On Isomorphic Decompositions of Graphs

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ON ISOMORPHIC DECOMPOSITIONS
OF GRAPHS

by

Sergio Ruiz

A Dissertation
Submitted to the
Faculty of The Graduate College
in partial fulfillment of the
requirements for the
Degree of Doctor of Philosophy
Department of Mathematics

Western Michigan University
Kalamazoo, Michigan
December 1983
ON ISOMORPHIC DECOMPOSITIONS OF GRAPHS

Sergio Ruiz, Ph.D.
Western Michigan University, 1983

A decomposition of a nonempty graph $G$ is a collection of subgraphs $G_1, G_2, \ldots, G_k$ of $G$ such that their edge sets form a partition of the edge set of $G$. If $G_i$ is isomorphic to a fixed graph $H$ for each $i$, then $G$ has an isomorphic decomposition into the graph $H$ or, equivalently, $G$ is $H$-decomposable. Several topics, each concerning isomorphic decompositions, are investigated in this dissertation.

An historical introduction to the subject of isomorphic decompositions is given in Chapter I. We also present there some new information on finding regular graphs that are $H$-decomposable for a prescribed graph $H$. These results are also extended to digraphs.

In order to illustrate more fully the main concepts of this dissertation, we study in Chapter II the problem of finding isomorphic decompositions of specific graphs, including the Petersen graph, for which all isomorphic decompositions are determined.

Chapter III is primarily devoted to isomorphic decompositions of complete and complete bipartite graphs into linear forests (graphs that are unions of paths). It is shown that if $F$ is any linear forest of size $n$ having no isolates, then $K_{2n}$ is $F$-decomposable.
Chapter IV is concerned with decomposing complete graphs of prime order into vertex-symmetric (regular) graphs called circulants. Results obtained here give some bounds for the so-called isomorphic Ramsey numbers.

In Chapter V we define the concept of randomly H-decomposable graphs and characterize these graphs in two cases.

Previous results have determined extremal regular graphs that do not contain 1-factors and, in some instances, these graphs have been shown to possess a "near 1-factorization". Following in this direction, we find in Chapter VI those extremal regular graphs that fail to possess a near 1-factor. In a few cases we verify that these extremal graphs nevertheless contain a natural isomorphic factorization.
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ACKNOWLEDGEMENTS

For his inspiration, encouragement, guidance and above all, unlimited patience during the preparation of this dissertation, I am ever thankful to my advisor Dr. Gary Chartrand. My recognition is extended to all my teachers, in particular to Drs. Joseph Buckley, Kung-Wei Yang, Arthur White and Anthony Gioia for sharing part of their time in constructive discussions. Thanks to Dr. Roberto Frucht from Universidad Técnica Federico Santa María for teaching me the first steps in graph theory in such an interesting way and for his valuable remarks on this dissertation.

I also want to express my gratitude to Eliana López for her careful drawing of all the figures, to Jorge Hidalgo for his expert opinion on computer programming and to my typist, Margo Johnson, who was great for her speed, accuracy, patience and good humor.

My deep appreciation to the Laeder family. They took the role of my foster family while I was an exchange student in Michigan fourteen years ago, and now they still provide me and my actual family with the love and support of real relatives.

I am indebted to all of those who made my visit to Western Michigan University possible and pleasant, namely Jorge Galbiati, Henry Levinson, Yousef Alavi, Eric and Pat Pratt, and Troy and Martha Anderegg. Finally, I want to acknowledge the financial support of Universidad Católica de Valparaíso, the Organization of American States and Western Michigan University.

Sergio Ruiz

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CHAPTER I

INTRODUCTION

Section 1.1

Historical Background

In this section an historical background of the isomorphic decomposition problem is presented along with some pertinent definitions. For basic terminology and notation, we follow Behzad, Chartrand and Lesniak-Foster [5].

By a decomposition of a nonempty graph $G$ is meant a family of subgraphs $G_1, G_2, \ldots, G_k$ of $G$ such that their edge sets form a partition of the edge set of $G$. Any member of the family is called a part (of the decomposition). This decomposition is usually denoted by $G = G_1 \oplus G_2 \oplus \ldots \oplus G_k$.

A graph $G$ is said to be $H$-decomposable (or has an $H$-decomposition) if $G$ has a decomposition in which all of its parts are isomorphic to the graph $H$. We also say that $G$ has an isomorphic decomposition into the graph $H$. If $H$ is a spanning subgraph of $G$, then $H$ is called an isofactor of $G$ and $G$ has an isomorphic factorization into the graph $H$. Clearly every nonempty graph $G$ has a $K_2$-decomposition and a $G$-decomposition (as well as $G$-factorization). These are referred to as the trivial decompositions of $G$. 

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A great deal of research has been done on the study of isomorphic decompositions of complete graphs. This appears to have been motivated by its close relationship to the theory of block designs. For this reason, an $H$-decomposition of $K_n$ is sometimes called an $H$-design on $K_n$. Nevertheless, the fundamental question of determining whether $K_n$ has an $H$-decomposition for a given graph $H$ has not been solved in general.

The $K_3$-designs are known as Steiner triple systems, the existence of which was established by Kirkman [39] in 1847. A few other $K_n$-designs have been studied as well.

It has been proved [67] that $K_n$ can be decomposed into stars with $k$ edges if and only if $n(n-1) \equiv 0 \pmod{2k}$ and $n \geq 2k$. Recently, Tarsi [60] showed that $K_n$ is $P_{k+1}$-decomposable if and only if $n(n-1) \equiv 0 \pmod{2k}$ and $n \geq k+1 \geq 2$. More will be said about decomposing complete graphs into linear forests (vertex disjoint union of paths) in a later chapter.

An extensive literature exists on isomorphic decompositions of complete graphs into cycles. Clearly, if $K_n$ has a $C_k$-decomposition, then $n$ is odd and the number of edges of $K_n$ is a multiple of $k$. An old conjecture states that these conditions are also sufficient for $K_n$ to be $C_k$-decomposable. The conjecture has been proved for many values of $n$ and $k$. The theses of Sotteau [57] and Gabel [27] contain a detailed account of this problem.

The following is easy to see.
Let $H$ be a graph of order $p$ and size $q$. If $K_n$ has an $H$-decomposition then

(i) $n \geq p$,
(ii) $n(n-1) \equiv 0 \pmod{2q}$, and
(iii) $n - 1 \equiv 0 \pmod{d}$,

where $d$ is the greatest common divisor of the degrees of the vertices of $H$.

While some authors have studied $H$-designs for graphs $H$ having no more than five vertices (see [7], [9]), Wilson [56] proved that the above necessary conditions are asymptotically sufficient, that is, there exists an integer $\lambda$ (depending on $H$) such that if $n \geq \lambda$ and $n$ satisfies the conditions (i) - (iii), then $K_n$ is $H$-decomposable. It follows that every graph $H$ gives rise to an $H$-design for infinitely many complete graphs.

If a graph $G$ of order $n$ and its complement $\overline{G}$ are isomorphic, they form an isomorphic decomposition of $K_n$ into two parts. In this case, $G$ is called a self-complementary graph. The structural properties of these graphs have been studied since 1962 [55] although many questions about them remain unanswered. Using Pólya's counting technique, Read [49] was able to find a formula for the number of self-complementary graphs of order $n$. Several references can be found in the papers of Gibbs [31] and Rao [47].

In 1963, Ringel [51] conjectured that if a tree $T$ has size $n$, then $K_{2n+1}$ is $T$-decomposable. Kotzig later conjectured that
$K_{2n+1}$ has what is referred to as a "cyclic $T$-decomposition" for any tree $T$ with $n$ edges. Ringel's conjecture has been solved for only a few families of trees.

An important tool for finding certain $G$-designs is the concept of $\rho$-valuation. A graph $G$ of size $n$ has a $\rho$-valuation if there exists a one-to-one labeling $f: V(G) \rightarrow \{0, 1, \ldots, 2n\}$ such that the set $\overline{f}(E) = \{|f(i) - f(j)|/ij \in E(G)\} = \{x_1, x_2, \ldots, x_n\}$, where $x_i = i$ or $x_i = 2n + 1 - i$ for each $i = 1, 2, \ldots, n$.

Rosa [54] noted that a graph $G$ with $n$ edges has a $\rho$-valuation if and only if $K_{2n+1}$ can be "cyclically decomposed" into $2n + 1$ copies of $G$.

The theory of permutation groups has been touched by our subject. Suppose that $K_n$ has been isomorphically decomposed into some parts. The group of permutations of $V(K_n)$ that preserve the edge-partition is called the full symmetry group. The symmetry group of the decomposition is the action of the full symmetry group on the members of the partition. References and results on the topic are found in a paper of Robinson [53] along with the following conjecture:

Every finite group is isomorphic to the symmetry group of some isomorphic decomposition of a complete graph.

Harary, Robinson and Wormald [37] showed that if $k$ is a divisor of the number of edges of $K_n$, then there exists an $H$-design on $K_n$ for some graph $H$ of size $k$. Recently, the same property was extended to complete equipartite graphs by Quinn [46].

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Sugiyama [59] and Kühler [40] noted that $K_n$ can be isomorphically decomposed into an $r$-regular graph $H$ if $r$ divides $n - 1$ and both $r$ and $n$ are not odd.

The case of decomposing complete graphs into isomorphic factors with a given diameter has been considered by Kotzig and Rosa [42] and Tomasta [61]. Let $G_m(d)$ be the smallest integer $n$ such that the complete graph $K_n$ can be decomposed into $m$ isomorphic factors with diameter $d$. Let $H_m(d)$ be the smallest positive integer such that whenever $p \geq H_m(d)$ and $m$ divides the size of $K_p$, then $K_p$ can be decomposed into $m$ isomorphic factors with diameter $d$. They conjectured that $G_m(d) = H_m(d)$ but verified it for only a few values of $m$.

So far we have only been concerned with decomposition of complete graphs. Historically, this has been the development of the topic with some isolated exceptions. In recent years, however, interest has grown in isomorphically decomposing graphs other than complete graphs. This pattern of evolution resembles the one of Ramsey theory through the years. In the beginning only complete graphs were considered, but actually the general Ramsey theory is highly regarded as a subject of research.

Among the aforementioned exceptions, there are two which deserve special attention, namely decompositions of graphs into two kinds of regular spanning subgraphs: hamiltonian cycles and 1-factors. The existence of hamiltonian decompositions has been decided for several families of graphs and for some operations between

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certain graphs such as cartesian product, lexicographic product and conjunction [8]. Graphs that can be decomposed into 1-factors are called 1-factorable. It is well known that $K_{2n}$, $n \geq 1$, and non-empty regular bipartite graphs are 1-factorable. Nevertheless, as in the case of hamiltonian decompositions, 1-factorable graphs have not been characterized. Generalizing in this direction, Alon [2] discovered that for graphs having sufficiently large size, there is a characterization of $tK_2$-decomposable graphs. On the other hand, there appears to be no result of this type for $C_t$-decomposable graphs.

One innate feature of mathematics is the tendency to generalize concepts. Graph theory is not an exception as can be seen by the following definition. Let $\mathcal{F}$ be a set of graphs. An $\mathcal{F}$-decomposition of a graph $G$ is a decomposition $G_1, G_2, \ldots, G_k$ of $G$ such that

$$G = G_1 \Theta G_2 \Theta \ldots \Theta G_k$$

and $G_i \in \mathcal{F}$ for $1 \leq i \leq k$.

Accordingly, our original $H$-decompositions are $\mathcal{F}$-decompositions where the family $\mathcal{F}$ has the single graph $H$. In this context, Schönhheim and Bialostocki [56] determined all those families $\mathcal{F}$ consisting of regular graphs of degree 1 for which $K_n$ has an $\mathcal{F}$-decomposition.

The last chapter of Fink's thesis [26] is dedicated to isomorphic decompositions. There it is proved that for every nonempty graph $H$, there exists a regular graph $G$ such that $G$ is

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H-decomposable and both $G$ and $H$ have the same chromatic number. For any nonempty graph $H$ three decomposition parameters were defined: $r^H_0$ is the smallest integer $r$ for which there exists an $r$-regular, connected, $H$-decomposable graph; $p^H_0$ is the minimum order among all regular, connected $H$-decomposable graphs and $f^H_0$ is the least integer $t$ such that there exists a $t$-regular, connected graph with an $H$-decomposition comprising $t$ parts. These parameters were computed for complete graphs, cycles and stars. Partial results were given for trees.

Isomorphic decompositions of complete multipartite graphs into two factors have been studied by Gangopadhyay and Rao Hebbare under the name of multipartite self-complementary graphs (see [28], [29], [30]). For other isomorphic decompositions of multipartite graphs see [25], [36], [46], [58], [64]. Information concerning isomorphic decomposition of trees can be found in [3], [4], [14], [15], [16].

Section 1.2

Labelings and Isomorphic Decompositions of Graphs

Let $n \geq 3$ be an integer and $S$ a nonempty subset of \{1, 2, ..., $\lfloor n/2 \rfloor$\}. The circulant graph or, more simply, circulant $G = G(n; S)$ has vertex set $V(G) = \{v_0, v_1, \ldots, v_{n-1}\}$ and $v_i v_j \in E(G)$ if and only if either $j - i$ or $i - j$ is congruent, modulo $n$, to an element of $S$. The set $S$ is called the length set of $G$ and the length of any pair $v_i, v_j$ of vertices is defined as $\lambda(v_i, v_j) = \min\{|i-j|, |n-(i-j)|\}$. When referring to
the length $\ell(e)$ of an edge $e = v_i v_j$ we mean $\ell(e) = \ell(v_i, v_j)$.

Circulants can be drawn in the Euclidean plane with its $n$ vertices $v_0, v_1, \ldots, v_{n-1}$ regularly distributed counterclockwise about a circle, where the edges are represented by chords joining the appropriate vertices.

Example 1.1

![Figure 1.1 The circulant $G(6; \{2,3\})$](image)

When $S = \{1\}$ then $G(n; S) = C_n$ (the $n$-cycle). If $S = \{1, 2, \ldots, \lfloor n/2 \rfloor\}$, then we have all possible chords joining the vertices of the circulant and so $G(n; S) = K_n$.

Note that in $G = G(n; S)$, for any vertex $v_i \in V(G)$ and any $s \in S$, the vertex $v_i$ is adjacent to both $v_{i+s}$ and $v_{i-s}$ (where the subscripts are expressed modulo $n$). Moreover, $v_{i+s} \neq v_{i-s}$ unless $s = n/2$. Therefore if $n/2 \notin S$, then $G(n; S)$ is regular of degree $2|S| - 1$; otherwise, it is regular of degree $2|S|$. The cyclic permutation $\varphi = (v_0 \ v_1 \ldots \ v_{n-1})$ is an automorphism of $G$.

Associated with $\varphi$ we have an induced permutation $\varphi_E$ defined on
the edges of $G$ as follows: The image of an edge $xy$ of $G$ under $\varphi_E$ is the edge $\varphi(x)\varphi(y)$. Considering the action of the permutation group

$$\{\varphi, \varphi^2, \ldots, \varphi^n\}$$
on $E(G)$, we observe that this action partitions the edge set of $G$ into $|S|$ orbits. Two edges belong to the same orbit $E_S$ if and only if they have the same length $s$. If $\lambda(e) = n/2$ then the orbit containing $e$ has $n/2$ members; otherwise it has $n$ members.

**Example 1.2**

In the circulant of Figure 1.1 there are two orbits of edges; namely: $E_2 = \{v_0v_2, v_1v_3, v_2v_4, v_3v_5, v_4v_0, v_5v_1\}$ and $E_3 = \{v_0v_3, v_1v_4, v_2v_5\}$.

The next lemma will enable us to construct some isomorphic decompositions from a given circulant. From now on $N_n$ will denote the set $\{0, 1, \ldots, n-1\}$, for $n \geq 1$.

**Lemma 1.1** Let $G = G(n; S)$ be a circulant such that $n/2 \notin S$. For each $s \in S$, let $e_s$ be an edge in the orbit $E_s$. If $H$ is the subgraph of $G$ induced by the edges $e_i$, $i \in S$, then $G$ is $H$-decomposable.

**Proof:** Let $E' = \{e_s | s \in S\}$. For each $k \in N_n$ let $G_k$ be the subgraph of $G$ induced by the set $\varphi_E^k(E')$ of edges. Thus, $G_0 = H$. 

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We now show that \( \varphi^k \) is an isomorphism between \( G_0 \) and \( G_k \), for each \( k \in \mathbb{N} \). Let \( v_s \) and \( v_t \) be two adjacent vertices of \( G_0 \). Note that \( \varphi^k(v_s) \) and \( \varphi^k(v_t) \) are adjacent in \( G_k \) since 
\[
\varphi^k(v_s v_t) = \varphi^k(v_s) \varphi^k(v_t) \in E(G_k); \quad \text{thus } G_0 \cong G_k.
\]

It remains only to prove that \( \{E(G_k)\}_{k \in \mathbb{N}} \) forms a partition of \( E \). If \( e \in E \), then \( \ell(e) \in S \), and by the way that \( G_0 \) was defined, there exists an edge \( e_o \) in \( G_o \) such that 
\[
\ell(e_o) = \ell(e). \quad \text{However, there exists } k \in \mathbb{N} \text{ such that } \varphi^k(e_o) = e;
\]
therefore \( e \in E(G_k) \) and consequently 
\[
E = \bigcup_{i=0}^{n-1} E(G_i).
\]

To show that the sets \( E(G_0), E(G_1), \ldots, E(G_{n-1}) \) are pairwise disjoint, we proceed by a counting argument. Since \( G \) is regular of degree \( 2|S| \), it has \( n|S| \) edges. If 
\[
E(G_1), E(G_1), \ldots, E(G_{n-1})
\]
were not pairwise disjoint, then
\[
n|S| = |E(G)| < \sum_{i=0}^{n-1} |E(G_i)| = n|S|,
\]
which is a contradiction.

Therefore \( G = G_0 \circ G_1 \circ \ldots \circ G_{n-1} \) and \( G \) is \( H \)-decomposable.

We say that \( G \) is cyclically \( H \)-decomposable if \( G \) possesses the kind of decomposition described in the preceding lemma. Loosely

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speaking, each part $G_{i+1}$ is obtained from $G_i$ by rotating $G_i$
counterclockwise through an angle of $2\pi/n$ radians about its
center. Now, the basic idea leading to the main results is to imbed
a preassigned graph $H$ into a circulant such that $H$ is a part of
the $H$-decomposition. To achieve this we will extend a labeling
technique introduced by Rosa [54] when he cyclically decomposed
$K_{2n+1}$ into a graph of size $n$ admitting a $\rho$-valuation.

A labeling of a graph $G$ is a one-to-one mapping from $V(G)$
into the set $\mathbb{N}$ of nonnegative integers. Given a labeling $f$ of
$G$, its induced edge labeling is the map $\overline{f}$ from $E(G)$ onto a set
$\mathbb{N}$ of positive integers defined by $\overline{f}(uv) = |f(u) - f(v)|$, where
$uv \in E(G)$. A graph $G$ is called graceful if $\mathbb{N} = \{1, 2, \ldots, |E(G)|\}$.

For the sake of completeness we provide a proof of the follow­
ing known result.

Lemma 1.2 Every nonempty graph has a labeling such that its
induced edge labeling is one-to-one.

Proof: Let $H$ be a nonempty graph, where $V(H) = \{v_1, v_2, \ldots, v_n\}$. Define $f(v_i) = 2^i$ for $i = 1, 2, \ldots, n$. We
show that the induced edge labeling $\overline{f}$ is one-to-one. Clearly, $f$
is a labeling of $G$. It suffices to show that $\overline{f}$ is one-to-one. If
$G$ has only one edge, $\overline{f}$ is certainly one-to-one. Let
$e = v_i v_j$ and $e' = v_s v_t$ be distinct edges of $G$ where $i < j$ and
$s < t$. Assume, to the contrary, that $\overline{f}(e) = \overline{f}(e')$, that is,

$$2^j - 2^i = 2^t - 2^s,$$
which implies that

\[ 2^i(2^{j_i-1}) = 2^s(2^{t-s-1}). \]  \hspace{1cm} (1.1)

Since \( 2^{j_i} \) and \( 2^{t-s} \) are different from 1, the numbers \( 2^{j_i-1} \) and \( 2^{t-s-1} \) are odd. If \( i \geq s \), then from (1.1) we have

\[ 2^{i-s}(2^{j_i-1}) = 2^{t-s-1} \]

so that \( i = s \) and \( j = t \), which leads us to the contradictory conclusion that \( e = e' \). If \( s \geq i \), the same contradiction is produced by dividing both terms in (1.1) by \( 2^i \).

Consequently \( \overline{f} \) is one-to-one as claimed.

If, in the above proof, we were to define \( g(v_i) = 2^{i-1}-1 \), \( i = 1, 2, \ldots, n \), then \( g \) is also a labeling of any nonempty graph \( H \) of order \( n \), for which the induced edge labeling \( \overline{g} \) is one-to-one. Acharya and Gill [1] also proved Lemma 1.2 using a different labeling with maximum edge value equal to \( (n-1)\chi(H) \), where \( \chi(H) \) represents the chromatic number of \( H \). The minimum value of the largest integer \( \theta(H) \) assigned to any vertex of \( H \) under any labeling \( \overline{f} \) of \( H \) for which \( \overline{f} \) is one-to-one is called the index of gracefulness of \( H \). As noted in [1],

\[ \theta(H) \leq \min\{(n-1)\chi(H), 2^{n-1} - 1\}. \]
This inequality may be strict since \( \theta(G) \leq \theta(K_n) \) for any graph \( G \) of order \( n \) and \( \lim \frac{\theta(K_n)}{n^2} = 1 \), as \( n \to \infty \), but no better general bounds are known.

Theorem 1.3. For every nonempty graph \( H \), there exists a connected circulant \( G \) such that \( G \) is cyclically \( H \)-decomposable.

**Proof:** We assume, without loss of generality, that \( V(H) = \{v_0, v_1, \ldots, v_m\} \) and that \( v_0v_1 \in E(H) \). By Lemma 1.2, there exists a labeling \( f \) of \( H \) for which \( f \) is one-to-one, \( f(v_0) = 1 \) and \( f(v_1) = 2 \).

Let \( S = \{ f(e) | e \in E(H) \} \) and \( r = \max S \). Then by Lemma 1.1, the circulant \( G(n;S) \) is \( H \)-decomposable, where \( n = \max\{2r+1, p(H)\} \).

Note that \( 1 \in S \) so that \( G \) contains a hamiltonian cycle and is connected. Moreover, from the proof of Lemma 1.1, it follows that \( G \) is cyclically \( H \)-decomposable. \( \square \)

With this result at hand we can prove an even stronger result.

Theorem 1.4. For every nonempty graph \( H \) there exists a connected circulant \( G \) such that \( G \) is cyclically \( H \)-decomposable and every part of such a decomposition is an induced subgraph of \( G \).

**Proof:** Let \( f \) be a labeling of \( H \) such that whenever we have \( x, y, w, z \in V(H) \) and \( \{x,y\} \neq \{w,z\} \), then \( |f(x) - f(y)| \neq |f(w) - f(z)| \). Notice that with this condition, \( f \) is also a labeling of a complete graph on \( |V(H)| \) vertices with a one-to-one
induced edge labeling. The existence of such a labeling is guaranteed by Lemma 1.2.

According to the proof of Theorem 1.3, a connected circulant $G$ which is cyclically $H$-decomposable exists, namely let $G$ be the circulant $G(n;S)$, where

$$ n = 1 + 2 \max \{ f(x) - f(y) | x, y \in V(H) \} \text{ and } S = \{ (e) | e \in E(H) \}. $$

Now let $G = G_0 \Theta G_1 \Theta \ldots \Theta G_{n-1}$ be such an $H$-decomposition of $G$, where $V(G) = \{ v_0, v_1, \ldots, v_{n-1} \}$ and $V(G_0) = \{ v_{f(x)} | x \in V(H) \}$.

It remains only to show that each of the parts $G_i$ is an induced subgraph of $G$. Choose an arbitrary integer in $\mathbb{N}^n$. We prove that $G_i$ is an induced subgraph of $G$.

Let $g$ be the isomorphism from $G_i$ to $G_o$ expressed as an appropriate power of the cycle $\varphi = (v_0 v_1 \ldots v_{n-1})$. Let $v_s$ and $v_t$ be two nonadjacent vertices in $G_i$. We claim that they are not adjacent in $G$. Suppose that $v_s v_t \in E(G)$. Then $\ell(v_s v_t) \in S$ and there exists an edge $v_j v_k$ in $G_o$ such that $\ell(v_j v_k) = \ell(v_s v_t)$. Moreover, the vertices $g(v_s) = v_{g(s)}$ and $g(v_t) = v_{g(t)}$ belong to $G_o$ but are not adjacent in $G_o$ because $g$ is an isomorphism. Therefore, there exist two distinct unordered pairs $\{ a, b \}$ and $\{ c, d \}$ of vertices of $H$ such that $f(a) = j$, $f(b) = k$, $f(c) = s$ and $f(d) = t$. However the equality $\ell(v_j v_k) = \ell(v_s v_t)$ implies that $|f(a) - f(b)| = |f(c) - f(d)|$, contradicting the property of the labeling $f$. Thus $v_s v_t$ is not an edge of $G$ as claimed.
Section 1.3

Isomorphic Decompositions and Digraphs

If $F$ is a digraph and $F_1, F_2, \ldots, F_n (n \geq 1)$ are nonempty arc-disjoint subdigraphs of $F$ satisfying the property that

$$E(F) = \bigcup_{i=1}^{n} E(F_i),$$

then we say that $F$ is the arc sum of the parts $F_1, F_2, \ldots, F_n$ and write $F = F_1 \oplus F_2 \oplus \ldots \oplus F_n$. If there is a digraph $D$ that is isomorphic to each of the parts $F_1, F_2, \ldots, F_n$ then we say that $F$ has an isomorphic decomposition or that $F$ is $D$-decomposable.

A digraph is $r$-regular (or simply regular) if each of its vertices has both indegree and outdegree equal to $r$. An example of a regular digraph is obtained from a complete graph $K_p$ by replacing each edge $uv$ by two arcs $(u,v)$ and $(v,u)$. This digraph is called the complete symmetric digraph $K^{*}_p$. Isomorphic decompositions of $K^{*}_p$ have been considered, among others, by Bermond and Sotteau [10]. Harary, Robinson and Wormald [35] showed that if $t$ divides $p(p-1)$, then $K^{*}_p$ has an isomorphic decomposition into $t$ copies of some digraph. In the same paper it was shown that the $n$-regular complete symmetric bipartite digraph $K^{*}(n,n)$ has an isomorphic decomposition into $t$ parts if and only if $t$ is odd and divides $2n^2$. Using algebraic techniques, Wilson [66] proved
that for every nonempty digraph $D$ there are infinitely many complete symmetric digraphs which are $D$-decomposable.

As a consequence of Theorem 1.4 we have the following.

**Theorem 1.5.** For every nonempty digraph $D$ there exist a connected regular digraph $F$ and a positive integer $n$ such that $F = F_0 \circ F_1 \circ \ldots \circ F_{n-1}$ and $\langle V(F_i) \rangle = D$ for all $i$, i.e., each part is an induced subdigraph of $F$.

**Proof.** Let $H$ be the underlying graph of the digraph $D$. According to Theorem 1.4 there exists a connected circulant $G$ such that $G$ is cyclically $H$-decomposable and every part of such a decomposition is an induced subgraph of $G$. With the aid of $G$, we will now construct the desired digraph $F$ so that $G$ is, in fact, the underlying graph of $F$.

Following the proof of Theorem 1.4, $H$ has a labeling $f$ and $G = G_0 \circ G_1 \circ \ldots \circ G_{n-1}$, where $V(G_0) = \{ v_f(x) \mid x \in V(H) \}$. For each part $G_i$ in the $H$-decomposition of $G$, let $\varphi_i$ be the $i$th power of the cycle $(v_0, v_1, \ldots, v_{n-1})$ such that $\varphi_i$ is the isomorphism from $G_0$ to $G_i$ ($\varphi_i$ preserves the length of each edge of $G_0$). Now, for each part $G_i$ and each edge $v_s v_t$ of $G_i$, we employ the following procedure.

There are unique vertices $a, b \in V(D) = V(H)$ such that $v_s = \varphi^i(v_f(a))$ and $v_t = \varphi^i(v_f(b))$. Let $E(ab)$ be the set of arcs of $D$ joining the vertices $a$ and $b$ of $D$. Replace the edge $v_s v_t$ of $G$ by the set of arcs $\{(v_s, v_t'), (v_t, v_s')\}$, $\{(v_t, v_s')\}$ or $\{(v_s', v_t)\}$.
according to whether $E(ab)$ is $\{(a,b),(b,a)\}$, $\{(a,b)\}$ or $\{(b,a)\}$, respectively. Then the digraph $F$ so obtained is connected, regular and has a $D$-decomposition where each part is induced.

We note, in closing this chapter, that a similar proof can be applied to multigraphs.
Chapter II

ISOMORPHIC DECOMPOSITIONS OF GRAPHS OF ORDER 10

In order to illustrate further some of the concepts introduced in the first chapter, we investigate various decompositions of certain regular graphs of order 10.

Section 2.1

Isomorphic Decompositions of the Petersen Graph

Suppose that a nonempty graph $G$ of size $kn$ has an automorphism $f$ such that the induced edge-automorphism $f^*$ is the product of $n$ $k$-cycles. Let $H$ be a subgraph of $G$ of size $n$ whose edges are obtained by selecting one edge from each of the $k$-cycles of $f^*$. Then $G$ is $H$-decomposable. This idea was used by Harary, Robinson and Wormald [37] to decompose complete graphs, but, as we will see in this section, it can be used for other graphs as well. In particular, it will be shown that this technique gives all isomorphic decompositions of the Petersen graph $P$. To find the isomorphic decompositions of $P$ into graphs of size 3, we proceed as follows.

By considering the drawing of $P$ in Figure 2.1, we see that the permutation

$$f = (v_1v_2v_3v_4v_5)(v_6v_7v_8v_9v_{10})$$

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is an automorphism of $P$. The induced edge-automorphism $f^*$ is composed of three 5-cycles, namely

$$f^* = (v_1v_2 v_2v_3 v_3v_4 v_4v_5 v_5v_1)(v_1v_6 v_2v_7 v_3v_8 v_4v_9 v_5v_{10})$$
$$ (v_6v_8 v_7v_9 v_8v_{10} v_9v_6 v_{10}v_7).$$

![Figure 2.1 The Petersen Graph](image)

From a geometric point of view, $f$ can be interpreted as a counterclockwise rotation of $2\pi/5$ radians of the drawing of $P$. The 5-cycles are formed by the outer edges, the spokes and the inner edges. Choosing one edge from each 5-cycle, we construct a subgraph $H$, so that $P$ is $H$-decomposable. The remaining copies of $H$ are obtained by applying the isomorphism $f$ (that is, by rotating Figure 2.1).

By taking the edges $v_1v_2, v_6v_8$ and $v_4v_9$, we conclude that $P$ is $3K_2$-decomposable. Choosing the edges $v_1v_6, v_7v_9$ and $v_1v_2$, we see that $P$ is $(P_3 \cup K_2)$-decomposable. Considering the edges
\[ v_1v_2, v_2v_7 \text{ and } v_7v_9, \] we obtain a \( P_4 \)-decomposition of \( P \). There are two other graphs with three edges and no isolated vertices, namely \( K_3 \) and \( K(1,3) \). Clearly \( P \) is not \( K_3 \)-decomposable since it does not contain triangles. To show that \( P \) is not \( K(1,3) \)-decomposable, we use the following lemma.

**Lemma 2.1** Let \( G \) be an \( r \)-regular graph. Then \( G \) is \( K(1,r) \)-decomposable if and only if \( G \) is bipartite.

**Proof:** If \( G \) is bipartite and \( r \)-regular, then clearly \( G \) is \( K(1,r) \)-decomposable.

Assume now that \( G \) is \( r \)-regular and that an isomorphic decomposition of \( G \) into \( K(1,r) \) is given. Let \( U \) be the set of vertices of \( G \) that have degree \( r \) in some copy of \( K(1,r) \) in the decomposition, and let \( W \) be the remaining vertices of \( G \). The copies of \( K(1,r) \) are edge-disjoint; therefore, no two vertices of \( U \) are adjacent. Every edge of \( G \) must be incident to a vertex of \( U \); therefore, no two vertices of \( W \) are adjacent. Since \( U \) and \( W \) are two independent sets that form a partition of \( V(G) \), the graph \( G \) is bipartite.

To find the \( H \)-decompositions of \( P \) where \( H \) has five edges, first note that \( \Delta(H) \leq 3 \) because \( \Delta(P) = 3 \) and \( H \) cannot contain \( C_3 \) or \( C_4 \) (since \( P \) has girth 5). Moreover, \( P \) is not \( C_3 \)-decomposable; otherwise the degrees of the vertices of \( P \) would be even. Therefore, if \( P \) is \( H \)-decomposable, then \( H \) must be a forest with \( \Delta(H) \leq 3 \). It turns out that from all such forests only

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three of them, namely $5K_2$, $P_6$, and $K(1,3) \cup P_3$, are not isomorphic parts of $P$. The remaining ten forests, exhibited in Figure 2.2, are isomorphic parts of $P$. We illustrate how to find one of the $H$-decompositions. The other cases can be handled similarly.

![Figure 2.2 The isomorphic parts of size 5 of the Petersen graph](image)

An alternative drawing of $P$ is shown in Figure 2.3. The automorphism $g = (v_1, v_9, v_3)(v_6, v_4, v_2)(v_5, v_7, v_8)(v_{10})$ can be obtained by rotating the figure about $v_{10}$ in a counterclockwise angle of $120^\circ$. Then the induced edge-automorphism $g^*$ is

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It remains only to show that $P$ does not have the $H$-decompositions for $5K_2$, $P_6$, and $K(1,3) \cup P_3$ previously mentioned. It is a well-known fact that $P$ is not 1-factorable, that is $P$ is not $5K_2$-decomposable.

To see that $P$ is not $P_6$-decomposable, we note that for each vertex $v$ of $P$, at most one copy of $P_6$ will have $v$ as an interior vertex, but three copies of $P_6$ have twelve interior vertices -- so $P$ is not $P_6$-decomposable. Finally we show that $P$ is not $(K(1,3) \cup P_3)$-decomposable. This is done by observing that the removal of the edges of any copy of $K(1,3) \cup P_3$ gives the graph in Figure 2.5, which does not contain two edge-disjoint copies of $K(1,3) \cup P_3$.

![Figure 2.5](image-url)
Section 2.2

On Isomorphic Decompositions of $K_{10}$ into Cubic Graphs

In 1977, G. Chartrand raised the problem of determining whether the Petersen graph is an isofactor of $K_{10}$. The answer turned out to be negative according to different proofs given by N.C. Wormald, B. Alspach and W. McCuaig, and A. Schwenk. Schwenk's proof is the shortest and it relies on the multiplicities of the three eigenvalues of the Petersen graph (see [22]). It is perhaps typical of the Petersen graph that this technique is not successful for any other connected cubic graph of order 10. In what follows we describe certain procedures which gives isomorphic decompositions of $K_{10}$.

When Sachs [55] and Ringel [52] constructed self-complementary graphs of order $n$ for $n(n-1)/2 \equiv 0 \pmod{2}$, they started with a permutation $f$ on the vertices of $K_n$ such that the induced edge permutation $f^*$ is composed of even cycles. Then from each cycle, half of the edges are chosen by selecting alternate edges. The graph $H$ so determined is self-complementary and $f$ is an isomorphism between the two isomorphic parts of $K_n$. In a similar way, to decompose $K_n$ into three isomorphic parts, where $3|q(K_n)$, we can use a permutation $g$ such that every cycle of $g^*$ has length equal to a multiple of 3. By selecting every third edge in each cycle beginning with any of the first three edges, an isomorphic part $G$ of $K_n$ is formed, and $g$ and $g^2$ are isomorphisms.
between $G$ and the other two parts. This procedure allows us to decompose $K_{10}$ into three isomorphic cubic graphs. Note that the only permutations, up to a labeling of the vertices, satisfying the above requirement are

$$f = (0)(123456789),$$
$$g = (0)(123)(456789),$$
$$h = (0)(123)(456)(789).$$ (2.1)

We will discuss in detail one particular construction for the permutation $f$. The steps are performed in Figure 2.6(a).

There are nine edges of $K_{10}$ incident with the vertex 0 and they are in a 9-cycle of $f^*$:

$$(01 02 03 04 05 06 07 08 09).$$

From this 9-cycle we choose an arbitrary edge $xy$ together with the edges $f^3(x)f^3(y)$ and $f^6(x)f^6(y)$. As an example, suppose $x = 0$ and $y = 1$, so that the edges 01, 04 and 07 have been taken from this edge cycle. There are four other cycles of $f^*$, each with nine edges. They are

$$(12 23 34 45 56 67 78 89 91),$$
$$(13 24 35 46 57 68 79 81 92),$$
$$(14 25 36 47 58 69 71 82 93),$$
$$(15 26 37 48 59 61 72 83 94).$$
Again from each of these cycles we choose an arbitrary edge $xy$ together with the edges $f^3(x)f^3(y)$ and $f^6(x)f^6(y)$. The graph obtained with all the selected edges need not be cubic; therefore special care on selecting the edges must be taken. For our example we have underlined those edges that we have taken from the graph of Figure 2.6(a), which is isomorphic to $K_4 \cup K(3,3)$. Only one more cubic graph can be found with this permutation $f$, which is exhibited in Figure 2.6(b). Notice that $f^3 = (0)(147)(258)(369)$ is an automorphism of these graphs.

![Figure 2.6 The cubic isofactors of $K_{10}$ constructed with $f = (0)(123456789)$](image)

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Using the permutation $g$, five cubic isofactors of $K_{10}$ can be constructed using the cyclic structure of $g^*$. They are presented in Figure 2.7. The permutation $g^3 = (0)(1)(2)(3)(47)(58)(69)$ is an automorphism of these graphs.

Figure 2.7 The cubic isofactors of $K_{10}$ constructed with $g = (0)(123)(456789)$
By considering the permutation $h$ defined in (2.1), we have constructed the nine cubic isofactors of $K_{10}$ shown in Figure 2.8. These graphs together with the cubic graph of Figure 2.7(c) comprise all ten cubic isofactors of $K_{10}$ that we currently know. Therefore, out of the 21 cubic graphs of order 10 (see [13]), there are at least ten that are isofactors of $K_{10}$.

![Figure 2.8 Cubic isofactors of $K_{10}$ constructed with $h = (0)(123)(456)(789)$](image)

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In addition to the Petersen graph, there are three other cubic graphs of order 10 that we know are not isofactors of $K_{10}$. Two of these graphs are bipartite and, according to a theorem of Kotzig [41], at least four bipartite graphs are required to decompose $K_{10}$. The other graph is $K_4 \cup \overline{C}_6$. In fact, $K_{10}$ can have at most one copy of this graph since its independence number is 3. Together with $K_4 \cup K(3,3)$, these are the only cubic disconnected graphs of order 10. We note the peculiarity that one of them is an isofactor of $K_{10}$ while the other can be present at most one time in a decomposition of $K_{10}$.

Of the ten cubic isofactors of $K_{10}$ we have presented, five are planar and five are not. This leads us to the following observation. It is known that the thickness of $K_{10}$ is 3, that is, $K_{10}$ can be decomposed into at least three planar graphs. The above statement shows that there exists a decomposition giving the thickness of $K_{10}$ which is an isomorphic decomposition. A similar question can be asked about other complete graphs for which their thickness divides its size.
CHAPTER III

ISOMORPHIC DECOMPOSITIONS OF GRAPHS INTO LINEAR FORESTS

Section 3.1

Isographs

Since every proper subgraph of a cycle or a path is a linear forest, our first goal here is to study the isomorphic decompositions of these two classes of graphs. This will prove to be useful in decomposing certain graphs whose decompositions into cycles or paths is already known.

Theorem 3.1 If G is a cycle of length \( q \geq 3 \) and F is any nonempty subgraph of G having size m, where \( m|q \), then G is F-decomposable.

Proof: Let \( F_1, F_2, \ldots, F_k \) be the nonempty components of F, where \( q(F_i) = q_i \), \( i = 1, 2, \ldots, k \), so that \( \sum_{i=1}^{k} q_i = m \). To each edge of G we assign one of the colors 1, 2, \ldots, k in the following way. Starting at any vertex, we proceed counterclockwise about the cycle and color the first \( q_1 \) edges 1, the next \( q_1 \) edges are colored 2 and so on until in the kth step, \( q_1 \) edges are colored with k. Thus far, \( kq_1 \) consecutive edges of the cycle have been colored. We now continue along the cycle as above.
where \( q_1 \) is replaced by \( q_2 \). We repeat this process \( k \) times until all edges are colored. Clearly, all those edges colored with \( i \) \((i = 1, 2, \ldots, k)\) account for the edges of a linear forest isomorphic to \( F \). Therefore \( G \) is \( F \)-decomposable.

This proof technique can also be used to verify the following result.

**Corollary 3.1a.** If \( G \) is a path of length \( q (> 1) \) and \( F \) is any nonempty subgraph of \( G \) having size \( m \), where \( m | q \), then \( G \) is \( F \)-decomposable.

The two preceding results suggest the following definition. If a graph \( G \) is \( H \)-decomposable for each subgraph \( H \) of \( G \) whose size divides the size of \( G \), then we call \( G \) an isograph. Trivially, any graph whose size is prime is an isograph. From the above results it follows that cycles and nontrivial paths are isographs. These are not the only nontrivial examples of connected isographs since it can be shown that \( K(3,3) \) is also an isograph.

Recall that if we are given a set \( \mathcal{F} \) of graphs, an \( \mathcal{F} \)-decomposition of a nonempty graph \( G \) is a decomposition of \( G \) such that each part is isomorphic to a member of \( \mathcal{F} \).

**Corollary 3.1b.** Let \( n \geq 3 \) be a fixed integer. If \( G \) is a graph having a \( \{C_n, P_n+1\} \)-decomposition, then \( G \) is \( F \)-decomposable for any nonempty linear forest \( F \) without isolates, whose size is a proper divisor of \( n \).
Proof: Suppose that the linear forest $F$ has $k$ edges and $k$ divides $n$ $(0 < k < n)$. First we decompose the graph $G$ into copies of $C_n$ and $P_{n+1}$. Next, each copy of either $C_n$ or $P_{n+1}$ in that decomposition is in turn isomorphically decomposed into the forest $F$, since $C_n$ and $P_{n+1}$ are isographs by the previous theorem and corollary.

Section 3.2

Isomorphic Decompositions of Complete Graphs into Linear Forests

In 1859, M. Reiss [50] discovered a combinatorial theorem which, when expressed in graph theory terminology, stipulates that the complete graph $K_{2n}$ is 1-factorable, or, equivalently, $K_{2n}$ is $nK_2$-decomposable for every $n \geq 1$. This is surely one of the first published result on isomorphic decompositions of graphs. In our next result we will see that Reiss's Theorem can be generalized to linear forests.

Theorem 3.2. If $F$ is a linear forest of size $n$ without isolates, then $K_{2n}$ is $F$-decomposable.

Proof: Since the result is obvious for $n = 1$, we assume that $n \geq 2$. Arrange $2n - 1$ of the vertices $v_1, v_2, \ldots, v_{2n-1}$ of $K_{2n}$ cyclically, in counterclockwise order about a regular
(2n-1)-gon. Since $K_{2n-1}$ is a circulant we recall from Section 1.2 that the length of an edge $v_iv_j$ is $d(v_iv_j) = \min(|i-j|, |2n-1-i+j|)$, for $i, j \in \{1, 2, \ldots, 2n-1\}$. Next, we place the remaining vertex $v_o$ in the center of the (2n-1)-gon.

For the purpose of this proof we define the length of every edge incident with $v_o$ as 0. Note that for each $i \in \{0, 1, \ldots, n-1\}$, our representation of $K_{2n}$ contains $2n - 1$ edges of length $i$. By an argument similar to the one used in proving Lemma 1.1, it follows that by selecting $n$ edges of distinct lengths (actually the length 0 has been added) an isomorphic part of $K_{2n}$ is formed. The other parts of the decomposition are obtained from $H$ by the consecutive powers of the permutation

$$f = (v_o)(v_1 v_2 \ldots v_{2n-1}),$$

or, equivalently, in our geometrical model, by counterclockwise rotations around $v_o$ in an angle of $2\pi/(2n-1)$ radians. With this background at hand, to complete the proof it suffices to show that the subgraph $H$ can be chosen so that it is isomorphic to our given linear forest $F$. We now describe two paths $P$ and $Q$ of length $n$ in $K_{2n}$. If $n$ is even, then

$$P: v_o, v_1, v_{2n-1}, v_2, v_{2n-2}, v_3, \ldots, v_{n/2}, v_{3n/2}$$

and

$$Q: v_o, v_n, v_{n+1}, v_{n-1}, v_{n+2}, v_{n-2}, \ldots, v_{(n+2)/2}, v_{3n/2}$$

while if $n$ is odd, then
Assume that the linear forest (without isolated vertices) is given by

\[ F = P_{n_1+1} \cup P_{n_2+1} \cup \ldots \cup P_{n_k+1}, \]

where \( \sum_{i=1}^{k} n_i = n \). Finally, a subgraph \( H \) isomorphic to \( F \) is induced by the following edges of \( K_{2n} \): the first \( n_1 \) edges of \( P \), edges \( n_1 + 1 \) through \( n_2 \) of \( Q \), edges \( n_2 + 1 \) through \( n_3 \) of \( P \), etc. until the last \( n_k \) edges of \( Q \) are taken if \( k \) is even or the last \( n_k \) edges of \( P \) if \( k \) is odd. The edges chosen have distinct lengths by the definition of \( P \) and \( Q \); thus the proof is complete.

Corollary 3.2a. If \( F \) is a nonempty linear forest of size \( k \) and without isolates, where \( k \) is a proper divisor of \( n \), then \( K_{2n} \) is \( F \)-decomposable.

Proof: From Theorem 3.2, \( K_{2n} \) is \( P_{n+1} \)-decomposable; therefore the proof is completed by applying Corollary 3.1b.
The preceding theorem and its proof are illustrated in Figure 3.1 for $2n = 12$ and $F = P_2 \cup P_3 \cup P_4$. The labeling of the vertices of $K_{12}$ is shown along with the edges that induce the initial copy of $F$.

**Figure 3.1** A step in the construction of a $F$-decomposition of $K_{12}$ for $F = P_2 \cup P_3 \cup P_4$

**Corollary 3.2b.** The complete graph $K_{2n+1}$ is cyclically $F$-decomposable for any linear forest $F$ of size $n(\geq 1)$ and without isolates.

**Proof:** By removing the central vertex $v_o$ in the proof of Theorem 3.2 along with the first edge in the paths $P$ and $Q$, the same construction applies, except that now we do not have edges of length 0.
Theorem 3.3 If \( k \) is a proper positive divisor of \( 2n + 1 \), then \( K_{2n+1} \) is \( F \)-decomposable for any linear forest \( F \) of size \( k \) and with no isolates.

**Proof:** It is well-known that \( K_{2n+1} \) can be decomposed into hamiltonian cycles; therefore, the theorem follows by applying Corollary 3.1b. \( \square \)

Theorem 3.4 If \( k \) is a positive divisor of \( 2n - 1 \), then \( K_{2n} \) is \( F \)-decomposable for any linear forest \( F \) of size \( k \) and without isolates.

**Proof:** Since \( K_{2n} \) is \( P_{2n} \)-decomposable (see [5] p. 168) the theorem follows from Corollary 3.1b. \( \square \)

Section 3.3

Isomorphic Decompositions of Complete Bipartite Graphs into Linear Forests

Sotteau [58] proved that given positive even integers \( m, n \) and \( k \) such that \( k \leq 2 \min(m, n) \) and \( k \mid mn \), then \( K(m, n) \) is \( C_k \)-decomposable. From this fact and Corollary 3.1b the next result is immediate.

**Theorem 3.5** Let \( \{m, n\} \) and \( k \) be positive even integers such that (i) \( k \leq 2 \min\{m, n\} \); (ii) \( k < m + n \) and (iii) \( k \) (properly) divides \( mn \). Then for each linear forest \( F \) of size \( k \) and
without isolates, the complete bipartite graph $K(m,n)$ is $F$-decomposable.

**Corollary 3.5** The graph $K(2n, 2n)$ is $F$-decomposable for any non-empty forest $F$ of size $k$ and without isolates, where $k$ divides $2n$.

The related problem of determining all isomorphic decompositions of $K(2n + 1, 2n + 1)$ into linear forests of size $2n + 1$ is open. A result in this direction, however, is the following.

**Theorem 3.6** The complete bipartite graph $K(2n + 1, 2n + 1)$ is $P_{2n+2}$-decomposable.

**Proof:** Let $A = \{a_1, a_2, \ldots, a_{2n+1}\}$ and $B = \{b_1, b_2, \ldots, b_{2n+1}\}$ be the partite sets of $K(2n+1, 2n+1)$. Consider the complete graph $K_{2n+1}$ whose vertices $v_1, v_2, \ldots, v_{2n+1}$ are arranged cyclically about a regular $(2n+1)$-gon. From Corollary 3.2b, $K_{2n+1}$ is cyclically $P_{n+1}$-decomposable. We use this $P_{n+1}$-decomposition of $K_{2n+1}$ to construct a $P_{2n+2}$-decomposition of $K(2n+1, 2n+1)$. Let the path $G_i$ described as

$v_{i_1}, v_{i_2}, \ldots, v_{i_{n+1}}$

be any copy of $P_{n+1}$ in the cyclic $P_{n+1}$-decomposition of $K_{2n+1}$. Corresponding to $G_i$ in $K_{2n+1}$, we define a path $G'_i$ in $K(2n+1, 2n+1)$ whose edge set consists of the single edge $a_{i_1} b_{i_1}$ together with the edges $a_{i_x} b_{i_y}$ and $a_{i_y} b_{i_x}$ for each edge $v_x v_y$ belonging
to $G_i$. Since the decomposition in $K_{2n+1}$ is cyclic, the permutation

$$g = (a_1 a_2 \ldots a_{2n+1})(b_1 b_2 \ldots b_{2n+1})$$

and its powers give the isomorphisms between $G_i$ and the other copies of $P_{2n+2}$ in the $P_{2n+2}$-decomposition of $K(2n+1, 2n+1)$. ■

In Figure 3.2 we illustrate the above theorem by showing a copy $G_i$ of $P_3$ in a cyclic $P_3$-decomposition of $K_5$ and its related copy $G_i'$ of $P_6$ in the $P_6$-decomposition of $K(5,5)$.

![Diagram of graph decomposition](image)

**Figure 3.2 The first step in the construction of a $P_6$-decomposition of $K(5,5)$**

If in the proof of the above theorem we consider, instead of the path $P_{n+1}$, any bipartite graph $H$ such that $K_{2n+1}$ is cyclically $H$-decomposable, then we can conclude, by the same construction, that $K(2n+1, 2n+1)$ is $(2H + e)$-decomposable, where $e$ is any edge joining two corresponding vertices of the two copies of $H$. By deleting from $K(2n+1, 2n+1)$ the edges $a_i b_i$ described in the proof we can state that the graph obtained from $K(2n+1, 2n+1)$
by removing the edges of a 1-factor is $H$-decomposable whenever $H$ is bipartite of size $n$ and $K_{2n+1}$ is $H$-decomposable.

Based on results obtained in working with graphs of small order, we propose the following conjecture.

**Conjecture 3.1** For every positive integer $n$, $K(n, n)$ is $F$-decomposable for any linear forest $F$ of size $n$ and having no isolates.

It may very well be the case that a result stronger than that suggested by the preceding conjecture holds.

**Conjecture 3.2** For every positive integer $n$, $K(n, n)$ is $H$-decomposable for any of its subgraphs $H$ of size $k$, where $k$ is a positive divisor of $n$. 
CHAPTER IV

ISOMORPHIC FACTORIZATION OF COMPLETE GRAPHS INTO CIRCULANTS

Section 4.1

The Existence Problem

Since the complete graph $K_n^{(n > 1)}$ is itself a circulant (see Section 1.2), a natural problem related to our subject is that of determining all isomorphic factorizations of $K_n$ into circulants. More precisely, we are seeking decompositions of $K_n$ of the type

$$K_n = H_1 \Theta H_2 \Theta \ldots \Theta H_k, \quad (4.1)$$

where each part $H_i$ is a circulant $G(n; S_i)$, and so that the sets $S_1, S_2, \ldots, S_k$ form a partition of $\{1, 2, \ldots, \lfloor n/2 \rfloor\}$ and all of the $k$ factors $H_i$ are isomorphic to some fixed circulant $H$. We refer to such a factorization in (4.1) as a circulant isomorphic factorization of $K_n$. It is apparent that for a nontrivial (i.e., $k > 1$) circulant isomorphic factorization of $K_n$ to exist, it is necessary that $n$ be odd, $n \geq 5$, and that all the length sets $S_i$ have the same cardinality $m$, where $2m$ divides $n - 1$. It follows that each factor $H_i$ in (4.1) must be regular of even degree $2m$ and that $(n-1)/2 = km$, where $k$ is the number of factors $H_i$. 

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Whenever a complete graph is isomorphically factored into circulants, then we have a factorization into vertex-symmetric graphs. On the other hand, a vertex-symmetric isofactor of a complete graph need not be a circulant, as we shall now see.

It is known that circulants are Cayley graphs of finite cyclic groups. The noncyclic group $Z_3 \oplus Z_3$ has the graph $G = K_3 \times K_3$ as a Cayley graph. By considering the cycle structure of $G$ and the generators of $Z_3$, we see that $G$ is not a circulant. On the other hand, $G$ is self-complementary ($K_3 = G \oplus G$). The graph $G$ belongs to the following general construction attributed to Paley [44]. Let $n$ be a positive integer power of a prime number $p$, so that $n \equiv 1 \pmod{4}$. Let $G$ be a graph whose vertices are the elements of the Galois field $GF(n)$ and two vertices $x$ and $y$ are adjacent if and only if $x - y$ is a quadratic residue (see [32], [12]). If $g$ is not a quadratic residue, then the permutation $x \mapsto gx$ is an isomorphism between $G$ and its complement $\overline{G}$, so that $G$ is an isofactor of $K_n$. These are the so-called Paley graphs. Since $K_3 \times K_3$ is a symmetric graph (vertex-symmetric and edge-symmetric), we conclude that in general not all symmetric isofactors of $K_n$ are circulants. Paley graphs have been of interest because they are examples of strongly regular graphs, that is, there exist constants $b$ and $c$ such that every pair of adjacent vertices possess $b$ common neighbors and every pair of non-adjacent vertices have $c$ common neighbors. In the next result we generalize the construction for Paley graphs.
Theorem 4.1. Let $H$ be a subgroup of even order of the multiplicative (cyclic) group $M$ of the Galois field $GF(n)$, where $n$ is a power of an odd prime $p$. Let $G$ be that graph whose $n$ vertices are the elements of $GF(n)$ and such that two vertices $x$ and $y$ are adjacent if and only if $x - y \in H$. Then $G$ is an isofactor of the complete graph $K_n$.

Proof: Since $H$ is a cyclic group of even order, it has a unique subgroup of order 2. On the other hand $\{1, -1\}$ is the unique subgroup of $M$ having order 2. Therefore $-1 \in H$, implying that $x - y \in H$ if and only if $y - x \in H$; therefore the edges of $G$ are well-defined. If $g$ is a generator of $M$, then $M$ can be expressed as the disjoint union of the cosets $H_0, H_1, \ldots, H_k$, where $H_i = g^iH$ ($i = 0, 1, \ldots, k$) and $k + 1 = (n-1)/|H|$. Now we proceed to color the edges of $K_n$ according to the following rule. Assign color $i$ to the edge $xy$ of $K_n$ if $x - y \in H_i$, for every $i = 0, 1, \ldots, k$. Denote by $G_i$ that subgraph of $K_n$ induced by the edges colored $i$, so that $G_0$ is in fact the graph $G$ defined in the statement of the theorem. If $k = 0$, then $G = K_n$ and $G$ is a trivial isofactor of $K_n$. Assume then that $k \geq 1$. Let $j$ be such that $0 < j \leq k$. Then the permutation on $V(K_n)$ defined by

$$x \rightarrow g^jx$$

is an isomorphism between $G$ and $G_j$. Since vertices $x$ and $y$ are adjacent in $G$ if and only if $x - y = h$, for some $h \in H$, it

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follows that $g^j(x - y) = g^j h$ which can be expressed as $g^j x - g^j y = g^j h$, that is, $g^j x$ and $g^j y$ are adjacent in $G_j$. ■

For the purpose of this chapter, we refer to the isofactors just constructed as generalized Paley graphs. As a consequence of this result we can solve the existence problem for circulant isomorphic factorizations of complete graphs of prime order.

Corollary 4.1. If $p = 2n + 1$ is a prime number and $r$ is an even divisor of $2n$, then there exists a circulant isomorphic factorization of $K_p$ into $2n/r$ circulants.

Proof: In this case the subgroup $H$ referred to in the previous theorem is the subgroup having order $r$ of the multiplicative group of $GF(p)$. The length set of the circulant isofactor $G$ is composed of half the elements of $H$, namely those numbers $x$ of $H$ satisfying $1 \leq x \leq n$. ■

Section 4.2

The Classification Problem

In 1967, Turner [62] showed that if a nonempty graph of prime order is vertex-symmetric, then it is a circulant. Therefore we can state the following result.

Theorem 4.2 Let $p$ be an odd prime. If $G$ is a vertex-symmetric isofactor of $K_p$, then $G$ is a circulant.
A deeper result in this direction is due to Chao [17] and it establishes, for an odd prime \( p \), the following equivalence of the statements (i) and (ii):

(i) \( G \) is a symmetric graph with \( p \) vertices, each having degree \( r \geq 1 \);

(ii) the integer \( r \) is an even divisor of \( p - 1 \) and \( G \) is isomorphic to the graph whose vertices are the elements of \( \text{GF}(p) \) and whose edges are the pairs \( \{a, a + h\} \), where \( a \in \text{GF}(p) \) and \( h \) belongs to the unique subgroup of order \( r \) of the multiplicative group of \( \text{GF}(p) \).

An alternative proof of this result was given later by Berggren [6]. As a consequence of this result and using the construction given in Theorem 4.1, we have our next theorem.

**Theorem 4.2.** Let \( G \) be a nonempty graph of prime order \( p \geq 3 \). Then \( G \) is symmetric if and only if \( G \) is isomorphic to a generalized Paley graph.

We know now that if we are given a circulant of prime order \( p \), we can determine (by considering its length set) whether the circulant is symmetric. Next we move to the more general problem of characterizing the circulant isofactors of prime order in terms of their length sets.

Let \( G(p; S) \) and \( G(p; S') \) be circulants. Then \( S \) and \( S' \) are said to be **multiplicatively related** if, when considered as subsets of the multiplicative group of \( \text{GF}(p) \), the following holds, (where the indicated operations are those in \( \text{GF}(p) \)):
for some nonzero element \( \mu \) of \( GF(p) \).

In general, the isomorphism problem for circulants remains open. Nevertheless, Turner [62] proved that when \( p \) is an odd prime, \( G(p; S) \) and \( G(p; S') \) are isomorphic if and only if \( S \) and \( S' \) are multiplicatively related. An immediate consequence of this result is the following characterization.

**Theorem 4.3.** Let \( p = 2n + 1 \) be a prime. The circulant \( G(p; S) \) is an isofactor of \( K_p \) if and only if the set \( \{1, 2, \ldots, n\} \) can be partitioned into sets, each being multiplicatively related to \( S \).

### Section 4.3

**Applications to Isomorphic Ramsey Numbers**

With the publication of Ramsey's theorem [48] in 1930, one of the more elusive branches of combinatorics was born. In a sense, Ramsey's theorem is a generalization of the well-known pigeonhole principle in combinatorics and it proves the existence of some numbers satisfying certain conditions. These are the so-called Ramsey numbers. Since then the search for those numbers has captivated the attention of many mathematicians, to the extent that in 1983 a complete issue of the *Journal of Graph Theory* was dedicated to Ramsey Theory honoring the memory of Frank Ramsey. A recent survey on the topic was done by F.R.K. Chung and C.M. Grinstead [23].

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For $k \geq 2$ and $m \geq 2$, the diagonal Ramsey number $r(k; m)$ is defined as the smallest positive integer $n$ such that if $p \geq n$, then for every decomposition of $K_p$ into $m$ parts, at least one of the parts contains $K_k$. There are other types of Ramsey numbers but for the purpose of relating the concept to isomorphic decompositions of complete graphs, the diagonal Ramsey number is most appropriate.

The isomorphic Ramsey number $ir(k; m)$ is defined as the smallest positive integer $n$ such that if $p \geq n$, then for every isomorphic decomposition of $K_p$ into $m$ parts, $K_k$ is contained in every part. This definition was first introduced by Harary and Robinson [34].

Trivially, $ir(2; m) = r(2; m) = 2$. For the case where $k > 2$, only three diagonal Ramsey numbers are actually known, namely, $r(3; 2) = 6$, $r(4; 2) = 18$ and $r(3; 3) = 17$, and in this three cases, $ir(k; m) = r(k; m)$.

In every case the search for a diagonal Ramsey number $t = r(k; m)$ is divided into two problems. On the one hand, lower bounds for $t$ are sought by finding a decomposition of some complete graph $K_n$ into $m$ parts with the property that every part does not contain $K_k$. This shows that $t > n$, so that $n + 1$ is a lower bound for the diagonal Ramsey number $t$. To show that $s$ is an upper bound for $t$, it becomes necessary to verify that for every decomposition of $K_s$ into $m$ parts, some part contains $K_k$. 

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In 1979, Clapham [24] proved that $ir(7; 2) \geq 114$ by finding a self-complementary graph of order 113 that does not contain $K_7$. In 1983, Guldan and Tomasta [33], extending Clapham's construction, were able to show that $ir(10; 2) \geq 458$ and $ir(11; 2) \geq 542$.

R. Mathon [43] has drastically improved the current bounds for some diagonal Ramsey numbers. His technique is based on association schemes.

We generalize a special case of the method developed by Guldan and Tomasta [33] to isomorphically decompose $K_p$, where $p$ is prime, into $m$ parts whenever certain divisibility conditions are satisfied. With the aid of computers, we have found nontrivial bounds for some numbers $ir(k; m)$. The algorithm is as follows.

Suppose $p = 2mt + 1$ is a prime number. Our goal is to decompose $K_p$ into $m$ isomorphic circulants of degree $2t$. Let $M$ be the multiplicative group of $GF(p)$. To simplify the computations and since the length set of a circulant is taken from the set $L = \{1, 2, \ldots, mt\}$, we assign to $L$ the structure of the quotient group $M/\{1, -1\}$ by identifying every member $\{x, -x\}$ of this quotient by $x$ if $x \in L$. Thus the multiplication $\ast$ in $L$ follows the rule

$$x \ast y = \min\{x \cdot y, p - x \cdot y\},$$

where the symbol $\cdot$ is the multiplication in $GF(p)$. 

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Let $H$ be the (cyclic) subgroup of $L$ having order $m$. Finally we choose one representative from each member of the quotient $L/H$, forming a set $S$. We claim that the circulant $G(p; S)$ is an isofactor of $K_p$ of degree $2t$. Let $h$ be a generator of $H$ and $g$ a generator of $L$. Then every coset of $L/H$ has the form

$$\{g^ih, g^ih^2, \ldots, g^ih^m = g^1\}$$

for some $i \in \{0, \ldots, t-1\}$. Therefore the sets $S, hS, \ldots, h^{m-1}S$ are pairwise multiplicatively related and they form a partition of $L$. Thus, by Theorem 4.3,

$$K_p = G(p; S) \oplus G(p; hS) \oplus \cdots \oplus G(p; h^{m-1}S)$$

is a circulant isomorphic decomposition of $K_p$.

If we place the elements of $L$ into the matrix $A = (a_{ij})$, where $a_{ij} = g^{i-1}h^{j-1}$, then in the first row we have the elements of $H$ and the other rows are formed by the cosets $g^iH$ for $i = 1, 2, \ldots, t-1$. Thus a length set $S$ of an isofactor of $K_p$ is formed by choosing one element from each row. Since each set $S$ has at most $p-1$ translates, this procedure gives at least $m^{t-1}/2t$ circulant isofactors of degree $2t$. Using a backtrack technique a computer program selects, step by step, an element from each row in such a way that the elements already selected do not form a complete subgraph of order $k$. A solution is found when $t$ elements are selected.
In Table 4.1 the \((k, m)\)-entry is the largest order \(p\) of any complete graph we have found using the above construction, such that \(K_p\) has a circulant isomorphic factorization and each factor does not contain \(K_k\) as a subgraph.

<table>
<thead>
<tr>
<th></th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>79</td>
<td>193</td>
<td>401</td>
</tr>
<tr>
<td>5</td>
<td>181</td>
<td>401</td>
<td>691</td>
</tr>
<tr>
<td>6</td>
<td>277</td>
<td>569</td>
<td></td>
</tr>
</tbody>
</table>

**TABLE 4.1**

Each of the values in the Table 4.1 gives rise to a strict lower bound for \(r(k; m)\) as well as for \(ir(k; m)\).

We note in closing this chapter that some of the pioneering work in the use of circulants to find lower bounds for Ramsey numbers was done by Kalbfleisch [38].

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CHAPTER V
RANDOMLY DECOMPOSABLE GRAPHS

Once it is determined that a certain graph $G$ is $H$-decomposable for some graph $H$, one may address the task of actually performing a decomposition of $G$ into copies of $H$. It may be that an $H$-decomposable graph $G$ has the property that for any copy $H_1$ of $H$, the graph $G - H_1$ is $H$-decomposable; for any copy $H_2$ of $H$ in $G - H_1$, the graph $G - H_1 - H_2$ is $H$-decomposable and so on. Certainly if $G$ has this property, then an $H$-decomposition of $G$ will be easier to find. We formalize this idea in the following definition. A graph $G$ is said to be randomly $H$-decomposable if every family of edge-disjoint subgraphs of $G$, each subgraph isomorphic to $H$, is a subfamily of an $H$-decomposition of $G$. We also say that $G$ can be randomly decomposed into the graph $H$.

Clearly, every nonempty graph $G$ is randomly $K_2$-decomposable as well as randomly $G$-decomposable. Less trivial examples are

(1) $K(n,n)$, $n \geq 2$, which is randomly $nK_2$-decomposable,
(2) the Petersen graph, which is randomly $F_2$-decomposable (where $F_2$ is displayed in Figure 2.2),
(3) $K_n$, where $n \equiv 0$ or $1 \pmod 4$, which is randomly $H$-decomposable, where $H$ is self-complementary of order $n$.

A characterization of randomly $H$-decomposable graphs for a given graph $H$ seems to be a difficult problem. As at least a step
in this direction, we will develop some theory which will lead to a description of all those graphs $G$ that are randomly $H$-decomposable whenever $H$ has size 2.

The following result [20] will prove to be useful.

**Theorem 5.1** (Chartrand, Polimeni and Stewart) Let $G$ be a connected graph with $2n$ odd vertices, $n \geq 1$. Then $E(G)$ can be partitioned into subsets $E_1, E_2, \ldots, E_n$ so that for each $i$, $\langle E_i \rangle$ is a trail connecting odd vertices and such that at most one of these trails has odd length.

**Theorem 5.2** Let $G$ be a nontrivial connected graph of size $q$. Then $G$ can be decomposed into $a$ copies of $K_2$ and $b$ copies of $P_3$ if and only if $q = a + 2b$.
It is convenient to determine those graphs of size \( 2n(\geq 2) \) that can be decomposed into one copy of \( 2K_2 \) and \( n - 1 \) copies of \( P_3 \). We begin with a few lemmas.

**Lemma 5.1** Every connected graph \( H \) with at least two edges has an edge \( f \) such that \( H - f \) contains exactly one nonempty component.

**Proof:** If \( H \) is a tree, then it has a terminal edge \( e \). The graph \( H - e \) contains exactly one nonempty component. If \( H \) is not a tree then it contains a cycle \( C \). Removing an edge \( f \) from \( C \) results in a connected graph. ■

**Lemma 5.2** If \( G \) is a graph of size \( 2n(\geq 2) \) that can be decomposed into \( k \) copies of \( 2K_2 \) and \( n - k \) copies of \( P_3 \), then \( G \) has at most \( 2k \) components of odd size.

**Proof:** Suppose that \( G \) is decomposed into \( k \) copies of \( 2K_2 \) and \( n - k \) copies of \( P_3 \) and assume that \( G \) has \( m \) components of odd size. Since \( G \) has a \( \{P_3, 2K_2\} \)-decomposition each of the \( m \) components contributes at least one edge to a copy of \( 2K_2 \). Since exactly \( 2k \) edges belong to the \( k \) copies of \( 2K_2 \), it follows that \( m \leq 2k \). ■

**Lemma 5.3** Let \( G \) be a connected graph of size at least 3 that is not a cycle or a star. Then \( G \) has two nonadjacent edges \( e \) and \( f \) such that \( G - e - f \) has exactly one nontrivial component.
Proof: We consider two cases, according to whether $G$ does or does not possess a bridge.

Case 1. Assume that $G$ has a bridge $b = uv$.

Then $G - b$ has two components $G_1$ and $G_2$, where $u \in V(G_1)$ and $v \in V(G_2)$. Now we consider two subcases.

Subcase 1.1. Suppose that $G_1$ and $G_2$ are nonempty graphs. Choose vertices $x_1 \in V(G_1)$ and $x_2 \in V(G_2)$ such that the distance $d(x_1, x_2)$ between $x_1$ and $x_2$ satisfies

$$d(x_1, x_2) = \max\{d(x, y) | x \in V(G_1), y \in V(G_2)\}.$$ 

Let $y_i \in V(G_i)$ such that $x_i$ is adjacent to $y_i$, for $i = 1, 2$. The existence of the bridge $b$ implies that $y_1 \neq y_2$.

Note that $e = x_1y_1$ and $f = x_2y_2$ satisfy the conclusion of the lemma. In fact, neither $x_1$ nor $x_2$ is a cut-vertex of $G$ by the way they were chosen. If $G - e - f$ has more than one nonempty component, then $x_1$ or $x_2$ is a cut-vertex.

Subcase 1.2. Assume that one of the components of $G - b$, say $G_1$, is a single vertex $u$. Therefore $\deg_G u = 1$. Let $x$ be a vertex of $G$ whose distance from $u$ is maximum. Since $G$ is not a star, there exists a vertex $y$ distinct from $v$ such that $x$ is adjacent to $y$. Then the edges $e = uv$ and $f = xy$ satisfy the property stated in the lemma.
Case 2. Assume that \( G \) does not contain bridges.

Therefore \( G \) contains at least four edges. Since \( G \) is not a cycle, it contains a cycle \( C \) and a vertex \( x \) not on \( C \) that is adjacent to a vertex \( y \) of \( C \). Then \( G - xy \) is a connected graph containing the cycle \( C \). Now in the graph \( G - xy \), remove an edge \( f \) of \( C \) that is not adjacent to \( xy \). It follows that \( G - xy - f \) is connected.

Lemma 5.4 Let \( G \) be a graph of size \( 2n(\geq 2) \) that has been decomposed into one copy \( H \) of \( 2K_2 \) and \( n - 1 \) copies of \( P_3 \), where \( E(H) = \{e, f\} \). Then \( e \) and \( f \) belong to different components of \( G \) if and only if \( G \) has exactly two components of odd size.

Proof: If \( e \) and \( f \) belong to different components in the given decomposition of \( G \), then the components containing \( e \) and \( f \) have odd size. The remaining components, if any, necessarily have even size; thus \( G \) has exactly two components of odd size.

Assume now that \( G \) has exactly two components \( G_1 \) and \( G_2 \) of odd size. Then neither \( G_1 \) nor \( G_2 \) is \( P_3 \)-decomposable. Thus \( G_1 \) and \( G_2 \) must contain the edges \( e \) and \( f \).

Lemma 5.5. Let \( G \) be a graph of size \( 2n(\geq 2) \). Then \( G \) can be decomposed into one copy of \( 2K_2 \) and \( n - 1 \) copies of \( P_3 \) if and only if

(i) exactly two components of \( G \) have odd size, or
(ii) all components of \( G \) have even size and at least one nonempty component is different from a star or the cycle \( C_4 \).

Proof: Assume that \( G \) has been decomposed into one copy of \( 2K_2 \) and \( n-1 \) copies of \( P_3 \). By Lemma 5.2, \( G \) has at most two components of odd size. If \( e \) and \( f \) lie in different components of odd size, then (i) follows by Lemma 5.4. Assume then that \( e \) and \( f \) belong to the same component \( C \) but (ii) does not hold. By Lemma 5.4, all components of \( G \) have even size; therefore every component of \( G \) must be a star or the graph \( C_4 \). Certainly if \( C \) is a star, it cannot contain two independent edges and if \( C \) is isomorphic to \( C_4 \), then taking a copy of \( 2K_2 \) from \( C \) results in a graph that cannot be decomposed into \( n-1 \) copies of \( P_3 \), producing the desired contradiction.

For the converse, suppose that \( G \) satisfies condition (i), that is, suppose that \( G \) contains exactly two components \( C_1 \) and \( C_2 \) of odd size. Using Lemma 5.1 we can remove one edge \( e \) from \( C_1 \) and one edge \( f \) from \( C_2 \) so that \( G - e - f \) has only components of even size. Therefore by Theorem 5.1, \( G - e - f \) is \( P_3 \)-decomposable and the edges \( e \) and \( f \) account for the copy of \( 2K_2 \).

Suppose next that \( G \) satisfies (ii), that is, assume that all components of \( G \) have even size and that there is a nonempty component \( C \) that is not a star or \( C_4 \). If \( C \) is a cycle of size \( 2k \geq 6 \), then we can choose two nonadjacent edges \( e \) and \( f \) from \( C \) such that each component of \( C - e - f \) has even size. Hence
C - e - f is \( P_3 \)-decomposable and \( G \) has the required decomposition. If \( G \) is not a cycle, then, by Lemma 5.3, \( G \) has two non-adjacent edges \( e \) and \( f \) such that \( G - e - f \) has exactly one nontrivial component, necessarily of even size. So by Theorem 5.1, \( G - e - f \) is \( P_3 \)-decomposable and the theorem is proved.

**Theorem 5.3** A graph \( G \) is randomly \( P_3 \)-decomposable if and only if each component of \( G \) is isomorphic to \( C_4 \) or a star of even size.

**Proof:** It is clear that if each component of \( G \) is isomorphic to \( C_4 \) or a star of even size, then \( G \) is randomly \( P_3 \)-decomposable.

Suppose now that \( G \) is randomly \( P_3 \)-decomposable. Since \( G \) is already \( P_3 \)-decomposable, each of its components has even size. Assume to the contrary, that some component of \( G \) is not isomorphic to \( C_4 \) or a star of even size. Then by Lemma 5.5, \( G \) can be decomposed into one copy of \( 2K_2 \) and \( n - 1 \) copies of \( P_3 \), where \( 2n \) is the size of \( G \). Certainly these \( n - 1 \) copies of \( P_3 \) do not belong to any \( P_3 \)-decomposition of \( G \), contradicting the hypothesis that \( G \) is randomly \( P_3 \)-decomposable.

In a private communication, Y. Caro states that he has established the following theorem in 1979. Since no published proof of this result exists, we include a verification in order to be complete.

**Theorem 5.4** Let \( G \) be a graph of size \( 2m > 0 \) and without isolated vertices. Then \( G \) is \( 2K_2 \)-decomposable if and only if \( \Delta(G) \leq m \) and \( G \) is not isomorphic to \( K_3 \cup K_2 \).

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Proof: The necessity is clear. It suffices to show that whenever $G \not\cong K_3 \cup K_2$ and $\Delta(G) \leq m$ then $G$ is $2K_2$-decomposable.

Suppose the result is false. Then among all counterexamples, let $G$ be the family of those graphs having minimum (even) size and let $G$ be a member of $G$ of minimum order. Therefore

(i) $G \not\cong K_3 \cup K_2$,
(ii) $G$ has $q = 2n$ edges for some integer $n \geq 1$,
(iii) $\Delta(G) \leq q/2 = n$,
(iv) $G$ does not contain isolated vertices, and
(v) $G$ is not $2K_2$-decomposable.

Let $W$ be the set of those vertices, if any, of $G$ having degree $n$. If $|W| = k \geq 1$, then let $W = \{w_1, w_2, \ldots, w_k\}$ and $t_i = \deg_{\langle W \rangle} w_i$ for $1 \leq i \leq k$. The subgraph $\langle W \rangle$ then has

$$\frac{1}{2} \sum_{i=1}^{k} t_i$$
edges, and $\sum_{i=1}^{k} (n-t_i)$ edges join $\langle W \rangle$ and $G - W$.

Therefore

$$q = 2n \geq \frac{1}{2} \sum_{i=1}^{k} t_i + \sum_{i=1}^{k} (n-t_i) = kn - \frac{1}{2} \sum_{i=1}^{k} t_i$$
or

$$\frac{1}{2} \sum_{i=1}^{k} t_i \geq (k-2)n. \quad (5.1)$$

However,

$$q = 2n \geq \frac{1}{2} \sum_{i=1}^{k} t_i. \quad (5.2)$$

Combining (5.1) and (5.2), we obtain $2n \geq (k-2)n$ so that

$$k \leq 4. \quad (5.3)$$

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Also, the number of edges in $\langle W \rangle$ is at most $k(k-1)/2$, that is,
\[ k \frac{k(k-1)}{2} \geq \frac{1}{2} \sum_{i=1}^{k} t_i. \] (5.4)

Inequalities (5.1) and (5.4) imply that
\[ n \leq \frac{k(k-1)}{2(k-2)}, \quad \text{when } k > 2. \] (5.5)

We now consider five cases, depending on the cardinality $k$ of $W$, which satisfies $0 \leq k \leq 4$ by (5.3). In each case it is shown that $G$ is $2K_2$-decomposable or that $G$ is isomorphic to $K_3 \cup K_2$, which produces a contradiction in either case. This will show that no such counterexample $G$ exists, completing the proof.

**Case 1. Assume** $k = 4$.

Inequalities (5.1) and (5.2) imply that
\[ \frac{1}{2} \sum_{i=1}^{4} t_i = 2n = q; \]
therefore $G = \langle W \rangle$. Further, inequality (5.5) gives us $n \leq 3$. For $n = 3$, the size of $G$ is 6 and its order is 4; so $G \cong K_4$, which is $2K_2$-decomposable. For $n = 2$, each vertex of $G$ has degree 2; therefore $G \cong C_4$, which is also $2K_2$-decomposable. For $n = 1$, we must have $G \cong 2K_2$, which is trivially $2K_2$-decomposable.

**Case 2. Assume** $k = 3$.

From (5.1) it follows that $\langle W \rangle$ contains at least $n$ edges but as $W$ has only three vertices, we must have $1 \leq n \leq 3$. 

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Subcase 2a. Suppose that \( n = 1 \). It is not possible to have a graph \( G \) of size 2 such that \( \Delta(G) = 1 \) and \( G \) has exactly three vertices of degree 1. Therefore, this subcase cannot occur.

Subcase 2b. Suppose that \( n = 2 \). Here \( \langle W \rangle \) contains at least two edges. The graph \( G \) has four edges and a vertex \( w \) of \( G \) has degree 2 if and only if \( w \in W \). If \( \langle W \rangle \) contains exactly two edges, then \( G = P_5 \), which is \( 2K_2 \)-decomposable. If \( \langle W \rangle \) contains exactly three edges, then \( G \cong K_3 \cup K_2 \).

Subcase 2c. Suppose that \( n = 3 \). From (5.1), we see that \( \langle W \rangle = K_3 \), \( G \) has six edges and \( W = \{ w \in V(G) | \deg_G w = 3 \} \). Under these conditions \( G \) is isomorphic to one of the two graphs shown in Figure 5.1. Both of these graphs, however, are \( 2K_2 \)-decomposable.

![Figure 5.1](image-url)
Case 3. Assume \( k = 2 \).

In this case it is not possible to have \( n = 1 \). For \( n = 2 \), the only possibilities for \( G \) are \( G = P_4 \cup K_2 \) and \( G = 2P_3 \), depending on whether the vertices \( w_1 \) and \( w_2 \) are or are not adjacent. In both cases the graphs are \( 2K_2 \)-decomposable.

Assume \( n \geq 3 \). Then there exist edges \( xw_1 \) and \( yw_2 \) such that \( x \neq y \). Let \( G' \) be that graph obtained by deleting from \( G \) the edges \( xw_1 \) and \( yw_2 \) together with any resulting isolated vertices. If \( G' = K_3 \cup K_2 \), then \( G \) is isomorphic to the graph of Figure 5.2, which has a \( 2K_2 \)-decomposition. In any other case, by

![Figure 5.2](image)

the minimality of \( G \) and because \( G' \) satisfies the conditions (i) -(iv), the graph \( G \) itself has a \( 2K_2 \)-decomposition. This decomposition together with the independent edges \( xw_1 \) and \( yw_2 \) form a \( 2K_2 \)-decomposition of \( G \).
Case 4. Assume \( k = 1 \).

Let \( W = \{w\} \). It is not possible to have \( n = 1 \); otherwise \( |W| = 4 \). Therefore \( n \geq 2 \). Let \( wx \) be an edge of \( G \). Since \( \text{deg } w = n \) and \( \text{deg } x < n \), there must exist some edge \( uv \) incident with neither \( w \) nor \( x \), so that the edges \( uv \) and \( wx \) are independent. The graph \( G' \) obtained by deleting from \( G \) edges \( uv \) and \( wx \) and any resulting isolated vertices is isomorphic to \( K_3 \cup K_2 \) only when \( G \) is isomorphic to one of the four graphs of Figure 5.3, each of which is \( 2K_2 \)-decomposable. If \( G' \neq K_3 \cup K_2 \) then \( G' \)

![Figure 5.3](image)

satisfies conditions (i) - (iv), implying that \( G' \) is \( 2K_2 \)-decomposable, as is \( G \).

Case 5. Assume \( k = 0 \).

Since \( G \) is not a star, it has two independent edges \( e \) and \( e' \) such that either \( G - e - e' \) is itself \( 2K_2 \)-decomposable or

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G - e - e' less any isolated vertices is isomorphic to $K_3 \cup K_2$.

In the latter case, $G$ is isomorphic to one of the three graphs shown in Figure 5.4. Thus in any case $G$ is $2K_2$-decomposable. 

![Figure 5.4](image)

We note that $3K_2$-decomposable graphs have recently been characterized by Bialostocki and Roditty [11].

Let $\mathcal{F}$ be the set of graphs defined by

$$\mathcal{F} = \{K_4, C_4, 2K_3, K_3 \cup K(1,3)\} \cup \{2nK_2, 2K(1,n)|n \geq 2\}. \quad (5.6)$$

Note that these graphs cannot be decomposed into one copy of $P_3$ and the remaining copies of $2K_2$. We will see that $\mathcal{F}$ is an exceptional family of graphs.

**Lemma 5.6.** Let $G$ be a graph of size $2n(\geq 2)$ and without isolates. Suppose that $G \notin \mathcal{F}$ (described in (5.6)). Then $G$ can be decomposed into one copy of $P_3$ and $n - 1$ copies of $2K_2$ if and only if $\Delta(G) \leq n + 1$. 

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Proof: First we note that the lemma holds for all graphs with at most six edges; so we assume henceforth that $G$ has at least eight edges, that is, $n \geq 4$.

Suppose that $G$ can be decomposed into $n - 1$ copies of $2K_2$ and one copy of $P_3$. If $G$ contains a vertex $v$ of degree $\Delta(G) \geq n + 2$, then at least $n$ of the edges incident with $v$ must belong to different copies of $2K_2$. This produces a contradiction, so that $\Delta(G) \leq n + 1$.

Suppose now that (1) $G \not\cong \emptyset$, (2) $G$ has no isolates and (3) $\Delta(G) \leq n + 1$. If $\Delta(G) \leq n - 1$, then since $G \not\cong 2nK_2$, the graph $G$ contains two adjacent edges whose removal results in a graph that can be decomposed into $2K_2$ (see Theorem 5.4), so the proof is complete. If $\Delta(G) = n$, then at most two adjacent vertices $x$ and $y$ have degree $n$. Since $G$ is not isomorphic to $2K(1,n)$ and the size of $G$ is $2n$, we can remove the edge $xy$ and any other edge adjacent to $xy$. The graph so obtained has maximum degree $n - 1$ and size $2(n - 1)$. The proof is completed by an application of Theorem 5.4.

If $\Delta(G) = n + 1$, then there exists only one vertex $x$ with degree $n + 1$. Delete two edges $xy$ and $xz$ from $G$, where $y$ and $z$ have the largest two degrees among those vertices adjacent with $x$. Since $\Delta(G - xy - xz) \leq n - 1$, we proceed as before to obtain the required decomposition of $G$. ■
Theorem 5.5 Let $G$ be a $2K_2$-decomposable graph without isolates and $G \neq 2K_2$. Then $G$ is randomly $2K_2$-decomposable if and only if $G \in \mathcal{F}$, where $\mathcal{F}$ is described in (5.6).

Proof: It is easy to see that every graph in $\mathcal{F}$ is randomly $2K_2$-decomposable.

Assume now that $G$ is randomly $2K_2$-decomposable. We claim that $G \in \mathcal{F}$. Suppose, to the contrary, that $G \notin \mathcal{F}$. Since $G$ is $2K_2$-decomposable, $\Delta(G) \leq n$, where $n$ is the size of $G$. By Lemma 5.6, it follows that $G$ has a decomposition into one copy of $P_3$ and $n-1$ copies of $2K_2$. These $n-1$ copies of $2K_2$ do not belong to any $2K_2$-decomposition of $G$, contradicting the fact that $G$ is randomly $2K_2$-decomposable. Therefore $G \in \mathcal{F}$. ■

As we have seen, for a given graph $H$, not all $H$-decomposable graphs $G$ are randomly $H$-decomposable. However, it is possible to define a measure of how close an $H$-decomposable graph is to being randomly $H$-decomposable.

Let $G$ be a graph that can be decomposed into $m(\geq 2)$ copies of a fixed graph $H$. We say that the random decomposition index $i(G; H)$ of $G$ with respect to $H$ is $(k+1)/m$ if $k$ is the largest integer less than $m$ such that any $k$ pairwise edge-disjoint copies of $H$ in $G$ belong to some $H$-decomposition of $G$.

The random decomposition index is always a rational number in the interval $(0,1]$. It has the value 1 when the decomposition is
random. Note that these indices can attain arbitrarily small positive values since

\[ i(P_{2m+1}; P_3) = \frac{1}{m}. \]

We close this chapter by proposing the following question. Given a nonempty graph \( H \) and a rational number \( \alpha \), such that \( 0 < \alpha \leq 1 \), does there exist a regular \( H \)-decomposable graph \( G \) for which \( i(G; H) = \alpha \)?
CHAPTER VI
EXTREMAL REGULAR GRAPHS WITHOUT NEAR 1-FACTORS

Section 6.1

Introduction

A great deal of research has involved the existence or non-existence of 1-factors in graphs. Graphs that contain 1-factors have, in fact, been characterized by Tutte [63].

Without question, the isomorphic decomposition problem that has received the most attention is that of decomposing (necessarily regular) graphs into 1-factors, i.e., of determining which regular graphs are 1-factorable. We have already mentioned several known results on this topic and have presented a number of extensions of these results. A number of conditions have been found that guarantee the existence of 1-factors in regular graphs. In some instances extremal graphs not satisfying such a condition have been shown to possess an isomorphic decomposition into graphs called "near 1-factors". (In [21], Chartrand, Saba and Zou introduced a metric on graphs having a given fixed order and size. With this metric, the only graph at distance 1 from a 1-factor is a near 1-factor.) The goal of this chapter is to establish the sharpness of sufficient conditions for a regular graph to possess a near 1-factor so that it will be known which extremal graphs need to be studied for the
possible existence of isomorphic parts (actually, isofactors) that are, in some sense, close to near 1-factors.

**Theorem 6.** A (Tutte) A graph $G$ has a 1-factor if and only if for every proper subset $S$ of $V(G)$, the number of odd components of $G - S$ does not exceed $|S|$.

Of particular interest has been the determination of those $r$-regular graphs, $r \geq 3$, that contain 1-factors. Of course, any graph having a 1-factor must have even order. An important and famous result along this line for 3-regular graphs is due to Petersen [45].

**Theorem 6.B** (Petersen) Every cubic graph with a most two bridges contains a 1-factor.

This result is best possible since cubic graphs with three bridges and no 1-factors exist. The graph of Figure 6.1 is the unique smallest such graph. Theorem 6.B also implies that every 3-regular, 2-edge-connected graph contains a 1-factor.
Figure 6.1 The smallest cubic graph without a 1-factor

All smallest connected r-regular graphs of even order without 1-factors have been studied and characterized for every $r \geq 3$ by Wallis [65]. This result has no connectivity condition (except that the graph be connected). It is well-known that an r-regular, $(r-1)$-edge-connected graph ($r \geq 3$) of even order contains a 1-factor (see [18], for example). This result is best possible in the sense that an r-regular, $(r-2)$-edge-connected graph, $r \geq 3$, of...
even order need not contain a \(l\)-factor. Chartrand, Goldsmith and Schuster [18] determined the smallest order of an \(r\)-regular, \((r-2)\)-edge-connected graph \((r \geq 3)\) of even order and containing no \(l\)-factor.

In [19] the concept of a near \(l\)-factor was introduced. A near \(l\)-factor of a graph \(G\) of order \(2n \geq 4\) is a factor of the type \((n-2)K_2 \cup P_3 \cup K_1\). (Note that a \(l\)-factor of \(G\) is a factor of the type \(nK_2\).) Therefore, a \(l\)-factor and a near \(l\)-factor differ in the placement of a single edge - more formally, the replacement of any edge \(uv\) in a \(l\)-factor \(F\) of \(G\) by an edge \(uw\) in the complement of \(F\) produces a near \(l\)-factor of \(G\).

Chartrand, Kapoor, Lesniak and Schuster [19] showed that the smallest \(r\)-regular, \((r-2)\)-edge-connected graphs of even order and not containing a \(l\)-factor can, however, be isomorphically factored into near \(l\)-factors.

It is rather easy to see that every \(r\)-regular, \((r-1)\)-edge-connected graph of even order contains a near \(l\)-factor. The following result was obtained in [19].

**Theorem 6.C** (Chartrand, Kapoor, Lesniak and Schuster) If \(G\) is an \(r\)-regular, \((r-2)\)-edge-connected graph \((r \geq 3)\) of even order \(p\) containing less than \(2r\) distinct edge cut sets of cardinality \(r - 2\), then \(G\) contains a near \(l\)-factor.

The following extension of Tutte's Theorem [19] was also obtained and will prove to be useful.
Theorem 6.D (Chartrand, Kapoor, Lesniak and Schuster) Let $G$ be a graph of even order $2n \geq 4$ with at most one isolated vertex such that $G \not\cong nK_2$. Then $G$ has a near 1-factor if and only if for every proper subset $S$ of $V(G)$, the number of odd components of $G - S$ does not exceed $|S| + 2$.

Section 6.2

Smallest Connected Regular Graphs Without Near 1-Factors.

The following elementary fact will be useful in this and the subsequent section.

Lemma 6.1 If $G$ is a $r$-regular graph and $H$ is a subgraph of $G$ induced by a nonempty subset $T$ of vertices of $G$, where $|T| \leq r$, then the number of edges joining $H$ and $G - H$ is at least $r$.

Proof: Let $t = |T|$. If $v$ is a vertex of $H$, then $\text{deg}_H v \leq t - 1$ and $\text{deg}_G v = r$; therefore $v$ is adjacent to at least $r - (t - 1)$ vertices of $G - T$. Then the number of edges joining $H$ and $G - T$ is at least $t(r - t + 1)$. Since $1 \leq t \leq r$, we have

$$(t-1)(r-t) = rt - r - (t^2 - t) \geq 0,$$

which implies that $t(r-t+1) \geq r$, thereby proving the lemma. \hfill \blacksquare
Theorem 6.1 If $G$ is a connected $r$-regular graph of even order $p$ and $G$ does not contain a near 1-factor, then $p \geq f(r)$, where

$$f(r) = \begin{cases} 
34 & \text{for } r = 3 \\
44 & \text{for } r = 4 \text{ and } r = 6 \\
56 & \text{for } r = 8 \\
5r + 11 & \text{for } r \geq 5 \text{ and } r \text{ odd} \\
5r + 6 & \text{for } r \geq 10 \text{ and } r \text{ even}.
\end{cases}$$

Proof: If $G$ does not contain a near 1-factor, then according to Theorem 6.D, there is a set $S \subseteq V(G)$, with $|S| = k$, such that $G - S$ has $n$ odd components and $n > k + 2$. The fact that $p$ is even implies that $n$ and $k$ have the same parity; therefore, the former inequality can be improved to

$$n \geq k + 4. \quad (6.1)$$

Let $G_1, G_2, \ldots, G_n$ be the odd components of $G - S$ and let $a_i$ denote the number of edges joining $G_i$ and $S$. Further, let $m$ be the number of odd components of $G - S$ with more than $r$ vertices, so that $n - m$ is the number of odd components of $G - S$ with at most $r$ vertices. Note that if $r$ is odd and $G_i$ is a component of $G - S$ with more than $r$ vertices, then $G_i$ has at least $r + 2$ vertices.

The next step will be to show that $m \geq 5$. This is done by finding upper and lower bounds on the number of edges joining $S$ and the $n$ odd components of $G - S$. In our setting, this number is
equal to \( \sum_{i=1}^{n} a_i \). Each vertex of \( S \) contributes at most \( r \) edges to this sum because \( G \) is \( r \)-regular; therefore

\[
\sum_{i=1}^{n} a_i \leq kr. \tag{6.2}
\]

On the other hand, there is at least one edge joining each component to \( S \) because \( G \) is connected. But we can say more about these \( n - m \) components of order at most \( r \). By Lemma 6.1, each such component is joined to \( S \) by at least \( r \) edges; therefore

\[
\sum_{i=1}^{n} a_i \geq (n-m)r + m. \tag{6.3}
\]

Comparing the inequalities (6.2) and (6.3), we obtain

\[
k \geq \frac{(n-m)r + m}{r}, \tag{6.4}
\]

but \( m/r \geq 0 \) so that

\[
k \geq \frac{m}{r}. \tag{6.5}
\]

Using inequality (6.1), we have

\[
k \geq \frac{n+m}{r+4} - m.
\]

Since \( m > 0 \), it follows that

\[
k > n - m. \tag{6.6}
\]

A lower bound for the order \( p \) of \( G \) is obtained by observing that (1) each of the \( n - m \) components of \( G - S \) having order at most \( r \) has at least one vertex, (2) each of the \( m \) remaining

\[
m \geq 5.
\]
components has at least \( r + 1 \) vertices if \( r \) is even and at least \( r + 2 \) vertices if \( r \) is odd, and (3) the vertices of \( S \) are in none of these components. Therefore,

\[
p \geq (n-m) + m(r+1) + k \quad \text{if } r \text{ is even} \tag{6.7}
\]

and

\[
p \geq (n-m) + m(r+2) + k \quad \text{if } r \text{ is odd}. \tag{6.8}
\]

However, \( m \geq 5 \), \( n-m \geq 0 \) and \( k \geq 1 \); thus

\[
p \geq 5r + 6 \quad \text{if } r \text{ is even} \tag{6.9}
\]

and

\[
p \geq 5r + 11 \quad \text{if } r \text{ is odd}. \tag{6.10}
\]

The bounds indicated in (6.9) and (6.10) can be improved, though, for the cases \( r = 3, 4, 6 \) and 8. We study these cases separately.

**Case 1. Assume** \( r = 3 \).

We know by (6.6) that \( m \geq 5 \). Now \( k \geq 2 \), for if \( k \) were equal to 1, then the vertex in \( S \) cannot be joined to five or more components. From (6.2) it follows that \( \sum_{i=1}^{k} a_i \leq 3k \), and from (6.3) we have

\[
m \cdot 1 + (n-m) \cdot 3 \leq \sum_{i=1}^{k} a_i.
\]
From (6.1) and the last two inequalities, we have

$$3n - 2m \leq 3k \leq 3(n-4).$$

Therefore $m \geq 6$.

Using the fact that $k \geq 2$ and inequality (6.8) we obtain

$p \geq 32$. If $p = 32$, then $m = 6$, $n - m = 0$ and $k = 2$.

The graph $G$ is 3-regular, $S$ has two vertices and $G - S$ has six components, from which it follows that $G$ is disconnected - a contradiction. Thus $p \geq 34$.

Case 2. Assume that $r = 4, 6$ or 8.

We will show when $r$ is even that

$$m \geq 4 + \frac{8}{r-2} = \frac{4r}{r-2} \quad (6.11)$$

and

$$k \geq \frac{8}{r-2} + (n-m). \quad (6.12)$$

If $p'$ and $q'$ are the order and size of an odd component $G_i$ of $G - S$, then

$$\sum_{v \in V(G_i)} \deg_G v = 2q' + a_i = 4p',$$

which implies that $a_i$ is even. However $a_i \neq 0$ since $G$ is connected, so that $a_i \geq 2$.

Now inequality (6.3) can be modified to produce

$$\sum_{i=1}^{k} a_i \geq (n-m)r + 2m.$$
From (6.2) we have

\[ kr \geq (n-m)r + 2m \]

\[ = rn - (r-2)m \]

\[ \geq r(k+4) - (r-2)m \]

\[ = rk + 4r - (r-2)m. \quad (6.13) \]

Hence \((r-2)m \geq 4r\) and finally

\[ m \geq \frac{4r}{r-2} = 4 + \frac{8}{r-2}, \]

thereby proving (6.11).

From (6.13) it follows that

\[ k \geq (n-m) + \frac{2m}{r} \]

\[ \geq (n-m) + \frac{2}{r} \cdot \frac{4r}{r-2} \]

\[ = (n-m) + \frac{8}{r-2}, \]

which verifies (6.12).

Using (6.11), (6.12) and (6.7), we conclude that

(i) if \( r = 4\) then \( m \geq 8, \ k \geq 4 \) and \( p \geq 44; \)

(ii) if \( r = 6\) then \( m \geq 6, \ k \geq 2 \) and \( p \geq 44; \) and

(iii) if \( r = 8\) then \( m \geq 6, \ k \geq 2 \) and \( p \geq 56. \)

This completes the proof of Theorem 6.1.

In the remainder of this section it is shown that the bounds presented in Theorem 6.1 are best possible.

Theorem 6.2 The smallest order \( p \) of a connected \( r \)-regular graph without a near \( 1 \)-factor is \( p = f(r), \) where
\[
\begin{align*}
\text{f}(r) = \begin{cases} 
34 & \text{for } r = 3 \\
44 & \text{for } r = 4, 6 \\
56 & \text{for } r = 8 \\
5r + 11 & \text{for } r \geq 5 \text{ and } r \text{ odd} \\
5r + 6 & \text{for } r \geq 10 \text{ and } r \text{ even.}
\end{cases}
\end{align*}
\]

Proof: According to Theorem 6.1, it is enough to show the existence of connected \(r\)-regular graphs of order \(f(r)\) having no near 1-factor. We now consider a number of cases.

Case 1. Assume \(r\) is odd, \(r \geq 5\) and \(p = 5r + 11\).

Using the terminology presented in the preceding theorem, we have in this case, that \(k = 1\) and \(m = 5\).

The five odd components are described as

\[
G_1 = \frac{(r-1)\times 2}{2} \cup P_3 \quad \text{for } 1 \leq i \leq 4
\]

and

\[
G_5 = P_{r-2} \cup 2K_2.
\]

Note that \(G_5\) has \(r - 4\) vertices of degree \(r - 1\) and all others of degree \(r\). The graphs \(G_i (1 \leq i \leq 4)\) have exactly one vertex of degree \(r - 1\) while all five graphs \(G_i (1 \leq i \leq 5)\) have order \(r + 2\). A graph \(G\) is constructed by adding an extra vertex \(u\) and joining \(u\) to each vertex of degree \(r - 1\) in the graph \(G_1 \cup G_2 \cup G_3 \cup G_4 \cup G_5\). In this case \(S = \{u\}\), and \(G - S\) has five odd components so, by Theorem 6.D, \(G\) has no near 1-factor. Figures 6.2 and 6.3 illustrate the construction for \(r = 5\) and \(r = 7\), respectively.
Figure 6.2 A smallest connected 5-regular graph without a near 1-factor
Figure 6.3 A smallest connected 7-regular graph without a near 1-factor

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Case 2. Assume \( r \) is even, \( r \geq 10 \) and \( p = 5r + 6 \).

Now we have \( k = 1 \) and \( m = 5 \) as in the former case. Here \( S \) consists of a single vertex \( u \) and five components \( G_1, G_2, \ldots, G_5 \) of order \( r + 1 \), where

\[
G_i = K_2 \cup (r-1)K_1 = K_{r+1} - e \quad \text{for} \quad i = 1, 2, 3, 4
\]

and

\[
G_5 = \frac{(r-8)}{2}K_2 \cup 9K_1.
\]

Figure 6.4 illustrates the construction and the circles represent the components.

![Figure 6.4 Construction of the smallest \( r \)-regular connected graph without a near 1-factor for \( r \) even and \( r \geq 10 \)]
Case 3. Assume $r = 3$.

For $k = 4$ and $m = 6$ we obtain the graph of order 34 in Figure 6.5. The set $S = \{a, b, c, d\}$ and the six odd components guarantee that this graph does not contain a near 1-factor by Theorem 6.6.

Figure 6.5 A smallest connected cubic graph without a near 1-factor

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Case 4. Assume $r = 4$.

For $r = 4$, we obtain $k = 4$, $m = 8$ and $p = 44$ (see Figure 6.6 where $S = \{a, b, c, d\}$).

Figure 6.6 A smallest connected 4-regular graph without a near 1-factor

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Case 5. Assume \( r = 6 \).

Here we have \( k = 2, m = 6 \) and \( p = 44 \). Take six components isomorphic to \( K_7 - e \), and let \( S = \{ u, v \} \). Join each vertex of degree 5 in the components to one of the vertices in \( S \). The graph obtained is indicated in Figure 6.7, where each circle represents \( K_7 - e \).

Figure 6.7 A smallest connected 6-regular graph without a near 1-factor
Case 6. Assume \( r = 8 \).

In this case we have, \( k = 2, m = 6 \) and \( p = 56 \). Let \( S = \{u,v\} \). For \( 1 \leq i \leq 5 \) let \( G_i = K_9 - e \) and define \( G_6 \) to be \( K_9 \) with three independent edges removed. A graph \( G \) is obtained by joining the vertices of \( S \) to each vertex of degree 7 in the components \( G_i, 1 \leq i \leq 5 \), and by joining \( u \) to three vertices of degree 7 in \( G_6 \) and \( v \) to the remaining three vertices of degree 7 in \( G_6 \), as shown in Figure 6.8.

This completes the proof of the theorem.

\[\text{Figure 6.8 Smallest connected 8-regular graph without a near 1-factor}\]
Section 6.3
Smallest r-Regular, (r-2)-Edge-Connected Graphs
Without Near 1-Factor

In the next two results we will be concerned with the minimality problem but now with restrictions on the edge-connectivity. As we saw in the introduction to this chapter, an r-regular, (r-1)-edge-connected graph must have a near 1-factor, but an r-regular, (r-2)-edge-connected graph need not. However for such graphs of small order, they must contain a near 1-factor, as we verify.

Theorem 6.3 Let G be an r-regular, (r-2)-edge-connected graph, r ≥ 3, of even order p. If p < 4(⌈r/2⌉ + r - 1), then G has a near 1-factor.

Proof: Assume, to the contrary, that G has no near 1-factor. Then, by Theorem 6.D, there exists a proper subset S of V(G) such that G - S has odd components G₁, G₂, ..., Gₙ, where |S| = k and n > k + 2. However, n and k have the same parity; therefore,

\[ n \geq k + 4. \tag{6.14} \]

Let \( a_i \) be the number of edges joining \( G_i \) and S. By Lemma 6.1, if \( |V(G_i)| \leq r \) then \( a_i \geq r \). Let m denote the number of odd components \( G_i \) of \( G - S \) having more than r vertices.

Because G is (r-2)-edge-connected then \( a_i \geq r - 2 \) whenever \( |V(G_i)| \geq r + 1 \); therefore
Further,
\[
\sum_{i=1}^{n} a_i \leq \sum_{v \in S} \deg v = kr. \tag{6.16}
\]
By (6.15) and (6.16), \( nr - 2m \leq kr \) so that
\[
2m \geq nr - kr = (n-k)r \geq 4r;
\]
thus
\[
m \geq 2r. \tag{6.17}
\]
Using (6.16) and the fact that \( n \geq k + 4 \), we have
\[
kr \geq \sum_{i=1}^{n} a_i \geq n(r-2) \geq (k+4)(r-2),
\]
so that
\[
k \geq 2r - 4. \tag{6.18}
\]
Hence for \( r \) even, we have by (6.17) and (6.18) that
\[
p \geq m(r+1) + k \geq 2r(r+1) + 2r - 4
\]
\[= 2r^2 + 4r - 4 \]
\[= 4(r\lceil r/2 \rceil + r - 1),
\]

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which contradicts the hypothesis.

For \( r \) odd,

\[
p \geq m(r + 2) + k \geq 2r(r + 2) + 2r - 4
= 2r^2 + 6r - 4
= 4(r\lfloor r/2\rfloor + r - 1),
\]

again producing a contradiction.

The smallest order of a connected cubic graph without a near 1-factor is 34 as was seen in Theorem 6.2. For \( r = 3 \),

\[4(r\lfloor r/2\rfloor + r - 1) = 32;\] therefore Theorem 6.3 does not give the best result for cubic graphs. For \( r > 3 \), the bound given in Theorem 6.3 is sharp.

**Theorem 6.4** For every integer \( r > 3 \), there exists an \( r \)-regular, \((r-2)\)-edge-connected graph of order \( 4(r\lfloor r/2\rfloor + r - 1) \) containing no near 1-factor.

**Proof:** We present different constructions according to the parity of \( r \).

**Case 1.** Assume \( r = 2n \) for some \( n \geq 2 \).

Let \( H = (n-1)K_2 \cup 3K_1 \). Note that \( H \) has (odd) order \( r + 1 \), three of its vertices have degree \( r \) and the remaining \( r - 2 \) vertices have degree \( r - 1 \). For \( i = 1, 2, 3, 4 \), let \( A_i = (n-1)K_1 \) and \( B_i = nH \). An \( r \)-regular graph \( G \) is produced by adding some edges among the graphs \( A_i \) and \( B_i \) for \( i = 1, 2, 3, 4 \), so that...
the order of $G$ will be $4|V(A_1) \cup V(B_1)| = 4|V((n-1)K_r) \cup V(nH)| = 4(n-1 + n(r+1)) = 4(r\lceil r/2 \rceil + r - 1)$.

For $i = 1, 2, 3$, in each component $H$ of $B_i$ join each of the $r - 2$ vertices of degree $r - 1$ to one vertex of $A_i \cup A_i + 1$ in such a way that each vertex of $A_i \cup A_i + 1$ is adjacent to exactly one vertex of each component of $B_i$. In a similar way, add the corresponding edges between $B_i$ and $A_i \cup A_i$. The graph $G$ obtained is $r$-regular and it has the property that for every two nonadjacent vertices $u$ and $v$, there exist at least $r - 2$ edge-disjoint $u - v$ paths. Therefore, $G$ is $(r-2)$-edge-connected. By removing from $G$ the $4n-4$ vertices of $A_1 \cup A_2 \cup A_3 \cup A_4$, we obtain the union of $4n$ components isomorphic to $H$, where $H$ has odd order $r + 1$. Therefore, by Theorem 6.D, $G$ has no near 1-factor. Figure 6.6 also illustrates this theorem for $r = 4$.

Case 2. Assume $r = 2n + 1$ for some integer $n \geq 2$.

Let $H = \overline{P_r} \cup \overline{P_2}$, so that $H$ has odd order $r + 2$, four vertices of degree $r$, and $r - 2$ vertices of degree $r - 1$. Let $A_1$ and $A_3$ be isomorphic to $nK_1$, and let $A_2$ and $A_4$ be isomorphic to $(n-1)K_1$. Further, let $B_1$ be isomorphic to the union of $n + 1$ copies of $H$ or $n$ copies of $H$ depending on whether $i \in \{1, 3\}$ or $i \in \{2, 4\}$, respectively. We assume that the graphs $A_i$ and $B_i$, $i = 1, 2, 3, 4$, have pairwise disjoint vertex sets. The union of these sets will give the vertex set of the $r$-regular graph $G$ without a near 1-factor, that is, $G$ has
2(n+1)(r+2) + 2n(r+2) + 2n + 2(n-1) = 4\lceil r/2 \rceil + r - 1

vertices.

For \( i = 1, 2, 3 \) and for each of those components in \( B_i \) isomorphic to \( H \), join each of the \( r - 2 \) vertices of degree \( r - 1 \) to a vertex of \( A_i \cup A_{i+1} \) in such a way that every vertex of \( A_i \cup A_{i+1} \) is adjacent to exactly one vertex of each component of \( B_i \). Analogously, add the appropriate edges between \( B_4 \) and \( A_4 \cup A_1 \).

The graph \( G \) so obtained is \( r \)-regular and \((r-2)\)-edge-connected. By removing from \( G \) the \( 2n + 2(n-1) = 4n-2 \) vertices of \( A_1 \cup A_2 \cup A_3 \cup A_4 \), we obtain the union of \( 2(n+1) + 2n = 4n + 2 \) odd components \( H \); so by Theorem 6.D, \( G \) has no near 1-factor. Moreover, it is easily seen that \( G \) is \((r-2)\)-edge-connected.

Figure 6.9 illustrates the theorem for \( r = 5 \), where dotted circle represents a copy of \( H \).
Figure 6.9. A smallest 3-edge-connected 5-regular graph without a near 1-factor
Section 6.4

A Smallest Connected Cubic Graph With Six Bridges and Without a Near 1-Factor.

If we apply Theorem 6.6 to cubic graphs, we see that every cubic graph with less than six bridges contains a near 1-factor. A cubic graph of order 44 with six bridges and without a near 1-factor was found by Chartrand, Kapoor, Lesniak and Schuster [19]. A natural question arises. What is the minimum order of a cubic graph G with six bridges and containing no near 1-factor? The answer to this question is provided in our next result.

Theorem 6.4 The smallest order of a connected cubic graph of even order with six bridges and containing no near 1-factor is 40.

Proof: Suppose, to the contrary, that there exists a graph G that is connected and cubic, has six bridges but no near 1-factor and has even order p, where p < 40.

By Theorem 6.6, there exists a proper subset S of V(G) having cardinality k such that the number n of odd components of G - S is at least k + 4.

From the proof of Theorem 6.1, we know that k ≥ 2 and p ≥ (n-m) + 5m + k, where m(≥ 6) is the number of odd components with at least five vertices.

Suppose that k = 2. The removal of the two vertices of S will produce at most five components - a contradiction with n ≥ 6.
Therefore \( k \geq 3 \).

Suppose that \( k = 3 \), so that \( n \geq 7 \). The removal of the three vertices of \( S \) produces at most seven components, but since \( n \geq 7 \) we conclude that \( n = 7 \). Therefore each vertex of \( S \) must be incident with three bridges, necessarily all of which are distinct bridges of \( G \); therefore \( G \) has at least nine bridges - a contradiction. Hence \( k \geq 4 \).

Suppose that \( k = 4 \), so that \( n \geq 8 \) and \( p \geq 36 \). First assume that \( p = 36 \). In this case \( G - S \) must have six components of order 5 and two trivial components with vertices \( u \) and \( w \). By a straightforward argument it follows that \( S \) is an independent set of vertices and that at most two vertices of \( S \) are joined to both \( u \) and \( w \).

Suppose exactly \( v_1, v_2 \in S \) are joined to \( u \) and \( w \). Then each of \( v_1 \) and \( v_2 \) is incident with exactly one bridge while the other two vertices of \( S \) are incident with three bridges, which implies that \( G \) has eight bridges - a contradiction.

Suppose now that exactly one vertex of \( S \) is adjacent with \( u \) and \( w \). But then \( G \) is disconnected - a contradiction.

Suppose \( k = 4 \) but \( p = 38 \). Then \( m = 6 \) and \( n - m = 4 \). In this case the ten components of \( G - S \) are joined to \( S \) with at least 18 edges, which is impossible since \( |S| = 4 \). Therefore, \( k \geq 5 \).

Suppose next that \( k = 5 \); then \( p \geq 38 \). Since \( p < 40 \), we have \( p = 38 \), which implies that \( m = 6 \) and \( n = 9 \). Hence \( G - S \)
has six components of order 5 and three trivial components. The number of edges joining $G - S$ with $S$ is at least 15, but since $|S| = 5$, the number of edges joining $S$ to $G - S$ is at most 15. This implies that the number of such edges is exactly 15. The three trivial components are joined to $S$ with nine edges; so there are six edges between $S$ and the six components of order 5. Therefore each component of $G - S$ of order 5 is joined to $S$ by exactly one edge, necessarily each such edge being a bridge. Since $G$ is connected, no vertex of $S$ can be joined to three components of order 5. Therefore, there is a vertex of $S$ that is joined to two components of $G - S$ having order 5 and to one trivial component. These three edges are necessarily bridges. Each of the other components of order 5 contributes one bridge. So $G$ has at least seven bridges - a contradiction. Therefore $k \geq 6$, but this implies that $p \geq 40$, producing the final contradiction.

Following a case by case argument we were able to show that there are exactly two cubic graphs of order 40 with six bridges and having no near 1-factor. Both graphs have an isomorphic factorization into the graph $H$ where $H = 16K_2 \cup 2P_3 \cup 2K_1$. Note that $H$ differs from a near 1-factor only in the location of one edge, so that a near 1-factor has edge independence number equal to 19 while the corresponding value for $H$ is 18. Figures 6.10 and 6.11 show both cubic graphs with their factorizations indicated by different ways to represent their edges. By studying the possible factorizations of the components of $G - S$ having order 5, one can conclude
that neither graph of Figure 6.10 and 6.11 can be isomorphically factored into any other graph having edge independence number 18.

Figure 6.10. A cubic graph of order 40 with six bridges and no near 1-factor

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Figure 6.11. A cubic graph of order 40 with six bridges and no near 1-factor

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