \,\}.$$

Furthermore, this bound is sharp.

Proof: By Theorem 5.1, we need only show that for any tree T of order $n \geq 2$ and $s \leq r + 1$,

$$\gamma_{r,s}(T) \leq \max\{2n / (r + s + 1),\; 2\}.$$

The proof is by induction on n. Let $T = (V, E)$ be a tree of order $n \geq 2$. If $\operatorname{rad} T \leq r$, then by Lemma 5.1,

$$\gamma_{r,s}(T) = 2 \leq \max\{2n / (r + s + 1),\; 2\}.$$

Consequently,

$$\gamma_{r,s}(T) = 2,$$

for any nontrivial tree of order at most $2r + 1$.

Now suppose that for any tree T′ of order m, $2 \leq m < n$,

$$\gamma_{r,s}(T') \leq \max\{2m / (r + s + 1),\; 2\},$$

and T is a tree of order n such that $\operatorname{rad} T > r$. Let P be a longest path in T, u and v be the end-vertices of P, and k be the length of P. Since $\operatorname{rad} T > r$, $k \geq 2r + 1$.

Let x and y be the vertices of P such that $d(x, u) = r$ and $d(y, u) = r + s$, and the x-y subpath of P be:

$$x = v_0, v_1, \ldots, v_s = y.$$

In the following, the tree T is treated as a rooted tree with root v.

Use Algorithm 5.1 to find a minimum (r, s)-dominating set U of some subtree T_{v_j} of T with root v_j, where j is the integer returned from Algorithm 5.1. For each vertex v in U, there is a set S_v of vertices such that $|S_v| \geq r + 1$ for $v \neq v_s$, and $|S_v| \geq s$ for $v = v_s$ if $v_s \in U$. Each vertex in S_v is within distance r from v, and $S_v \cap S_{v'} = \emptyset$ if the corresponding vertices v and v' in U are different. Let $t = d_T(U, v_j)$ and $d = |U| = \gamma_{r,s}(T_{v_j})$. If $t = 0$, then $|V(T_{v_j})| \geq (d - 1) \cdot (r + 1) + s$; otherwise $|V(T_{v_j})| \geq d(r + 1) + t$.

Let T' be the subtree of T obtained from T by removing the subtree rooted at v_j (including vertex v_j) from T, and let n' be the order of T'. Then $n' < n$, by the inductive hypotheses,

$$\gamma_{r,s}(T') \leq \max\{ 2n' / (r + s + 1),\ 2 \}.$$

We consider three cases:

Case 1: $2n' / (r + s + 1) < 1$.

Since $s \leq r + 1$, $n' < (r + s + 1) / 2 \leq r + 1$, it follows that $n' \leq r$.

If $t + n' \leq r$, then U is an (r, s)-dominating set of T. Since U is a minimum (r, s)-dominating set of T_{v_j}, U is necessarily a minimum (r, s)-dominating set of T. By the inductive hypotheses,

$$\begin{aligned}\gamma_{r,s}(T) &= \gamma_{r,s}(T_{v_j}) \\ &\le 2(n - n') / (r + s + 1) \\ &< 2n / (r + s + 1).\end{aligned}$$

Otherwise, $t + n' \ge r + 1$. Since $n' \le r$, $t \ge 1$. This implies that $|V(T_{v_j})| \ge d \cdot (r + 1) + t$. Therefore $n = |V(T)| = |V(T_{v_j})| + n' \ge (d + 1) \cdot (r + 1)$. Note that $U' = U \cup \{v_j\}$ is an (r, s)-dominating set of T. Since $s \le r+1$, we have

$$\begin{aligned}\gamma_{r,s}(T) &\le |U'| \\ &= d + 1 \\ &= (d + 1)(r + s + 1) / (r + s + 1) \\ &\le 2(d + 1)(r + 1) / (r + s + 1) \\ &\le 2n / (r + s + 1).\end{aligned}$$

Case 2: $1 \le 2n' / (r + s + 1) < 2$.

In this case $n' < r + s + 1$. Since n' is a positive integer, $n' \le r + s$.

If $t + n' \le r + s$, let $S' = \{ v \in V(T') \mid d_T(U, v) = s \}$. If $S' \ne \varnothing$, then let u' be a vertex in S', otherwise let u' be any fixed vertex in T'. Then $U \cup \{u'\}$ is an (r, s)-dominating set of T, thus

$$\begin{aligned}\gamma_{r,s}(T) &\le 1 + |U| \\ &\le 2n' / (r + s + 1) + |U| \\ &\le 2n'/(r + s + 1) + 2(n - n') / (r + s + 1) \\ &= 2n / (r + s + 1).\end{aligned}$$

Otherwise, $t + n' \ge r + s + 1$. Since $n' \le r + s$, $t \ge 1$. Therefore $|V(T_{v_j})| \ge d\,(r + 1) + t$. It follows that $n = |V(T)| = |V(T_{v_j})| + n' \ge (d+1)\cdot(r + 1) + s$. Let $S' = \{ v \in V(T') \mid d_T(v_j, v) = s\}$. If $S' \ne \varnothing$, then let u' be a vertex in S', otherwise let u' be a

fixed vertex in the center of T'. Then $U \cup \{v_j, u'\}$ is an (r, s)-dominating set of T, thus

$$\begin{aligned}\gamma_{r,s}(T) &\le |U| + 2\\ &= d + 2\\ &= d(r + s + 1) / (r + s + 1) + 2\\ &\le 2d \cdot (r + 1) / (r + s + 1) + 2\\ &= 2\{(d + 1) \cdot (r + 1) + s\} / (r + s + 1)\\ &\le 2n / (r + s + 1),\end{aligned}$$

where $s \le r+1$.

Case 3: $2n' / (r + s + 1) \ge 2$.

In this case $\gamma_{r,s}(T') \le 2n' / (r + s + 1)$. Note that the union of an (r, s)-dominating set of T' and an (r, s)-dominating set of T_{v_j} is an (r, s)-dominating set of T. Thus, by induction

$$\begin{aligned}\gamma_{r,s}(T) &\le \gamma_{r,s}(T') + \gamma_{r,s}(T_{v_j})\\ &\le 2n' / (r + s + 1) + 2(n - n') / (r + s + 1)\\ &= 2n / (r + s + 1).\end{aligned}$$

By mathematical induction,

$$\gamma_{r,s}(T) \le \max\{2n / (r + s + 1),\ 2\},$$

for all trees T of order $n \ge 2$ and $s \le r + 1$. Thus

$$\gamma_{r,s}(G) \le \max\{2n / (r + s + 1),\ 2\},$$

for all connected graphs G of order $n \ge 2$ and positive integers r and s such that $s \le r + 1$.

Now we show that this bound is sharp. We need only show that the bound $2n / (r + s + 1)$ is obtainable under the assumption that $n \ge r + s + 1$. Let β_{r+s+2} be the set of graphs each of which

can be obtained by taking an end-vertex from an $(r+s+2)$-star graph. By observation, for $n \geq r+s+1$, the upper bound $2n/(r+s+1)$ is obtainable by all the graphs in $\mathcal{S}_{r+s+2}$. Thus the bound is sharp. □

The graph G represented by Figure 5.2 is a graph in $\mathcal{S}_{r+s+2}$, where $r=3$ and $s=2$. G has order $n=24$. The set of solid vertices is an (r,s)-dominating set of G of cardinality $\gamma_{r,s}(G)$ which is equal to $2n/(r+s+1)$.

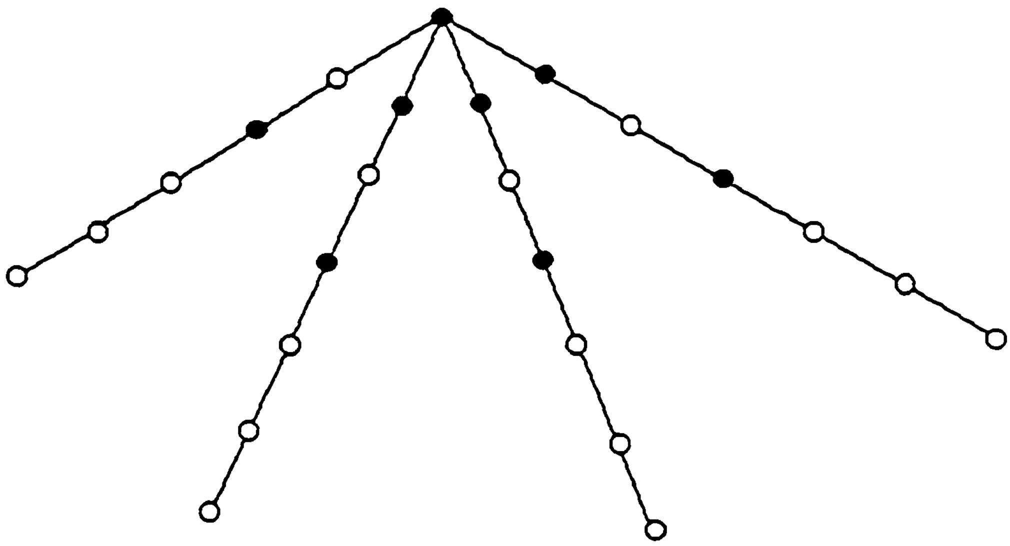


Figure 5.2

As a consequence, we have that for any $(r+s+2)$-star G of order n and positive integers r and s such that $s \leq r+1$,

$$\gamma_{r,s}(G) = 2(n-1)/(r+s+1).$$

Theorem 5.4 Let G be a connected graph of order $n \geq 2$, and r and s be two positive integers such that $s \geq r+1$. Then

$$\gamma_{r,s}(G) \leq \max\{n/(r+1),\ 2\}.$$

Furthermore, this bound is sharp.

Proof: Since $s \geq r + 1$, by Lemma 5.2 and Theorem 5.3, we have

$$\begin{aligned}\gamma_{r,s}(G) &\leq \gamma_{r,r+1}(G)\\ &\leq \max\{n/(r+1),\ 2\}.\end{aligned}$$

To show this bound is sharp, we need only show that the bound $n/(r+1)$ is obtainable under the assumption that $n \geq 2(r+1)$. Let $\mathcal{B}_{r+1}$ be the set of graphs each of which can be obtained by taking an end-vertex from an $(r+1)$-star graph. By observation, for $n \geq 2(r+1)$, the upper bound $n/(r+1)$ is obtainable by all the graphs in $\mathcal{B}_{r+1}$. Thus the bound is sharp. □

Let $G = (V, E)$ be a graph and r be a nonnegative integer. Define $End_r(G) = \{v \in V \mid \exists$ an end-vertex $u \in V$ such that $d(u, v) < r\}$. Note that $End_1(G)$ is the set of end-vertices in G.

Theorem 5.5 Let G be a connected graph of order $n \geq 2$, and r and s be two positive integers. Then

$$\gamma_{r,s}(G) \leq \max\{2, \min\{n - |End_r(T)|\}\}$$

where the minimum is taken over all spanning trees T of G.

Proof: Let T be a spanning tree of $G = (V, E)$ and $U = V - End_r(T)$. Then $|U| = n - |End_r(T)|$. If $|U| \geq 2$, define U' to be the set U, otherwise define U' to be the union of U and $2 - |U|$ end-vertices of T. Clearly, U' is an (r, s)-dominating set of T. Therefore,

$$\gamma_{r,s}(T) \leq |U'| \leq \max\{2,\ n - |End_r(T)|\}.$$

Thus

$$\begin{aligned}\gamma_{r,s}(G) &\leq \min \gamma_{r,s}(T)\\ &\leq \min\max\{2,\ n - |End_r(T)|\}\\ &= \max\{2, \min\{n - |End_r(T)|\}\},\end{aligned}$$

where the minimum is taken over all spanning trees T of G. □

The following result has been obtained by Cockayne et al. [9]: If G is a connected graph of order n such that $\Delta(G) < n - 1$, then $\gamma_t(G) \leq n - \Delta$. This result is generalized by Theorem 5.6.

Denote the set of end-vertices of a tree T by End(T). By observation, $|\mathrm{End}(T)| \geq \Delta(T)$. Theorem 5.6 relates the (r, s)-domination number and the maximum degree of a graph.

Theorem 5.6 Let G be a connected graph of order $n \geq 2$ with maximum degree $\Delta = \Delta(G)$, and r and s be two positive integers such that $s \leq r + 1$. Then

$$\gamma_{r,s}(G) \leq \max\{2, n - (r + s + \Delta) + 2\}.$$

Proof: Let r and s be two positive integers such that $s \leq r + 1$. By Theorem 5.1, it is sufficient to show that

$$\gamma_{r,s}(T) \leq \max\{2, n - (r + s + \Delta) + 2\},$$

for any tree T of order $n \geq 2$ with maximum degree $\Delta = \Delta(T)$.

If $\mathrm{rad}\, T \leq r$, then by Lemma 5.1, $\gamma_{r,s}(T) = 2$. So we may assume that $\mathrm{rad}\, T > r$. Let P be a longest path in T with end-vertices u and v. Then there exist vertices x and y of P such that $d(x, u) = r$ and $d(y, u) = r + s$. Let P' be the u–y subpath of P, $V' = V(P') - \{x, y\}$, and $U = V(T) - (V' \cup \mathrm{End}(T))$. Then $\{x, y\} \subseteq U$, it follows that $|U| \geq 2$. Thus U is an (r, s)-dominating set of T. Since $u \in V' \cap \mathrm{End}(T)$ and $|\mathrm{End}(T)| \geq \Delta(T)$, where End(T) is the set of end-vertices of T, we have

$$\begin{aligned}\gamma_{r,s}(T) &\leq |V(T)| - |V' \cup \mathrm{End}(T)| \\ &\leq |V(T)| - |V'| - |\mathrm{End}(T)| + 1 \\ &\leq n - (r + s - 1) - \Delta(T) + 1\end{aligned}$$

$$= n - (r + s + \Delta(T)) + 2 \qquad \square$$

Corollary 5.6 Let G be a graph of order n which contains no isolated vertices and there exists a component C of G such that $\Delta(C) = \Delta(G)$ and $|V(C)| \geq r + s + \Delta(G)$, where r and s are positive integers such that $s \leq r + 1$. Then

$$\gamma_{r,s}(G) \leq n - (r + s + \Delta) + 2.$$

Proof: Let U be a minimum (r, s)-dominating set in a component C of G such that $\Delta(C) = \Delta(G)$ and $n' = |V(C)| \geq r + s + \Delta(G)$. Since G contains no isolated vertices, $(V(G) - V(C)) \cup U$ is an (r, s)-dominating set of G. By Theorem 5.5,

$$\begin{aligned} \gamma_{r,s}(G) &\leq |V(G) - V(C)| + |U| \\ &= n - n' + \gamma_{r,s}(C) \\ &\leq n - n' + n' - (r + s + \Delta(C)) + 2 \\ &= n - (r + s + \Delta(G)) + 2 \qquad \square \end{aligned}$$

The following theorem gives a lower bound for $\gamma_{r,s}(G)$ in terms of the diameter of a graph G and r, s.

Theorem 5.7 Let G be a graph which contains no isolated vertices, and r and s be two positive integers. Then

$$\gamma_{r,s}(G) \geq 2\lfloor(\mathrm{diam}(G) + 1) / (2r + s + 1)\rfloor.$$

Proof: Let u be a vertex in $G = (V, E)$ such that $e(u) = \mathrm{diam}(G)$ and U be a minimum (r, s)-dominating set of G. Then $\gamma_{r,s}(G) = |U|$. Define $L_i = \{v \in V \mid d(u, v) = i\}$, $0 \leq i \leq \mathrm{diam}(G)$, and $L_j = \varnothing$, $j > \mathrm{diam}(G)$. A set L_i is said to be dominated by some set S if each element in L_i is dominated by S. Observe that any two vertices in U alone can dominate at most $2r + s + 1$ L_i's in G within distance r. Therefore, $|U \cap (L_k \cup L_{k+1} \cup \ldots \cup L_{k+2r+s})| \geq 2$, for k

$= 0,\ 2r+s+1, \ldots, (\lfloor(\mathrm{diam}(G)+1)/(2r+s+1)\rfloor - 1)\cdot(2r+s+1)$.

It follows that

$$\begin{aligned}\gamma_{r,s}(G) &= |U| \\ &\geq 2\lfloor(\mathrm{diam}(G)+1)/(2r+s+1)\rfloor . \quad \square\end{aligned}$$

Computational Complexities

(r, s)-Dominating Set

Instance: Graph $G = (V, E)$ which does not contain isolated vertices, positive integers r, s, and K such that $s \leq 2r + 1$ and $K \leq |V|$.

Question: Is there an (r, s)-dominating set of size K or less, i.e., A subset $U \subseteq V$ such that for any $v \in V - U$, there exists $u \in U$ for which $d(u, v) \leq r$, and for any $u_1 \in U$, there exists $u_2 \in U$ $(u_2 \neq u_1)$ such that $d(u_2, u_1) \leq s$?

Digraph (r, s)-Dominating Set

Instance: Digraph $D = (V, A)$ each vertex of which has in-degree > 0, positive integers r, s, and K such that $K \leq |V|$.

Question: Is there an (r, s)-dominating set of size K or less, i.e., A subset $U \subseteq V$ such that for any $v \in V - U$, there exists $u \in U$ for which $d(u, v) \leq r$, and for any $u_1 \in U$, there exists $u_2 \in U$ $(u_2 \neq u_1)$ such that $d(u_2, u_1) \leq s$?

m-Centrix (r, s)-Domination Augmentation

Instance: Graph $G = (V, E)$, a subset C of V of cardinality $m \geq 2$, called an m-centrix, positive integers B, r, and s such that $s \leq 2r + 1$.

Question: Is there a set E' of unordered pairs of vertices from V

such that $|E'| \le B$ and C is an (r, s)-dominating set of $G' = (V, E \cup E')$, i.e., $d_{G'}(C, v) \le r$ for all $v \in V$, and for any $c \in C$, there exists $c' \in C$ $(c' \ne c)$ such that $d_{G'}(c', c) \le s$?

Digraph m-Centrix (r, s)-Domination Augmentation

Instance: Digraph $D = (V, A)$, a subset C of V of cardinality $m \ge 2$, called an m-centrix, positive integers B, r, and s.

Question: Is there a set A' of ordered pairs of vertices from V such that $|A'| \le B$ and C is an (r, s)-dominating set of $D' = (V, A \cup A')$, i.e., $d_{D'}(C, v) \le r$ for all $v \in V$, and for any $c \in C$, there exists $c' \in C$ $(c' \ne c)$ such that $d_{D'}(c', c) \le s$?

We consider (r, s)-Dominating Set and Digraph (r, s)-Dominating Set problems first.

Theorem 5.8 Let $G=(V,E)$ be a connected graph, r be a positive integer, and $U \subseteq V$ such that $|U| \ge 2$. Then U is a dominating set of radius r of G if and only if U is an $(r, 2r+1)$-dominating set of G.

Proof: Clearly if U is an $(r, 2r+1)$-dominating set of G, then U is a dominating set of radius r of G.

On the other hand, assume that G is a connected graph, r is a positive integer, and U is a dominating set of radius r of G such that $|U| \ge 2$. We show that U is an $(r, 2r+1)$-dominating set of G. Suppose, to the contrary, that there exists $u_1 \in U$ such that $d_G(U-\{u_1\}, u_1) > 2r+1$. Since $|U| \ge 2$, there exists $u_2 \in U$ such that $d_G(U-\{u_1\}, u_1) = d_G(u_2, u_1)$. Denote $d_G(u_2, u_1)$ by n. Let P be a u_2-u_1 path of length n in G and let $v \in V(P)$ such that $d_G(v, u_1) = \lfloor n/2 \rfloor$. Then $d_G(U, v) > r$, contradicting that U is a dominating

set of radius r of G. Therefore $d_G(U-\{u\}, u) \le 2r+1$ for all $u \in U$. Thus G is an $(r, 2r+1)$-dominating set of G. □

Corollary 5.8 Let $G = (V, E)$ be a connected graph and r be a positive integer such that $\gamma_r(G) \ge 2$. Then $\gamma_{r,2r+1}(G) = \gamma_r(G)$. □

Let G be a connected graph or digraph and r be an arbitrary positive integer. Observe that $\gamma_r(G) = 1$ if and only if rad $G \le r$. It can easily be shown that the problem of determining whether rad $G \le r$ can be solved in polynomial time. Thus we have

Lemma 5.4 Let G be a connected graph or digraph and r be an arbitrary positive integer. The problem of determining whether $\gamma_r(G) = 1$ can be solved in polynomial time. □

By Corollary 5.8, Lemma 5.4, and Theorem 2.1, we have

Theorem 5.9 For any fixed positive integer r, the (r, s)-Dominating Set problem is NP-complete even in the case that the graph is a connected planar graph of maximum degree 3 and of minimum degree 1. □

By an argument similar to that of Theorem 5.8, it can be shown that for a connected symmetric digraph $D = (V, A)$, a positive integer r, and $U \subseteq V$ such that $|U| \ge 2$, U is a dominating set of radius r of D if and only if U is an $(r, 2r+1)$-dominating set of D. This result together with Lemma 5.4 and Theorem 2.2 give

Theorem 5.10 For any fixed positive integer r, the Digraph (r, s)-Dominating Set problem is NP-complete even in the case that the digraph is symmetric and its underlying graph is connected planar of maximum degree 3 and of minimum degree 1. □

Let T be an oriented tree no vertex of which has in-degree 0. An algorithm for finding a minimum (r, s)-dominating set of T can be designed in a manner similar to Algorithm 2.1.

Theorem 5.11 There is a linear time algorithm which finds a minimum (r, s)-dominating set of an oriented tree for which no vertex has in-degree 0. □

Similarly,

Theorem 5.12 There is a linear time algorithm which finds a minimum (r, s)-dominating set of an arbitrary nontrivial tree.

□

Now we consider m-Centrix (r, s)-Domination Augmentation and Digraph m-Centrix (r, s)-Domination Augmentation problems.

Theorem 5.13 For $r = 1$, the problem of finding a minimum digraph m-centrix (r, s)-domination augmentation can be solved in $O(m^2 \cdot |V|^2)$ time.

Proof: Let $D = (V, A)$ be a nontrivial digraph and $C = \{c_1, c_2, \ldots, c_m\} \subseteq V$ be an m-centrix, $m \geq 2$. Denote the minimum number of arcs to be added to the digraph D so that C is a $(1, s)$-dominating set of the resulting digraph by Aug(D). We consider two cases:

Case 1: $s = 1$.

In this case, the problem is to find a minimum set of arcs A' to be added to D so that in the resulting digraph $D' = (V, A \cup A')$, every vertex in V is adjacent from some vertex in the m-centrix C.

For each vertex c_i, $1 \le i \le m$, consider the set V_i of vertices adjacent from c_i. Let $V' = V_1 \cup V_2 \cup \ldots \cup V_m$ and $V" = V - V'$. Then $V"$ is the set of vertices which are not adjacent from vertices in C. For each vertex v in $V"$, find a vertex c_v in C such that $v \neq c_v$. Let $A' = \{(c_v, v) \mid v \in V"\}$. Then in the digraph $D' = (V, A \cup A')$, every vertex in V is adjacent from some vertex in C and $|A'| = |V"|$.

On the other hand, to ensure that each vertex in $V"$ is adjacent from some vertex in C, at least $|V"|$ number of arcs need to be added to D. Therefore $Aug(D) \ge |V"|$. Thus A' is a desired set of arcs.

Case 2: $s \ge 2$.

For each i, $1 \le i \le m$, find the set V_i of vertices adjacent from c_i. Let $V' = V_1 \cup V_2 \cup \ldots \cup V_m$ and $V" = V - (V' \cup C) = \{v_1, v_2, \ldots, v_k\}$. Then $V"$ is the set of vertices in $V - C$ which are not adjacent from C. Clearly $Aug(D) \ge |V"|$. We consider three subcases.

Subcase 2. 1 : For each $v_i \in V"$, $1 \le i \le k$, there exists $c_{j_i} \in C$ such that $d_D(v_i, c_{j_i}) \ge s$.

Let $A_1 = \{ (c_{j_i}, v_i) \mid 1 \le i \le k\}$, $D_1 = (V, A \cup A_1)$, and let C' be the subset of C such that for each $c' \in C'$, $d_{D_1}(C - c', c') > s$. Note that C' can be found in $O(m^2 \cdot |V|^2)$ time. Clearly, in the digraph D_1, every vertex in $V - C$ is adjacent from some vertex in C. If $C' = \emptyset$, then $Aug(D) = |V"| = |A_1|$. So we may assume that $C' = \{c_1', c_2', \ldots, c_t'\} \neq \emptyset$. For each j, $1 \le j \le t$, let $c_j"$ be a vertex in C such that $c_j" \neq c_j'$. Then for the digraph $D' =$

$(V, A \cup A')$, where $A' = A_1 \cup \{(c_j", c_j') \mid 1 \leq j \leq t\}$, C is a $(1, s)$-dominating set of D'.

We show that $|A'| = Aug(D)$. Suppose that A_2 is a set of arcs such that C is a $(1, s)$-dominating set of $D_2 = (V, A \cup A_2)$. Then A_2 contains a set A_3 of arcs of cardinality k ($= |V"|$) such that each arc of A_3 joins a central vertex with some v_i, $1 \leq i \leq k$, and no arc in A_3 is incident to more than one v_i, $1 \leq i \leq k$. We consider the digraph $D_3 = (V, A \cup A_3)$. Let $C"$ be the subset of C such that for each $c" \in C"$, $d_{D_3}(C - c", c") > s$. By the way in which the arcs in A_1 were chosen, we have $|C'| \leq |C"|$. Note that in the digraph D_3, each vertex $v \in V - C$ is adjacent from some vertex in C. This implies that for each $c" \in C"$, there exists an arc $e" = (u", c") \in A_2 - A_3$, for some $u" \in V$; for otherwise, we would have $d_{D_2}(C - c", c") > s$, which is a contradiction. Therefore $|A_2 - A_3| \geq |C"|$. Since A_3 is a subset of A_2, we have

$$\begin{aligned} |A'| &= |A_1| + |C'| \\ &\leq |A_3| + |C"| \\ &\leq |A_3| + |A_2 - A_3| \\ &= |A_2|. \end{aligned}$$

Thus $|A'| = Aug(D)$.

In the following subcases, there exists $v_{i_0} \in V"$, $1 \leq i_0 \leq k$, such that $d_D(v_{i_0}, c) \leq s - 1$, for all $c \in C$.

Subcase 2.2: There exist $v_{i_0}, v_{j_0} \in V"$ and $c' \in C$, $1 \leq i_0, j_0 \leq k$, $i_0 \neq j_0$, such that $d_D(v_{i_0}, c) \leq s - 1$ for all $c \in C$, and $d_D(v_{j_0}, c') \leq s - 1$.

Let $c"$ be a vertex in C such that $c" \neq c'$ and let $A' = (c", v_{j_0}) \cup$

$\{(c', v_i) \mid i \neq j_0, 1 \le i \le k\}$, then $|A'| = |V"| \le Aug(D)$. We show that C is a $(1, s)$-dominating set of D', where $D' = (V, A \cup A')$. Clearly C is a dominating set of radius 1 of D'. Since for all $c \in C$ $(c \neq c')$, $d_{D'}(c', c) \le d_{D'}(c', v_{i_0}) + d_{D'}(v_{i_0}, c) \le 1 + (s - 1) = s$, and $d_{D'}(c", c') \le d_{D'}(c", v_{j_0}) + d_{D'}(v_{j_0}, c') \le s$, $c" \neq c'$, C is a $(1, s)$-dominating set of D'. Therefore A' is a desired set of arcs.

Subcase 2.3: For any $v_{i_0} \in V"$, $1 \le i_0 \le k$, such that $d_D(v_{i_0}, c) \le s - 1$ for all $c \in C$, there does not exist i, $i \neq i_0$, $1 \le i \le k$, such that $d_D(v_i, C) \le s - 1$.

Note that in this subcase, there is only one vertex $v_{i_0} \in V"$, $1 \le i_0 \le k$, such that $d_D(v_{i_0}, c) \le s - 1$ for all $c \in C$. Let $A' = \{(c_1, v_i) \mid 1 \le i \le k\} \cup (c_2, c_1)$ and $D' = (V, A \cup A')$. Then $|A'| = |V"| + 1$ and C is a $(1, s)$-dominating set of D'. We show that $Aug(D) = |A'|$. Suppose that A_1 is a set of arcs such that C is a $(1, s)$-dominating set of $D_1 = (V, A \cup A_1)$. Then A_1 contains a set A_2 of arcs of cardinality k ($= |V"|$) such that each arc of A_2 joins a central vertex with some v_i, $1 \le i \le k$, and no arc in A_2 is incident to more than one v_i, $1 \le i \le k$. We consider the digraph $D_2 = (V, A \cup A_2)$. Let (c', v_{i_0}) be the arc in A_2 incident to v_{i_0}. Since for all $i \neq i_0$, $1 \le i \le k$, $d_D(v_i, C) \ge s$, we have $d_{D_2}(C - c', c') > s$, it follows that $|A_1| \ge |A_2| + 1 = |A'|$. Thus $Aug(D) = |A'|$.

The time complexity to find A' is $O(m^2 \cdot |V|^2)$. □

Note that if the distance matrix of the digraph is given, then the problem of finding a minimum digraph m-centrix $(1, s)$-domination augmentation can be solved in $O(m^2 \cdot |V|)$ time.

Corollary 5.13 For $r = 1$, the Digraph m-Centrix (r, s)-Domination Augmentation problem can be solved in $O(m^2 \cdot |V|^2)$ time. □

The next theorem presents a corresponding result for graphs.

Theorem 5.14 For $r = 1$, the problem of finding a minimum m-centrix (r, s)-domination augmentation can be solved in $O(m^2 \cdot |V|^2)$ time.

Proof: Let $G=(V, E)$ be a nontrivial graph and $C= \{c_1, c_2, \ldots, c_m\} \subseteq V$ be an m-centrix, and let $Aug(G)$ be the $(1, s)$ augmentation number of G with respect to C. We consider two cases.

Case 1: $s = 1$.

In this case, the problem is equivalent to finding a minimum set of edges E' to be added to G so that C is a total dominating set of the resulting graph $G' = (V, E \cup E')$.

For each i, $1 \le i \le m$, we consider the set V_i of vertices adjacent to c_i. Let $V' = V_1 \cup V_2 \cup \ldots \cup V_m$ and $V'' = V - V'$. Then V'' is the set of vertices which are not adjacent to C. For each vertex v in V'', find a vertex c_v in C such that $v \neq c_v$. Let $E' = \{c_v v \mid v \in V''\}$. Then $|E'| = |V''|$ and C is a total dominating set of $G' = (V, E \cup E')$.

To ensure that each vertex in V'' is adjacent to some vertex in C, at least $|V''|$ number of edges must be added to G. Therefore E' is a minimum augmentation. Note that E' can be found in $O(m \cdot |V|)$ time.

Case 2: $s \ge 2$.

For each i, $1 \le i \le m$, find the set V_i of vertices adjacent from c_i. Let $V' = V_1 \cup V_2 \cup \ldots \cup V_m$ and $V" = V - (V' \cup C) = \{v_1, v_2, \ldots, v_k\}$. Then V" is the set of vertices in V - C which are not adjacent from C. Clearly $Aug(G) \ge |V"|$. We consider three subcases.

Subcase 2.1 : For each $v_i \in V"$, $1 \le i \le k$, there exists $c_{j_i} \in C$ such that $d_G(v_i, c_{j_i}) \ge s$.

Let $E_1 = \{ c_{j_i} v_i \mid 1 \le i \le k\}$, $G_1 = (V, E \cup E_1)$, and let C′ be the subset of C such that for each $c' \in C'$, $d_{G_1}(C - c', c') > s$. Note that C′ can be found in $O(m^2 \cdot |V|^2)$ time. Clearly, in the graph G_1, every vertex in V - C is adjacent from some vertex in C. If $C' = \varnothing$, then $Aug(G) = |V"| = |E_1|$. So we may assume that $C' = \{c_1', c_2', \ldots, c_t'\} \ne \varnothing$. Define c_{t+1}' to be some fixed vertex in C other than $c_{\lceil t/2 \rceil}'$ and let $E' = E_1 \cup \{c_i' c_{i+\lceil t/2 \rceil}' \mid 1 \le i \le \lceil t/2 \rceil\}$. Then C is a (1, s)-dominating set of $G' = (V, E \cup E')$.

We show that $|E'| = Aug(G)$. Suppose that E_2 is a set of edges such that C is a (1, s)-dominating set of $G_2 = (V, E \cup E_2)$. Then E_2 contains a set E_3 of edges of cardinality k (= |V"|) such that each edge of E_3 joins a central vertex with some v_i, $1 \le i \le k$, and no edge in E_3 is incident to more than one v_i, $1 \le i \le k$. We consider the graph $G_3 = (V, E \cup E_3)$. Let C" be the subset of C such that for each $c" \in C"$, $d_{G_3}(C - c", c") > s$. By the way in which the edges in E_1 were chosen, we have $|C'| \le |C"|$. Note that in the graph E_3, each vertex $v \in V - C$ is adjacent from some vertex in C. This implies that for each $c" \in C"$, there exists an edge $e" = u"c" \in E_2 - E_3$, for some $u" \in V$; for otherwise, we

would have $d_{G_2}(C - c", c") > s$, which is a contradiction. Therefore $|E_2 - E_3| \geq \lceil |C"|/2 \rceil$. By a similar augument as in Subcase 2.1 of Theorem 5.13, we have $|E'| \leq |E_2|$. Thus $|E'| = Aug(G)$.

Subcases 2.2 and 2.3 correspond to Subcases 2.2 and 2.3 in Theorem 5.13 and can be proved similarly.

In each case above, the desired set E′ of edges can be found in $O(m^2 \cdot |V|^2)$ time. □

Corollary 5.14a For $r = 1$, the m-Centrix (r, s)-Domination Augmentation problem can be solved in $O(m^2 \cdot |V|^2)$ time. □

Corollary 5.14b Let $G = (V, E)$ be a nontrivial graph and let C be a subset of V, $|C| = m \geq 2$. The problem of finding a minimum set of edges E′ to be added to G so that C is a total dominating set of the resulting graph $G' = (V, E \cup E')$ can be solved in $O(m \cdot |V|)$ time. □

By Corollary 5.8, Lemma 5.4, Theorem 2.9 and Theorem 2.10, we have

Theorem 5.15 For any fixed positive integers m and r, $r \geq 2$, the m-Centrix (r, s)-Domination Augmentation problem is NP-complete even in the case that the graph is connected planar of maximum degree 3. □

Theorem 5.16 For any fixed positive integers m and r, $r \geq 2$, the Digraph m-Centrix (r, s)-Domination Augmentation problem is NP-complete even in the case that the digraph is symmetric and its underlying graph is connected planar of maximum degree 3. □

CHAPTER VI

ON BLOCK-COMPLETE GRAPHS

The class of graphs whose blocks are complete subgraphs is denoted by $\mathcal{B}_c$. Graphs in $\mathcal{B}_c$ are called <u>block-complete graphs</u>.

There are a number of graph problems which are not known to be polynomial, but if the graphs are restricted to be trees (or forests), many of these problems can be solved in polynomial time. In this chapter, polynomial time algorithms are presented for solving some graph problems on the domain $\mathcal{B}_c$.

A graph H of order n is complete if and only if H has $n(n-1)/2$ edges. Thus the completeness of a graph can be determined in linear time. Therefore, given a graph G and a block B of G, the completeness of the block B can be determined in linear time. Hopcroft and Tarjan [32] have proved that the components of a graph and the blocks of a connected graph $G = (V, E)$ can be found in linear time $O(|E| + |V|)$. Thus we have

Theorem 6. 1 There is a linear time algorithm which recognizes the class $\mathcal{B}_c$ of graphs. □

Since each block in a forest is either K_1 or K_2, the class of forests is a proper subclass of $\mathcal{B}_c$. A triangulated graph is a graph which does not contain a cycle of length at least four as an induced subgraph. It is clear that $\mathcal{B}_c$ is a proper subclass of the

class of triangulated graphs.

The block-cutvertex graph of a graph G, denoted bc(G), is defined as follows. Each block and each cut-vertex of G is represented by a vertex of bc(G). We call the vertices of bc(G) which represent blocks its b-vertices, and those representing cut-vertices its c-vertices. The vertices u, v of bc(G) are adjacent if and only if u is a cut-vertex contained in the block of G corresponding to v, or vice versa. It has been shown [29] that bc(G) is always a forest; it will be known as the bc-tree of G when G is connected.

Lemma 6.1 Let G be a graph and bc(G) be the block-cutvertex graph of G. Then

(1) Every end-vertex of bc(G) is a b-vertex.

(2) Two vertices in bc(G) have odd distance if and only if one is a b-vertex and the other is a c-vertex.

(3) If G is a block-complete graph, then for each pair of c-vertices having distance two in bc(G), the corresponding pair of vertices in G are adjacent.

Proof: (1) Observe that each end-vertex of bc(G) corresponds to an end-block of G.

(2) Let F = bc(G), and u and v be two vertices in F such that $d_F(u, v)$ is odd, say $d_F(u, v) = 2n + 1$. By the definition of block-complete graphs, each b-vertex in F is adjacent to c-vertices only and vice versa. Let P: $u = v_0, v_1, \ldots, v_{2n}, v_{2n+1} = v$ be a u-v path in F. If u is a b-vertex, then $v_1, v_3, \ldots, v_{2n+1}$ are c-vertices and $v_2, v_4, \ldots, v_{2n}$ are b-vertices.

This implies that v is a c-vertex. Similarly, if u is a c-vertex, then v is a b-vertex. Therefore, if two vertices in T have odd distance, then one is a b-vertex and the other is a c-vertex. By a similar proof, it can be shown that if two vertices in F have even distance, then either both vertices are b-vertices or both vertices are c-vertices.

(3) Let G be a block-complete graph, and u and v be two c-vertices in bc(G) having distance two. Then there exists a b-vertex w in bc(G) such that w is adjacent to both u and v. Let u′ and v′ be the vertices in G corresponding to u and v in bc(G) respectively and let B_w be the block in G corresponding to w in bc(G). Then u′ and v′ are vertices in B_w. Since G is a block-complete graph, u′ and v′ are adjacent in G. □

Block-Complete Augmentation

Instance: Graph $G = (V, E)$, positive integer $B \leq |V|^2 - |V|$.

Question: Is there a set E′ of unordered pairs of vertices from V such that $|E'| \leq B$ and the graph $G' = (V, E \cup E')$ is block-complete?

Theorem 6.2 The Block-Complete Augmentation problem can be solved in linear time.

Proof: Let G be a graph. The blocks of G can be found in linear time [32]. For each block which is not complete, add those edges which are not present in the block to obtain a complete block. Clearly, the number of edges to be added to G is minimum so that the resulting graph is block-complete. The complete process requires only linear time. □

The technique for designing Algorithm 6.1 and Algorithm 6.2 comes from Kariv and Hakimi [36].

We know that the vertex cover number for the complete graph K_n is $\alpha(K_n) = n - 1$. Algorithm 6.1 finds a minimum vertex cover U in an arbitrary connected block-complete graph G. The algorithm is carried out through a search on the blocks of G, starting from the end-blocks and moving toward the "middle." During this search, we locate the vertices of the desired vertex cover set in an "optimal fashion" until all edges in the graph are covered by some minimum set of vertices.

Algorithm 6.1 finds a minimum vertex cover U in a connected block-complete graph G.

begin

1) $G' \leftarrow G$; $U \leftarrow \emptyset$

2) while G' is not complete loop

3)   Find an end-block B of G'.

4)   Let v be the cut-vertex of B in G'.

5)   if $(|V(B) \cap U| < |V(B)| - 1)$ then

6)   Select a vertex, say u, in $V(B) - \{U \cup \{v\}\}$.

7) $U \leftarrow U \cup \{V(B) - \{u\}\}$

8)   end if

9) $G' \leftarrow G' - \{V(B) - \{v\}\}$

10) end loop

/* At this point, G' is a complete graph */

11) if $(|U \cap V(G')| < |V(G')| - 1)$ then

12) Find a set, say W, consisting of $|V(G')| - 1 - |U \cap V(G')|$ vertices in $V(G') - U$.

13) $U \leftarrow U \cup W$

14) end if

end Algorithm 6.1

Theorem 6.3 Algorithm 6.1 finds a minimum vertex cover U in a connected block-complete graph G.

Proof: If G is a complete graph of order n, then $|U| = |V(G)| - 1$ by steps 11 to 13. So we assume that G is not complete. An end-block B is found with cut vertex v.

If $|V(B) \cap U| \geq |V(B)| - 1$, then all edges in the block B are already covered by vertices in the vertex cover which have been located so far, so there is no need to select vertices of B into U. Otherwise, $|V(B)| - 1 - |V(B) \cap U|$ additional vertices of B must be added to U. By steps 11 to 13, the vertex that is not selected into U is not the cut-vertex v of B. This implies that the number of new vertices selected into U is required and the set of vertices selected is "optimal". □

Recall that Algorithm 2.1 finds a minimum dominating set of radius r in a weighted oriented tree and that Algorithm 2.1 may also be used to find a minimum dominating set of radius r in a weighted tree. Next we present a linear time algorithm for finding a minimum dominating set of radius r in an (unweighted) block-complete graph. The algorithm finds a minimum dominating set of radius r, U, in an arbitrary connected block-complete graph G. The process is carried out through a search on the

blocks of G, starting from the end-blocks and moving toward the "middle." During this search, we locate the vertices of the desired dominating set of radius r in an "optimal fashion" until all vertices in the graph are dominated by vertices in U within distance r. To do this, we use a copy G′ of the original graph as an auxiliary graph on which the algorithm is carried out, and we attach two variables C(v) and R(v) to each vertex v of G′. C(v) is a boolean variable which has value TRUE if v is already dominated (within distance r) by some vertex in U has value FALSE otherwise. (The interpretation of R(v) will be given later.) If B is an end-block of the auxiliary graph G′ and v_c is the cut-vertex of B in G′, then we update the variables $C(v_c)$ and $R(v_c)$ and remove all vertices in $V(B) - v_c$ (and incident edges) from G′. As a result, a new block may become an end-block of G′, and the process is repeated until the graph G′ becomes a complete graph. Then an appropriate set of vertices is added to U if necessary.

The variable R(v) has the following interpretation (based on C(v)):

Case 1: If the vertex v is already dominated by one of the vertices in U which have been located so far, then R(v) is the distance between v and the nearest located vertex in U.

Case 2: If the vertex is not yet dominated, then let S(v) be the set of all the vertices of the original graph G which are not yet dominated, and for which v is the nearest vertex in the auxiliary graph G′. Notice that v is the only vertex in S(v) which belongs to G′; in fact, S(v) is the set consisting of the vertex v

and all those vertices which have already been removed from G' and are to be dominated by the same vertex of the dominating set of radius r as v. $R(v) = \max \{d_G(u, v) \mid u \in S(v)\}$.

Algorithm 6.2 finds a minimum dominating set of radius r for a connected block-complete graph G.

begin

    /* initializations */

1) $G' \leftarrow G$; $U \leftarrow \emptyset$

    for each vertex v in G' loop

2) $C(v) \leftarrow$ FALSE; $R(v) \leftarrow 0$

    end loop

3)   while (G' is not complete) loop

4)   Find an end-block B in G'.

5)   Let v_c be the cut-vertex of B in G'.

        /* For convenience, define $\min \{R(v) \mid v \in \emptyset\} = r + 1$ and $\max \{R(v) \mid v \in \emptyset\} = 0$ */

        $R_t \leftarrow \min \{R(v) \mid v \in V(B)$ and $C(v) =$ TRUE$\}$

        $R_f \leftarrow \max \{R(v) \mid v \in V(B)$ and $C(v) =$ FALSE$\}$

        $R' \leftarrow \min \{R(v) \mid v(\neq v_c) \in V(B)$ and $C(v) =$ TRUE$\}$

        $R'' \leftarrow \max \{R(v) \mid v(\neq v_c) \in V(B)$ and $C(v) =$ FALSE$\}$

        /* Update $C(v_c)$ and $R(v_c)$ */

6)   if $C(v_c)$ then

            if ($R_t + R_f + 1 \le r$) then

                $R(v_c) \leftarrow \min \{R' + 1, R(v_c)\}$

            else

                $C(v_c) \leftarrow$ FALSE; $R(v_c) \leftarrow R'' + 1$

```
                end if
            else
                if (R_t + R_f + 1 ≤ r ) then
                    C(v_c) ← TRUE;  R(v_c) ← R' + 1
                else
                    R(v_c) ← max {R" + 1, R(v_c)}
                end if
 7)         end if
            /* Select vertex into U in an optimal way */
 8)         if (C(v_c) = FALSE) and (R(v_c) = r) then
                U ← U ∪ {v_c};  C(v_c) ← TRUE;  R(v_c) ← 0
 9)         end if
            /* R(v_c) = r+1 only if C(v_c) = TRUE */
10)         if (R(v_c) = r+1) then
                C(v_c) ← FALSE;  R(v_c) ← 0
11)         end if
        end if
        G' ← G' -{V(B) -{v_c}}
        end loop
12)     R_t ← min {R(v) | v ∈ V(G') and C(v) = TRUE}
        R_f ← max {R(v) | v ∈ V(G') and C(v) = FALSE}
        if (R_t + R_f ≥ r ) then
            Select a vertex u in G'
            U ← U ∪ {u}
13)     end if
end Algorithm 6.2
```

Theorem 6.4 Algorithm 6.2 finds a minimum dominating set of radius r in an (unweighted) connected block-complete graph G in linear time $O(|E|+|V|)$.

Proof: Steps 1 and 2 initialize G′, U, C(v), and R(v) for $v \in V(G')$. The interpretations for these symbols have already been given. If G is a complete graph, then steps 12 to 13 find a desired set U. Suppose that G is not complete. Steps 4 and 5 find a block in G′ with cut-vertex v_c. In is easy to verify that the if statement from steps 6 to 7 updates the two variables $C(v_c)$ and $R(v_c)$ correctly. Steps 8 to 9 select vertex into U in an "optimal fashion". The if statement from steps 10 to step 11 resets $C(v_c)$ and $R(v_c)$ since the vertex v_c is not yet dominated. Finally, steps 12 to 13 handle the resulting complete graph.

It can be seen that the time complexity of Algorithm 6.2 is $O(|E|+|V|)$. □

Algorithm 6.3

Given a block-complete graph G = (V, E) with m-centrix $C = \{c_1, c_2, \ldots, c_m\} \subseteq V$, and positive integer r. This algorithm finds a minimum m-centrix radius r augmentation of G.

begin

Step 1 Find the set $V_1 = \{v \in V \mid d_G(C, v) \le r\}$, $V_2 = V - V_1$ and $H = \langle V_2 \rangle_G$.

Step 2 If r = 1, $E' = \{c_1 v \mid v \in V_2\}$ is a desired set of edges. Go to step 5.

/* Let the components of H be $C_1, C_2, \ldots, C_k$. The blocks of C_i are complete, $1 \le i \le k$. */

Step 3 For each component C_i of H , use Algorithm 6.2 to find a minimum dominating set, U_i, of radius r in H.

Step 4 $U \leftarrow \cup \{U_i \mid 1 \le i \le k\}$

$E' = \{c_1 v \mid v \in U\}$ is a desired set of edges.

Step 5 Stop.

end Algorithm 6.3

By a proof similar to Theorem 2.11, we have

Theorem 6.5 Algorithm 6.3 finds a minimum m-centrix radius r augmentation of G. □

CHAPTER VII

SUMMARY

This dissertation has initiated the study of a number of graph and directed graph augmentation problems and and has provided solutions for most of the augmentation problems and related problems investigated.

In Chapter II, the study of m-Centrix Radius r Augmentation problem and the Digraph m-Centrix Radius r Augmentation problem was initiated. These two problems were shown to be NP-complete for any fixed positive integers m and r such that $r \geq 2$. Both problems can be solved in $O(m \cdot |V|)$ time for $r = 1$ and for trees (oriented trees). The related Graph and Digraph Dominating Set of Radius r problems were investigated and were proved to be NP-complete for any fixed r. A linear time algorithm was presented which can be used to find a minimum dominating set of radius r of a weighted tree or oriented tree.

Chapter III has explored the computational complexity of the following problems: r-Factorability, K Disjoint Maximum Matchings, and K Disjoint 1-Factors Augmentation. It was shown that these three problems are NP-complete. Results obtained include a linear time algorithm for deciding whether a forest contains K disjoint maximum matchings and for finding them if they exist, and an $O(n^{5/2})$ time algorithm which finds a

minimum K disjoint 1-factors augmentation of a forest of even order where $K \le \max\{5, 2n-1\}$.

Chapter IV has introduced the generalization of (M, N)-transitivity to (M, N, R_1, R_2)-transitivity for graphs and digraphs in order to better model redundancy constraints on networks. The study of (M, N, R_1, R_2)-transitivity has been extended through provision of a number of characterization results on tournaments as well as on graphs and digraphs. This chapter has provided an efficient minimum $(M, 1, R_1, =)$-transitive augmentation algorithms for graphs.

Chapter V has extended the definition of total dominating sets to (r, s)-dominating sets in graphs and digraphs. Various bounds on the (r, s)-domination number of a graph have been investigated. Computational complexity results concerning the (r, s)-Dominating Set and m-Centrix (r, s)-Domination Augmentation problems have been obtained in parallel to Chapter II.

Finally, Chapter VI has provided linear time algorithms for solving some graph problems for graphs whose blocks are complete.

BIBLIOGRAPHY

[1] G. Andreatta and F. M. Mason, Properties of the k-centra in a tree network. Networks 15 (1985) 21-25.

[2] T. Asano, An application of duality to edge-deletion problems. SIAM J. Comput. 16 (1987) 312-331.

[3] M. Behzad, G. Chartrand and L. Lesniak-Foster, Graphs & Digraphs. Wadsworth, Monterey (1979).

[4] A. J. Boals and K. L. Williams, Optimizing subcase solutions for (M, N)-transitivity. Proceedings - Sixth International Conf. on Mathematics Modeling, 1987.

[5] A. J. Boals, K. L. Williams, and Z. Mo, Notes on (M, N, R_1, R_2)-transitive directed graphs. Submitted for publication.

[6] F. T. Boesch, S. Chen and J. A. M. McHugh, On covering the points of a graph with point disjoint paths, presented at the Capitol Conf. on Graph Theory and Combinatorics, Washington, D.C., 1973.

[7] F. T. Boesch, C. Suffel and R. Tindell, The spanning subgraphs of eulerian graphs. J. Graph Theory 1 (1977) 79-84.

[8] F. Camion, Chemins et circuits hamiltoniens des graphes complets. C. R. Acad. Sci. Paris 249 (1959) 2151-2152.

[9] E. J. Cockayne, R. M. Dawes and S. T. Hedetniemi, Total domination in graphs. Networks 10 (1980) 211-219.

[10] E. J. Cockayne, B. L. Hartnell and S. T. Hedetniemi, A linear algorithm for disjoint matchings in trees. Discrete Math. 21 (1978) 129-136.

[11] D. G. Corneil, Y. Perl and L. K. Stewart, A linear recognition algorithm for cographs. SIAM J. Comput. 14 (1985) 926-934.

[12] J. Edmonds, Path, trees, and flowers. Canad. J. Math. 17 (1965) 449-467.

[13] J. Edmonds, Maximum matching and a polyhedron with 0, 1-vertices. J. Res. NBS (Math. and Math. Phys.) 69B (1965) 125-130.

[14] P. Erdős, S. Fajtlowicz, and A. J. Hoffman, Maximum degree in graphs of diameter 2. Networks 10 (1980) 87-90.

[15] K. Eswaran, Representation of graphs and minimally augmented Eulerian graphs with applications in data base management. Rep. RJ 1305, IBM Research, Yorktown Heights, N.Y., 1973.

[16] K. P. Eswaran and R. E. Tarjan, Augmentation problems. SIAM J. Comput. 5 (1976) 653-665.

[17] J. Folkman and D. R. Fulkerson, Edge colorings in bipartite graphs. Combinatorial Mathematics and its Applications. R. C. Bose and T. A. Dowling, Eds. University of North Carolina Press, Chapel Hill (1969) 561-577.

[18] G. N. Frederickson and J. Ja'Ja, Approximation algorithms for several graph augmentation problems. SIAM J. Comput. 10 (1981) 270-283.

[19] H. N. Gabow and E. W. Myers, Finding all spanning trees of directed and undirected graphs. SIAM J. Comput. 7 (1978) 280-287.

[20] Z. Galil, S. Micali and H. Gabows, An O(EVlogV) algorithm for finding a maximal weighted matching in general graphs. SIAM J. Comput. 15 (1986) 120-130.

[21] M. R. Garey and D. S. Johnson, The rectilinear Steiner tree problem is NP-complete. Networks. 32 (1977) 826-834.

[22] M. R. Garey and D. S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, W. H. Freeman, San Francisco (1979).

[23] M. C. Golumbic, Algorithmic Graph Theory and Perfect Graphs. Academic Press, New York (1983).

[24] S. Goodman and S. Hedetniemi, On the hamiltonian completion problem, presented at the Capitol Conf. on Graph Theory and Combinatorics, Washington, D.C., 1973.

[25] D. Gouyou-Beauchamps, The hamiltonian circuit problem is polynomial for 4-connected planar graphs. SIAM J. Comput. 11 (1982) 529-539.

[26] D. Gusfield, Optimal mixed graph augmentation. SIAM J. Comput. 16 (1987) 599-612.

[27] A. Gyárfás, M. S. Jacobson and L. F. Kinch, On a generalization of transitivity for digraphs. Discrete Math. To appear, 1987.

[28] P. Hall, On representation of subsets. J. London Math. Soc. 10 (1935) 26-30.

[29] F. Harary, Graph Theory. Addison-Wesley, Reading, Mass., 1969.

[30] I. Holyer, The NP-completeness of edge-coloring. SIAM J. Comput. 10 (1981) 718-720.

[31] J. E. Hopcroft and R. M. Karp, A $N^{5/2}$ algorithm for maximum matchings in bipartite graphs. Conference Record 1971 Twelfth Annual Symposium on Switching and Automata Theory, IEEE Computer Society, New York (1971).

[32] J. E. Hopcroft and R. E. Tarjan, Efficient algorithms for graph manipulation, Comm. ACM, 16 (1973) 372-378.

[33] M. Jeger and O. Kariv, Algorithms for finding P-centers on a weighted tree (for relatively small P). Networks 15 (1985) 381-389.

[34] D. S. Johnson, Approximation algorithms for combinatorial problems. J. Comput. System Sci. 9 (1974) 256-278

[35] Y. Kajitani and S. Ueno, The minimum augmentation of a directed tree to a k-edge-connected directed graph. Networks 16 (1986) 181-197.

[36] O. Kariv and S. L. Hakimi, An algorithmic approach to network location problems. I: The p-centers. SIAM J. Appl. Math. 37 (1979) 513-538.

[37] D. König, Über Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre. Math Ann. 77 (1916) 453-465.

[38] D. König, Graphen und Matrizen. Mat. Fiz. Lapok. 38 (1931) 116-119.

[39] E. Minieka, The m-center problem. SIAM Review 12 (1970) 138-139.

[40] E. Minieka, The optimal location of a path or tree in a tree network. Networks 15 (1985) 309-321.

[41] T. Ohtsuki, L. K. Cheung and T. Fujisawa, Minimal triangulation of a graph and optimal pivoting order in a sparse matrix. J. Math. Anal. Appl. 54 (1976) 622-633.

[42] T. Ohtsuki, H. Mori, T. Kashiwabara and T. Fujisawa, On minimal augmentation of a graph to obtain an interval graph. J. Comput. System Sci. 22 (1981) 60-97.

[43] J. Petersen, Die Theorie der regulären Graphs. Acta Math. 15 (1891) 193-220.

[44] D. J. Rose, Triangulated graphs and the elimination process. J. Math. Anal. Appl. 32 (1970) 597-609.

[45] D. J. Rose, R. E. Tarjan and G. S. Lueker, Algorithmic aspects of vertex elimination on graphs. SIAM J. Comput. 5 (1976) 266-283.

[46] A. Rosenthal and A. Goldner, Smallest augmentations to biconnect a graph. SIAM J. Comput. 6 (1977) 55-66.

[47] U. Schumacher, An algorithm for construction of a k-connected graph with minimum number of edges and quasiminimal diameter. Networks 14 (1984) 63-74.

[48] P. J. Slater, R-domination in graphs. J. Assoc. Comput. Mach. 23 (1976) 446-450.

[49] R. Tarjan, Depth-first search and linear graph algorithms. SIAM J. Comput. 1 (1972) 146-159.

[50] R. Tarjan, Finding optimum branchings. Networks 7 (1977) 25-35.

[51] V. G. Vizing, On an estimate of the chromatic class of a p-graph. Diskret. Analiz. 3 (1964) 25-30.

[52] K. L. Williams and A. J. Boals, On algorithms for (M, N)-transitivity. Currently under review for publication.

[53] S. Win, A sufficient condition for a graph to contain three disjoint 1-factors. J. Graph Theory 6 (1982) 489-492.

[54] C. Witzgall and C. T. Zahn, Jr., Modification of Edmonds' maximum matching algorithm. J. Res. NBS (Math. and Math. Phys.) 69B (1965) 91-98.