On the Genus of a Block Design

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ON THE GENUS OF A BLOCK DESIGN

by

Joan Marie Rahn

A Dissertation
Submitted to the
Faculty of The Graduate College
in partial fulfillment of the
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Degree of Doctor of Philosophy
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ON THE GENUS OF A BLOCK DESIGN

Joan Marie Rahn, Ph.D.
Western Michigan University, 1984

The genus of a design (BIBD or PBIBD) is defined to be the genus of its corresponding hypergraph (objects as vertices, blocks as edges); that is, the genus of the bipartite graph associated with the hypergraph in a natural way. The Euler formula is used to establish a lower bound γ for the genus of a block design. An imbedding of the design on the surface of genus γ is then described by a voltage hypergraph or voltage graph. Use of the lower bound formula leads to a characterization of planar BIBDs. A connection between a block design derived from a graph imbedding and the hypergraph imbedding of the design is established. This leads to the determination of genus formulas for several infinite families of designs. The concept of the generalized pseudocharacteristic of a design is developed along with formulas for infinite families of designs.
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For
Paul
Mom and Dad
Cutz
ACKNOWLEDGEMENTS

I wish to thank Professor Arthur T. White for introducing me to the concept of block designs and challenging me to explain how I was able to construct them. His advice, encouragement, patience and example were invaluable. I am indebted to Professor Linda Kraai of Western Michigan University, Kalamazoo and Professor Henry Levinson of Rutgers University, Newark for their careful reading of this dissertation and helpful suggestions. I would like to thank Professor Gary Chartrand of Western Michigan University who introduced me to Graph Theory. Thanks are due to Professor Joseph Buckley for serving on my committee. I would also like to express my gratitude to Margo Johnson for her excellent typing of the manuscript and to the Mathematics Department for financial assistance.

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CHAPTER 1

INTRODUCTION

1.1 Examples

Block designs are abstract mathematical structures in which objects are arranged in sets so that certain conditions are satisfied. As such, they are useful in solving "real world" problems involving schedules and experimental designs. Two easily understood applications of block designs are given below.

Problem 1.1.1: Seven nurses must be assigned to work seven evenings so that: (i) each nurse works exactly three evenings; (ii) there are exactly three nurses working each evening; (iii) each pair of nurses work together exactly one evening.

Solution: The nurses are assigned numbers from 0 through 6. One schedule is:

\[
\begin{align*}
S &: 0 \ 3 \ 2 \\
M &: 1 \ 4 \ 3 \\
T &: 2 \ 5 \ 4 \\
W &: 3 \ 6 \ 5 \\
T &: 4 \ 0 \ 6 \\
F &: 5 \ 1 \ 0 \\
S &: 6 \ 2 \ 1 \\
\end{align*}
\]

This is an example of a balanced incomplete block design (BIBD) on parameters \((7, 7, 3, 3, 1)\). Observe that there are seven numbers (each representing a nurse) arranged in seven sets (each
representing an evening) such that each number occurs in exactly three sets, each set has exactly three numbers, and each pair of distinct numbers occur together in exactly one set.

**Problem 1.1.2:** An advertising firm is hired to compare grape and strawberry jams which are produced by three companies A, B, C. Due to budgetary constraints, only four people can be employed to taste the six jams. Each person can taste at most three jams. Each pair of jams from different companies must be compared (i.e., tasted by the same person). There is no interest in comparing two jams from the same company.

**Solution:** The six jams are assigned numbers as indicated in the chart.

<table>
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<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strawberry</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Grape</td>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
</tbody>
</table>

The pairs which must be compared are:

1 and 2  1 and 3  1 and 5  1 and 6  2 and 3  2 and 6
4 and 2  4 and 5  4 and 3  4 and 6  5 and 3  5 and 6.

The pairs which need not be compared are:

1 and 4  2 and 5  3 and 6.
One experimental design satisfying the conditions is given below.

Taster 1: 1 2 3
Taster 2: 1 5 6
Taster 3: 4 2 6
Taster 4: 4 5 3

This is a partially balanced incomplete block design (PBIBD) on parameters (6, 4, 2, 3; 0, 1). The six numbers are arranged in four sets so that: (i) each number appears in exactly two sets; (ii) each set has exactly three numbers; (iii) each pair of numbers which need not be compared appear together in no set and each pair of numbers which must be compared appear together in exactly one set.

1.2 Tools for Construction and Analysis

Diverse branches of mathematics are blended in the study of block designs. Combinatorics is, of course, basic since a block design is a combinatorial structure. Finite field theory is instrumental in the generation of sets of mutually orthogonal Latin squares which form the bases of Latin square partially balanced incomplete block designs. Finite geometries and perfect difference sets are also used to generate block designs. Two-fold triple systems (both (v, b, r, 3, 2)-BIBDs and (v, b, r, 3; 0, 2)-PBIBDs) arise from the triangular imbeddings of graphs on surfaces. Thus graph theory, surface topology and algebra have roles in the construction and analysis of block designs. Even the analysis of
polynomial functions will be used in this thesis for the determi-
nation of planar block designs.

The close relationship between graph imbeddings and block de-
signs can be illustrated by the triangular imbedding of the complete
graph on seven vertices, $K_7$, on the torus (doughnut).

![Figure 1.1](image)

(The torus may be formed from this figure as follows: the hori-
zontal lines are identified, forming a cylinder; then the circles of
the ends of the cylinder are identified, forming the doughnut
shape.) A $(7, 14, 6, 3, 2)$-BIBD is generated in which the seven
vertices are arranged into fourteen sets (triangular regions) of
these vertices so that each vertex belongs to six sets and each pair
of vertices belong to exactly two sets. The actual design can be
written down as follows.
1.3 History

The relationship between graph imbeddings and block designs was first observed by Heffter in a paper [16] which appeared in 1891. In this work he gave a partial solution to the Heawood Map Coloring Problem by finding imbeddings for the complete graph on v vertices for v ≤ 12. Heffter noticed the correspondence between (v, b, r, 3, 2)-BIBDs and triangular imbeddings of \( K_v \) for v = 3, 4, 7, 12. In a work [17] which appeared in 1897, he observed a relationship between Steiner triple systems ((v, b, r, 3, 1)-BIBDs) and triangular imbeddings of certain graphs.

Apparently little work was done between 1897 and 1929 on developing this relationship. In 1929, Emch [13] observed the BIBDs associated with triangular imbeddings of \( K_6, K_7 \) and \( K_9 \).

The next major breakthrough in research came in 1968 with the final solution of the Heawood Map Coloring Problem by Ringel and Youngs along with other mathematicians. (See [34] for a summary.) Part of this solution involved determining the "most efficient" drawings of complete graphs on surfaces. Alpert, in a work [1] that appeared in 1975, examined these efficient triangular imbeddings for complete graphs \( K_v \), v = 0, 1 (mod 3) and the
(v, b, r, 3, 2)-BIBDs associated with them. He established the existence of a bijection between (v, b, r, 3, 2)-BIBDs and triangular imbeddings of $K_v$ on generalized pseudosurfaces.

Alpert's work inspired White and Garman to investigate the relationship between block designs and imbeddings of graphs other than complete graphs. In 1976 Garman completed his Ph.D. thesis on block designs associated with imbeddings of Cayley graphs [14]. White explored the designs arising from imbeddings of strongly regular graphs in a paper [31] appearing in 1978. He proved that the partially balanced incomplete two-fold triple systems are in 1-1 correspondence with triangular imbeddings of strongly regular graphs on generalized pseudosurfaces.

Anderson and White examined bi-embeddings of certain complete graphs in a paper [5] published in 1978. They observed the balanced incomplete block designs arising from bi-embeddings of $K_{13}, K_{16}, K_{21}, K_{25}, K_{28}, K_{37}$ and others. (See also [2].) In 1981 and 1982 Anderson also noted designs arising in the study of the toroidal thickness of graphs ([3] and [4]).

Until 1980, most of the work on block designs and graph imbeddings dealt with either generating designs from graph imbeddings or imbedding a graph by means of a block design (see [31]). There had been no attempt to find a geometric realization of a block design. In 1980, Jungerman, Stahl and White, in a paper [19] discussing geometric realizations of hypergraphs, mentioned that a design could be represented by a hypergraph. They then gave an
example of how the $(7, 7, 3, 3, 1)$-BIBD given in Problem 1.1.1 would appear as a hypergraph drawn on the torus. This concept, however, was not further developed.

1.4 Significance

Surprisingly, since then, little attention has been focused on such geometric realizations of block designs. The purpose of this research is to extend the work on block design realizations begun by Jungerman, Stahl and White. Specifically, the abstract mathematical concept of a block design is given a concrete geometric realization. This is accomplished by representing the design as a hypergraph which is then concretely realized by drawing, or imbedding, it on some surface. Particular attention is paid to "most efficient", or genus, imbeddings of designs.

In order to carry out this program, definitions and background material are necessary and are presented in Chapter 2. In Chapter 3 imbedding techniques are discussed, along with a general approach for determining genus imbeddings. Questions concerning surface imbeddings of designs seem to fall into two main categories. Chapter 4 deals with some questions of the first type: on what kinds of surfaces can a given design be imbedded? Chapter 5 deals with the second type: what kinds of designs can be imbedded on a given surface? (The surface under consideration is the sphere.) The connection between graph imbeddings yielding block designs and
hypergraph imbeddings of designs is established in Chapter 6.
Chapter 7 includes several open questions pointing out new directions of research.
CHAPTER 2
DEFINITIONS AND BACKGROUND

In the introductory chapter several examples of block designs were given without actually defining what they were. Graphs and graph imbeddings were used. References were made to the concepts of hypergraph, genus imbedding and planarity. All of these terms and concepts, plus many others, are precisely defined in this chapter.

2.1 Graphs

A graph $G$ consists of a finite nonempty set $V(G)$ of objects called vertices and a collection $E(G)$ of distinct 2-element subsets of $V(G)$ called edges. A graph may be depicted by drawing (on some locally 2-dimensional drawing board) its vertices as hollow dots and its edges by line segments connecting pairs of dots. Two distinct vertices are adjacent if they are joined by an edge. Two distinct edges are adjacent if they have a common vertex. If a vertex $v$ belongs to an edge $e$, then $v$ and $e$ are incident. If $v$ is incident with exactly $r$ edges in $G$, then the degree of $v$ in $G$, $\deg_G v$, is $r$. A graph $G$ is $r$-regular if $\deg_G v = r$ for each $v \in V(G)$. The "First Theorem of Graph Theory" relates the degrees of the vertices of a graph and the number of edges.
Theorem 2.1.1: If $G$ is a graph with $\mathcal{V}(G) = \{v_1, v_2, \ldots, v_p\}$ and $q$ edges, then $2q = \sum \deg_G v_i$.

Some special graphs encountered in this research are complete graphs and cycles. The complete graph on $n$ vertices, $K_n$, consists of $n$ vertices and all $\binom{n}{2}$ possible edges. A cycle of length $n$ (at least 3), $C_n$, consists of $n$ distinct vertices which may be labeled $v_1, v_2, \ldots, v_n$ such that the edges are $v_1v_2, v_2v_3, \ldots, v_{i}v_{i+1}, \ldots, v_nv_1$. See Figure 2.1 for examples of these graphs.

\begin{figure}[h]
\centering
\begin{tabular}{cc}
$C_3$ & $K_3$
\end{tabular}
\begin{tabular}{cc}
$C_4$ & $K_4$
\end{tabular}
\begin{tabular}{cc}
$C_5$ & $K_5$
\end{tabular}
\caption{Figure 2.1}
\end{figure}

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A graph $H$ is a subgraph of a graph $G$, $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. In Figure 2.1, $C_3 \subseteq K_3$, $C_4 \subseteq K_4$, and $C_5 \subseteq K_5$. Note also that $C_n \subseteq K_n$ for $n \geq 3$. If a graph $G$ contains a cycle, then the girth of $G$, $g(G)$, is the length of the smallest cycle in $G$. Thus $g(K_n) = 3$ if $n \geq 3$ and $g(C_n) = n$.

A graph $H$ is isomorphic to a graph $G$, $H \cong G$, if there exists a bijection $\alpha: V(H) \rightarrow V(G)$ such that $vw \in E(H)$ if and only if $\alpha(v)\alpha(w) \in E(G)$. A graph $H$ is an elementary subdivision of a graph $G$ if $H$ is derived from $G$ by removing an edge $vw$ from $G$ and adding a new vertex $x$ and two new edges $vx$ and $xw$. Again referring to Figure 2.1, $C_4$ is an elementary subdivision of $C_3$. A graph $H$ is a subdivision of $G$ if $H$ is obtained from $G$ by a finite succession of elementary subdivisions. So $C_n$, $n \geq 4$, are subdivisions of $C_3$. A graph $H$ is homeomorphic to $G$ if either $H \cong G$ or $H$ is isomorphic to a subdivision of $G$. In Figure 2.2 $H_1$ is homeomorphic from $G$ but $H_2$ is not homeomorphic from $G$. Note that all new vertices of a subdivision $H$ of $G$ must be of degree 2 in $H$.

A $u-v$ walk $W$ in a graph $G$ is a listing of vertices of $G$, $W: u = v_1, v_2, \ldots, v_n = v$, such that $v_i v_{i+1}$ is an edge of $G$ for $1 \leq i \leq n - 1$. If there is a $u-v$ walk in $G$ for each pair of vertices $u$ and $v$ of $G$, then $G$ is connected. A component of $G$ is a maximal connected subgraph of $G$. 

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FIGURE 2.2

The complement of a graph $G$, $\overline{G}$, consists of the vertices of $G$ and all of the edges that are not in $G$; that is, $V(G) = V(\overline{G})$ and $E(\overline{G}) = \{vw|v, w \in V(G), v \neq w \text{ and } vw \notin E(G)\}$. Thus the complement of $K_n$, $\overline{K}_n$, is the empty graph on $n$ vertices.

Several families of graphs are important in this study of block designs: bipartite graphs, complete multipartite graphs and strongly regular graphs. A graph $G$ is bipartite if the vertex set of $G$ can be partitioned into two nonempty sets $V_1$ and $V_2$ (called partite sets) such that if $e = vw \in E(G)$, then $v \in V_1$ and $w \in V_2$ or vice versa. Examples of bipartite graphs are $\overline{K}_n$, $n \geq 2$, $K_2$ and $C_n$ for $n$ an even positive integer at least 4. The complete bipartite graph $K_{n,m}$ consists of $V(K_{n,m}) = V_1 \cup V_2$.
such that $|V_1| = n$, $|V_2| = m$ and $E(K_{n,m}) = \{vw | v \in V_1, w \in V_2\}$.

The graph $K_{n,n}$ is denoted by $K_{2(n)}$.

The following theorems and corollary are pertinent in this research.

**Theorem 2.1.2:** For each graph $G$ of order at least two there is a bipartite graph $H$ homeomorphic from $G$.

**Proof:** If $G = K_n$, $n \geq 2$, then take $H = G$. If $G$ is nonempty, then construct a subdivision of $G$ by removing each edge $e = vw \in E(G)$ and then replacing it with a new vertex $x_e$ and edges $vx_e$ and $x_ew$. The original vertices of $G$ form one partite set. The new "edge" vertices form the other partite set. □

**Theorem 2.1.3:** (Characterization of Bipartite Graphs) A graph $G$ is a bipartite graph if and only if it contains no cycles of odd length as subgraphs. (See [6].)

**Corollary 2.1.4:** If $G$ is a bipartite graph containing cycles as subgraphs, then $g(G) \geq 4$.

Bipartite graphs comprise a subfamily of a larger family of graphs known as multipartite graphs. A graph $G$ is said to be an $n$-partite graph if the vertex set of $G$ can be partitioned into $n$ nonempty subsets $V_1, V_2, \ldots, V_n$ such that if $e = vw \in E(G)$,
then there exist $i, j$ such that $1 \leq i < j \leq n$ and $v \in V_i$ and $w \in V_j$. The complete $n$-partite graph is denoted by $K_{m_1, m_2, \ldots, m_n}$ and its edge set is $\{vw | v \in V_i, w \in V_j,\ 1 \leq i < j \leq n \}$. If $m_i = m$ for $1 \leq i \leq n$, then $K_{m_1, m_2, \ldots, m_n}$ is denoted by $K_n(m)$.

The graphs $K_{n(m)}$, for $m \geq 2$, $n \geq 2$, in turn are examples of strongly regular graphs. Let $G$ be a graph and $x, y \in V(G)$. The parameters $p^h(x, y)$ are defined as follows:

- if $x, y$ are not adjacent, then $h = 1$;
- if $x, y$ are adjacent, then $h = 2$;
- $p_{11}^h(x, y)$: number of vertices not adjacent to both $x$ and $y$;
- $p_{22}^h(x, y)$: number of vertices adjacent to both $x$ and $y$;
- $p_{12}^h(x, y)$: number of vertices not adjacent to $x$ but adjacent to $y$;
- $p_{21}^h(x, y)$: number of vertices adjacent to $x$ but not adjacent to $y$.

A graph $G$ is strongly regular if it satisfies the following conditions:

(i) $G \not\cong K_n$, $G \not\cong K_2^n$;
(ii) $G$ is $n_2$-regular;
(iii) $p_{ij}^h(x, y)$ is independent of the choice of $x$ and $y$ for $h, i, j = 1, 2$.

The eight conditions in part (iii) can be condensed into just two conditions.
Theorem 2.1.5: If $G$ is a regular graph which is neither empty nor complete, then $G$ is strongly regular if and only if $p_h(x,y)$ is independent of $x$ and $y$ for $h = 1, 2$. (See [31].)

In order to prove that $K_{n(m)}$, $n, m \geq 2$, is strongly regular, it is necessary to show that $p_1(x,y)$ and $p_2(x,y)$ are constant. If $x$ and $y$ are not adjacent, then they must belong to the same partite set. Hence $\deg x = \deg y = m(n-1)$. Therefore $p_1(x,y) = m(n-1)$. If $x, y$ are adjacent, then they must belong to different partite sets. Thus $p_2(x,y) = m(n-2)$.

In a strongly regular graph $G$, if two vertices are not adjacent, they are said to be first associates. They are second associates if they are adjacent. The number of $i$th associates of a vertex is denoted by $n_i$ for $i = 1, 2$.

Two other important classes of strongly regular graphs are the line graphs of complete graphs, $L(K_n)$, and the line graph of $K_2(n)$, $L(K_2(n)) = K_n \times K_n$. The line graph of a nonempty graph $G$, $L(G)$, is defined as follows:

$$V(L(G)) = E(G)$$

$$E(L(G)) = \{ef | e, f \in E(G) \text{ and } e, f \text{ are adjacent in } G\}.$$  

See Figure 2.3 for an example of $L(K_4)$. 

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The cartesian product of graphs $G_1$ and $G_2$, $G_1 \times G_2$, is a graph with $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and $E(G_1 \times G_2) = \{(u_1, u_2)(v_1, v_2) | u_1 = v_1$ and $u_2v_2 \in E(G_2)$ or $u_2 = v_2$ and $u_1v_1 \in E(G_1)\}$. See Figure 2.4.
For additional details and definitions concerning graphs, see [6]. Consult [7], [10] or [31] for more information about strongly regular graphs.

2.2 Hypergraphs

The concept of a graph \( G \) may be generalized by relaxing the conditions on the edge set of the graph. If finitely many repetitions of the 2-element subsets in the edge set are allowed (i.e., the edge set is a multi-set), then \( G \) becomes a multigraph. If, in addition, 1-element subsets (i.e., loops) are permitted, then \( G \) is a pseudograph. If \( E(G) \) is allowed to contain \( k \)-element subsets of \( V(G) \) (with possible repetitions of the subsets), for \( k \) possibly varying from 1 to \( |V(G)| \), then \( G \) is a hypergraph. If \( k \) is constant, then \( G \) is a \textit{k-uniform hypergraph}.

As in the case of graphs, multigraphs and pseudographs are easily drawn by the use of hollow dots, line segments and loops. See Figure 2.5.

![Graph](image1.png)  ![Multigraph](image2.png)  ![Pseudograph](image3.png)

FIGURE 2.5

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Depicting hypergraphs in this fashion is, unfortunately, not possible, since an edge could consist of three or more vertices. One method of drawing a hypergraph $H$ indicates the edges of $H$ by drawing closed curves around the appropriate vertices, as in Figure 2.6. Needless to say, this method can be quite cumbersome.

$V(H) = \{1, 2, 3, 4\}$

$E(H) = \{\{1\}, \{2, 3\}, \{1, 2, 4\}, \{2, 3, 4\}\}$

FIGURE 2.6

Jungerman, Stahl and White [19] devised a much clearer method of drawing hypergraphs, which will be examined in the imbedding section of this chapter. This method involves a bipartite graph which is associated with the hypergraph in a natural way. In 1975, Walsh [30] indicated the existence of a bijection between hypergraphs and bipartite graphs. Let $H$ be a hypergraph with vertex
set $V(H)$ and edge set $E(H)$. Define the bipartite graph associated with $H$, $G(H)$, as follows:

$V(G(H)) = V(H) \cup E(H)$ and

$E(G(H)) = \{ve|v \in V(H), e \in E(H), v \in e\}$.

See Figure 2.7 for an example.

$V(H) = \{1,2,3,4\}$

$E(H) = \{\{1\}, \{2,3\}, \{1,2,4\}, \{2,3,4\}\}$

![Figure 2.7](image)

**FIGURE 2.7**

### 2.3 Block Designs

The family of block designs may be considered as a subfamily of the family of hypergraphs. Two types of block designs are studied in this work.

A $(v, b, r, k, \lambda)$-balanced incomplete block design (BIBD) is an arrangement of $v$ objects into $b$ subsets called blocks such that:
(i) each object belongs to exactly $r$ blocks;
(ii) each block contains exactly $k$ objects.
(iii) each pair of distinct objects appear together in exactly $\lambda$ blocks.

The solution to Problem 1.1.1 in Chapter 1 is a $(7, 7, 3, 3, 1)$-BIBD.

A $(G, (v, b, r, k; \lambda_1, \lambda_2))$-partially balanced incomplete block design (PBIBD) is an arrangement of $v$ objects, first by pairs into two associate classes as determined by vertex (object) adjacency in the strongly regular graph $G$, and then into $b$ blocks such that:

(i) each object belongs to exactly $r$ blocks;
(ii) each block contains exactly $k$ objects;
(iii) each pair of ith associates appear together in exactly $\lambda_i$ blocks for $i = 1, 2$.

In Problem 1.1.2 of Chapter 1, the solution is a $(K_3(2), (6, 4, 2, 3; 0, 1))$-PBIBD. Pairs of jams from the same company are first associates. Each company forms a partite set. A pair of jams from different companies are second associates with an edge connecting them. See Figure 2.8.
Several relations among the parameters of a block design now follow. They greatly simplify calculations and are well-known. Thus they are stated without proof. If further details are desired, consult [9], [15] or [27].

**Theorem 2.3.1:** If $D$ is a $(v, b, r, k, \lambda)$-BIBD, then

(i) $vr = bk$;

(ii) $\lambda(v-1) = r(k-1)$;

(iii) (Fisher's Inequality) $b \geq v$. 

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Theorem 2.3.2: For any \((G) (v, b, r, k; \lambda_1, \lambda_2)-\text{PBIBD}\)

(i) \(vr = bk\);

(ii) \(\lambda_1 n_1 + \lambda_2 n_2 = r(k-1)\), where \(n_i\) is the number of \(i\)th associates possessed by each vertex in \(G\) for \(i = 1, 2\);

(iii) \(n_1 + n_2 = v - 1\).

A design is considered to be incomplete if \(k < v\) and balanced if \(\lambda_1 = \lambda_2\). In a \((G)\) PBIBD it is also assumed that \(\lambda_1 < \lambda_2\). (If \(\lambda_1 > \lambda_2\), then the strongly regular graph is actually \(G\).)

To view a design \(D\) (BIBD or PBIBD) as a hypergraph, simply consider the objects of \(D\) as vertices and the blocks of \(D\) as edges [19]. The hypergraph thus associated with design \(D\) is denoted by \(H(D)\). The bipartite graph associated with design \(D\), \(G(D)\), is \(G(H(D))\).

A block design \(D\) is said to be connected if, given any pair of objects \(u\) and \(v\) in \(D\), there is a sequence of objects and blocks \(W: u = v_1, b_1, v_2, b_2, \ldots, v_i, b_i, \ldots, v_{n-1}, b_{n-1}, v_n = v\) such that \(v_i, v_{i+1} \in b_i\) for \(i = 1, \ldots, n - 1\). This definition of a connected block design \(D\) may also be expressed in terms of \(G(D)\).

Theorem 2.3.3: A block design \(D\) is connected if and only if \(G(D)\) is connected.
2.4 Surfaces

Graphs, hypergraphs and block designs are geometrically realized by "drawing" them on various locally 2-dimensional "drawing boards". A surface is a connected, closed 2-manifold. A surface is orientable if a clockwise sense of rotation is maintained upon traveling once about each simple closed curve on the surface. If this is not the case, then the surface is nonorientable.

An orientable surface $S_k$ may be pictured as a sphere with $k$ handles attached to it, or equivalently as a sphere with $k$ holes punched through it. Such a surface is said to have genus $k$, which is denoted by $\gamma(S_k) = k$. Examples of surfaces are the sphere $S_0$, the torus $S_1$ and the double torus $S_2$. A nonorientable surface is formed from an orientable surface by the addition of cross-caps. A sphere with $k$ cross-caps is denoted by $N_k$ and has nonorientable genus $k$, $\gamma(N_k) = k$, for $k \geq 0$. ($N_0$ is the sphere with 0 cross-caps; i.e., $N_0 = S_0$.)

Every nonorientable surface is homeomorphic to a sphere with $k$ cross-caps (see [32]). Moreover:

Theorem 2.4.1 (Classification Theorem) Two closed 2-manifolds are homeomorphic if and only if they have the same characteristic and are both orientable or nonorientable.

Thus every nonorientable surface has a nonorientable genus. The characteristic parameter will be defined in the imbeddings section of Chapter 2.
In Chapter 1, the torus was pictured as a rectangle with
directions given to form it into a doughnut. It turns out that
every surface has a similar polygonal representation [32].

Theorem 2.4.2: Each closed 2-manifold has a corresponding polygonal
representation such that its sides may be labeled in one of the
following ways:

(i) $x \times x^{-1}(S_0)$;
(ii) $x_1y_1x^{-1}y^{-1} \cdots x_ky_kx^{-1}y^{-1}(S_k \text{ for } k > 0)$;
(iii) $x_1x_1^x_2x_2^2 \cdots x_kx_k^x(N_k \text{ for } k > 0)$.

The concept of a surface is now broadened to included pseudo-
surfaces and generalized pseudosurfaces; i.e., "surfaces" which fail
to be closed 2-manifolds at a finite number of points. A pseudo-
surface is constructed from a surface by making finitely many
identifications of finitely many points. A generalized pseudo-
surface is constructed from finitely many pseudosurfaces by again
making finitely many identifications of finitely many points to form
a connected topological space. For examples, see Figure 2.9.
These points of identification are called singular points and are precisely the points where these "surfaces" fail to be closed 2-manifolds. As with surfaces, generalized pseudosurfaces are either orientable or nonorientable depending upon the starting surface. For further details, consult [14], [31] or [32].

2.5 Imbeddings

Hypergraphs and the block designs are drawn, or imbedded, on various surfaces by means of their associated bipartite graphs. A graph is said to be imbedded on a generalized pseudosurface $M$ if the geometric realization of $G$ as a one-dimensional simplicial complex is homeomorphic to a subspace of $M$. More intuitively, $G$ is imbedded on $M$ if it is drawn on $M$ so that its edges do not intersect each other. In Figure 2.10, the first representation of $C_5$ is imbedded on $S_0$ but the second is not.
A graph $G$ is 2-cell imbedded on a generalized pseudosurface $M$, $G \sqsubset M$, if each of the components of $M - G$, called regions of $G \sqsubset M$, is homeomorphic to an open unit disk in $\mathbb{R}^2$. If $M$ has singular points, then it is assumed that each singular point is occupied by a vertex of $G$. If $G$ is disconnected, then $G$ has no 2-cell imbedding on any generalized pseudosurface.

Let $G$ be a connected pseudograph that is 2-cell imbedded on some generalized pseudosurface $M$. The dual of $G$ (relative to the imbedding), $G^*$, is a pseudograph with the regions of $G \sqsubset M$ as vertices. Two vertices of $G^*$ are joined by an edge if and only if their corresponding regions in $G \sqsubset M$ share an edge. Thus loops and multiple edges may occur in $G^*$. Furthermore, each edge of $G$ corresponds with an edge of $G^*$. See Figure 2.11.
In [19], Jungerman, Stahl and White described a method of geometrically realizing a hypergraph $H$ by 2-cell imbedding $G(H)$ on a generalized pseudosurface $M$ and then modifying the imbedding. This method requires that "object" vertices, rather than "edge" vertices, occupy any singular points that $M$ might have. Each edge vertex $e$ is encircled by object vertices in some order $v_1$, $v_2$, $\cdots$, $v_k$, which belong to edge $e$ in $H$. In the Jungerman, Stahl and White modification of $G(H) \triangleleft M$, the edges $v_i v_{i+1}$, $1 \leq i \leq k-1$, and $v_k v_1$ are added and then the vertex $e$ and edges $v_i e$, $1 \leq i \leq k$, are removed. The result is a picture of $H$ 2-cell imbedded on $M$ with two types of regions: "edges" that had contained edge vertices of $G(H)$ and bona fide regions of $H \triangleleft M$. (See Figure 2.12.)
$V(H) = \{1, 2, 3, 4\}; \ E(H) = \{\{1\}, \{2, 3\}, \{1, 2, 4\}, \{2, 3, 4\}\}$

Figure 2.12

Let $H$ be imbedded on a generalized pseudosurface $M$ by the Jungerman, Stahl and White method. The line segments separating the edges of $H$ from the regions of $H \triangleleft M$ along with the vertices of $H$ form a pseudograph called the 1-skeleton of $H$, $H'$. Hence $H'$ is imbedded on $M$ with bichromatic dual; i.e., the regions of $H' \triangleleft M$ may be colored with two colors so that adjacent regions receive different colors. In this case, the colors assume the names...
edge of $H$ and bona fide region of $H$. The $1$-skeleton of the hypergraph in Figure 2.12 is shown in Figure 2.13.

![Diagram of $H'$]

FIGURE 2.13

It should be pointed out that $H'$ depends upon how $G(H)$ was imbedded upon $M$.

A block design $D$ can now be 2-cell imbedded on $M$ in the form of its associated hypergraph $H(D)$ (i.e., $G(D) \triangleleft M$ and is then modified by the Jungerman, Stahl and White method to obtain $H(D) \triangleleft M$). The $(K_{3(2)})(6, 4, 2, 3; 0, 1)$-PBIRD $D$ of Problem 1.1.2 is imbedded on the plane in Figure 2.14.
The blocks of \( D \) are the unshaded regions and the shaded regions are the bona fide regions of \( D \subseteq M \).

Many connected graphs may be 2-cell imbedded on more than one surface. The (orientable) genus of \( G \), \( \gamma(G) \), is the minimum genus among all orientable surfaces upon which \( G \) is imbeddable. Such a minimum imbedding for a connected graph \( G \) is always 2-cell. The (orientable) genus of a hypergraph \( H \), \( \gamma(H) \), and of a block design \( D \), \( \gamma(D) \), is defined by \( \gamma(H) = \gamma(G(H)) \) and \( \gamma(D) = \gamma(G(D)) \).
respectively. Since the \((K^3(2))\) \((6, 4, 2, 3; 0, 1)\)-PBIBD of Problem 1.1.2 is imbeddable on the sphere, \(\gamma(D) = 0\).

If \(G\) is a connected graph, then the maximum (orientable) genus of \(G\), \(\gamma_M(G)\), is defined as the maximum genus among all orientable surfaces upon which \(G\) can be 2-cell imbedded. Thus, if a hypergraph \(H\) or a block design \(D\) has a connected associated bipartite graph, then \(\gamma_M(H) = \gamma_M(G(H))\) and \(\gamma_M(D) = \gamma_M(G(D))\).

Duke's Theorem makes the following statement about 2-cell imbeddings of a connected graph \(G\).

**Theorem 2.5.1: (Duke)** If there exists a 2-cell imbedding of a connected graph \(G\) on the orientable surfaces \(S_m\) and \(S_n\), where \(m < n\), and \(k\) is any integer such that \(m \leq k \leq n\), then there exists a 2-cell imbedding of \(G\) on the surfaces \(S_k\).

For a proof of this theorem, consult [12]. The following are corollaries of Duke's Theorem.

**Corollary 2.5.2:** If \(G\) is a connected graph, then \(G\) is 2-cell imbeddable on \(S_k\) for \(\gamma(G) \leq k \leq \gamma_M(G)\).

**Corollary 2.5.3:** If \(D\) is a block design with connected \(G(D)\), then \(D\) is 2-cell imbeddable on \(S_k\) for \(\gamma(D) \leq k \leq \gamma_M(D)\).

There are nonorientable analogues of the previous definitions and theorems. The nonorientable genus of a graph \(G\), \(\gamma(G)\), is
the minimum \( k \) for which \( G \) can be imbedded on \( N_k \) \((k \geq 0)\). Thus for a hypergraph \( H \) or a block design \( D \), \( \gamma(H) = \gamma(G(H)) \) and \( \gamma(D) = \gamma(G(D)) \). If \( G \) is a connected graph, then the maximum nonorientable genus of \( G \), \( \gamma^N(G) \), is the maximum \( k \) for which \( G \) can be 2-cell imbedded on \( N_k \). For a hypergraph \( H \) and a block design \( D \) having a connected associated bipartite graph, the definitions are \( \gamma^N(H) = \gamma^M(G(H)) \) and \( \gamma^M(D) = \gamma^M(G(D)) \) respectively.

The nonorientable analogues of Duke's Theorem and corollaries now follow.

**Theorem 2.5.4:** If \( G \) is a connected graph which has 2-cell imbeddings on \( N_m \) and \( N_n \) for \( m < n \), then \( G \) has a 2-cell imbedding on \( N_k \) for \( m \leq k \leq n \).

**Corollary 2.5.5:** If \( G \) is a connected graph, then \( G \) has a 2-cell imbedding on \( N_k \) for \( \gamma(G) \leq k \leq \gamma^M(G) \).

**Corollary 2.5.6:** If \( D \) is a connected block design, then \( D \) has a 2-cell imbedding on \( N_k \) for \( \gamma(D) \leq k \leq \gamma^M(D) \).

(For more information concerning genus and maximum genus of a graph, consult [6] or [29].)

In the Classification Theorem for connected, closed 2-manifolds, the term "characteristic" was introduced. The characteristic...
of a generalized pseudosurface $M$, $\chi(M)$, is defined in terms of a 2-cell imbedding of a connected graph $G$. Specifically, let $G$ be a connected graph with $p$ vertices and $q$ edges, which is 2-cell imbedded on the generalized pseudosurface $M$ with $r$ regions. Then $\chi(G \triangle M) = p - q + r$. Since this is actually independent of the graph $G$ [32], we may write $\chi(M) = p - q + r$. The parameter $\chi(M)$ is also known as the Euler characteristic of $M$.

If $M$ is an orientable surface of genus $k$, then $\chi(M) = 2 - 2k$. If $M = N_k$, then $\chi(M) = 2 - k$. Thus, if the characteristic of $M$ is known, along with the nature of "orientation", the genus of $M$ may be easily calculated. If $M$ has singular points, however, $\chi(M)$ does not "characterize" the generalized pseudosurface; i.e., two nonhomeomorphic generalized pseudosurfaces may have the same Euler characteristic.

The characteristic of a connected graph $G$, $\chi(G)$, is the maximum $\chi(M)$ among all surfaces $M$ upon which $G$ is 2-cell imbeddable. For hypergraphs $H$ and block designs $D$ with connected associated bipartite graphs, the definitions are $\chi(H) = \chi(G(H))$ and $\chi(D) = \chi(G(D))$. If pseudosurface imbeddings are included, then the pseudocharacteristic of a connected graph $G$, $\chi'(G)$, is the maximum $\chi(M)$ among all pseudosurfaces $M$ such that $G \triangle M$. For connected hypergraphs and block designs the corresponding definitions are $\chi'(H) = \chi'(G(H))$ and $\chi'(D) = \chi'(G(D))$. The generalized pseudocharacteristic of a connected graph $G$, $\chi''(G)$, is the maximum $\chi(M)$ among all generalized pseudosurfaces.
M such that $G < M$. Analogously, $x''(H) = x''(G(H))$ and $x''(D) = x''(G(D))$. Observe that $x''(G) \geq x'(G) \geq x(G)$.

It should be pointed out that to find $x''(G) = p - q + r$, the number of regions of $G < M$ is maximized. $x''(G)$ could be made arbitrarily large by attaching as many spheres as desired to a vertex of $G < M$. Therefore regions with 0 sides are excluded. Since $G$ has no loops, there are no one-sided regions. A generalized pseudosurface imbedding of $G$ in which each edge of $G$ is on a different surface is also regarded as trivial. Thus two-sided regions are excluded. Therefore triangular imbeddings of a graph $G$ are optimal. In this dissertation, work is concentrated upon the determination of the most efficient (optimal) imbeddings of designs and, in the case of surfaces, on the genus imbeddings of designs.
CHAPTER 3

OPTIMAL IMBEDDINGS

In this chapter techniques are developed to find optimal imbeddings of block designs on generalized pseudosurfaces. These techniques are based upon graph imbedding methods which are examined through examples.

3.1 General Approach for Determining Orientable Genus

The general approach for finding the orientable genus of a graph \( G \) is first to determine a lower bound \( h \) for \( \gamma(G) \) and then to construct an imbedding of \( G \) on \( S_h \). The following theorem gives a lower bound for \( \gamma(G) \) if \( G \) is a connected graph (see [6 p.96]).

**Theorem 3.1.1:** If \( G \) is a connected graph with \( p \) vertices, \( q \) edges and girth \( n \), then
\[
\gamma(G) \geq \frac{q}{2} \left(1 - \frac{2}{n}\right) - \frac{p}{2} + 1.
\]

This approach is interpreted in terms of block designs: a lower bound \( h \) is found for \( \gamma(D) \) and then \( G(D) \) is imbedded on \( S_h \).

**Theorem 3.1.2:** If \( D \) is a \((v, b, r, k, \lambda)\)-BIBD or a connected \((v, b, r, k; \lambda_1, \lambda_2)\)-PBIBD with \( g(G(D)) = n \), then
\[
\gamma(D) \geq \frac{bk}{2} \left(1 - \frac{2}{n}\right) - \frac{v+b}{2} + 1.
\]
Proof: The graph $G(D)$ has $p = v + b$ vertices, $q = \frac{vr + bk}{2}$ vertices, $q = vr$ edges and girth $n$ at least 4. By Theorem 3.1.1.,

$$\gamma(D) = \gamma(G(D)) \geq \frac{bk}{2} (1 - \frac{2}{n}) - \frac{v+b}{2} + 1. \quad \Box$$

In order to determine a lower bound for the genus of the $(7, 7, 3, 3, 1)$-BIBD $D$ given in Problem 1.1.1, $g(G(D))$ must be found. The parameter $\lambda = 1$ implies that $g(G(D)) \geq 6$. If $g(G(D)) = 4$, then two object vertices must be adjacent to the same two block vertices. This implies that two distinct objects appear together in two blocks, a contradiction to $\lambda = 1$. Therefore $g(G(D)) \geq 6$. A lower bound for $\gamma(D)$ may now be calculated:

$$\gamma(D) \geq \frac{7 \times 3}{2} (1 - \frac{2}{6}) - \frac{7 + 7}{2} + 1. \quad \Box$$

Thus $\gamma(D) \geq 1$. A toroidal imbedding of $G(D)$ should be attempted.

3.2 Graph Imbedding Techniques

There are several methods of describing a graph imbedding: Edmonds' permutation technique, current graphs, voltage graphs, surgery. The Edmonds' permutation technique algebraically describes a graph imbedding. Current graphs and their duals, voltage graphs, utilize covering spaces. Surgery "physically" transforms the "surface" upon which a given graph is imbedded to achieve a new imbedding of either the same graph or a different graph.
Explanations, by way of examples, of these techniques are presented in this section.

To illustrate the Edmonds' approach, consider the 2-cell imbedding of $K_7$ on the torus in Figure 3.1.

For each $i \in V(K_7)$, let $V_i = \{j | ij \in E(K_7)\}$. Note that a cyclic permutation $\pi_i : V_i \to V_i$ is determined by this imbedding of $K_7$ on $S_1$ as follows: $\pi_i(j) = k$ where $ik$ immediately succeeds $ij$ (in the counterclockwise direction) in the imbedding. For example, $\pi_5$ is the cyclic permutation $(0 \ 4 \ 2 \ 3 \ 6 \ 1)$ where $\pi_5(0) = 4$, $\pi_5(4) = 2$, $\pi_5(2) = 3$, etc. As $i$ varies over the vertices of $K_7$, the permutations are given as follows:
\[ \pi_0 = (1 \ 3 \ 2 \ 6 \ 4 \ 5) \]
\[ \pi_1 = (0 \ 5 \ 6 \ 2 \ 4 \ 3) \]
\[ \pi_2 = (0 \ 3 \ 5 \ 4 \ 1 \ 6) \]
\[ \pi_3 = (0 \ 1 \ 4 \ 6 \ 5 \ 2) \]
\[ \pi_4 = (0 \ 6 \ 3 \ 1 \ 2 \ 5) \]
\[ \pi_5 = (0 \ 4 \ 2 \ 3 \ 6 \ 1) \]
\[ \pi_6 = (0 \ 2 \ 1 \ 5 \ 3 \ 4). \]

The regions of \( K_7 \triangleleft S_1 \) can be determined from these \( \pi_i \)'s.

Let \( D(K_7) = \{(i,j)|ij \in E(K_7)\} \), the set of directed edges of \( K_7 \).

A permutation (which depends upon \( \pi_1, \pi_2, \cdots, \pi_7 \)) \( \pi:D(K_7) \rightarrow D(K_7) \)
is defined as follows:

\[ \pi(i,j) = (j, \pi_j(i)). \]

The orbits of \( \pi \) yield the regions of \( K_7 \triangleleft S_1 \) (whose boundary edges are traced in the clockwise direction). For example, one orbit of \( \pi \) is determined:

\[ \pi(3,2) = (2, \pi_2(3)) = (2,5) \]
\[ \pi(2,5) = (5, \pi_5(2)) = (5,3) \]
\[ \pi(5,3) = (3, \pi_3(5)) = (3,2). \]

Thus the region traced out is

\[ R_1: (3,2),(2,5),(5,3) \]
or more briefly denoted by

\[ R_1: 3 - 2 - 5 - 3. \]

All of the orbits of \( \pi \) are thus listed:

\[
\begin{align*}
0 - 1 - 5 - 0 & \quad 1 - 2 - 6 - 1 \\
0 - 3 - 1 - 0 & \quad 1 - 4 - 2 - 1 \\
0 - 2 - 3 - 0 & \quad 1 - 3 - 4 - 1 \\
0 - 6 - 2 - 0 & \quad 2 - 4 - 5 - 2 \\
0 - 4 - 6 - 0 & \quad 2 - 5 - 3 - 2 \\
0 - 5 - 4 - 0 & \quad 3 - 5 - 6 - 3 \\
1 - 6 - 5 - 1 & \quad 3 - 6 - 4 - 3.
\end{align*}
\]

Note that each of the directed edges in \( D(K_r) \) is listed exactly once among the orbits of \( \pi \) (regions of \( K_r \)). If these regions are "sewn together" so that each directed edge \((i,j)\) is matched with its opposite \((j,i)\), then \( S_1 \) is constructed.

In general, for any given nonempty connected graph \( G \), there is a 1-1 correspondence between 2-cell imbeddings of \( G \) and permutations \( \pi \), depending upon \( \pi_1, \pi_2, \ldots, \pi_p \), where \( V(G) = \{1, 2, \ldots, p\} \) and \( \pi_i \) is a cyclic permutation of \( V_i \) for each \( i \in V(G) \). For an exact statement and proof of Edmonds' Theorem, see Theorem 6-36 of [32].

Although Edmonds' permutation technique is a powerful tool for constructing 2-cell imbeddings of a connected graph \( G \), it can be cumbersome for large graphs. Current graphs, in some instances,
provide a "short hand" notation for these cyclic permutations of vertices. A current graph is a triple \((K, \Gamma, \lambda)\), where \(K\) is a pseudograph, \(\Gamma\) is a finite group with identity \(e\) and \(\lambda: K^* \rightarrow \Gamma - \{e\}\) is a map which assigns currents from \(\Gamma\) to the directed edges of \(K\), \(K^* = \{(v_i, v_j)|v_i, v_j \in D(K)\}\), such that \(\lambda(v_i, v_j) = -\lambda(v_j, v_i)\). If \(K\) is 2-cell imbedded on some (orientable) surface \(S\) and \(v \in V(K)\) such that the product of currents directed away from \(v\), in the order given by the imbedding of \(K\) on \(S\), is \(e\), then Kirchoff's Current Law (KCL) is said to be satisfied. An example of a current graph is given in Figure 3.2. It will be used to find an imbedding of \(K\) on \(S\).

![Figure 3.2](image_url)

Note that the KCL holds at each vertex of \(K\). There is one hexagonal region in the imbedding. The currents listed in counterclockwise order about this region yield the cyclic permutation of the vertices adjacent with vertex 0 in the imbedding of \(K\) on \(S\) in Figure 3.1. Specifically, the permutation is
\( \pi_0 = (1 \ 3 \ 2 \ 6 \ 4 \ 5). \)

To generate the other \( \pi_i \)'s simply add 1 to each element of the cycle, as follows:

\[ \pi_i = (1+i \ 3+i \ 2+i \ 6+i \ 4+i \ 5+i) \]

where addition takes place in \( Z_7 \).

In the theory of quotient graphs (see [32 p. 102]), this imbedding of \( K \triangleleft S_1 \) is "covered by" another graph hexagonally imbedded on \( S_1 \) with 14 vertices of degree 3 and 21 edges as in Figure 3.3. Note that the dual of this graph is \( K_7 \triangleleft S_1 \).

For additional details about current graphs and quotient graphs, see [28], [32] or [36].

\[ \text{FIGURE 3.3} \]
Voltage graphs are the duals of current graphs and, as such, eliminate the necessity of "taking the dual" of the covering graph as the final step to achieve the desired graph imbedding. A voltage graph is a triple \((K, \Gamma, \phi)\) where \(K\) is a pseudograph, \(\Gamma\) is a finite group and \(\phi: K^* \to \Gamma\) satisfies \(\phi(v, w) = -\phi(w, v)\) where, again, \(K^* = \{(v, w) | vw \in E(K)\}\). The covering graph \(K \times_{\phi} \Gamma\) for \((K, \Gamma, \phi)\) has \(V(K \times_{\phi} \Gamma) = V(K) \times \Gamma\) and \(E(K \times_{\phi} \Gamma) = \{(u, g)(v, g\phi(u, v)) | uv \in E(K)\} \times \Gamma\). Thus each vertex \(v\) of \(K\) is covered by \(|\Gamma|\) vertices \((v, g)\) and each edge \(uv\) of \(K\) is covered by \(|\Gamma|\) edges \((u, g)(v, g\phi(u, v))\). The graph \(K \times_{\phi} \Gamma\) is said to be a \(|\Gamma|\)-fold covering of \(K\) if \(K\) is considered as a topological space.

A 2-cell imbedding of \(K\) on a surface \(S\) with Edmonds' permutation scheme \(\pi\) (depending upon \(\pi_1, \pi_2, \ldots, \pi_{|V(K)|}\)) determines a permutation scheme \(\pi'\) for \(K \times_{\phi} \Gamma \triangleleft S\)' as follows:

\[\text{if } \pi_v(u) = w, \text{ then } \pi'_{(v, g)}((u, g\phi(v, u)) = (w, g\phi(v, w)).\]

The scheme \(\pi'\) is said to be the lift of the scheme \(\pi\).

If \(R\) is a region of \(K \triangleleft S\), then \(|R|_{\phi}\) is the order of \(\phi(c)\) where \(c\) is the boundary of the region \(R\) and \(\phi(c)\) is the product of the voltages on that boundary (in clockwise order). If \(|R|_{\phi} = 1\), then Kirchoff's Voltage Law (KVL) holds for region \(R\). The following theorem (Theorem 2.2 of [33]) is the heart of voltage graph theory.
Theorem 3.2.1: Let \((K, \Gamma, \phi)\) be a voltage graph with rotation scheme \(\pi\) and \(\pi'\) the lift of \(\pi\) to \(K \times \Gamma\). Let \(\pi\) and \(\pi'\) determine 2-cell imbeddings of \(K\) and \(K \times \Gamma\) on orientable surfaces \(S\) and \(S'\) respectively. Then there exists a branched covering projection \(\rho: S' \rightarrow S\) such that:

(i) \(\rho^{-1}(K) = K \times \Gamma\); 

(ii) if \(b\) is a branch point of multiplicity \(n\), then \(b\) is in the interior of a region \(R\) such that \(|R|_\phi = n\); 

(iii) if \(R\) is a region of the imbedding of \(K\) which is a \(k\)-gon, then \(\rho^{-1}(R)\) has \(\frac{|\Gamma|}{|R|_\phi}\) components, each of which is a \(k|R|_\phi\)-gon region of \(K \times \Gamma \subset S'\).

Note that if the KVL holds for each region of \(K \subset S\), then \(\rho\) is actually a covering projection. For further details see [33].

As an example of how voltage graph theory works, consider \((K, Z_7, \phi)\) as given in Figure 3.4. The covering graph is \(K_7\) triangularly imbedded on \(S_1\).

\[ (K, Z_7, \phi) \subset S_1 \]

\[ K_7 = K \times \phi Z_7 \subset S_1 \]

**FIGURE 3.4**
The surgical technique is especially suited to finding optimal imbeddings of graphs on generalized pseudosurfaces. The following example was inspired by Anderson's and White's work on bi-embeddings of complete graphs [5]. Consider the two voltage graphs and their covering graphs given in Figure 3.5.

Observe that $K_2 \times \phi Z_{13} \triangleleft S_1$ is actually the complement of $G$. By identifying the vertices of $G$ and $\overline{G}$ receiving the same label, an optimal (triangular) imbedding of $K_{13}$ on a generalized pseudosurface $M$ is achieved. Note that $\chi''(K_{13}) = p - q + r = 13 - 78 + 52 = -13$. 

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3.3 Voltage Hypergraphs

Voltage graph theory is now extended to hypergraphs. There are several approaches to such extensions. Method 1 relies upon the bipartite graph associated with a given hypergraph. Method 2 applies voltage graph theory to a modified 1-skeleton of the hypergraph. Method 3 uses voltage graph theory directly on the 1-skeleton of the hypergraph.

In Method 1 the imbedding of $G(K)$ on $S$ is determined from the imbedding of the hypergraph $K$ on $S$. Voltages from $\Gamma$ are then assigned to the directed edges of $G(K)$ and voltage graph theory is applied to find the covering graph $G(K) \times_{\phi} \Gamma \triangleleft S'$. The Jungerman, Stahl and White modification then transforms $G(K) \times_{\phi} \Gamma \triangleleft S$ into the hypergraph covering $K, K \times_{\phi} \Gamma \triangleleft S'$.

To see how this applies to block designs, recall the $(7, 7, 3, 3, 1)$-BIBD $D$. It was determined that $\gamma(D) \geq 1$. A toroidal imbedding of $D$ is now attempted with the aid of the hypergraph $K$ in Figure 3.6. Note that $K$ has one vertex, one triangular edge (unshaded) and one triangular region (shaded). The next step (shown in Figure 3.6) is to use the Jungerman, Stahl and White modification "in reverse" to find $G(K) \triangleleft S_1$. Voltages from $Z_7$ are assigned to the edges of $G(K) \triangleleft S_1$ so that the KVL is satisfied. The covering graph $G(K) \times_{\phi} Z_7$ is then constructed. Observe that $G(K) \times_{\phi} Z_7 = G(D)$. Finally $G(D) \triangleleft S_1$ is modified to achieve the toroidal imbedding of $H(D)$. Thus
FIGURE 3.6

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\(\gamma(D) = 1\). It should be pointed out that the 1-skeleton of \(H(D)\) is nothing more than \(K_7 \triangleleft S_1\).

Method 2 modifies the 1-skeleton \(K'\) of \(K \triangleleft S\) by inserting into each edge of \(K'\) a vertex of degree 2. Voltages are then assigned to the edges of the "new" graph \(K''\) and the covering graph \(K'' \times \phi \Gamma\) is found. Since \(K' \triangleleft S\) has bichromatic dual (edges of \(K\) and regions of \(K\)), \(K'' \times \phi \Gamma\) also has bichromatic dual with edge-regions of \(K'' \times \phi \Gamma\) covering edge-regions of \(K\) and regional-regions of \(K'' \times \phi \Gamma\) covering regional-regions of \(K\). To find the covering hypergraph, \(K \times \phi \Gamma\), remove the vertices of degree 2 from the edges of \(K'' \times \phi \Gamma\) covering the vertices that were inserted into each edge of \(K'\). The resulting multigraph is actually \(K' \times \phi \Gamma\), which is the 1-skeleton of \(K \times \phi \Gamma\).

In special cases, if voltages are properly assigned, the covering hypergraph of Method 1 coincides with the covering graph of Method 2. Compare the two treatments of \(K \triangleleft S_1\) given in Figures 3.6 and 3.7. Note that upon removal of the vertices of degree 2 from \(K'' \times \phi Z_7\), the resulting graph \(K_7\) is precisely the 1-skeleton of the covering hypergraph \(K \times \phi Z_7\).
\( K \triangleleft S_1 \)

\( (G(K), \gamma_7, \phi) \triangleleft S_1 \)

\( (K'', \gamma_7, \phi) \triangleleft S_1 \)

\( K'' \times_\phi \gamma_7 \triangleleft S_1 \)

\( K \times_\phi \gamma_7 \triangleleft S_1 \)

FIGURE 3.7

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Method 3 simply assigns voltages to the edges of the 1-skeleton $K'$ of $K \triangleleft S$. The covering graph $K' \times_\phi \Gamma$ has bichromatic dual, with "unshaded regions" covering edges of $K \triangleleft S$ and "shaded regions" covering the bona fide regions of $K \triangleleft S$. Again, in some cases, the covering hypergraph of Method 3 coincides with that of Method 1. Compare the treatments of $K \triangleleft S_1$ given in Figure 3.6 and 3.8. The 1-skeleton of the covering hypergraph $K \times_\phi Z_7$ is $K' \times_\phi Z_7 = K_7$.

![Diagram](image)

**Figure 3.8**

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Methods 2 and 3 have a slight advantage, if they can be used, since the Jungerman, Stahl and White modification need not be carried out. But, unfortunately, these two methods are not always consistent with Method 1. Consider the hypergraph $K \triangleleft S_1$ in Figure 3.9 with voltages assigned to the edges of the 1-skeleton $K'$ as indicated. (Note that the KVL does not hold.)

In Method 3, the unshaded region $E$ is covered by $\frac{|Z_5|}{|E|_\phi} = 1$ region, which is a $4 \times 5 = 20$-gon. $R_1$ is covered by $\frac{|Z_5|}{|R_1|_\phi} = 5$ regions, which are digons. $R_2$ is covered by $\frac{|Z_5|}{|R_2|_\phi} = 1$ region,
which is a $2 \cdot 5 = 10$-gon. But note that no matter how voltages are assigned to the edges of $G(K)$ (shown in Figure 3.9), the edge vertex will always be covered by 5 edge vertices of degree 4. Hence, upon applying the Jungerman, Stahl and White modification to $G(K) \times \Gamma \triangleleft S'$, there will always be five edge-regions with four sides.

Since $\gamma(H)$ is defined to be $\gamma(G(H))$, Method 1 will be the preferred extension of voltage graph theory to hypergraphs, although on some occasions Methods 2 and 3 will also be suitable.
CHAPTER 4

GENUS AND CHARACTERISTIC FORMULAS

4.1 Helpful Theorems

Certain block designs have easily-identified associated bipartite graphs or graphs describing relationships among the blocks. The genus and/or characteristic of such designs can be calculated with a minimum amount of effort. This certainly is true for the "complete" block designs on parameters \((v, b, b, v, b)\).

Theorem 4.1.1: If \(D\) is a complete design on parameters \((v, b, b, v, b)\) for \(v \geq 2\) and \(b \geq 2\), then \(G(D) \cong K_{v,b}\) and \(\gamma(D) = \frac{(v-2)(b-2)}{4}\).

Proof: Let \(D\) be a complete design on parameters \((v, b, b, v, b)\) for \(v \geq 2\) and \(b \geq 2\). Since \(k = v\), each block contains all of the objects of \(D\). Therefore, for each object \(u\) and each block \(e\), \(u \in e\). But this implies \(u \in E(G(D))\). Therefore \(G(D) \cong K_{v,b}\) and \(\gamma(D) = \gamma(K_{v,b}) = \frac{(v-2)(b-2)}{4}\). (See [25].)

The following theorem is proved in a similar fashion.

Theorem 4.1.2: If \(D\) is a \((v, 1, 1, v, 1)\) complete design, then \(G(D) \cong K_{v,1}\) and \(\gamma(D) = 0\).
For each complete design, the value of the genus parameter is known. Complete designs are therefore of little interest in this work.

The following theorems will be useful in determining the genus or characteristic of incomplete block designs which have parameter $k = 2$. The definition of $\lambda$-fold $G$ must first be given. The multigraph $\lambda$-fold $G$, $G^{(\lambda)}$, is formed from a nonempty graph $G$ by replacing each edge $uv$ of $G$ by $\lambda$ such edges $uv$. Necessarily, $\gamma(G) = \gamma(G^{(\lambda)})$.

**Theorem 4.1.3:** If $D$ is a $(v, b, r, 2, \lambda)$-BIBD, then $G(D)$ is homeomorphic from $\lambda$-fold $K_v$, $K_v^{(\lambda)}$.

**Proof:** Let $D$ be a $(v, b, r, 2, \lambda)$-BIBD. $K_v^{(\lambda)}$ may be thought of as a representation of $D$ if the vertices of $K_v^{(\lambda)}$ are the objects of $D$. The edges of $K_v^{(\lambda)}$ yield the blocks of $D$ since each block is a 2-element subset of the object set (vertex set) of $D$. Note that there are exactly $\lambda$ such edges (or blocks) for each pair of distinct vertices (objects) in $K_v^{(\lambda)}$. By inserting a "block" vertex of degree two into each edge $uw$ of $K_v^{(\lambda)}$ and labeling it $\{u, w\}$, we construct a new graph, which is actually $G(D)$. Thus $G(D)$ is homeomorphic from $K_v^{(\lambda)}$. □

**Corollary 4.1.4:** If $D$ is a $(v, b, r, 2, \lambda)$-BIBD, then

$$\gamma(D) = \gamma(K_v) = \frac{(v-3)(v-4)}{12}.$$ (See [26].)
Theorem 4.1.5: If D is a (G) (v, b, r, 2; \lambda_1, \lambda_2)\text{-PBIBD}, then G(D) is homeomorphic from G (\lambda_1, \lambda_2) in which there are exactly \lambda_i edges between each pair of objects which are ith associates, i = 1, 2.

Proof: Let D be a (G) (v, b, r, 2; \lambda_1, \lambda_2)\text{-PBIBD}. As in the proof of Theorem 4.1.3, G (\lambda_1, \lambda_2) may be considered as representing D if the vertices are the objects of D. The edges of G (\lambda_1, \lambda_2) yield the blocks of D since each block is a 2-element subset of the object set of D. By inserting a "block" vertex into each edge uw of G (\lambda_1, \lambda_2) and labeling it \{u, w\}, we obtain the graph G(D), (\lambda_1, \lambda_2), which is homeomorphic from G (\lambda_1, \lambda_2). □

Corollary 4.1.6: If D is a (G) (v, b, r, 2; \lambda_1, \lambda_2)\text{-PBIBD such that } \lambda_1 \geq 1, then G(D) is homeomorphic from K_v (\lambda_1, \lambda_2) and \gamma(D) = \frac{(v-3)(v-4)}{12}.

Corollary 4.1.7: If D is a (G) (v, b, r, 2; 0, \lambda_2)\text{-PBIBD, then } \gamma(D) = \gamma(G).

If D is a (G) (v, b, r, 2; 0, \lambda_2)\text{-PBIBD, then } \gamma(D) may be found if \gamma(G) is known. PBIBDs based on the strongly regular graphs K_{n(m)}, L(K_n) and L(K_2(n)) will be examined next.
4.2 Group Divisible Designs

Group divisible PBIBDs have strongly regular graphs $K_{n(m)}$. In Chapter 2, it was shown that $p_{22}^1 = m(n-1)$ and $p_{22}^2 = m(n-2)$. Furthermore $n_2 = m(n-1)$ and $n_1 = m - 1$. Thus by Theorem 2.3.2, the following relation holds for a $(K_{n(m)})^D_{n(m)}$:

$$(m-1)\lambda_1 + m(n-1)\lambda_2 = r(k-1).$$

If $k = 2$ and $\lambda_1 = 0$, then $D_{n(m)}$ must be on parameters

$$\left(\frac{m^2n(n-1)\lambda_2}{2}, \frac{m(n-1)\lambda_2}{2}, 2, 0, \lambda_2\right).$$

By Theorem 4.1.5, $G(D_{n(m)})$ is homeomorphic from $K_{n(m)}$, whose genus is known in many cases (see [8]). Thus follows

Corollary 4.2.1: If $D_{n(m)}$ is a $(K_{n(m)})^D_{n(m)}$ $(v, b, r, 2; 0, \lambda_2)$-PBIBD, then $\gamma(D_{n(m)}) = \gamma(K_{n(m)})$.

4.3 Triangular Designs

Triangular PBIBDs are also commonly encountered block designs. A triangular scheme $T_n$ yields the two associate classes for the objects of the design. Scheme $T_n$ is described as follows [7]. An $n \times n$ matrix whose entries are the objects of $D$ (often represented by the numbers $1, 2, \ldots, \binom{n}{2}$) is set up so that:

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(1) the \( \binom{n}{2} \) positions above the leading diagonal are filled by the objects of \( D \);
(2) the positions on the main diagonal are left blank;
(3) the positions below the leading diagonal are filled so that the matrix is symmetrical with respect to the leading diagonal.

Two distinct numbers are **second associates** if they occur in the same row (or same column). They are **first associates** otherwise.

For example, consider the \( n=5 \) triangular scheme given below.

\[
\begin{array}{cccc}
* & 1 & 2 & 3 & 4 \\
1 & * & 5 & 6 & 7 \\
2 & 5 & * & 8 & 9 \\
3 & 6 & 8 & * & 10 \\
4 & 7 & 9 & 10 & *
\end{array}
\]

The second associates of object 6 are 1, 5, 7, 3, 8, 10. The first associates of 6 are 2, 4, 9.

A graph \( G_n \) may be associated with scheme \( T_n \) if the vertices of \( G_n \) are the \( \binom{n}{2} \) objects in \( T_n \). An edge \( e = ij \) is in \( E(G_n) \) if and only if \( i \) and \( j \) are second associates. In fact \( G_n \) = \( L(K_n) \), which is strongly regular (see [10]). Each vertex in \( L(K_n) \) has \( n_1 = \frac{(n-2)(n-3)}{2} \) first associates and \( n_2 = 2n - 4 \) second associates. Furthermore, \( p_1^{12} = 4 \) and \( p_2^{12} = n - 2 \).

Any PBIBD based upon \( L(K_n) \) is called a **triangular design**.
If \( D_n \) is a \((L(K_n)) (v, b, r, 2; 0, 1)\)-PBIBD, then the objects of \( D \) may be considered as the vertices of \( L(K_n) \) and the blocks of \( D \) may be considered as the edges of \( L(K_n) \). The following must also be true:

\[
\begin{align*}
v &= \frac{n(n-1)}{2}; \\
r &= 2n - 4 = \deg_{L(K_n)} w \text{ for each } w \in V(L(K_n)); \\
b &= \frac{n(n-1)(n-2)}{2}.
\end{align*}
\]

By Corollary 4.1.7, \( \gamma(D_n) = \gamma(L(K_n)) \). Unfortunately, determining \( \gamma(L(K_n)) \) is a difficult problem. A more tractable problem is the determination of \( \chi''(L(K_n)) \). In fact, we have the following theorem.

**Theorem 4.3.1:** \( \chi''(L(K_n)) = -\frac{n(n-1)(n-5)}{6} \) for \( n - 1 \equiv 0, 1 \pmod{3} \), \( n \geq 5 \).

**Proof:** A triangular generalized pseudosurface imbedding of \( L(K_n) \) is attempted. Let \( n \geq 5 \) be such that \( n - 1 \equiv 0, 1 \pmod{3} \). Consider \( K_n \) with vertices labeled \( R_i \), \( 1 \leq i \leq n \). Label the edges of \( K_n \) with \( 1, 2, \ldots, \binom{n}{2} \). (These will become the vertex labels upon forming \( L(K_n) \).) Note that each edge \( R_i R_j \) of \( K_n \) is incident with exactly two vertices, \( R_i \) and \( R_j \), and each vertex \( R_i \) is incident with each of the \( n - 1 \) other vertices \( R_j \) exactly once. Upon forming \( L(K_n) \), the \( n - 1 \) edges incident with
vertex $R_i$ form $K_{n-1}$ whose vertices receive the edge labels from $K_n$, $1 \leq i \leq n$. Since $n - 1 \equiv 0,1 \pmod{3}$, $K_{n-1}$ may be triangularly imbedded on some surface $M$. (See [1].) Take $n$ disjoint copies of $K_{n-1} \triangleleft M$ triangularly. Label $K_{n-1} \triangleleft M$ with $R_i$ and label the vertices with the labels of the edges incident with $R_i$, $1 \leq i \leq n$. Now observe that each edge label appears exactly twice among the $K_{n-1} \triangleleft M (R_i)$, $1 \leq i \leq n$. Identify vertices receiving the same edge label. Since each $R_i$ is adjacent with the $n - 1$ other $R_j$'s in $K_n$, the resulting generalized pseudosurface $M'$ must be connected. The graph thus formed is $L(K_n)$ and $L(K_n) \triangleleft M'$ triangularly. Thus

$$x'(L(K_n)) = \frac{n(n-1)}{2} \cdot \frac{n(n-1)(n-2)}{2} + \frac{n(n-1)(n-2)}{3} =$$

$$= \frac{n(n-1)(n-5)}{6}.$$ 

See Figure 4.1 for the case $n = 5$. 

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FIGURE 4.1
Thus, by Theorem 4.1.5, $\chi''(D_n)$ may be calculated since $G(D_n)$ is homeomorphic to $L(K_n)$. Specifically, we have the following corollary.

**Corollary 4.3.2:** If $D_n$ is a $(L(K_n)), (\binom{n}{2}, \binom{n}{2}(n-2), 2(n-2), 2; 0, 1)$-PBIBD for $n-1 \equiv 0, 1 \pmod{3}$, $n \geq 5$, then

$$\chi''(D_n) = -\frac{n(n-1)(n-5)}{6}.$$

There is another type of $L(K_n)$-PBIBD whose genus can be readily calculated. Let $D'_n$ be the $(L(K_n)) (v, b, 2, k; \lambda_1, \lambda_2)$-PBIBD whose objects are the vertices of $L(K_n)$. A block of $D'_n$ consists of the $n-1$ edges incident with vertex $v$ in $K_n$ (before the $L(K_n)$ is formed). Alternately, the blocks of $D'_n$ are the rows of scheme $T_n$ (see [7]). $D'_n$ is therefore on parameters $\binom{n}{2}, n, 2, n-1; 0, 1$. ($b = n$ since there are just $n$ vertices in $K_n'$; $r = 2$ since each edge of $K_n$ is incident with exactly two vertices of $K_n$.)

Consider, for example, the design $D'_5$ on parameters $(10, 5, 2, 4; 0, 1)$, whose blocks are given in Figure 4.2.
FIGURE 4.2

Note that $G(D'_3)$ is homeomorphic from $K_5$. For the general case we have the following theorem.

**Theorem 4.3.3:** Let $D'_n$ be the $(L(K_n))((\begin{array}{c} n \\ 2 \end{array}), n, 2, n-1; 0, 1)$-PBIBD as described previously. Thus $G(D'_n)$ is homeomorphic from $K_n$ and $|\gamma(D')| = \left\{ \frac{(n-3)(n-4)}{12} \right\}$.

**Proof:** Let $D'_n$ be the $(L(K_n))$-PBIBD as specified for $n \geq 4$. Label the object and block vertices of $G(D'_n)$ using scheme $T_n$; i.e., the objects are the numbers $1, 2, \cdots, (\begin{array}{c} n \\ 2 \end{array})$ and the blocks are $R'_i, 1 \leq j \leq n$. ($R'_i$ refers to row $i$ of $T_n$.) Observe the following:

1. if $i$ is an object vertex, then $\deg_{G(D'_n)}i = 2$;
2. if \( R_j = \{i_1, i_2, \ldots, i_{n-1}\} \) is a block vertex, then
\[
\deg_{G(D'_n)} R_j = n - 1;
\]
3. there are exactly \( n \) block vertices, each of which has degree \( n - 1 \);
4. there are exactly \( \binom{n}{2} \) object vertices each of which has degree 2.

If \( R_j \) and \( R_k \) are distinct block vertices, then we claim that \( |R_j \cap R_k| = 1 \). If \( |R_j \cap R_k| > 1 \), then there must exist distinct objects \( h \) and \( i \) such that \( h, i \in R_j \cap R_k \). Now \( \lambda_1 = 0 \) implies that \( h \) and \( i \) are second associates. But this contradicts \( \lambda_2 = 1 \). Therefore \( |R_j \cap R_k| \leq 1 \). There is no loss of generality in assuming \( j < k \). Recall that \( T_n \) is symmetric with respect to its leading diagonal. Therefore the entry in the \((j,k)\) position is equal to the entry in the \((k,j)\) position. But entry \((j,k)\) is in \( R_j \) and entry \((k,j)\) is in \( R_k \). Therefore
\[
|R_j \cap R_k| = 1.
\]

Carry out the following process on \( G(D'_n) \) for each pair of distinct block vertices \( R_i \) and \( R_j \):
1. \( R_j \cap R_k = \{i\} \) (since \( \deg_{G(D'_n)} i = 2 \), \( i \) does not belong to any other blocks of \( D'_n \));
2. remove edges \( iR_k \) and \( iR_j \) and vertex \( i \) from \( G(D'_n) \);
3. add edge \( R_k R_j \).

The new graph \( K \) thus constructed has vertex set \( V(K) = \{R_j | 1 \leq j \leq n\} \) and edge set \( E(K) = \{R_j R_k | 1 \leq j < k \leq n\} \). (There are no...
multiple edges joining $R_j$ and $R_k$ since $|R_j \cap R_k| = 1$.) Therefore $K = K_n$ and, indeed, $G(D'_n)$ is homeomorphic from $K_n$. Thus $\gamma(D'_n) = \frac{(n-3)(n-4)}{12}$.

### 4.4 Latin Square Designs

A third class of commonly encountered PBIBDs is based upon collections of mutually orthogonal Latin squares. An $n \times n$ Latin square LS is an arrangement of $n$ objects (usually the numbers 1, 2, ..., $n$) in an $n \times n$ array so that each object appears exactly once in each row and each column of the array. Two $n \times n$ Latin squares LS(1) and LS(2) are said to be orthogonal if the $n^2$ ordered pairs $(x_{ij}, y_{ij})$, $1 \leq i, j \leq n$, where $x_{ij}$ is the entry in the $(i,j)$ position of LS(1) and $y_{ij}$ is the entry in the $(i,j)$ position of LS(2), are distinct. A collection of $r \geq 2$ $n \times n$ Latin squares is mutually orthogonal if each pair of distinct Latin squares are orthogonal. If $n$ is a prime power greater than two, then finite field theory assures the existence of a collection of $n - 1$ mutually orthogonal $n \times n$ Latin squares. For more information about Latin squares and their construction, see [9] or [15].

Bose [7] and Clatworthy [11] define the two-associate class scheme $L_r(n)$ for $2 \leq r \leq n + 1$ as follows. The scheme $L_r(n)$ has $n^2$ objects (usually the numbers $h$, $1 \leq h \leq n^2$) arranged in an $n \times n$ array $A$. It is assumed that for some $r$, $3 \leq r \leq n + 1$,
there exists a collection of \( r - 2 \) mutually orthogonal \( n \times n \) Latin squares \( \text{LS}(i), 1 \leq i \leq r - 2 \). (If \( r = 2 \), then this collection is empty and there is just the array \( A \).) Two distinct objects are second associates if either
1. they occur in the same row or same column of \( A \) or
2. they correspond to the same symbol \( a \) of \( \text{LS}(i) \) when \( \text{LS}(i) \) is superimposed upon array \( A, 1 \leq i \leq r - 2 \).

They are first associates otherwise.

For example, consider the \( L_4(4) \) scheme given below.

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16 \\
\end{array}
\]

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
2 & 1 & 4 & 3 \\
3 & 4 & 1 & 2 \\
4 & 3 & 2 & 1 \\
\end{array}
\]

Object 7 has second associates 5, 6, 8 and 3, 11, 15 from array \( A \), 4, 10, 13 (corresponding to the symbol 4) from \( \text{LS}(1) \) and 1, 12, 14 (corresponding to symbol 1) from \( \text{LS}(2) \).

Bose [7] (see also [11]) established that for two objects \( x \) and \( y \) which are first associates, the number of objects which are second associates with both \( x \) and \( y \) is \( p_{22}(x,y) = r(r-1) \). If \( x \) and \( y \) are second associates, then \( p_{22}(x,.) = (n-2) + (r-1)(r-2) \). Furthermore, \( n_1 = (n-1)(n-r+1) \) and \( n_2 = r(n-1) \). In
The previous example of $L^4(4)$, $n_1 = 3 \times 1 = 3$, $n_2 = 4 \times 3 = 12$, $p_{22}^1 = 4 \times 3 = 12$ and $p_{22}^2 = 2 + 3 \times 2 = 8$.

The association scheme $L_r(n)$ defines a graph, also denoted by $L_r(n)$, as follows:

$$\mathcal{V}(L_r(n)) = \{h | 1 \leq h \leq n^2\}$$
$$\mathcal{E}(L_r(n)) = \{h_1h_2 | h_1 \text{ and } h_2 \text{ are second associates in } L_r(n)\}.$$ 

The graphs $L_2(n)$ and $L_{n+1}(n)$ are "familiar" graphs as the next two theorems demonstrate.

**Theorem 4.4.1:** If there exists a collection of $n - 1$ mutually orthogonal $n \times n$ Latin squares, then $L_{n+1}(n) = K_{n^2}$.

**Proof:** Assume that $n - 1$ mutually orthogonal $n \times n$ Latin squares exist. Let $h$ be an object of the scheme $L_{n+1}(n)$. Therefore $h$ must have $n_2 = (n+1)(n-1)$ second associates. Thus $\deg_{L_{n+1}(n)}h = (n+1)(n-1) = n^2 - 1$. Since the collection of Latin squares is mutually orthogonal, $h$ must be adjacent to each of the $n^2 - 1$ other vertices of $L_{n+1}(n)$. So $L_{n+1}(n) = K_{n^2}$. 

Cameron, in [10], noted the following theorem.
Theorem 4.4.2: If \( r = 2 \), then \( L_2(n) = L(K_n^2) \) (\( = K_n \times K_n \)).

The following must therefore be true about the graphs \( L_r(n) \) if there exist \( r - 2 \) mutually orthogonal \( n \times n \) Latin squares:

\[
K_n \times K_n \subset L_r(n) \subset K_n^2.
\]

Since \( p_1 \) and \( p_2 \) are constant, \( L_r(n) \) is strongly regular for \( 2 \leq r \leq n \). The graphs \( L_2(n) \) and \( L_3(n) \) always exist. Moreover, if \( n \) is a prime power \( p^t \) greater than 2, then the graphs \( L_r(p^t) \), \( 2 \leq r \leq p^t + 1 \), always exist.

Since \( L_r(n) \) is strongly regular, it determines a \( (L_r(n)) \)

\[
(n^2, \frac{r \cdot n \cdot (n-1)}{2}, r(n-1), 2; 0, 1)\text{-PBIBD} \ D_r(n) \text{ with vertices as objects and edges as blocks (provided that } r - 2 \text{ mutually orthogonal } n \times n \text{ Latin squares exist). The following theorem gives the values of } \chi''(L_r(n)).
\]

Theorem 4.4.3: If there exist \( r - 2 \) mutually orthogonal \( n \times n \) Latin squares \( LS(i) \) for \( n \equiv 0, 1 \pmod{3} \), then \( \chi''(L_r(n)) = \frac{n^2 \cdot (6 - r(n-1))}{6} \).

Proof: Cases \( r = 2 \) and \( r = 3 \) will be examined before proceeding to the general case.

White, in [31], constructed the triangular imbedding of \( L_2(n) = K_n \times K_n \) on a generalized pseudosurface. This construction is
repeated here because it provides a base for the cases $r \geq 3$. Recall that there exists a triangular surface imbedding of $K_n \nabla M$ if $n \equiv 0, 1 \pmod{3}$. Take $n$ disjoint copies of $K_n \nabla M$, labeling them $R_i$ for $1 \leq i \leq n$. Label the vertices of $K_n \nabla M(R_i)$ by $(i,j)$ for $1 \leq j \leq n$. Label the vertex set for $K_n \nabla M(R_i)$ by $V_i$.

Note that this labeling coincides with the coordinates of the $i$th row of array $A$.

Take $n$ additional disjoint copies of $K_n \nabla M$ (triangularly), labeling them $C_j$ for $1 \leq j \leq n$. Label the vertices of $K_n \nabla M(C_j)$ by $(i,j)$ for $1 \leq i \leq n$. Label the vertex set $V_j$.

The labeling coincides with the coordinates of the $j$th column of array $A$.

Observe that each ordered pair $(i,j)$ appears exactly once among the $K_n \nabla M(R_i)$ imbeddings and exactly once among the $K_n \nabla M(C_j)$ imbeddings. If vertices receiving the same label $(i,j)$ are identified, then the result is a triangular generalized pseudosurface imbedding of $K_n \times K_n = L_2(n)$. Thus $\chi''(L_2(n)) = n^2 - n^2(n-1) + \frac{2n^2(n-1)}{3} = \frac{n^2(4-n)}{3}$.

In the case $r = 3$, there is exactly one $n \times n$ Latin square.

To construct a triangular generalized pseudosurface imbedding of $L_3(n)$, take three disjoint sets of $n$ disjoint copies of $K_n \nabla M$ (triangularly) where $M$ is a surface. Now each $K_n \nabla M$ has $n$ vertices, $\frac{n(n-1)}{2}$ edges and $\frac{n(n-1)}{3}$ regions (which are triangular).
Label the \( n \) disjoint \( K_n \triangle M \) of the first set by \( R_i \)'s with vertex sets \( V_i = \{(i,j)|1 \leq j \leq n\} \) for \( i = 1, \ldots, n \) (corresponding with the rows of array \( A \)). Label the \( n \) disjoint copies of \( K_n \triangle M \) of the second set by \( C_j \)'s with vertex sets \( V_j = \{(i,j)|1 \leq i \leq n\} \) for \( j = 1, \ldots, n \) (corresponding with the columns of array \( A \)). Label the vertex sets of \( n \) disjoint copies of \( K_n \triangle M \) of the third set by \( V_j(k) = \{(i,j)|k \text{ is the } (i,j) \text{ entry of } LS(j)\} \) for \( k = 1, \ldots, n \).

If the vertices receiving the same labels are identified, a triangular imbedding of \( L_3(n) \) on a generalized pseudosurface \( M \) is achieved. Thus we have

\[
\chi''(L_3(n)) = n^2 - \frac{3n^2(n-1)}{2} + \frac{3n^2(n-1)}{3} = \frac{n^2(3-n)}{2}
\]

In general, assume that there exist \( r - 2 \) mutually orthogonal \( n \times n \) Latin squares for some \( r \) such that \( 2 \leq r \leq n + 1 \). Take \( r \) disjoint sets of \( n \) disjoint copies of \( K_n \triangle M \) (triangularly) where \( M \) is a surface and \( n \equiv 0, 1 \pmod{3} \). For set 1 (rows of \( A \)), label the vertices of the \( i \)th copy of \( K_n \triangle M \) by \( V_i = \{(i,j)|1 \leq j \leq n\}, \ i = 1, \ldots, n \). For set 2 (columns of \( A \)), label the vertices of the \( j \)th copy of \( K_n \triangle M \) by \( V_j = \{(i,j)|1 \leq i \leq n\}, \ j = 1, \ldots, n \). For set \( k \) (3 \leq k \leq r if \( r \geq 3 \)), label the vertices of the \( k \)th copy of \( K_n \triangle M \) by \( V_j(k) = \{(i,j)|k \text{ is the } (i,j) \text{ entry of } LS(k)\}, \ k = 1, \ldots, n \).

If vertices receiving the same label are identified, then a
triangular imbedding of $L_r(n)$ on a generalized pseudosurface is achieved. Thus we have

$$\chi''(L_r(n)) = n^2 - \frac{rn^2(n-1)}{2} + \frac{rn^2(n-1)}{3} = \frac{n^2[6-r(n-1)]}{6}.$$  

For the labeling used to construct a triangular generalized pseudosurface imbedding of $L_4(4)$, see Figure 4.3.

![Diagram of triangular imbedding](image)  

**FIGURE 4.3**  

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By Theorem 4.1.5, the design $D_r(n)$ arising from $L_r(n)$ must have $G(D_r(n))$, which is homeomorphic from $L_r(n)$. Therefore $\chi''(D_r(n)) = \chi''(L_r(n))$. Specifically we have the following corollary.

**Corollary 4.4.4:** If there exist $r - 2$ mutually orthogonal $n \times n$ Latin square designs and $n \equiv 0,1 \pmod{3}$, then $D_r(n)$, which is a $\chi''(D_r(n)) = \chi''(L_r(n))$. Specifically we have the following corollary.

Another easily constructed $(L_r(n))$-PBIBD is the lattice design. A lattice design $D^{'r}(n)$ is a $(L_r(n))$ $(n^2, nr, r, n; 0, 1)$-PBIBD with blocks of the following type:

1. the rows of array $A$;
2. the columns of array $A$;
3. for each $LS(i), 1 \leq i \leq r - 2$ (if $r \geq 3$), all $n$ objects corresponding to the same symbol in $LS(i)$ when $LS(i)$ is superimposed upon array $A$. (Thus each $LS(i)$ generates $n$ blocks.)

Clearly, there are $n^2$ objects (in array $A$). There are $2n$ blocks coming from array $A$ and $n$ blocks from each $LS(i)$ for a total of $nr$ blocks. Each object belongs to exactly one row in $A$, one column in $A$ and is represented by exactly one symbol in each $LS(i), 1 \leq i \leq r - 2$. Thus each object belongs to exactly $r$ blocks. Each block certainly contains $n$ objects. By definition of the associate classes, pairs of first associates do not appear
together in any of the blocks. Because the LS(i)'s are mutually orthogonal, pairs of second associates appear together exactly once.

For example consider the $D_4'(4)$ design based on the $L_4(4)$ scheme previously given. The blocks of $D_4'(4)$ are listed below.

\begin{align*}
1 & 2 & 3 & 4 & 1 & 5 & 9 & 13 \\
5 & 6 & 7 & 8 & 2 & 6 & 10 & 14 \\
9 & 10 & 11 & 12 & 3 & 7 & 11 & 15 \\
13 & 14 & 15 & 16 & 4 & 8 & 12 & 16 \\
\end{align*}

(Rows of A)

\begin{align*}
1 & 6 & 11 & 16 & 1 & 7 & 12 & 14 \\
2 & 5 & 12 & 15 & 2 & 8 & 11 & 13 \\
3 & 8 & 9 & 14 & 3 & 5 & 10 & 16 \\
4 & 7 & 10 & 13 & 4 & 6 & 9 & 15 \\
\end{align*}

(Columns of A)

The genus of a lattice design $D_2(n)$ is quite easy to establish, as can be seen in the next theorem.

**Theorem 4.4.5:** If $D_2'(n)$ is the lattice design based on the parameters $(n^2, 2n, 2, n; 0, 1)$, then $G(D_2(n))$ is homeomorphic from $K_2(n)$.

**Proof:** Let $D_2'(n)$ be a lattice design on parameters $(n^2, 2n, 2, n; 0, 1)$. Array $A$ is given below.
The blocks of $D_2'(n)$ fall into two categories: rows and columns of $A$. They are listed below.
It will be shown that $G(D'_2(n))$ is homeomorphic from $K_{2n}$ with one partite set consisting of the row blocks and the other partite set consisting of the column blocks.

Observe that each row block $R_i$ has exactly one element in common with each column block $C_j$, namely $(i-1)n+j$. Define a graph $G$ as follows:

$$V(G) = \{\text{row blocks of } D'_2(n)\} \cup \{\text{column blocks of } D'_2(n)\}$$

$$E(G) = \{R_iC_j | 1 \leq i \leq n, 1 \leq j \leq n\}. \text{ Clearly } G \cong K_{2n}.\$$

Construct a new graph from $G$ by inserting a new vertex of degree 2 into each edge $R_iC_j$ of $G$ and labeling it $(i-1)n+j$, which is the element shared by $R_i$ and $C_j$. This new graph is $G(D'_2(n))$ and is clearly homeomorphic from $K_{2n}$. $\triangleright$
Thus, $\gamma(D_2'(n))$ may be determined as follows.

**Corollary 4.4.6:** $\gamma(D_2'(n)) = \left\{ \frac{(n-2)^2}{4} \right\}$. 
CHAPTER 5

PLANAR DESIGNS

As noted in the introductory chapter, block design imbedding problems fall into two categories. Chapter 4 dealt with some of the problems of the first category: on what kinds of surfaces can a given design be imbedded? Chapter 5 deals with a problem in the second category: what kinds of designs can be imbedded on the sphere? In other words, can the planar block designs be characterized?

Again, certain "trivial" designs are briefly considered and then are excluded from the rest of the discussion. Recall that the complete design $D_{v,b}$ on parameters $(v, b, b, v, b)$ has associated bipartite graph $G(D_{v,b}) = K_{v,b}$ and that $\gamma(D_{v,b}) = \frac{(v-2)(b-2)}{4}$.

(See Theorem 4.1.1.) Hence $D_{v,b}$ is planar if and only if one of the parameters is at most 2. So we assume that in a $(v, b, r, k)$ incomplete block design (BIBD or PBIBD) $k < v$. Since $k \geq 2$, the parameter $v$ must be at least 3.

If, in a $(v, b, r, k; \lambda_1, \lambda_2)$-PBIBD $D$, $r = 1$, then each object must belong to exactly one block. Hence the components of $G(D)$ must be isomorphic to the star graph $K_{1,k}$. Thus $G(D)$ is planar. We therefore assume, from this point on, that $r \geq 2$.

If there is just one block in a design $D$, then the design $D$ is planar since $G(D) = K_{1,k}$. Thus we assume that $b \geq 2$. 

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5.1 Planar BIBDs

We present a characterization of the planar balanced incomplete block designs. The characterization is based on the lower bound formula for the genus of a connected block design (Theorem 3.1.2). The following theorem shows that every BIBD is connected.

Theorem 5.1.1: If \( D \) is a \((v, b, r, k, \lambda)\)-BIBD, then \( D \) is connected.

Proof: Let \( D \) be a \((v, b, r, k, \lambda)\)-BIBD. To prove that \( D \) is connected, we show that \( G(D) \) is a connected graph. Let \( x \) and \( y \) be distinct object vertices of \( G(D) \). Since \( \lambda \geq 1 \), \( x \) and \( y \) must appear together in \( \lambda \) blocks. If \( e \) is one of these blocks, then \( xe \) and \( ye \) are edges in \( G(D) \). Thus there is an \( x - y \) walk \( x, e, y \) in \( G(D) \). If \( x \) is an object vertex and \( e \) is a block vertex, then pick some \( y \in e \) such that \( x \neq y \). Again, \( \lambda \geq 1 \) implies that there must be some block \( e' \) containing both \( x \) and \( y \). Thus there is an \( x - e \) walk \( x, e', y, e \) in \( G(D) \). Finally, let \( e \) and \( f \) be distinct block vertices of \( G(D) \). Let \( x \in e \) and \( y \in f \) such that \( x \neq y \). There must be a block \( e' \) such that \( x \), \( y \in e' \). Thus there exists an \( e - f \) walk \( e, x, e', y, f \) in \( G(D) \). Since, for any two distinct \( u, v \in V(G(D)) \), there is a \( u - v \) walk in \( G(D) \), \( G(D) \) is connected.

The characterization also depends on Fisher's Inequality, which is stated in the next theorem.
Theorem 5.1.2: (Fisher's Inequality) If \( D \) is a \((v, b, r, k, \lambda)\)-BIBD, then \( b \geq v \).

For background information and proof see either [15] or [27].

Theorems 5.1.1 and 5.1.2 lead to the following restriction on the values that the \( k \) parameter of a planar BIBD may assume.

Lemma 5.1.3: If \( D \) is a planar \((v, b, r, k, \lambda)\)-BIBD, then \( k \leq 2 \) or \( k = 3 \).

Proof: Let \( D \) be a planar \((v, b, r, k, \lambda)\)-BIBD. By Theorem 5.1.1, \( D \) must be connected. Applying Theorem 3.1.2, we have

\[
\gamma(D) \geq \frac{bk}{2}(1 - \frac{2}{n}) = \frac{v+b}{2} + 1,
\]

where \( n = g(G(D)) \). Note that \( \gamma(D) = 0 \) and \( g(G(D)) \geq 4 \). Thus

\[
0 \geq \frac{bk}{4} - \frac{v+b}{2} + 1 \quad \text{which simplifies to}
\]

\[
0 \geq bk - 2(v+b) + 4.
\]

After some mild algebraic manipulation of this last inequality, we have

\[
2v + 2b - bk \geq 4.
\]

By Theorem 5.1.2, \( b \geq v \). Thus we have

\[
b(4 - k) \geq 4.
\]

Since \( b \geq 2 \), \( 4 - k \) must also be positive. Thus \( 2 \leq k \leq 3 \). □
Lemma 5.1.3 implies that only two cases need to be considered in the determination of the planar BIBDs: $k = 2$ and $k = 3$. The characterization of the planar BIBDs now follows.

**Theorem 5.1.4:** $D$ is a planar $(v, b, r, k, \lambda)$-BIBD if and only if

(i) $k = 2$ and $v = 3$ or $v = 4$, or

(ii) $k = 3$ and $D$ is on parameters $(4, 4, 3, 3, 2)$.

**Proof:** Let $D$ be a $(v, b, r, k, \lambda)$-BIBD. If $D$ is planar, then $k = 2$ or $k = 3$ by Lemma 5.1.3. If $k = 2$, then $G(D)$ is homeomorphic from $K_v^{(\lambda)}$ (by Theorem 4.1.3), which is planar if and only if $v = 3$ or $v = 4$.

We now assume that $k = 3$. If $\lambda = 1$, then $g(G(D)) \geq 6$ and $D$ is on parameters $(v, \frac{v(v-1)}{6}, \frac{v-1}{2}, 3, 1)$. By Theorem 3.1.2, we have the following inequality

$$0 \geq \frac{v(v-1)}{2} \cdot \frac{2}{6} - \frac{6v+v(v-1)}{12} + 1,$$

which may be simplified to

$$0 \geq v^2 - 7v + 12.$$

Define a function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2 - 7x + 12$, which is a parabola that "opens up" and has zeros at $x = 3$ and $x = 4$. Thus $f(x) > 0$ on the interval $(4, \infty)$. But this implies that, for each integer $v \geq 5$, $f(v) = v^2 - 7v + 12 > 0$. Thus $D$ must be nonplanar.
for each \( v \geq 5 \). Since \( k = 3, \ v > 3 \). Thus the only other value of \( v \) that must be considered is \( v = 4 \). If \( v = 4 \), then \( r = \frac{4-1}{2} = \frac{3}{2} \), which is not an integer. Therefore there is no \((4, b, r, 3, 1)\)-BIBD.

If \( \lambda \geq 2 \), then \( g(G(D)) \geq 4 \) and \( D \) is on parameters \((v, \frac{\lambda v(v-1)}{6}, \frac{\lambda(v-1)}{2}, 3, \lambda)\). Again, by Theorem 3.1.2, we have

\[
0 \geq \frac{\lambda v(v-1)}{2 \cdot 2} (1 - \frac{2}{4}) - \frac{6v + \lambda v(v-1)}{2 \cdot 6} + 1,
\]

which may be simplified to

\[
0 \geq \lambda v^2 - (12 + \lambda)v + 24.
\]

The right-hand side of this inequality is used to define a family of polynomial functions \( f^\lambda: \mathbb{R} \to \mathbb{R} \) for each \( \lambda \geq 2 \). Specifically, \( f^\lambda(x) = \lambda x^2 - (12 + \lambda)x + 24 \), which is a parabola that "opens up". If \( v \) is an integer such that \( f^\lambda(v) > 0 \) and a \((v, b, r, 3, \lambda)\)-BIBD \( D \) exists, then \( D \) must be nonplanar. If \( v \) is an integer such that \( f^\lambda(v) \leq 0 \) and a \((v, b, r, 3, \lambda)\)-BIBD \( D \) exists, then \( D \) must be examined separately to determine if \( \gamma(D) = 0 \).

The derivative of \( f^\lambda \) with respect to \( x \), \( f^\lambda_1 \), may be used to determine values of \( x \) for which \( f^\lambda(x) > 0 \). Specifically, we know that

\[
f^\lambda_1(x) = 2\lambda x - (12 + \lambda) > 0
\]

if and only if
Observe that

\[
\frac{12 + \lambda}{2\lambda} = \frac{6}{\lambda} + \frac{1}{2} \leq \frac{6}{2} + \frac{1}{2} = \frac{7}{2}
\]

since \( \lambda \geq 2 \). Thus, for each \( x > \frac{7}{2} \), \( f_\lambda'(x) > 0 \). This implies that \( f_\lambda \) is increasing on the interval \( \left( \frac{7}{2}, \infty \right) \). Now \( f_\lambda(5) = 25\lambda - 5(12 + \lambda) + 24 = 20\lambda - 36 > 0 \) if \( \lambda \geq 2 \). Thus for each \( \lambda \geq 2 \), \( f_\lambda(x) > 0 \) on the interval \( \left( 5, \infty \right) \). This implies that for each integer \( v \geq 5 \) and each \( \lambda \geq 2 \) such that a \((v, b, r, 3, \lambda)\)-BIBD \( D \) exists, \( D \) must be nonplanar. The case \( v = 4 \) and \( \lambda \geq 2 \) must still be analyzed. So \( f_\lambda(4) = 16\lambda - 4(12 + \lambda) + 24 = 12\lambda - 24 > 0 \) if \( \lambda \geq 3 \). Thus if \( D \) is a \((4, b, r, 3, \lambda)\)-BIBD with \( \lambda \geq 3 \), then \( D \) is nonplanar. The only case remaining is \( v = 4 \) and \( \lambda = 2 \); i.e., a \((4, 4, 3, 3, 2)\)-BIBD. Thus if \( D \) is planar with \( k = 3 \), it must be on parameters \((4, 4, 3, 3, 2)\).

Conversely, let \( D \) be a \((4, 4, 3, 3, 2)\)-BIBD. Without loss of generality, the objects of \( D \) may be denoted by the numbers 1, 2, 3, 4. Each block must consist of exactly 3 objects. Each object must belong to exactly three blocks. There is only one design \( D \) on this parameter set:

\[
\begin{array}{ccc}
1 & 2 & 3 \\
1 & 2 & 4 \\
1 & 3 & 4 \\
2 & 3 & 4 \\
\end{array}
\]
A planar imbedding of this design is given in Figure 5.1. Thus \( \gamma(D) = 0 \).

The techniques used in the proof of Theorem 5.1.4 will be used in the characterization of several PBIBDs. It is therefore important to summarize them. If \( D \) is a \((v, b, r, k, \lambda)\)-BIBD or a connected \((G)(v, b, r, k; \lambda_1, \lambda_2)\)-PBIBD which is planar, then 
\[ \gamma(D) = 0 \text{ and, by Theorem 3.1.2, } 0 \geq \frac{bk}{2} \left(1 - \frac{2}{n}\right) - \frac{v + b}{2} + 1. \]

The right-hand side of the inequality is used to define a family of polynomial functions \( f: \mathbb{R} \rightarrow \mathbb{R} \), which are analyzed by means of derivatives to determine on which intervals \( f \) is positive. For integer values in these intervals, the design \( D \) must be nonplanar. For integer values outside these intervals, the corresponding designs must be analyzed separately.
5.2 Disconnected PBIBDs

The techniques used to determine planar BIBDs depend upon the connectedness of the designs. Unfortunately, however, not all partially balanced incomplete block designs are connected. For example, consider the \((nK_m)^{(nm, \frac{nm(m-1)}{2}, m-1, 2; 0, 1)}-PBIBD D\) for some \(n \geq 2\) and \(m \geq 2\). The graph \(nK_m\) is strongly regular since it is \((m-1)\)-regular, it is neither complete nor empty, and \(p_{11}^1 = 0\) and \(p_{22}^2 = m - 2\) are constant. By Theorem 4.1.3, \(G(D)\) is homeomorphic from \(nK_m\). Thus \(\gamma(D) = \gamma(nK_m) = n\left(\frac{(m-3)(m-4)}{12}\right)\).

(See [32].)

In general, we have the following theorem, which establishes that all disconnected PBIBDs are based upon \(nK_m\).

**Theorem 5.2.1:** If \(D\) is a disconnected \((G)(v, b, r, k; \lambda_1, \lambda_2)\)-PBIBD, then \(G \sim nK_m\) for some \(n \geq 2\) and \(m \geq 2\), \(\lambda_1 = 0\), and \(G(D)\) consists of the \(n\) components \(G_i\) such that \(G_i = G(D_i)\) where \(D_i\) is a \((\frac{v}{n}, \frac{b}{n}, r, k, \lambda_2)\) balanced design (either complete or incomplete).

**Proof:** Let \(D\) be a disconnected \((G)(v, b, r, k; \lambda_1, \lambda_2)\)-PBIBD. If \(\lambda_1 \geq 1\), then each pair of first associates must appear together in exactly \(\lambda_1(\neq 0)\) blocks. But it can be shown, as in the proof of Theorem 5.1 that this implies \(G(D)\) is connected. Therefore \(\lambda_1 = 0\).
Let the components of $G(D)$ be denoted by $G_i$ with vertex set consisting of the partite sets $V_i$ (objects of $D$) and $B_i$ (blocks of $D$). We claim that $G \cong nK_m$ for some $n \geq 2$ and $m \geq 2$. Let $x \in V_i$ and $y \in V_j$ such that $i \neq j$. If $x$ and $y$ were second associates, then there must be some block $e$ such that $x, y \in e$. Thus there is an $x-y$ walk $x, e, y$ in $G(D)$, which implies that $x$ and $y$ are in the same component of $G(D)$, a contradiction. Therefore $x$ and $y$ are first associates; i.e., $xy \notin E(G)$. Moreover, there is no $z \in V(G)$ such that $xz, yz \in E(G)$. (If this were the case, then there would be blocks $e$ and $f$ such that $x, z, e \in e$ and $y, z \in f$. This implies the existence of an $x-y$ walk $x, e, z, f, y$ in $G(D)$.) Since $G$ is strongly regular with $p_{22}^1 = 0$, $G \cong nK_m$ by Theorem 2.3 of [10 p. 339].

We next claim that if $x$ and $y$ are first associates, then they must belong to different components of $G(D)$. If this were not the case, then there exist two distinct objects $x$ and $y$ of $D$ which are first associates such that $x, y \in V_i$. There must exist a sequence $W$ of objects and blocks in $G_i$ given by $x = u_1, e_1, u_2, e_2, \ldots, u_t, e_t, u_{t+1} = y$ where $u_i, u_{i+1} \in e_i$ for $1 \leq i \leq t$. Note that if $x$ and $u_i$ are second associates, then $x$ and $u_{i+1}$ are also second associates: if $x$ and $u_{i+1}$ were first associates, then they would have a common second associate $u_i$. (Recall that $\lambda_1 = 0$ implies $u_i$ and $u_{i+1}$ are second associates since $u_i, u_{i+1} \in e_i$.) This contradicts $p_{22}^1 = 0$. Since $x$ and $u_2$ are clearly second associates, then by iterating the above argument we see that $x$ and
y must be second associates, a contradiction. Thus x and y must be in different components.

In summary, x and y are first associates if and only if they belong to different components of G(D). Alternatively, x and y are second associates if and only if they belong to the same component of G(D). Thus there must be n components of G(D), each corresponding to a component of $n \times K_m^r$, $n \geq 2$, m $\geq 2$.

We are now ready to prove that each $G_i$ is the bipartite graph associated with some $(m, \frac{mr}{k}, r, k, \lambda_2)$ balanced design $D_i$. Let the objects of $D_i$ be the m object vertices in $V_i$. Let the blocks of $D_i$ be the blocks of D corresponding to the vertices in $B_i$. Note that, from D, we know that each object of $D_i$ belongs to exactly r blocks (in $B_i$) and each block contains exactly k objects (in $V_i$). Furthermore, each pair of distinct objects (in $V_i$) belong to exactly $\lambda_2$ blocks (in $B_i$) since they are second associates in D. Thus

$D_i$ is an $(m, \frac{mr}{k}, r, k, \lambda_2)$ balanced design. Moreover, D is an $(n \times K_m^r)(nm, \frac{nmr}{k}, r, k; 0, \lambda_2)$-PBIBD.

Clearly, a disconnected PBIBD D is planar if and only if each of the components of G(D) is planar. We may now characterize the disconnected planar PBIBDs.
Theorem 5.2.2: A disconnected design $D$ is a planar $(n \ K_m)$

$(nm, \frac{nmr}{k}, r, k; 0, \lambda_2)$-PBIBD if and only if either

(i) each of the components of $G(D)$ is of the form $G(D_i)$ where $D_i$
is an $(m, r, r, m, r)$ balanced complete design with either $m = 2$
or $r = 2$; or

(ii) each of the components of $G(D)$ is of the form $G(D_i)$ where $D_i$
is an $(m, \frac{mr}{k}, r, k, \lambda_2)$-BIBD with either

(a) $k = 2$ and $m = 3$ or $m = 4$; or

(b) $k = 3$, $m = 4$ and $\lambda_2 = 2$.

Proof: Let $D$ be a disconnected $(n \ K_m) (nm, \frac{nmr}{k}, r, k; 0, \lambda_2)$-PBIBD.
Thus each component of $G(D)$ is of the form $G(D_i)$ where $D_i$ is an
$(m, \frac{mr}{k}, r, k, \lambda_2)$ balanced design (by Theorem 5.2.1).

If each $D_i$ is complete, then $D_i$ is on parameters
$(m, r, r, m, r)$ with $m \geq 2$ and $r \geq 2$. Recall that $G(D_i) = K_{m,r}$,
which is planar if and only if one of $m$ and $r$ is at most 2. Since
$m \geq 2$ and $r \geq 2$, $G(D_i)$ is planar if and only if either $m = 2$ or
$r = 2$.

If each $D_i$ is an $(m, \frac{mr}{k}, r, k, \lambda_2)$-BIBD, then, by Theorem
5.1.4, $D_i$ is planar if and only if either

(a) $k = 2$ and $m = 3$ or $m = 4$, or

(b) $k = 3$, $m = 4$ and $\lambda_2 = 2$.
5.3 Connected PBIBDs

If D is a disconnected (G)-PBIBD, then by Theorem 5.2.1 we know that \( G \cong nK_m \) for \( n \geq 2 \) and \( m \geq 2 \). Thus if D is a (G)-PBIBD and \( G \cong nK_m \) for any \( n \geq 2 \) and \( m \geq 2 \), then D must be a connected design. The techniques of Section 5.1 may therefore be applied to these PBIBDs. A restriction on the parameters of a planar PBIBD is first found. Unfortunately, there is no Fisher's Inequality for the partially balanced designs. Thus the restriction involves two parameters, as is seen in the next theorem.

Theorem 5.3.1: Let D be a planar (connected) \((v, b, r, k; \lambda_1, \lambda_2)\)-PBIBD. (a) If \( b \geq v \), then \( k = 2 \) or \( k = 3 \). (b) If \( b < v \), then \( r = 2 \) or \( r = 3 \).

Proof: Let D be a planar (connected) \((v, b, r, k; \lambda_1, \lambda_2)\)-PBIBD. (a) Assume that \( b \geq v \). Then, by Theorem 3.1.2, we have

\[
0 \geq \frac{bk}{2}(1 - \frac{2}{4}) - \frac{v + b}{2} + 1.
\]

Upon simplifying this inequality, we have

\[
0 \geq bk - 2v - 2b + 4
\]

or

\[
2v + 2b - bk \geq 4.
\]
Since \( b \geq v \), we know that

\[
b(v - k) \geq 4.
\]

Thus either \( k = 2 \) or \( k = 3 \), as seen previously.

(b) Assume that \( b < v \). Thus we may write

\[
0 \geq \frac{vr}{4} - \frac{v + b}{2} + 1
\]

or, upon simplifying,

\[
0 \geq vr - 2v - 2b + 4.
\]

Since \( b < v \), we have

\[
v(4 - r) \geq 4.
\]

So either \( r = 2 \) or \( r = 3 \).

One immediate result of Theorem 5.3.1 is the following corollary.

**Corollary 5.3.2:** If \( D \) is a (connected) \((v, b, r, k; \lambda_1, \lambda_2)\)-PBIBD with \( r \geq 4 \) and \( k \geq 4 \), then \( D \) is nonplanar.

Much of the structure of a \((G)\)-PBIBD apparently depends upon the strongly regular graph \( G \). The characterization of planar PBIBDs is therefore intimately connected with the structure of the graph \( G \). Analysis is now carried out for designs based on the strongly regular graphs \( L(K_n), K_{n(m)} \) and \( L_r(n) \).
5.4 Planar Triangular PBIBDs

Let $D_n^{\lambda_1, \lambda_2}$ be a $(L(K_n))^{(v, b, r, k; \lambda_1, \lambda_2)}$-PBIBD. Recall that $v = \frac{n(n-1)}{2}$, $n_1 = \frac{(n-2)(n-3)}{2}$ and $n_2 = 2n - 4$. Thus we may find the following expressions for $r$ and $b$ by using Theorem 2.3.2:

$$r = \frac{(n-2)(n-3)\lambda_1 + 4(n-2)\lambda_2}{2(k-1)}$$

and

$$b = \frac{n(n-1)(n-2)[(n-3)\lambda_1 + r \lambda_2]}{4k(k-1)}.$$

(It is assumed that $n \geq 4$ since $L(K_3) = K_3$.)

If $D$ is a $(L(K_n))^{(\lambda_1, \lambda_2)}$-PBIBD with parameter $k = 2$, then by Theorem 4.1.3, $G(D)$ is homeomorphic from $L(K_n)^{(\lambda_1, \lambda_2)}$. In fact we have the following theorem.

**Theorem 5.4.1:** The design $D_n^{\lambda_1, \lambda_2}$ is a planar $(L(K_n))^{(\lambda_1, \lambda_2)}$ if and only if $n = 4$.

**Proof:** Let $D_n^{\lambda_1, \lambda_2}$ be a $(L(K_n))^{(\frac{n(n-1)}{2}, b, r, 2; \lambda_1, \lambda_2)}$-PBIBD.

By Theorem 4.1.3, $G(D_n^{\lambda_1, \lambda_2})$ is homeomorphic from $L(K_n)^{(\lambda_1, \lambda_2)}$. Thus $D_n^{\lambda_1, \lambda_2}$ is planar if and only if $L(K_n)$ is planar. Observe that $K_{n-1} \subset L(K_n)$. Thus for each $n \geq 6$, $L(K_n)$ must be nonplanar.
by Kuratowski's Theorem (see [21] or [6]). If \( n = 5 \), then by Theorem 4.3.2, \( \chi''(L(K_5)) = 0 \). Recall that \( \chi''(L(K_5)) \geq \chi(L(K_5)) \).

Therefore

\[
0 \geq 2 - 2\gamma(L(K_5)),
\]

which, in turn, implies

\[
\gamma(L(K_5)) \geq 1
\]

Thus \( L(K_5) \) is nonplanar. The graph \( L(K_4) \) is given in Figure 5.2.

Clearly, \( \gamma(L(K_4)) = 0 \). Thus \( D_{4,1,4,2} \) is planar.

\[FIGURE 5.2\]
We now consider those $(L(K_n))$-PBIBDs with parameters $k = 3$.

The analysis will be broken up into several cases: $\lambda_1 = 0$ and $\lambda_2 = 1$ (which implies $g(G(D)) \geq 6$), $\lambda_1 = 0$ and $\lambda_2 \geq 2$ (which implies $g(G(D)) = 4$) and $\lambda_2 > \lambda_1 \geq 1$.

**Theorem 5.4.2:** If $D_{n}^{0,1}$ is a $(L(K_n))(n(n-1), \frac{n(n-1)(n-2)}{2}, n-2, 3; 0, 1)$-PBIBD for $n > 5$, then $D_{n}^{0,1}$ is nonplanar.

**Proof:** Let $D_{n}^{0,1}$ be a planar $(L(K_n))(n(n-1), \frac{n(n-1)(n-2)}{2}, n-2, 3; 0, 1)$-PBIBD. Now since $\lambda_1 = 0$ and $\lambda_2 = 1$, $g(G(D_{n}^{0,1})) \geq 6$. Thus by Theorem 3.1.2 we have

$$0 \geq \frac{n(n-1)(n-2)}{2} \left(1 - \frac{2}{6}\right) - \frac{3n(n-1)+n(n-1)(n-2)}{2} + 1,$$

which simplifies to

$$0 \geq n(n-1)(n-2) - 3n(n-1) + 12$$

or $0 \geq n^3 - 6n^2 + 5n + 12$. Define a function $f: \mathbb{R} \to \mathbb{R}$ by

$$f(x) = x^3 - 6x^2 + 5x + 12.$$  We are interested in determining the intervals on which $f$ is positive. Taking the derivative we have

$$f'(x) = 3x^2 - 12x + 5,$$

which has zeros at

$$\frac{12 \pm \sqrt{84}}{6}.$$
The largest zero is

\[ \frac{12 + \sqrt{84}}{6} < \frac{12 + 10}{6} = \frac{22}{6} \]

Thus \( f'(x) > 0 \) on the interval \( \left[ \frac{22}{6}, \infty \right) \). This implies that \( f(x) \) is increasing on the interval \( \left[ \frac{22}{6}, \infty \right) \). Consider

\[ f(5) = 125 - 150 + 25 + 12 > 0. \]

Thus \( f(x) > 0 \) on \( [5, \infty) \). Hence \( D_{n}^{0,1} \) is nonplanar for each \( n \geq 5 \).

If \( n = 4 \), then there is a planar \( D_{4}^{0,1} \) on parameters \( (6, 4, 2, 3; 0, 1) \). Two such designs are given by the imbedding of \( L(K_{4}) \) on the sphere in Figure 5.2. One design has the blocks: 126, 234, 456, 153, which are the shaded regions of the imbedding. The other design is given by the unshaded regions of the imbedding: 123, 435, 156, 246. Both of these designs have planar associated bipartite graphs. It is interesting to note that \( L(K_{4}) \cong K_{3}(2) \) and that the second design is the solution to Problem 1.1.2.

We now let \( \lambda_{2} \geq 2 \) and \( \lambda_{1} = 0 \). The girth of the bipartite graph associated with the design \( D_{n}^{0,\lambda_{2}} \) is 4. We have the following theorem.
Theorem 5.4.3: Let $D_n^{0,\lambda_2}$ be a $(L(K_n))$

\[
\frac{n(n-1)}{2}, \frac{\lambda_2 n(n-1)(n-2)}{6}, \lambda_2(n-1), 3; 0, \lambda_2\text{-PBIBD.}
\]

The design $D_n^{0,\lambda_2}$ is nonplanar if $n \geq 5$ and $\lambda_2 = 2$ or if $\lambda_2 \geq 3$.

Proof: Let $D_n^{0,\lambda_2}$ be a planar $(L(K_n))$

\[
\frac{n(n-1)}{2}, \frac{\lambda_2 n(n-1)(n-2)}{6}, \lambda_2(n-2), 3; 0, \lambda_2\text{-PBIBD.}
\]

By Theorem 3.1.2, we know that

\[
0 \geq \frac{\lambda_2 n(n-1)(n-2)}{2 \cdot 2} \left(1 - \frac{2}{6}\right) - \frac{3n(n-1) + \lambda_2 n(n-1)(n-2)}{2 \cdot 6} + 1,
\]

which may be reduced to

\[
0 \geq \lambda_2 n(n-1)(n-2) - 6n(n-1) + 24
\]

or

\[
0 \geq \lambda_2 n^3 - 3(\lambda_2 + 2)n^2 + 2(\lambda_2 + 3)n + 24.
\]

Define the family of functions $f_{\lambda_2} : \mathbb{R} \rightarrow \mathbb{R}$ by

\[
f_{\lambda_2}(x) = \lambda_2 x^3 - 3(\lambda_2 + 2)x^2 + 2(\lambda_2 + 3)x + 24
\]

for each $\lambda_2$ an integer at least 2. We now determine the intervals for which each $f_{\lambda_2}$ is positive. Consider the derivative

\[
f'_{\lambda_2}(x) = 3\lambda_2 x^2 - 6(\lambda_2 + 2)x + 2(\lambda_2 + 3).
\]

If we can determine the intervals for which $f'_{\lambda_2}$ is positive, then we know the intervals on which $f_{\lambda_2}$ is increasing. Consider the second derivative
\[ f''_\lambda(x) = 6\lambda x - 6(\lambda^2 + 2), \]

which is positive if and only if

\[ x > \frac{\lambda^2 + 2}{\lambda}. \]

Now observe the following

\[ \frac{\lambda^2 + 2}{\lambda} = \frac{\lambda^2}{\lambda^2} + \frac{2}{\lambda^2} \leq 1 + 1 = 2 \]

if \( \lambda \geq 2 \). Thus \( f''_\lambda(x) > 0 \) on the interval \((2, \infty)\) for each \( \lambda \geq 2 \). Therefore \( f'_\lambda \) is increasing on \((2, \infty)\) for each \( \lambda \geq 2 \).

Consider \( f'_\lambda(4) = 3\lambda(16) - 6(\lambda^2 + 2)(4) + 2(\lambda^2 + 3) = 26\lambda - 42 > 0 \) for each \( \lambda \geq 2 \). Thus \( f'_\lambda(x) > 0 \) on the interval \([4, \infty)\) for each \( \lambda \geq 2 \). So \( f'_\lambda \) is increasing on \([4, \infty)\) for each \( \lambda \geq 2 \). Consider \( f'_\lambda(5) = 125\lambda - 75(\lambda^2 + 2) + 10(\lambda^2 + 3) + 24 = 60\lambda - 96 > 0 \) for each \( \lambda \geq 2 \). Thus \( f'_\lambda(x) > 0 \) on the interval \([5, \infty)\) for each \( \lambda \geq 2 \). Therefore, for each integer \( n \geq 5 \) and each \( \lambda \geq 2, D_{n,\lambda} \) is nonplanar.

There is still one more case to be considered: \( n = 4 \) and \( \lambda \geq 3 \). The design \( D_{4,\lambda} \) has the polynomial function \( f_{\lambda,\lambda} = 24\lambda^2 - 48 \), which is positive if \( \lambda \geq 3 \). Hence \( D_{4,\lambda} \) is nonplanar for each \( \lambda \geq 3 \).

The above proof encompasses all cases of \( \lambda_1 = 0 \) and \( \lambda_2 \geq 2 \) except the case \( n = 4 \) and \( \lambda_2 = 2 \). There does exist a planar design \( D_{4,0} \) on parameters \((6, 8, 4, 3; 0, 2)\) whose blocks are given by

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the triangular imbedding of \( L(K_4) \) on the sphere which is shown in Figure 5.2. A planar imbedding of \( G(D_4^{0,2}) \) may be achieved by inserting a block vertex \( e \) into the interior of each region of \( L(K_4) \backslash S_0 \), adding the edges \( xe \) for each vertex \( x \) incident with the region labeled \( e \), and then removing the edges of \( L(K_4) \). See Figure 5.3.
We now consider those designs $D_n$ with $k = 3$, $\lambda_2 > \lambda_1 \geq 1$. Note that $g(G(D_n^{\lambda_1, \lambda_2})) = 4$.

Theorem 5.4.4: Let $D_n$ be a $(L(K_n))$-PBIBD for $\lambda_2 > \lambda_1 \geq 1$. Then $D_n$ is nonplanar.

Proof: Let $D_n$ be a planar $(L(K_n))$-PBIBD with $k = 3$ and $\lambda_2 > \lambda_1 \geq 1$. By Theorem 3.1.2, we know

$$0 \geq \frac{n(n-1)(n-2)[(n-3)\lambda_1 + 4\lambda_2]}{2 \cdot 8} \left(1 - \frac{2}{4}\right) - \frac{12n(n-1) + n(n-1)(n-2)[(n-3)\lambda_1 + 4\lambda_2]}{2 \cdot 24} + 1,$$

which simplifies to

$$0 \geq \frac{n(n-1)(n-2)[(n-3)\lambda_1 + 4\lambda_2]}{32} - \frac{12n(n-1) + n(n-1)(n-2)[(n-3)\lambda_1 + 4\lambda_2]}{48} + 1$$

or $0 \geq n(n-1)(n-2)[(n-3)\lambda_1 + 4\lambda_2] - 24n(n-1) + 96$. Carrying out the indicated multiplications and collecting like terms, we have
Define the doubly-indexed family of polynomial functions

\[ f_{\lambda_1, \lambda_2} : \mathbb{R} \to \mathbb{R} \text{ by } f_{\lambda_1, \lambda_2}(x) = \lambda_1 x^4 - 2(3\lambda_1 - 2\lambda_2)x^3 + (11\lambda_1 - 12\lambda_2 - 24)x^2 - 2(3\lambda_1 - 4\lambda_2 - 12)x + 96. \]

Again, by use of derivatives, we determine the intervals on which \( f_{\lambda_1, \lambda_2} \) is positive.

\[
f'_{\lambda_1, \lambda_2}(x) = 4\lambda_1 x^3 - 6(3\lambda_1 - 2\lambda_2)x^2 + 2(11\lambda_1 - 12\lambda_2 - 24)x - 2(3\lambda_1 - 4\lambda_2 - 12)
\]

\[
f''_{\lambda_1, \lambda_2}(x) = 12\lambda_1 x^2 - 12(3\lambda_1 - 2\lambda_2)x + 2(11\lambda_1 - 12\lambda_2 - 24)
\]

\[
f'''_{\lambda_1, \lambda_2}(x) = 24\lambda_1 x - 12(3\lambda_1 - 2\lambda_2), \text{ which is positive if and only if } x > \frac{12(3\lambda_1 - 2\lambda_2)}{24\lambda_1}.
\]

Consider \( \frac{12(3\lambda_1 - 2\lambda_2)}{24\lambda_1} = \frac{3\lambda_1 - 2\lambda_2}{2\lambda_1} < \frac{3}{2} \) for each pair \((\lambda_1, \lambda_2)\) such that \( \lambda_2 > \lambda_1 \geq 1 \). Thus \( f''_{\lambda_1, \lambda_2}(x) > 0 \) on the interval \((\frac{3}{2}, \infty)\).

This implies that \( f''_{\lambda_1, \lambda_2} \) is increasing on the interval \((\frac{3}{2}, \infty)\).

Consider \( f'''_{\lambda_1, \lambda_2}(3) = 108\lambda_1 - 36(3\lambda_1 - 2\lambda_2) + 2(11\lambda_1 - 12\lambda_2 - 24) = 22\lambda_1 + 48\lambda_2 - 48 > 0 \) since \( \lambda_2 > \lambda_1 \geq 1 \). So \( f'''_{\lambda_1, \lambda_2}(x) > 0 \) on the
interval \([3, \infty)\). This implies that \(f'_{\lambda_1, \lambda_2}\) is increasing on the interval \([3, \infty)\). Consider \(f'_{\lambda_1, \lambda_2}(4) = 256\lambda_1 - 96(3\lambda_1 - 2\lambda_2) + 8(11\lambda_1 - 12\lambda_2 - 24) - 2(3\lambda_1 - 4\lambda_2 - 12) = 50\lambda_1 + 104\lambda_2 - 168 > 0\) for each \(\lambda_2 > \lambda_1 \geq 1\). Thus \(f'_{\lambda_1, \lambda_2}(x) > 0\) on the interval \([4, \infty)\). This implies that \(f'_{\lambda_1, \lambda_2}\) is increasing on \([4, \infty)\). Consider \(f_{\lambda_1, \lambda_2}(4) = 256\lambda_1 - 128(3\lambda_1 - 2\lambda_2) + 16(11\lambda_1 - 12\lambda_2 - 24) - 8(3\lambda_1 - 4\lambda_2 - 12) + 96 = 24\lambda_1 + 96\lambda_2 - 192 > 0\) for each \(\lambda_2 > \lambda_1 \geq 1\). Thus \(f_{\lambda_1, \lambda_2}(x) > 0\) on \([4, \infty)\) for each \(\lambda_2 > \lambda_1 \geq 1\). Hence \(D_n\) is nonplanar for each \(\lambda_2 > \lambda_1 \geq 1\) and each \(n \geq 4\).

Theorem 5.3.1 indicates that the cases \(r = 2\) and \(r = 3\) should also be analyzed. If \(r = 2\), each object appears in exactly two blocks. This puts a restriction on the values of the \(\lambda_i\) parameters. Specifically, either \(\lambda_1 = 0\) and \(\lambda_2 = 1\) or \(2\) or \(\lambda_1 = 1\) and \(\lambda_2 = 2\). Similarly, if \(r = 3\), then either \(\lambda_1 = 0\) and \(\lambda_2 = 1, 2\) or \(3\), \(\lambda_1 = 1\) and \(\lambda_2 = 2\) or \(3\), or \(\lambda_1 = 2\) and \(\lambda_2 = 3\). Thus the analysis is broken down into two major cases: \(r = 2\) and \(r = 3\).

The case \(r = 2\) is considered first. If \(\lambda_1 = 0\), then pairs of first associates cannot appear in the same block. This puts a restriction on the "block size" \(k\) of a \(D_n\) design. We know that \(p_2 = n - 2\). Therefore there can be at most \(n\) objects in a block; i.e., \(k \leq n\). From Theorem 2.3.2, we may derive the following expression for \(k\) (recall that \(n_2 = 2(n-2)\):

\[k = \frac{n(n-1)}{2} - p_2 = \frac{n(n-1)}{2} - (n-2) = \frac{(n-1)(n-2)}{2}\]
Thus we have

\[(n-2)\lambda_2 + 1 \leq n\]

or

\[(\lambda_2 - 1)n - 2\lambda_2 + 1 \leq 0.\]

If \(\lambda_2 = 2\), then \(n \leq \frac{2\lambda_2 - 1}{\lambda_2 - 1} = 3\). But we are assuming that \(n \geq 4\).

Thus there are no \(D_n^{0,2}\) designs with \(r = 2\). There are, however, \(D_n^{0,1}\) designs with \(r = 2\) (see Theorem 4.3.3).

Theorem 5.4.5: If \(D_n^{0,1}\) is a \((L(K_n))(n(n-1)/2, n, 2, n-1; 0, 1)\)-PBIBD, then \(D_n^{0,1}\) is nonplanar for each \(n \geq 5\).

Proof: If we apply Theorem 3.1.2, we have

\[0 \geq \frac{n(n-1)}{2} \left(1 - \frac{2}{6}\right) - \frac{n(n-1) + 2n}{2 \cdot 2} + 1\]

or \(0 \geq n^2 - 7n + 12\).

The expression on the right-hand side of the inequality is positive for each \(n \geq 5\). Thus \(D_n^{0,1}\) is nonplanar for each \(n \geq 5\). 

There do exist planar \((L(K_n))(6, 4, 2, 3; 0, 1)\)-PBIBDs. (See the examples immediately following Theorem 5.4.2.)

We now consider the case \(\lambda_1 = 1\) and \(\lambda_2 = 2\) \((r = 2)\). The design \(D_n^{1,2}\) is on parameters \((\frac{n(n-1)}{2}, b, 2, k; 1, 2)\) with
\[ k = \frac{(n-2)[(n-3)(1) + 4(2)] + 4}{4} \] (by Theorem 2.3.2) or, in simplified form,
\[ k = \frac{n^2 + 3n - 6}{4} . \]

The parameter \( b = \frac{vr}{k} \) is given next:
\[ b = \frac{n(n-1)4}{n^2 + 3n - 6} \]
or
\[ b = \frac{4n^2 - 4n}{n^2 + 3n - 6} \]

Write \( b \) as follows:
\[ b = 4 - \frac{8(2n-3)}{n^2 + 3n - 6} \]

Since \( n \geq 4 \), \( \frac{8(2n-3)}{n^2 + 3n - 6} \) must be a positive integer \( t \) such that
\[ b = 4 - t \geq 2. \] Thus either \( \frac{8(2n-3)}{n^2 + 3n - 6} = 1 \) or \( \frac{8(2n-3)}{n^2 + 3n - 6} = 2 \).

If \( \frac{8(2n-3)}{n^2 + 3n - 6} = 1 \), then \( 8(2n-3) = n^2 + 3n - 6 \)
or \( 0 = n^2 - 13n + 18 \),
which has no integer solutions.

If \( \frac{8(2n-3)}{n^2 + 3n - 6} = 2 \), then
\[ 16n - 24 = 2n^2 + 6n - 12 \]
or \( 0 = n^2 - 5n + 6 \),
which has integer solutions 2 and 3. But we are assuming that 
\( n \geq 4 \). Hence there are no values of \( n \) such that 
\[
\frac{4n(n-1)}{n^2 + 3n - 6}
\]
is an integer at least 2. Therefore there are no \( D_n^{1,2} \) designs.

The second major case is \( r = 3 \). Here the expression for the 
parameter \( k \) is given as follows (Theorem 2.3.2):

\[
k = \frac{(n-2)(n-3)\lambda_1 + 4\lambda_2}{6} + 6.
\]

If \( \lambda_1 = 0 \), first associates cannot appear together in a block, which 
puts restrictions on the parameter \( k \); i.e., \( k \leq n \), as before. 
Specifically, we have

\[
\frac{4(n-2)\lambda_2 + 6}{6} \leq n,
\]
or \((2\lambda_2 - 3)n - 4\lambda_2 + 3 \leq 0\). If \( \lambda_2 \geq 2 \), then \( n \leq \frac{4\lambda_2 - 3}{2\lambda_2 - 3} \);

i.e., if \( 2\lambda = 2 \), then \( n \leq 5 \);

if \( \lambda_2 = 3 \), then \( n \leq 3 \).

The second case is ruled out since \( n \geq 4 \). In the first case, if 
\( n = 4 \), then \( D_4^{0,2} \) has parameter \( k = \frac{22}{6} \), which can't happen. If 
\( n = 5 \), then \( D_5^{0,2} \) is on parameters \((10, 6, 3, 5; 0, 2)\). Now by

Theorem 3.1.2 \( \gamma(D_5^{0,2}) \geq \frac{20}{4} - \frac{16}{2} + 1 = \frac{1}{2} > 0 \); i.e., if \( D_5^{0,2} \) exists,
it is nonplanar. If \( \lambda_2 = 1 \), then \( k = \frac{4(n-2) + 6}{6} = \frac{2(n-2) + 3}{3} \).

The expression for \( b \) is found with the assistance of Theorem 2.3.2:
Rewriting $b$, we have

$$b = \frac{3n(n-1)}{2[2(n-2)+3]} = \frac{9n(n-1)}{2(2n-1)}.$$ 

Since $b$ is an integer, $4b$ must also be an integer; i.e.,

$$\frac{36n^2 - 36n}{4n - 2}$$

is an integer. If the indicated division is performed, the result is

$$4b = 9n - \frac{18n}{4n - 2} = 9n - \frac{9n}{2n - 1}.$$ 

Now $\frac{9n}{2n-1}$ must be an integer. Therefore $\frac{18n}{2n-1}$ is also an integer.

Carrying out the division, we have

$$\frac{18n}{2n - 1} = 9 + \frac{9}{2n - 1}.$$ 

So $\frac{9}{2n - 1}$ is an integer. This implies that $n = 5$ (recall $n \geq 4$).

Hence there is only one case, $n = 5$, to be considered. By Theorem 5.4.2, $D_5^{0,1}$ is nonplanar.

Let $\lambda_1 = 1$ and $\lambda_2 = 2$ or 3 ($r = 3$). The parameter $k$ is

$$k = \frac{(n-2)(n-3+4\lambda_2) + 6}{6}$$

and the parameter $b$ is

$$b = \frac{9n(n-1)}{(n-2)(n-3+4\lambda_2) + 6}.$$
If $\lambda_2 = 2$, then $k = \frac{(n-2)(n+5) + 6}{6}$ and $b = \frac{9n(n-1)}{(n-2)(n+5) + 6} = \frac{9n(n-1)}{n^2 + 3n - 4}$; i.e., $b = \frac{9n(n-1)}{(n+4)(n-1)} = \frac{9n}{n + 4}$. If the indicated division is carried out, the result is

$$b = 9 - \frac{36}{n + 4}.$$

Since $r = 3$, $b \geq 3$. Therefore $\frac{36}{n + 4}$ must be an integer at least 1 and at most 6. We now test the following cases:

- $\frac{36}{n + 4} = 1$ or $n = 32$, $b = 9 - 1 = 8$, $k = 186$;
- $\frac{36}{n + 4} = 2$ or $n = 14$, $b = 9 - 2 = 7$, $k = 39$;
- $\frac{36}{n + 4} = 3$ or $n = 8$, $b = 9 - 3 = 6$, $k = 14$;
- $\frac{36}{n + 4} = 4$ or $n = 5$, $b = 9 - 4 = 5$, $k = 6$;
- $\frac{36}{n + 4} = 5$ or $n = \frac{16}{5}$; not possible;
- $\frac{36}{n + 4} = 6$ or $n = 2$: discounted since $n \geq 4$.

We now assume that designs exist for these values of $n$ and apply Theorem 3.1.2:

$$\gamma(D_5^{1,2}) \geq \frac{5(6)}{4} - \frac{10 + 5}{2} + 1 = 1 > 0;$$
$$\gamma(D_8^{1,2}) \geq \frac{6(14)}{4} - \frac{28 + 6}{2} + 1 = 5 > 0;$$
$$\gamma(D_{14}^{1,2}) \geq \frac{7(39)}{4} - \frac{91 + 7}{2} + 1 = \frac{81}{4} > 0;$$
$$\gamma(D_{32}^{1,2}) \geq \frac{8(186)}{4} - \frac{496 + 8}{2} + 1 = 121 > 0.$$
Each of these designs (if it exists) is nonplanar.

Now assume that \( \lambda_1 = 1 \) and \( \lambda_2 = 3 \). In this case

\[
k = \frac{(n-2)(n+9) + 6}{6}
\]

and

\[
b = \frac{9n(n-1)}{n^2 + 7n - 12}.
\]

If we perform the indicated division, we have

\[
b = 9 - \frac{36(2n-3)}{n^2 + 7n - 12}.
\]

So, again \( b \geq 3 \), which forces \( \frac{72n - 108}{n^2 + 7n - 12} \) to be an integer at least 1 and at most 6. We now test these cases:

\[
\frac{72n - 108}{n^2 + 7n - 12} = 1 \quad \text{or} \quad n = \frac{65 + \sqrt{3841}}{2}, \quad \text{not an integer;}
\]

\[
\frac{72n - 108}{n^2 + 7n - 12} = 2 \quad \text{or} \quad n = \frac{29 + \sqrt{673}}{2}, \quad \text{not an integer;}
\]

\[
\frac{72n - 108}{n^2 + 7n - 12} = 3 \quad \text{or} \quad n = \frac{17 + \sqrt{193}}{2}, \quad \text{not an integer;}
\]

\[
\frac{72n - 108}{n^2 + 7n - 12} = 4 \quad \text{or} \quad n = \frac{11 + \sqrt{61}}{2}, \quad \text{not an integer;}
\]

\[
\frac{72n - 108}{n^2 + 7n - 12} = 5 \quad \text{or} \quad n = \frac{37 + \sqrt{409}}{10}, \quad \text{not an integer;}
\]

\[
\frac{72n - 108}{n^2 + 7n - 12} = 6 \quad \text{or} \quad n = 2 \quad \text{or} \quad 3, \quad \text{discounted.}
\]

Thus there are no \( D_{n}^{1,2} \) designs with \( r = 3 \).
The last case to be considered is \( \lambda_1 = 2 \) and \( \lambda_2 = 3 \). Here we have

\[
k = \frac{(n-2)(2n+6) + 6}{6} = \frac{(n-2)(n+3) + 3}{3}
\]

and

\[
b = \frac{9n(n-1)}{2(n^2 + n - 3)} = \frac{9n^2 - 9n}{2n^2 + 2n - 6}.
\]

Therefore \( 2b = \frac{18n^2 - 18n}{2n^2 + 2n - 6} \) is an integer. If the division is performed, we have

\[
2b = 9 - \frac{36n - 54}{2n^2 + 2n - 6} = 9 - \frac{18n - 27}{n^2 + n - 3}.
\]

Since \( b \geq 3 \), \( 2b \geq 6 \). Therefore \( \frac{18n - 27}{n^2 + n - 3} \) must be an integer equal to 1, 2 or 3. Again we test to see if there are any integers \( n \geq 4 \) for which this is true:

\[
\frac{18n - 27}{n^2 + n - 3} = 1 \quad \text{or} \quad n = \frac{17 + \sqrt{193}}{2}, \text{ not an integer};
\]

\[
\frac{18n - 27}{n^2 + n - 3} = 2 \quad \text{or} \quad n = \frac{16 + \sqrt{88}}{4}, \text{ not an integer};
\]

\[
\frac{18n - 27}{n^2 + n - 3} = 3 \quad \text{or} \quad n = 2 \text{ or } 3, \text{ discounted}.
\]

Thus there are no \( D_n^{2,3} \) designs with \( r = 3 \).

In summary, if \( D_n^{\lambda_1,\lambda_2} \) is a planar \( (L(K_n),\lambda_2) \)-PBIBD, it must be on one of the following sets of parameters:
\((6, 4, 2, 3; 0, 1)\)
\((6, 8, 4, 3; 0, 2)\)
\((6, 6(3 \lambda_1 + 4\lambda_2), 2(3\lambda_1 + 4\lambda_2), 2; \lambda_1, \lambda_2)\)

for \(\lambda_2 > \lambda_1 \geq 0\).

5.5 Planar Group Divisible PBIBDs

The group divisible designs (based on the graph \(K_{n(m)}\) for \(n \geq 2\)
and \(m \geq 2\) may also be analyzed with the techniques used for the
triangular designs in Section 5.4. Recall that if \(D_{n(m)}^{\lambda_1, \lambda_2}\) is a
\((K_{n(m)}^{(nm, b, r, k; \lambda_1, \lambda_2)})\)-PBIBD, then \(n_1 = m - 1\) and
\(n_2 = m(n-1)\). From Theorem 2.3.2 we may find the following expres­
sions for \(r\) and \(b\):

\[
    r = \frac{\lambda_1(m-1) + \lambda_2 m(n-1)}{k - 1}
\]

and

\[
    b = \frac{mn[\lambda_1(m-1) + \lambda_2 m(n-1)]}{k(k-1)}.
\]

The planar designs for \(k = 2\) are easily determined by the next
theorem.

**Theorem 5.5.1:** The design \(D_{n(m)}^{\lambda_1, \lambda_2}\) is a planar \((K_{n(m)}^{(nm, b, r, 2; \lambda_1, \lambda_2)})\)-PBIBD if and only if \(K_{n(m)}^{(\lambda_1, \lambda_2)}\) is planar.

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$(i.e., \lambda_1 = 0, m = 2$ and $n = 2$ or $3$, or $\lambda_1 \geq 1, m = 2$ and $n = 2$).

The polynomial functions derived from the inequality

$$0 \geq \frac{bk}{2} (1 - \frac{2}{n}) - \frac{v + b}{2} + 1$$

can be used to prove the next theorem.

**Theorem 5.5.2:** The design $D_{n(m)}^{\lambda_1, \lambda_2}$ is a planar $(K_{n(m)})$

$(nm, b, r, 3; \lambda_1, \lambda_2)$-PBIBD if and only if $n = 3, m = 2, \lambda_1 = 0$ and $\lambda_2 = 1$ or $2$.

The proof is similar to the proofs in Section 5.4, but it is quite tedious. Therefore the proof of Theorem 5.5.2 is omitted. Theorem 5.5.3 is treated in like fashion.

**Theorem 5.5.3:** (a) Let $D_{n(m)}^{\lambda_1, \lambda_2}$ be a $(K_{n(m)})$

$(nm, b, 2, k; \lambda_1, \lambda_2)$-PBIBD. The design $D_{n(m)}^{\lambda_1, \lambda_2}$ is planar if and only if it is either a $(K_{2(2)}) (4, 4, 2, 2; 0, 1)$-PBIBD or a

$(K_{3(2)}) (6, 4, 2, 3; 0, 1)$-PBIBD. (b) If $D_{n(m)}^{\lambda_1, \lambda_2}$ is a $(K_{n(m)})$

$(nm, b, 3, k; \lambda_1, \lambda_2)$-PBIBD, then $D_{n(m)}^{\lambda_1, \lambda_2}$ is nonplanar.
5.6 Planar Latin Square PBIBDs

We now consider the Latin square PBIBDs. Recall that if

\[ D_{r(n)} \] is a \((L_r(n))(n, b, r', k; \lambda_1, \lambda_2)\)-PBIBD, then \( n_1 = (n-1)(n-r+1) \) and \( n_2 = r(n-1) \). Thus we know that (from Theorem 2.3.2)

\[
\frac{(n-1)(n-r+1)\lambda_1 + (n-1)r\lambda_2}{k-1}
\]

and

\[
\frac{n^2(n-1)[(n-r+1)\lambda_1 + r\lambda_2]}{k(k-1)}.
\]

Theorem 5.3.1 allows us to restrict our analysis to the cases

\( r' = 2 \) or \( 3 \) and \( k = 2 \) or \( 3 \).

If \( k = 2 \), then \( G(D_{r(n)}) \) is homeomorphic from the multigraph \( L_{r(n)}^{(\lambda_1, \lambda_2)} \). This fact leads us to the following theorem.

**Theorem 5.6.1:** The design \( D_{r(n)}^{(\lambda_1, \lambda_2)} \) is a planar \((L_r(n)) \)

\((n^2, b, r', 2; \lambda_1, \lambda_2)\)-PBIBD if and only if \( n = 2 \).

**Proof:** Let \( D_{r(n)}^{(\lambda_1, \lambda_2)} \) be a \((L_r(n))(n^2, b, r', 2; \lambda_1, \lambda_2)\)-PBIBD.

Therefore \( G(D_{r(n)}^{(\lambda_1, \lambda_2)}) \) is homeomorphic from \( L_{r(n)}^{(\lambda_1, \lambda_2)} \). Recall that
$K_n \times K_n \subseteq L_r(n) \subseteq L_r(n)_{(\lambda_1, \lambda_2)}$. (See the comment following Theorem 4.4.2.) Thus $L_r(n)_{(\lambda_1, \lambda_2)}$ is planar if and only if $n = 2$. So $G(D(n))_{(\lambda_1, \lambda_2)}$ is planar if and only if $n = 2$. □

The proof of the next theorem is straightforward, using the polynomial functions described in the previous sections of this chapter. However, the proof is long and somewhat tedious. Therefore it is omitted.

Theorem 5.6.2: If the design $D(n)_{(\lambda_1, \lambda_2)}$ is a $(L_r(n))_{(n^2, b, r', 3; \lambda_1, \lambda_2)}$-PBIBD, then it is nonplanar.

The proofs of the next two theorems are also omitted for similar reasons.

Theorem 5.6.3: A design $D(n)_{(\lambda_1, \lambda_2)}$ is a planar $(L_r(n))_{(n^2, b, 2, k; \lambda_1, \lambda_2)}$-PBIBD if and only if it is on the parameter set $(4, 4, 2, 2; 0, 1)$.

Theorem 5.6.4: The $D(n)_{(\lambda_1, \lambda_2)}$ $(L_r(n))$-PBIBDs with $r' = 3$ are nonplanar.
CHAPTER 6

CONNECTIONS

6.1 Triple Systems Arising From Graph Imbeddings

Until the work of Jungerman, Stahl and White on hypergraph realizations in 1980, the bulk of the research relating graph imbeddings with block designs concerned designs derived from graph imbeddings. Alpert, in [1], examined the balanced two-fold triple systems arising from triangular imbeddings of complete graphs on generalized pseudosurfaces. Specifically, he proved

Theorem 6.1.1: The balanced two-fold triple systems are in one-to-one correspondence with triangular imbeddings of complete graphs on generalized pseudosurfaces.

White explored the partially balanced two-fold triple systems arising from triangular imbeddings of strongly regular graphs (see [31]). He established

Theorem 6.1.2: The partially balanced two-fold triple systems are in one-to-one correspondence with triangular imbeddings of strongly regular graphs on generalized pseudosurfaces.
Thus, each triangular imbedding of $L(K_n)$, $n - 1 \equiv 0,1 \pmod{3}$, as described in the proof of Theorem 4.3.1, yields not only a $(L(K_n))^{n(n-1)/2, n(n-1)(n-2)/2, 2(n-2), 2; 0, 1}$-PBIBD but also a $(L(K_n))^{n(n-1)/2, n(n-1)(n-2)/3, 2(n-2), 3; 0, 2}$-PBIBD. Similarly, each triangular imbedding of the graph $L_r(n)$ described in Theorem 4.4.4 also yields a $(L_r(n))^{n^2, n^2r(n-1)/3, r(n-1), 3; 0, 2}$-PBIBD.

In addition, White also introduced the next two theorems concerning pairs of one-fold (or Steiner) triple systems and graph imbeddings (in [31]).

**Theorem 6.1.3:** Each triangular imbedding of the complete graph $K_v$ with bichromatic dual on a generalized pseudosurface yields a pair of Steiner triple systems on $v$ objects. Conversely, each pair of Steiner triple systems on $v$ objects is representable as a triangular imbedding of $K_v$ with bichromatic dual on a generalized pseudosurface.

**Theorem 6.1.4:** Each triangular imbedding of a strongly regular graph of order $v$ with bichromatic dual on a generalized pseudosurface yields a pair of $(v, b, r, 3; 0, 1)$-PBIBDs. Conversely, each pair of $(v, b, r, 3; 0, 1)$-PBIBDs defined using the same strongly regular graph $G$ of order $v$ is representable as a triangular imbedding of $G$ with bichromatic dual in a generalized pseudosurface.
Thus the triangular imbedding of $K_7$ on the torus in Figure 1.1 yields not only the indicated $(7, 14, 6, 3, 2)$-BIBD but also two $(7, 7, 3, 3, 1)$-BIBDs. If the regions of $K_7 \triangleleft S_1$ are colored black and white so that regions sharing a common edge receive different colors, then the black regions yield one of the $(7, 7, 3, 3, 1)$-BIBDs and the white regions yield the other $(7, 7, 3, 3, 1)$-BIBD.

Moreover, White established a criterion for determining when two designs $D_1$ and $D_2$ on the same parameters set ($k = 3$ and $\lambda_2 = 2$ if the $D_i$'s are both BIBDs or $k = 3$, $\lambda_1 = 0$, $\lambda_2 = 2$ if the $D_i$'s are both PBIBDs) are "truly different" based upon the generalized pseudosurfaces they generate (as in the work of Alpert and White). Two designs $D_1$ and $D_2$ on the same parameter set are isomorphic if there exists a bijection $\phi$ from the objects of $D_1$ onto the objects of $D_2$ such that $e$ is a block of $D_1$ if and only if $\phi(e)$ is a block of $D_2$. The theorem is stated as follows.

**Theorem 6.1.5:** Let $D_1$ and $D_2$ be two designs (both BIBDs with $k = 3$ and $\lambda = 2$ or both PBIBDs with $k = 3$, $\lambda_1 = 0$ and $\lambda_2 = 2$) on the same parameter set; if the generalized pseudosurfaces they determine are not homeomorphic, then the designs are not isomorphic.

Hence, there appears to be an intimate connection between a block design which is a two-fold triple system and the generalized pseudosurface that the design generates. It is therefore reasonable to
conjecture that the generalized pseudosurface which is generated by some design \( D \) is the "most efficient surface" upon which \( H(D) \) can be imbedded.

### 6.2 Geometric Realizations of Designs Arising from Graph Imbeddings

The conjecture about the "most efficient surface" on which a design \( D \) can be imbedded if \( D \) arises from a graph imbedding is true in many cases. Specifically, we have the following theorem.

**Theorem 6.2.1:** If \( D \) is a block design arising from a \( k \)-gonal \((k \geq 3)\) imbedding of a graph \( G \) on a generalized pseudosurface \( M \) (ie, the objects of \( D \) are the vertices of \( G \) and the blocks of \( D \) are all of the regions of \( G \ \cap M \) ), then \( \chi''(D) = \chi(M) \).

**Proof:** Let \( D \) be a \((v, b, r, k, \lambda)\)-BIBD or a \((v, b, r, k, \lambda_1, \lambda_2)\)-PBIBD yielded by a \( k \)-gonal imbedding of a graph \( G \) on a generalized pseudosurface \( M \) such that the objects of \( D \) are the vertices of \( G \) and the blocks of \( D \) are all of the regions of \( G \ \cap M \). An imbedding of the bipartite graph \( G(D) \) on \( M \) is constructed from \( G \ \cap M \) as follows. Into the interior of each region of \( G \ \cap M \), insert a new "regional" vertex \( e \). For each vertex \( u \) of \( G \) incident with the region labeled by the vertex \( e \), insert the edge \( ue \). Perform this for each region of \( G \ \cap M \). Observe that the resulting graph \( G' \) is triangularly imbedded on \( M \). Now remove the edges of \( G' \) belonging to \( G \). We claim that this new graph \( G'' \) is
G(D) and that G(D) is quadrilaterally imbedded on M. To see that $G'' = G(D)$, observe that $V(G'') = V(G) \cup \{e | e \text{ is a "regional" vertex}\}$ and $E(G'') = \{ue | u \text{ is incident with region } e\}$. Each of the regional vertices is actually a block vertex of G(D) by the definition of the design D. Thus $E(G'') = \{ue | u \in e\}$. To see that $G'' \triangleleft M$ quadrilaterally, note that each edge $u_1u_2$ of G belongs to exactly two triangular regions of $G' \triangleleft M$. Upon the removal of the edge $u_1u_2$ from G', the two triangular regions become one quadrilateral region. (See Figure 6.1.) Since G(D) is bipartite and contains cycles of length four, $g(G(D)) = 4$. Thus this imbedding of G(D) on M is most efficient, which implies $\chi''(G(D)) = \chi(M)$; i.e., $\chi''(D) = \chi(M)$.

![Diagram](image)

**FIGURE 6.1**

Observe that when the Jungerman, Stahl and White modification is performed on $G(D) \triangleleft M$, the "edges" of H(D) are k-gons and the "regions" of H(D) are digons. If the edges of these
digons (in the 1-skeleton of $\mathcal{H}(D)$) are identified, then the result is the original imbedding of $G \triangleleft M$. Hence there is a direct connection between the geometric realization of this type of block design and the work carried out by Alpert and White.

There is a similar connection between geometric realizations of Steiner triple systems and triangular imbeddings of graphs with bichromatic duals. In fact, we have the following theorem.

**Theorem 6.2.2:** If $G$ is a graph triangularly imbedded on some generalized pseudosurface $M$ with bichromatic dual which gives rise to a two-fold triple system $D$ (on parameters $(v, b, r, 3, 2)$ or $(v, b, r, 3; 0, 2)$) with at least four blocks, then $D$ splits into two Steiner triple systems $D_1$ and $D_2$ (on parameters $(v, \frac{b}{2}, \frac{r}{2}, 3, 1)$ or $(v, \frac{b}{2}, \frac{r}{2}, 3; 0, 1)$) and $\chi''(D) = \chi''(D_i) = \chi(M)$, $i = 1, 2$.

**Proof:** Let $G$ be a graph triangularly imbedded on generalized pseudosurface $M$ with bichromatic dual which gives rise to a two-fold triple system $D$ with at least four blocks. By Theorems 6.1.3 and 6.1.4, we know that $D$ splits into two Steiner triple systems as follows. If the regions of $G \triangleleft M$ are colored with the colors 1 and 2 so that adjacent regions receive different colors, then the regions colored $i$ give rise to Steiner triple system $D_i$, $i = 1, 2$. Construct the graph $G'$ from $G \triangleleft M$ as in the proof of Theorem 6.2.1: insert the "regional" vertex $e$ into the interior.
of each region \( e \) of \( G \triangle M \) and add the edge \( u_e \) for each vertex \( u \) of \( G \) incident with region \( e \) of \( G \triangle M \). Thus \( G' \triangle M \) triangularly. We now color the edges of \( G' \triangle M \) as follows:

- If \( u_1u_2 \in E(G) \), then color the edge \( u_1u_2 \) with color 3;
- If \( u \) is a vertex of \( G \) and \( e \) is a vertex inserted into the interior of a region of \( G \triangle M \) colored \( i \), then color the edge \( u_e \) with color \( i \) \((i = 1, 2)\).

We claim that each edge of \( G' \triangle M \) receives exactly one color: if \( f \) is an edge of \( G' \triangle M \), then either \( f \in E(G) \) or \( f = u_e \) where \( u \in V(G) \) and \( e \) is a vertex inserted into an \( i \)-colored region of \( G \triangle M \).

Fix \( i \) (either \( i = 1 \) or \( i = 2 \)). Observe that for each edge \( uw \) of \( G \triangle M \) there is exactly one vertex \( e \) in an \( i \)-colored region whose boundary contains \( uw \) and therefore exactly two \( i \)-colored edges \( u_e \) and \( w_e \) in \( G' \). Thus each region of \( G \triangle M \) not colored \( i \) is "encircled" by a cycle of length 6 whose edges are colored \( i \) (since there are at least four regions in the triangular imbedding of \( G \) on \( M \)). See Figure 6.2 for an example.

We define new graphs \( G''_i \), \( i = 1, 2 \), as follows:

\[
\begin{align*}
V(G''_i) &= V(G) \cup \{e|e \text{ is a vertex inserted into the interior of an } i \text{-colored region of } G \triangle M\}; \\
E(G''_i) &= \{f \in E(G')|f \text{ received color } i\}.
\end{align*}
\]

Thus \( G''_i \) is hexagonally imbedded on \( M \). Moreover, \( G''_i = G(D_i) \).
Recall that $D_4$ is either a $(v, \frac{b}{2}, \frac{k}{2}, 3; 0, 1)$-PBIBD or a $(v, \frac{b}{2}, \frac{k}{2}, 3, 1)$-BIBD. The parameters $\lambda_1 = 0$, $\lambda_2 = 1$ if $D_4$ is a PBIBD or $\lambda = 1$ if $D_1$ is a BIBD imply that $g(G(D_4)) \geq 6$.

Since we have a hexagonal imbedding of $G(D_4)$ on $M$, this must be a most efficient imbedding. So $\chi''(D_4) = \chi(M)$ for $i = 1, 2$. By Theorem 6.2.1, $\chi''(D) = \chi(M)$.

Again, it is interesting to note that when the Jungerman, Stahl and White modification is performed on $G(D_4) \triangle M$, the 1-skeleton of the hypergraph $H(D_4)$ is precisely $G \triangle M$ with bichromatic dual as originally given.

6.3 Applications of the Connections

Theorems 6.2.1 and 6.2.2 can be used to find the genus and/or characteristic of many of the designs examined by Alpert [1]
and White [31]. Note that if a design $D$ arises from an imbedding of a graph $G$ on a surface $M$, then the genus (or nonorientable genus) of $D$ may be calculated from the formula $\chi(M) = 2 - 2\gamma(D)$ (or from $\chi(M) = 2 - \gamma(D)$). Specifically, we have the following theorems.

**Theorem 6.3.1:** If $D$ is a $(v, b, r, 3, 2)$-BIBD arising from $K_v \triangleleft M$ triangularly, then $\chi''(D) = -\frac{v(v-7)}{6}$. If $D$ splits into two Steiner triple systems $D_1$ and $D_2$, then $\chi''(D_i) = -\frac{v(v-7)}{6}$, $i = 1, 2$. If $M$ is an orientable surface, then $\gamma(D) = \gamma(D_i) = \frac{(v-3)(v-4)}{12}$, $i = 1, 2$. If $M$ is a nonorientable surface, then

$$\gamma(D) = \gamma(D_i) = \frac{(v-3)(v-4)}{6}, \quad i = 1, 2.$$

**Proof:** Let $D$ be a $(v, b, r, 3, 2)$-BIBD arising from $K_v \triangleleft M$ (triangularly). Thus $r = v - 1$ and $b = \frac{v(v-1)}{3}$ (by Theorem 2.3.1).

By Theorem 6.2.1, $\chi''(D) = \chi(M) = v - \frac{v(v-1)}{2} + \frac{v(v-1)}{3} = -\frac{v(v-7)}{6}$.

If $D$ splits into two Steiner triple systems $D_1$, then $\chi''(D_1) = -\frac{v(v-7)}{6}$ by Theorem 6.2.2. If $M$ is an orientable surface, then $\chi(M) = 2 - 2\gamma(D)$ or, upon solving for $\gamma(D)$,

$$\gamma(D) = \gamma(D_1) = 1 + \frac{v(v-7)}{12} = \frac{(v-3)(v-4)}{12}.$$

If $M$ is a nonorientable surface, then $\chi(M) = 2 - \gamma(D)$ implies

$$\gamma(D) = \gamma(D_1) = 2 + \frac{v(v-7)}{12} = \frac{(v-3)(v-4)}{6}.$$
Theorem 6.3.2: If $D$ is a $(G)(v, b, r, 3; 0, 2)$-PBIBD (arising from $G \triangleleft M$ triangularly), then \( \chi''(D) = -\frac{v(r-6)}{6} \). If $D$ splits into two Steiner triple systems $D_1$, then \( \chi''(D_1) = -\frac{v(r-6)}{6} \), $i = 1, 2$. If $M$ is an orientable surface, then \( \gamma(D) = \gamma(D_1) = 1 + \frac{v(r-6)}{12}, i = 1, 2 $, if $M$ is an nonorientable surface, then \( \gamma(D) = \gamma(D_1) = 2 + \frac{v(r-6)}{6}, i = 1, 2 $.

Proof: Let $D$ be a $(v, b, r, 3; 0, 2)$-PBIBD arising from $G \triangleleft M$ triangularly. Now $b = \frac{v \cdot r}{3}$. By Theorem 6.2.1, \( \chi''(D) = \chi(M) = v - \frac{v \cdot r}{2} + \frac{v \cdot r}{3} = -\frac{v(r-6)}{6} \). The rest of the proof is analogous to the proof of Theorem 6.3.1.

Corollary 6.3.3: If $D_n$ is a $(L(K_n))$
\( \left( \frac{n(n-1)}{2}, \frac{n(n-1)(n-2)}{3}, 2(n-2), 3; 0, 2 \right)$-PBIBD, then \( \chi''(D_n) = -\frac{n(n-1)(n-5)}{6} \).

Corollary 6.3.4: If $D_{n(m)}$ is a $(K_{n(m)})$
\( \left( nm, \frac{nm^2(n-1)}{3}, m(n-1), 3; 0, 2 \right)$-PBIBD, then \( \chi''(D_{n(m)}) = -\frac{nm[m(n-1) - 6]}{6} \).

Corollary 6.3.5: If $D_r(n)$ is a $(L_r(n))$
\( \left( n^2, \frac{n^2 r(n-1)}{3}, r(n-1), 3; 0, 2 \right)$-PBIBD, then \( \chi''(D_r(n)) = -\frac{n^2 [r(n-1) - 6]}{6} \).
Up to this point, all of the examples of designs arising from graph imbeddings have had the parameter $k = 3$. Theorem 6.2.1, however, can be applied to designs arising from $k$-gonal imbeddings of graphs in which $k > 3$. Consider the imbedding of the Petersen graph, denoted by $P$, on the projective plane $N_1$ given in Figure 6.3. This imbedding yields a $(10, 6, 3, 5; 1, 2)$-PBIBD $D$ if the vertices of $P$ are considered as objects and the regions of $P \triangleleft N_1$ are considered as blocks (see [31]). Thus, by Theorem 6.2.1, $\gamma(D) = 1$.

There is an infinite family of designs $D_n$ with $n$ a prime power $p^r \equiv 1 \pmod{8}$ called Paley designs. White examined these
designs in [35]. Specifically, a Paley graph $G_n$, $n = p^r$ ($n \equiv 1 \pmod{8}$) is a Cayley graph (see [32]) $G_{\Delta_n}(\Gamma_n)$ where $\Gamma_n = (\mathbb{Z}_p^r)$ (the additive group of the Galois field $\text{GF}(p^r)$) and $\Delta_n = \{1, x^2, x^4, \ldots, x^{n-3}\}$ (set of all squares in $\text{GF}(p^r)$ where $x$ is a primitive element for $\text{GF}(p^r)$); i.e., $[u,v] \in E(G_n)$ if and only if $v - u$ is a square in $\text{GF}(p^r)$. White showed that $G_n$ can be $4m$-gonally imbedded on an orientable surface of genus $8m^2 - 7m$ so that the imbedding (or map) is "strongly symmetrical." (Again see [35] for more information.) Moreover, $G_n$ is strongly regular with $p_1 = 2m$ and $p_2 = 2m - 1$. Thus $G_n \triangleleft S_{8m^2 - 7m}$ yields a $(G_n)(8m+1, 8m+1, 4m, 4m; \lambda_1, \lambda_2)$-PBIBD $D_n$ where $\lambda_1 + \lambda_2 = 4m - 1$ and $\lambda_1 = 2m$ if $x^2 + 1$ is a square in $\text{GF}(p^r)$ and $\lambda_1 = 2m - 1$ if $x^2 + 1$ is a non-square. (See Theorem 4.4 of [35].) By Theorem 6.2.1 we now know that $\gamma(D_n) = 8m^2 - 7m$, where $n = p^r \equiv 8m+1$.

6.4 Designs Arising from Bi-embeddings of Graphs

The generalized characteristic of many of the designs found by Anderson and White in their study of bi-embedding numbers can also be determined with the help of Theorems 6.2.1 and 6.2.2. The first eight examples of such designs appear in [5].

Example 6.4.1: The complete graph $K_{6r+1}$ can be "edge-partitioned" into $r$ toroidal subgraphs using the following method. Ringel
demonstrated that it is possible to find \( r \) disjoint triples \( \{a_i, b_i, c_i\}, \ i = 1, \ldots, r \), in the group \( \mathbb{Z}_{6r+1} \) such that:

(i) \( a_i + b_i = c_i \);

(ii) the greatest common divisor of \( a_i, b_i, c_i \) and \( 6r+1 \) is 1 (i.e., \( a_i, b_i, c_i \) and \( 6r+1 \) are relatively prime); (iii) the \( 3r \) numbers, with their negatives, are all of the nonzero elements of \( \mathbb{Z}_{6r+1} \). These triples are used as currents on the current graph given in Figure 6.4. The hollow vertices indicate counter-clockwise rotation. The solid vertices indicate clockwise rotation.

The assignment of currents as indicated in Figure 6.4, along with the above described properties of the triples, ensures that the graph \( G_i \) (for each \( i = 1, \ldots, r \)) "generated" is triangularly imbedded on the torus with bichromatic dual. Moreover, each \( G_i \) has order \( 6r+1 \) and is \( 6 \)-regular. Furthermore, the "edge union" of all of these \( G_i \)'s is \( K_{6r+1} \). If vertices receiving the same label are identified, then a triangular imbedding of \( K_{6r+1} \) is achieved, which yields a \((6r+1, 2r(6r+1), 6r, 3, 2)\)-BIBD \( D_r \) that splits into
two Steiner triple systems $D^1_r$ and $D^2_r$ on parameters $(6r+1, r(6r+1), 3r, 3, 1)$. By Theorems 6.2.1 and 6.2.2, we now know

$$\chi''(D_r) = \chi''(D_r) = (6r+1) - 3r(6r+1) + 2r(6r+1)$$

$$= (1-r)(6r+1).$$

![Diagram](image)

**FIGURE 6.5**

**Example 6.4.2:** Consider the two current graphs given in Figure 6.5 with currents in $Z_{13}$. Current graph A yields a triangulation $G_A$ of the torus with 13 vertices of degree 6. The vertex rotation scheme at vertex 0 is $\pi_0: (1, 4, 3, 12, 9, 10)$ and the rotation at vertex $i$ is $\pi_i: (1+i, 4+i, 3+i, 12+i, 9+i, 10+i)$.
Current graph $B$ has "vortices" at vertices $x$, $y$ and $z$ (see [34] for additional information). The graph $G_B$ generated by current graph $B$ is triangularly imbedded on $S_6$ with 13 vertices of degree 9 and 3 vertices of degree 13. The rotation at vertex 0 is $\pi_0: (2 \times 11 6 y 7 5 z 8)$ and at vertex $i$,

$$\pi_i: (2+i \times 11+i 6+i y 7+i 5+i z 8+i)$$

for $i = 1, \ldots, 12$. The vertices $x$, $y$ and $z$ appear in $G_B$ and have rotations

$$\pi_x: (2 4 6 8 10 12 1 3 5 7 9 11 0)$$
$$\pi_y: (6 12 5 11 4 10 3 9 2 8 1 7 0)$$
$$\pi_z: (5 10 2 7 12 4 9 1 6 11 3 8 0).$$

The "edge union" of $G_A$ and $G_B$ is $K_{16} - K_3$ (with triangle $xyz$ missing).

If vertices receiving the same labels in $G_A \triangleleft S_6$ and $G_B \triangleleft S_6$ are identified, then a triangular imbedding of $K_{16} - K_3$ is achieved. Now imbed $K_3: xyz$ on another sphere and identify the vertices $x$, $y$ and $z$ with their "partners" in the imbedding of $K_{16} - K_3$. We now have a triangular imbedding of $K_{16}$ on a generalized pseudosurface, which yields a $(16, 80, 15, 3, 2)$-BIBD $D$ with block $xyz$ appearing twice. We compute

$$\chi'(D) = 16 - 120 + 80 = -24.$$
Example 6.4.3: The currents assigned to the current graphs A and B in Figure 6.6 are in the group $\mathbb{Z}_{25}$. Current graph A yields a graph $G_A$, which is triangularly imbedded on $S_1$ with bichromatic dual. The graph $G_A$ has 25 vertices, each of degree 6. The rotation scheme at vertex 0 is $\pi_0: (2 \ 9 \ 7 \ 23 \ 16 \ 18)$. Current graph B yields a graph $G_B$, which is triangularly imbedded on $S_{26}$ with bichromatic dual. The graph $G_B$ has 25 vertices, each of degree 18. The rotation scheme at vertex 0 is $\pi_0: (1 \ 14 \ 6 \ 21 \ 20 \ 17 \ 11 \ 12 \ 15 \ 19 \ 8 \ 3 \ 13 \ 24 \ 4 \ 10 \ 22 \ 5)$.

The "edge union" of $G_A$ and $G_B$ is $K_{25}$. So if vertices receiving the same labels in $G_A \triangleleft S_1$ and $G_B \triangleleft S_{26}$ are identified, a
triangular imbedding of $K_{25}$ is achieved, which yields a $(25, 200, 24, 3, 2)$-BIBD and two $(25, 100, 12, 3, 1)$-BIBDs $D_1$ and $D_2$. Thus

$$x''(D) = x''(D_1) = 25 - 300 + 200 = -75, \quad i = 1, 2.$$ 

Example 6.4.4: The currents in Figure 6.7 are in the group $Z_{25}$. Current graph A yields a graph $G_A$ triangularly imbedded on $S_{12}$ with 25 vertices of degree 9 and 3 vertices of degree 25. The rotation at vertex 0 is $\pi_0$: $(2 \times 23 \ 16 \ y \ 9 \ 7 \ z \ 18)$. The rotation at vertex $i$ (for each $i = 1, \ldots, 24$) is

$$\pi_i: (2+i \times 23+i \ 16+i \ y \ 9+i \ 7+i \ z \ 18+i).$$

The rotations at vertices $x, y$ and $z$ are
Current graph $B$ generates a graph $G_B$ triangularly imbedded on $S_{26}$ with 25 vertices of degree 18. (See Example 6.4.3.) The "edge union" of the two graphs $G_A$ and $G_B$ is $K_{28} - K_3$ (with triangle $xyz$ missing). If vertices receiving the same labels in $G_A \triangleleft S_{12}$ and $G_B \triangleleft S_{26}$ are identified, then a triangular imbedding of $K_{28} - K_3$ on $M$ is achieved. Imbed $K_3: xyz$ on a sphere and identify the vertices $x, y$ and $z$ in $K_3 \triangleleft S_0$ with their counterparts in $K_{28} - K_3 \triangleleft N$. The resulting graph is a triangular imbedding of $K_{28}$, which yields a (28, 252, 27, 3, 2)-BIBD with the block $xyz$ appearing twice. Thus

$$\chi''(D) = 28 - 378 + 252 = -98.$$ 

**Example 6.4.5:** The currents in Figure 6.8 are in the group $Z_{37}$. Current graph $A$ generates a graph $G_A$ triangularly imbedded on $S_{38}$ with bichromatic dual, which has 37 vertices of degree 18. The rotation at vertex 0 is

$$\pi_0: (1, 17, 19, 14, 13, 2, 20, 21, 32, 18, 35, 11, 16, 36, 23, 5, 26, 24).$$
The current graph $B$ yields a graph $G_B$ triangularly imbedded on $S_{38}$ with bichromatic dual, which has 37 vertices of degree 18. The rotation at vertex 0 is

\[ \pi_0: (12 \ 9 \ 15 \ 8 \ 33 \ 6 \ 28 \ 3 \ 30 \ 22 \ 31 \ 27 \ 34 \ 25 \ 29 \ 7 \ 10 \ 4) \]

The "edge union" of $G_A$ and $G_B$ is $K_{37}$. If the vertices receiving the same labels in $G_A \triangleleft S_{38}$ and $G_B \triangleleft S_{38}$ are identified, then a triangular imbedding of $K_{37}$ on $M$ is achieved. A $(37, 444, 36, 3, 2)$-BIBD result, which splits into two $(37, 222, 18, 3, 1)$-BIBDs $D_1$ and $D_2$. Thus $\chi''(D) = \chi''(D_1) = 37 - 666 + 444 = -185$. 

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Example 6.4.6: The currents in Figure 6.9 are from the group $\mathbb{Z}_{37}$.

Current graph A yields a graph $G_A$ triangularly imbedded on $S_{55}$
with 37 vertices of degree 21 and 3 vertices of degree 37. The vertex rotations are

$$
\pi_0: (23 \ x \ 14 \ 1 \ 17 \ 19 \ y \ 18 \ 35 \ 11 \ 16 \ 36 \ 13 \\
2 \ 20 \ 21 \ 32 \ z \ 5 \ 26 \ 24)
$$

$$
\pi_x: (23 \ 2(23) \ 3(23) \ \cdots \ m(23) \ \cdots \ 14 \ 0)
$$

$$
\pi_y: (19 \ 2(19) \ 3(19) \ \cdots \ m(19) \ \cdots \ 18 \ 0)
$$

$$
\pi_z: (32 \ 2(32) \ 3(32) \ \cdots \ m(32) \ \cdots \ 5 \ 0)
$$

Current graph B yields a graph $G_B$ triangularly imbedded on $S_{38}$
with 37 vertices of degree 18 (see Example 6.4.5). If vertices
receiving the same labels in $G_A \triangleleft S_{55}$ and $G_B \triangleleft S_{38}$ are

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identified, then a triangle imbedding of $K_{40} - K_3$ on $M$ is achieved (with triangle xyz missing). Now imbed $K_3$: xyz on a sphere and identify the vertices x, y and z with their counterparts in $K_{40} - K_3 \triangle M$. The result is a triangular imbedding of $K_{40}$, which yields a $(40, 520, 39, 3, 2)$-BIBD $D$ with block xyz appearing twice. So

$$\chi''(D) = 40 - 780 + 520 = -220.$$ 

![Diagram](image)

**FIGURE 6.10**

**Example 6.4.7:** The currents in Figure 6.10 are chosen from the group $Z_{17}$. Current graph $A$ generates a graph $G_A$ triangularly imbedded on $S_1$ with 17 vertices of degree 6. The rotation at vertex 0 is $\pi_0: (1 8 7 16 9 10)$. Current graph $B$ generates a graph $G_B$ triangularly imbedded on $S_{16}$ with 17 vertices of degree 14 and 4 vertices of degree 17. The rotations are.
If the vertices receiving the same labels in $G_A \triangleleft S_1$ and $G_B \triangleleft S_{16}$ are identified, then a triangular imbedding of $K_{21} - K_4$ on $M$ is achieved (with $K_4$: $xyzw$ missing). Imbed $K_4$: $xyzw$ (triangularly) on $S_0$ and identify the vertices $x, y, z$ and $w$ with their counterparts in $K_{21} - K_4 \triangleleft M$. This yields a triangular imbedding of $K_{21}$, which in turn gives a $(21, 140, 20, 3, 2)$-BIBD $D$ with

$$x''(D) = 21 - 210 + 140 = -49.$$
Example 6.4.8: The currents in Figure 6.11 are chosen from the group $Z_{29}$. Current graph A yields a triangulation $G_A$ of $S_{30}$ with 29 vertices of degree 18. The rotation at vertex 0 is

$$\pi_0: (1 \ 4 \ 22 \ 3 \ 28 \ 23 \ 13 \ 18 \ 25 \ 26 \ 19 \ 6 \ 5 \ 16 \ 10 \ 7 \ 11 \ 24).$$

The current graph B yields a triangulation $G_B$ of $S_{28}$ with 29 vertices of degree 14 and 4 vertices of degree 29. The rotations are

$$\pi_0: (2 \ 27 \ 15 \ y \ 14 \ 12 \ 21 \ z \ 8 \ 20 \ w \ 9 \ 17)$$

$$\pi_x: (2 \ 2(2) \ \ldots \ m(2) \ \ldots \ 27 \ 0)$$

$$\pi_y: (15 \ 2(15) \ \ldots \ m(15) \ \ldots \ 14 \ 0)$$

$$\pi_z: (21 \ 2(21) \ \ldots \ m(21) \ \ldots \ 8 \ 0)$$

$$\pi_w: (20 \ 2(20) \ \ldots \ m(20) \ \ldots \ 9 \ 0).$$

If vertices receiving the same labels in $G_A \triangleleft S_{30}$ and $G_B \triangleleft S_{28}$ are identified, then a triangular imbedding of $K_{33} - K_4$ on $M$ is achieved (with $K_4$: xyzw missing). Imbed $K_4$: xyzw on the sphere and identify the vertices $x$, $y$, $z$ and $w$ with their counterparts in $K_{33} - K_4 \triangleleft M$. The result is a triangular imbedding of $K_{33}$, which yields a $(33, 352, 32, 3, 2)$-BIBD $D$ with

$$\chi''(D) = 33 - 528 + 352 = -143.$$
Example 6.4.9: (Due to Anderson [2]) The currents in Figure 6.12 are taken from the group $Z_{37}$. Current graph A yields a triangulation $G_A$ of $S_{75}$ with 37 vertices of degree 30. Current graph B yields a triangulation $G_B$ of $S_1$ with 37 vertices of degree 6. If vertices receiving the same labels in $G_A \triangleright S_{75}$ and $G_B \triangleleft S_1$ are identified, a triangular imbedding of $K_{37}$ with bichromatic dual is achieved. This results in a $(37, 444, 36, 3, 2)$-BIBD $D$, which splits into two $(37, 222, 18, 3, 1)$-BIBDs $D_1$ and $D_2$ with

$$\chi''(D) = \chi''(D_1) = 37 - 666 + 444 = -185.$$  

It is interesting to note that these designs are not isomorphic to the designs of Example 6.4.5 since the generalized pseudosurfaces that they generate are not homeomorphic. (See Theorem 6.1.5.)
CHAPTER 7

OPEN QUESTIONS

Although some questions about geometric realizations of block designs have been answered, many more still remain to be examined. Some of the questions concern the maximum genus of a connected design. There are questions about which designs can "topologically cover" other designs. Other questions can be asked about geometric realizations of partially balanced incomplete block designs with three or more associate classes. In this chapter, these "open areas" of research are examined. Examples and some preliminary results are presented.

7.1 Maximum Genus of a Design

The maximum genus questions are first considered. The maximum (orientable) genus of a (connected) design \( D \), \( \gamma_M(D) \), is defined to be the maximum genus of its associated bipartite graph \( G(D) \); i.e.,

\[ \gamma_M(D) = \gamma_M(G(D)). \]

The maximum nonorientable genus of a (connected) design \( D \), \( \gamma_M^*(D) \), is defined to be the maximum nonorientable genus of \( G(D) \); i.e.,

\[ \gamma_M^*(D) = \gamma_M^*(G(D)). \]

The Betti number of a connected graph \( G \) is defined as follows:

\[ \beta(G) = q - p + 1 \]

where \( q \) is the number of edges of \( G \) and \( p \) is the number of vertices of \( G \).
The maximum nonorientable genus of a connected graph is easily determined as demonstrated by the following theorem (see [24] and [29]).

**Theorem 7.1.1:** (Ringel; Stahl) If $G$ is a connected graph, then $\gamma_M(G) = \Delta(G)$.

From this we are able to find the maximum nonorientable genus of any connected design. Specifically, we have:

**Corollary 7.1.2:** If $D$ is a connected $(v, b, r, k, \lambda)$-BIBD or $(v, b, r, k; \lambda_1, \lambda_2)$-PBIBD, then $\gamma_M(D) = bk - (v + b) + 1$.

**Proof:** If $D$ is a connected design on parameters $(v, b, r, k)$, then $G(D)$ is also connected. Now $G(D)$ has $v + b$ vertices and $bk$ edges. If Theorem 7.1.1 is applied to $G(D)$, then $\gamma_M(G(D)) = \gamma_M(G(D)) = \Delta(G(D)) = bk - (v + b) + 1$. 

The relationship between the maximum orientable genus and the Betti number is not as straightforward. There are, however, a number of theorems relating maximum genus and Betti number. A few of these, along with some definitions, are listed below: See [6] or [32] for additional details and proofs.
Theorem 7.1.3: (Nordhaus, Stewart and White) If $G$ is a connected graph, then

$$\gamma_{N}(G) \leq \left\lfloor \frac{\kappa(G)}{2} \right\rfloor.$$

Furthermore, equality holds if and only if there exists a 2-cell imbedding of $G$ on the (orientable) surface of genus $\gamma_{N}(G)$ with exactly one or two regions according to whether $\kappa(G)$ is even or odd, respectively. ([22])

A graph $G$ is said to be upper imbeddable if $\gamma_{N}(G) = \left\lfloor \frac{\kappa(G)}{2} \right\rfloor$. A tree is a connected graph with no cycles. A splitting tree $T$ of a connected graph $G$ is a subgraph of $G$ which contains all of the vertices of $G$ such that $G - E(T)$ has at most one component with an odd number of edges.

Theorem 7.1.4: (Jungerman - Xuong) A graph $G$ is upper imbeddable if and only if $G$ has a splitting tree. ([18] or [37])

Theorem 7.1.5: (Kronlc, Ringeisen and White) Every complete $n$-partite graph is upper imbeddable. ([20])

Corollary 7.1.6: (Nordhaus, Stewart and White) The maximum genus of $K_p$ is given by $\gamma_{N}(K_p) = \left\lfloor \frac{(p-1)(p-2)}{4} \right\rfloor$. ([22])

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Corollary 7.1.7: (Ringeisen) The maximum genus of $K_{m,n}$ is
given by $\gamma_M(K_{m,n}) = \left[ \frac{(m-1)(n-1)}{2} \right]$. ([23])

These theorems and corollaries can be applied to several families of block designs. For example, if $D$ is a complete design on parameters $(v, b, b, v, b)$, then $G(D) = K_{v,b}$. Thus we have

Corollary 7.1.8: The maximum genus of a complete $(v, b, b, v, b)$ design $D_{v,b}$ is given by

$$\gamma_M(D_{v,b}) = \left[ \frac{(v-1)(b-1)}{2} \right].$$

Proof: Apply Corollary 7.1.7 to $G(D_{v,b})$.

Knowledge of the structure of bipartite graphs associated with designs having parameter $k = 2$ leads to the following corollaries.

Corollary 7.1.9: If $D$ is a $(v, b, r, 2, 1)$-BIBD, then

$$\gamma_M(D) = \left[ \frac{(v-1)(v-2)}{4} \right].$$

Proof: By Theorem 4.1.3, the graph $G(D)$ is homeomorphic from $K_v$. Thus $\gamma_M(D) = \gamma_M(K_v)$. Now apply Corollary 7.1.6.

Corollary 7.1.10: If $D_{n(m)}$ is a $(K_{n(m)}^{n(m)}, b, r, 2; 0, 1)$-PBIBD, then

$$\gamma_M(D_{n(m)}) = \left[ \frac{n^2(m-1) - 2nm + 2}{4} \right].$$
Proof: Observe that \(G(D_{n(m)})\) is homeomorphic from \(K_{n(m)}\) by Theorem 4.1.5. Thus \(\gamma_M(G(D_{n(m)})) = \gamma_M(K_{n(m)})\). By Theorem 7.1.5,
\[
\gamma_M(K_{n(m)}) = \left\lfloor \frac{m}{2} \right\rfloor.
\]

Corollary 7.1.11: Let \(D_n\) be the \((L(K_n))\)
\[
\left(\frac{n(n-1)}{2}, n, 2, n-1; 0, 1\right)-PBIBD \text{ described in Theorem 4.3.3. Then}
\]
\[
\gamma_M(D_n) = \left\lfloor \frac{(n-1)(n-2)}{4} \right\rfloor.
\]

Proof: By Theorem 4.3.3, \(G(D_n)\) is homeomorphic from \(K_n\). Now apply Corollary 7.1.6.

There are relatively few families of designs which have easily identified associated bipartite graphs homeomorphic from \(K_p\) or some other complete \(n\)-partite graph. For most designs, we must fall back upon determining whether the associated bipartite graph has a splitting tree and then use Theorem 7.1.4 if applicable. Perhaps one direction for further research would be in determining necessary or sufficient conditions for the bipartite graph associated with a design to be upper imbeddable.

7.2 Covering Designs

In Chapter 3 we considered the problem of imbedding a design on an orientable surface by means of a voltage hypergraph. In this method, voltage hypergraphs are (topologically) covered by design hypergraphs. Are there any examples in which the voltage hypergraph is actually a design hypergraph?
Recall that, in the approach adopted in Chapter 3, voltage graph theory is applied to the bipartite graph associated with a given voltage hypergraph \( H \). Thus if \( \Gamma \) is a finite group of order \( n \) such that \( G(H) \times \phi \Gamma \) is an \( n \)-fold covering of \( G(H) \), then each object vertex of \( G(H) \) is covered by \( n \) object vertices in \( G(H) \times \phi \Gamma \) and each edge vertex of \( G(H) \) is covered by \( n \) edge vertices in \( G(H) \times \phi \Gamma \). The degrees of object and edge vertices are preserved in the covering graph. So if \( D \) is a \( (v, b, r, k) \) design (balanced or partially balanced) which is covered by a design \( D \times \phi \Gamma \), then \( D \times \phi \Gamma \) is on parameters \( (nv, nb, r, k) \).

Consider the bipartite graph associated with the planar \((3, 12, 8, 2, 4)\)-BIBD \( D \) shown in Figure 7.1. Voltages from the group \( \mathbb{Z}_3 \) are assigned to the edges of \( G(D) \) as indicated.

![Figure 7.1](image)

The covering graph \( G(D) \times \phi \mathbb{Z}_3 \) consists of 9 object vertices, 36 block vertices and 72 edges. It is imbedded on an
orientable surface of genus \( h \) with six hexagonal regions and nine 12-gonal regions. (See Theorem 3.2.1.) Thus we may calculate \( h \):

\[
\chi(S_h) = 45 - 72 + 15 = -12;
- 12 = 2 - 2h \text{ or } h = 7.
\]

So \( G(D) \times \phi Z_3 \) is imbedded on \( S_7 \). This covering graph \( G(D) \times \phi Z_3 \), shown in Figure 7.2 not properly imbedded, is actually the bipartite graph associated with the \((G) (9; 36, 8, 2; 0, 2)\)-PBIBD \( D' \); i.e., \( G(D) \times \phi Z_3 \cong G(0,2) \) (see Theorem 4.1.5). The graph \( G \), also given in Figure 7.2, is strongly regular with \( p_{22}^1 = 2 \) and \( p_{22}^2 = 1 \). Thus the design \( D' = D \times \phi Z_3 \) covers the design \( D \). There is, therefore, at least one example of one design (a PBIBD) covering another design (a BIBD).
FIGURE 7.2
If an \((nv, nb, r, k, \lambda')\)-BIBD \(D'\) is an \(n\)-fold covering of a \((v, b, r, k, \lambda)\)-BIBD \(D\), then by Theorem 2.3.1 we know that 
\[
\lambda(v-1) = r(k-1) \quad \text{and} \quad \lambda'(nv - 1) = r(k-1).
\]
So \(\lambda(v-1) = \lambda'(nv -1)\), which implies 
\[
\frac{\lambda}{\lambda'} = n + \frac{n-1}{v-1}.
\]
The covering design must have a smaller \(\lambda'\) value than the voltage design; i.e., \(\lambda\) is more than \(n\) times the \(\lambda'\) value. Are there any sets of parameters which satisfy these conditions? Let \(v = n\). Then \(\lambda = (n+1)\lambda'\). Thus the voltage design is on parameters \((n, \frac{nr}{k}, r, k, (n+1)\lambda')\) and the covering design is on parameters \((n^2, \frac{n^2r}{k}, r, k, \lambda')\). So there are parameters satisfying the conditions. It should be pointed out, though, that \(\Gamma(D) \times \mathcal{G}\) need not be the bipartite graph of a design which covers \(D\).

### 7.3 Geometric Realizations of Designs with Three or More Associate Classes

There exist partially balanced incomplete block designs with three or more associate classes. Bose [7] defines such designs as follows. The design \(D\) is a \((v, b, r, k; \lambda_1, \lambda_2, \ldots, \lambda_m)\)-PBIBD if the \(v\) objects are first arranged by pairs into \(m\) associate classes satisfying the following conditions:

1. any two distinct objects are either 1st, 2nd, \(\ldots\), or \(m\)th associates, the relation of association being symmetric;
(ii) each object has \( n_i \) \( i \)th associates, the number \( n_i \) being independent of the object;

(iii) if any two distinct objects \( x \) and \( y \) are \( i \)th associates, then the number of objects which are \( j \)th associates of \( x \) and \( k \)th associates of \( y \) is \( P_{ijk} \) and is independent of the pair of \( i \)th associates \( x \) and \( y \).

The \( v \) objects are then arranged into \( b \) blocks so that:

(i) each object appears in exactly \( r \) blocks;

(ii) each block contains exactly \( k \) objects;

(iii) each pair of \( i \)th associates appear together in exactly \( \lambda_i \) blocks, \( i = 1, 2, \ldots, m \).

The next theorem presents several relationships among the parameters of a PBIBD with \( m \) associate classes. (Again see [7].)

**Theorem 7.3.1:** If \( D \) is a \( (v, b, r, k; \lambda_1, \lambda_2, \ldots, \lambda_m) \)-PBIBD, then

(i) \( vr = bk \); 

(ii) \( n_1 \lambda_1 + n_2 \lambda_2 + \cdots + n_m \lambda_m = r(k-1) \); 

(iii) \( n_1 + n_2 + \cdots + n_m = v - 1 \).

White, in [31], gives an example of an \( (8, 6, 3, 4; 0, 1, 2) \)-PBIBD \( D \) with association scheme based upon the graph \( Q_3 \) given in Figure 7.3. The vertices of \( Q_3 \) are assigned elements in the group \( Z_2 \times Z_2 \times Z_2 \). Two vertices \( v_1 \) and \( v_2 \) are first associates if \( v_2 = v_1 + (1, 1, 1) \) (addition taking place in \( Z_2 \times Z_2 \times Z_2 \)).
second associates if nonadjacent but not first associates and third associates if adjacent.

FIGURE 7.3

The regions of $Q_3 \triangleleft S_0$ yield the blocks of the design. We define $G(D)$ in the usual way and note that an imbedding of $G(D)$ on $S_0$ can be constructed from $Q_3 \triangleleft S_0$: insert a vertex $e$ into the interior of each region of $Q_3 \triangleleft S_0$ and insert the edges $v_1 e$ for each vertex $v_1$ incident with region $e$. Thus $\gamma(D) = 0$. The hypergraph realization of $D$ is shown in Figure 7.4.
FIGURE 7.4

Apparently, many of the definitions and theorems developed for balanced designs and two-associate class partially balanced designs can be extended to designs on three or more associate classes. The determination of planar PBIBDs on $m \geq 3$ associate classes could prove to be more complicated, however. These designs do not have association schemes given by strongly regular graphs.
7.4 Conclusion

The open questions and extensions presented in this chapter indicate just a few of the many possible directions that future research on geometric realizations of block designs could take. The geometric realization of a block design, suggested by Jungerman, Stahl and White [19], is consistent with previous work done on relating block designs and graph imbeddings as demonstrated in Chapter 6. The imbedding techniques described in Chapters 3 and 4 not only provide a "concrete" tool for describing an abstract design but also can lead to the construction of new designs.
BIBLIOGRAPHY


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