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Generalized Connectivity in Graphs

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The connectivity of a graph $G$ is the minimum number of vertices in $G$ whose deletion produces a disconnected or trivial graph, while the edge-connectivity of $G$ is the minimum number of edges having this property. In this dissertation several generalizations and variations of these two parameters are introduced and studied.

Chapter I is an overview to the history of connectivity and provides a background for the chapters that follow. In Chapter II major $n$-connected subgraphs are introduced. Through this concept, the connectivities (of subgraphs) that are most representative in a given graph are studied.

Chapter III is devoted to the study of determining the minimum cardinality of a vertex cutset $S$ of a graph $G$, with given induced subgraph $F$, such that either each component of $G - S$ is a subgraph of $F$ or each component is $F$-free. The corresponding edge versions of these concepts are also introduced and studied. Among the results presented are analogues of the well-known connectivity theorem of Whitney.

Two variations of connectivity are studied in Chapter IV. First, the $j$-connectivity ($j \geq 2$) of a graph $G$ is considered,
which is the minimum number of vertices which need to be deleted from $G$ to produce a disconnected graph with at least $\lambda$ components or a graph with fewer than $\lambda$ vertices. A graph is $(n, \lambda)$-connected if its $\lambda$-connectivity is at least $n$. Several sufficient conditions for a graph to be $(n, \lambda)$-connected are established. Second, the connected cutset connectivity of a graph $G$ is introduced and defined as the minimum cardinality of a vertex cutset $S$ of $G$ such that $S$ induces a connected subgraph. The connected edge-cutset connectivity is defined similarly. The connected edge-cutset connectivity is bounded below by the edge-connectivity and above by the minimum degree of a graph. As an analogue to a well-known problem in connectivity, a sufficient condition is presented for which the minimum degree equals the connected edge-cutset connectivity but does not necessarily equal the edge-connectivity.
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GENERALIZED CONNECTIVITY IN GRAPHS

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To the memory of

Siegrid, Landolf and Johannes
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CHAPTER I

An Introduction to Connectivity

Section 1.1 Connectivity and Edge-Connectivity

Perhaps the most important and basic property that a graph may possess is that of being connected. However, some graphs are "more connected" than others. The parameters "connectivity" and "edge-connectivity" were introduced in an attempt to measure the degree of connectedness in a graph.

The connectivity $\kappa(G)$ of a graph $G$ is the minimum number of vertices in $G$ whose deletion produces a disconnected or trivial graph, while the edge-connectivity $\kappa_1(G)$ of $G$ is the smallest number of edges whose removal produces a disconnected or trivial graph. (For basic terminology in graph theory, see [12].) These two global parameters, as well as many results dealing with connectivity, hinge on the 1927 theorem of Menger [44], which was discovered in the course of research he conducted on curve theory in point-set topology (see [45]). Let's begin by introducing some terminology. For a pair $u$, $v$ of distinct vertices in a graph $G$, let $\kappa(u, v)$ (respectively, $\kappa_1(u, v)$) denote the maximum number of internally disjoint (edge-disjoint) $u - v$ paths in $G$.

Theorem 1A (Menger's Theorem) If $u$ and $v$ are distinct non-adjacent vertices in a graph $G$, then $\kappa(u, v)$ equals the minimum
number of vertices of \( G \) that separate \( u \) and \( v \).

The edge analogue to Theorem 1A states that for distinct vertices \( u \) and \( v \) in a graph \( G \), \( \kappa_1(u, v) \) equals the smallest number of edges that separate \( u \) and \( v \).

For a nonnegative integer \( n \), a graph \( G \) is said to be \( n \)-connected if \( \kappa(G) \geq n \). Whitney [47] established a very useful characterization of \( n \)-connected graphs.

**Theorem 1B** A nontrivial graph \( G \) is \( n \)-connected if and only if \( \kappa(u, v) \geq n \) for each pair \( u, v \) of distinct vertices of \( G \).

Similarly, for a nonnegative integer \( n \), a graph \( G \) is said to be \( n \)-edge-connected if \( \kappa_1(G) \geq n \). An analogue to Theorem 1B states that a graph \( G \) is \( n \)-edge-connected if and only if \( \kappa_1(u, v) \geq n \) for every pair \( u, v \) of distinct vertices of \( G \).

A characterization of \( n \)-connected graphs of a different character was conjectured independently by A. Frank and S. Maurer (see [38]), and verified independently by Lovász [30] and Győri [22].

**Theorem 1C** Let \( G \) be a graph of order \( p \geq n + 1 \). Then \( G \) is \( n \)-connected if and only if for any distinct vertices \( v_1, v_2, \ldots, v_n \) of \( G \) and any partition of \( p \) into positive integers \( m_1, m_2, \ldots, m_n \), there is a partition \( V_1, V_2, \ldots, V_n \) of \( V(G) \) such that for every \( i \ (1 \leq i \leq n) \), \( v_i \in V_i \), \( |V_i| = m_i \) and \( \langle V_i \rangle \) is connected.
The well-known max-flow min-cut theorem, due to Ford and Fulkerson [16], may be regarded as a variation of Theorem 1A and states that in any network \( N \) the value of a maximum flow is equal to the capacity of a minimum cut.

A fundamental relationship involving the connectivity, edge-connectivity and minimum degree of a graph was given by Whitney [47].

**Theorem 1D** For any graph \( G \),

\[
\kappa(G) \leq \kappa_1(G) \leq \delta(G).
\]

Chartrand and Harary [8] showed that this result is best possible in the sense that if \( a, b \) and \( c \) are positive integers with \( a \leq b \leq c \), then there exists a graph \( G \) with \( \kappa(G) = a \), \( \kappa_1(G) = b \) and \( \delta(G) = c \).

Boesch and Suffel [3] defined a quadruple \((a, b, c, d)\) of nonnegative integers to be \((p, \Delta, \delta, \kappa_1)\) realizable if there is a graph of order \( a \), maximum degree \( b \), minimum degree \( c \) and edge-connectivity \( d \). They then established necessary and sufficient conditions for a quadruple of nonnegative integers to be \((p, \Delta, \delta, \kappa_1)\) realizable. In [4] the problem of characterizing those quadruples of nonnegative integers that are \((p, \Delta, \delta, \kappa)\) realizable (where \( \kappa \) denotes the connectivity parameter) was solved, while in [5] necessary and sufficient conditions were given.
for a triple of nonnegative integers to be \((p, q, r), (p, q, r_1)\) or \((p, q, \delta)\) realizable.

Conditions under which the rightmost inequality in Theorem 1D is an equality have been studied extensively. Probably the most useful of this type is due to Plesník [46], who showed that if the diameter of a graph \(G\) is at most 2, then \(r_1(G) = \delta(G)\). Other sufficient conditions for the edge-connectivity of a graph to equal its minimum degree have been given in [7], [19], [21] and [28].

Harary [24] determined an upper bound for the connectivity of any \((p, q)\) graph, while Esfahanian [14] provided lower bounds for the connectivity and edge-connectivity of a graph in terms of its order, minimum degree, maximum degree and diameter.

By Theorem 1D, \(r(G) \leq r_1(G) \leq \delta(G)\) for any graph \(G\). For every pair \(u, v\) of distinct vertices in a graph \(G\), we have the following analogous, easily verified inequalities

\[
r(u, v) \leq r_1(u, v) \leq \min \{\deg u, \deg v\}.
\]

One might suspect that every system of \(r_1(u, v)\) edge-disjoint \(u - v\) paths in \(G\) also contains \(r(u, v)\) internally disjoint \(u - v\) paths. However, Beineke and Harary [1] showed that this need not be the case. They referred to a pair \((k, \lambda)\) of non-negative integers as a connectivity pair for distinct vertices \(u\) and \(v\) of a graph \(G\) if there is a set of \(k\) vertices and \(\lambda\) edges whose removal separates \(u\) and \(v\), but no set of \(k - 1\)
vertices and \(l\) edges or \(k\) vertices and \(l - 1\) edges has this property. Similarly, a pair \((k, l)\) of nonnegative integers is a connectivity pair for a (noncomplete) graph \(G\) if there exist \(k\) vertices and \(l\) edges whose removal disconnects \(G\) but no set of \(k - 1\) vertices and \(l\) edges or \(k\) vertices and \(l - 1\) edges has this property. For each integer \(k\) \((0 \leq k \leq r(G))\), there is a unique connectivity pair \((k, l_k)\); thus \(G\) has exactly \(r(G) + 1\) connectivity pairs. This observation motivated Beineke and Harary to define the connectivity function \(f\) of a graph \(G\), where \(f\) maps the set \(\{0, 1, \ldots, r(G)\}\) into the nonnegative integers such that \(f(r(G)) = 0\). They further observed that the connectivity function of a graph \(G\) is strictly decreasing and verified that these conditions are also sufficient for a function to be the connectivity function of a graph. In addition, they generalized Theorem 1A to obtain the next result.

**Theorem 1E** If \((k, l)\) is a connectivity pair for distinct vertices \(u\) and \(v\) of a graph \(G\), then there are \(k + l\) edge-disjoint paths connecting \(u\) and \(v\), of which \(k\) are internally disjoint.

Mader [38], pointed out, however, that if \((k, l)\) is a connectivity pair for distinct vertices \(u\) and \(v\) in a graph \(G\), then there may be more than \(k + l\) edge-disjoint \(u - v\) paths, of which more than \(k\) are internally disjoint.
Even though for a pair $u, v$ of distinct vertices in a graph $\min\{\deg u, \deg v\}$ may exceed $r(u, v)$ by an arbitrarily large quantity, Mader [34] showed that every graph contains a pair $u, v$ of distinct vertices for which equality holds.

Section 1.2 Some Connectivity Concepts

In order to gain a better understanding into the structure of a graph, several variations of the concept of connectivity have been introduced and investigated.

Much research has revolved around the study of "minimally" and "critically" $n$-connected and $n$-edge-connected graphs. A graph $G$ is defined to be minimally $n$-connected if $r(G) \geq n$, and $r(G - e) < n$ for every edge $e$ of $G$. Halin [23] verified that every minimally $n$-connected graph contains a vertex of degree $n$. A stronger result was obtained by Mader [33] when he showed that every such graph contains at least $n + 1$ vertices of degree $n$. Furthermore, Mader proved that every cycle in a minimally $n$-connected graph $G$ contains a vertex of degree $n$, which implies that the deletion of the vertices of degree $n$ from a non-regular minimally $n$-connected graph $G$ produces a forest $F$. Moreover, it was shown in [37] that if $T$ is a component of $F$ and $v$ is a vertex of degree $n$ in a minimally $n$-connected graph $G$, then $v$ is adjacent to at most $n - 2$ vertices of $T$. 

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An n-edge-connected graph $G$ with the property that $\gamma_1(G - e) < n$ for every edge $e$ of $G$ is called a minimally n-edge-connected graph. Lick [29] showed that such a graph contains at least one vertex of degree $n$, while Mader [31] strengthened this result by proving that a minimally n-edge-connected graph has at least $n + 1$ vertices of degree $n$.

A graph $G$ is defined to be critically n-connected if $G$ is n-connected, but $\gamma(G - v) < n$ for every vertex $v$ of $G$. Chartrand, Kaugars and Lick [11] showed that every critically n-connected graph contains a vertex with degree at most $\lceil 3n/2 \rceil - 1$, and they further showed that this bound is attainable (see also [32]).

If a graph $G$ is n-edge-connected but every vertex-deleted subgraph has edge-connectivity less than $n$, then $G$ is said to be critically n-edge-connected. Mader [39] verified that such graphs $G$ contain at least two vertices of degree $n$.

Thus far, the main interest has been to determine a smallest set $S$ of vertices or of edges in a graph $G$ such that $G - S$ is disconnected or trivial, with no conditions imposed on the components or the number of components of $G - S$. In [25] Harary introduced the idea of "conditional connectivity", where for a graphical property $P$, he defined the $P$-connectivity of a graph $G$ as the minimum cardinality of a set $S \subseteq V(G)$ such that $G - S$ is disconnected with each of its components having property $P$. A
graph is said to be \( n-(P\text{-connected}) \) if its \( P \)-connectivity is at least \( n \). The concepts of \( P\text{-edge-connectivity} \) and \( n-(P\text{-edge-connected}) \) are defined analogously. Whereas the connectivity and edge-connectivity of a graph are always defined, the \( P \)-connectivity and \( P\text{-edge-connectivity} \) of a graph need not be defined. For example, if \( G \in K(n, n) \) and \( P \) is the property "2-connected", then the \( P \)-connectivity of \( G \) is not defined.

As mentioned earlier, the connectivity of a graph \( G \) gives no information on the number of components that are produced when deleting a vertex cutset of cardinality \( \kappa(G) \). For example, \( K(1, n) \) and \( P_{n+1} \) \((n \geq 3)\) are both trees of order \( n + 1 \) and therefore have connectivity 1 (in fact, these graphs are both minimally \( n \)-connected), but the deletion of a cut-vertex from \( K(1, n) \) produces \( n \) components, while the deletion of a cut-vertex from \( P_{n+1} \) produces only two components. These observations motivated the next generalization of the connectivity of a graph. For a graph \( G \) of order \( p \geq 2 \), the \( \lambda \)-connectivity \( \kappa_\lambda(G) \) of \( G \), \( 2 \leq \lambda \leq p \), is the least number of vertices whose removal from \( G \) results in a graph with at least \( \lambda \) components or a graph of order less than \( \lambda \). Hence if \( \lambda = 2 \), then the \( \lambda \)-connectivity of a graph coincides with its connectivity. Chartrand, Kapoor, Lesniak and Lick [10] defined the connectivity sequence of a graph \( G \) as the sequence

\[ S: \kappa_2(G), \kappa_3(G), \ldots, \kappa_p(G), \]
and showed that $S$ attains its maximum value, namely $p - \beta(G)$, at the $(\beta(G) + 1)$st term of $S$, where $\beta(G)$ denotes the independence number of $G$. Further, they characterized those sequences of positive integers that can be realized as the connectivity sequence of some graph.

The $\lambda$-edge-connectivity ($\lambda \geq 2$) of a graph $G$ is the least number of edges whose removal from $G$ produces a graph with at least $n$ components. This concept has been extensively studied in [2], [10], [17], [18] and [20].

Although, of course, the $\lambda$-connectivity is bounded above by the $\lambda$-edge-connectivity when $\lambda = 2$, the $\lambda$-connectivity is neither bounded above nor below by the $\lambda$-edge-connectivity when $\lambda \geq 3$.

Some concepts related to connectivity that have been introduced in the literature involve consideration of the connectivities of subgraphs of the given graph. We conclude this chapter by describing two of these concepts.

In [13] Chartrand and Pippert defined a graph to be **locally connected** if the subgraph induced by the neighborhood of every vertex of the graph is connected. Neither the property of being connected nor the property of being locally connected implies the other. For example, $K(n, n)$, for $n \geq 1$, is connected, in fact $n$-connected, but not locally connected. Conversely, $2K_{n+1}$, for $n \geq 2$, is a disconnected graph that is locally connected.
Chartrand and Pippert observed that every vertex $v$ of a locally connected graph belongs to at least $\deg v - 1$ triangles, so that a bipartite graph cannot be locally connected. Further, they defined a graph to be locally $n$-connected if the neighborhood of every vertex induces an $n$-connected graph. They then showed that every component of a locally $n$-connected graph, $n \geq 1$ is $(n + 1)$-connected, while for a graph $G$ of order $p$, the condition

$$\deg u + \deg v > \frac{4}{3} [p + (n - 3)/2]$$

for every pair $u, v$ of vertices of $G$ ($1 \leq n \leq p - 2$) implies that $G$ is locally $n$-connected.

Matula [43] defined the bondage $\delta(G)$ of a graph $G$ to be $\max\{\delta(H)|H \subseteq G\}$, while the maximum subgraph connectivity is defined by $\kappa(G) = \max\{\kappa(H)|H \subseteq G\}$ and the maximum subgraph edge-connectivity by $\kappa_1(G) = \max\{\kappa_1(H)|H \subseteq G\}$. Matula proved the following analogue to Theorem 1D.

Theorem 1F For any graph $G$,

$$\omega(G) - 1 \leq \kappa(G) \leq \kappa_1(G) \leq \delta(G).$$

These parameters have been studied further in [40], [41] and [42].
MAJOR \textit{n}-CONNECTED GRAPHS

Section 2.1. Introduction

Although the connectivity of a graph is considered a global parameter, it need not reveal much information about the structure of the graph or, in fact, say much about connectedness throughout the graph. Indeed, every subgraph of a graph with small connectivity may also have small connectivity, such as \( \overline{K_p} \), or a graph with small connectivity may contain subgraphs having large connectivity. For example, \( K_n \cup K_1 \) \((n \geq 2)\) has connectivity 0 but contains a subgraph with connectivity \( n - 1 \).

In 1978 Matula [43] defined the \textit{maximum subgraph connectivity}

\[ \varkappa(G) = \max \{ \kappa(H) | H \subseteq G \} . \]

This parameter then gives the largest connectivity of a subgraph of \( G \). It may happen, however, that a graph containing subgraphs with large connectivity also contains subgraphs of large order having small connectivity. Consider, for example, the graph \( G_n \) obtained by identifying an end-vertex of a path \( H_n \) of order \( n^2 \) with a vertex in the complete graph of order \( n + 1 \geq 2 \). Then
so that \( \lim_{n \to \infty} \frac{p(H_n)}{p(G_n)} = 1 \). However, \( \rho(G_n) = n \) and \( \kappa(H_n) = 1 \).

Thus, neither the connectivity nor the maximum subgraph connectivity gives a good indication, in general, of the structure of a graph. With these observations in mind, we introduce the major concept of this chapter.

**Section 2.2 Major n-Connected Graphs**

An induced subgraph \( H \) of connectivity \( n \) in a graph \( G \) is a **major n-connected subgraph** of \( G \) if \( H \) contains no subgraph with connectivity exceeding \( n \) and \( H \) has maximum order with respect to this property. An induced subgraph is a **major subgraph** if it is a major \( n \)-connected subgraph for some \( n \). Let \( m \) be the maximum order among all major subgraphs of \( G \). Then the **major connectivity set** of \( G \) is defined by

\[
K(G) = \{n|\text{there exists a major subgraph } H \subseteq G \text{ with } \kappa(H) = n \text{ and } p(H) = m}\.
\]

By definition, \( K(G) \neq \emptyset \).

To illustrate the above definitions, consider the graph \( G \) shown in Figure 2.1. For \( i = 0, 1, 2, 3 \), the subgraphs \( H_i \) defined below are major \( i \)-connected subgraphs. For \( i \geq 4 \),
The following lemma will prove to be a basic tool in verifying upcoming results. A graph \( G \) is called critically \( n \)-connected \((n \geq 1)\), if \( \kappa(G) = n \) and \( \kappa(H) < n \) for every proper induced subgraph \( H \) of \( G \).

**Lemma 2.1** If \( G \) is a graph with connectivity \( n \), then \( G \) contains a major \( n \)-connected subgraph.

**Proof.** Let \( H \) be an induced critically \( n \)-connected subgraph of minimum order in \( G \). Then \( H \) does not contain subgraphs whose connectivity exceeds \( n \). To see this, suppose that \( H' \prec H \) with \( \kappa(H') > n \). Then \( H' \) contains an induced critically \( n \)-connected
subgraph $H''$ with $p(H'') < p(H') < p(H)$. This contradicts our assumption that $H$ is a critically $n$-connected subgraph having minimum order.

Hence every graph with connectivity $n$ contains an induced subgraph having connectivity $n$ and such that each of its subgraphs has connectivity at most $n$. Consequently, every graph with connectivity $n$ contains a major $n$-connected subgraph.

The following corollary is now immediate.

**Corollary** A graph $G$ contains a major $k$-connected subgraph if and only if $0 \leq k \leq \hat{r}(G)$.

In the proof of Lemma 2.1, we showed that in an $n$-connected graph $G$, every critically $n$-connected subgraph of minimum order contains no (induced) subgraph with connectivity exceeding $n$.

We now describe a class of graphs that are critically $n$-connected but which contain (induced) subgraphs whose connectivity exceeds $n$. Let $m, n \geq 2$ be positive integers. Define $H_0 = K_{mn}$ and $H_1 \cong H_2 \cong \cdots \cong H_m \cong K_n$, with $V(H_0) = \bigcup_{i=1}^{m} \{u_{ij} \mid 1 \leq j \leq n\}$ and $V(H_k) = \{v_{kj} \mid 1 \leq j \leq n\}$ for $1 \leq k \leq m$. Let $G$ be obtained from $H_0 \cup H_1 \cup \cdots \cup H_k$ by adding the edges in $\bigcup_{i=1}^{m} \{u_{ij} \mid 1 \leq j \leq n\}$.
We show first that \(\kappa(G) = n\). Since \(\delta(G) = n\), it follows that \(\kappa(G) \leq n\). Let \(S\) be a set of \(n - 1\) vertices of \(G\). Then \(V(H_i) - S \neq \emptyset\) and \(H_i - S\) is connected for \(0 \leq i \leq n\).

Further, for every \(k\) \((1 \leq k \leq m)\), there exists at least one \(i\) \((1 \leq i \leq n)\) such that \(u_{ik}, v_{ik} \notin S\). Hence \(G - S\) is connected, so that \(\kappa(G) \geq n\). Consequently, \(\kappa(G) = n\).

It remains to be shown for every vertex \(v\) of \(G\) that \(\kappa(G - v) = n - 1\). Clearly, \(\kappa(G - v) \geq n - 1\) for every vertex \(v\) of \(G\). However, \(\delta(G - v) = n - 1\) for all vertices \(v\) of \(G\) so that \(\kappa(G - v) \leq n - 1\). Hence \(\kappa(G - v) = n - 1\) for every vertex \(v\) of \(G\). However, \(G\) contains an induced subgraph (namely \(H_0\)) having connectivity \(mn - 1\).

Clearly, if \(k \in \kappa(G)\), then \(k \leq \kappa(G)\). In general, the connectivity of a graph need not be an upper or lower bound for the elements belonging to the major connectivity set. The following two lemmas aid us in proving this result.

**Lemma 2.2** Let \(n_1\) and \(n_2\) be positive integers. If \(H\) is a graph having connectivity \(k\) and order at least \(n_1 + n_2 + k\), then \(H + (K_{n_1} \cup K_{n_2})\) has connectivity \(n_1 + n_2 + k\).

**Proof.** Let \(H_1 \cong K_{n_1}\) and \(H_2 \cong K_{n_2}\) and \(G = H + (H_1 \cup H_2)\).

Since \(H\) has order \(n_1 + n_2 + k\) and connectivity \(k\), there is a set \(S\) of \(k\) vertices in \(V(H)\) such that \(H - S\) is disconnected. Hence
\[ G - (V(H_1) \cup V(H_2) \cup S) = H - S \]

is disconnected, so that \( \kappa(G) \leq n_1 + n_2 + k \).

Suppose now that \( S' \) is a set of at most \( n_1 + n_2 + k - 1 \) vertices. Then \( V(H) - S' \neq \emptyset \). If \( V(H_1) \cup V(H_2) \subseteq S' \), then \( S' \) contains at most \( k - 1 \) vertices of \( H \). Therefore, \( G - S' = H - S' \) is connected. Suppose that \( (V(H_1) \cup V(H_2)) - S' \neq \emptyset \).

Because \( V(H) - S' \neq \emptyset \) and since every vertex in \( (V(H_1) \cup V(H_2)) - S' \) is joined to every vertex in \( V(H) - S' \), the graph \( G - S' \) is connected. Hence \( \kappa(G) > n_1 + n_2 + k - 1 \).

Therefore, \( \kappa(G) = n_1 + n_2 + k \).

Jolivet [27] showed that if \( G \) is a connected graph of order at least \( k + 1 \), then the \( k \)th power \( G^k \) of \( G \) is \( k \)-connected. This result then implies our next lemma.

**Lemma 2.3** If \( T \) is a tree of order at least \( k + 1 \), then the \( k \)th power \( T^k \) of \( T \) is \( k \)-connected.

We are now prepared to present our next theorem.

**Theorem 2.4** For every pair \( n, k \) of positive integers, there is a graph \( G \) with \( \kappa(G) = n \) and \( K(G) = \{k\} \).

**Proof.** If \( n = k \), then \( G = K_{n+1} \) has the desired properties.

To prove the remainder of the theorem, we consider two cases.
Case 1. Assume that $0 < n < k$.

Let $H_1 \subseteq H_2 \subseteq K_{k+1}$, and let $V(H_1) = \{v_1, v_2, \ldots, v_{k+1}\}$ and $V(H_2) = \{u_1, u_2, \ldots, u_{k+1}\}$. Define $H$ to be the graph obtained from $H_1 \cup H_2$ by adding the edges in $\{u_i v_i | 1 \leq i \leq k\}$. Let $G$ now be obtained from $H$ by joining a new vertex $v$ to $u_1, u_2, \ldots, u_n$. Since $\delta(G) = n$, the connectivity of $G$ is at most $n$. Let $S$ be a set of $n-1$ vertices of $G$. Then both $V(H_1) - S$ and $V(H_2) - S$ are nonempty. Further, $H_1 - S$ is connected ($i = 1, 2$) and there is at least one $j (1 \leq j \leq k)$ such that $u_j, v_j \notin S$. Consequently, $H - S$ is connected.

Because $V(H) - S$ contains at least one vertex that is adjacent to $v$, we may conclude that $G - S$ is connected. Therefore $\kappa(G) \geq n$, so that $\kappa(G) = n$.

We show next that $\delta(G) = k$. Clearly, if a subgraph of $G$ contains $v$, then the connectivity of this subgraph is at most $n \leq k$. Assume thus that $G'$ is a subgraph of $G$ that does not contain $v$. If $G'$ is contained in either $H_1$ or $H_2$, then $\delta(G') \leq k$. Suppose, therefore, that $G'$ contains vertices of both $H_1$ and $H_2$. If $v_{k+1}$ or $u_{k+1}$ belongs to $G'$, then $\delta(G') \leq k$, so that $\kappa(G') \leq k$. However, if neither $v_{k+1}$ nor $u_{k+1}$ belongs to $G'$, then $\delta(G') \leq k$, implying that $\kappa(G') \leq k$. In particular, we may conclude that $\delta(G) \leq k$ and $\kappa(H) \leq k$. We show now that $\kappa(H) \geq k$. Let $S$ be a set of $k-1$ vertices of $H$. Then both $V(H_1) - S$ and $V(H_2) - S$ are nonempty and $H_1 - S$ is
connected \((i = 1, 2)\). Further, there exists at least one 
\(j (1 \leq j \leq k)\) such that \(v_j, u_j \notin S\). Hence \(H - S\) is con-
nected, implying that \(\kappa(H) \geq k\). Therefore \(\kappa(H) = k\).

Since every subgraph of \(H\) has minimum degree at most \(k\),
it follows that every subgraph of \(H\) has connectivity at most \(k\).
Consequently, every major \(k\)-connected subgraph of \(G\) has order at
least \(p(H) = p(G) - 1\). However, since \(\kappa(G) = n < k\), every
major \(k\)-connected subgraph of \(G\) has order less than \(p(G)\).
Hence every major \(k\)-connected subgraph of \(G\) has order \(p(G) - 1\).

To show that \(\Delta(G) = \{k\}\), we need only verify that every
major \(m\)-connected subgraph of \(G\), where \(0 \leq m < k\), has order
less than \(p(G) - 1\). Since \(\kappa(H_i) = k\) for \(i = 1, 2\), it follows
that a major \(m\)-connected subgraph of \(G (0 \leq m < k)\) contains at
most \(p(H_i) - 1\) vertices of \(H_i (i = 1, 2)\). Hence a major \(m\)-con-
nected subgraph of \(G (0 \leq m < k)\) has order at most \(p(G) - 2\).

**Case 2. Assume that** \(0 < k < n\).

Let \(m = (2n - 2k + 1)(2n - k + 1) + 1\) and let
\(P_m : v_1, v_2, \ldots, v_m\) be a path of order \(m\). Suppose that \(H\) is
the \(k\)th power of \(P_m\). Then, by Lemma 2.3, \(H\) is \(k\)-connected.
Further, every induced subgraph of \(H\) has connectivity at most
\(k\) because such a subgraph always contains a vertex having degree
at most \(k\).

Let
\[
\lambda = (2n - 2k + 1)(2n - k + 1) - n + k.
\]

Next let \( H_1 \) and \( H_2 \) be two copies of \( K_{n-k} \) and let \( G \) be obtained from \( H, H_1 \) and \( H_2 \) by joining every vertex in \( H_1 \) to \( v_1, v_2, \ldots, v_k \) (that is, join every vertex in \( H_1 \) to every vertex in \( H \) that belongs to the path \( P_m \) except to the last \( n - k + 1 \) vertices of this path) and then join every vertex in \( H_2 \) to the vertices \( v_{n-k+2}, v_{n-k+3}, \ldots, v_m \).

We show first that \( r(G) = n \). Because \( \deg_G v_1 = n \), it follows that \( r(G) \leq n \). Let \( S \) be a set of at most \( n - 1 \) vertices of \( G \). If \( V(H_1) \cup V(H_2) \subseteq S \), then \( n - 1 \geq 2n - 2k \) and \( S \) contains at most \( 2k - n - 1 (< k) \) vertices of \( H \).

Since \( H \) is \( k \)-connected and \( G - S = H - S \), it follows that \( G - S \) is connected. Suppose now that \( V(H_1) \subseteq S \) and \( V(H_2) \not\subseteq S \). Then \( S \) contains at most \( k - 1 \) vertices of \( H \). Therefore, \( H - S \) is connected. Further, \( H - S \) contains at least

\[
(2n - 2k + 1)(2n - k + 1) - n + 1 > 0
\]

vertices that are adjacent to every vertex in \( V(H_2) - S \). Hence \( G - S \) is connected. We may now assume that \( V(H_1) - S \neq 0 \) and \( V(H_2) - S \neq 0 \). Then at least

\[
(2n - 2k + 1)(2n - k) - n + 1 > 0
\]

vertices of the set \( S' = \{v_{n-k+2}, v_{n-k+3}, \ldots, v_k\} \) do not belong
to $S$. If $v \in S' - S$, then every vertex in $(V(H_1) \cup V(H_2)) - S$ is joined to $v$. Furthermore, every vertex in $V(H) - S$ is joined to at least one vertex in $(V(H_1) \cup V(H_2)) - S$, so that $G - S$ is connected. Hence $\kappa(G) > n - 1$ and therefore $\kappa(G) = n$.

By Lemma 2.2, the subgraph of $G$ induced by $V(H_1) \cup V(H_2) \cup S'$ has connectivity $2n - k$. Since every induced subgraph of $G$ contains a vertex which has degree at most $2n - k$ (in that subgraph), it follows that $\rho(G) = 2n - k$.

We show next that every major $k$-connected subgraph of $G$ has order $p(G) - 2n + 2k$. Since $H$ is a $k$-connected subgraph of $G$ (by Lemma 2.3) and since all subgraphs of $H$ have connectivity at most $k$, it follows that every major $k$-connected subgraph has order at least $p(H) = p(G) - 2n + 2k$. Suppose that $H^*$ is a major $k$-connected subgraph such that $V(H^*) \cap (V(H_1) \cup V(H_2)) \neq \emptyset$. We may assume that there is a vertex $v \in V(H^*) \cap V(H_1)$. Since $H^*$ does not contain a $(k+1)$-connected subgraph, it follows that $H^*$ does not contain more than $k$ consecutive vertices in the sequence $v_1', v_2', \ldots, v_k'$. But then $V(G) - V(H^*)$ contains at least

$$\left\lfloor \frac{k}{k+1} \right\rfloor = \left\lfloor \frac{(2n - 2k + 1)(2n - k + 1) - n + k}{k + 1} \right\rfloor$$

$$= \left\lfloor \frac{2(n - k)(2n - k + 1) + n + 1}{k + 1} \right\rfloor > \left\lfloor \frac{2(n - k)n}{k + 1} \right\rfloor \geq 2(n - k)$$

vertices of $\{v_1, v_2, \ldots, v_k\}$, so that $H$ contains fewer
than \( p(G) - 2(n - k) \) vertices, which is not possible. Hence \( H \) is the unique major \( k \)-connected subgraph of \( G \) and its order is \( p(G) - 2n + 2k \).

We show next that if \( 0 \leq t \leq \overline{\gamma}(G) \) and if \( t \neq k \), then every major \( t \)-connected subgraph of \( G \) has order less than \( p(G) - 2n + 2k \). If \( t > n \), then a major \( t \)-connected subgraph \( H_t \) cannot contain the vertices \( v_1, v_2, \ldots, v_{n-k+1} \); for otherwise, if \( i \) is the smallest integer with \( 1 \leq i \leq n - k + 1 \) such that \( v_i \in V(H_t) \), then \( v_i \) has degree at most \( n \) in \( H_t \), which is not possible. Similarly, \( v_{k}, v_{k+1}, \ldots, v_{m} \) do not belong to \( V(H_t) \). Hence \( p(H_t) \leq p(G) - 2n + 2k - 2 \).

Suppose now that \( k < t \leq n \) and let \( H_t \) be an induced subgraph of \( G \) having connectivity \( t \) and which contains no subgraph with connectivity exceeding \( t \). Let

\[
S'' = \{v_1, v_2, \ldots, v_{n-k+1}\} \cup \{v_k, v_{k+1}, \ldots, v_m\}.
\]

If \( V(H_t) \cap S'' = \emptyset \) then \( p(H_t) \leq p(G) - 2n + 2k - 2 \). Suppose now that \( V(H_t) \cap S'' \neq \emptyset \) and that \( V(H_t) \) contains \( n_1 \) vertices of \( V(H_i) \), \( i = 1, 2 \). If \( n_1 = 0 \), then \( H_t \) does not contain any vertex of \( \{v_1, v_2, \ldots, v_{n-k+1}\} \); otherwise, \( H_t \) contains a vertex with degree less than \( t \), which is not possible because \( \gamma(H_t) = t \). However, then, \( p(H_t) \leq p(G) - 2n + 2k - 1 \). Similarly, if \( n_2 = 0 \), then \( p(H_t) \leq p(G) - 2n + 2k - 1 \). Assume thus that \( n_1 > 0 \) and \( n_2 > 0 \). Since \( \delta(H_t) \geq t \) and
\( V(H_t) \cap S'' \neq \emptyset \), it follows that \( n_1 + k \geq t \) or \( n_2 + k \geq t \), so that \( n_1 + n_2 + k > t \). From the fact that \( H_t \) contains no subgraphs having connectivity greater than \( t \), we may deduce, by Lemma 2.2, that \( V(H_t) \) contains no more than \( n_1 + n_2 + k - 1 \) consecutive vertices from the sequence \( v_{n-k+2}, v_{n-k+3}, \ldots, v_k \).

Since \( n_1 + n_2 + k \leq 2n - k \), the set \( V(G) - V(H_t) \) contains at least

\[
\left\lfloor \frac{(2n - k)(2n - 2k + 1)}{2n - k} \right\rfloor = 2n - 2k + 1
\]

vertices, that is, \( p(H_t) \leq p(G) - 2n + 2k - 1 \). Consequently, for \( k < t \leq n \), a major \( t \)-connected subgraph of \( G \) has order at most \( p(G) - 2n + 2k - 1 \).

Finally, suppose that \( 0 \leq t < k \). Let

\[
G' = \langle V(H_1) \cup V(H_2) \cup \{v_{n-k+2}, v_{n-k+3}, \ldots, v_k\} \rangle.
\]

We have already observed that \( \kappa(G') = 2n - k \). Therefore, if \( H_t \) is a major \( t \)-connected subgraph, then at least \( 2n - 2k + 1 \) vertices of \( G' \) do not belong to \( H_t \), that is, \( p(H_t) \leq p(G) - 2n + 2k - 1 \). Hence every major \( t \)-connected subgraph of \( G \), \( 0 \leq t < k \), contains at most \( p(G) - 2n + 2k - 1 \) vertices.

Therefore \( K(G) = \{k\} \).
Let $k$ be a fixed integer, and let $n$ be an arbitrary integer exceeding $k$. The second part of the proof of the preceding theorem shows that there is a graph $G$ having connectivity $n$ and order $(2n - 2k + 1)(2n - k + 2)$ containing a major $k$-connected subgraph $H$ of order $(2n - 2k + 1)(2n - k + 1) + 1$. Since $\lim_{n \to \infty} \frac{p(H)}{p(G)} = 1$, it follows that for fixed $k$ and given $\epsilon$ ($0 < \epsilon < 1$), there is a positive integer $N = N(\epsilon)$ such that if $n \geq \max\{N, k\}$, then there is a graph $G$ with $r(G) = n$ and a major $k$-connected subgraph $H$ of $G$ with $1 - \epsilon < \frac{p(H)}{p(G)} < 1$.

In the preceding discussion, we focused our attention on graphs whose major connectivity sets consist of a single element. The example of Figure 2.1 shows that the major connectivity set of a graph may contain more than one element. Observe that if $K(G) = \{n_1, n_2, \ldots, n_k\}$ is the major connectivity set of some graph, where $n_i < n_{i+1}$ ($1 \leq i \leq k - 1$), then the order of every major $n_i$-connected subgraph is at least $n_k + 1$. The next result shows that every set $S$ of positive integers is the major connectivity set of some graph $G$. In addition, by specifying a positive integer $m$ exceeding every element of $S$, one can choose $G$ in such a way that every major $k$-connected subgraph of $G$, where $k \in K(G)$, has order $m$.

**Theorem 2.5** Let $S = \{n_1, n_2, \ldots, n_k\}$ be a set of positive integers with $n_i < n_{i+1}$ ($1 \leq i \leq k - 1$). If $m \geq n_k + 1$ is a
positive integer, then there is a graph $G$ with $K(G) = S$ and
such that every major $n_i$-connected subgraph $(1 \leq i \leq k)$ of $G$
has order $m$. Further, the minimum order of such a graph $G$ is
$m + n_k - n_i$.

Proof. Since the result is obvious for $k = 1$, we assume $k \geq 2$.
Let $G_1 \cong K_{n_k+1}$ and $G_2 \cong \overline{K_{m-n_i-1}}$, where $V(G_1) =$
$\{v_1, v_2, \ldots, v_{n_k+1}\}$ and $V(G_2) = \{u_1, u_2, \ldots, u_{m-n_i-1}\}$.
Set $H = G_1 \cup G_2$. Let $G$ be that graph obtained from $H$ by
joining every vertex in $\{u_1, u_2, \ldots, u_{m-n_i-1}\}$ to every vertex
in $\{v_1, v_2, \ldots, v_{n_k}\}$, for $1 \leq i \leq k - 1$, and, in addition,
if $m > n_k + 1$, by joining every vertex of $\{u_1, u_2, \ldots, u_{m-n_k-1}\}$
to every vertex in $\{v_1, v_2, \ldots, v_{n_k}\}$.

First we show that every major $n_i$-connected subgraph,
$1 \leq i \leq m$, has order $m$. For $i = 1, 2, \ldots, k$, let

$$H_i = \left\langle \{v_1, v_2, \ldots, v_{n_i}\} \cup \{v_{n_k+1}\} \cup \{u_1, u_2, \ldots, u_{m-n_i-1}\} \right\rangle.$$  

Then $p(H_i) = m$. Because $H_i - \{v_1, v_2, \ldots, v_{n_i}\}$ is disconnected
for $1 \leq i \leq k - 1$ and since $H_k - \{v_1, v_2, \ldots, v_{n_k}\}$ is
either trivial if $m = n_k + 1$ or disconnected if $m > n_k + 1$,
we conclude that $r(H_i) \leq n_i$, $1 \leq i \leq k$. However, if $U$ is
a set of at most $n_i - 1$ vertices of $H_i$, then $V(H_i) - U$
contains at least one vertex of \( \{v_1, v_2, \ldots, v_{n_i}\} \). Let 
\( v \in \{v_1, v_2, \ldots, v_{n_i}\} - U \). Then \( v \) is joined to every vertex
in \( V(H^i) \), other than itself, implying that \( H^i - U \) is connected.
Further, \( r(H^i - U) \geq 2 \), so that \( r(H^i) \geq n_i \); hence \( r(H^i) = n_i \).
Let \( H_0 \) be an induced subgraph of \( H^i \). Then

\[ H_0 \cong \tilde{K}_h \ (1 \leq h \leq m - n_i) \], \( H_0 \cong K_h \ (1 \leq h \leq n_i + 1) \) or

\[ H_0 \cong K_r + K_s \], where \( 1 \leq r \leq n_i \) and \( 1 \leq s \leq m - n_i \). Consequently, \( H^i \ (1 \leq i \leq k) \) contains no subgraphs with connectivity exceeding \( n_i \). Hence every major \( n_i \)-connected subgraph has
order at least \( m \).

Next we show that every major \( n_i \)-connected subgraph has order
at most \( m \). A major \( n_i \)-connected subgraph contains at most
\( n_i + 1 \) vertices that belong to \( V(G) \) because a major \( n_i \)-connected subgraph does not contain subgraphs whose connectivity exceeds \( n_i \). Further, a major \( n_i \)-connected subgraph, \( 2 \leq i \leq k \), does not contain vertices from \( \{u_{m-n_i}, u_{m-n_i+1}, \ldots, u_{m-n_i-1}\} \)
because these vertices have degree at most \( n_i-1 < n_i \) in \( G \) and thus in every subgraph of \( G \). Hence, every major \( n_i \)-connected subgraph has order at most \( m \), so that every major \( n_i \)-connected subgraph has order \( m \).

Since \( \bar{r}(G) = n_k \), the graph \( G \) does not contain major
t-connected subgraphs for \( t > n_k \). It remains to be shown that if
\( 0 \leq t < n_k \) and \( t \notin S \), then every major \( t \)-connected subgraph

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of $G$ has order less than $m$. If $t < n_1$, then every major $t$-connected subgraph of $G$ contains at most $t + 1(\leq n_1)$ vertices of $V(G_1)$, so that a major $t$-connected subgraph has order at most $|V(G_2)| + t + 1 = m - n_1 + t < m$. Suppose now that $n_i < t < n_{i+1}$ for some $i \in \{1, 2, \ldots, k - 1\}$. Then every major $t$-connected subgraph contains at most $t + 1$ vertices of $G$. Since the vertices in $\{u_{m-n_{i+1}}^{m-n_{i+1}+1}, \ldots, u_{m-n_i}^{m-n_i-1}\}$ have degree at most $n_i(< t)$ in $G$ (and therefore in every induced subgraph of $G$), these vertices do not belong to any major $t$-connected subgraph. Hence a major $t$-connected subgraph contains at most $m - n_{i+1} - 1$ vertices of $G_2$ and thus at most $m - n_{i+1} + t (\leq m)$ vertices of $G$. Therefore, $K(G) = \{n_1, n_2, \ldots, n_k\}$ and every major $n_i$-connected subgraph of $G$ has order $m$ ($1 \leq i \leq k$).

Since $p(G) = m + n_k - n_1$, it follows that the minimum order of a graph having $S$ as its major connectivity set and for which every major $n_i$-connected subgraph ($1 \leq i \leq k$) has order $m$ is at most $m + n_k - n_1$. Suppose now that $H$ is any graph with $K(H) = S$ such that every major $n_i$-connected subgraph ($1 \leq i \leq k$) has order $m$. Then a major $n_i$-connected subgraph of $H$ contains at most $m - (n_k - n_1)$ vertices of any major $n_k$-connected subgraph. However, then, $H$ contains at least $m + n_k - n_1$ vertices. Hence $p(H) = m + n_k - n_1$.  \[ \square \]
If, in the preceding theorem, we allow $0 \in S$, so that $n_1 = 0$, then a minor modification of the proof yields the same conclusion. Consequently, if $G$ is a graph with $K(G) = \{n_1, n_2, \ldots, n_k\}$, where $0 \leq n_1 < n_2 < \cdots < n_k$, every major $n_i$-connected subgraph $(1 \leq i \leq k)$ of which has order $m$, then $p(G) \geq m + n_k - n_1$. However, the order of such a graph $G$ cannot be arbitrarily large. Clearly if $m = 1$, then $K(G) = \{0\}$, implying that $p(G) = 1$. For $m \geq 2$, we present the following result.

**Theorem 2.6** Let $G$ be a graph of order $p$, every major $n$-connected subgraph of which has order $m \geq 2$ for $n \in K(G)$. Then

$$p \leq \frac{m + 1 - m}{m - 1}.$$

**Proof.** Observe first that

$$\frac{m + 1 - m}{m - 1} = m + m^{-1} + \cdots + m.$$

Let $G$ be a graph satisfying the hypothesis of the theorem. If $H$ is a subgraph of $G$ having connectivity $k$, then $k \leq m - 1$; otherwise, $G$ contains a subgraph $H$ with connectivity $k \geq m$. However, then, by Lemma 2.1, $H$ and therefore $G$ contains a major $k$-connected subgraph of order at least $k + 1 \geq m + 1$, which is not possible. In particular, this implies that $v(G) \leq m - 1$. Assume, to the contrary, that

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\[
p \geq m^m + m^{m-1} + \ldots + m.
\]

Suppose \( S_0 \subseteq V(G) \), where \(|S_0| = r(G)\) and \( G - S_0 \) is disconnected. Then, by a preceding observation, \(|S_0| \leq m - 1\).

If \( G - S_0 \) contains at least \( m + 1 \) components, then \( \beta(G) \geq m + 1 \), which is not possible since every major \( \Omega \)-connected subgraph of \( G \) has order at most \( m \). Further, \( p > m \) so that \( G \) is not complete. Hence \( G - S_0 \) has at least two but at most \( m \) components. Therefore, \( G - S_0 \) contains at least one component \( G_1 \) with order at least

\[
p - \frac{(m-1)}{m} > \frac{p - m}{m} \geq m^{m-1} + m^{m-2} + \ldots + m. \tag{2.1}
\]

Let \( v_1 \in V(G - S_0) - V(G_1) \). Then \( v_1 \) is not adjacent to any vertex of \( G_1 \).

Since \( m \geq 2 \), it follows from (2.1) that \( p(G_1) > m \).

Further, because \( G_1 \subseteq G \), the connectivity of \( G_1 \) is at most \( m - 1 \). Let \( S_1 \subseteq V(G_1) \), where \(|S_1| = r(G_1)\) and \( G_1 - S_1 \) is disconnected. As in the case of \( G - S_0 \), the graph \( G_1 - S_1 \) has at most \( m \) components. Therefore, \( G_1 - S_1 \) contains a component \( G_2 \) having order at least

\[
\frac{p(G_1) - (m - 1)}{m} > \frac{p(G_1) - m}{m} \geq m^{m-2} + m^{m-3} + \ldots + m.
\]

Let \( v_2 \in V(G_1 - S_1) - V(G_2) \). Then \( \{v_1, v_2\} \) is an independent
set of vertices disjoint from $V(G_2)$ and such that no vertex in $G_2$ is adjacent to a vertex in $\{v_1, v_2\}$.

Continuing in this fashion for $k$ steps, where $k < m$, we obtain a sequence $G_1, G_2, \ldots, G_k$ of subgraphs of $G$, where

$$p(G_i) > m^{m-1} + m^{m-2} + \cdots + m,$$

$1 \leq i \leq k$, and a set $\{v_1, v_2, \ldots, v_k\}$ of independent vertices of $G$ disjoint from $V(G_k)$ such that no vertex of $G_k$ is adjacent to a vertex in $\{v_1, v_2, \ldots, v_k\}$. In particular, if $k = m - 1$, then $\{v_1, v_2, \ldots, v_{m-1}\}$ is an independent set of vertices disjoint from $V(G_{m-1})$ and such that no vertex in $G_{m-1}$ is adjacent to any vertex in $\{v_1, v_2, \ldots, v_{m-1}\}$. Since $p(G_{m-1}) > m$ and $\kappa(G_{m-1}) \leq m - 1$, it follows that $G_{m-1}$ is not complete and therefore contains a pair $v_m, v_{m+1}$ of nonadjacent vertices. Then $\{v_1, v_2, \ldots, v_{m+1}\}$ is an independent set of vertices of $G$, implying that $\beta(G) \geq m + 1$. However, this contradicts the hypothesis that every major $O$-connected subgraph of $G$ has order less than $m$.

Section 2.3 Major $n$-Edge-Connected Graphs

In this section we consider the edge analogue of major $n$-connected graphs. An induced subgraph $H$ of a graph $G$ with $\kappa_1(H) = n$ is a major $n$-edge-connected subgraph of $G$ if
H contains no (induced) subgraph with edge-connectivity exceeding n and H has maximum order with respect to this property. An induced subgraph is a major edge-subgraph if it is a major n-edge-connected subgraph for some n. Let m be the maximum order among all major edge-subgraphs of G. Then the major edge-connectivity set of G is defined by

\[ K_1(G) = \{ n | \text{there exists a major edge-subgraph } H < G \text{ with } \kappa_1(H) = n \text{ and } p(H) = m \} \].

To illustrate these definitions, consider the graph G of Figure 2.1. For \( i = 0, 1, 2, 3 \), the subgraphs \( H_i \) defined in Figure 2.1 are major \( i \)-edge-connected subgraphs. For \( i \geq 4 \), G contains no major \( i \)-edge-connected subgraphs. We observe that \( K_1(G) = K(G) = \{ 1, 2, 3 \} \).

Mader [39] defined a graph G to be critically n-edge-connected (\( n \geq 1 \)) if G is n-edge-connected and if for every vertex v of G, the graph \( G - v \) is not n-edge-connected. The following two results are due to Mader and will aid us in this section.

**Theorem 2A** For a positive integer n, every critically n-edge-connected graph G contains at least two vertices of degree n.

**Theorem 2B** For a positive integer n, every n-edge-connected graph G contains at least \( \min \{ p(G), 2 \lceil n/2 \rceil + 2 \} \) vertices v such that \( \kappa_1(G - v) \geq n - 1 \).
From Theorem 2A, it now follows that a graph $G$ is critically $n$-edge-connected if and only if $\kappa_1(G) = n$ and $\kappa_1(G - v) < n$ for every vertex $v$ of $G$.

**Lemma 2.7** If $G$ is a graph with edge-connectivity $n \geq 1$, then $G$ contains an induced critically $n$-edge-connected subgraph.

**Proof.** If $\kappa_1(G - v) < n$ for all $v \in V(G)$, then $G$ is itself critically $n$-edge-connected. Suppose, therefore, that $G$ contains a vertex $v_0$ such that $\kappa_1(G - v_0) \geq n$. Let $G_1 = G - v_0$. If $\kappa_1(G_1 - v) < n$ for all $v \in V(G_1)$, then $G_1$ is an induced critically $n$-edge-connected subgraph of $G$. Otherwise, $G_1$ contains a vertex $v_1$ such that $\kappa_1(G_1 - v_1) \geq n$. Let $G_2 = G_1 - v_1$. If $G_2$ is critically $n$-edge-connected, then the proof is complete; otherwise, we continue in this fashion to produce an induced subgraph $G_4$ of $G$ such that $G_4$ is critically $n$-edge-connected.

**Lemma 2.8** If $G$ is a critically $n$-edge-connected graph ($n \geq 1$), then $G$ contains a vertex $v$ such that $\kappa_1(G - v) = n - 1$.

**Proof.** Since $G$ is critically $n$-edge-connected, $\kappa_1(G - v) \leq n - 1$ for every vertex $v$ of $G$. By Theorem 2B, $G$ contains a vertex $u$ such that $\kappa_1(G - u) \geq n - 1$. Consequently, $\kappa_1(G - u) = n - 1$. 

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Lemma 2.9 If \( G \) is a graph with edge-connectivity \( n \geq 1 \), then for every nonnegative integer \( k \) \( (0 \leq k \leq n) \), \( G \) contains an induced subgraph \( G_k \) with edge-connectivity \( k \).

Proof. Clearly \( G_n = G \) is an induced subgraph of \( G \) having edge-connectivity \( n \). By Lemma 2.7, \( G_n \) contains an induced critically \( n \)-edge-connected subgraph \( G'_n \). By Lemma 2.8, \( G_n \) contains a vertex \( v_n \) such that \( r_1(G'_n - v_n) = n - 1 \). Let \( G_{n-1} = G'_n - v_n \). If \( n - 1 \geq 1 \), then, by Lemma 2.7, \( G_{n-1} \) contains a critically \( (n - 1) \)-edge-connected subgraph \( G'_{n-1} \).

By Lemma 2.8, \( G_n \) contains a vertex \( v_{n-1} \) such that \( r_1(G'_{n-1} - v_{n-1}) = n - 2 \). Let \( G_{n-2} = G'_{n-1} - v_{n-1} \). Proceeding in this fashion, we produce induced subgraphs \( G_n, G_{n-1}, \ldots, G_0 \) with \( r_1(G_i) = i \) \( (0 \leq i \leq n) \).

Theorem 2.10 If \( G \) is a graph with edge-connectivity \( n \), then \( G \) contains a major \( n \)-edge-connected subgraph.

Proof. To prove the theorem we need only show that every graph \( G \) with edge-connectivity \( n \) contains an induced subgraph having edge-connectivity \( n \) and which contains no subgraphs whose edge-connectivity exceeds \( n \).

If \( n = 0 \), then a maximum independent set of vertices induces a major \( 0 \)-connected subgraph. Assume then that \( n \geq 1 \). By Lemma 2.7, \( G \) contains an induced critically \( n \)-edge-connected subgraph. We show that if \( H \) is an induced critically \( n \)-edge connected
subgraph of $G$ having minimum order, then $\kappa_1(H) = n$ and every (induced) subgraph of $H$ has edge-connectivity at most $n$. The observation preceding Lemma 2.7 shows that $\kappa_1(H) = n$. It remains to be shown that $H$ contains no subgraph whose edge-connectivity exceeds $n$. Assume, to the contrary, that $H$ contains an induced subgraph $H'$ with $\kappa_1(H') = m > n$. By Lemma 2.9, for every $k \ (0 \leq k \leq m)$, $H'$ contains an induced subgraph with edge-connectivity $k$. In particular, $H'$ contains an induced subgraph $F$ with $\kappa_1(F) = n$. However, by Lemma 2.7, $F$ contains an induced critically $n$-edge-connected subgraph $F'$. Clearly $p(F') < p(H)$, which contradicts our choice of $H$.

Consequently, $G$ contains an induced subgraph whose edge-connectivity is $n$ and which contains no subgraphs with edge-connectivity exceeding $n$. This implies that $G$ contains a major $n$-edge-connected subgraph.

The construction immediately following the corollary to Lemma 2.1 also illustrates that there are critically $n$-edge-connected graphs containing (induced) subgraphs with edge-connectivity exceeding $n$.

Hatula [43] defined the maximum subgraph edge-connectivity of a graph $G$ by

$$\tilde{\kappa}_1(G) = \max\{\kappa_1(H) \mid H \subseteq G\}$$
We now show for which integers $k$, a graph $G$ contains a major $k$-edge-connected subgraph.

**Theorem 2.11** A graph $G$ contains a major $k$-edge-connected subgraph if and only if $0 \leq k \leq \kappa_1(G)$.

**Proof.** Let $H$ be an induced subgraph of $G$ having edge-connectivity $\kappa_1(H) = m$. By Lemma 2.9, $H$ contains induced subgraphs $G_0, G_1, \ldots, G_m$ such that $\kappa_1(G_k) = k$ for $0 \leq k \leq m$. By Theorem 2.10, $G_k$ contains a major $k$-edge-connected subgraph for every $k$ ($0 \leq k \leq m$), implying that for every $k$ ($0 \leq k \leq \kappa_1(G)$), the graph $G$ contains an induced subgraph having edge-connectivity $k$ and which contains no subgraph with edge-connectivity exceeding $k$ ($0 \leq k \leq \kappa_1(G)$). Consequently, $G$ contains a major $k$-edge-connected subgraph for every $k$ satisfying $0 \leq k \leq \kappa_1(G)$.

Clearly, since $G$ contains no (induced) subgraphs whose edge-connectivity exceeds $\kappa_1(G)$, $G$ does not contain major $k$-edge-connected subgraphs for $k > \kappa_1(G)$. $\square$

The following result may be regarded as the edge analogue to Theorem 2.4.

**Theorem 2.12** For every pair $n, k$ of positive integers, there is a graph $G$ with $\kappa_1(G) = n$ and $K_1(G) = \{k\}$.

We omit the proof of this theorem, since the graphs described in the proof of Theorem 2.4 have the desired properties.
Theorem 2.5 shows that every set of positive integers can be realized as the major connectivity set of some graph. Similarly the next result shows that every set of positive integers can also be realized as a major edge-connectivity set of some graph. This proof too is omitted since the graphs described in Theorem 2.5 have the desired properties.

Theorem 2.13 Let \( S = \{n_1, n_2, \ldots, n_k\} \) be a set of positive integers with \( n_i < n_{i+1} \) (\( 1 \leq i \leq k - 1 \)). If \( m \geq n_k + 1 \) is a positive integer, then there is a graph \( G \) with \( k_1(G) = n \) and \( K_1(G) = \{k\} \).

The next result parallels Theorem 2.6.

Theorem 2.14 Let \( G \) be a graph of order \( p \), every major \( n \)-edge-connected subgraph of which has order \( m \geq 2 \), where \( n \in K_1(G) \). Then

\[
p < \frac{m+1}{m-1} \cdot m.
\]

The proof of this result is very similar to the proof of Theorem 2.6. We need only observe, by Theorem 2.10, that if \( G \) is a graph satisfying the hypothesis of the theorem, then every (induced) subgraph of \( G \) has edge-connectivity at most \( m - 1 \), implying that every (induced) subgraph of \( G \) has connectivity at most \( m - 1 \).
We show next that there is no relationship in general between an element of the major connectivity set of a graph and an element of its major edge-connectivity set.

**Theorem 2.15** For every pair \( m, n \) of positive integers, there exists a graph \( G \) with \( K(G) = \{m\} \) and \( K_1(G) = \{n\} \).

**Proof.** Notice that if \( m = n \), then \( G \subseteq K(n,n) \) has the desired properties. Thus we assume that \( m \neq n \) and consider two cases.

**Case 1. Assume that \( n > m \).**

Let \( G_1 \subseteq G_2 \subseteq K(n,n) \) and suppose that \( V_1 \) and \( V_2 \) are the partite sets of \( G_1 \) while \( W_1 \) and \( W_2 \) are the partite sets of \( G_2 \), where \( V_i = \{v_{i1}, v_{i2}, \ldots, v_{in}\} \) and \( W_i = \{w_{i1}, w_{i2}, \ldots, w_{in}\}, i = 1, 2 \). Let \( G \) be obtained from \( G_1 \cup G_2 \) by adding the edges of

\[
E = \{v_{2j}w_{1j} | j = 1, 2, \ldots, m\} \cup \{v_{2m}w_{1j} | j = m+1, m+2, \ldots, n\}.
\]

Since \( \delta(G) = n \), it follows that \( \kappa_1(G) \leq n \). Suppose now that \( E_0 \) is any set of at most \( n - 1 \) edges. Then \( G_i - E_0 \) is connected for \( i = 1, 2 \) and \( E - E_0 \neq \emptyset \). Consequently, \( G - E_0 \) is connected so that \( \kappa_1(G) \geq n \). Hence \( \kappa_1(G) = n \).

Since every induced subgraph of \( G \) contains a vertex having degree at most \( n \), we may conclude that \( G \) itself is a major \( n \)-edge-connected graph and, therefore, that \( K_1(G) = \{n\} \).
Let

\[ H = G - (\{v_{2j}| j = m+1, m+2, \ldots, n\} \cup \{w_{1j}| j = m+1, m+2, \ldots, n\}). \]

Then \( H \) is a connected induced subgraph of \( G \) having order \( p(G) - 2n + 2m \) and \( H - \{v_{2j}| j = 1, 2, \ldots, m\} \) is disconnected. Hence \( \kappa(H) \leq m \). If \( V \) is a set of at most \( m - 1 \) vertices of \( H \), then \( \{v_{21}, v_{22}, \ldots, v_{2m}\} - V \neq \emptyset \) and

\[ \langle (V_1 \cup \{v_{2j}| j = 1, 2, \ldots, m\}) - V \rangle_H \]

as well as

\[ \langle (W_1 \cup \{w_{1j}| j = 1, 2, \ldots, m\}) - V \rangle_H \]

are connected, implying that \( H - V \) is connected. Hence \( \kappa(H) = m \). Since every induced subgraph of \( H \) contains a vertex having degree at most \( m \), it follows that every major \( m \)-connected subgraph of \( G \) has order at least \( p(G) - 2n + 2m \). However, no major \( m \)-connected subgraph \( H_m \) of \( G \) can contain more than \( p(G) - 2n + 2m \) vertices; for otherwise, \( H_m \) contains at least \( n + m + 1 \) vertices of \( V(G_1) \) or \( V(G_2) \), which is impossible, since \( H_m \) has no subgraph with connectivity exceeding \( m \). Hence every major \( m \)-connected subgraph of \( G \) has order \( 2n + 2m \).

Suppose that \( k \) is an integer satisfying \( m < k \leq \overline{\kappa}(G) = n \). Then a major \( k \)-connected subgraph \( H_k \) of \( G \) cannot contain
vertices belonging to both $V(G_1)$ and $V(G_2)$ since such a subgraph has connectivity at most $m$. Consequently, a major $k$-connected subgraph of $G$ with $m < k \leq n$ has order at most $2n$.

Suppose now that $0 \leq k < m$. Then a major $k$-connected subgraph contains at most $n + k$ vertices of $V(G_1)$, $i = 1, 2$; that is, a major $k$-connected subgraph has order at most $2n + 2k < 2n + 2m$.

We may now conclude that $K(G) = \{m\}$.

**Case 2. Assume that** $1 \leq n < m$.

**Subcase 2.1. Let** $2 \leq n \leq m - 2$. Further, let $G' = K(m,m)$ and $G'' = P^m_{m-1}$, that is, $G''$ is the $n$th power of $P_{m-1}$. Let $V_1$ and $V_2$ denote the partite sets of $G'$, where $V_1 = \{v_{i1}, v_{i2}, \ldots, v_{im}\}$, $i = 1, 2$, and let $V(G'') = \{u_1, u_2, \ldots, u_{m-1}\}$. Moreover, let $G$ be the graph obtained from $G' \cup G''$ by adding the edges in $E = \{v_{2j}u_1 | j = 1, 2, \ldots, n\}$.

Define $G_n$ by

$$G_n = \left( V_1 \cup \{v_{2j} | j = 1, 2, \ldots, n\} \cup V(G'') \right)_G.$$

Then $p(G_n) = p(G) - m + n = 2m + n - 1$. Since $G_n - E$ is disconnected, it follows that $\kappa_1(G_n) \leq n$. Let $E_0$ be a set of at most $n - 1$ edges of $G_n$. Since $m - 1 \leq n + 1$, it follows, by Lemma 2.3, that $\kappa_1(G'') = n$. Hence $G'' - E_0$ is connected.

Further, since
we see that $G'_n - E_0$ is connected. Because $E - E_0 \neq \emptyset$, we may now conclude that $G_n - E_0$ is connected. Hence $\kappa_1(G_n) = n$.

The fact that every induced subgraph of $G_n$ contains a vertex with degree at most $n$ implies that every major $n$-edge-connected subgraph of $G$ has order at least $2m + n - 1$. However, a major $n$-edge-connected subgraph $G_n$ of $G$ contains at most $m + n$ vertices of $G'$; otherwise, $G_n$ contains a subgraph with edge-connectivity exceeding $n$. Thus every major $n$-edge-connected subgraph of $G$ has order at most $2m + n - 1$, so that every major $n$-edge-connected subgraph has order $2m + n - 1$.

Suppose now that $k$ is such that $n < k \leq \kappa_1(G) = m$. Then a major $k$-edge-connected subgraph $G_k$ of $G$ does not contain vertices of $G''$ because this implies that $G_k$ contains a vertex having degree at most $n$, which is not possible. Thus a major $k$-edge-connected subgraph has order at most $2m < 2m + n - 1$. If $0 \leq k < n$, then a major $k$-edge-connected subgraph of $G$ contains at most $m + k$ vertices of $G'$. Hence every major $k$-connected subgraph of $G$ has order at most $2m + k - 1 < 2m + n - 1$. Consequently, $K_1(G) = \{n\}$.

We show next that $K(G) = \{m\}$. Since $\kappa(G') = m$, it follows that every major $m$-connected subgraph of $G$ has order at least $2m$. However, every major $m$-connected subgraph $H_m$ of $G$ does
not contain vertices of \( G'' \) since this would imply that \( H_m \) contains a vertex having degree at most \( n \), which is not possible. Consequently, a major \( m \)-connected subgraph of \( G \) has order \( 2m \).

Assume now that \( 2 \leq k < m \). Then every major \( k \)-connected subgraph \( H_k \) of \( G \) is contained in \( G' \) or \( G'' \), because \( H_k \) contains no cut-vertex. Further, \( H_k \) contains at most \( m + k \) vertices of \( G' \). Hence every major \( k \)-connected subgraph of \( G \) has order at most \( m + k < 2m \).

Suppose now that \( k = 1 \). Then every major 1-connected subgraph \( G_1 \) of \( G \) contains at most \( m + 1 \) vertices of \( G' \) and at most \( m - 2 \) vertices of \( G'' \); otherwise, if \( G_1 \) contains every vertex of \( G'' \), then \( G_1 \) contains a cycle. Hence a major 1-connected subgraph of \( G \) has order less than \( 2m \). If \( k = 0 \), then a major 0-connected subgraph contains at most \( m \) vertices of \( G' \) and, therefore, less than \( 2m \) vertices of \( G \). Hence a major 0-connected subgraph of \( G \) has order less than \( 2m - 1 \). Therefore, \( K(G) = \{m\} \).

Subcase 2.2. Suppose that \( m = n + 1 \), where \( n \geq 2 \).

Let \( G' \equiv K_n \), \( G'' \equiv K_n \) and \( G^* \equiv K_m \). Suppose that \( V(G') = \{v\} \), \( V(G'') = \{v_1, v_2, \ldots, v_n\} \) and \( V(G^*) = \{w_1, w_2, \ldots, w_m\} \).
Define \( G = G' + (G'' \cup G^*) \).

We show first that \( K_1(G) = \{n\} \). Let \( G_n = G - w_m \). Then \( G_n - \{v \in G_n \mid j = 1, 2, \ldots, n\} \) is disconnected, so that \( k_1(G) \leq n \). Let \( E \) be a set of \( n - 1 \) edges of \( G_n \). Because
\[ H' = \langle V(G') \cup \{v\} \rangle_G \cong K_{n+1} \]

and

\[ H'' = \langle (V(G') - \{v\}) \cup \{v\} \rangle_G \cong K_{n+1} \]

it follows that both \( H' - E \) and \( H'' - E \) are connected. Since \( V(H') \cap V(H'') = \{v\} \) and \( G_n = \langle V(H') \cup V(H'') \rangle \), we conclude that \( G_n - E \) is connected. Hence \( \kappa_1(G_n) \geq n \) so that \( \kappa_1(G_n) = n \).

Further, every induced subgraph of \( G_n \) has a vertex with degree at most \( n \). Consequently, every major \( n \)-edge-connected subgraph of \( G \) has order at least \( \pi(G) - 1 \). However, because

\[ \langle V(G') \cup \{v\} \rangle_G \cong K_{n+2} \]

a major \( n \)-edge-connected subgraph of \( G \) cannot have order \( \pi(G) \).

If \( 0 \leq t < n \), then every major \( t \)-edge-connected subgraph \( G_t \) of \( G \) contains at most \( t + 1 \) vertices in \( V(G') \cup \{v\} \), so that

\[ \pi(G_t) \leq n + t + 1 \leq 2n = 2m - 2 = \pi(G) - 2. \]

If \( t = m \), then every major \( t \)-edge-connected subgraph cannot contain vertices of \( G'' \) because these vertices have degree \( n \) in \( G \). Hence a major \( m \)-edge-connected subgraph of \( G \) has order at most \( \pi(G) - n < \pi(G) - 1 \). Finally, because every induced subgraph of \( G \) contains a vertex with degree at most \( m \)
and \( r_f(\langle V(G^*) \cup \{v\} \rangle) = m \), it follows that \( r_f(G) = m \). We may now deduce that every major \( t \)-edge-connected subgraph \( G_t \) of \( G \) has order at most \( p(G) - 2 \), unless \( t = n \), in which case \( p(G_t) = p(G) - 1 \). Hence \( K_1(G) = \{n\} \).

We show next that \( K(G) = \{m\} \). Since \( \langle V(G^*) \cup \{v\} \rangle \not\cong K_{m+1} \), it follows that \( G \) contains a major \( m \)-connected subgraph, which clearly has order at least \( m + 1 \). However, because \( \deg_{G^*i} v_i = n \) (\( 1 \leq i \leq n \)), it follows that a major \( m \)-connected subgraph of \( G \) has order at most \( p(G) - n = m + 1 \). Hence every major \( m \)-connected subgraph of \( G \) has order \( m + 1 \). Since every induced subgraph of \( G \) contains a vertex with degree at most \( m \), it follows that \( r(G) = m \). It remains to be shown that every major \( t \)-connected subgraph of \( G \) (\( 0 \leq t < m \)) has order less than \( p(G) - n \). If \( 2 \leq t < n \), then every major \( t \)-connected subgraph \( H_t \) is contained in \( H' = \langle V(G^*) \cup \{v\} \rangle_G \) or in \( H'' = \langle V(G^*) \cup \{v\} \rangle_G \). If \( H_t \subset H' \), then \( p(H_t) \leq p(G) - n - 1 \). If \( H_t \subset H'' \), then \( H_t \) contains exactly \( t + 1 \) vertices of \( H'' \). Since \( t + 1 \leq m \), it now follows that \( p(H_t) \leq p(G) - n - 1 \).

Suppose now that \( t = 1 \). If a major \( t \)-connected subgraph \( H_t \) contains \( v \), then \( H_t \) contains at most one vertex of \( G'' \) and at most one vertex of \( G^* \). Hence \( p(H_t) \leq 3 \leq m \). If \( H_t \) does not contain \( v \), then a major \( t \)-connected subgraph is contained in \( G'' \) or \( G^* \) and, therefore, has order at most 2.

Suppose now that \( t = 0 \). Since \( \varphi(G) = 2 \), it follows that a
major 0-connected subgraph has order at most 2. Consequently, \( K(G) = \{m\} \).

Subcase 2.3. Suppose now that \( n = 1 \) and that \( m \geq 1 \).

Let \( G'' = F_1 \cup H'' \cup F_2 \cup K_m \), \( G' = H' \cup \overline{K}_m \), and \( P = P_{4m-3} \).

Let \( V(G'') = \{v_{11}, v_{12}, \ldots, v_{1m}\} \), \( V(G') = \{v_{21}, v_{22}, \ldots, v_{2m}\} \), \( V(F_1) = \{v_{31}, v_{32}, \ldots, v_{3m}\} \), \( V(H'') = \{w_{11}, w_{12}, \ldots, w_{1m}\} \), \( V(H') = \{w_{21}, w_{22}, \ldots, w_{2m}\} \), \( V(F_2) = \{w_{31}, w_{32}, \ldots, w_{3m}\} \), and \( V(P) = \{u_1, u_2, \ldots, u_{4m-3}\} \). Set \( G^* = G' + (G'' \cup F_1) \) and \( H^* = H' + (H'' \cup F_2) \) and let \( G \) be obtained from \( G^* \), \( H^* \), and \( P \) by joining \( v_{3j} \) to \( w_{1j} \) (\( 1 \leq j \leq m \)) and then joining \( u_1 \) to \( w_{1m} \).

We show first that \( \lambda_1(G) = \{1\} \). Let

\[
G_1 = \langle V(G') \cup \{v_{3m}, v_{1m}\} \cup V(H') \cup V(P) \rangle_G.
\]

Then \( p(G_1) = 6m - 1 \) and \( G_1 \) is a tree. Hence every major 1-edge-connected subgraph of \( G \) has order at least \( 6m - 1 \). We now show that every major 1-edge-connected subgraph of \( G \) has order at most \( 6m - 1 \). This is verified by showing that every induced subtree \( T \) of \( G^* \) has order at most \( m + 1 \). Observe first that \( T \) contains most two vertices of \( G'' \) or \( F_1 \). However, if \( T \) contains two vertices of \( G'' \) or \( F_1 \), then \( T \) does not contain a vertex of \( G' \); otherwise, \( T \) contains a triangle. Hence, in this case, \( p(T) = 2 \). Suppose now that \( T \) contains at
most one vertex from either \( G'' \) or \( F_1 \). If 
\[ V(G'') \cap V(T) \neq \emptyset \text{ and } V(F_1) \cap V(T) \neq \emptyset, \]
then \( T \) contains at least one vertex of \( G' \); otherwise, \( T \) is disconnected.
However, \( T \) cannot contain more than two vertices of \( G' \) because \( T \) is acyclic. Hence, in this case, \( p(T) = 3 \). Assume, therefore, that \( |V(G'') \cap V(T)| = 1 \) and that \( V(F_1) \cap V(T) = \emptyset \).
Then \( |V(T) \cap V(G')| \leq m \), implying that \( p(T) \leq m + 1 \). Hence, every induced subtree of \( G^* \) has order at most \( m + 1 \). Similarly, every induced subtree of \( H^* \) has order at most \( m + 1 \). Consequently, every induced subtree of \( G \) has order at most 
\[ p(P) + 2m + 2 = 6m - 1, \]
so that every major 1-edge-connected subgraph of \( G \) has order \( 6m - 1 \).

Next we show that if \( 0 \leq k \leq \delta_1(G) \) and \( k \neq 1 \), then every major \( k \)-edge-connected subgraph \( G_k \) of \( G \) has order less than \( 6m - 1 \). Suppose first that \( m < k \leq \delta_1(G) \). Because \( G - \{v_{3j} \mid j = 1, 2, \ldots, m\} \) is disconnected, \( G_k \) cannot contain vertices of both \( G^* \) and \( H^* \). Further, \( G_k \) cannot contain vertices of \( P \); otherwise, \( G_k \) contains a bridge. Therefore \( G_k \) is contained in \( G^* \) or \( H^* \). But then, 
\[ p(G_k) \leq 3m < 6m - 1. \]

Assume now that \( 1 \leq k \leq m \). Since \( \text{diam } G^* = \text{diam } H^* = 2 \), it follows that \( \delta(G^*) = \delta(H^*) \) and \( \delta_1(G^*) = \delta_1(H^*) \). Therefore, 
\[ \delta_1(G^*) = \delta_1(H^*) = 2m - 1 > k. \]
But then \( G_k \) contains at most \( 3m - 1 \) vertices of \( G^* \) and at most \( 3m - 1 \) vertices of \( H^* \). Further, \( G_k \) contains no vertices of \( P \). Therefore
\( p(G_k) \leq 6m - 2 \) for \( 1 < k \leq m \). Since \( \beta(G^*) = \beta(H^*) = m \) and \( \beta(P) = 2m - 1 \), it follows that every major \( \beta \)-edge-connected subgraph of \( G \) has order at most \( 4m - 1 < 6m - 1 \). Hence \( K_1(G) = \{1\} \).

We proceed now to show that \( K(G) = \{m\} \). Let \( H_m = \left( V(G^*) \cup V(H^*) \right)_G \). Since \( H_m - \{v_{3j} | j = 1, 2, \ldots, m\} \) is disconnected, \( k(H_m) \leq m \). If \( S \) is a set of \( m - 1 \) vertices of \( H_m \), then both \( G^* - (V(G^*) \cap S) \) and \( H^* - (V(H^*) - S) \) are connected, and for at least one \( j \in \{1, 2, \ldots, m\} \), we have \( v_{3j}, w_{1j} \notin S \). Therefore, \( k(H_m) \geq m \), so that \( k(H_m) = m \). Let \( H \) be a subgraph of \( H_m \). If \( H \) contains vertices of both \( G^* \) and \( H^* \), then \( k(H) \leq m \). Suppose now that \( H \) contains vertices of both \( G^* \) or \( H^* \). Because \( G^* \not\cong H^* \), we may assume \( H \subset G^* \). Since \( G^* - V(G') \) is disconnected, it follows that if \( H \) contains vertices of both \( G'' \) and \( F_1 \), then \( k(H) \leq m \). We may therefore assume that

\[ H \subset G'' + G' \cup K_m + \overline{K}_m. \]

Then \( H \) contains a vertex having degree at most \( m \), implying that \( k(H) \leq m \). Consequently, every induced subgraph of \( H_m \) has connectivity at most \( m \). Thus every major \( m \)-connected subgraph has order at least \( 6m \). However, since a major \( m \)-connected subgraph of \( G \) does not contain vertices of \( P \), it follows that a major \( m \)-connected subgraph of \( G \) has order at most \( p(G) - 4m + 3 = 6m \). Hence a major \( m \)-connected subgraph has order \( 6m \).
Therefore $H_m$ is a major $m$-connected subgraph. Hence $\hat{\chi}(G) = m$.

Suppose now that $2 \leq k < m$, and let $H_k$ be a major $k$-connected subgraph of $G$. Then $V(H_k) \cap V(P) = \emptyset$ and since $\kappa(H_m) = m$, it follows that $H_k$ contains fewer than $6m$ vertices of $H_m$. Therefore $p(H_k) < 6m$.

Because a major 1-edge-connected subgraph of $G$ is also a major 1-connected subgraph of $G$, we conclude from the preceding part of the proof, that a major 1-connected subgraph of $G$ has order $6m - 1$. Finally, because $\beta(G) \leq 4m - 1$, every major 0-connected subgraph of $G$ has order less than $6m$. Hence $K(G) = \{m\}$.\hfill \Box
CHAPTER III

CONDITIONAL CONNECTIVITIES DEFINED BY
INDUCED SUBGRAPHS

Section 3.1 Introduction

In 1983 Harary [25] introduced conditional connectivity when he defined the P-connectivity of a graph G, for a graphical property P, as the smallest number of vertices whose removal from G produces a disconnected graph each component of which has property P. The P-edge-connectivity is defined analogously.

From the above definition, it is evident that the P-connectivity of a graph may not be defined for some properties P. For example, suppose that P is the property of being 2-connected and that G is a tree. Then the P-connectivity of G is not defined. Further, Harary's definition clearly requires the graph to be noncomplete if the P-connectivity is to exist.

In this chapter we study the conditional connectivity of a graph G with respect to two properties, each defined in terms of induced subgraphs of G.

Section 3.2 The F-Connectivity of a Graph

For an induced subgraph F of a graph G, the F-connectivity Fr(G) of G is defined as the smallest number of vertices whose removal from G produces the trivial graph or a disconnected graph, each of whose components is a subgraph of F. Hence if
we let $P$ be the property of being a subgraph of $F$, then the $P$-connectivity of a noncomplete graph $G$ is its $F$-connectivity. Further, for such graphs $G$ and $F$, the $F$-connectivity always exists.

If $F \subseteq G$, then $Fr(G) = r(G)$. That the $F$-connectivity of $G$ is not always the same as the connectivity of $G$ can be seen by considering the graphs $G$ and $F$ of Figure 3.1. Since $G$ is connected and contains $u_2$ as a cut-vertex, it follows that $r(G) = 1$. However, since $u_2$ is the only cut-vertex of $G$ and as $G - u_2$ contains cycles, it follows, since $F$ is acyclic, that $Fr(G) \geq 2$. On the other hand, $G - \{u_2, w\}$ consists of two components both of which are subgraphs of $F$, so that $Fr(G) \leq 2$. Hence $Fr(G) = 2$.

For a graph $G$ having order at least 2 and for every $v \in V(G)$, $G - v$ is an induced subgraph of $G$. If $r(G) = n$ and $S$ is a set of $n$ vertices such that $G - S$ is disconnected or the trivial graph, then for every $v \in S$, the $(G - v)$-connectivity of $G$ coincides with the connectivity of $G$. That is,
whenever a vertex \( v \) of \( G \) belongs to some set \( S \) of \( n \) vertices so that \( G - S \) is trivial or disconnected, then
\[(G - v)\kappa(G) = \kappa(G).\]
Consequently, there are at most \( p(G) - n \) vertices \( v \) in \( G \) such that \((G - v)\kappa(G) \neq \kappa(G)\). This result is best possible in the sense described below.

**Theorem 3.1** For every positive integer \( n \), there exists a graph \( G \) with \( \kappa(G) = n \) and having \( p(G) - n \) vertices \( v \) of \( G \) for which \((G - v)\kappa(G) \neq \kappa(G)\).

**Proof.** Let \( H, H_1 \) and \( H_2 \) be isomorphic to \( \overline{K}_n, K_{n+3} \) and \( K(n+2, n+2) \), respectively. Further, let \( S_i \) be a subset of \( n + 1 \) vertices of \( H_1 \) and \( S_2 \) a subset of \( n + 1 \) independent vertices of \( H_2 \). Let \( G \) be the graph obtained from \( H, H_1 \) and \( H_2 \) by joining every vertex of \( H \) to every vertex of
\[S_i \ (i = 1, 2).\]
Observe that \( G - V(H) \) is disconnected, so that \( \kappa(G) \leq n \). If \( S \) is a set of at most \( n - 1 \) vertices, then it follows, since both \( H_1 \) and \( H_2 \) are \((n+2)\)-connected, that \( H_i - S \ (i = 1, 2) \) is connected and nontrivial. Further, each of \( S_1, S_2 \) and \( H \) contains a vertex that does not belong to \( S \). Since every vertex of \( H \) is adjacent to every vertex of
\[S_i \ (i = 1, 2), \ G - S \ is connected and \( \kappa(G) \geq n \). Hence \( \kappa(G) = n \). By similar arguments, we can see that \( V(H) \) is the unique set of \( n \) vertices whose removal from \( G \) produces a disconnected graph.
We show next that for every \( v \in V(G) - V(H) \), the \((G - v)\)-connectivity of \( G \) exceeds the connectivity of \( G \). Suppose \( v \in S_1 \). Then the clique number \( \omega(G - v) \) of \( G - v \) is \( n + 2 \). Since \( \omega(G - V(H)) = n + 3 \), it follows that \((G - v)\kappa(G) \geq n + 1\) so that \((G - v)\kappa(G) \neq \kappa(G)\). Suppose now that \( v \in V(H_1) - S_1 \). Then \( \omega(G - v) = n + 2 \) for such a vertex \( v \) as well; however, \( \omega(G - V(H)) = n + 3 \) so that \((G - v)\kappa(G) \neq \kappa(G)\). If \( v \in V(H_2) \), then every complete bipartite subgraph of \( G - v \) is a subgraph of \( K(n+1, 2n+2) \). Since \( G - V(H) \) contains \( H_2 \neq K(n+2, n+2) \) as a subgraph and \( H_2 \) is not a subgraph of \( K(n+1, 2n+2) \), it follows that \((G - v)\kappa(G) \neq \kappa(G)\).

The next result provides bounds on the \( F \)-connectivity of a graph \( G \).

**Theorem 3.2** Let \( F \) be an induced subgraph of a graph \( G \). Then

\[ \kappa(G) \leq Fr(G) \leq \kappa(G) + p(G) - p(F). \]

**Proof.** Since \( F \) is an induced subgraph of \( G \), there is a set \( S' \) of vertices of \( G \) such that \( F \neq G - S' \) and \( |S'| = p(G) - p(F) \).

Let \( S_0 \) be a subset of \( V(G) \) having cardinality \( \kappa(G) \), where \( G - S_0 \) is disconnected or the trivial graph. If \( G - S_0 \) is the trivial graph, then \( G \) is complete so that \( \kappa(G) = Fr(G) \).
Suppose, therefore, that $G - S_0$ has components $G_1, G_2, \ldots, G_k$, where $k \geq 2$. If $V(G_i) \cap S' = V(G_i)$, let $S_i$ be any subset of $|V(G_i)| - 1$ vertices of $V(G_i)$; otherwise, let

$$S_i = V(G_i) \cap S', \quad (1 \leq i \leq k).$$

Then $G_i - S_i$ $(1 \leq i \leq k)$ is a subgraph of $F$ and, furthermore,

$$G = \bigcup_{j=0}^{k} S_j$$

is disconnected, so that

$$\sum_{j=0}^{k} V(S_j) \leq \kappa(G) + p(G) - p(F).$$

Since we need to delete at least $\kappa(G)$ vertices of $G$ to disconnect $G$ or to produce the trivial graph, in case $G$ is complete, it follows that $\kappa(G) \leq Fr(G)$. Hence

$$\kappa(G) \leq Fr(G) \leq \kappa(G) + p(G) - p(F).$$

We now show that the bounds given in the preceding theorem are best possible.

**Theorem 3.3** Let $n$ and $k$ be given positive integers and $j$ an integer such that $0 \leq j \leq k$. Then there is a graph $G$ and an induced subgraph $F$ of $G$ such that $\kappa(G) = n$, $p(G) - p(F) = k$ and $Fr(G) = n + j$. 

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Proof. Let \( q \) and \( r \) be integers such that \( k = q(n + j) + r \), where \( 0 \leq r < n + j \).

We now describe the construction of a graph \( G \) containing an induced subgraph \( F \) such that \( r(G) = n \), \( p(G) - p(F) = k \) and \( Fx(G) = n + j \) \((0 \leq j \leq k)\). Suppose that \( H_1 \cong K_{n+j+r+1} \), \( H_2 \cong K(n+j+1,n+j+1) \) and \( H \cong K_n \). For \( q > 0 \), let \( G_1 \cong G_2 \cong \cdots \cong G_q \cong K_{n+j} \), where \( V(G_i) = S_i \), \( 1 \leq i \leq q \). Let \( A_1 \) denote a set of \( n + j \) vertices of \( H_1 \) and \( A_2 \) a set of \( n + j \) independent vertices of \( H_2 \). If \( q = 0 \), define \( H_1 = H_1' \); otherwise, define \( H_1 \) to be that graph obtained by joining every vertex of \( \bigcup_{i=1}^{q} S_i \) to every vertex of \( A_1 \). Finally, \( G \) is produced from \( H \cup H_1 \cup H_2 \) by joining every vertex of \( H \) to every vertex of \( A_1 \cup A_2 \) (see Figure 3.2).

![Figure 3.2](image-url)
Since \( G \) is not complete, neither the connectivity nor the \( F \)-connectivity of \( G \) equals \( p(G) - 1 \). Furthermore, both \( H_1 \) and \( H_2 \) have connectivity at least \( n + j \).

Because \( G - V(H) \) is disconnected, it follows that 
\[
k(G) \leq |V(H)| = n.
\]
However, if \( S' \) is any set of at most \( n - 1 \) vertices of \( G \), then the subgraph induced by \( V(H_1) - S' \) is connected and contains a vertex of \( A_i \) (\( i = 1, 2 \)) that is joined to every vertex of the nonempty set \( V(H) - S' \). Consequently, \( G - S' \) is connected, implying that \( k(G) \geq n \). Hence \( k(G) = n \).

Let \( S_0 \) be a subset of \( r \) vertices in \( V(H_1) - A_1 \), and let 
\[
F = G - \bigcup_{i=0}^{q} S_i.
\]
Then \( F \) is an induced subgraph of \( G \) and
\[
p(G) - p(F) = \left| \bigcup_{i=0}^{q} S_i \right| = k.
\]
Observe that \( F \) is isomorphic to the graph obtained by joining a set of \( n \) independent vertices to both \( n + j \) vertices of \( K_{n+j+1} \) and \( n + j \) independent vertices of \( K(n+j+1, n+j+1) \).

We proceed now to prove that \( F_k(G) = n + j \). Note first that \( G - A_1 \) is disconnected and that each component of \( G - A_1 \) is isomorphic to \( K_{n+j} \), \( K_{r+1} \) or the graph \( H_1' \) obtained from \( H \cup H_2 \) by joining each vertex of \( H \) to every vertex of \( A_2 \). Since \( \omega(F) = n + j + 1 \) and \( r < n + j \), every component of \( G - A_1 \) that is isomorphic to \( K_{n+j} \) or to \( K_{r+1} \) is also a subgraph of \( F \).
Further, $H'_2$ is a subgraph of $F$, so that $Fr(G) \leq n + j$.

It remains to be shown that $Fr(G) \geq n + j$. Let $S'$ be any set of at most $n + j - 1$ vertices such that $G - S'$ is disconnected. Then $V(H) \subseteq S'$; for otherwise, it follows, since $\chi(H_1) \geq n + j$ and $\chi(H_2) \geq n + j$, that the subgraph induced by $V(H_i) - S'$ $(i = 1, 2)$ is connected and contains a vertex of $A_i$ that is adjacent to every vertex in the nonempty set $V(H) - S'$. Hence $G - S'$ is connected, contrary to our assumption. Therefore $V(H) \subseteq S'$ and $S'$ contains at most $j - 1$ vertices of $H_1$.

Now if $q \geq 1$, then $G - S'$ contains a subgraph isomorphic to $K_{2n + j + 1}$. However, since $\omega(F) = n + j + 1$ and $2n + j + 1 > n + j + 1$, it follows that $G - S'$ contains a component that is not a subgraph of $F$.

Suppose now that $q = 0$. Then $k = r$ and $G - S'$ contains a subgraph isomorphic to $K_{n + k + 2}$. Since $0 \leq j \leq k$ and as $\omega(F) = n + j + 1$, it follows that $G - S'$ contains a component that is not a subgraph of $F$. Hence $Fr(G) \geq n + j$, so that $Fr(G) = n + j$.

Suppose now that $H < F < G$, and let $S$ be a set of $Hr(G)$ vertices such that $G - S$ is either the trivial graph or a disconnected graph with each of its components a subgraph of $H$. In the latter case, every component of $G - S$ is also a subgraph of $F$, so that $Fr(G) \leq Hr(G)$. Consequently, if for some positive
integer \( k \geq 2 \),

\[
F_k < F_{k-1} < \cdots < F_1 = G,
\]

then

\[
F_k \kappa(G) \geq F_{k-1} \kappa(G) \geq \cdots \geq F_1 \kappa(G) = \kappa(G).
\]

We next show that these conditional connectivities can attain arbitrary distinct values.

**Theorem 3.4** If \( n_1, n_2, \ldots, n_k \) are \( k \geq 2 \) positive integers such that \( n_1 < n_2 < \cdots < n_k \), then there is a graph \( G \) and subgraphs \( F_1, F_2, \ldots, F_k \) of \( G \) such that \( F_k < F_{k-1} < \cdots < F_1 = G \) and \( F_i \kappa(G) = n_i (1 \leq i \leq k) \).

**Proof.** Let \( H_1 = K_{n_1}, H_2 = K_{2n_k-n_1+1} \) and \( H = \overline{K_{n_1}} \). Now let

\[
G = F_1 = H + (H_1 \cup H_2)
\]

and \( F_i = K_{2n_k-n_i+1} \) for \( 2 \leq i \leq k \). Then \( F_k < F_{k-1} < \cdots < F_1 = G \). Since \( G - V(H) \) is disconnected, \( \kappa(G) \leq n_1 \). Further, if \( S \) is a set of at most \( n_1 - 1 \) vertices, then, since \( \kappa(H_2) > n_1 \) and every vertex of the nonempty set \( V(H) - S \) is joined to every vertex of \( V(G) - V(H) \), the graph \( G - S \) is connected. Hence \( \kappa(G) \geq n_1 \), so that \( \kappa(G) = F_1 \kappa(G) = n_1 \).
We show now that for every $i$ ($2 \leq i \leq k$), $F_i \kappa(G) = n_i$.

Let $S_i'$ be a set of $n_i - n_1$ vertices of $V(H)$, and set $S_i = V(H) \cup S_i'$. Then $G - S_i \cong K_1 \cup K_{2n_k - n_i + 1}$. Consequently, $G - S_i$ is disconnected and each of its components is a subgraph of $F_i$. Hence $F_i \kappa(G) \leq |S_i| = n_i$. Now let $S$ be a set of at most $n_i - 1$ vertices such that $G - S$ is disconnected. Then $V(H) \subseteq S$; otherwise, since $r(H) > n_1$ ($1 \leq i \leq k$) and every vertex of $V(H) - S$ is joined to every vertex of $V(G) - V(H)$, the graph $G - S$ is connected. Hence $S$ contains at most $n_i - n_i - 1$ vertices of $V(H)$ so that $G - S$ contains a subgraph isomorphic to $K_{2n_k - n_i + 2}$ which is not a subgraph of $F_i$. Hence $F_i \kappa(G) \geq n_i$, so that $F_i \kappa(G) = n_i$.

Suppose that $G$ is a graph and $F$ an induced subgraph of $G$. Then $G$ is said to be $F$-n-connected if $Fr(G) \geq n$. There are several well-known results in the literature dealing with n-connected graphs. We now discuss extensions of four of these to $F$ n-connected graphs.

If $G$ is an n-connected graph, then $G + K_1$ is $(n+1)$-connected (see [12]). This has a natural extension to $F$ n-connectivity.

**Theorem 3.5** Let $G$ be a graph and $F$ an induced subgraph of $G$. If $Fr(G) \geq n$, then $Fr(G + K_1) \geq n + 1$. 

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Proof. Let $H_1 \neq K_1$, where $v$ is the vertex of $H_1$. Further, let $H = G + H_1$ and suppose that $F_r(H) \leq n$. Then $H$ contains a set $S$ of at most $n$ vertices such that $H - S$ is disconnected, where each component is a subgraph of $F$. Observe that $v \in S$; otherwise, since every vertex of $G$ is joined to $v$, the graph $H - S$ is connected. Hence $S' = S - \{v\}$ is a set of at most $n - 1$ vertices of $G$ such that $G - S'$ is disconnected with each component a subgraph of $F$. Hence $F_r(G) \leq n - 1$. This contradiction now establishes the fact that $F_r(H) \geq n$.

Suppose that $G$ is $n$-connected and that $H$ is the graph obtained from $G$ by joining a new vertex to $n$ distinct vertices of $G$. Then $H$ is known to be $n$-connected (see [12]). This result also has an analogue for $F$ $n$-connected graphs.

Theorem 3.6 Let $G$ be a graph and $F$ an induced subgraph of $G$ such that $F_r(G) \geq n$. If $H$ is a graph obtained from $G$ by adding a new vertex $v$ and joining it to $n$ vertices of $G$, then $F_r(H) \geq n$.

Proof. Suppose that $F_r(H) \leq n - 1$. Then $H$ contains a set $S$ of at most $n - 1$ vertices such that $H - S$ is disconnected with each of its components a subgraph of $F$. If $v \in S$, then $S' = S - \{v\}$ is a set of at most $n - 2$ vertices such that $G - S'$ is disconnected with each component a subgraph of $F$. This contradicts the fact that $F_r(G) \geq n$. We may thus assume that $v \notin S$. 

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Since |S| ≤ n - 1, the set \( N(v) - S \) is nonempty, so that \( v \) belongs to a nontrivial component of \( H - S \). Hence
\( H - (S \cup \{v\}) = G - S \) is disconnected with each component a subgraph of \( F \), implying that \( Fr(G) \leq n - 1 \). This contradiction now proves the fact that \( Fr(H) \geq n \).

The next result is another analogue of a property of \( n \)-connected graphs.

**Theorem 3.7** Let \( G_1 \) and \( G_2 \) be \( F \)-\( n \)-connected graphs. Further, let \( u_1, u_2, \ldots, u_n \) be \( n \) vertices of \( G_1 \) and let \( v_1, v_2, \ldots, v_n \) be \( n \) vertices of \( G_2 \). If \( G \) is obtained from \( G_1 \) and \( G_2 \) by adding the edges \( u_i v_i \), \( 1 \leq i \leq n \), then \( G \) is \( F \)-\( n \)-connected.

**Proof.** Suppose, to the contrary, that \( Fr(G) \leq n - 1 \). Then there is a set \( S \) of at most \( n - 1 \) vertices such that \( G - S \) is disconnected with each component a subgraph of \( F \). If \( G_1 - S \) or \( G_2 - S \) contains a component that is not a subgraph of \( F \), then \( G - S \) also contains such a component. We may therefore assume that \( G_1 - S (i = 1, 2) \) is connected. This implies, however, that \( G - S \) is connected, contrary to our assumption. Hence \( Fr(G) \geq n \).

We know that if \( G \) is \( n \)-connected and if \( u \) and \( v \) are two nonadjacent vertices of \( G \), then \( G + uv \) is also \( n \)-connected (see
(12). As an analogue to this result we have the following.

**Theorem 3.8** Let \( G \) be a graph and \( F \) an induced subgraph of \( G \). If \( u \) and \( v \) are nonadjacent vertices of \( G \) such that \( G + uv \) contains \( F \) as an induced subgraph, then \( Fr(G) \leq Fr(G + uv) \).

**Proof.** Since \( Fr(G) \geq n \), the order of \( G \) is at least \( n + 1 \). Suppose now that \( Fr(G + uv) \leq n - 1 \). Then there is a set of at most \( n - 1 \) vertices of \( G + uv \) such that \( G + uv - S \) is disconnected with each component a subgraph of \( F \). Since \( G - S \) is a spanning subgraph of \( G + uv - S \), it follows that \( G - S \) is disconnected with each component a subgraph of \( F \) so that \( Fr(G) \leq |S| \leq n - 1 \). This contradicts the fact that \( Fr(G) \geq n \). Hence \( Fr(G + uv) \geq n \).

For a connected graph \( G \) and an induced subgraph \( F \) of \( G \) we define the **F-edge-connectivity** \( Fr_1(G) \) of \( G \) as the minimum cardinality of a set \( E \) of edges of \( G \) such that \( G - E \) is either the trivial graph or a disconnected graph with each of its components a subgraph of \( F \). If \( F \not\subseteq G \), then \( Fr_1(G) = r_1(G) \). That \( Fr_1(G) \) does not always equal \( r_1(G) \) can be seen by considering the graphs \( G \) and \( F \) of Figure 3.1.

For the graph \( G \) of Figure 3.1, observe that \( r_1(G) = 1 \) since \( G \) is connected and contains a bridge. Because \( q(G) = 9 \) and \( p(G) = 6 \), we need to delete at least four edges from \( G \) to produce an acyclic subgraph of \( G \). If we wish to produce a
disconnected acyclic subgraph of $G$ by deleting some edges of $G$, we need to delete at least five edges, so that $\tau_{1}(G) \geq 5$.

However, since

$$G - \{u_{2}u_{3}, u_{2}u_{4}, u_{2}w, u_{3}w, u_{4}w\}$$

is a disconnected graph with each of its components a subgraph of $F$, it follows that $\tau_{1}(G) \leq 5$. Consequently, $\tau_{1}(G) = 5$.

The following theorem gives bounds on the $F$-edge-connectivity and may be regarded as an analogue to Theorem 3.2.

**Theorem 3.9** Let $G$ be a nontrivial graph and $F$ a proper induced subgraph of $G$. Then

$$\tau_{1}(G) \leq \tau_{1}(G) \leq q(G) - q(F).$$

**Proof.** Since $G$ is nontrivial, we need to delete at least $\tau_{1}(G)$ edges from $G$ to produce a disconnected graph, so that we need to delete at least $\tau_{1}(G)$ edges of $G$ to produce a disconnected graph each of whose components is a subgraph of $F$. Hence $\tau_{1}(G) \leq \tau_{1}(G)$. Further, since $F$ is a proper induced subgraph of $G$, deletion of the edges in $E(G) - E(F)$ from $G$ produces a disconnected graph each of whose components is a subgraph of $G$, implying that $\tau_{1}(G) \leq q(G) - q(F)$.

Hence

$$\tau_{1}(G) \leq \tau_{1}(G) \leq q(G) - q(F).$$
To see that the lower bound given in the preceding theorem is sharp, let \( G_1 \otimes G_2 \otimes K_{n+1} \) and let \( G \) be the graph obtained from \( G_1 \) and \( G_2 \) by joining \( v \in V(G_1) \) to \( n \) distinct vertices \( u_1, u_2, \ldots, u_n \) of \( G_2 \). Further, let \( F \otimes K_{n+1} \). Then \( \chi_1(G) = n \) and since
\[
G = \{vu_i | 1 \leq i \leq n\} \otimes 2K_{n+1},
\]
it follows that \( F \chi_1(G) = n \), so that the lower bound is sharp.

To see that the upper bound on \( F \chi_1(G) \) given above is best possible in general, we consider, for each pair \( n, k \) of positive integers with \( n \leq k \), the graph \( G \) obtained by joining \( n \) vertices of \( K_{k+1} \) to a new vertex \( v \). Let \( F \) be the induced subgraph of \( G \) obtained by deleting a vertex of \( G \) that is not adjacent to \( v \). Then \( \chi_1(G) = n \) and \( q(G) - q(F) = k \). Let \( E \) be a set of \( F \chi_1(G) \) edges such that \( G - E \) is disconnected with each of its components a subgraph of \( F \). If \( F \chi_1(G) \leq k - 1 \), then it follows, since \( G - E \) is disconnected and \( K_{k+1} \) is \( k \)-edge-connected, that \( E \) contains the \( n \) edges incident with \( v \) in \( G \). Hence \( G - E \) consists of two components, one of which is trivial and the other contains at least
\[
\left( \begin{array}{c} k+1 \\ 2 \end{array} \right) - (k - 1 - n) = \frac{k^2 - k + 2n + 2}{2}
\]
edges. However, \( F \) has
\[
\binom{k}{2} + n = \frac{k^2 - k + 2n}{2}
\]

edges. This contradicts the fact that every component of \( G - E \) is a subgraph of \( F \), so that \( \delta_1(G) \geq k \). By the preceding theorem, \( \delta_1(G) \leq k \). Hence \( \delta_1(G) = k \).

Whitney [47] established the following relationship between the connectivity and edge-connectivity of a graph.

**Theorem 3A** (Whitney) For any graph \( G \),

\[
\tau(G) \leq \delta_1(G) \leq \delta(G).
\]

The next result may be regarded as an analogue of Whitney's theorem with respect to \( F \)-connectivity.

**Theorem 3.10** Let \( G \) be a graph and \( F \) an induced subgraph of \( G \). Then

\[
\delta_1(G) \leq \delta_1(G).
\]

**Proof.** Suppose first that \( \delta_1(G) = 0 \). Then \( G \) is either trivial or disconnected with each component a subgraph of \( F \). In either case, \( \delta(G) = 0 \) and the theorem follows.

Assume now that \( \delta_1(G) \geq 1 \). Consider first the case where \( G \) is disconnected, and let \( u \) and \( v \) be vertices of \( G \) that belong to distinct components of \( G \). Let \( E = \{e_1, e_2, \ldots, e_k\} \) be a
set of \( F_1(G) = k \) edges of \( G \) such that \( G - E \) is disconnected with each component a subgraph of \( F \). For each \( e_i \) (\( 1 \leq i \leq k \)), select an incident vertex different from \( u \) and \( v \). Then the removal of these vertices results in a disconnected graph with each component a subgraph of \( F \), so that the theorem also follows in this case.

Consider next the case where \( G \) is connected. If \( F_r(G) = 1 \), then \( G \) contains a bridge whose deletion produces a disconnected graph with each component a subgraph of \( F \). If \( G \not\subseteq K_2 \), then \( F_r(G) = 1 \); otherwise, \( G \) has a cut-vertex whose removal from \( G \) produces a disconnected graph with each component a subgraph of \( F \), so that \( F_r(G) = 1 \). Suppose now that \( F_r(G) = k \geq 2 \). Let \( E = \{e_1, e_2, \ldots, e_k\} \) be a set of \( F_r(G) \) edges of \( G \) such that \( G - E \) is disconnected with each component a subgraph of \( F \). If \( e_1 = u_1v_1 \) is a bridge of \( G \), then it follows, since \( G \not\subseteq K_2 \), that at least one of \( u_1 \) and \( v_1 \) is a cut-vertex of \( G \). Suppose that \( u_1 \) is a cut-vertex of \( G \). Let \( w \) be a vertex different from \( v_1 \) that is adjacent to \( u_1 \). Then \( v_1 \) and \( w \) belong to distinct components of \( G - u_1 \). For each \( j \) (\( 2 \leq j \leq k \)), select a vertex \( w_j \) incident with \( e_j \) such that \( w_j \not\in \{v_1, w\} \). Further, let \( w_1 = u_1 \) and \( S = \{w_1, w_2, \ldots, w_k\} \). Then \( G - S \) is disconnected with each component a subgraph of \( F \); so that in this case \( F_r(G) \leq F_r(G) \).
Assume now that $e_1$ is not a bridge of $G$. Let $\lambda$ be the smallest integer ($2 \leq \lambda \leq k$) such that $G - \{e_1, e_2, \ldots, e_{\lambda-1}\}$ is connected and $G - \{e_1, e_2, \ldots, e_{\lambda}\}$ is disconnected. Let $e_{\lambda} = u_{\lambda}v_{\lambda}$. For each $e_i$ ($1 \leq i \leq \lambda - 1$), select an incident vertex $w_i$ such that $w_i \notin \{u_{\lambda}, v_{\lambda}\}$. If the removal of the vertices in $S = \{w_1, w_2, \ldots, w_{\lambda-1}\}$ produces a disconnected graph, then let $w$ be a vertex in a component of $G - S$ that does not contain $e_{\lambda}$. For each $j$ ($\lambda + 1 \leq j \leq k$), select a vertex $w_j$ incident with $e_j$ that is different from $v_{\lambda}$ and $w$. Let $w_j = u_{\lambda}$ and $S' = \{w_1, w_2, \ldots, w_{\lambda}\}$. Then $G - S'$ is disconnected with each component a subgraph of $F$. Hence

$$\text{Fr}(G) \leq |S'| \leq k = \text{Fr}_1(G).$$

If $G - S$ is connected, then $e_{\lambda}$ is a bridge of $G - S$ so that $G - S \cong K_2$ or $G - S$ contains a cut-vertex. If $G - S \cong K_2$, then removing one additional vertex produces the trivial graph so that $\text{Fr}(G) \leq \lambda \leq k$. Suppose, finally, that $e_{\lambda}$ is a bridge of $G - S$ and that $u_{\lambda}$ is a cut-vertex of $G - S$. Let $w$ be a vertex of $G - S$ different from $v_{\lambda}$ that is adjacent to $u_{\lambda}$. Then $v_{\lambda}$ and $w$ belong to distinct components of $G - (S \cup \{u_{\lambda}\})$. For each $j$ ($\lambda + 1 \leq j \leq k$), choose a vertex $w_j$ incident with $e_j$ such that $w_j \notin \{v_{\lambda}, w\}$. Further, let $w_j = u_{\lambda}$ and $S' = \{w_1, w_2, \ldots, w_{\lambda}\}$. Then $G - S'$ is disconnected with each
component a subgraph of $F$, so that
\[ Fr(G) \leq |S'| \leq k = Fr_1(G). \]

Chartrand and Harary [8] showed that Whitney's theorem (Theorem 3A) is best possible.

Theorem 3B (Chartrand - Harary) For positive integers $m$, $n$ and $k$, where $m \leq n \leq k$, there exists a graph $G$ with $r(G) = m$, $r_1(G) = m$ and $\delta(G) = k$.

If we let $F = G$, then it follows from Theorem 3B that Theorem 3.10 is best possible. Notice, however, that $p(F)/p(G) \neq 1$ for this choice of $F$. The next theorem illustrates that we may choose $G$ and $F$ in such a way that $p(F)/p(G) \neq 1$.

Theorem 3.11 Let $m$ and $n$ be positive integers such that $m \leq n$. Then there exist a connected graph $G$ and a proper induced subgraph $F$ of $G$ such that $Fr(G) = m$ and $Fr_1(G) = n$.

Proof. Let $H_1 \cong H_2 \cong K_{n+2}$ and suppose that $V(H_1) = \{v_1, v_2, \ldots, v_{n+2}\}$ and $V(H_2) = \{w_1, w_2, \ldots, w_{n+2}\}$. Let $G$ be obtained from $H_1$ and $H_2$ by adding the edges in
\[ E = \{v_iw_i | 1 \leq i \leq m\} \cup \{w_jw_i | m + 1 \leq j \leq n\} \]
and let $F \cong K_{n+2}$.
We show first that \( Fr(G) = m \). Let \( S = \{v_1, v_2, \ldots, v_m\} \) and \( G - S \cong K_{n-m+2} \cup K_{n+2} \), so that \( G - S \) is disconnected with each component a subgraph of \( F \). Hence \( Fr(G) \leq m \). Suppose now that \( S' \) is a set of \( m - 1 \) vertices of \( G \). Since \( \kappa(H_i) = n + 1 \geq m - 1 \) \( (i = 1, 2) \), both \( H_1 - S' \) and \( H_2 - S' \) are connected. Further, there is at least one \( i \in \{1, 2, \ldots, m\} \) such that \( v_i', w_i \notin S' \). Hence \( G - S' \) is connected, so that \( Fr(G) > m - 1 \). Consequently, \( Fr(G) = m \).

We show next that \( Fr_1(G) = n \). Since \( G - E \cong 2K_{n+2} \), each of the two components of \( G - E \) is a subgraph of \( F \). Hence \( Fr_1(G) \leq n \). Suppose now that \( E' \) is a set of \( n - 1 \) edges. Then \( H_1 - E' \) and \( H_2 - E' \) are connected. Further, \( E - E' \neq \emptyset \), implying that \( G - E' \) is connected. Hence \( Fr_1(G) \geq n \), so that \( Fr_1(G) = n \).

From Theorem 3.11 we now deduce that for every pair \( m, n \) of positive integers where \( m \leq n \) and every real number \( \varepsilon \) with \( 0 < \varepsilon \leq 1 \), there exist a graph \( H \) and induced subgraph \( F \) of \( H \) such that \( Fr(H) = m \), \( Fr_1(H) = n \) and \( \frac{p(F)}{p(H)} < \varepsilon \). To see this, let \( G \) and \( F \) be as in Theorem 3.11 and set \( H = G \cup rF \), where \( r \) is a positive integer chosen sufficiently large so that \( \frac{p(F)}{p(H)} < \varepsilon \). Notice, however, that in this case \( H \) is not connected. For a connected graph \( G \) and induced subgraph \( F \) of \( G \) having fixed \( F \)-edge-connectivity, the ratio \( \frac{p(F)}{p(G)} \) cannot be too small. In fact, if \( G \) is a nontrivial connected graph and
If F is an induced subgraph of G with \( \text{Fr}(G) = m \) and \( \text{Fr}_1(G) = n \), then

\[
\frac{1}{n+1} \leq \frac{p(F)}{p(G)} \leq 1.
\]

Clearly \( p(F)/p(G) \leq 1 \).

To establish the lower bound, let \( E = \{e_1, e_2, \ldots, e_n\} \) be a set of \( n = \text{Fr}_1(G) \) edges of G such that \( G - E \) is disconnected with each component a subgraph of \( F \). Let \( G_0 = G \) and \( G_1 = G - \{e_1, e_2, \ldots, e_i\} \) (\( 1 \leq i \leq n \)), and observe that \( G_{j+1} \) has at most one more component than \( G_j \) (\( 0 \leq j \leq n - 1 \)). Hence \( G - S = G_n \) has at most \( n + 1 \) components and each of these components has at most \( p(F) \) vertices. Therefore \( p(G) \leq (n + 1)p(F) \) so that \( p(F)/p(G) \geq \frac{1}{n+1} \).

We observe next that the lower bound on \( p(F)/p(G) \) given in the preceding discussion is best possible by showing that for every pair \( m, n \) of positive integers such that \( m \leq n \), there is a connected graph \( G \) and an induced subgraph \( F \) of \( G \) such that \( \text{Fr}(G) = m \), \( \text{Fr}_1(G) = n \) and \( p(F)/p(G) = \frac{1}{n+1} \).

Let \( k \) be an integer such that \( k \geq n + 1 \). Let \( H_0 \leq H_1 \leq \cdots \leq H_n \leq K_k \). For \( i \in \{1, 2, \ldots, n\} \), let \( v_i \in V(H_i) \) and let \( u_1, u_2, \ldots, u_m \) be \( m \) vertices of \( H_0 \). Define \( G \) to be the graph obtained from \( \bigcup_{j=0}^{n} H_j \) by adding the edges in
\[ E' = \{u_i v_i \mid 1 \leq i \leq m\} \cup \{u_m v_j \mid m + 1 \leq j \leq n\} \]

and let \( F \neq F_m \). Then \( p(F)/p(G) = \frac{1}{n+1} \).

We show now that \( Fr(G) = m \). Since \( G - \{u_1, u_2, \ldots, u_m\} \)

is a disconnected graph with each component a subgraph of \( F \), it

follows that \( Fr(G) \leq m \). Suppose now that \( S \) is a set of at most

\( m - 1 \) vertices. Then there exists some \( i \in \{1, 2, \ldots, m\} \) such

that

\[(V(H_i) \cup \{u_i\}) \cap S = \emptyset.\]

Hence \( G - S \) contains a component (possibly \( G - S \) itself) that

is not a subgraph of \( F \). Therefore \( Fr(G) \geq m \), implying that

\( Fr(G) = m \).

It remains to be shown that \( Fr_1(G) = n \). Since \( G - E' \)

is disconnected with each component a subgraph of \( F \), it follows

that \( Fr_1(G) \leq n \). Suppose that \( E \) is a set of \( n - 1 \) edges.

Then \( H_i - E \) is connected for every \( i \) \((0 \leq i \leq n) \). Further,

there exists at least one edge \( e \in E' \) such that \( e \notin E \). Hence

\( e = u_i v_j \) where \( j = i \) or \( i = m \), and \( \langle V(H_j) \cup \{u_i\} \rangle - E \)

is a connected subgraph of \( G - E \) having order \( p(F) + 1 \). Con­

sequently, \( Fr_1(G) \geq n \), so that \( Fr_1(G) = n \).

Section 3.3 The F-Free-Connectivity of a Graph.

Let \( F \) and \( G \) be graphs. Then \( G \) is said to be \( F \)-free if
G does not contain F as an induced subgraph. For a nontrivial graph G and a nontrivial induced subgraph F of G, define the F-free-connectivity \( \hat{\kappa}(G) \) as the minimum number of vertices whose removal from G produces the trivial graph or a disconnected graph with each component F-free.

As in the case of F-connectivity, it is true that if G is a nontrivial complete graph of order p, then for every nontrivial induced subgraph F of G, \( \kappa(G) = p - 1 = \hat{\kappa}(G) \). Assume thus that G is not complete and that F is a nontrivial induced subgraph of G. Since at least \( \kappa(G) \) vertices need to be deleted from G to produce a disconnected graph, \( \hat{\kappa}(G) \geq \kappa(G) \). To see that the F-free-connectivity need not equal the F-connectivity, consider the graphs F and G of Figure 3.3. Observe that \( \kappa(G) = 1 \) since G is connected and because \( u_2 \) and \( u_4 \) are cut-vertices of G. However, both \( G - u_2 \) and \( G - u_4 \) have a component containing F as an induced subgraph, so that \( \hat{\kappa}(G) \geq 2 \). On the other hand, \( G - \{u_2, u_4\} \) is disconnected with each component F-free, so that \( \hat{\kappa}(G) \leq 2 \). Hence \( \hat{\kappa}(G) = 2 \neq \kappa(G) \).

![Figure 3.3](image-url)
If \( G \) is a nonempty graph and \( F \neq K_2 \), then \( \text{Fr}(G) = p - \beta(G) \), so that \( \text{Fr}(G) = \alpha(G) \).

Suppose now that \( G \) is a nontrivial graph and \( F \) a nontrivial induced subgraph of \( G \). Then \( F = G - S \) for some set \( S \) of vertices of \( G \) with \( 0 \leq |S| \leq p - 2 \). If \( F = G \) or if \( G \) is complete, then \( \text{Fr}(G) = \kappa(G) \). Hence, if \( \text{Fr}(G) \neq \kappa(G) \), then \( G \) is not complete and \( 1 \leq |S| \leq p - 2 \). The following theorem provides bounds on \( \kappa(G) \) and \( \text{Fr}(G) \) whenever \( \text{Fr}(G) \neq \kappa(G) \).

**Theorem 3.12** Let \( G \) be a noncomplete graph and \( F \) a proper, nontrivial induced subgraph of \( G \) such that \( \text{Fr}(G) \neq \kappa(G) \). Then

\[
\kappa(G) + 1 \leq \text{Fr}(G) \leq p(G) - p(F).
\]

**Proof.** We show first that \( \kappa(G) \leq p(G) - p(F) - 1 \). Assume, to the contrary, that \( \kappa(G) > p(G) - p(F) \), and let \( S \) be a set of \( \kappa(G) \) vertices such that \( G - S \) is disconnected. Since \( \kappa(G) \geq p(G) - p(F) \), every component of \( G - S \) has order at most \( p(F) - 1 \). This implies that every component of \( G - S \) is \( F \)-free and that \( \text{Fr}(G) \leq |S| = \kappa(G) \). Since \( \kappa(G) \leq \text{Fr}(G) \), it follows that \( \kappa(G) = \text{Fr}(G) \), contrary to our assumption that \( \text{Fr}(G) \neq \kappa(G) \). Consequently, \( \kappa(G) \leq p(G) - p(F) - 1 \).

Next we show that \( \text{Fr}(G) \leq p(G) - p(F) \). Suppose to the contrary, that \( \text{Fr}(G) > p(G) - p(F) + 1 \). Then \( G \) has at least \( p(G) - p(F) + 2 \) vertices. To see this, observe first, since \( \kappa(G) \leq p(G) - p(F) - 1 \), that there exists a set \( S \) of
p(G) - p(F) vertices such that G - S is disconnected. Since p(G - S) = p(F), it now follows that every component of G - S contains at most p(F) - 1 vertices, so that every component of G - S is F-free. Hence \( \tilde{\chi}(G) \leq p(G) - p(F) \), contrary to our assumption that \( \tilde{\chi}(G) \geq p(G) - p(F) + 1 \). Therefore \( \tilde{\chi}(G) \leq p(G) - p(F) \).

The preceding theorem is best possible, as we now show.

**Theorem 3.13** Let \( n \) and \( k \) be integers with \( 0 \leq k < n \). For every \( j \) (\( 0 \leq j \leq n - k - 1 \)), there exists a noncomplete graph \( G \) and a proper induced subgraph \( F \) of \( G \) such that \( p(G) - p(F) = n \), \( \chi(G) = k \) and \( \tilde{\chi}(G) = n - j \).

**Proof.** Suppose first that \( k \geq 1 \), so that \( n \geq 2 \). Let \( H_1 \cong \overline{K}_{j+1} \), \( H_2 \cong K_{2n-k-j-1} \) and \( H \cong \overline{K}_k \). Set \( G = H + (H_1 \cup H_2) \) and observe that \( p(G) = 2n \). Let \( F \cong \overline{K}_n \) and notice that \( p(G) - p(F) = n \). Since \( G - V(H) \) is disconnected, \( \chi(G) \leq k \). If \( S \) is a set of at most \( k - 1 \) vertices, then \( V(H) - S \) is non-empty. As every vertex of \( V(H) \) is joined to every vertex of \( V(H_i) - S \) (\( i = 1, 2 \)) and since \( \chi(K_{2n-k-j-1}) \geq n - 1 \geq k \), it follows that \( G - S \) is connected, so that \( \chi(G) \geq k \). Hence \( \chi(G) = k \).

It remains to be shown that \( \tilde{\chi}(G) = n - j \). Let \( S_2 \) be a set of \( n - k - j \) vertices of \( H_2 \) and let \( S = V(H) \cup S_2 \). Then \( G - S \cong \overline{K}_{j+1} \cup \overline{K}_{n-1} \), so that \( G - S \) is disconnected with
each component $F$-free. Hence $\hat{r}(G) \leq n - j$. Suppose now that $S$ is a set of at most $n - j - 1$ vertices such that $G - S$ is disconnected. Then $V(H) \subseteq S$; for otherwise, since $H_2 - S$ is connected and every vertex of $V(H_1) - S$ ($i = 1, 2$) is joined to every vertex of $V(H) - S$, it follows that $G - S$ is connected. Consequently, $S$ contains at most $n - k - j - 1$ vertices of $H_2$, implying that $K_n$ is an induced subgraph of a component of $G - S$, so that $\hat{r}(G) \geq n - j$. Hence $\hat{r}(G) = n - j$.

Suppose next that $k = 0$. Let $H_1 \equiv K_{j+1}$ and $H_2 \equiv K_{2n-j-1}$. Define $G = H_1 \cup H_2$, and $F \equiv K_n$. Then $p(G) - p(F) = n$ and $r(G) = 0$. Since $H_2 \equiv K_{2n-j-1}$, we need to delete at least $n - j$ vertices from $G$ to produce a disconnected graph each of whose components is $F$-free. Hence $\hat{r}(G) \geq n - j$. However, if $S$ is a set of $n - j$ vertices of $H_2$, then $G - S$ is disconnected and each component is $F$-free. Hence $\hat{r}(G) \leq n - j$. Consequently, $\hat{r}(G) = n - j$.

Suppose now that $K_1 \not\subseteq H < F < G$, and let $S$ be a set of $\hat{r}(G)$ vertices such that $G - S$ is disconnected with each component $H$-free. Then every component of $G - S$ is also $F$-free, so that $\hat{r}(G) \leq \hat{r}(G)$. Hence if $F_1, F_2, \ldots, F_k$ are subgraphs of $G$ such that $K_1 \not\subseteq F_1 < F_2 < \ldots < F_k = G$, then

$\hat{r}(G) \geq \hat{r}(G) \geq \ldots \geq \hat{r}(G) = r(G)$.
This observation leads to our next result.

Theorem 3.14 Let $n_1, n_2, \ldots, n_k$ be positive integers such that $n_1 > n_2 > \ldots > n_k$. Then there exists a graph $G$ and induced subgraphs $F_1, F_2, \ldots, F_k (= G)$ of $G$ such that

$$K_1 \leq F_1 < F_2 < \cdots < F_k (= G)$$

and $\hat{\kappa}_i(G) = n_i$ $(1 \leq i \leq k)$.

Proof. Let $H \neq K_{n_1-n_k+2}$, $H_2 \neq K(n_1+1,n_1+1)$ and $H \neq K_{n_k}$.
Let $G = H + (H_1 \cup H_2)$, $F_k = G$ and $F_i \neq K_{n_i-n_j+3}$ $(1 \leq i \leq k-1)$.
Observe that

$$F_1 < F_2 < \cdots < F_k (= G).$$

Since $G - V(H)$ is disconnected, it follows that $\kappa(G) = \hat{\kappa}_k(G) \leq n_k$. If $S$ is a set of at most $n_k - 1$ vertices of $G$, then $V(H) - S$ is nonempty and $H_2 - S$ is connected, implying that $G - S$ is connected. Hence $\kappa(G) \geq n_k$. Let $S'$ be a set of $n_i - n_k$ vertices of $H_1$ $(1 \leq i \leq k-1)$ and let $S = V(H) \cup S'$.
Then

$$G - S \neq K(n_1+1,n_1+1) \cup K_{n_1-n_1+2}.$$ 

Since both $K(n_1+1,n_1+1)$ and $K_{n_1-n_1+2}$ are $F_i$-free, it
follows that \( F_r(G) \leq n \). If \( S \) is a set of at most \( n - 1 \) vertices of \( G \) such that \( G - S \) is disconnected, then \( \mathcal{V}(H) \subseteq S \); otherwise, \( G - S \) is connected. Consequently, \( S \) contains at most \( n - 1 - n_k \) vertices of \( H \), so that \( G - S \) contains an induced subgraph isomorphic to \( K_{n_1 - n_1 + 3} \), implying that \( G - S \) is not \( F \)-free. Hence \( F(G) \geq n_1 \), so that \( F_k(G) = n_1 \).

Define a graph \( G \) to be \( F \)-free \( n \)-connected if \( F_r(G) \geq n \).

We now consider several well-known results for \( n \)-connected graphs which have analogues in the theory of \( F \)-free \( n \)-connected graphs.

It is known that if \( G \) is \( n \)-connected, then \( G + K_1 \) is \((n+1)\)-connected. Corresponding to this result, we have the following.

**Theorem 3.15** Let \( G \) be a nontrivial graph and \( F \) a nontrivial induced subgraph of \( G \). If \( G \) is \( F \)-free \( n \)-connected, then the graph \( H = G + K_1 \) is \( F \)-free \((n+1)\)-connected.

**Proof.** If \( G \) is complete, then \( H \) is complete and the result follows. Since \( F_r(G) \geq n \), it follows that \( p(G) \geq n + 1 \), so that \( p(H) \geq n + 2 \). Suppose that \( F_r(H) \leq n \). Then there exists a set \( S \) of at most \( n \) vertices of \( H \) such that \( H - S \) is disconnected, each component of which is \( F \)-free. Since \( H - S \) is disconnected, the vertex \( v \) of \( V(H) - V(G) \) is contained in \( S \).
Hence \( G - (S - \{v\}) \) is disconnected with each component \( F \)-free, so that

\[
\widehat{Fr}(G) \leq |S - \{v\}| = n - 1,
\]

contrary to our hypothesis. Consequently, \( \widehat{Fr}(H) \geq n + 1 \).

Recall that if \( H \) is a graph obtained from an \( n \)-connected graph \( G \) by joining a new vertex to \( n \) distinct vertices of \( G \), then \( H \) is also \( n \)-connected. The next theorem may be regarded as an analogue to this result with respect to \( F \)-free \( n \)-connectivity.

**Theorem 3.16** Let \( G \) be a nontrivial graph and \( F \) a nontrivial induced subgraph of \( G \) such that \( \widehat{Fr}(G) \geq n \). Let \( v_1, v_2, \ldots, v_n \) be \( n \) distinct vertices of \( G \) and let \( H \) be the graph obtained from \( G \) by adding a new vertex \( v \) joined to each \( v_i \) (\( 1 \leq i \leq n \)). Then \( H \) is \( F \)-free \( n \)-connected.

**Proof.** Suppose that \( \widehat{Fr}(G) \leq n - 1 \). Then there exists a set \( S \) of \( n - 1 \) vertices of \( H \) such that \( H - S \) is disconnected with each component \( F \)-free. If \( v \in S \), then \( G - (S - \{v\}) \) is disconnected with each component \( F \)-free so that

\[
\widehat{Fr}(G) \leq |S - \{v\}| = n - 2,
\]

contrary to our hypothesis. Assume thus that \( v \notin S \). Since \( |S| = n - 1 \) and as \( \deg_H v = n \), it follows that \( v \) is contained
in a nontrivial component of $H - S$. Hence $H - (S \cup \{v\})$ is disconnected with each component $F$-free. Consequently, $G - S$ is disconnected with each component $F$-free, so that $\hat{Fr}(G) \leq |S| \leq n-1$. This contradiction establishes the fact that $\hat{Fr}(H) \geq n$. \hfill \Box

The following result parallels yet another property of $n$-connected graphs and is stated without proof.

**Theorem 3.17** Let $G_1$ and $G_2$ be $F$-free $n$-connected graphs. Suppose that $u_i$ ($i = 1, 2, \cdots, n$) are $n$ distinct vertices of $G_1$ and $v_i$ ($i = 1, 2, \cdots, n$) are $n$ distinct vertices of $G_2$. Then the graph $G$, defined by adding the edges $u_i v_i$ ($1 \leq i \leq n$) to $G_1 \cup G_2$, is $F$-free $n$-connected.

We now illustrate the fact that not all results that hold for $n$-connected graphs have analogues for $F$-free $n$-connected graphs. If $G$ is $n$-connected, then for every pair $u, v$ of nonadjacent vertices of $G$, the graph $G + uv$ is also $n$-connected. However, if $G$ is $F$-free $n$-connected, than $G + uv$ need not be $F$-free $n$-connected. Observe first that the graph obtained by joining a pair of nonadjacent vertices $u$ and $v$ of $G$ need no longer contain $F$ as an induced subgraph so that the $F$-free connectivity of $G + uv$ may not be defined. However, even if $G + uv$ contains $F$ as an induced subgraph, then it need not be the case that $\hat{Fr}(G + uv) \geq n$, as we shall now see.
For $n \geq 3$, let $H = K_{n-2}$ and $H_1 \equiv H_2 \equiv K(n,n)$. Let $G = H + (H_1 \cup H_2)$ and let $F \equiv K(n,n)$. Suppose that $v_i \in V(H_i)$ ($i = 1, 2$). Then $G - V(H) \cup \{v_1, v_2\}$ is isomorphic to $K(n, n-1)$, so that $G - (V(H) \cup \{v_1, v_2\})$ is disconnected with each component $F$-free. Hence $\tilde{\nu}(G) \leq n$. Suppose now that $S$ is a set of $n - 1$ vertices such that $G - S$ is disconnected. Then $V(H) \subseteq S$; otherwise, since $V(H_1) - S$ is nonempty and every vertex of $V(H) - S$ is joined to every vertex of $V(G) - V(H)$, the graph $G - S$ is connected. Hence $S$ contains exactly one vertex from either $H_1$ or $H_2$ so that $G - S \equiv K(n, n-1) \cup K(n,n)$, implying that $G - S$ contains a component that is not $F$-free. Therefore $\tilde{\nu}(G) \geq n$, so that $\tilde{\nu}(G) = n$. Let $u$ and $v$ be distinct vertices belonging to the same partite set of $H_2$, and consider $G + uv$. Let $w \in V(H_1)$. Then $G + uv - (V(H) \cup \{w\})$ is disconnected with each component $F$-free. Hence $\tilde{\nu}(G + uv) \leq n - 1$. Consequently,

$$n = \tilde{\nu}(G) > \tilde{\nu}(G + uv) = n - 1.$$ 

If we assume that the $F$-free connectivity of $G + uv$ is defined, then the following theorem provides bounds on the $F$-free connectivity of $G + uv$ in terms of the $F$-free connectivity of $G$. 

**Theorem 3.18** Let $G$ be a nontrivial graph and $F$ a nontrivial induced subgraph of $G$. If $G$ contains a pair $u, v$ of non-adjacent vertices such that $G + uv$ has an induced subgraph
isomorphic to $F$, then

$$\overline{Fr}(G) - 1 \leq \overline{Fr}(G + uv) \leq \overline{Fr}(G) + 1.$$  

Proof. Let $\overline{Fr}(G) = n$ and suppose, to the contrary, that $\overline{Fr}(G + uv) \leq n - 2$. Then $G + uv$ contains a set $S$ of at most $n - 2$ vertices such that $G + uv - S$ is disconnected with each component $F$-free. If at least one of $u$ and $v$ belongs to $S$, then $G + uv - S = G - S$, so that $\overline{Fr}(G) \leq n - 2$, contrary to the fact that $\overline{Fr}(G) = n$. Suppose thus that neither $u$ nor $v$ belongs to $S$. Then $u$ and $v$ belong to the same component of $G + uv - S$, so that

$$G + uv - (S \cup \{v\}) = G - (S \cup \{v\})$$

is disconnected with each component $F$-free. Hence $\overline{Fr}(G) \leq n - 1$, contrary to our assumption that $\overline{Fr}(G) = n$. We conclude then that $\overline{Fr}(G) - 1 \leq \overline{Fr}(G + uv)$.

To show that $\overline{Fr}(G + uv) \leq \overline{Fr}(G) + 1$, we consider two cases. Suppose first that $G$ contains a set $S$ of $\overline{Fr}(G) = n$ vertices such that at least one of $u$ and $v$ belongs to $S$ and such that $G - S$ is disconnected with each component $F$-free. Then $G + uv - S = G - S$ is disconnected with each component $F$-free so that $\overline{Fr}(G + uv) \leq \overline{Fr}(G)$.

Assume next that every set $S$ of $\overline{Fr}(G)$ vertices of $G$, for which $G - S$ is disconnected with each component $F$-free, contains
neither u nor v. Let S' be such a set S. If
\( V(G - S') = \{u, v\} \), then \( G + uv - (S' \cup \{v\}) \) is the trivial
graph; otherwise, \( G + uv - (S' \cup \{v\}) \) or \( G + uv - (S' \cup \{u\}) \)
is disconnected with each component F-free. Hence \( \tilde{\chi}(G + uv) \leq \tilde{\chi}(G) + 1 \). We may now conclude that
\[
\tilde{\chi}(G) - 1 \leq \tilde{\chi}(G + uv) \leq \tilde{\chi}(G) + 1.
\]

The discussion preceding the theorem shows that the lower
bound on \( \tilde{\chi}(G + uv) \) given in Theorem 3.18 is sharp. We see now
that the upper bound on \( \tilde{\chi}(G + uv) \) given in Theorem 3.18 is also
sharp. For \( n \geq 2 \), let \( H = K_{n-1} \) and \( H_1 = K(n,n) \), and let \( H_2 \)
be a copy of \( K(n,n) \) from which an edge \( uv \), say, has been
deleted. Let \( G = H + (H_1 \cup H_2) \) and \( F = K(n,n) \). If \( v_1 \in V(H_1) \),
then \( G - (V(H) \cup \{v_1\}) \) is disconnected with each component
F-free. Hence \( \tilde{\chi}(G) \leq n \). If \( S \) is a set of \( n - 1 \) vertices of
G such that \( G - S \) is disconnected, then \( V(H) \subseteq S \); otherwise,
since \( V(H_1) - S \) is nonempty and every vertex of \( V(H) - S \) is
joined to every vertex of \( V(G) - V(H) \), the graph \( G - S \) is
connected. Consequently, \( G - S \) contains a component isomorphic
to \( K(n,n) \), so that not every component of \( G - S \) is F-free.
Hence \( \tilde{\chi}(G) \geq n \), implying that \( \tilde{\chi}(G) = n \).

Consider now the graph \( G + uv \). Observe that \( F \subseteq G + uv \).
Let \( v_i \in V(H_i), i = 1, 2 \). Then
\[
G + uv - (V(H) \cup \{v_1, v_2\}) \cong 2K(n, n - 1).
\]
Hence \( \hat{Fr}(G + uv) \leq n + 1 \). Let \( S \) be a set of \( n \) vertices such that \( G - S \) is disconnected. Then \( V(H) \subseteq S \); for otherwise, as before, \( G - S \) is connected. Consequently, \( S \) contains exactly one vertex from either \( H_1 \) or \( H_2 \). This implies that

\[
G - S \cong K(n, n - 1) \cup K(n, n)
\]

so that \( G - S \) contains a component that is not \( F \)-free. Hence \( \hat{Fr}(G + uv) \geq n + 1 \). We now conclude that

\[
\hat{Fr}(G + uv) = n + 1 = \hat{Fr}(G) + 1,
\]

so that the upper bound of Theorem 3.17 is sharp.

As a direct consequence of Theorem 3.17, we also have the following.

**Corollary** Let \( G \) be a nontrivial graph and \( F \) a nontrivial induced subgraph of \( G \). Suppose that \( \{u_1, v_1\}, \{u_2, v_2\}, \ldots, \{u_k, v_k\} \) are \( k \) sets of pairwise nonadjacent vertices of \( G \) such that \( G + \{u_1v_1, u_2v_2, \ldots, u_kv_k\} \) contains \( F \) as an induced subgraph. Then

\[
\hat{Fr}(G) - k \leq \hat{Fr}(G + \{u_1v_1, u_2v_2, \ldots, u_kv_k\}) \leq \hat{Fr}(G) + k.
\]

We show next that the bounds given in the preceding corollary are sharp. We verify first that the lower bound can be attained. Let \( n \) and \( k \) be positive integers. For \( i = 1, 2, \ldots, k + 1 \),
let $H_i = K(n + 1, n + 1)$ and let $H = K_n$. Set $G = H + (H_1 \cup H_2 \cup \cdots \cup H_{k+1})$ and let $F = K(n + 1, n + 1)$. For every $i = 1, 2, \cdots, k + 1$, let $v_i \in V(H_i)$ and let $S' = \{v_1, v_2, \cdots, v_{k+1}\}$. Since $G - (V(H) \cup S') \cong (k + 1)K(n, n + 1)$, it follows that $G - (V(H) \cup S')$ is disconnected with each component $F$-free so that $\hat{F}_r(G) \leq n + k + 1$. Suppose now that $S$ is a set of at most $n + k$ vertices such that $G - S$ is disconnected. Then $V(H) \subseteq S$; otherwise, since $\bigcup_{i=1}^{k+1} V(H_i) - S$ is nonempty and every vertex of $V(H) - S$ is joined to every vertex of $V(G) - V(H)$, the graph $G - S$ is connected. Hence $S$ contains at most $k$ vertices of $\bigcup_{i=1}^{k+1} V(H_i)$, so that there is at least one integer $i = 1, 2, \cdots, k + 1$ such that $V(H_i) \cap S = \emptyset$. However, then, $G - S$ contains a component that is not $F$-free. Therefore, $\hat{F}_r(G) \geq n + k + 1$. We may now conclude that $\hat{F}_r(G) = n + k + 1.$

For every $i = 1, 2, \cdots, k$, let $u_i$ and $w_i$ be two nonadjacent vertices of $H_i$ and consider $G' = G + \{u_1w_1, u_2w_2, \cdots, u_kw_k\}$. Observe that $H_i + u_iw_i$ is $F$-free for every $i (1 \leq i \leq k)$. 

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Since \( G' - (V(H) \cup \{v_{k+1}\}) \) is disconnected with each component \( F \)-free, \( \hat{\text{Fr}}(G') \leq n + 1 \). Suppose now that \( S \) is a set of \( n \) vertices such that \( G - S \) is disconnected. Then \( V(H) \subseteq S \); otherwise since \( V(H_1) - S \) is nonempty and every vertex of \( V(H) - S \) is joined to every vertex of \( V(G') - V(H) \), it follows that \( G - S \) is connected. Consequently, \( V(H) = S \) so that \( G - S \) contains a component that is not \( F \)-free. Hence \( \hat{\text{Fr}}(G') \geq n + 1 \), which implies that \( \hat{\text{Fr}}(G') = n + 1 \). Therefore,

\[
\hat{\text{Fr}}(G + \{u_1w_1, u_2w_2, \ldots, u_kw_k\}) = \hat{\text{Fr}}(G) - k.
\]

We show next that the upper bound in the preceding corollary is sharp. Let \( n \) and \( k \) be positive integers. Let \( H \cong K_n \), and for \( i = 1, 2, \ldots, k + 1 \), let \( H_i \cong K_{n+2} \). Suppose that \( u_i, v_i \in V(H_i) \), where \( 1 \leq i \leq k + 1 \). For \( i = 1, 2, \ldots, k + 1 \), define \( H'_i = H_i - u_iv_i \). Define

\[
G = H + (H'_1 \cup H'_2 \cup \cdots \cup H'_{k+1})
\]

and \( F \cong K_{n+2} \). Since \( G - V(H) \) is disconnected with each component \( F \)-free, \( \hat{\text{Fr}}(G) \leq n \). Suppose that \( S \) is a set of at most \( n - 1 \) vertices. Then, since \( V(H) - S \) and \( V(H'_1) - S \) are both nonempty and every vertex of \( V(H) - S \) is joined to every vertex of \( V(G) - V(H) \), the graph \( G - S \) is connected and non-trivial. Hence \( \hat{\text{Fr}}(G) \geq n \), so that \( \hat{\text{Fr}}(G) = n \). Let

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Let $S' = \{u_i | 1 \leq i \leq k\}$ and observe that $G' - (V(H) \cup S')$ is disconnected with each component $F$-free. Hence $\hat{F}(G') \leq n + k$. Suppose now that $S$ is a set of at most $n + k - 1$ vertices of $G'$ such that $G - S$ is disconnected. Then $V(H) \subseteq S$; otherwise, since $V(G) - (V(H) \cup S)$ is nonempty and every vertex of $V(H) - S$ is joined to every vertex of $V(G) - V(H)$, the graph $G - S$ is connected. Consequently, $S$ contains at most $k - 1$ vertices of $k = \bigcup_{i=1}^{k} V(H_i)$, implying that there exists at least one $i \in \{1, 2, \ldots, k\}$ such that $S \cap V(H_i) = \emptyset$. However, then, $G - S$ contains a component that is not $F$-free. Hence $\hat{F}(G') \geq n + k$. We conclude therefore that

$$\hat{F}(G + \{u_1v_1, u_2v_2, \ldots, u_kv_k\}) = \hat{F}(G) + k.$$ 

We now turn our attention to $F$-free edge-connectivity. Let $G$ be a nontrivial graph and $F$ a nontrivial induced subgraph of $G$. The $F$-free edge-connectivity of $G$ is the minimum cardinality of a set $E$ of edges of $G$ such that $G - E$ is disconnected with each of its components $F$-free.

Since we define $F$-free edge-connectivity only for nontrivial graphs $G$, the deletion of $\hat{F}_{1}(G)$ edges from $G$ always produces a disconnected graph. Hence $\hat{F}_{1}(G) \geq r_1(G)$. If we let $F = G$, then $\hat{F}_{1}(G) = r_1(G)$. That the $F$-free edge-connectivity does not
always equal the edge-connectivity of \( G \) can be seen by consider-
ing the graphs \( G \) and \( F \) of Figure 3.3. Since \( u_1u_2 \) and \( u_4u_5 \) are bridges of \( G \) and since \( G \) is connected, it follows that
\[ \kappa_1(G) = 1. \]
However, neither \( G - u_1u_2 \) nor \( G - u_4u_5 \) is \( F \)-free. Hence \( \tilde{\kappa}_1(G) > 1 \) so that \( \tilde{\kappa}_1(G) \neq \kappa_1(G) \). Since \( G \) contains no
triangles and \( F \neq P_3 \), we need to delete at least two edges
incident with \( u_i \) (\( i = 2, 4 \)) to produce a graph, each of whose
components is \( F \)-free. Hence \( \tilde{\kappa}_1(G) \geq 4 \). Since
\[ G - \{u_2u_6, u_2u_3, u_4u_6, u_4u_3\} \]
is disconnected with each component \( F \)-free, it follows that
\[ \tilde{\kappa}_1(G) \leq 4. \] Hence \( \tilde{\kappa}_1(G) = 4. \)

The following result may be regarded as an analogue to
Whitney's theorem.

**Theorem 3.19** If \( G \) is a nontrivial graph and \( F \) a nontrivial
induced subgraph of \( G \), then
\[ \tilde{\kappa}_r(G) \leq \tilde{\kappa}_r(G). \]

**Proof.** Suppose first that \( \tilde{\kappa}_r(G) = 0 \). Then \( G \) is disconnected
and each component is \( F \)-free so that \( \tilde{\kappa}(G) = 0 \). Assume thus that
\[ \tilde{\kappa}_r(G) \geq 1. \]

Suppose that \( G \) is disconnected. Let \( u \) and \( v \) be vertices
of \( G \) that belong to distinct components of \( G \), and let

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E = \{e_1, e_2, \ldots, e_k\} be a set of \(\tilde{\chi}_1(G) = k\) edges of G such that each component of \(G - E\) is F-free. For each \(e_i (1 \leq i \leq k)\), select an incident vertex different from \(u\) and \(v\). Then the removal of these vertices results in a disconnected graph, each of whose components is F-free, producing the desired result.

Consider now the case where G is connected. If \(\tilde{\chi}_1(G) = 1\), then G contains a bridge whose deletion produces a disconnected graph each of whose components is F-free. If \(G \neq K_2\), then \(F = G\) and \(\tilde{\chi}(G) = 1\); otherwise, G has a cut-vertex whose removal from G produces a disconnected graph each of whose components is F-free, so that \(\tilde{\chi}(G) = 1\). Assume, henceforth, that \(\tilde{\chi}_1(G) = k \geq 2\). Let \(E = \{e_1, e_2, \ldots, e_k\}\) be a set of \(\tilde{\chi}_1(G)\) edges such that \(G - E\) is a disconnected graph with each of its components F-free. If \(e_1 = u_1v_1\) is a bridge of G, then since \(G \neq K_2\), it follows that at least one of \(u_1\) and \(v_1\) is a cut-vertex of G. Suppose that \(u_1\) is a cut-vertex of G. Let \(w\) be a vertex different from \(v_1\) that is adjacent to \(u_1\). Then \(v_1\) and \(w\) belong to distinct components of \(G - u_1\). For each \(j (2 \leq j \leq k)\), select a vertex \(w_j\) incident with \(e_j\) such that \(w_j \notin \{v_1, w\}\). Further, let \(w_1 = u_1\) and \(S = \{v_1, w_2, \ldots, w_k\}\). Then \(G - S\) is disconnected and each component of \(G - S\) is F-free, so that \(\tilde{\chi}(G) \leq |S| \leq k = \tilde{\chi}_1(G)\).

Assume now that \(e_1\) is not a bridge, and suppose that \(\lambda\) is the smallest integer \((2 \leq \lambda \leq k)\) such that
G - \{e_1, e_2, \ldots, e_{\lambda - 1}\} is connected and G - \{e_1, e_2, \ldots, e_{\lambda}\} is disconnected. Let e_{\lambda} = u_{\lambda} v_{\lambda}. For each e_i (1 \leq i \leq \lambda - 1), select an incident vertex w_i such that w_i \notin \{u_{\lambda}, v_{\lambda}\}. If the removal of the vertices in \{w_1, w_2, \ldots, w_{\lambda - 1}\} produces a disconnected graph, then let w be a vertex in a component of G - S that does not contain e_{\lambda}. For each j (\lambda + 1 \leq j \leq \lambda), select a vertex incident with e_j that is different from v_{\lambda} and w. Let w_{\lambda} = u_{\lambda} and S' = \{w_1, w_2, \ldots, w_{\lambda}\}. Then G - S' is disconnected and each of its components is F-free. Hence \(F_{r}(G) \leq |S'| \leq \lambda = F_{r}(G)\).

If G - S is connected, then e_{\lambda} is a bridge of G - S so that G - S \notin K_2 or G - S contains a cut-vertex. If G - S \notin K_2, then removing one additional vertex produces the trivial graph so that \(F_{r}(G) \leq \lambda \leq \lambda\). Suppose, finally, that e_{\lambda} is a bridge of G - S and u_{\lambda} is a cut-vertex of G - S. Let w be a vertex of G - S different from v_{\lambda} that is adjacent to u_{\lambda}. Then v_{\lambda} and w belong to distinct components of G - (S \cup \{u_{\lambda}\}). For each j (\lambda + 1 \leq j \leq \lambda), choose a vertex w_j incident with e_j such that w_j \notin \{v_{\lambda}, w\}. Further, let w_{\lambda} = u_{\lambda} and S' = \{w_1, w_2, \ldots, w_{\lambda}\}. Then G - S' is disconnected and each of its components is F-free, so that

\(F_{r}(G) \leq |S'| \leq \lambda \leq F_{r}(G)\).
If $\hat{F}(G) = 0$ for some nontrivial induced subgraph $F$ of $G$, then $G$ is disconnected with each component $F$-free. Consequently, $\hat{F}_1(G) = 0$. We may thus restrict our attention to nontrivial graphs $G$ and nontrivial induced subgraphs $F$ of $G$ such that $\hat{F}(G) \geq 1$. Theorem 3.19 is best possible, for if we let $F = G$, then it follows from Theorem 3B that for every pair $m, n$ of positive integers, where $1 \leq m \leq n$, there is a graph $G$ with $\hat{F}(G) = m$ and $\hat{F}_1(G) = n$. Note, however, that in this case, the ratio $p(F)/p(G)$ is 1. The next result shows that the ratio $p(F)/p(G)$ can be made arbitrarily small.

**Theorem 3.20** For every pair $m, n$ of positive integers such that $1 \leq m \leq n$ and every $\epsilon > 0$, there is a connected graph $G$ and an induced subgraph $F$ of $G$ such that $\hat{F}(G) = m$, $\hat{F}_1(G) = n$ and $p(F)/p(G) < \epsilon$.

**Proof.** Let $H_1 \cong K_{n+2}$ and $H_2 \cong K_{rn, rn}$, where $r$ is a positive integer. Let $v_1, v_2, \ldots, v_n$ be $n$ vertices of $H_1$ and $w_1, w_2, \ldots, w_m$ be $m$ vertices from one of the partite sets of $H_2$. Define $G$ to be the graph having $V(G) = V(H_1) \cup V(H_2)$ and

$$E(G) = E(H_1) \cup E(H_2) \cup \{v_i w_i | 1 \leq i \leq m\} \cup \{v_j w_m | m + 1 \leq j \leq n\},$$

and let $F = \langle V(H_1) \cup \{w_m\} \rangle$. Then $p(F) = n + 3$ and $p(G) = 2rn + n + 2$, so that
Let \( r \) be the smallest positive integer for which

\[
    r \geq \frac{1}{2} \left( \frac{n + 3}{n} - 1 \right) .
\]

Then \( n(2r + 1) \geq n + 3 \), implying that

\[
    \varepsilon \geq \frac{n + 3}{n(2r + 1)} > \frac{n + 3}{n(2r + 1) + 2} .
\]

Hence \( p(F)/p(G) < \varepsilon \).

We show next that \( \mathcal{F}_r(G) = m \). Since \( G - \{w_1, w_2, \ldots, w_m\} \) is disconnected with each component \( F \)-free, it follows that \( \mathcal{F}_r(G) \leq m \). Suppose now that \( S \) is a set of at most \( m - 1 \) vertices of \( G \). Since \( H_1 - S \) and \( H_2 - S \) are connected and since \( v_i, w_i \notin S \) for at least one \( i \in \{1, 2, \ldots, m\} \), it follows that \( G - S \) is connected. Hence \( \mathcal{F}_r(G) \geq m \). We conclude now that \( \mathcal{F}_r(G) = m \).

To see that \( \mathcal{F}_{r_1}(G) = n \), observe first that

\[
    G - (\{v_iw_i | 1 \leq i \leq m\} \cup \{v_jw_m | m + 1 \leq j \leq n\})
\]

is disconnected with each component \( F \)-free, so that \( \mathcal{F}_{r_1}(G) \leq n \).

Let \( E \) be a set of at most \( n - 1 \) edges. Since both \( H_1 - E \) and \( H_2 - E \) are connected and since \( v_iw_i \notin E \) for at least one
Section 3.4 Relationships between F-Connectivity and F-Free-Connectivity

For a nontrivial graph $G$ and a nontrivial induced subgraph $F$ of $G$, we observed in the preceding two sections that $\hat{r}(G) \geq r(G)$ and $\hat{r}(G) \geq r(G)$. The next two results illustrate that neither the $F$-connectivity nor the $F$-free connectivity is bounded, in general, by the other.

**Theorem 3.21** Let $\lambda, m$ and $n$ be positive integers such that $\lambda \leq m \leq n$. Then there is a graph $G$ and an induced subgraph $F$ of $G$ such that $r(G) = \lambda$, $\hat{r}(G) = m$ and $\hat{r}(G) = n$.

**Proof.** Suppose first that $\lambda < m < n$. Let $G_1 \cong K_{n-1}$, $G_2 \cong K_{n+m-\lambda}$, $H \cong K_{\lambda}$ and $H_1 \cong H_2 \cong \ldots \cong H_{n-m-1} \cong K_n$. Let

$$G = H + (G_1 \cup G_2 \cup H_1 \cup H_2 \cup \ldots \cup H_{n-m-1})$$

and $F \cong K_n$. Since $G - V(H)$ is disconnected, it follows that $r(G) \leq \lambda$. If $S$ is a set of at most $\lambda - 1$ vertices, then $V(H) - S$ is nonempty. Further, $V(G_2) - S$ is nonempty. Consequently, since every vertex of $G - V(H)$ is joined to every vertex of $H$, it follows that $G - S$ is connected. Hence $r(G) \geq \lambda$, so that $r(G) = \lambda$. 

$\square$
We show next that $\text{Fr}(G) = m$. Let $S'$ be a set of $m - \lambda$ vertices of $G_2$. If $S = V(H) \cup S'$, then

$$G - S \cong (n-m)K_n \cup K_{n-1},$$

so that $G - S$ is disconnected with each of its components a subgraph of $F$. Hence $\text{Fr}(G) \leq |S| = m$. Suppose now that $S$ is a set of at most $m - 1$ vertices of $G$ such that $G - S$ is disconnected. Then $V(H) \subseteq S$; otherwise, since $G_2 - S$ is connected and since every vertex of $V(H) - S$ is joined to every vertex in $V(G) - V(H)$, it follows that $G - S$ is connected. Consequently, $S$ contains at most $m - \lambda - 1$ vertices of $G_2$ so that $K_{n+1}$ (which is not a subgraph of $F$) is a component of $G - S$. Hence $\text{Fr}(G) \geq m$, implying that $\text{Fr}(G) = m$.

Finally, we show that $\text{Fr}(G) = n$. For $i = 1, 2, \ldots, n-m-1$, let $v_i$ be a vertex of $H_i$ and let $S' = \{v_1, v_2, \ldots, v_{n-m-1}\}$. Further, let $S''$ be a set of $m - \lambda + 1$ vertices of $G_2$. If $S = V(H) \cup S' \cup S''$, then $G - S \cong (n-m+1)K_{n-1}$, so that $G - S$ is disconnected with each of its components $F$-free. Hence $\text{Fr}(G) \leq n$. Suppose now that $S$ is a set of at most $n - 1$ vertices such that $G - S$ is disconnected. Then $V(H) \subseteq S$; otherwise, the facts that $V(G_2) - S$ is nonempty and that every vertex of $V(H) - S$ is joined to every vertex of $V(G) - V(H)$ imply that $G - S$ is connected. Consequently, $S$ contains at most $n - \lambda - 1$ vertices of
so that $G_2 - S$ or $H_i - S$ ($1 \leq i \leq n - m - 1$) contains $F$ as an induced subgraph. Hence $\hat{\tau}(G) \geq n$, so that $\hat{\tau}(G) = n$.

Assume now that $\lambda = m < n$. Let $H \subseteq K_{\lambda}$, $H_1 \supseteq H_2 \supseteq \ldots \supseteq H_{n-m} \supseteq K_n$ and $G_1 \supseteq K_{n-1}$. Define

$$G = H + (G_1 \cup H_1 \cup H_2 \cup \ldots \cup H_{n-m})$$

and let $F \subseteq K_n$. Since $G - V(H) \supseteq (n-m)K_n \cup K_{n-1}$, it follows that $G - V(H)$ is disconnected with each component a subgraph of $F$. Hence $\hat{\tau}(G) \leq |V(H)| = \lambda = m$, so that $\tau(G) \leq \lambda$. Suppose now that $S$ is a set of at most $\lambda - 1$ vertices. Then $V(H) - S$ is nonempty, and since $H_1 - S$ is connected and every vertex of $V(H) - S$ is joined to every vertex of $V(G) - V(H)$, it follows that $G - S$ is connected. Hence $\tau(G) \geq \lambda - m$. Consequently, $\tau(G) = \lambda = m = \hat{\tau}(G)$.

We show next that $\hat{\tau}(G) = n$. For $i = 1, 2, \ldots, n - m$, let $v_i$ be a vertex of $H_i$ and let $S' = \{v_1, v_2, \ldots, v_{n-m}\}$. Let $S = V(H) \cup S'$. Then $G - S \supseteq (n - m + 1)K_{n-1}$, so that $G - S$ is disconnected with each component $F$-free. Hence $\hat{\tau}(G) \leq n$.

Suppose now that $S$ is a set of at most $n - 1$ vertices of $G$ such that $G - S$ is disconnected. Then $V(H) \subseteq S$; otherwise, since $H_1 - S$ is connected and every vertex of $V(H) - S$ is joined to every vertex of $V(G) - V(H)$, it follows that $G - S$
is connected. Hence $S$ contains at most $n - m - 1$ vertices of $H_1 \cup H_2 \cup \cdots \cup H_{n-m}$, implying that $K_n$ is a subgraph of $G - S$. Therefore, $G - S$ is not $F$-free. Hence $\widehat{\chi}(G) \geq n$, so that $\widehat{\chi}(G) = n$.

Suppose now that $\lambda < m = n$. Let $H = K_\lambda$, $G_1 = K(n+1, n-\lambda)$ and $H_1 \subset H_2 \subset \cdots \subset H_{n-\lambda} \subset K_{n+1}$. Let

$$G = H + (G_1 \cup H_1 \cup H_2 \cup \cdots \cup H_{n-m})$$

and let $F \subset K_{n+1}$. Observe that $n + 1 \geq 3$ and therefore $G_1$ is $F$-free. Since $G - V(H)$ is disconnected, it follows that $\chi(G) \leq \lambda$. Suppose that $S$ is a set of at most $\lambda - 1$ vertices. Then $V(H) - S$ is nonempty. Since $V(H_1) - S$ is nonempty and every vertex in $V(H) - S$ is joined to every vertex in $V(G) - V(H)$, the graph $G - S$ is connected. Hence $\chi(G) \geq \lambda$, so that $\chi(G) = \lambda$.

We show now that $\text{Fr}(G) = \widehat{\text{Fr}}(G) = n$. Let $S'$ be a set of $n - \lambda$ independent vertices of $G_1$ and let $S = V(H) \cup S'$. Then

$$G - S \equiv (n - \lambda) K_{n+1} \cup \overline{K_{n+1}}$$

so that $G - S$ is disconnected with each component a subgraph of $F$. Hence $\text{Fr}(G) \leq |S| = n$. Suppose now that $S$ is a set of at most $n - 1$ vertices of $G$ such that $G - S$ is disconnected. Then $V(H) \subset S$; otherwise, since $V(H_1) - S$ is nonempty and every vertex of $V(H) - S$ is joined to every vertex of $V(H)$.
$V(G) - V(H)$, it follows that $G - S$ is connected. Hence $S$ contains at most $n - \lambda - 1$ vertices of $G_1$, and $G_1 - S$ is connected of order at least $n + 2$. Hence $G_1 - S$ is not a subgraph of $F$, implying that $Fr(G) \geq n$. Consequently, $Fr(G) = n$.

To see that $\hat{Fr}(G) = n$, let $v_i$ be a vertex of $H_i$, where $1 \leq i \leq n - \lambda$, and set $S' = \{v_1, v_2, \ldots, v_{n-\lambda}\}$. Let $S = V(H) \cup S'$. Then

$$G - S \cong (n - \lambda)K_n \cup K(n+1, n-\lambda),$$

so that $G - S$ is disconnected with each of its components $F$-free. Hence $\hat{Fr}(G) \leq |S| = n$. Suppose now that $S$ is a set of at most $n - 1$ vertices so that $G - S$ is disconnected. Then $V(H) \subseteq S$; otherwise, since $H_1 - S$ is connected and every vertex of $V(H) - S$ is joined to every vertex of $V(G) - V(H)$, it follows that $G - S$ is connected. Hence $S$ contains at most $n - \lambda - 1$ vertices of $H_1 \cup H_2 \cup \cdots \cup H_{n-\lambda}$, implying that $G - S$ contains a component, and therefore an induced subgraph, isomorphic to $K_{n+1}$. Thus, $G - S$ is not $F$-free, so that $\hat{Fr}(G) \geq n$. Hence $\hat{Fr}(G) = n$.

If $\lambda = m = n$, then let $G \cong F \cong K(n, n)$ and observe that $k(G) = Fr(G) = \hat{Fr}(G) = n$.

**Theorem 3.22** For positive integers $\lambda$, $m$ and $n$ where $\lambda \leq m \leq n$, there is a graph $G$ and an induced subgraph $F$ of $G$ such that $k(G) = \lambda$, $\hat{Fr}(G) = m$ and $Fr(G) = n$. 

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Proof. In Theorem 3.21, we already considered the cases where \( \lambda = m = n \) and where \( \lambda < m = n \). Suppose now that \( \lambda < m < n \), so that \( n \geq 3 \). Let \( H_1 = K_{n+m-\lambda-1} \), \( H_2 = K(n, n-m+1) \) and \( H = \overline{K}_\lambda \). Further, let \( G = H + (H_1 \cup H_2) \) and \( F = K_n \). Since \( G - V(H) \) is disconnected, it follows that \( r(G) \leq \lambda \). Suppose now that \( S \) is a set of at most \( \lambda - 1 \) vertices. Then both \( V(H) - S \) and \( V(H_1) - S \) are nonempty. Since every vertex of \( V(H) - S \) is joined to every vertex of \( V(G) - V(H) \), it follows that \( G - S \) is connected. Hence \( r(G) > \lambda - 1 \), implying that \( r(G) = \lambda \).

We show next that \( \bar{F}r(G) = m \). Let \( S' \) be a set of \( m - \lambda \) vertices of \( H_1 \) and let \( S = S' \cup V(H) \). Then

\[
G - S = K_{n-1} \cup K(n-m+1, n),
\]

so that \( G - S \) is disconnected and \( F \)-free. Consequently, \( \bar{F}r(G) \leq m \). Suppose now that \( S \) is a set of at most \( m - 1 \) vertices such that \( G - S \) is disconnected. Then \( V(H) \subseteq S \); otherwise, since \( H_1 - S \) is connected and every vertex of \( V(H) - S \) is joined to every vertex of \( V(G) - V(H) \), it follows that \( G - S \) is connected. Hence \( S \) contains at most \( m - \lambda - 1 \) vertices of \( V(H_1) \), so that \( G - S \) contains a component that is not \( F \)-free. Therefore, \( \bar{F}r(G) \geq m \), so that \( \bar{F}r(G) = m \).

It remains to be shown that \( Fr(G) = n \). Let \( S' \) be a set of \( m - \lambda - 1 \) vertices of \( H_1 \) and \( S'' \) a set of \( n - m + 1 \) independent vertices of \( H_2 \) such that \( H_2 - S'' \) is disconnected. Let
$S = S' \cup S'' \cup V(H)$. Then $G - S \cong K_n \cup \overline{K}_{n'}$, so that $G - S$ is disconnected with each component a subgraph of $F$. Hence $Fr(G) \leq n$. Suppose now that $S$ is a set of at most $n - 1$ vertices such that $G - S$ is disconnected. Then $V(H) \subseteq S$; otherwise, since $V(H_1) - S$ is nonempty and every vertex of $V(H) - S$ is joined to every vertex in $V(G) - V(H)$, it follows that $G - S$ is connected. Hence $S$ contains at most $n - \lambda - 1$ vertices of $V(H_1) \cup V(H_2)$. If $S$ contains fewer than $m - \lambda - 1$ vertices of $H_1$, then $G - S$ contains a component that is not a subgraph of $F$. If $S$ contains at least $m - \lambda - 1$ vertices of $H_1$, then $S$ contains at most $n - m$ vertices of $H_2$, so that $G - S$ contains a component that is not a subgraph of $F$. Hence $Fr(G) \leq n$, implying that $Fr(G) = n$.

Suppose now that $\lambda = m < n$. Let $H_1 \cong K_n$, $H_2 \cong K(n - \lambda, n)$ and $H \cong \overline{K}_{\lambda}$. Let $G = H + (H_1 \cup H_2)$ and let $F \cong K_{n+1}$. Since $n + 1 \geq 3$ and $G - V(H) \cong K_n \cup K(n - \lambda, n)$, it follows that $G - V(H)$ is disconnected with each of its components $F$-free. Hence $Fr(G) \leq \lambda$, so that $\kappa(G) \leq \lambda$. Suppose now that $S$ is a set of at most $\lambda - 1$ vertices. Since $V(H) - S$ and $V(H_1) - S$ are both nonempty and since every vertex of $V(H) - S$ is joined to every vertex of $V(G) - V(H)$, we conclude that $G - S$ is connected. Hence $\kappa(G) \geq \lambda$. Consequently, it follows that $\kappa(G) = \lambda$ and that $Fr(G) = \lambda$. 

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It remains to be shown that $\text{Fr}(G) = n$. Let $S'$ be a set of $n - \ell$ independent vertices of $H_2$ such that $H_2 - S'$ is connected and set $S = S' \cup V(H)$. Then $G - S \leq K_n \cup \overline{K}_n$ so that $G - S$ is disconnected with each component a subgraph of $F$.

Hence $\text{Fr}(G) \leq n$. Suppose now that $S$ is a set of at most $n - 1$ vertices such that $G - S$ is disconnected. Then $V(H) \subseteq S$; otherwise, since $H_1 - S$ is connected and every vertex of $V(H) - S$ is joined to every vertex of $V(G) - V(H)$, it follows that $G - S$ is connected. Hence $S$ contains at most $n - \ell - 1$ vertices of $H_2$, implying that $H_2 - S$ is a component of $G - S$ having order at least $n + 1$. Consequently, $\text{Fr}(G) \geq n$ and so $\text{Fr}(G) = n$.

We now derive bounds on the $F$-free connectivity of a graph in terms of the $F$-connectivity and the connectivity of $G$.

For a nontrivial complete graph $G$ and a nontrivial induced subgraph $F$ of $G$, we know that

$$\text{Fr}(G) = \text{Fr}(G) = \kappa(G) = \rho(G) - 1.$$ 

Assume thus for the remainder of this section that $G$ is not complete and that $F$ is a nontrivial induced subgraph of $G$. Let $S$ be a set of $\text{Fr}(G)$ vertices of $G$ such that $G - S$ is disconnected with each of its components a subgraph of $F$. Suppose that $G_1, G_2, \ldots, G_\ell$ are the components of $G - S$. Then it follows that $\rho(G_i) \leq \rho(F)$ for $1 \leq i \leq \ell$. If $\rho(G_i) < \rho(F)$, then $G_i$
is $F$-free and if $p(G_i) = p(F)$, then the deletion of at most one vertex from $G_i$ produces an $F$-free graph. Let

$$k = \min k(G - S), \quad (3.1)$$

where the minimum is taken over all subsets $S$ of $V(G)$ having cardinality $Fr(G)$ and such that $G - S$ is disconnected. Then

$$r(G) \leq Fr(G) \leq Fr(G) + k. \quad (3.2)$$

In the next theorem, we show that the bounds given in (3.2) are best possible.

**Theorem 3.23** Let $\lambda, m, n$ and $k$ be positive integers such that $k \geq 2$, $\lambda \leq n$ and $\lambda \leq m \leq n + k$. Then there exists a graph $G$ and an induced subgraph $F$ of $G$ such that $r(G) = \lambda$, $Fr(G) = m$ and $Fr(G) = n$, where $k$ is defined as in equation (3.1).

**Proof.** First consider the case where $m > n$, so that $m = n + j$ for some $j$, $1 \leq j \leq k$. Let $H = \bar{K}_\lambda$ and $G' = K_{m+n-j+1}$. If $j > 1$, let $H_1 \subseteq H_2 \subseteq \cdots \subseteq H_{j-1} \subseteq K_{m+1}$, while if $j < k$, let $G_1 \subseteq G_2 \subseteq \cdots \subseteq G_{k-j} \subseteq K_m$. Define

$$G = H + (G' \cup H_1 \cup H_2 \cup \cdots \cup H_{j-1} \cup G_1 \cup G_2 \cup \cdots \cup G_{k-j})$$

and let $F = K_{m+1}$. Since $G - V(H)$ is disconnected, it follows that $r(G) \leq \lambda$. Suppose now that $S$ is a set of at most $\lambda - 1$
vertices. Then $V(H) - S$ is nonempty and $G' - S$ is connected.

Since every vertex of $V(H) - S$ is joined to every vertex of $G - V(H)$, it follows that $G - S$ is connected. Hence $r(G) \geq \lambda$, implying that $r(G) = \lambda$.

We show now that $Fr(G) = n$. Let $S'$ be a set of $n - \lambda$ vertices of $G'$ and set $S = S' \cup V(H)$. Then

$$G - S' \not\subseteq jK_{m+1} \cup (k - j)K_m$$

so that $G$ is disconnected with each component a subgraph of $F$. Hence $Fr(G) \leq |S| = n$. Suppose now that $S$ is a set of at most $n - 1$ vertices such that $G - S$ is disconnected. Then $V(H) \subseteq S$; otherwise, since $G' - S$ is connected and every vertex of $V(H) - S$ is joined to every vertex of $V(G) - V(H)$, it follows that $G - S$ is connected. Hence $S$ contains at most $n - \lambda - 1$ vertices of $G'$, so that $G - S$ contains a component having order at least $m + 2$, that is, $G - S$ contains a component that is not a subgraph of $F$. Consequently, $Fr(G) \geq n$, implying that $Fr(G) = n$.

It remains to be shown that $Fr(G) = m = n + j (1 \leq j \leq k)$.

If $j \geq 2$, then for every $i (1 \leq i \leq j - 1)$, let $v_i$ be a vertex of $H_i$ and let $S' = \{v_1, v_2, \ldots, v_{j-1}\}$. If $j = 1$, define $S' = \emptyset$. Let $S''$ be a set of $n - \lambda + 1$ vertices of $G'$. If $S = S' \cup S'' \cup V(H)$, then $G - S \not\subseteq k K_m$, so that $G - S$ is disconnected with each of its components $F$-free. Hence
Suppose now that $S$ is a set of at most $n + j - 1$ vertices such that $G - S$ is disconnected. Then $V(H) \subseteq S$; otherwise, since $V(G') - S$ is nonempty and every vertex of $V(H) - S$ is joined to every vertex of $V(G) - V(H)$, it follows that $G - S$ is connected. Hence $S$ contains at most $n + j - 1$ vertices of $G'$. If $S$ contains fewer than $n - \lambda + 1$ vertices of $G'$, then $G - S$ contains a component that is not $F$-free. If $S$ contains at least $n - \lambda + 1$ vertices of $G'$, then, since $n + j - 1 - \lambda \geq n - \lambda + 1$, we have $j \geq 2$. Therefore, for at least one $i (1 \leq i \leq j - 1)$, $S$ contains no vertex of $H_i$, so that $G - S$ is not $F$-free. Hence $\hat{\chi}(G) \geq n + j$ so that $\hat{\chi}G = n + j$.

To see that $k$ satisfies (3.1), suppose that $S$ is a set of $n$ vertices such that (3.1) is satisfied. Since $G - S$ is disconnected, it follows that $V(H) \subseteq S$; otherwise, since $V(G') - S$ nonempty and every vertex of $V(H) - S$ is joined to every vertex of $V(G) - V(H)$, it follows that $G - S$ is connected. Note that

$$G - V(H) \cong K_{m+n-\lambda+1} \cup (j - 1)K_{m+1} \cup (k - j)K_{m}.$$ 

Since $G - S$ has the property that each of its components is a subgraph of $F$, it follows that $S$ contains at least $n - \lambda$ vertices of $G'$. Since $|S| = n$, the set $S$ contains exactly $n - \lambda$ vertices of $G'$. Hence every set $S$ for which $G - S$ is
disconnected with each component a subgraph of \( F \) is uniquely determined, so that \( k \) satisfies (3.1).

We consider now the case where \( \lambda \leq m \leq n \), so that \( m = \lambda + j \) for some \( j \) (\( 0 \leq j \leq n - \lambda \)). Let \( H \equiv K_\lambda \), \( G_1 \equiv K(n, n - \lambda - j + 3) \), \( G_2 \equiv K_{n+j+1} \) and \( H_1 \equiv H_2 \equiv \ldots \equiv H_{k-2} \equiv K_{n+1} \). Let

\[
G = H + (G_1 \cup G_2 \cup H_1 \cup H_2 \cup \ldots \cup H_{k-2})
\]

and let \( F \equiv K_{n+2} \). Since \( G - V(H) \) is disconnected, it follows that \( \kappa(G) \leq \lambda \). If \( S \) is a set of at most \( \lambda - 1 \) vertices, then \( V(G_2) - S \) and \( V(H) - S \) are both nonempty. Since every vertex of \( V(H) - S \) is joined to every vertex of \( V(G) - V(H) \), the graph \( G - S \) is connected. Hence \( \kappa(G) \geq \lambda \).

We show now that \( \overline{\kappa}(G) = \lambda + j \). Let \( S' \) be a set of \( j \) vertices of \( G_2 \) and set \( S = S' \cup V(H) \). Then \( G - S' \equiv G_1 \cup (k-1)K_{n+1} \), so that \( G - S' \) is disconnected with each component \( F \)-free. Hence \( \overline{\kappa}(G) \leq \lambda + j \). Suppose now that \( S \) is a set of at most \( \lambda + j - 1 \) vertices such that \( G - S \) is disconnected. Then \( V(H) \subseteq S \); otherwise, since \( G_2 - S \) is connected and every vertex of \( V(H) - S \) is joined to every vertex in \( V(G) - V(H) \), it follows that \( G - S \) is connected. Hence \( S \) contains at most \( j - 1 \) vertices of \( G_2 \) so that \( G - S \) is not \( F \)-free. Hence \( \overline{\kappa}(G) \geq \lambda + j \), implying that \( \overline{\kappa}(G) = \lambda + j \).

We show next that \( \overline{\kappa}(G) = n \). Let \( S' \) be a set of \( j - 1 \) vertices of \( G_2 \) and \( S'' \) a set of \( n - \lambda - j + 1 \) vertices of \( G_1 \).
Let $S = S' \cup S'' \cup V(H)$. Then

$$G - S = (k - 2)K_{n+1} \cup K_{n+2} \cup (G_1 - S'')$$

so that $G - S$ is disconnected with each component a subgraph of $F$. Hence $Fr(G) \leq |S| = n$. Suppose now that $S$ is a set of at most $n - 1$ vertices of $G$ such that $G - S$ is disconnected. Then $V(H) \subseteq S$; otherwise, since $V(G_2) - S$ is nonempty and every vertex of $V(H) - S$ is joined to every vertex of $V(G) - V(H)$, it follows that $G - S$ is connected. Hence $S$ contains at most $n - j - 1$ vertices of $V(G_1) \cup V(G_2)$. If $S$ contains fewer than $n - j - j + 1$ vertices of $G_1$, then $G_1 - S$ is connected and contains at least $n + 3$ vertices, so that $G - S$ contains a component that is not a subgraph of $F$. Suppose now that $G_1 - S$ contains at least $n - j - j + 1$ vertices of $G_1$. Then $S$ contains at most $j - 2$ vertices of $G_2$ so that $p(G_2 - S) \geq n + 3$. Hence $G - S$ contains a component that is not a subgraph of $F$. Consequently, $Fr(G) \geq n$ and, thus, $Fr(G) = n$.

It remains to be shown that $k = \min k(G - S)$, where the minimum is taken over all subsets $S$ of $V(G)$ having cardinality $n$ and such that $G - S$ is disconnected with each component a subgraph of $F$. Let $S$ be a set of $n$ vertices such that $G - S$ is disconnected with each component a subgraph of $F$. Then $V(H) \subseteq S$; otherwise, since $V(G_2) - S$ is nonempty and every vertex of $V(H) - S$ is joined to every vertex of $V(G) - V(H)$, it
follows that $G - S$ is connected. Hence $S$ contains $n - \lambda$
vertices of

$$G - V(H) = G_1 U G_2 U H_1 U H_2 U \cdots U H_{k-2}.$$\]

Since every component of $G - S$ is a subgraph of $K_{n+2}$, the set
$S$ contains at least $j - 1$ vertices of $G_2$ and at least
$n - \lambda - j + 1$ vertices of $G_1$. Hence $|S| \geq \lambda + (j - 1) +
n - \lambda - j + 1 = n$, so that $S$ contains exactly $j - 1$ vertices
of $G_2$ and exactly $n - \lambda - j + 1$ vertices of $G_1$. Let $S'$ be
a set of $n - \lambda - j + 1$ independent vertices of $G_1$ having
degree $n$ in $G_1$ and let $S''$ be a set of $j - 1$ vertices of $G_2$.
Since $\kappa(G_2) > j - 1$, it follows that $G_2 - S''$ is connected.
Further, since $S'$ is a set of $n - \lambda - j + 1$ independent
vertices of degree $n$ in $G_1$, it follows that they belong to the
same partite set and that such a partite set contains at least two
vertices other than those in $S_1'$. Hence $G_1 - S'$ is connected.
Since $G_1 - S'$ and $G_2 - S''$ are both subgraphs of $F$, it
follows that $S$ is a set of $n$ vertices such that $G - S$ is
disconnected with each component a subgraph of $F$. Also, $G - S$
has $k$ components. Since $V(H) \subseteq S$ and the removal of $n - \lambda$
vertices from $G - V(H)$ cannot decrease the number of components,
we conclude that $k = \min k(G - S)$, where the minimum is taken
over all sets $S$ of vertices of cardinality $n$ such that $G - S$
is disconnected with each component a subgraph of $F$. \q
The following two results show that neither the F-edge-connectivity nor the F-free edge-connectivity bounds the other in general.

**Theorem 3.24** Let \( m \) and \( n \) be positive integers such that \( m \leq n \). Then there exists a graph \( G \) and an induced subgraph \( F \) of \( G \) such that \( F_{\text{F}}(G) = m \) and \( F_{\text{F}}(G) = n \).

**Proof.** Let \( H = K(n, n) \) with partite sets \( V_1 \) and \( V_2 \), where \( V_1 = \{v_1, v_2, \ldots, v_n\} \). Let \( H_1 \) be the graph obtained from \( H \) by adding the edges of \( E_1 = \{v_i v_{i+1} | 1 \leq i \leq n - m + 1 \} \) (indices expressed modulo \( n \)), and let \( H_2 = K(n, n) \). Suppose that \( W_1 \) and \( W_2 \) are the partite sets of \( H_2 \), where \( W_1 = \{w_1, w_2, \ldots, w_n\} \). Define \( G \) to be the graph obtained from \( H_1 \cup H_2 \) by adding the edges of \( E_2 = \{v_i w_i | 1 \leq i \leq m - 1 \} \) and let \( F = K(n, n) \).

We show first that \( F_{\text{F}}(G) \leq m \). Let \( e \) be any edge of \( H_2 \) and observe that \( G - (E_2 \cup \{e\}) \) is disconnected with each component \( F \)-free. Hence \( F_{\text{F}}(G) \leq m \). Suppose that \( E \) is a set of at most \( m - 1 \) edges such that \( G - E \) is disconnected. Then \( E_2 \subseteq E \); otherwise, since \( H_1 - E \) and \( H_2 - E \) are both connected it follows that \( G - E \) is connected. Hence \( E = E_2 \) and \( G - E = H_1 \cup H_2 \), so that \( G - E \) contains a component that is not \( F \)-free. Therefore, \( F_{\text{F}}(G) \geq m \), implying that \( F_{\text{F}}(G) = m \).

To see that \( F_{\text{F}}(G) = n \), observe first that \( G - (E_1 \cup E_2) \cong K(n, n) \), so that \( F_{\text{F}}(G) \leq n \). Suppose now that \( E \) is a set of
at most $n - 1$ edges such that $G - E$ is disconnected. Then $E_2 \subseteq E$; otherwise, since both $H_1 - E$ and $H_2 - E$ are connected, the graph $G - E$ is connected. Thus, $E$ contains at most $n - m$ edges of $H_1$. Therefore, $q(H_1 - E) \geq n^2 + 1 > q(F)$, implying that $H_1 - E$ is a component of $G - E$ that is not a subgraph of $F$. Hence $\text{Fr}_1(G) \geq n$, so that we may conclude that $\text{Fr}_1(G) = n$.

Theorem 3.25 For positive integers $m$ and $n$ where $m \leq n$, there exists a graph $G$ and an induced subgraph $F$ of $G$ such that $\text{Fr}_1(G) = m$ and $\text{Fr}_1(G) = n$.

Proof. We have already considered the case where $m = n$. Suppose now that $m < n$ and thus that $n \geq 2$. Let $H_1 \subseteq H_2 \subseteq \ldots \subseteq H_n \subseteq K_{n+2}$ and $G_1 = K_{n+2} - e$, where $e$ denotes an edge of $K_{n+2}$. Suppose that $w_1, w_2, \ldots, w_m$ are $m$ distinct vertices of $H_1$ and that $v_1, v_2, \ldots, v_m$ are $m$ distinct vertices of $G_1$. Let $H$ be obtained from $H_1 \cup G_1$ by adding the edges of $E = \{e_{ij} | 1 \leq i \leq m\}$. Define $G = H \cup H_2 \cup \ldots \cup H_n$ and $F = K_{n+2} - e$. We show now that $\text{Fr}_1(G) = m$. Since $G - E$ is disconnected with each component a subgraph of $F$, it follows that $\text{Fr}_1(G) \leq m$. Suppose now that $E'$ is a set of at most $m - 1$ edges. Then both $H_1 - E'$ and $G_1 - E'$ are connected. Further, since $E - E'$ is nonempty, it follows that $H - E'$ is connected. Since $q(H - E') \geq 2q(F) > q(F)$, the graph $G - E'$ contains a component
that is not a subgraph of $F$. Hence $F_{x_1}(G) \geq m$ so that

$F_{x_1}(G) = m$.

We prove next that $F_{x_1}(G) = n$. For $i \in \{1, 2, \ldots, n\}$, let

$e_i \in E(H_i)$. Then $G - \{e_1, e_2, \ldots, e_n\}$ is disconnected with

each component $F$-free. Hence $F_{x_1}(G) \leq n$. Suppose now that $E'$
is a set of at most $n - 1$ edges. Then there exists at least one

$i \in \{1, 2, \ldots, n\}$ such that $E' \cap E(H_i) = \emptyset$. Hence $G - E'$
contains a component that is not $F$-free. Therefore, $F_{x_1}(G) \geq n$,

which proves that $F_{x_1}(G) = n$. 

\[\square\]
CHAPTER IV

VARlATIONS OF CONNECTIVITY

Section 4.1 Introduction

In the preceding chapter we considered the problem of finding the minimum cardinality of a set $S$ of vertices in a graph $G$ for which $G - S$ is trivial or $G - S$ is disconnected each component of which has a specified property. Here we vary the connectivity concept in two additional ways, namely finding the minimum cardinality of a set $S$ of vertices of $G$ such that:

1. $G - S$ has at least $\lambda$ components or fewer than $\lambda$ vertices, for a given integer $\lambda \geq 2$; and

2. $\langle S \rangle$ is connected and $G - S$ is disconnected or trivial.

Section 4.2 $\lambda$-Connectivity in Graphs

Chartrand, Kapoor, Lesniak and Lick [10] defined the $\lambda$-connectivity $\kappa_{\lambda}(G)$ of a graph $G$, for an integer $\lambda \geq 2$, as the minimum cardinality of a set $S$ of vertices of $G$ such that $G - S$ is disconnected with at least $\lambda$ components or $G - S$ has order less than $\lambda$. Hence for $\lambda = 2$, the $\lambda$-connectivity of a graph is simply its connectivity. If a graph $G$ contains a set $S$ of vertices such that $G - S$ has at least $\lambda(\geq 2)$ components, then necessarily $\beta(G) \geq \lambda$. However, if $\beta(G) < \lambda$, then for every proper subset $S$ of vertices of $G$, the number of components of
Let \( p \geq 2 \) be integers. A graph \( G \) is said to be \((n, \lambda)-connected\) if \( \kappa_{\lambda}(G) \geq n \). Consequently, a graph is \((n, 2)-connected\) if and only if it is \( n\)-connected. Chartrand, Kapoor, Lesniak and Lick [10] provided a sufficient condition for a graph to be \((n, \lambda)-connected\).

**Theorem 4A** Let \( G \) be a graph of order \( p \) with \( \beta(G) \geq \lambda \geq 2 \). If

\[
\delta(G) \geq \left\lceil \frac{p + (\lambda - 1)(n - 2)}{\lambda} \right\rceil,
\]

then \( G \) is \((n, \lambda)-connected\).

We present here a stronger sufficient condition for a graph to be \((n, \lambda)-connected\).

**Theorem 4.1** Let \( G \) be a graph of order \( p \geq 2 \) the degrees \( d_i \) of whose vertices satisfy \( d_1 \leq d_2 \leq \cdots \leq d_p \). Suppose that \( n \) and \( \lambda \geq 2 \) are integers with \( 1 \leq n \leq p - \lambda + 1 \). If

\[
d_k \leq k + n - 2 \Rightarrow d_{p-n+1} \geq p - k(\lambda - 1)
\]

for each \( k \) such that \( 1 \leq k \leq \lfloor (p - n + 1)/\lambda \rfloor \), then \( G \) is \((n, \lambda)-connected\).

**Proof.** Suppose that the \( \lambda \)-connectivity of \( G \) is less than \( n \). Then there is a set \( S \) of fewer than \( n \) vertices of \( G \) such that
G - S has either at least \( \lambda \) components or order less than \( \lambda \).

However, since \( n \leq p - \lambda + 1 \), it follows that \( |S| \leq p - \lambda \), so that \( p(G - S) \geq \lambda \). Hence \( G - S \) contains at least \( \lambda \) components.

Let \( H \) be a component of \( G - S \) of minimum order \( k \). Then

\[
k \leq \lfloor (p - n + 1)/\lambda \rfloor,
\]

which implies that \( k + n - 2 < p - k(\lambda - 1) \). Clearly, each vertex of \( H \) has degree at most \( k + n - 2 \) in \( G \). Since \( H \) has order \( k \), this implies that \( d_k \leq k + n - 2 \). By hypothesis,

\[
d_{p-n+1} \geq p - k(\lambda - 1).
\]

If \( u \in V(G) - (V(H) \cup S) \), then \( u \) is not adjacent to any vertex in at least \( \lambda - 1 \) components of \( G - S \). Since each such component has order at least \( k \),

\[
\deg u \leq p - k(\lambda - 1) - 1.
\]

Hence only the vertices of \( S \) have degree at least \( p - k(\lambda - 1) \).

Since \( d_{p-n+1} \geq p - k(\lambda - 1) \) and \( d_p \geq d_{p-1} \geq \cdots \geq d_{p-n+1} \), it follows that \( S \) contains at least \( n \) vertices, which is a contradiction.

By letting \( \lambda = 2 \) in Theorem 4.1, we obtain the following result of Bondy [6].

**Corollary** Let \( G \) be a graph of order \( p \geq 2 \), the degrees \( d_i \) of whose vertices satisfy \( d_1 \leq d_2 \leq \cdots \leq d_p \), and let \( n \) be an integer such that \( 1 \leq n \leq p - 1 \). If
for each \( k \) such that \( 1 \leq k \leq \lfloor (p - n + 1)/2 \rfloor \), then \( G \) is \( n \)-connected.

In a certain sense, which we now describe, the result given in Theorem 4.1 is sharp. Let \( p, n, m \) and \( \lambda \geq 2 \) be nonnegative integers satisfying

\[
2 \leq n \leq p - \lambda + 1 \quad \text{and} \quad 1 \leq m \leq \lfloor (p - n + 1)/\lambda \rfloor.
\]

Let \( H \cong K_{n-1}, G_{0} \cong G_{1} \cong G_{2} \cong \cdots \cong G_{\lambda-2} \cong K_{m} \) and

\[ G_{\lambda-1} \cong K_{p-n-m(\lambda-1)+1}. \]

Since \( m \leq \lfloor (p - n + 1)/\lambda \rfloor \), it follows that \( p - n - m(\lambda - 1) + 1 \geq m \). Define \( G = H + (G_{0} \cup G_{1} \cup \cdots \cup G_{\lambda-1}) \)

and denote the degrees of the vertices of \( G \) by \( d_{1}, d_{2}, \ldots, d_{p} \), where \( d_{1} \leq d_{2} \leq \cdots \leq d_{p} \). Then

\[
d_{k} = \begin{cases} 
    m + n - 2 & \text{for } 1 \leq k \leq m(\lambda - 1) \\
    p - m(\lambda - 1) - 1 & \text{for } m(\lambda - 1) + 1 \leq k \leq p - n + 1 \\
    p - 1 & \text{for } p - n + 2 \leq k \leq p.
\end{cases}
\]

Since \( d_{p-(n-1)+1} = d_{p-n+2} = p - 1 \geq p - k(\lambda - 1) \) for every \( k \) satisfying \( 1 \leq k \leq \lfloor (p - (n - 1) + 1)/\lambda \rfloor \), it follows from Theorem 4.1 that \( G \) is \((n - 1, \lambda)\)-connected. Suppose now that \( 1 \leq k \leq \lfloor (p - n + 1)/\lambda \rfloor \) and \( k \neq m \). If \( 1 \leq k \leq m - 1 \), then
\[ d_k = m + n - 2 > k + n - 2. \] If \( m + 1 \leq k \leq \lfloor (p - n + 1)/\lambda \rfloor, \) then

\[ d_{p-n+1} = p - m(\lambda - 1) - 1 \geq p - k(\lambda - 1). \]

Thus

\[ d_k \leq k + n - 2 \Rightarrow d_{p-n+1} \geq p - k(\lambda - 1) \]

for \( 1 \leq k \leq \lfloor (p - n + 1)/\lambda \rfloor \) and \( k \neq m. \) However, for \( k = m, \)
\[ d_k = k + n - 2 \] and \[ d_{p-n+1} = p - k(\lambda - 1) - 1. \] Hence the
hypothesis of the preceding theorem is not quite satisfied. Since
\( G - V(H) \) is disconnected with \( \lambda \)-components, it follows that \( G \)
is not \((n, \lambda)\) - connected.

The preceding theorem provides us with another sufficient
condition for a graph to be \((n, \lambda)\) - connected.

**Theorem 4.2** Let \( G \) be a graph of order \( p \) such that for some
integers \( n \) and \( \lambda \geq 2, \) with \( 1 \leq n < p - \lambda + 1, \) the follow­ing conditions hold:

(i) for each integer \( k, 1 \leq k \leq (p - n)/\lambda, \) the number of
vertices with degree at most \( k + n - 2 \) is less than \( k; \) and

(ii) the number of vertices with degree at most
\[ [p + n(\lambda - 1) - \lambda - 1]/\lambda \] is at most \( p - n. \)

Then \( G \) is \((n, \lambda)\) - connected.
Proof. To prove that $G$ is $(n, \lambda)$-connected, we show that the hypothesis of Theorem 4.1 is satisfied. Suppose, to the contrary, that there is some integer $k$, $1 \leq k \leq \lfloor (p - n + 1)/\lambda \rfloor$, such that
\[ d_k \leq k + n - 2 \quad \text{and} \quad d_{p-n+1} < p - k(\lambda - 1). \]

By condition (i), $d_m \geq m + n - 1$ for every integer $m$ with $1 \leq m \leq (p - n)/\lambda$. Consequently, $k = (p - n + 1)/\lambda$, implying that $p - n + 1$ is divisible by $\lambda$ and
\[ d_{p-n+1} < p - \left(\frac{p - n + 1}{\lambda}\right) (\lambda - 1), \]
that is,
\[ d_{p-n+1} \leq p - \left(\frac{p - n + 1}{\lambda}\right) (\lambda - 1) - 1 = \frac{p + n(\lambda - 1) - 2\lambda + 1}{\lambda}. \]

By condition (ii),
\[ d_{p-n+1} \geq \frac{p + n(\lambda - 1) - \lambda}{\lambda}. \]

However, since $\lambda \geq 2$, this produces a contradiction.

In the case where $\lambda = 2$, we have the following result due to Chartrand, Kapoor and Kronk [9].

Corollary For a graph $G$ of order $p \geq 2$ and an integer $n$, $1 \leq n < p - 1$, the conditions (i) and (ii) together are
sufficient for \( G \) to be \( n \)-connected:

(i) for every \( k \) such that \( 1 \leq k \leq (p - n)/2 \), the number of vertices of degree at most \( k + n - 2 \) is less than \( k \); and

(ii) the number of vertices of degree at most \( (p + n - 3)/2 \) is at most \( p - n \).

Theorem 4A may also be deduced from Theorem 4.1. To see this, suppose that \( G \) is a graph of order \( p \) with \( \beta(G) \geq \lambda \geq 2 \), the degrees \( d^1 \) of whose vertices satisfy \( d_1 \leq d_2 \leq \cdots \leq d_p \) and

\[
d_1 \geq \left\lfloor \frac{p + (\lambda - 1)(n - 2)}{\lambda} \right\rfloor.
\]

If the hypothesis of Theorem 4.1 is not satisfied, then there exists some integer \( k \), \( 1 \leq k \leq \lfloor (p - n + 1)/\lambda \rfloor \), such that

\[
d_k \leq k + n - 2 \quad \text{and} \quad d_{p-n+1} < p - k(\lambda - 1).
\]

However, then, \( d_k \leq \frac{p + (\lambda - 1)(n - 2) - 1}{\lambda} \), which contradicts the assumption that

\[
d_k \geq d_1 \geq \frac{p + (\lambda - 1)(n - 2)}{\lambda}.
\]

We know from Menger's theorem that if \( S = \{u, v\} \) is a set of two independent vertices in a graph \( G \), then the maximum number of internally disjoint \( u - v \) paths in \( G \) equals the minimum number of vertices that separate \( u \) and \( v \).
For a set $S = \{v_1, v_2, \ldots, v_k\}$ of $k \geq 2$ vertices in a graph $G$, an $S$-path is defined as a path between a pair of vertices of $S$ that contains no other vertices of $S$. Two $S$-paths $P_1$ and $P_2$ are said to be internally disjoint if they are vertex-disjoint except possibly for their end-vertices. If $S = \{v_1, v_2, \ldots, v_k\}$ is a set of $k \geq 2$ independent vertices of a graph $G$, then a set $V$ of vertices of $G$ with $V \cap S = \emptyset$ is said to separate $S$ if every two vertices of $S$ belong to distinct components of $G - V$. A natural extension of Menger's theorem may well be suggested, namely: If $S = \{v_1, v_2, \ldots, v_k\}$ is a set of $k(\geq 3)$ independent vertices of a graph $G$, then the maximum number of internally disjoint $S$-paths equals the minimum number of vertices that separate $S$. However, this statement is not true in general, as can be seen from the graph $G$ in Figure 4.1, where $S = \{v_1, v_2, v_3\}$. Observe that we must delete at least two vertices to separate $S$, but that there is only one (internally disjoint) $S$-path. In fact, $G$ is $(2, 3)$-connected.

![Figure 4.1](image)
Let \( S \) be a set of at least three independent vertices in a graph \( G \). If \( M \) denotes the maximum number of internally disjoint \( S \)-paths and \( m \) the minimum number of vertices that separate \( S \), then Mader \([36]\) proved that \( M \geq \frac{1}{2} m \). Further, he showed that this bound is sharp. Mader also proved an edge analogue of this result in \([35]\).

Recall that Whitney \([47]\) showed a graph \( G \) is \( n \)-connected if and only if for every pair \( u, v \) of distinct vertices of \( G \), there exist at least \( n \)-internally disjoint \( u - v \) paths in \( G \). Further, suppose that \( G \) is a graph of order \( p \) and \( n \) an integer, \( 1 \leq n \leq p - 1 \). Then it follows from Theorem 4A, with \( \lambda = 2 \), that if \( \delta(G) \geq \lceil (p + n - 2) / 2 \rceil \), then \( G \) is \( n \)-connected, implying, for every pair \( u, v \) of vertices of \( G \), that there exist at least \( n \) internally disjoint \( u - v \) paths. Hedman \([26]\) verified for integers \( n \) and \( p \), where \( p \geq 2 \) and \( 1 \leq n \leq p - 1 \), that every graph \( G \) of order \( p \) with \( \delta(G) \geq \lceil (p + n - 1) / 2 \rceil \) has the property that for every pair \( u, v \) of distinct vertices of \( G \), there exist at least \( n \) internally disjoint \( u - v \) paths each of length at most 2.

Let \( G \) be a graph of order \( p \geq 2 \) with \( \delta(G) \geq \lambda \geq 2 \) and \( n \) an integer with \( 1 \leq n \leq p - \lambda + 1 \). From Theorem 4A it follows, that if

\[
\delta(G) \geq \left\lceil \frac{p + (\lambda - 1)(n - 2)}{\lambda} \right\rceil,
\]
then G is \((n, \lambda)\) - connected. However, as remarked above, if G is an \((n, \lambda)\) - connected graph, \(\lambda \geq 3\), and \(S\) is a set of \(\lambda\) vertices of G, then there do not, in general, exists \(n\) S-paths. On the other hand, we do have the following analogue to the result of Hedman.

**Theorem 4.3** Let G be a graph of order \(p \geq 2\), and let \(\lambda \geq 2\) and \(n\) be integers satisfying \(1 \leq n \leq p - \lambda + 1\). If

\[
\deg v \geq \left\lceil \frac{p + (n - 2)(\lambda - 1) + 1}{\lambda} \right\rceil
\]

for every vertex \(v\) of G, then for each set \(S = \{v_1, v_2, \ldots, v_\lambda\}\) of \(\lambda\) vertices of G there exist \(n\) internally disjoint S-paths each of length at most 2.

**Proof.** Let \(H = \langle S \rangle_G\) and suppose that \(d_i = \deg_{H} v_i\) \((1 \leq i \leq \lambda)\).

Then \(2q = 2q(H) = \sum_{i=1}^{\lambda} d_i\). Observe that the \(q\) edges of \(H\) account for \(q\) internally disjoint paths of length 1 between pairs of vertices of \(S\).

It remains therefore to be shown that \(V(G) - S\) contains at least \(n - q\) vertices each of which is adjacent to at least two vertices of \(S\). Suppose, to the contrary, that \(V(G) - S\) contains \(t < n - q\) vertices which are adjacent to at least two vertices of \(S\). Every vertex \(v_i\) \((1 \leq i \leq \lambda)\) is adjacent to \(d_i\) vertices of \(S\) so that \(v_i\) is adjacent to
\[ t_i \geq \left( n - 2 \right) \left( \frac{1}{2} - 1 \right) + 1 \]

vertices in \( V(G) - S \). Of these \( t_i \) vertices of \( V(G) - S \), at most \( t \) vertices are adjacent to some vertex in \( S - \{ v_i \} \). Hence there are at least \( t_i - t \) vertices of \( V(G) - S \) belonging to \( N(v_i) \) that are adjacent to no other vertices of \( S \). Hence

\[ p \geq \sum_{i=1}^{t} \left( t_i - t \right) + t + \frac{1}{2} \]

\[ \geq \sum_{i=1}^{t} \left( \left( n - 2 \right) \left( \frac{1}{2} - 1 \right) + 1 \right) + \left( 1 - t \right) \left( n - q - 1 \right) + \frac{1}{2} \]

\[ \geq p + \left( n - 2 \right) \left( \frac{1}{2} - 1 \right) + 1 - 2q + \left( 1 - t \right) \left( n - q - 1 \right) + \frac{1}{2} \]

\[ = p + \left( \frac{1}{2} - 3 \right) q + 2. \]

If \( \frac{1}{2} = 2 \), then \( q = 0 \) or \( q = 1 \). If \( q = 0 \), then the preceding inequality implies that \( p \geq p + 2 \), which is not possible. If \( q = 1 \), then the above inequality yields \( p \geq p + 1 \), which also produces a contradiction. Further, if \( \frac{1}{2} \geq 3 \), then \( q \left( \frac{1}{2} - 3 \right) \geq 0 \), implying by the preceding inequality that \( p \geq p + 2 \), which again is not possible. Hence our assumption, that there are fewer than \( n - q \) vertices of \( V(G) - S \) that are adjacent to at least two vertices of \( S \), is false. Consequently,
there are at least $n$ internally disjoint paths between pairs of vertices of $S$, each of which has length at most 2.

Section 4.3 Connected Cutset Connectivity

In Chapter 3 and earlier in this chapter, our study of connectivity led us to investigate properties of $G - S$ for a vertex cutset $S$ of a graph $G$. In this section our emphasis shifts to characteristics of the set $S$ itself. This type of connectivity was considered when Esfahanian [15] studied, for a graph $G$, the problem of determining the minimum cardinality of an edge cutset $S$ of $G$ having the property that $S$ does not contain all the edges incident with any vertex of $G$. For a nontrivial connected graph $G$, however, this type of edge-connectivity can be rephrased in terms of the conditional connectivity described in Chapter 3, that is, it is equal to the minimum cardinality of an edge cutset $S$ such that $G - S$ has no nontrivial components. Esfahanian also investigated the corresponding vertex connectivity. We now consider a problem of this type where $S$ is required to satisfy another condition.

In the current section, we take a vertex cutset of a graph $G$ to mean a set $S$ of vertices such that $G - S$ is disconnected or trivial. For a nontrivial connected graph $G$, the connected cutset connectivity $\text{cr}(G)$ is the minimum cardinality of a vertex
cutset $S$ of $G$ for which $\langle S \rangle$ is connected. Further, if $G$ is disconnected or trivial, we define $\kappa(G)$ to be 0. Clearly, $\kappa(G) \leq \kappa(G)$ for every graph $G$. If $\kappa(G) = 0$ or 1, then $\kappa(G) = \kappa(G)$. However, the connectivity of a graph certainly need not always equal its connected cutset connectivity. For example, if $G = K(n, n)$, $n \geq 2$, then $\kappa(G) = n$ and $\kappa(G) = n + 1$. If $G = K_p$, $p \geq 2$, then $\kappa(G) = \kappa(G) = p - 1$. The complete graphs are the only graphs $G$ of order $p$ for which $\kappa(G) = p - 1$. We now characterize those graphs $G$ of order $p$ for which $\kappa(G) = p - 1$.

**Proposition 4.4** Let $G$ be a graph of order $p \geq 1$. Then $\kappa(G) = p - 1$ if and only if $G \cong K_p$ or $G \cong C_p$.

**Proof.** If $G \cong K_p$ or $G \cong C_p$, then $\kappa(G) = p - 1$. Conversely, suppose that $G$ is a graph of order $p$ such that $\kappa(G) = p - 1$. As we noted earlier, if $\kappa(G) = 0$ or 1, then $\kappa(G) = \kappa(G)$ so that $G \cong K_1$ or $G \cong K_2$. We may assume, therefore, that $G$ is 2-connected. Suppose $G \not\cong K_p$.

If $G$ is 3-connected, then $G$ contains a pair $u, v$ of nonadjacent vertices. Since $G - \{u, v\}$ is connected and $G - (V(G) - \{u, v\})$ is disconnected, $\kappa(G) \leq p - 2$. Hence $\kappa(G) = 2$. If $G$ contains a pair $u, v$ of nonadjacent vertices such that $G - \{u, v\}$ is connected, then $\kappa(G) \leq p - 2$, which is not possible. This further implies that $G$ contains no vertex
of degree \( p - 1 \). Thus the removal of every pair of nonadjacent vertices of \( G \) produces a disconnected graph. Therefore, every vertex of \( G \) belongs to a vertex cutset of cardinality 2, implying that \( G \) is critically 2-connected. However, then, \( G \) contains a vertex \( v \) of degree 2 (see [11]). Let \( u \) and \( w \) be the neighbors of \( v \) in \( G \). Since every pair of nonadjacent vertices is a cutset of \( G \), every vertex \( x \) in \( V(G) - \{u, v, w\} \) is a cut-vertex of \( G - v \).

We show next that \( G - v \not\cong P_{p-1} \). If \( G - v \) is a tree, then \( G - v \cong P_{p-1} \); otherwise, \( G \) has at least three end-vertices and therefore at least three vertices that are non-cut-vertices.

Assume, therefore, that \( G - v \) is not a tree. Then every spanning tree of \( G - v \) has exactly two non-cut-vertices, implying that every spanning tree of \( G \) is a path of order \( p - 1 \) having \( u \) and \( v \) as end-vertices. However, since \( G - v \) is not a tree, it follows, if \( T \) is a spanning tree of \( G - v \), that there is an edge \( xy \) in \( G - v \) that joins nonadjacent vertices of \( T \). Let \( z \) be the vertex immediately following \( x \) on the \( x - y \) path in \( T \). Then \( (T + xy) - xz \) is a spanning tree of \( G - v \) that contains three end-vertices, namely \( u, v \) and \( z \). But then \( G - v \) contains at least three non-cut-vertices, which is not possible. Hence \( G - v \not\cong P_{p-1} \), so that \( G \cong C_p \).

As remarked earlier, \( \kappa(G) \leq \chi(G) \) for every graph \( G \). Further, we know that \( \kappa(G) \leq \delta(G) \). However, neither the connected
cutset connectivity nor the minimum degree of a graph bounds the other in general, as we now verify.

Theorem 4.5 Let $a \geq 2$, $b$ and $c$ be integers with $a \leq b$ and $a \leq c$. Then there exists a graph $G$ with $\kappa(G) = a$, $\chi'(G) = b$ and $\delta(G) = c$.

Proof. If $a = b = c$, then $G \cong K_{a+1}$ has the desired properties.
Suppose now that $a = b < c$. Let $H_1 \cong H_2 \cong K(c, c)$ and $H \cong K_a$.
Denote the partite sets of $H_1$ by $V_1$ and $V_2$ and the partite sets of $H_2$ by $W_1$ and $W_2$. Let $G$ be obtained from
$H_1 \cup H_2 \cup H$ by joining every vertex of $H$ to every vertex of $V_2 \cup W_1$. Clearly $\delta(G) = c$. Since $G - V(H)$ is disconnected, $\kappa(G) \leq a$. However, if $S$ is a set of $a - 1$ vertices of $G$, then $V(H) - S$, $V_2 - S$ and $W_1 - S$ are nonempty. Further, both $H_1$ and $H_2$ are $c$-connected, so that the graphs $H_1 - S$ and $H_2 - S$ are connected. Since every vertex of $H - S$ is joined to every vertex of $(V_2 \cup W_1) - S$, it follows that $G - S$ is connected. Hence $\kappa(G) \geq a$, implying that $\kappa(G) = a$. Therefore $\chi'(G) \geq a$. But since $H$ is connected and $G - V(H)$ is disconnected, $\kappa(G) \leq a$ so that $\chi'(G) = a$.

For the remainder of the theorem we consider two cases.

Case 1 Suppose $a < b \leq c$.
Let $H_1 \cong H_2 \cong \cdots \cong H_{b-a+1} \cong K_c \cong G_1 \cong G_2 \cong \cdots \cong G_{b-a+1}$ and suppose that $V(H_1) = \{v_{i,1}, v_{i,2}, \ldots, v_{i,c}\}$ and

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\( V(G_1) = \{u_{i,1}, u_{i,2}, \ldots, u_{i,c}\} \). Further, let \( H' \) be obtained from \( H_1 \cup H_2 \cup \cdots \cup H_{b-a+1} \) by adding the edges in 
\[ \{v_{i,j}, v_{i+1,j} \mid 1 \leq i \leq b-a, 1 \leq j \leq c\}, \] 
and let \( G' \) be the graph obtained from \( G_1 \cup G_2 \cup \cdots \cup G_{b-a+1} \) by adding the edges 
in \( \{u_{i,j}, u_{i+1,j} \mid 1 \leq i \leq b-a, 1 \leq j \leq c\} \). Now let \( F \neq \mathbb{F}_{a-1} \) with \( V(F) = \{w_1, w_2, \ldots, w_{a-1}\} \). Define \( G \) to be the graph 
obtained from \( H' \cup G' \cup F \) by joining \( v_{1,1} \) to \( u_{1,1} \) and then 
joining every vertex of \( F \) to every vertex in \( H_{b-a+1} \cup G_{b-a+1} \).

Since \( c \geq 2 \), it follows that \( \deg v_{1,c} = c \), implying that 
\( \delta(G) = c \). Further, since \( G - (V(F) \cup \{v_{1,1}\}) \) is disconnected, 
\( r(G) \leq a \). Let \( S \) be a set of \( a - 1 \) vertices of \( G \). Since 
a \leq c and both \( H' \) and \( G' \) are \( c \)-connected, the graphs 
\( H' - S \) and \( G' - S \) are connected. If \( V(F) \subseteq S \), then 
\( V(F) = S \) so that \( v_{1,1}, u_{1,1} \notin S \). Hence, in this case \( G - S \) 
is connected. Assume therefore that \( V(F) - S \neq \emptyset \). Then since 
both \( V(H_{b-a+1}) - S \) and \( V(G_{b-a+1}) - S \) are nonempty and every 
vertex of \( V(F) - S \) is joined to every vertex of 
\( (V(H_{b-a+1}) \cup V(G_{b-a+1})) - S \), it follows that \( G - S \) is connected. 
Hence \( r(G) \geq a \) and therefore \( r(G) = a \).

It remains to be shown that \( \chi(G) = b \). Let \( S = 
\{v_{1,1}, v_{2,1}, \ldots, v_{b-a+1,1}\} \cup V(F) \). Then \( |S| = b \), the graph 
\( \langle S \rangle \) is connected and \( G - S \) is disconnected. Hence \( \chi(G) \leq b \).

We show that \( \chi(G) \geq b \). Assume, to the contrary, that there 
exists a set \( V \) of at most \( b - 1 \) vertices such that \( \langle V \rangle \) is
connected and $G - V$ is disconnected. Since $b - 1 < c$, both $H' - V$ and $H'' - V$ are connected, and $V(H_{b-a+1}) - V$ as well as $V(G_{b-a+1}) - V$ are nonempty. Consequently, $V(F) \subseteq V$ and either $v_{1,1}$ or $u_{1,1}$ belongs to $V$; otherwise, $G - V$ is connected. However, $d(u_{1,1}, x) = d(v_{1,1}, x) = b - a + 1$ for each $x \in V(F)$; so because $\langle V \rangle$ is connected, $V$ must contain at least $b - a$ vertices that are distinct from $v_{1,1}$, $u_{1,1}$ and the vertices of $F$. Hence $|V| \geq b$, contrary to our assumption. Therefore $\sigma(G) = b$.

Case 2 Suppose $a \leq c < b$.

Let $H_1 \supseteq H_2 \supseteq \cdots \supseteq H_{b-a+1} \supseteq K_b \supseteq G_1 \supseteq G_2 \supseteq \cdots \supseteq G_{b-a+1}$, where $V(H_i) = \{v_{i,1}, v_{i,2}, \ldots, v_{i,b}\}$ and $V(G_i) = \{u_{i,1}, u_{i,2}, \ldots, u_{i,b}\}$ for $1 \leq i \leq b - a + 1$. Let $H'$ be the graph obtained from $H_1 \cup H_2 \cup \cdots \cup H_{b-a+1}$ by adding the edges in $\{v_{i,j}, v_{i+1,j} \mid 2 \leq i \leq b - a, 1 \leq j \leq b\}$ and $G'$ the graph obtained from $G_1 \cup G_2 \cup \cdots \cup G_{b-a+1}$ by adding the edges in $\{u_{i,j}, u_{i+1,j} \mid 1 \leq i \leq b - a, 1 \leq j \leq b\}$. Let $F \supseteq \mathbb{K}_{a-1}$ with $V(F) = \{w_1, w_2, \ldots, w_{a-1}\}$ and $F' \supseteq K_1$ with $V(F') = \{w_a\}$. Define $G$ to be the graph obtained from $H' \cup G' \cup F \cup F'$ by joining every vertex of $F$ to every vertex of $H_1 \cup G_1$ and then by joining $w_a$ to $v_{b-a+1,1}$, $v_{b-a+2,1}$, $\ldots$, $v_{b-a+1,c-1}$ and $u_{b-a+1,1}$. Clearly, $\delta(G) = \deg w_a = c$. Since $G - (V(F) \cup V(F'))$ is disconnected, $\tau(G) \leq a$. If $S$ is a set of $a - 1$ vertices, then both $H' - S$ and $G' - S$ are connected,
because \( H' \) and \( G' \) are \( b \)-connected. Further, \( V(H'_1) - S \) and \( V(G'_1) - S \) are nonempty. If \( V(F) \subseteq S \), then \( V(F) = S \) so that neither \( w_a \) nor its neighbors belong to \( S \). Hence, in this case, \( G - S \) is connected. Suppose therefore that \( V(F) - S \) is nonempty. Then since every vertex of \( V(F) - S \) is joined to every vertex in \( V(H'_1) - S \) and \( V(G'_1) - S \), it follows that \( G - S \) is connected. Consequently, \( \kappa(G) \geq a \) so that \( \kappa(G) = a \).

It remains to show that \( \kappa(G) = b \). Let \( S = \{ u_{i,1} \mid 1 \leq i \leq b - a + 1 \} \cup V(F) \). Then \( \langle S \rangle \) is connected and \( G - S \) is disconnected. Hence \( \kappa(G) \leq b \). We show next that \( \kappa(G) \geq b \). Assume, to the contrary, that there is a set \( V \) of at most \( b - 1 \) vertices such that \( \langle V \rangle \) is connected and \( G - V \) is disconnected. Then \( V(F) \subseteq V \); otherwise, \( G - V \) is connected. Further, either \( v_a, u_{b-a+1,1} \) or the vertices in
\[ \{ v_{b-a+1,1}, v_{b-a+1,2}, \ldots, v_{b-a+1,c-1} \} \]
belong to \( V \); otherwise, \( G - V \) is connected. However, the distance between a vertex of \( F \) and any of the vertices \( v_a, u_{b-a+1,1}, v_{b-a+1,1}, v_{b-a+1,2}, \ldots, v_{b-a+1,c-1} \) is at least \( b - a + 1 \), implying, since \( \langle V \rangle \) is connected, that \( |V| \geq b \), contrary to our assumption. Hence \( \kappa(G) \geq b \) so that \( \kappa(G) = b \).

We now consider the edge analogue of the connected cutset connectivity. The connected edge-cutset connectivity \( \kappa_1(G) \) of a nontrivial connected graph \( G \) is the minimum
cardinality of an edge cutset $E$ of $G$ such that $\langle E \rangle$ is connected. Further, if $G$ is trivial or disconnected, then $\alpha_r^1(G) = 0$. The graph of Figure 4.2 has edge-connectivity 2 while its connected edge-cutset connectivity is 3.

Figure 4.2

If $G$ is a nontrivial connected graph and $E$ is an edge cutset of $G$ such that $G' = \langle E \rangle$ is connected and $|E| = \alpha_r^1(G)$, then $G - E$ contains exactly two components. To see this, suppose first that $G'$ is a tree. Let $e$ be an edge of $E$ incident with an end-vertex of $G'$, so that $\langle E - \{e\} \rangle$ is connected. Observe that $G - (E - \{e\})$ is connected; for otherwise $\alpha_r^1(G) \leq |E| - 1$, which produces a contradiction. Hence $e$ is a bridge of $G - (E - \{e\})$, so that $G - E$ has two components. On the other hand, if $G'$ contains cycles and $e$ denotes a cycle edge of $G'$, then $\langle E - \{e\} \rangle$ is connected as is the graph $G - (E - \{e\})$. Then $e$ is a bridge of $G - (E - \{e\})$ and $G - E$ has exactly two components.
Clearly \( r_1(G) \leq cr_1(G) \). Further, since the edges incident with a vertex induce a connected subgraph, \( cr_1(G) \leq \delta(G) \) for every graph \( G \). If \( r_1(G) = 0 \) or \( r_1(G) = 1 \), then \( cr_1(G) = r_1(G) \). Hence only graphs \( G \) for which \( cr_1(G) \geq 2 \) need to be considered. We show that the inequalities

\[
r_1(G) \leq cr_1(G) \leq \delta(G)
\]

are best possible.

**Theorem 4.6** Let \( a, b \) and \( c \) be integers with \( 2 \leq a \leq b \leq c \). Then there is a graph \( G \) such that \( r_1(G) = a, cr_1(G) = b \) and \( \delta(G) = c \).

**Proof.** If \( a = b = c \), then \( K_{a+1} \) has the desired properties. If \( a = b < c \), let \( G_1 \equiv G_2 \equiv K_{c+1} \) with \( V(G_1) = \{u_1, u_2, \ldots, u_{c+1}\} \) and \( V(G_2) = \{v_1, v_2, \ldots, v_{c+1}\} \). Define \( G \) to be the graph obtained from \( G_1 \cup G_2 \) by adding the edges in \( E = \{u_iv_i \mid 1 \leq i \leq a\} \). Then \( r_1(G) = a \) because \( r_1(G_1) = r_1(G_2) = c \). Further, \( E \) is an edge cutset of cardinality \( a = b \) and \( \langle E \rangle \) is connected. Therefore, \( cr_1(G) \leq b \). However, since \( b = a \leq r_1(G) \leq cr_1(G) \), it follows that \( cr_1(G) = b \). Clearly \( \delta(G) = \text{deg} u_2 = c \), so that \( G \) has the desired properties.

Suppose now that \( a < b \leq c \). Let \( H_1 \equiv H_2 \equiv \cdots \equiv H_{b-a+1} \equiv K_c \equiv G_1 \equiv G_2 \equiv \cdots \equiv G_{b-a+1}, \) where...
\[ V(H_1) = \{v_{i,1}, v_{i,2}, \ldots, v_{i,c}\} \text{ and } V(G_1) = \{u_{i,1}, u_{i,2}, \ldots, u_{i,c}\} \text{ for } 1 \leq i \leq c. \] Define \( H' \) to be the graph obtained from \( H_1 \cup H_2 \cup \cdots \cup H_{b-a+1} \) by adding the edges in \( \{v_{i,j} v_{i+1,j} | 1 \leq i \leq b-a, 1 \leq j \leq c\} \) and let \( G' \) be the graph obtained from \( G_1 \cup G_2 \cup \cdots \cup G_{b-a+1} \) by adding the edges \( \{u_{i,j} u_{i+1,j} | 1 \leq i \leq b-a, 1 \leq j \leq c\} \). Then \( \kappa_1(G') = \kappa_1(H') = c \).

Let \( G \) be obtained from \( G' \cup H' \) by adding the edges in \( E = \{v_{i,1} u_{i+1,1} | 1 \leq i \leq b-a\} \). Clearly \( \delta(G) = \deg v_{1,2} = c \). Since \( G - E \) is disconnected, \( \kappa_1(G) \leq a \).

Suppose now that \( S \) is a set of \( a-1 \) edges. Then both \( G' - S \) and \( H' - S \) are connected, and since \( S - E \neq \emptyset \), the graph \( G - S \) is connected. Therefore \( \kappa_1(G) \geq a \), implying that \( \kappa_1(G) = a \).

It remains to show that \( \chi_1(G) = b \). Let \( E' = E \cup \{v_{i,1} v_{i+1,1} | 1 \leq i \leq b-a\} \). Then \( \langle E' \rangle \) is connected and \( G - E' \) is disconnected, so that \( \chi_1(G) \leq b \). We show next that \( \chi_1(G) \geq b \). Assume, to the contrary, that \( \chi_1(G) < b - 1 \).

Then there exists a set \( E'' \) of at most \( b - 1 \) edges such that \( \langle E'' \rangle \) is connected and \( G - E'' \) is disconnected. Since \( b - 1 < c \), both \( G' - E'' \) and \( H' - E'' \) are connected. Hence \( E \subseteq E'' \); otherwise, \( G - E'' \) is connected. However, since the distance between the vertices \( v_{1,1} \) or \( u_{1,1} \) and the end-vertices of the edges in \( E - \{e\} \) is at least \( b - a \), it follows that \( E'' - E \) contains at least \( b - a \) edges. Thus \( |E''| \geq b \), contrary to our assumption. Consequently, \( \chi_1(G) \geq b \) and therefore \( \chi_1(G) = b \). \[ \square \]
As remarked earlier, for a graph $G$, $\kappa(G) = 0$ if and only if $\kappa_1(G) = 0$, and $\kappa_1(G) = 0$ if and only if $\kappa_1'(G) = 0$. Since $\kappa(G) = 0$ if and only if $\kappa_1(G) = 0$, it follows that $\kappa(G) = 0$ if and only if $\kappa_1'(G) = 0$. Further, if $\kappa_1'(G) = 1$, then $\kappa_1' = 1$ so that $\kappa(G) = 1$ and hence $\kappa(G) = 1$. The next result shows, however, that neither the connected cutset connectivity nor the connected edge-cutset connectivity bounds the other in general.

Theorem 4.7 For every pair $a$, $b$ of positive integers with $b \neq 1$, there exists a graph $G$ with $\kappa_1(G) = a$ and $\kappa_1'(G) = b$.

Proof. Clearly if $a = b$, then $G \equiv K_{a+1}$ has the desired properties. For the remainder of the proof, we consider two cases.

Case 1 Suppose $a < b$.

If $a = 1$, then let $G_1 \equiv G_2 \equiv K_{b+1}$ and define $G$ to be the graph obtained from $G_1 \cup G_2$ by joining a vertex $v$ of $G_1$ to $b$ vertices, say $v_1, v_2, \ldots, v_b$, of $G_2$. Then $G$ is connected and because $G - v$ is disconnected, $\kappa(G) = 1$, implying that $\kappa_1(G) = 1$. Let $E = \{v v_i | 1 \leq i \leq b\}$. Then $\langle E \rangle$ is connected and $G - E$ is disconnected so that $\kappa_1'(G) \leq b$. If $S$ is a set of $b - 1$ edges, then $G_1 - S$ and $G_2 - S$ are connected and because $E - S \neq \emptyset$, the graph $G - S$ is also connected. Hence $\kappa_1(G) \geq b$, implying that $\kappa_1'(G) \geq b$. Therefore $\kappa_1'(G) = b$. 

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Assume therefore that 1 \leq a < b. Let
\[ H_1, H_2, \ldots, H_a, K_b, G_1, G_2, \ldots, G_a, \text{ where } V(H_i) = \{v_{i,1}, v_{i,1}' , \ldots, v_{i,b}\} \text{ and } V(G_i) = \{u_{i,1}, u_{i,2}, \ldots, u_{i,b}\} \]
for 1 \leq i \leq a. Let \( H' \) be the graph obtained from
\[ H_1 \cup H_2 \cup \ldots \cup H_a \]
by adding the edges in
\[ \{v_{i,j}, v_{i+1,j} \mid 1 \leq i \leq a - 1, 1 \leq j \leq b\}\]
and let \( G' \) be obtained from
\[ G_1 \cup G_2 \cup \ldots \cup G_a \]
by adding the edges in
\[ \{u_{i,j}, u_{i+1,j} \mid 1 \leq i \leq a - 1, 1 \leq j \leq b\}. \]
Define \( G \) to be the graph obtained from \( H' \cup G' \) by adding the edges in
\[ E = \{v_{1,1}, v_{1,1}'\} \cup \{v_{a,1}, v_{a,1}' \mid 1 \leq i \leq b - a\}. \]

We show first that \( c(G) = a \). Let \( V = \{v_{1,1}, v_{2,1}', \ldots, v_{a,1}\} \). Then \( \langle V \rangle \) is connected and \( G - V \) is disconnected. Hence \( c(G) \leq a \). To see that \( c(G) \geq a \), assume, to the contrary, that there exists a set \( S \) of at most \( a - 1 \) vertices of \( G \) such that \( \langle S \rangle \) is connected and \( G - S \) is disconnected. Since \( r(H') = r(G') = b \), both \( H' - S \) and \( G' - S \) are connected. Therefore, one of the vertices \( v_{1,1} \) and \( u_{1,1}' \) as well as one of the vertices \( v_{a,1} \) and \( u_{a,1}' \) belong to \( S \). Let \( u \) and \( v \) be two such vertices. Since \( d_G(u, v) \geq a - 1 \) and because \( \langle S \rangle \) is connected, \( S - \{u, v\} \) contains at least \( a - 2 \) vertices. Hence \( |S| \geq a \), contrary to our assumption. Therefore \( c(G) = a \).

It remains to show that \( c_1(G) = b \). Let \( E' = \{v_{i,1}, v_{i+1,1} \mid 1 \leq i \leq a - 1\} \cup E \). Then \( \langle E' \rangle \) is connected,
|E'| = b and G - E' is disconnected, implying that cr_1(G) ≤ b.
We show next that cr_1(G) ≥ b. Assume, to the contrary that there exists a set E'' of at most b - 1 edges such that \( \langle E'' \rangle \) is connected and G - E'' is disconnected. Since r_1(H') = r_1(G') = b, both H' - E'' and G' - E'' are connected. Hence E ⊆ E''; otherwise, G - E'' is connected. However, since the distance between v_1,1 or u_1,1 and the end-vertices of the edges in E - {e} is at least a - 1, it follows that E'' - E contains at least a - 1 edges. Therefore |E''| ≥ b, contrary to our assumption. Hence cr_1(G) ≥ b and consequently cr_1(G) = b.

**Case 2** Suppose b < a.
Then 1 ≤ b ≤ a. Let G_1, G_2, ... G_{a-b+2} ≅ K_a with V(G_1) = \{v_{1,1}, v_{1,2}, ..., v_{1,a}\}. Let G' be obtained from G_1 U G_2 U ... U G_{a-b+2} by adding the edges in \( \{v_{1,j}, v_{i+1,j} \mid 1 \leq i \leq b - a + 1, 1 \leq j \leq a\} \). Define G to be the graph obtained from G' by joining a new vertex v to v_{1,1}, v_{1,2}, ..., v_{1,b-1} and v_{a-b+2,1}.

Let \( V = \{v_{1,1}, v_{1,2}, ..., v_{1,b-1}\} \cup \{v_{i,1} \mid 2 \leq i \leq a - b + 2\} \).
Then \( \langle V \rangle \) is connected and G - V is disconnected. Therefore cr(G) ≤ a. We show next that cr(G) ≥ a. Assume, to the contrary, that there exists a set S of at most a - 1 vertices such that \( \langle S \rangle \) is connected and G - S is disconnected. Since r(G') = a, the graph G' - S is connected. Therefore v ∉ S; otherwise, G - S = G' - S, implying that G - S is connected. However, then
N(v) ⊆ S; otherwise, v is joined to a vertex of G' - S, implying that G - S is connected. Thus v ∈ S and N(v) ⊆ S.

However, since the distance in G - v between v_{a-b+2,1} and each vertex of N(v) - \{v_{a-b+2}\} is at least a - b + 1, it follows that |S - N(v)| ≥ a - b, so that |S| ≥ a, contradicting our assumption. Hence \(\chi(G) > a\), so that \(\chi(G) = a\).

We show next that \(\chi_1(G) = b\). Let E = \{v_{1,1} \mid 1 ≤ i ≤ b - 1\} ∪ \{v_{1,b-a+2}\}. Then \(E'\) is connected and G - E is disconnected. Consequently, \(\chi_1(G) ≤ b\).

To see that \(\chi_1(G) ≥ b\), assume, to the contrary, that there exists a set E' of at most b - 1 edges such that \(E'\) is connected and G - E' is disconnected. Since \(\chi_1(G') = a\), the graph G' - E' is connected. However, \(|E| = b\) so that G - E' contains at least one edge of E, implying that G - E' is connected and producing a contradiction. Hence \(\chi_1(G) ≥ b\) so that \(\chi_1(G) = b\).

The preceding result shows that for every pair \(a, b\) of integers with \(1 < b < a\), there is a graph G with \(\chi(G) = a\) and \(\chi_1(G) = b\). We next show that the minimum degree of such a graph G is determined.

**Theorem 4.8** If G is a graph with \(1 < \chi_1(G) ≤ \chi(G)\), then \(\chi_1(G) = \delta(G)\).
Proof. Let \( k = \text{cr}_1(G) \) and suppose that \( E \) is a set of \( k \) edges of \( G \) such that \( \langle E \rangle \) is connected and \( G - E \) is disconnected.

Then \( G' = \langle E \rangle \) has order at most \( k + 1 \).

Assume, to the contrary, that \( \delta(G) \neq k \). Then \( k < \delta(G) \).

Let \( C_1 \) and \( C_2 \) be the two components of \( G - E \). Then each of \( C_1 \) and \( C_2 \) contains at least one vertex of \( G' \). Suppose \( v_i \in V(C_i) \cap V(G'), \ i = 1, 2 \). Observe that every edge incident with \( v_i (i = 1, 2) \) that is not an edge of \( C_i \) necessarily is an edge of \( G' \). From this observation and the facts that \( \deg_G v_i \geq k + 1 \) and \( \deg_G v_i \leq k, \ i = 1, 2 \), it follows that there exists a vertex \( v_i' \in V(C_i) - V(G') \) that is adjacent to \( v_i \). However, then \( v_1 \) and \( v_2 \) belong to distinct components of \( G - V(G') \). Since \( \langle V(G') \rangle \) is connected, we therefore have

\[
k = \text{cr}(G) \leq |V(G')| \leq k + 1.
\]

Suppose first that \( |V(G')| = k \). Then \( G' \) is unicyclic. Let \( v \) be a non-cut-vertex of \( G' \) (that is, \( G' - v \) is connected) and let \( C' \) be the component of \( G - E \) containing \( v \). Suppose that \( C'' \) is the second component of \( G - E \). By a previous observation, \( V(C'') - V(G') \neq \emptyset \). Let \( u \in V(C'') - V(G') \). Then \( u \) and \( v \) belong to distinct components of \( G - (V(G') - \{v\}) \). However, \( G' - v \) is connected, so that \( \langle V(G') - \{v\} \rangle \) is connected and
therefore that \(\text{cr}(G) \leq |V(G')| - 1 = k - 1\), contrary to hypothesis.

Hence \(|V(G')| = k + 1\), so that \(G'\) is a tree. Let \(v\) and \(w\) be two end-vertices of \(G'\). Since \(p(G') = k + 1 \geq 3\), \(V(G') - \{v, w\} \neq \emptyset\). Further, \(G' - \{v, w\}\) is connected.

Suppose \(v\) belongs to the component \(C'\) of \(G - E\), and let \(C''\) be the other component of \(G - E\). By a previous remark, \(V(C'') - V(G')\) contains a vertex \(u\), say. However, then \(u\) and \(v\) belong to distinct components of \(G - (V(G') - \{v, w\})\). Since \(\langle V(G') - \{v, w\}\rangle\) is disconnected, it follows that \(\text{cr}(G) \leq |V(G')| - 2 = k - 1\), contrary to hypothesis.

Therefore \(\text{cr}_1(G) = \delta(G)\).

The result obtained in the preceding theorem is best possible in the following sense. For every pair \(b, d\) of integers with \(2 \leq b < d\), there is a graph \(G\) with \(\text{cr}_1(G) = b\), \(\delta(G) = d\) and \(\text{cr}(G) = b - 1\). To see this, suppose first that \(b = 2\). Let \(G_1 \equiv G_2 \equiv K_{d+1}\), where \(V(G_1) = \{u_1, u_2, \ldots, u_{d+1}\}\) and \(V(G_2) = \{v_1, v_2, \ldots, v_{d+1}\}\). Define \(G\) to be the graph obtained from \(G_1 \cup G_2\) by adding the edges \(e_1 = u_1v_1\) and \(e_2 = u_1v_2\). Let \(E = \{e_1, e_2\}\). Then \(\langle E\rangle\) is connected and \(G - E\) is disconnected. Hence \(\text{cr}_1(G) \leq 2\). Observe that \(\kappa_1(G) = 2\) and, therefore, that \(\text{cr}_1(G) \geq 2\). Since \(G - u_1\) is disconnected, \(\text{cr}(G) \leq 1\). Clearly,
$G$ is connected, so that $\phi(G) \geq 1$. Hence $\phi(G) = \phi_1(G) - 1 = 1$.

Further, $\delta(G) = d$.

Suppose now that $3 \leq b < d$. Let $G_1 \sqcup G_2 \sqcup \cdots \sqcup G_{b-1} \sqcup K_d \sqcup H_1 \sqcup H_2 \sqcup \cdots \sqcup H_{b-1}$, where $V(G_i) = \{u_{i,1}, u_{i,2}, \ldots, u_{i,d}\}$ and $V(H_i) = \{v_{i,1}, v_{i,2}, \ldots, v_{i,d}\}$ for $1 \leq i \leq b - 1$. Let $G'$ denote the graph obtained from $G_1 \sqcup G_2 \sqcup \cdots \sqcup G_{b-1}$ by adding the edges in $\{u_{i,j}, u_{i+1,j} \mid 1 \leq i \leq b - 2, 1 \leq j \leq d\}$ and define $H'$ to be the graph obtained from $H_1 \sqcup H_2 \sqcup \cdots \sqcup H_{b-1}$ by adding the edges in $\{v_{i,j}, v_{i+1,j} \mid 1 \leq i \leq b - 2, 1 \leq j \leq d\}$. Define $G$ to be the graph obtained from $G' \cup H'$ by adding the edges $e_1 = u_{1,1}v_{1,1}$ and $e_2 = u_{b-1,1}v_{b-1,1}$.

Let $E = \{v_{i,1}, v_{i+1,1} \mid 1 \leq i \leq b - 2\} \cup \{e_1, e_2\}$. Then $\langle E \rangle$ is connected and $G - E$ is disconnected. Hence $\phi_1(G) \leq b$.

To see that $\phi_1(G) \geq b$, assume, to the contrary, that $\phi_1(G) \leq b - 1$. Then there exists a set $S$ of at most $b - 1$ edges such that $\langle S \rangle$ is connected and $G - S$ is disconnected. Since $\phi_1(G') = \phi_1(H') = d$, both $G' - S$ and $H' - S$ are connected. Hence $\{e_1, e_2\} \subseteq S$. Since the distance between the end-vertices of $e_2$ is at least $b - 2$ and because $\langle S \rangle$ is connected, it follows that $S - \{e_1, e_2\}$ contains at least $b - 2$ edges. However, then $|S| \geq b$, contrary to assumption. Hence $\phi_1(G) \leq b$, so that $\phi_1(G) = b$. 

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We show next that $\kappa(G) = b - 1$. Let $V = \{v_{1,1}, v_{2,1}, \ldots, v_{b-1,1}\}$. Then $\langle V \rangle$ is connected and $G - V$ is disconnected. Hence $\kappa(G) \leq b - 1$. To see that $\kappa(G) \geq b - 1$, assume, to the contrary, that there is a vertex cutset $S$ of at most $b - 2$ vertices such that $\langle S \rangle$ is connected. Since $\kappa(G') = \kappa(H') = d$, both $G' - S$ and $H' - S$ are connected. Hence $S$ contains at least one of the vertices in $V_1 = \{u_{1,1}, v_{1,1}\}$ and one of the vertices in $V_2 = \{u_{b-1,1}, v_{b-1,1}\}$. However, since the distance between any vertex of $V_1$ and any vertex of $V_2$ is at least $b - 2$, the set $S - (V_1 \cup V_2)$ contains at least $b - 3$ vertices, so that $|S| \geq b - 1$, contrary to assumption. Hence $\kappa(G) \geq b - 1$ so that $\kappa(G) = b - 1$. Clearly $\delta(G) = d$.

A well-known problem in the area of connectivity deals with the development of conditions under which $\kappa_1(G) = \delta(G)$. Among these are the results (1) by Chartrand [7] who showed that if $G$ is a graph of order $p$ and $\delta(G) \geq (p - 1)/2$, then $\kappa_1(G) = \delta(G)$ and (2) by Plesnık [46] who verified that $\kappa_1(G) = \delta(G)$ for every graph $G$ with $\text{diam } G \leq 2$. Other conditions implying that $\kappa_1(G) = \delta(G)$ are given in [19], [21] and [27]. Since $\kappa_1(G) \leq \kappa_1(G) \leq \delta(G)$, these conditions also imply that $\kappa_1(G) = \delta(G)$. Theorem 4.8 provides, of course, another sufficient condition for $\kappa_1(G)$ to equal $\delta(G)$. This condition does not, however, imply that $\kappa_1(G) = \delta(G)$. To see this, let a
and \( b \) be integers with \( 3 \leq b \leq a \). We show that there exists a graph \( G \) such that \( \chi(G) = a, \ \chi_1(G) = \delta(G) = b \) and \( \chi_1(G) \neq \delta(G) \). (Note that the lower bound 3 given for \( b \) is necessary for the existence of such a graph \( G \), since if \( \chi_1(G) = b = 2 \), then \( \chi(G) = a \) only for \( a \leq 2 \).)

Let \( G_1 \cong G_2 \cong \cdots \cong G_a \cong H_1 \cong H_2 \cong \cdots \cong H_a \), where
\[
V(G_i) = \{u_{i,1}, u_{i,2}, \ldots, u_{i,a}\} \quad \text{and} \quad V(H_i) = \{v_{i,1}, v_{i,2}, \ldots, v_{i,a}\} \quad \text{for} \quad 1 \leq i \leq a.
\]

Let \( G' \) be obtained from \( G_1 \cup G_2 \cup \cdots \cup G_a \) by adding the edges in
\[
\{u_{i,j}, u_{i+1,j} \mid 1 \leq i \leq a - 1, 1 \leq j \leq a\}, \quad \text{and let} \quad H' \quad \text{be obtained from} \quad H_1 \cup H_2 \cup \cdots \cup H_a \quad \text{by adding the edges in}
\]
\[
\{v_{i,j}, v_{i+1,j} \mid 1 \leq i \leq a - 1, 1 \leq j \leq a\}.
\]

Define \( G \) to be the graph obtained from \( G' \cup H' \) by adding a new vertex \( v \) and joining it to \( v_{1,1}, v_{1,2}, \ldots, v_{1,b-1} \) and \( u_{1,1} \) and joining \( v_{a,1} \) and \( u_{a,1} \).

Let \( E \) denote the edges that are incident with \( v \). Then \( E \) is an edge cutset and \( E \) is connected. Hence \( \chi_1(G) \leq b \). To see that \( \chi_1(G) \geq b \), assume, to the contrary, that there is an edge cutset \( S \) such that \( |S| \leq b - 1 \) and \( S \) is connected.

Since \( |E| = b \), \( E \not\subseteq S \). Therefore, \( e = v_{a,1}u_{a,1} \in S \). Further, since \( G - S \) is disconnected, either \( E - \{v_{1,1}\} \subseteq S - \{e\} \) or \( v_{1,1}u_{1,1} \in S - \{e\} \). However, since \( |S| \leq b - 1 \), \( E - \{v_{1,1}\} \not\subseteq S - \{e\} \). Therefore, both \( e \) and \( v_{1,1}u_{1,1} \) are edges of \( S \). But the distance between the end-vertices of \( e \) and
\( v u_{1,1} \) is at least \( a - 1 \). Hence \(|S| \geq a + 1 > b\), which is impossible. Hence \( \text{cr}_1(G) \leq b \), so that \( \text{cr}_1(G) = b \).

We show next that \( \text{cr}(G) = a \). Let \( V = \{u_{1,1}, u_{2,1}, \ldots, u_{a,1}\} \). Then \( \langle V \rangle \) is connected and \( G - V \) is disconnected. Hence \( \text{cr}(G) \leq a \). To see that \( \text{cr}(G) \geq a \), assume, to the contrary, that there exists a vertex cutset \( S \) of at most \( a - 1 \) vertices such that \( \langle S \rangle \) is connected. Observe that \( G' - S \) and \( H' - S \) are connected. If \( v \in S \), then either \( v_a,1 \) or \( u_a,1 \) also belongs to \( S \). Since the distance between \( v \) and \( v_a,1 \) or \( u_a,1 \) is \( a \), the set \( S - \{v\} \) contains at least \( a \) vertices, contrary to assumption. Hence \( v \notin S \). If, in addition, \( u_{1,1} \) and at least one vertex of \( N(v) - \{u_{1,1}\} \) are not in \( S \), then \( G - S \) is connected. Therefore, either \( u_{1,1} \) or all the vertices of \( N(v) - \{u_{1,1}\} \) belong to \( S \). Further, one of the vertices \( v_a,1 \) and \( u_a,1 \) belongs to \( S \); otherwise, \( G - S \) is connected. But the distance between any vertex of \( N(v) \) and either of the vertices \( v_a,1 \) and \( u_a,1 \) is at least \( a - 1 \). Hence \( S \) contains at least \( a \) vertices, contrary to assumption. Hence \( \text{cr}(G) \geq a \), so that \( \text{cr}(G) = a \).

Clearly \( \delta(G) = b \). However, since \( G \) contains no bridge and \( G - \{v u_{1,1}, v_a,1 u_a,1\} \) is disconnected, \( k_1(G) = 2 \). Therefore \( k_1(G) \neq \delta(G) \).
BIBLIOGRAPHY


