Traceability in Graphs

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TRACEABILITY IN GRAPHS

by

Ronald J. Gould

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in partial fulfillment
of the
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Ronald J. Gould
To my family,

for missing me when I wasn't there,

yet always being there for me.
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CHAPTER I

PRELIMINARIES

In the first section of this chapter we present some of the basic definitions, notation, and conventions that will be used throughout this dissertation. The second section provides a brief historical sketch of the problem of traceability in graphs for the purpose of highlighting some of the areas of interest related to this topic. The third section begins our discussion of traceability in graphs.

Section 1.1

Definitions and Notation

We begin by presenting some of the basic definitions and notation which are fundamental to this dissertation. In addition, more specialized definitions will be introduced as needed. All other terms will be defined as in Bezhad, Chartrand, and Lesniak-Foster [2].

As usual, \(|S|\) denotes the cardinality of a set \(S\). We shall use \(V(G)\) and \(E(G)\) to denote the vertex set and edge set of a graph \(G\), respectively; \(|V(G)|\) is called its order and \(|E(G)|\) its size.

The neighborhood \(N(v)\) of the vertex \(v\) is the set of all vertices adjacent to \(v\), while the closed neighborhood is \(N[v] = N(v) \cup \{v\}\).

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If \( e = vw \) is an edge of a graph \( G \) and \( v \notin V(G) \), we say the edge \( e \) is **subdivided** when it is replaced by the vertex \( v \) and edges \( uw \) and \( vw \). If every edge of \( G \) is subdivided, the resulting graph is called the **subdivision graph** \( S(G) \) of \( G \).

The **connectivity** \( \kappa(G) \) of a graph \( G \) is the minimum number of vertices whose removal results in a disconnected or trivial graph. Analogously, the **edge connectivity** \( \kappa_1(G) \) is the minimum number of edges whose removal disconnects \( G \). Any vertex whose removal disconnects \( G \) is called a **cut vertex** of \( G \), while any edge whose removal disconnects \( G \) is termed a **bridge**.

A connected graph is **eulerian** if it possesses a circuit which contains every edge of \( G \).

The **join** \( G_1 + G_2 \) of two graphs \( G_1 \) and \( G_2 \) with disjoint vertex sets is that graph with vertex set \( V(G_1) \cup V(G_2) \) and edge set 
\[
E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{uv | u \in V(G_1) \text{ and } v \in V(G_2) \}.
\]

**Section 1.2**

**Historical Background**

The first general discussion of traceability in graphs was given in 1855 by Thomas Penyngton Kirkman (see [1]). He considered the question of whether the graph of a polyhedron always contains a circuit which passes through each vertex once and only once. He claimed the condition he presented was sufficient to guarantee such a circuit could always be found. Unfortunately, his claim was false.
At approximately the same time, another mathematician, William Rowan Hamilton, was interested in a similar question. Hamilton reportedly introduced a game that consisted of a solid dodecahedron with each vertex representing an important city of the time. The object of the game was to produce a round trip through the city by following the edges of the dodecahedron, the only restriction being that each city could be visited only once. This is equivalent to finding a cycle containing every vertex in the graph of the dodecahedron. In honor of Hamilton, a graph containing a cycle through all its vertices is called a Hamiltonian graph.

Thus began one of the most widely investigated concepts in graph theory. Although many sufficient conditions have been determined, there is no known characterization of Hamiltonian graphs. This is considered by many to be one of the most famous unsolved problems in graph theory today. Thus it is not surprising that certain subclasses of Hamiltonian graphs and certain classes of non-Hamiltonian, "nearly" Hamiltonian, and "highly" Hamiltonian graphs have been investigated.

One of the best known variations on the Hamiltonian theme is that of traceable graphs. A graph is traceable if it contains a Hamiltonian path, that is, a path containing each vertex of the graph. Since every Hamiltonian graph is traceable, it was natural that these graphs would gain attention.

Recently, interest has grown in a class of "nearly" Hamiltonian graphs called hypohamiltonian graphs. A graph $G$ is hypohamiltonian if it is not Hamiltonian but if $G - v$ is Hamiltonian for
every vertex $v$ of $G$. Herz, Goudin, and Rossi [16] showed, for example, that hypohamiltonian graphs exist and that the Petersen Graph (see Figure 1.1) is the hypohamiltonian graph of smallest order. A great deal of additional work has been done on this topic.

![Figure 1.1: The Petersen Graph](image)

In one sense, a more general class of graphs was introduced by Skupien [22]. A graph $G$ is called homogeneously traceable if for each $v \in V(G)$, there is a hamiltonian path in $G$ with initial vertex $v$. Every hamiltonian graph is homogeneously traceable, thus hamiltonian graphs are a subclass of the family of homogeneously traceable graphs. Further, since every hypohamiltonian graph is homogeneously traceable, we see that hamiltonian graphs form a proper subclass of homogeneously traceable graphs.

Recently, Gimbel [11] has developed a computer algorithm to determine homogeneously traceable graphs. Also, Simões-Pereira and Zamfirescu [21] have investigated the existence of homogeneously traceable nonhamiltonian digraphs without 2-cycles.

Another variation involves "highly" hamiltonian graphs. A graph $G$ is hamiltonian connected if for every two vertices $u$ and $v$ of $G$ there is a $u - v$ hamiltonian path in $G$. The close ties between hamiltonian connected, hamiltonian, and traceable graphs can be seen.
in the following results, the last two of which are due to Ore [18] and the first is a consequence of his work.

Theorem 1A (Ore)

Let $G$ be a graph of order $p \geq 3$. If for all distinct non-adjacent vertices $u$ and $v$

(a) $\deg u + \deg v \geq p - 1$, then $G$ is traceable,
(b) $\deg u + \deg v \geq p$, then $G$ is hamiltonian, or
(c) $\deg u + \deg v \geq p + 1$, then $G$ is hamiltonian connected.

We begin our investigation of traceability in graphs by examining the structure of homogeneously traceable nonhamiltonian graphs.

Section 1.3

The Degree Set of a Homogeneously Traceable Nonhamiltonian Graph

In [4] it was shown that homogeneously traceable nonhamiltonian graphs exist for all orders $p$ except $3 \leq p \leq 8$. In the construction presented, every homogeneously traceable nonhamiltonian graph of order 9 and greater contained a vertex of degree two. R. Frucht (personal communication) asked if there exist homogeneously traceable nonhamiltonian graphs with only large degrees. Of course the Petersen Graph is cubic. In this section we give a complete answer to this question.

The following result was established in [22] and will be useful.
Lemma 1B (Skupién)

If \( G \) is homogeneously traceable of order \( p \geq 3 \), then \( G \) is 2-connected.

It is convenient to construct a class of graphs for use in the remainder of this section. Define the graphs \( H_{2n+1}, n \geq 1 \), to consist of two disjoint cycles, \( C: u_1, u_2, \ldots, u_{2n+1}, u_1 \) and \( C': v_1, v_2, \ldots, v_{2n+1}, v_1 \) and for each \( i = 1, 2, \ldots, 2n+1 \) join \( u_i \) and \( v_i \) by a path \( P_i \) of length two. Denote the vertex of degree two on \( P_i \) by \( t_i \). These graphs (see Figure 1.2) are homogeneously traceable and nonhamiltonian for each \( n \geq 1 \).

Figure 1.2: The graph \( H_{2n+1} \)

The degree set of a graph \( G \) is the set of degrees of the vertices of \( G \).

Theorem 1.1

Suppose \( S = \{n_0, n_1, \ldots, n_k\} \) is a set of \( k + 1 \) (\( \geq 1 \)) positive integers and \( n_i \geq 2 \) for all \( i \) (\( 0 \leq i \leq k \)). Then \( S \) is the degree set of a homogeneously traceable nonhamiltonian graph unless \( S = \{2\} \).

Proof: Without loss of generality we assume that \( n_0 < n_1 < \cdots < n_k \). Suppose \( S = \{2\} \). Then by Lemma 1B, \( G \) is 2-connected. Since \( G \) is

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2-regular of order at least 3, the graph \( G \) is a cycle and hence is hamiltonian. Thus \( \{2\} \) is not the degree set of a homogeneously traceable nonhamiltonian graph. We now consider the converse. Suppose \( S \neq \{2\} \). We distinguish three cases.

**Case 1:** Suppose \( 3 \in S \). If \( S = \{3\} \), the Petersen Graph satisfies the theorem. If \( S = \{2, 3\} \) the graph \( H_3 \) suffices. Now if \( 3 \in S \) but \( S \neq \{3\} \) and \( S \neq \{2, 3\} \) consider the graph \( H_\ell \) where \( \ell \) is odd and \( \ell \geq \max(3, k) \). We now construct a graph \( H \) from \( H_\ell \).

If \( n_0 \neq 2 \), then \( n_0 = 3 \) and each \( u_i \) and \( v_i \), \( i = 1, 2, \ldots, k \), has degree 3. For each \( i = 1, 2, \ldots, k \), replace the vertex \( t_i \) (and its incident edges) by \( M_i = K_{n_i+1} - e \), where \( e = x_iy_i \in E(K_{n_i+1}) \), that is, a copy of the complete graph on \( n_i + 1 \) vertices minus one edge. Then insert the edges \( u_iw_i \) and \( v_iy_i \). If \( \ell > k \), repeat this argument with \( M_k \) replacing each \( t_j \), \( k + 1 \leq j \leq \ell \). Then \( \deg_H u_i = \deg_H v_i = 3 \) and \( \deg_H x = n_i \) for each \( x \in V(M_i) \), \( i = 1, 2, \ldots, k \), and if \( \ell > k \), \( \deg_H x = n_k \) if \( x \in V(M_i) \), \( k + 1 \leq i \leq \ell \). Thus \( H \) has degree set \( S \).

To see that \( H \) is homogeneously traceable, note that each \( M_i \) is hamiltonian connected since \( \deg_H v \geq (|V(M_i)| + 1)/2 \) for each \( v \in V(M_i) \), \( i = 1, 2, \ldots, \ell \). Thus by Theorem 1A [18], \( M_i \) is hamiltonian connected. To find a hamiltonian path beginning with vertex \( u_i \) or \( v_i \), \( i = 1, 2, \ldots, \ell \), consider the path \( P \) in \( H_\ell \) beginning at \( u_i \) (or respectively \( v_i \)) with vertex \( t_i \) replaced by a hamiltonian path through \( M_i \), \( i = 1, 2, \ldots, \ell \). Further, we can find a hamiltonian path with initial vertex \( x \in V(M_i) \), \( i = 1, 2, \ldots, \ell \), by beginning with the hamiltonian path \( P_1 \) in \( H_\ell \) with initial vertex \( t_i \). If \( t_i \) is
followed by $u_i$ on $P_1$, then replace $t_i$ by a hamiltonian $x - x_i$ path in $M_i$; similarly, if $t_i$ is followed by $v_i$, replace $t_i$ by a hamiltonian $x - y_i$ path in $M_i$. Replace each $t_j$, $j \neq i$, by a hamiltonian $x_j - y_j$ or $y_j - x_j$ path in $M_j$, the replacement matching the order of $u_j$ and $v_j$ on $P_1$. Since each $M_i$, $i = 1, 2, \ldots, \ell$, is hamiltonian connected and since there are hamiltonian paths in $H_\ell$ with initial vertex $t_i$ and second vertex either $u_i$ or $v_i$, such substitutions yield a hamiltonian path in $H$ with initial vertex $t_i$. Thus $H$ is homogeneously traceable.

To see that $H$ is not hamiltonian, suppose to the contrary that $H$ is hamiltonian. Then we could start and end some hamiltonian cycle with some vertex $x \in V(M_i)$. Note that the vertices of any $M_i$ must be consecutive (although their particular order may vary) in any hamiltonian cycle of $H$, since the edges $u_i x_i$ and $y_i v_i$ must be used. However, replacing the subsequence of vertices in $M_i$ with we produce a hamiltonian cycle in $H_\ell$, which is impossible since $H_\ell$ is nonhamiltonian. Thus $H$ is homogeneously traceable nonhamiltonian with degree set $S$.

In $n_0 = 2$, we repeat the last argument with vertices $t_j$ ($j = 2, 3, \ldots, \ell$), leaving $t_1$ unchanged. Then $\deg_H t_1 = 2$ and again $H$ has a degree set $S$. An analogous argument shows $H$ is homogeneously traceable and nonhamiltonian.

Case 2: Suppose $S = \{n_0, n_1, \ldots, n_\ell\}$ and $n_i \geq 4$ for $i = 0, 1, \ldots, k$. Again consider the graph $H_\ell$ where $\ell$ is odd and $\ell \geq \max(3, k)$. We next construct a graph $H$ from $H_\ell$.
Remove vertex $u_i$ $(i = 1, 2, \ldots, \lambda)$ and its incident edges and in each case insert a copy of the graph $L_i = K_{n+1} - \{x_i f_i, f_i w_i\}$, $x_i, f_i, w_i \in V(L_i)$. Then remove each vertex $v_i$, $i = 1, 2, \ldots, \lambda$, and replace it with a copy of $M_i = K_{n+1} - \{r_i s_i, s_i z_i\}$ for $r_i, s_i, z_i \in V(M_i)$. Now insert the edges $x_i w_{i+1}$ and $r_i s_{i+1}$ for $i = 1, 2, \ldots, \lambda - 1$ and $x_i w_1$ and $r_i s_1$.

Remove each vertex $t_i$ and its incident edges, $i = 1, 2, \ldots, \lambda$, and insert a copy of $G_i = K_{n+1} - \{a_i b_i, c_i d_i\}$ where $a_i, b_i, c_i, d_i \in V(G_i)$. Then add the edges $f_i a_i, f_i c_i, s_i b_i$, and $s_i d_i$ $(i = 1, 2, \ldots, \lambda)$. If $\lambda > k$, then let $G_i = G_k$ for each $i = k + 1, k + 2, \ldots, \lambda$.

As before, the graphs $G_i, M_i,$ and $L_i$, $i = 1, 2, \ldots, \lambda$, are hamiltonian connected. An argument analogous to that used in the last case shows that $H$ is homogeneously traceable. Further, since at most one of the edges $f_i a_i$ and $f_i c_i$ (and, similarly, $s_i b_i$ and $s_i d_i$) can appear on any hamiltonian path or cycle for each $i$, an analogous argument shows that $H$ is not hamiltonian. Since $H$ has degree set $S$, Case 2 is completed.

**Case 3:** Suppose $S = \{2, n_1, n_2, \ldots, n_k\}$ and $n_i \geq 4$ for $i = 1, 2, \ldots, \lambda$. Here we proceed exactly as in Case 2 except vertex $t_1$ is not changed and the additional edge $f_1 s_1$ is inserted. The graph $H$ (see Figure 1.3) so constructed has degree set $S$. However, the edge $f_1 s_1$ can appear on no hamiltonian path; so the argument of Case 2 shows $H$ to be homogeneously traceable and nonhamiltonian.
Figure 1.3: The graph $\mathcal{H}$ (dashed lines represent missing edges in a complete graph).
CHAPTER II

SOME SUFFICIENT CONDITIONS FOR TRACEABILITY IN GRAPHS

Section 2.1

$k$-Traceable Graphs

Homogeneously traceable graphs can be generalized in a variety of ways. We consider one of those ways in this section. A graph $G$ of order $p \geq 2$ is $k$-traceable ($1 \leq k \leq p$) if $G$ is traceable from at least $k$ of its vertices. Thus a graph $G$ of order $p$ is homogeneously traceable if and only if $G$ is $p$-traceable. Every traceable graph is 2-traceable and hence we shall primarily consider $k$-traceable graphs for $k \geq 2$. Further, a graph is termed strictly $k$-traceable if it is $k$-traceable but not $(k+1)$-traceable.

For each integer $p \geq 2$ and each integer $k$, $2 \leq k \leq p$, the existence of a strictly $k$-traceable graph of order $p$ is shown by considering the graph formed by identifying an end vertex of the path $P_{p-k+1}$ of order $p - k + 1 (\geq 2)$ with a vertex of the complete graph $K_k$ of order $k$. The graph thus formed is traceable from one vertex of $P_{p-k+1}$ and $k-1$ vertices of $K_k$, but from no other vertices. It is also possible to find 2-connected graphs that are strictly $k$-traceable for each $k \geq 3$.  

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Example 2.1

The graph $G_1$ of Figure 2.1(a) has order $p_1 = n + \lambda + 5 \geq 8$ ($n \geq 3, \lambda \geq 0$) and is traceable from exactly the vertices of $V(G_1) = \{w, u_1, u_2, \ldots, u_\lambda\}$. Varying the values of $n$ and $\lambda$ yields graphs of order $p_1 \geq 8$ traceable from exactly $k$ vertices for each $k$, $7 \leq k \leq p_1 - 1$. The graph $G_2$ (with $\lambda \geq 0$) of Figure 2.1(b) has order $p_2 \geq 6$ and is traceable from exactly four vertices, namely $x_1$, $x_2$, $x_3$, and $x_4$. Variations of $G_2$ are possible to produce graphs traceable from exactly five and six vertices. This is done by subdividing the edge $\omega_0x_1$ and both $\omega_0x_1$ and $\omega_0x_2$, respectively.

![Diagram](image.png)

(a)

(b)

Figure 2.1: (a) $G_1$, a 2-connected strictly $k$-traceable graph, $k \geq 7$, of order $p_1 = n + \lambda + 5$ ($n \geq 3$, $\lambda \geq 0$). (b) $G_2$, a 2-connected strictly 4-traceable graph of order $p_2 = \lambda + 6$ ($\lambda \geq 0$).
We now examine some sufficient conditions for a graph to be $k$-traceable.

**Theorem 2.1**

Let $G$ be a graph of order $p \geq 3$. If $G$ is traceable from a vertex of degree $k$ then $G$ is $(k+1)$-traceable.

**Proof:** Let $G$ be a graph of order $p \geq 3$ and suppose that $G$ is traceable from a vertex $v_1$ of degree $k$. If $k = 1$ the result is clear; so assume $k \geq 2$ and let $P:v_1, v_2, \ldots, v_p$ be a hamiltonian path with initial vertex $v_1$. Suppose that $N(v_1) = \{v_2, v_{i_2}, v_{i_3}, \ldots, v_{i_k}\}$ where $2 < i_2 < \cdots < i_k$. If $v_p \in N(v_1)$ then $G$ is hamiltonian and the result is clear; thus assume $v_p \notin N(v_1)$.

For each edge $v_rv_{i_r}$, $2 \leq r \leq k$, it is possible to find a vertex, namely $v_{i_r-1}$, that is the initial vertex of a hamiltonian path, namely $P':v_{i_r-1}, v_{i_r-2}, \ldots, v_1, v_{i_r}, v_{i_r+1}, \ldots, v_p$.

Thus $G$ is traceable from each of the vertices $v_{i_r-1}$, $r = 2, 3, \ldots, k$, as well as from $v_1$ and $v_p$ and hence $G$ is $(k+1)$-traceable. ■

**Corollary 2.2**

If $G$ is traceable and $\delta(G) \geq n \geq 1$, then $G$ is $(n+1)$-traceable.

We note that Corollary 2.2 implies that there are no 2-connected graphs that are traceable from exactly two vertices.

We next turn our attention to graphs that are traceable from as few vertices as possible.
Theorem 2.3

If \( G \) is a graph with \( \delta(G) = n \geq 2 \) that is strictly \((n+1)\)-traceable, then either \( G = K_{n+1} \) or \( G = H + \overline{K}_{n+1} \) for some graph \( H \) of order \( n \).

Proof: Let \( G \) be a graph of order \( p \) with \( \delta(G) = n \) that is traceable from exactly \( n + 1 \) vertices, \( n \geq 2 \). Clearly \( p \geq n + 1 \). If \( p = n + 1 \), then \( G = K_{n+1} \).

Now suppose \( p > n + 1 \) and let \( P : v_1, v_2, \ldots, v_p \) be a hamiltonian path in \( G \). Since \( G \) is traceable from exactly \( n + 1 \) vertices and \( \delta(G) = n \), it follows by Theorem 2.1 that \( \deg v_1 = \deg v_p = n \).

Let \( N(v_1) = \{v_2, v_{i_2}, v_{i_3}, \ldots, v_{i_n}\} \) where \( 2 < i_2 < i_3 < \ldots < i_n \). By the proof technique of Theorem 2.1, it follows that \( G \) is traceable from the vertices \( v_1, v_{i_2}, v_{i_3}, \ldots, v_{i_n}, \) and \( v_p \). Since \( \deg v_p = n \), an argument analogous to the one above shows that if \( v_p v_s \in E(G) \) then \( v_{s+1} \) is the initial vertex of a hamiltonian path in \( G \). However, if \( s \neq p - 1 \) or \( s \neq 1 \), we have that \( s + 1 = i_j - 1 \) for some \( j \), \( 2 \leq j \leq n \), for otherwise \( G \) would be \((n+2)\)-traceable. Further, suppose \( v_1 v_j, v_p v_s \in E(G) \) where \( s \leq j - 2 \), and \( v_1 \) and \( v_p \) are not adjacent to any of the vertices \( v_{s+1}, v_{s+2}, \ldots, v_{j-1} \). Then \( G \) is traceable from \( v_{s+1} \) and unless \( s = j - 2 \), \( G \) would be \((n+2)\)-traceable. Because \( s \) was arbitrary, it follows that for each \( k > 2 \) such that \( v_1 v_k \in E(G), v_p v_{k-2} \) is an edge of \( G \). Next note that with this configuration \( G \) is traceable from \( v_{k-3} \) and \( v_{k+1} \) as well, since \( v_{k-3}, v_{k-4}, \ldots, v_1, v_k, v_{k-1}, v_{k-2}, v_p, v_{p-1}, \ldots, v_{k+1} \) is a hamiltonian path. Thus \( v_p v_2 \) and \( v_1 v_{p-1} \) are edges of \( G \) and \( i_2 + 2 = i_3 \).
\[ i_3 + 2 = i_4, \ldots, i_{n-1} + 2 = i_n \] and \[ i_n = p - 1 \] so that \[ i_2 = 4. \]

Hence, \[ N(v_i) = \{v_2, v_4, v_6, \ldots, v_{2n}\} = N(v_p) \] and \[ p = 2n + 1. \]

Further, \( G \) is traceable from each of the vertices \( v_1, v_3, v_5, \ldots, v_p \). Now applying the same arguments to the hamiltonian path with initial vertex \( v_r \) \((r = 3, 5, 7, \ldots, p - 2)\) and terminal vertex \( v_p \), we see that \( N(v_i) = N(v_r) = N(v_p) \) for each \( r = 3, 5, 7, \ldots, p - 2; \) that is, each vertex in the set \( V_1 = \{v_1, v_3, \ldots, v_p\} \) is adjacent to each vertex in the set \( V_2 = \{v_2, v_4, \ldots, v_{p-1}\} \). If any two vertices of \( V_1 \) are adjacent, then a vertex of \( V_1 \) has degree greater than \( n \); thus \( \langle V_1 \rangle = \overline{K}_{n+1} \). No hamiltonian path begins with a vertex of \( V_2 \), for otherwise \( K(n, n+1) \) is hamiltonian. Thus, additional edges among the vertices of \( V_2 \) induce no new hamiltonian paths so that \( G = H + \overline{K}_{n+1} \), where \( \overline{K}_{n+1} = \langle V_1 \rangle \) and \( H = \langle V_2 \rangle. \]

We now observe that if \( G \) is 2-connected and traceable from exactly three vertices, then \( G = K_3, G = K(2, 3), \) or \( G = K_2 + \overline{K}_3; \) that is, there are only three graphs that are 2-connected and strictly 3-traceable.

We complete this section by noting that among graphs of order 6 or less that are 2-connected, various restrictions exist as to the number of vertices from which the graph is traceable (see [13, pp. 215-24]). However, we have demonstrated that for graphs of order \( p \geq 7 \), there exist 2-connected strictly \( k \)-traceable graphs for each \( k, 4 \leq k \leq p \). The table on page 16 summarizes these possibilities.
Order | Possible Values of \( k \) for 2-Connected Strictly \( k \)-Traceable Graphs
-------|---------------------------------------------------------------
3      | 3                                                             
4      | 4                                                             
5      | 3, 5                                                          
6      | 4, 6                                                          
p \( \geq 7 \) | 4 \leq k \leq p

Section 2.2

Hamiltonian Graphs of Small Diameter

Graph theory literature abounds with sufficient conditions for hamiltonian graphs. Since every hamiltonian graph is homogeneously traceable, the aforementioned conditions are also sufficient for a graph to be homogeneously traceable. In this section we present a new sufficient condition for a graph to be hamiltonian and thus for a graph to be homogeneously traceable.

We begin by stating two well-known sufficient conditions for a graph to be hamiltonian.

**Theorem 2A (Ore [18])**

If \( G \) is a graph of order \( p \geq 3 \) such that \( \deg u + \deg v \geq p \) for all nonadjacent vertices \( u \) and \( v \), then \( G \) is hamiltonian.

**Theorem 2B (Dirac [9])**

If \( G \) is a graph of order \( p \geq 3 \) such that \( \deg v \geq p/2 \) for every vertex \( v \) of \( G \), then \( G \) is hamiltonian.

Although Dirac's theorem precedes Ore's theorem, it follows as a corollary of Ore's theorem. These results have another common
element. If \( \deg v \geq p/2 \) for every vertex \( v \) of a graph \( G \) or if \( \deg u + \deg v \geq p \) for every pair of nonadjacent vertices \( u \) and \( v \) of \( G \), then \( \text{diam } G \leq 2 \). That is, both conditions imply the graph has a small diameter. However, not all hamiltonian graphs of diameter 2 satisfy the conditions of Ore's theorem.

**Example 2.2**

Consider the graph \( G \) obtained by taking a copy of \( K_3 \) with vertices \( x_1, x_2, \) and \( x_3 \) and a copy of \( K_{p-3} \) (\( p \geq 7 \)) and joining each \( x_i \) (\( i = 1, 2, 3 \)) of \( K_3 \) to a distinct vertex \( w_i \) (\( i = 1, 2, 3 \)) of \( K_{p-3} \).

The graph \( G \) thus formed is hamiltonian. However, \( G \) fails to satisfy the conditions of Ore's theorem, since \( \deg x_1 = 3 \) and if \( w \in V(G) - \{x_1, x_2, x_3, w_1, w_2, w_3\} \), then \( \deg w = p - 4 \). Further, \( x_1 \) and \( w \) are nonadjacent and \( \deg x_1 + \deg w = p - 1 < p \).

We now investigate another sufficient condition for a graph of small diameter to be hamiltonian. We begin with a lemma.

**Lemma 2.4**

If \( G \) is a 2-connected graph of order \( p \), \( 3 \leq p \leq 8 \), containing no induced subgraph isomorphic to \( K(1, 3) \), then \( G \) is hamiltonian. Further, if \( p \geq 9 \), then \( G \) contains a cycle of length at least 8.

**Proof:** For \( 3 \leq p \leq 6 \), the result follows by investigating each graph individually (see [13], pp. 215-24, for example). If \( p = 7 \) and \( G \) is not hamiltonian, let \( C: u_0, u_1, \ldots, u_k, u_0 \) be a longest cycle in \( G \). (Such a cycle \( C \) exists since \( G \) is 2-connected.)
Further, because $G$ is connected there exists $x \in V(G) - V(C)$ such that $x$ is adjacent to a vertex of $C$. Without loss of generality, assume $xu_1 \in E(G)$ [otherwise, relabel the cycle so that $xu_1 \in E(G)$]. Since $C$ is a longest cycle of $G$, the edges $xu_0$ and $xu_2$ are not in $G$ or a longer cycle would result. But then since $G$ contains no induced subgraph isomorphic to $K(1, 3)$, the edge $u_0u_2$ must be in $G$.

Since $G$ is 2-connected, there exists a path $P$ from $x$ to $u_k$ that avoids $u_1$. Let $u_i$ be the first vertex of $P$ on $C$ and let $P'$ denote the $x - u_i$ subpath of $P$. Clearly $i \neq 0, 2$ or a cycle longer than $C$ would be obtained. Also note that the vertex preceding $u_i$ on $P'$ is not a vertex of $C$ and so it cannot be adjacent to $u_{i-1}$ or $u_{i+1}$, or again a cycle longer than $C$ would result. But then $u_{i-1}u_{i+1} \in E(C)$, for otherwise $G$ would contain an induced $K(1, 3)$.

If $k = 3$, then $u_i = u_3$ and the graph $G$ contains a cycle longer than $C$, namely the cycle $u_0, u_1, P', u_2, u_0$. Therefore, $k > 3$ and we now have two possibilities, namely $k = 4$ and $k = 5$.

If $k = 4$, then $u_i = u_3$ or $u_i = u_4$. If $u_i = u_3$, then the cycle $u_0, u_2, u_1, P', u_4, u_0$ is longer than $C$. If $u_i = u_4$, then note that the edge $u_{i-1}u_{i+1}$ is actually $u_3u_0$. Hence the cycle $u_0, u_2, u_1, P', u_4, u_3, u_0$ is longer than $C$. In each case, a contradiction is obtained, so $k = 5$.

Since $k = 5$ and $p = 7$, the path $P'$ is actually the edge $xu_5$.

Again we have choices for $u_i$. If $u_i = u_3$, then $u_4, u_2, u_1, x, u_3, u_4, u_5, u_0$ is a hamiltonian cycle. If $u_i = u_4$, then $u_3u_5 \in E(G)$ so that $u_0, u_2, u_1, x, u_4, u_3, u_5, u_0$ is a hamiltonian cycle. If $u_i = u_5$, then $u_4u_0 \in E(G)$ so that $u_0, u_5, x, u_1, u_2, u_3, u_4, u_0$ is
a hamiltonian cycle. In all cases we have a contradiction. Thus $G$ is hamiltonian.

Now suppose that $p = 8$ and that $G$ is not hamiltonian. Let $C: u_0, u_1, u_2, \ldots, u_k, u_0$ be a longest cycle of $G$. If $k < 6$, the techniques of the case $p = 7$ show a longer cycle can be obtained. Thus suppose $k = 6$, i.e., $C: u_0, u_1, u_2, \ldots, u_6, u_0$. Without loss of generality assume $x \in V(G) - V(C)$ is adjacent to $u_3$. Since $\delta(G) \geq 2$, $x$ is adjacent to some other vertex as well. If $x$ is adjacent to $u_2$ or $u_4$, then a hamiltonian cycle is easily obtained. Since $x$ is adjacent to neither $u_2$ nor $u_4$, $u_2u_4 \in E(G)$ or $G$ would contain an induced $K(1, 3)$.

Now suppose $x$ is adjacent to $u_1$. Then $x$ is not adjacent to $u_0$ and hence $u_0u_2 \in E(G)$. But then $u_0, u_2, u_1, x, u_3, u_4, u_5, u_6, u_0$ is a hamiltonian cycle of $G$. Thus $xu_1 \notin E(G)$ and by symmetry $xu_5 \notin E(G)$.

Finally, suppose $xu_0 \in E(G)$. Then $xu_1$ and $xu_6$ are not edges of $G$ and $u_1u_6$ is an edge of $G$. However, $u_0, u_1, u_6, u_5, u_4, u_2, u_3$, $x, u_0$ is then a hamiltonian cycle of $G$. By symmetry $xu_6 \notin E(G)$.

But then $x$ cannot be adjacent to any vertex of $G$ different from $u_3$. Since $\delta(G) \geq 2$, a contradiction is produced.

Hence if $p = 8$, $G$ must be hamiltonian, and so for each $p$, $3 \leq p \leq 8$, $G$ is hamiltonian.

Now suppose $p \geq 9$. The arguments given for $p = 8$ can be used to show that a cycle of maximum length in $G$ must have at least eight vertices. This is done by replacing the edge $xu_5$ by an $x - u_5$ path $P'$ where $V(P') \cap V(C) = \{u_5\}$. ■
With the aid of Lemma 2.4 we now prove the main result of this section.

**Theorem 2.5**

If $G$ is a 2-connected graph of order $p \geq 3$ with $\text{diam } G \leq 2$ and $G$ contains no induced subgraph isomorphic to $K(1, 3)$, then $G$ is hamiltonian.

**Proof:** If $3 \leq p \leq 8$, we apply Lemma 2.4. If $p \geq 9$, then by Lemma 2.4 we know $G$ contains a cycle of length at least 8. Let $C: u_0, u_1, u_2, \ldots, u_k, u_0$ be a cycle of maximum length in $G$. (Thus $k \geq 7$).

If $C$ is a hamiltonian cycle of $G$, then the proof is complete; so assume $C$ is not a hamiltonian cycle. Since $G$ is connected, there exists a vertex $x \in V(G) - V(C)$ such that $x$ is adjacent to a vertex $u_i$ of $C$. Without loss of generality we may assume $4 \leq i < k$.

Note that because $C$ is a cycle of maximum length in $G$, the vertex $x$ is not adjacent to consecutive vertices of $C$. Further, the edge $u_{i-1}u_{i+1}$ must be in $G$ or else $G$ would contain an induced $K(1, 3)$.

Now consider the distance $d(x, u_{i-2})$ between $x$ and $u_{i-2}$. If $d(x, u_{i-2}) = 1$, then the edges $xu_{i-3}$ and $xu_{i-1}$ are not in $G$ and because $G$ contains no induced $K(1, 3)$, the edge $u_{i-3}u_{i-1}$ is in $G$ (see Figure 2.2).

![Figure 2.2](image-url)
But then

\[ u_0, u_1, u_2, \ldots, u_{i-1}, u_{i-2}, x, u_i, u_{i+1}, \ldots, u_k, u_0 \]

is a cycle that includes each vertex of \( C \) and \( x \), contradicting the fact that \( C \) is a cycle of maximum length. Thus \( d(x, u_{i-2}) = 2 \). A similar argument shows no \( x - u_{i-2} \) path containing a vertex of \( V(G) - V(C) \) different from \( x \) exists in \( G \). Thus any \( x - u_{i-2} \) distance path must be of the form \( x, u_j, u_{i-2}, \) where \( u_j \in V(C) \). (Note that \( j \neq i - 1 \) and \( j \neq i + 1 \).)

**Case 1:** Assume that \( i + 1 < j < k \). Since \( x \) is not adjacent to consecutive vertices of \( C \), \( xu_{j-1} \) and \( xu_{j+1} \) are not edges of \( G \) (subscripts expressed modulo \( k + 1 \)). Then since \( G \) contains no induced \( K(1, 3) \), the edge \( u_{j-1}u_{j+1} \) belongs to \( G \). However, if \( i + 1 < j < k \), then

\[ u_0, u_1, u_2, \ldots, u_{i-2}, u_j, x, u_i, u_{i-1}, u_{i+1}, u_{i+2}, \ldots, u_{j-1}, u_{j+1}, u_{j+2}, \ldots, u_k, u_0 \]

is a cycle in \( G \) longer than \( C \), which is a contradiction, while if \( j = k \), then \( u_{k-1}u_0 \in E(G) \) and

\[ u_0, u_1, u_2, \ldots, u_{i-2}, u_k, x, u_i, u_{i-1}, u_{i+1}, u_{i+2}, \ldots, u_{k-1}, u_0 \]

is a cycle in \( G \) longer than \( C \), once again producing a contradiction.

**Case 2:** Suppose that \( 0 \leq j < i - 1 \). We again note that the edges \( xu_{j-1} \) and \( xu_{j+1} \) are not in \( G \) so that the edge \( u_{i-1}u_{j+1} \) must be in \( G \) since \( G \) contains no induced \( K(1, 3) \). If \( 0 < j < i - 1 \), then

\[ u_0, u_1, u_2, \ldots, u_{j-1}, u_j, u_{j+1}, u_{j+2}, \ldots, u_{i-2}, u_j, x, u_i, u_{i-1}, u_{i+1}, u_{i+2}, \ldots, u_k, u_0 \]
is a cycle in $G$ longer than $C$, which is contradictory. If $j = 0$, then

$$u_0, x, u_i, u_{i-1}, u_{i+1}, u_{i+2}, \ldots, u_k, u_1, u_2, \ldots, u_{i-2}, u_0$$

is a cycle in $G$ longer than $C$, again producing a contradiction. Thus the only remaining possibility is $j = i$.

Case 3: Assume that $x, u_i, u_{i-2}$ is the only $x - u_{i-2}$ path of length 2. By symmetry, $x, u_i, u_{i+2}$ is the only $x - u_{i+2}$ path of length 2. Since $G$ is 2-connected, there exists an $x - u_0$ path that does not contain $u_i$. Let $u_\ell$ be the first vertex of $C$ on this path and let $P$ denote the $x - u_\ell$ subpath. Assume $0 \leq \ell < i - 2$. If $\ell = i - 3$, then $u_{i-4}u_{i-2} \in E(G)$ and

(1)  $$u_0, u_1, u_2, \ldots, u_{i-4}, u_{i-2}, P, u_i, u_{i-1}, u_{i+1}, u_{i+2}, \ldots, u_k, u_0$$

is a cycle in $G$ longer than $C$. Thus $0 \leq \ell < i - 3$. But then $d(x, u_{i-3}) = 2$ for otherwise we may replace $P$ in (1) by $u_{i-3}, x$ to obtain a contradiction.

Again we note that any $x - u_{i-3}$ distance path must be of the form $x, u_j, u_{i-3}$ where $u_j \in V(C)$ and that $u_{j-1}u_{j+1} \in E(G)$. If $0 \leq j < i - 3$, we then see that

$$u_0, u_1, u_2, \ldots, u_{j-1}, u_{j+1}, u_{j+2}, \ldots, u_{i-3}, u_j, x, u_i, u_{i-2}, u_{i-1}, u_{i+1}, u_{i+2}, \ldots, u_k, u_0$$

is a cycle in $G$ longer than $C$ [see Figure 2.3(a)]. Further, if $i + 1 < j \leq k$, then the cycle

$$u_0, u_1, u_2, \ldots, u_{i-3}, u_j, x, u_i, u_{i-2}, u_{i-1}, u_{i+1}, u_{i+2}, \ldots, u_{j-1}, u_{j+1}, u_{j+2}, \ldots, u_k, u_0$$

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is longer than $C$ [see Figure 2.3(b)]. Since $j \neq i - 1$ and $j \neq i + 1$, and all other possibilities have led to a contradiction, we conclude that the only $x - u_{i-3}$ distance path must be of the form $x, u_i, u_{i-3}$. (Note that if $i + 2 < \ell \leq k$, a similar argument would hold by symmetry.)

![Diagram](image)

Figure 2.3

The above process can be repeated showing $u_{\ell}u_r \in E(G)$ for each $r, \ell + 1 \leq r \leq i - 2$. But then,
\[ u_0, u_1, \ldots, u_{i-1}, P, u_i, u_{i+1}, u_{i+2}, \]
\[ \ldots, u_{j-1}, u_{j+1}, u_{j+2}, \ldots, u_k, u_0 \]
is a cycle in \( G \) longer than \( C \) [see Figure 2.3(c)]. Thus a contradiction is reached in all cases so that \( G \) is hamiltonian. □

We also note that Goodman and Hedetniemi [12] have shown that if \( G \) is 2-connected and contains no induced subgraph isomorphic to either \( K(1, 3) \) or \( K(1, 3) + x \) [\( x \in E(G) \)], then \( G \) is hamiltonian. The graph of Example 2.2 fails to satisfy the conditions of this result. However, the cycles \( C_n, n \geq 8 \), satisfy the conditions of the theorem of Goodman and Hedetniemi, but fail to satisfy the conditions of Theorem 2.5.

**Corollary 2.6**

If \( G \) is a 2-connected graph of order \( p \geq 3 \) containing no induced subgraph isomorphic to any tree of order 4, then \( G \) is hamiltonian.

**Proof:** Since \( G \) contains no induced subgraph isomorphic to \( P_n \), the diameter of \( G \) is at most 2. Further, \( G \) does not contain the tree \( K(1, 3) \) as an induced subgraph. Thus, by Theorem 2.5, \( G \) is hamiltonian. □

**Corollary 2.7**

If \( G \) is a connected graph of order \( p \) containing no induced subgraph isomorphic to \( K(1, 3) \) and \( \text{diam} \ G \leq 2 \), then \( G \) is \((p - 1)\)-traceable. In fact, for \( p \geq 3 \) either \( G \) is hamiltonian or \( G = K_1 + (K_r \cup K_s) \) for \( r, s \geq 1 \).
Proof: If \( p = 2 \), then since \( G \) is connected, \( G = K_2 \) and the result holds. Thus assume \( p \geq 3 \). If \( G \) is 2-connected, the result follows from Theorem 2.5.

Thus assume \( k(G) = 1 \) and that \( v \) is a cut-vertex of \( G \). Since \( G \) contains no induced \( K(1, 3) \), the graph \( G - v \) has exactly two components \( C_1 \) and \( C_2 \). If \( |V(C_1)| = |V(C_2)| = 1 \), then \( G = P_3 \) and \( G \) is \((p-1)\)-traceable. Thus assume that at least one of \( C_1 \) and \( C_2 \) has two or more vertices, say \( |V(C_2)| \geq 2 \). Let

\[ V(C_2) = \{w_1, w_2, \ldots, w_s\}, \ s \geq 2, \]

and let

\[ V(C_1) = \{u_1, u_2, \ldots, u_r\}, \ r \geq 1. \]

Note that each vertex of \( C_2 \) is adjacent to \( v \); for if \( w_i \ (1 \leq i \leq s) \) is not adjacent to \( v \), then \( d(u_i, w_i) > 2 \), yielding a contradiction.

Further, \( \langle V(C_2) \rangle = K_s \), for if any two vertices of \( C_2 \) are not adjacent, say \( w_i \) and \( w_j \), then \( \langle [u_i, v, w_i, w_j] \rangle = K(1, 3) \), again producing a contradiction. A similar argument holds if \( |V(C_1)| \geq 2 \). Thus the graph \( G \) has the form \( G = K_1 + (K_r \cup K_s) \), where \( r, s \geq 1 \), and so \( G \) is traceable from each vertex of \( V(G) - \{v\} \); thus \( G \) is \((p-1)\)-traceable.

We have already noted that the graph of Example 2.2 does not satisfy the conditions of Ore's theorem. However, it does satisfy those of Theorem 2.5. We next show that there exist graphs which satisfy the conditions of Ore's theorem but not those of Theorem 2.5.
Example 2.3

Let \( G = K(4, 5) \) with partite sets \{\( w_1, w_2, w_3, w_4, w_5 \)\} and \{\( u_1, u_2, u_3, u_4 \)\} and let \( H \) be the graph with \( V(H) = V(G) \) and \( E(H) = E(G) \cup \{w_1 w_2, w_3 w_4, w_4 w_5\} \). Note that \( \deg u_i = 5 \) (\( i = 1, 2, 3, 4 \)) and \( \deg w_j = 5 \) (\( j \neq 4 \)) and \( \deg w_4 = 6 \). Thus for any nonadjacent vertices \( r \) and \( s \) in \( H \), \( \deg r + \deg s > 10 > |V(H)| = 9 \), so \( H \) satisfies the conditions of Ore's theorem. However, \( \langle\{u_1, w_1, w_3, w_5\}\rangle = K(1, 3) \), so \( H \) does not satisfy the conditions of Theorem 2.5.

Finally, we note that many theorems in the literature imply Ore's theorem without implying that the diameter of the graph is at most 2. Among such well-known results are those of Posa [19] and Chvátal and Erdős [8].

We conclude this section by noting that the conditions of Theorem 2.5 cannot be relaxed to include graphs of diameter 3. However, another possibility still remains.

Conjecture: If \( G \) is a 2-connected graph of order \( p \geq 3 \) with \( \text{diam } G \leq 3 \) and \( G \) contains no induced subgraph isomorphic to \( K(1, 3) \), then \( G \) is homogeneously traceable.

Section 2.3

A Sufficient Condition for a Graph To Be \( k \)-Traceable

Corollary 2.6 of Section 2.2 gives a sufficient condition for a graph to be homogeneously traceable. In a certain sense, this result is not ideal, since the same condition also implies that a
graph is hamiltonian. In this section we present a sufficient condition for a graph to be homogeneously traceable that does not imply the graph is hamiltonian. We first present a sufficient condition for a graph to be \( k \)-traceable, \( k \geq 4 \).

**Theorem 2.8**

Let \( G \) be a graph of order \( p \geq 6 \). If the vertex set of \( G \) can be partitioned into two sets, \( S_0 \) and \( S_1 \), such that

1. \( \langle S_i \rangle \) is hamiltonian (\( i = 0, 1 \)), and
2. \( k_i > 0 \) vertices of \( S_i \) are adjacent to vertices of \( S_{i+1} \) (subscripts expressed modulo 2), where \( k_i < |S_i| \)

then the graph \( G \) is \((k_0 + k_1 + 2)\)-traceable.

**Proof:** Let \( G \) be a graph of order \( p \geq 6 \) such that \( V(G) = S_0 \cup S_1 \), \( S_0 \cap S_1 = \emptyset \), and \( \langle S_i \rangle \) is hamiltonian (\( i = 0, 1 \)), where \( k_0 > 0 \) vertices of \( S_0 \) are adjacent to vertices of \( S_1 \) while \( k_1 > 0 \) vertices of \( S_1 \) are adjacent to vertices of \( S_0 \).

Let \( C_0 : v_1, v_2, v_3, \ldots, v_k, v_1 (k \geq 3) \) and \( C_1 : w_1, w_2, w_3, \ldots, w_\ell, w_1 (\ell \geq 3) \) be hamiltonian cycles in \( \langle S_0 \rangle \) and \( \langle S_1 \rangle \), respectively. Suppose that the vertex \( v_i \) (\( 1 \leq i \leq k \)) is adjacent to the vertex \( w_j \) (\( 1 \leq j \leq \ell \)). Then we note that hamiltonian paths with initial vertex \( v_{i-1} \) and \( v_{i+1} \) exist in \( G \), namely
\[
\begin{align*}
v_{i-1}, v_{i-2}, v_{i-3}, \ldots, v_1, v_k, v_{k-1}, \ldots, v_i, w_j, w_{j+1}, \\
\ldots, w_{i}, w_1, w_2, \ldots, w_{j-1}
\end{align*}
\]
and
\[
\begin{align*}
v_{i+1}, v_{i+2}, \ldots, v_k, v_1, v_2, \ldots, v_i, w_j, w_{j+1}, \\
\ldots, w_{i}, w_1, w_2, \ldots, w_{j-1}
\end{align*}
\]
Thus, if \( k_0 \) distinct vertices of \( S_0 \) are adjacent to vertices of \( S_1 \), then \( G \) must be traceable from at least \( k_0 + 1 \) vertices. Since a similar argument holds for each vertex of \( S_1 \) adjacent to a vertex of \( S_0 \), \( k_1 + 1 \) additional vertices may serve as the initial vertices of a hamiltonian path. Thus \( G \) is traceable from at least \( k_0 + k_1 + 2 \) vertices. \\

A special case of this result yields a sufficient condition for a graph to be homogeneously traceable.

**Corollary 2.9**

If \( G \) is a graph of order \( p \geq 6 \) and if the vertex set of \( G \) can be partitioned into two sets, \( S_0 \) and \( S_1 \), such that

1. \( \langle S_i \rangle \) is hamiltonian (\( i = 0, 1 \)), and
2. at most one vertex of \( S_i \) is not adjacent to a vertex of \( S_{i+1} \) (subscripts expressed modulo 2),

then \( G \) is homogeneously traceable.

**Proof:** Let \( |S_0| = r \) and \( |S_1| = s \), where \( p = r + s \). By Theorem 2.8, \( G \) is traceable from at least \( (r - 1) + (s - 1) + 2 = p \) vertices. Thus \( G \) is homogeneously traceable. \\

Note that Corollary 2.9 does not imply that the graph in question is hamiltonian.

**Example 2.4**

Consider the graph \( G \) of Figure 2.4. Let \( S_0 = \{v_{11}, v_{12}, v_1, v_2, v_3\} \) and \( S_1 = V(G) - S_0 \). Then \( \langle S_0 \rangle = C_5 \) and \( \langle S_1 \rangle = C_8 \), and so
both are hamiltonian. Further, each vertex of $S_0$ is adjacent to a vertex in $S_1$, while only $v_7$ of $S_1$ is not adjacent to a vertex of $S_0$. Thus, by Corollary 2.9, we see that $G$ is homogeneously traceable.

However, the graph $G$ is hypohamiltonian and so is not hamiltonian.

![Figure 2.4: A Hypohamiltonian Graph of Order 13.](image-url)
CHAPTER III

LINE GRAPHS AND THE HAMILTONIAN INDEX

Section 3.1

Traceability in Line Graphs

With every nonempty graph $G$ there is associated a graph $L(G)$, called the line graph of $G$, having the property that there exists a one-to-one correspondence between $E(G)$ and $V(L(G))$ such that two vertices of $L(G)$ are adjacent if and only if the corresponding edges of $G$ are adjacent.

A set $F$ of edges of a graph $G$ is a dominating set of edges if every edge of $G$ either belongs to $F$ or is adjacent to an edge of $F$. If $C = \langle F \rangle$ is a circuit, then $C$ is called a dominating circuit of $G$. Thus, a circuit $C$ is a dominating circuit of $G$ if every edge of $G$ is incident with a vertex of $C$. Similarly, if $T = \langle F \rangle$ is a trail, then $T$ is a dominating trail if every edge of $G$ is incident to a vertex of $T$.

Harary and Nash-Williams [14] have characterized those graphs $G$ for which $L(G)$ is hamiltonian.

Theorem 3A (Harary and Nash-Williams)

Let $G$ be a graph without isolated vertices. Then $L(G)$ is hamiltonian if and only if $G = K(1, n)$ for some $n \geq 3$ or $G$ contains a dominating circuit.
Since a hamiltonian cycle of a graph $G$ is a dominating circuit for $G$, the property of a graph being hamiltonian is preserved under the line graph function. We will show that the property of being homogeneously traceable is also preserved under the line graph function. To this end the following theorem will be useful.

**Theorem 3.1**

Let $G$ be a traceable graph and let $X = \{v_1, v_2, ..., v_k\}$ ($k \geq 2$) be the set of all vertices from which $G$ is traceable. Then $L(G)$ is traceable from at least $\sum_{i=1}^{k} \deg v_i - |E(H)|$ distinct vertices, where $H = \langle X \rangle$.

**Proof:** Let $G$ be a graph and let $X = \{v_1, v_2, ..., v_k\}$ ($k \geq 2$) be the set of initial vertices of the set of all hamiltonian paths in $G$. Let $v$ be an arbitrary vertex of $X$ and let $T:v = x_0, x_1, ..., x_p$ be an arbitrary hamiltonian path in $G$ with initial vertex $v$. Further, let $e$ be an arbitrary edge incident with $v$. Define the path $P_0$ in $L(G)$ to be:

$$P_0 = \begin{cases} u_e, u_f, x_1 & \text{if } e \notin E(P) \\ u_e & \text{if } e \in E(P) \end{cases}$$

where $u_f$ is the vertex of $L(G)$ corresponding to the edge $f$ of $G$ under the line graph function. For each $i$, $1 \leq i \leq p - 1$, define the path $P_i$ in $L(G)$ to be $P_i : u_{e_1}, u_{e_2}, ..., u_{e_{i-1}}, u_{x_i}, x_{i+1}$ where $e_1, e_2, ..., e_{i-1}$ is an arbitrary ordering of the edges of $E(G) - E(P)$ incident with $x_i$, that are not incident with any $x_j$, $0 < j < i$. Note that for some $i$ there may be no such edges and in this case $P_i$ is
merely the vertex $u_{x_i x_{i+1}}$. Finally, define $P_p$ to be the vertex $u_{x_i x_{p}}$ if the edge $x_j x_p$ exists in $G$.

We can now define a path $P^*$ in $L(G)$ with initial vertex $u_v$ to be $P^* : P_0, P_1, P_2, \ldots, P_{p-1}, P_p$ if $P_p$ is defined and $P^* : P_0, P_1, P_2, \ldots, P_{p-1}$ otherwise. We note that $P^*$ is a hamiltonian path of $L(G)$, since each edge of $G$ is incident with one of the vertices and hence lies on some $P_i$.

Since $v$ was an arbitrary vertex of $X$ and $e$ was an arbitrary edge incident with $v$, we have that $\sum_{i=1}^{k} \deg v_i$ such hamiltonian paths exist in $L(G)$. However, corresponding to each edge of the form $v_i v_j$ ($v_i, v_j \in X$) we have constructed two hamiltonian paths in $L(G)$ with initial vertex $u_{v_i v_j}$. Thus, if $H = \langle X \rangle$, then $L(G)$ is traceable from at least $\sum_{i=1}^{k} \deg v_i - |E(H)|$ distinct vertices.

Remark: Theorem 3.1 is best possible in the sense that there exist graphs that have $\{v_1, v_2, \ldots, v_k\} \ (k \geq 2)$ as the set of initial vertices of hamiltonian paths and such that $L(G)$ is traceable from exactly $\sum_{i=1}^{k} \deg v_i - |E(H)|$ vertices. One such graph is that of Figure 3.1(a), which is traceable from $v_1$, $v_2$, and $v_3$. We note that $\sum_{i=1}^{3} \deg v_i - |E(H)| = 4$ in this case and that $L(G)$ [see Figure 3.1(b)] is traceable from exactly four vertices, namely $u_1$, $u_2$, $u_3$, and $u_4$.

![Figure 3.1](https://example.com/figure3.1.png)

Figure 3.1: (a) The Graph $G$; (b) $L(G)$
Remark: If a graph $F$ contains a dominating trail, then $L(F)$ is traceable, so one might naturally ask if a theorem similar to Theorem 3.1 exists under the hypothesis of dominating trails or dominating paths rather than hamiltonian paths. To see that this is not the case, consider the graph $M$ obtained by identifying endpoints of three copies of the path $P_7$ (see Figure 3.2). This graph has dominating paths with initial vertices $v_i$ ($i = 1, 2, \ldots, 12$) and $\sum_{i=1}^{12} v_i - |E(H)| = 18$. However, $L(M)$ is traceable from exactly 12 vertices, namely $w_i$ ($i = 1, 2, \ldots, 12$).

Figure 3.2

An immediate consequence of Theorem 3.1 is the following.

**Corollary 3.2**

If $G$ is a nontrivial homogeneously traceable graph, then $L(G)$ is homogeneously traceable.

**Proof:** Suppose $G$ is a nontrivial homogeneously traceable graph with $p \geq 2$ vertices and $q$ edges. Then $G$ is traceable from each vertex of

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$V(G)$ and by Theorem 3.1, $L(G)$ is traceable from at least

$$\sum_{i=1}^{p} \deg_{G} v_i - |E(G)| = 2q - q = |V(L(G))|$$

distinct vertices. That is, $L(G)$ is homogeneously traceable. ■

Section 3.2

An Extension of a Theorem of Chartrand and Wall

For each integer $n \geq 1$, the $n$th iterated line graph $L^n(G)$ of a graph $G$ is defined to be $L(L^{n-1}(G))$ where $L^1(G) = L(G)$.

Chartrand and Wall [7] determined conditions under which the second iterated line graph is hamiltonian.

Theorem 3B (Chartrand and Wall)

If $G$ is a connected graph with $\delta(G) \geq 3$, then $L(G)$ contains a dominating circuit and so $L(L(G))$ is hamiltonian.

It is evident from Theorem 3B that for many graphs $L^2(G)$ is hamiltonian. Following Chartrand and Wall [7], we define the hamiltonian index $h(G)$ of a graph $G$ to be the smallest nonnegative integer $n$ such that $L^n(G)$ is hamiltonian. Thus Theorem 3B can be restated to say, if $G$ is a connected graph with $\delta(G) \geq 3$, then $h(G) \leq 2$. Chartrand [3] showed that $h(G)$ exists for most connected graphs.

Theorem 3C (Chartrand)

If $G$ is a connected graph that is not a path, then $L^n(G)$ is hamiltonian for some $n \geq 0$. 

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Our goal is to extend Theorem 3B and, with it, gain some insight into the hamiltonian index of a homogeneously traceable graph.

It is a simple observation that $L(K(1, n)) = K_n$, for $n \geq 1$. Corresponding to any vertex in a graph $G$ with $\text{deg} \ v \geq 2$ is a complete subgraph $K(v)$ in $L(G)$, formed by taking the line graph of the star $K(1, \text{deg} \ v)$ at $v$. With this in mind, we now present a generalization of Theorem 3B.

**Theorem 3.3**

If $G$ is a connected graph of order at least 3 such that no bridge in $G$ is incident to a vertex of degree 2 and no path in $G$ contains three or more consecutive vertices of degree 2, then $L(L(G))$ is hamiltonian.

**Proof:** We show that $L(G)$ contains a dominating circuit. We proceed by induction on $n$, the number of vertices of degree 2. Assume first that $G$ contains no vertices of degree 1.

Suppose $n = 0$. Then our conditions reduce to those of Theorem 2B and hence $L(G)$ contains a dominating circuit.

Now suppose $n = 1$. Then we form a subgraph $H$ of $L(G)$ as follows. For each vertex $v$ of $G$, form $K(v)$. If $\text{deg}_G v = d \geq 3$, then $K(v) = K_d$ and hence contains a spanning cycle $C(v)$. If $\text{deg}_G v = 2$, then $K(v) = K_2$. Then let

$$V(H) = V(L(G)) \quad \text{and} \quad E(H) = \bigcup_{v \in V(G)} E(C^*(v)),$$

where

$$C^*(v) = \begin{cases} C(v) & \text{if } \text{deg}_G v \geq 3 \\ K(v) & \text{if } \text{deg}_G v = 2. \end{cases}$$
Note that each vertex of $H$ belongs to exactly two of the subgraphs $C^*(v)$, while each edge belongs to exactly one. Thus if vertex $u$ of $H$ belongs to $C^*(v_1)$ and $C^*(v_2)$ where $\deg_G v_1 \geq 3$ and $\deg_G v_2 \geq 3$, then $\deg_H u = 4$. Otherwise, exactly one of $v_1$ and $v_2$ has degree 2 in $G$ so that $\deg_H u = 3$. Further, since $G$ has exactly one vertex of degree 2, the graph $H$ contains exactly two vertices of degree 3 and these vertices are adjacent.

Let $\omega_1$ and $\omega_2$ be two vertices of $H$. Then for some vertices $v_1$ and $v_2$ of $G$ (not necessarily distinct), $\omega_1$ is a vertex of $C^*(v_1)$ while $\omega_2$ is a vertex of $C^*(v_2)$. Since $G$ is a connected graph, there is a $v_1$-$v_2$ path $P$ in $G$. Then the subgraph $H'$ of $H$ defined by

$$V(H') = \bigcup_{v \in V(P)} V(C^*(v)) \quad \text{and} \quad E(H') = \bigcup_{v \in V(P)} E(C^*(v))$$

is a connected subgraph of $H$ containing $\omega_1$ and $\omega_2$. Thus the edge-connectivity $\kappa_1(H)$ of $H$ is at least 1.

Suppose $\kappa_1(H) = 1$. Then $H$ contains a bridge $e = \omega x$.

Clearly $e \notin E(C^*(v))$ where $\deg_G v \geq 3$. Thus $e \in E(K(v))$ where $\deg_G v = 2$. But the only edges of $L(G)$ not in $H$ are diagonals of $C^*(v)$ where $\deg_G v \geq 3$. Thus, each $\omega - x$ path in $L(G)$ must contain $e$; that is, $e$ is a bridge of $L(G)$. But $L(G)$ contains a bridge if and only if $G$ contains a vertex of degree 2 lying on no cycle. However, by hypothesis, this is not possible; hence, $\kappa_1(H) \geq 2$.

We now form the graph $H^* = H - e$ and note that $\kappa_1(H^*) \geq 1$ and each vertex of $H^*$ now has even degree. Hence $H^*$ is eulerian and so contains a eulerian circuit $D$. Further, $V(D) = V(H^*) = V(H) = V(L(G))$ and so $D$ is a dominating circuit of $L(G)$. 

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Now suppose that if a graph satisfies the hypothesis of the theorem and has $t \geq 1$ vertices of degree 2, then its line graph contains a dominating circuit. Let $G$ satisfy the hypothesis and have $t + 1 \geq 2$ vertices of degree 2.

**Case 1:** Suppose $G$ contains two adjacent vertices $u_1$ and $u_2$ of degree 2. Form a new graph $G'$ from $G$ by deleting the edge $e = u_1u_2$ and identifying the vertices $u_1$ and $u_2$, denoting the new vertex by $u$. The graph $G'$ is connected and has no bridge incident to a vertex of degree 2 and no path with more than two consecutive vertices of degree 2. Further, $G'$ contains $t$ vertices of degree 2, so by the induction hypothesis, $L(G')$ contains a dominating circuit $D'$.

Suppose that $u_0$ is the other vertex of $G$ adjacent to $u_1$, while $u_3$ ($\neq u_1$) is adjacent to $u_2$. In $L(G)$, let $v_{i,j}$ correspond to the edge $u_iu_j$ of $G$ ($i, j \in \{0, 1, 2, 3\}, i \neq j$). In $G'$, vertex $u$ is adjacent to $u_0$ and $u_3$ so in $L(G')$ let $w_{0,1}$ and $w_{2,3}$ correspond to the edges $uu_0$ and $uu_3$ of $G$, respectively (see Figure 3.3).

![Figure 3.3](image_url)
If \(D\) contains the subpath \(P':w_0, w_2,\) then form a circuit \(D\) in \(L(G)\) from \(D'\) by replacing \(P'\) by the path \(v_0, v_1, v_2,\) and replacing all other vertices of \(D'\) by the corresponding vertices of \(D.\)

Since \(D'\) is a dominating circuit in \(L(G'),\) the circuit \(D\) is a dominating circuit in \(L(G)\).

If \(D'\) does not contain the subpath \(P',\) then since \(D'\) is a dominating circuit of \(L(G'),\) it follows that \(D'\) contains at least one of \(w_0, w_2,\) If \(D'\) contains both \(w_0, w_2,\) then let \(D\) be the circuit of \(G\) obtained from \(D'\) by replacing each vertex of \(D'\) by the corresponding vertex of \(D.\) Otherwise, without loss of generality, let \(D'\) contain \(w_2,\) but not \(w_0.\) Further, let \(w_1, w_2, \ldots, w_m (m \geq 2)\) be the vertices of \(L(G'),\) other than \(w_2,\) that are adjacent to \(w_0.\) Then \(\{w_1, w_2, \ldots, w_m\} = K_m.\)

Each of the edges \(w_kw_0 (k = 1, 2, \ldots, m)\) must be incident with a vertex of \(D.\) If some path \(w_i, w_j (1 \leq i < j \leq m)\) is a subpath of \(D',\) form a new circuit \(D''\) from \(D'\) by replacing \(w_i, w_j,\) by \(w_i, w_0, w_j,\) and replacing each of the other vertices of \(D'\) by the corresponding vertices of \(L(G).\) If no path \(w_i, w_j\) is a subpath of \(D',\) then each of the vertices \(w_k (k = 1, 2, \ldots, m)\) must lie on \(D'\) (since \(D'\) is a dominating circuit). Form the circuit \(D''\) by replacing some vertex \(w_i (1 \leq i \leq m)\) by the path \(w_i, w_0, w_i,\) and adding \(w_i, w_j, w_k\) to \(D''.\)

In all cases, \(D''\) now contains both \(w_0, w_2,\) and is a dominating circuit of \(L(G').\) We now form a circuit \(D\) from \(D''\) by replacing \(w_0\) by \(v_0, w_2\) by \(v_2,\) Then \(D\) is a dominating circuit of \(L(G)\) since each edge of \(L(G')\) is incident with a vertex of \(D''\) and the new edges \(v_0v_2\) and \(v_2v_3\) are now incident to vertices of \(D.\)
Case 2: Suppose no two vertices of degree 2 in $G$ are adjacent. Let $u_1$ be a vertex of $G$ having degree 2, adjacent to vertices $u_0$ and $u_2$. Form a new graph $G'$ from $G$ by removing $u_1$ and its incident edges and inserting the edge $u_0u_2$ if it does not already exist. Every vertex of degree 2 in $G'$ lies on some cycle since, by hypothesis, no bridge is incident with a vertex of degree 2. Further, $G'$ contains $t$ vertices of degree 2. Thus, by the induction hypothesis, $L(G')$ contains a dominating circuit $D'$. Let the vertices $w_1, w_2, \ldots, w_k$ ($k \geq 2$) in $L(G)$ correspond to the edges $e_1, e_2, \ldots, e_k$ incident to $u_0$ (and different from $u_0u_1$) in $G$ and let the vertices $r_1, r_2, \ldots, r_m$ ($m \geq 2$) in $L(G)$ correspond to the edges $f_1, f_2, \ldots, f_m$ incident to $u_2$ (and different from $u_1u_2$) in $G$. Let the vertex $w_{02}$ in $L(G')$ correspond to the edge $u_0u_2$ in $G'$ (see Figure 3.4).

![Diagram](image-url)  
(a) the structure in $G$  
(b) the corresponding structure in $L(G)$  
(c) the structure in $G'$  
(d) the corresponding structure in $L(G')$  

Figure 3.4

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Subcase 2a: Suppose the dominating circuit $D'$ of $L(G')$ contains a subpath of the form $w_i, w_{02}, w_j$ where $1 \leq i, j \leq k$, with $i \neq j$. Form a new circuit $D$ in $L(G)$ from $D'$ by replacing $w_{02}$ by $v_{01}$ and replacing each remaining vertex of $D'$ by the corresponding vertex in $L(G)$. If each of the edges $r_jv_{12}$ ($1 \leq j \leq m$) is on $D$ or is incident to a vertex of $D$, then $D$ is a dominating circuit of $L(G)$.

Otherwise, if one of the edges $r_i r_j$ ($i \neq j, 1 \leq i, j \leq m$) lies on $D$, then replace the subpath $r_i$, $r_j$ by the path $r_i$, $v_{12}$, $r_j$. If no edge $r_i r_j$ lies on $D$ then at least one of the vertices $r_i$ ($1 \leq i \leq m$) must lie on $D$, since $D'$ is a dominating circuit of $L(G')$; so replace $r_i$ by the path $r_i$, $v_{12}$, $r_j$ where $1 \leq j \leq m$ and $i \neq j$. Then $D$ is a dominating circuit of $L(G)$ since each of the edges of the form $r_i v_{12}$ ($1 \leq i \leq m$) and the edge $v_{01}v_{12}$ is incident to the vertex $v_{12}$ of $D$.

Further, all other edges of $L(G')$ are incident to a vertex of $D'$ so that the corresponding edges of $L(G)$ are incident to a vertex of $D$. Note that an analogous argument holds if a subpath of the type $r_i, w_{02}, r_j$ lies on $D'$.

Subcase 2b: Suppose $D'$ contains exactly one subpath of the form $w_i, w_{02}, r_j$ ($1 \leq i \leq k, 1 \leq j \leq m$). Form a circuit $D$ of $L(G)$ from the circuit $D'$ of $L(G')$ by replacing the subpath $w_i, w_{02}, r_j$ by the path $w_i, v_{01}, v_{12}, r_j$. Then $D$ is a dominating circuit of $L(G)$. A similar argument holds if $D'$ contains exactly one subpath of the form $r_j, w_{02}, w_i$.

Subcase 2c: Suppose $D'$ contains $s \geq 2$ subpaths of the form $w_i, w_{02}, r_j$ or $r_j, w_{02}, w_i$ ($1 \leq i \leq k, 1 \leq j \leq m$). Then with each such subpath, the vertex $w_{02}$, of course, lies on $D'$. These
occurrences of \( w_{q_2} \) divide \( D' \) into \( s + 1 \) subtrails \( T_1, T_2, \ldots, T_{s+1} \).

Without loss of generality, we can assume \( w_{q_2} \) is not the initial vertex of \( D' \) so that \( D' \) can be described as the circuit

\[
T_1, w_{q_2}, T_2, w_{q_2}, T_3, \ldots, w_{q_2}, T_{s+1}.
\]

The subtrails \( T_1 \) and \( T_{s+1} \) have, respectively, as terminal and initial vertices, either some \( w_i (1 \leq i \leq k) \) or some \( r_j (1 \leq j \leq m) \), while for \( 2 \leq t \leq s \), the trail \( T_t \) has both initial and terminal vertices from the set \( \{w_1, w_2, \ldots, w_k, r_1, r_2, \ldots, r_m\} \). There are \( 2s - 2 \) such initial and terminal vertices, \( s - 1 \) of which are chosen from \( \{w_1, w_2, \ldots, w_k\} \) and \( s - 1 \) of which are chosen from \( \{r_1, r_2, \ldots, r_m\} \).

Let \( T_k^{-1} \) denote the trail obtained by traversing \( T_k \) in reverse order. We now form a new circuit \( F^* \) in \( L(G') \) from \( D' \) as follows. Traverse \( T_1, w_{q_2}, T_2 \). Next traverse \( T_k \) or \( T_k^{-1} (k \geq 3) \) where the initial vertex of this trail is of the same type (either \( w_i \) or \( r_j \)) as the terminal vertex of \( T_2 \). Now repeat the above procedure using any trail \( T_k \) or \( T_k^{-1} \) that has not yet been traversed. We continue this procedure until all subtrails \( T_j (1 \leq j \leq s + 1) \) have been used. We have now created a circuit in \( L(G') \) having at most one subpath of the form \( w_i, w_{q_2}, r_j \) or \( r_j, w_{q_2}, w_i \); hence, we can apply Subcase 2b to the circuit \( D^* \) to obtain the desired dominating circuit in \( L(G) \).

**Subcase 2d:** Suppose \( w_{q_2} \notin V(D') \). Since \( D' \) is a dominating circuit of \( L(G') \), each of the edges \( w_i; w_{q_2} (1 \leq i \leq k) \) and \( r_j; w_{q_2} (1 \leq j \leq m) \) must be incident to a vertex of \( D' \). If \( D' \) contains a

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subpath \( w_i, w_j \) (or \( r_i, r_j \)) then form a circuit \( D \) in \( L(G) \) by replacing \( w_i, w_j \) by the path \( w_i, v_{01}, w_j \) (or replace \( r_i, r_j \) by \( r_i, v_{12}, r_j \)). Then \( D \) is a dominating circuit of \( L(G) \). If \( D' \) contains no subpaths of the form \( w_i, w_j \) or \( r_i, r_j \), then each of the vertices \( w_1, w_2, \ldots, w_k \) and \( r_1, r_2, \ldots, r_m \) must lie on \( D' \). Then replace some vertex \( w_i \) (\( 1 \leq i \leq k \)) on \( D' \) by the path \( w_i, v_{01}, w_j, w_k \) where \( 1 \leq j \leq k \), \( i \neq j \) or, similarly, replace some \( r_t \) with \( 1 \leq t \leq m \) by \( r_t, v_{12}, r_t, r_t \) where \( 1 \leq t \leq m \), \( t \neq t \). The circuit \( D \) is a dominating circuit of \( L(G) \) since the edge \( v_{01}v_{12} \) is incident with a vertex of \( D \) and since all edges of \( L(G') \) are incident with a vertex of \( D' \), the corresponding edges of \( L(G) \) are incident with a vertex of \( D \).

Thus in all cases \( L(G) \) contains a dominating circuit. Hence, if \( G \) satisfies the hypothesis of the theorem, but contains no vertices of degree 1, then \( L(G) \) contains a dominating circuit, so that by Theorem 3A, \( L(L(G)) \) is hamiltonian.

Assume now that \( G \) contains vertices of degree 1. Since \( G \) has no bridge incident to vertices of degree 2, if \( \text{deg}_G v = 1 \), then \( v \) is adjacent to a vertex \( u \) of degree 3 or more. Let vertices \( v_1, v_2, \ldots, v_k \) (\( k \geq 2 \)) also be adjacent to \( u \).

Suppose that \( H = G - v \) has no vertices of degree 1. Then, by the previous case of the proof, \( L(H) \) contains a dominating circuit \( C \). If \( x \in V(L(G)) \) where \( x \) corresponds to the edge \( e = uv \) in \( G \), we show \( x \) can be included on a dominating circuit of \( L(G) \).

Let the vertices \( x_1, x_2, \ldots, x_k \) in \( L(G) \) correspond to the edges \( uv_1, uv_2, \ldots, uv_k \) in \( G \). Note that \( x x_{j} \) is an edge of \( L(H) \) for \( j = 1, 2, \ldots, k \). If \( C \) contains the subpath \( x_i, x_{j} \) (\( i \neq j \), \( 1 \leq i \), \( 1 \leq j \),
\( j \leq k \), then replace this subpath by the path \( x_i, x, x_j \). If \( C \) contains no such subpath, then at least one of the vertices \( x_i \) must be on \( C \) (since the edges \( x_i x_j \) exist). Then replace the vertex \( x_i \) by the path \( x_i, x, x_j, x_i \). In either case, the new circuit formed is a dominating circuit of \( L(H) \).

Note that if more than one vertex of degree 1 is incident with vertex \( u \), then the vertices of \( L(G) \) corresponding to the edges incident with these vertices may be traversed consecutively. That is, in the dominating circuit constructed above, replace \( x \) by \( x, y_1, y_2, \ldots, y_n \) where the vertices \( x, y_1, y_2, \ldots, y_n \) correspond to the edges incident to \( u \) and the vertices of degree 1 adjacent to \( u \).

Repeating the above procedure for each such occurrence of vertices of degree 1 yields the desired dominating circuit of \( L(G) \). Thus, \( L(L(G)) \) is hamiltonian.

Several results can now be obtained as a consequence of Theorem 3.3. For the first result, we also employ a result of Chartrand and Stewart [6].

**Theorem 3.3 (Chartrand and Stewart)**

If \( G \) is a \( k \)-connected graph \( (k \geq 1) \), then \( L(G) \) is \( k \)-connected.

**Corollary 3.4**

If \( G \) is a 2-connected graph with at most \( n \geq 2 \) consecutive vertices of degree 2 on any path in \( G \), then \( L^n(G) \) is hamiltonian.
Proof: If \( G \) contains a path with at most \( n \) (\( \geq 1 \)) consecutive vertices of degree 2, then necessarily \( L(G) \) contains a path with at most \( n - 1 \) consecutive vertices of degree 2. Hence, \( L^{n-2}(G) \) has at most two consecutive vertices of degree 2 on any path. By repeatedly applying Theorem 3D, \( L^{n-2}(G) \) is 2-connected and hence has no bridges. Thus, by Theorem 3.3, \( L(\min(L^{n-2}(G))) = L^n(G) \) is hamiltonian. \( \blacksquare \)

This result is best possible in the sense that there exist 2-connected graphs with \( n \geq 2 \) consecutive vertices of degree 2 such that \( L^{n-1}(G) \) is not hamiltonian.

Example 3.2

Consider the graph \( G \) of Figure 3.5(a). The graph \( G \) is clearly 2-connected and contains at most two consecutive vertices of degree 2 on any path. However, \( L(G) = L^{n-1}(G) \) (\( n = 2 \)) is not hamiltonian, in fact, \( L(G) \) in Figure 3.5(b) is homogeneously traceable and non-hamiltonian.

Figure 3.5

We next note that Theorem 3.3 allows us to bound the hamiltonian index of any homogeneously traceable graph.
**Corollary 3.5**

If $G$ is a homogeneously traceable graph of order $p \geq 3$, then $L(L(G))$ is hamiltonian; that is, $h(G) \leq 2$.

**Proof:** Since $G$ is homogeneously traceable of order $p \geq 3$, $G$ is 2-connected. If $G$ is hamiltonian, then $L^2(G)$ is hamiltonian. If $G$ is not hamiltonian, then $G$ contains no path with three or more consecutive vertices of degree 2, because no vertex is adjacent to two or more vertices of degree 2. Hence, by Theorem 3.3, $L(L(G))$ is hamiltonian. 

Corollary 3.5 is best possible, since there exist homogeneously traceable nonhamiltonian graphs such that $L(G)$ is also homogeneously traceable and nonhamiltonian. The graph $H$ of Figure 3.6(a) is homogeneously traceable nonhamiltonian and has a homogeneously traceable line graph $L(H)$ [Figure 3.6(b)]. However, $L(H)$ is not hamiltonian.

![Diagram](image)

(a) $H$, a homogeneously traceable nonhamiltonian graph  
(b) $L(H)$, also nonhamiltonian

**Figure 3.6**

Finally, we note that Theorem 3B follows as a corollary to Theorem 3.3 as well.
Corollary 3.6 (Chartrand and Wall)

If $G$ is a connected graph with $\delta(G) \geq 3$, then $L(L(G))$ is hamiltonian.
CHAPTER IV

TRACEABILITY IN THE SQUARE OF A TREE

The square of a connected graph $G$ is a graph $G^2$ where $V(G^2) = V(G)$ and such that $uv \in E(G)$ if and only if $1 \leq d(u, v) \leq 2$ in $G$. Since $G^2$ contains $G$ as a subgraph, it follows that $G^2$ is hamiltonian ($k$-traceable for some $k$) whenever $G$ is hamiltonian ($k$-traceable).

Nash-Williams and Plummer independently conjectured that the square of a 2-connected graph is hamiltonian. In 1974 Fleischner [10] proved this conjecture to be correct.

Theorem 4A (Fleischner)

Let $G$ be a 2-connected graph. Then $G^2$ is hamiltonian.

Various theorems have employed Theorem 4A to obtain stronger results. For example, it has been shown [5] that the square of a 2-connected graph is hamiltonian connected.

Theorem 4B (Chartrand, Hobbs, Jung, Kapoor, Nash-Williams)

Let $G$ be a 2-connected graph. Then $G^2$ is hamiltonian connected.

Harary and Schwenk [14] determined all trees with a hamiltonian square and investigated some properties of such trees.
Theorem 4C (Harary and Schwenk)

Let $T$ be a tree of order $p \geq 3$. The following statements are equivalent:

1. $T^2$ is hamiltonian.
2. $T$ does not contain $S(K(1, 3))$ as a subgraph.
3. $T$ minus its end vertices is a path.

Theorem 4D (Harary and Schwenk)

If $T$ is a tree and $T^2$ is hamiltonian, then any hamiltonian cycle contains exactly two edges of $T$, and these are the terminal edges of a longest path. Moreover, the terminal edges of every longest path lie on some hamiltonian cycle.

The object of this chapter is to explore the concept of $k$-traceability in the square of a graph. By Theorem 4A, the square of any 2-connected graph of order $p$ is $k$-traceable for each $k$, $1 \leq k \leq p$. Thus, for the remainder of this chapter, we shall concentrate on $k$-traceability in the square of the most important class of graphs with connectivity 1, namely trees. To this end, some terminology will be helpful.

A tree $T = T(n_1, n_2, \ldots, n_j)$, $j \geq 3$, with $n_1 \geq n_2 \geq \ldots \geq n_j$ is called star-like if it is isomorphic to the graph formed by joining an end-vertex of each of the paths $P_{n_1}$, $P_{n_2}$, $\ldots$, $P_{n_j}$ to a vertex (which we commonly denote $r$).

If $T$ is a tree, a root of $T$ is any vertex $r$ such that $\deg r \geq 3$. A branch at $r$ is any component of $T - r$, for some root $r$. 

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The *initial vertex* of a branch is the unique vertex of the branch adjacent to the root. An *i-root* is a root with exactly *i* nontrivial branches, that is, an *i*-root is a root of an induced $S(K(1, i))$ but not an $S(K(1, i + 1))$. We say the tree $T$ has a *trunk* if there is a path $\mathcal{T}$ in $T$, whose initial and terminal vertices are *i*-roots, $i \geq 3$, and which contains each *i*-root with $i \geq 3$. The path $\mathcal{T}$ constitutes the trunk of $\mathcal{T}$. At this time, an example is appropriate.

**Example 4.1**

In the tree $T$ of Figure 4.1(a) are five roots $r_j$, $j = 1, 2, 3, 4, 5$. The root $r_5$ is a 1-root while $r_1$, $r_3$, and $r_4$ are 4-roots and $r_2$ is a 5-root. The tree $T$ also has a trunk since the $r_1 - r_4$ path contains every *i*-root of $T$, $i \geq 3$. We also note that $r_2$ has seven branches while $r_1$, $r_4$, and $r_5$ have four branches and $r_3$ has five branches. (See Figure 4.1 on the following page.)

We now begin our investigation with a lemma.

**Lemma 4.1**

If $T$ is star-like with *i*-root $r$ ($i \geq 3$), then no hamiltonian path in $T^2$ can begin and end in the same branch at $r$.

**Proof:** Let $T$ be star-like with nontrivial branches $B_1, B_2, \ldots, B_n$, $n \geq 3$. Without loss of generality, assume $T^2$ has a hamiltonian path beginning in $B_1$ which proceeds through $B_2, B_3, \ldots, B_n$ and eventually concludes in $B_1$. Note that each nontrivial branch after $B_1$ can be entered and exited only once. If the branches are encountered in the
order $B_1, B_2, \ldots, B_n, B_1$, there are $n$ branch changes and $n \geq 3$.
But in order to reenter $B_1$, either its initial vertex $u_1$ or $v$ must
not yet have been traversed. However, entering and exiting both of
the branches $B_2$ and $B_3$ requires the use of both the initial vertex
of the branch and the root $r$, and since $n > 3$ it will be impossible
to reenter $B_1$.

Clearly we cannot begin in a trivial branch for this would
imply $T^2$ was hamiltonian; however, this contradicts Theorem 4C.$\blacksquare$

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Corollary 4.2

If \( T \) is a tree and \( v \) is an \( i \)-root of \( T \) (\( i \geq 3 \)), then no Hamiltonian path in \( T^2 \) can begin and end in the same branch at \( v \).

Example 4.2

Now consider the tree \( T_1 \) of Figure 4.1(b). We note that \( T_1 \) contains four roots \( r_j \), \( j = 1, 2, 3, 4 \). However, \( T_1 \) has no trunk since no path can contain all of the roots \( r_1, r_3, \) and \( r_4 \). Further, since each is a 4-root, no Hamiltonian path in \( T_1^2 \) can begin and end in the same branch at \( r_1 \) (or \( r_3 \), or \( r_4 \)).

Lemma 4.3

If \( T \) is a tree containing an \( i \)-root \( r \) (\( i \geq 3 \)), then \( T^2 \) is not traceable from \( r \).

Proof: Let \( T \) be a tree with \( i \)-root \( r \) (\( i \geq 3 \)) and let \( B_1, B_2, \ldots, B_i \) be the nontrivial branches of \( T \) at \( r \). If there is a Hamiltonian path in \( T^2 \) with initial vertex \( r \), then each branch at \( r \) can be entered and exited exactly once. But entering and exiting any such branch requires traversing its initial vertex and \( r \). Hence, at most two nontrivial branches about \( r \) can be traversed. But \( r \) has at least three nontrivial branches. Thus \( T^2 \) is not traceable from \( r \).

Lemma 4.4

If \( T \) is star-like with root \( r \), then \( T^2 \) is traceable from:

(1) every vertex, if \( T \) has no \( i \)-root, \( i \geq 3 \).
(2) \( V(T) = \{r\} \), if \( r \) is a 3-root.

(3) \( V(T) = N[r] \), if \( r \) is a 4-root.

(4) no vertex, if \( r \) is an \( i \)-root, \( i \geq 5 \).

**Proof:** Let \( T \) be star-like with root \( r \). If \( T \) contains no \( i \)-root, \( i \geq 3 \), then \( T \) contains no induced \( S(K(1, 3)) \), so by Theorem 4C, \( T^2 \) is hamiltonian. Therefore, \( T^2 \) is traceable from each vertex; thus (1) holds.

If \( r \) is a 3-root, let \( B_1, B_2, \) and \( B_3 \) be the nontrivial branches at \( r \) and let \( v_1, v_2, \ldots, v_z \) be the trivial branches at \( r \), if any exist. Let \( u_{i,j} \) be on the branch \( B_i \) and such that \( d(r, u_{i,j}) = j \).

By Lemma 4.3, \( T^2 \) is not traceable from \( r \). Let \( u_{1,s} \) be an arbitrary vertex on \( B_1 \). We construct a hamiltonian path in \( T^2 \) with initial vertex \( u_{1,s} \) in the following manner.

Let the path \( P_1 \) in \( B_1 \) be

\[
\begin{align*}
&u_{1,s}, u_{1(s+2)}, u_{1(s+4)}, \ldots, u_{1(n_1)}, u_{1(n_1-1)}, u_{1(n_1-3)}, \\
&\phantom{u_{1,s},} u_{1(s-1)}, u_{1(s-3)}, \ldots, u_{11}
\end{align*}
\]

if \( s \) and \( n_1 \) have the same parity, and let \( P_1 \) be

\[
\begin{align*}
&u_{1,s}, u_{1(s+2)}, u_{1(s+4)}, \ldots, u_{1(n_1-1)}, u_{1(n_1)}, u_{1(n_1-2)}, u_{1(n_1-4)}, \\
&\phantom{u_{1,s},} u_{1(s-1)}, u_{1(s-3)}, \ldots, u_{11}
\end{align*}
\]

if \( s \) and \( n_1 \) are of opposite parity. Similarly, define a path \( P_2 \) in \( B_2 \) to be

\[
\begin{align*}
&u_{2,1}, u_{2,3}, u_{2,5}, \ldots, u_{2(n_2)}, u_{2(n_2-1)}, u_{2(n_2-3)}, \ldots, u_{22}
\end{align*}
\]

if \( n_2 \) is odd, and let \( P_2 \) be defined as
if \( n_2 \) is even. Note that an analogous path \( P_3 \) with initial vertex \( u_{31} \) and terminal vertex \( u_{32} \) exists in \( E_3 \).

We now construct a hamiltonian path in \( T^2 \) with initial vertex \( u_{1,0} \) to be

\[ P; P_1, P_2, r, v_1, v_2, \ldots, v_t, P_3 \]

if there are nontrivial branches and to be

\[ P; P_1, P_2, r, P_3, \] otherwise.

Since \( u_1 \) was an arbitrary vertex in \( B_1 \), \( T^2 \) is traceable from each vertex of \( B_1 \) and because \( B_1 \) was an arbitrary nontrivial branch, \( T^2 \) is traceable from each vertex in \( B_1, B_2, \) and \( B_3 \). Note that \( P \) also shows \( T^2 \) is traceable from \( v_i \) and since, if trivial branches exist, they may be initially traversed in any order, followed by the vertices of \( B_1, r \), and the vertices of \( B_2 \) and \( B_3 \), we conclude that \( T^2 \) is traceable from each vertex of \( V(T) - \{ r \} \).

Next suppose \( r \) is a 4-root with nontrivial branches \( B_1, B_2, B_3, \) and \( B_4 \) and let \( u_{i,j} \) be the vertex on branch \( B_i \) a distance \( j \) from \( r \). Again let \( v_1, v_2, \ldots, v_t \) \((t \geq 0)\) be the vertices which constitute the trivial branches at \( r \), if any. Again, by Lemma 4.3, no hamiltonian path can begin at \( r \). Now suppose a hamiltonian path begins with a vertex in \( N(r) \), then before a vertex in the second nontrivial branch is traversed, the vertex \( r \) must be traversed. However, it is then impossible to enter and exit the three remaining nontrivial branches. Thus \( G \) is not traceable from any vertex in \( N[r] \).
An argument similar to that given when \( r \) is a 3-root shows any other vertex of \( T \) can be the initial vertex of a hamiltonian path in \( T^2 \).

Finally, suppose \( r \) is an \( i \)-root (\( i \geq 5 \)), with nontrivial branches \( B_1, B_2, \ldots, B_i \). Since \( r \) must be traversed within the first three branch changes, it is impossible to traverse vertices in five or more nontrivial branches. Thus \( T^2 \) is not traceable if \( r \) is an \( i \)-root where \( i \geq 5 \).

**Corollary 4.5**

If \( T \) is a tree and \( T \) contains \( S(K(1, 5)) \) as an induced subgraph, then \( T^2 \) is not traceable.

We next turn our attention to exactly which trees \( T \) have a traceable square and which vertices can be the initial vertices of a hamiltonian path in \( T^2 \).

Define the graph \( F_0 \) to be the graph obtained by taking two copies of \( S(K(1, 3)) \) and joining the roots in each copy by an edge [see Figure 4.2(a)]. Define the graph \( F_i, i \geq 1 \), to be the graph obtained by taking a copy of \( F_0 \) and \( i \) vertices, \( w_1, w_2, \ldots, w_i \). Subdivide the edge joining the two roots of \( F_0 \) a total of \( i \) times and join each of the vertices to one and only one of \( w_1, w_2, \ldots, w_i \) [see Figure 4.2(b)].
Theorem 4.6

If $T$ is a tree, then $T^2$ is traceable if and only if $T$ contains a trunk and $T$ does not contain an $S(K(1, 5))$ or any $F_i$, $i = 0, 1, 2, ...$ as a subgraph.

Proof: Let $T$ be a tree such that $T^2$ is traceable and suppose $T$ does not contain a trunk. Then there exist roots $r_1, r_2, r_3$, each with three or more nontrivial branches, which do not lie on a path. Hence, there exists a root $r$ such that the $r - r_i$ paths $P_i$ ($i = 1, 2, 3$) are all disjoint, except for $r$. Let $B_2$ be the branch at $r$ containing $r_2$ and without loss of generality assume $B_2$ is intermediate to a hamiltonian path $P$ in $T^2$; that is, the path $P$ neither begins nor ends in $B_2$.

Then either the vertices of $B_2$ must all be traversed after $B_2$ is entered, in which case Corollary 4.2 is violated, or $B_2$ must be exited, entered, and exited again. But this is also impossible, as no branch can be entered and exited twice. Hence $T$ contains a trunk, which we denote by $T$. 

Figure 4.2

(a) the graph $F_0$  
(b) the graph $F_i$, $i \geq 1$
Now note that \( T \) contains no \( S(K(1, 5)) \), by Corollary 4.5, so suppose \( T \) contains \( F_i \), for some \( i \geq 0 \).

**Case 1:** Suppose \( T \) contains \( F_0 \). Let \( r_1 \) and \( r_2 \) be the roots of \( F_0 \) and suppose \( B_1, B_2, \) and \( B_3 \) are branches at \( r_1 \) not containing \( r_2 \), while \( B_4, B_5, \) and \( B_6 \) are branches at \( r_2 \) not containing \( r_1 \).

Without loss of generality, suppose vertices in \( B_1 \) are traversed prior to those in \( B_2, B_3, \ldots, B_6 \). In particular, then, \( P \) cannot end in \( B_1, B_2, B_3, \) or at \( r_1 \); otherwise, \( P \) would begin and end in the same branch at \( r_2 \) (that containing \( r_1, B_1, B_2, B_3 \)), contradicting Corollary 4.2.

Also, the branch \( B^* \) at \( r_1 \) containing \( B_4, B_5, \) and \( B_6 \) cannot be entered twice; hence, the branches \( B_1, B_2, B_3, \) and the vertex \( r_1 \) must be completely traversed before \( B^* \) is entered. However, as \( B_1, B_2, B_3, \) and \( B^* \) are all branches at \( r_1 \), this implies that \( r_1 \) is not the last vertex traversed before \( B^* \) is entered. Hence, \( r_2 \) is the first vertex traversed in \( B^* \). But then it is impossible to traverse the remaining three branches at \( r_2 \), which implies \( T^2 \) is not traceable, contradicting the hypothesis. Thus, \( F_0 \) is not a subgraph of \( T \).

**Case 2:** Suppose \( T \) contains some \( F_i, i \geq 1 \). Let \( r_1 \) and \( r_2 \) be the 4-roots of \( F_i \), while \( v_1, v_2, \ldots, v_i \) are the 2-roots of \( F_i \) lying consecutively on the \( r_1 - r_2 \) path in \( F_i \). Let \( \omega_j \) be adjacent to \( v_j \) in \( F_i \) \((j = 1, 2, \ldots, i)\). An argument analogous to that of Case 1 shows \( v_1 \) is the first vertex traversed in the branch at \( r_1 \) containing \( r_2 \).

But then \( \omega_1, \omega_2, \omega_3, \ldots, \omega_i, \omega_i \) must be traversed in order. Hence, \( r_2 \) must be traversed prior to any vertex of \( B_4, B_5, \) or \( B_6 \).

But then it is impossible to completely traverse these branches,
contradicting the fact that $T^2$ is traceable. Hence, $T$ does not contain any $F_i$, $i \geq 1$.

Now suppose $T$ is a tree containing a trunk that contains no $S(K(1, 5))$ or $F'$, $i \geq 0$. Let $\mathcal{J}; v_1, v_2, \ldots, v_k$ be the trunk of $T$. Since $T$ has a trunk, each branch $B$ disjoint from $\mathcal{J}$ contains no $S(K(1, 3))$. Hence, by Theorem 4C, $B^2$ is hamiltonian. Further, by Theorem 4D, each terminal edge is contained in some hamiltonian cycle; that is, we can find a hamiltonian cycle of $B^2$ containing an edge incident with the initial vertex of $B$. Thus, each such branch can be entered, traversed, and exited as the end vertices of the hamiltonian path in $B^2$, created by removing the terminal edges that lie on a hamiltonian cycle, are at a distance 1 or 2 from the root of $B$.

Let $B_1$ be a branch at $v_1$. Let $x_1$ be a vertex in $B_1$ a distance 2 from $v_1$. By previous remarks there exists a hamiltonian path in $B_1^2$ beginning at $x_1$ and ending at $x_2$, the initial vertex of $F_1$. Next, traverse all trivial branches at $v_1$ and $x_2$, the initial vertex of $B_2$. Again, there is a hamiltonian path in $B_2^2$ with initial vertex $x_3$ and terminal vertex $x_4$, where $d(x_4, v_1) = 2$. Next, traverse $v_1$. Finally, if a third branch, disjoint from $\mathcal{J}$, exists at $v_1$, we traverse it, ending at the initial vertex and then traverse $v_2$. If no third branch exists at $v_1$, we again traverse $v_2$. Since $T$ contains no $F'$, there are at most two nontrivial branches at $v_2$ which can be traversed. Continue these arguments at $v_3, v_4, \ldots, v_k$. Since $T$ contains no $F'$, we may enter and traverse one of the branches at $v_k$ before traversing $v_k$ itself. This leaves at most two remaining nontrivial branches which clearly can be traversed. Hence $T^2$ is traceable.
Lemma 4.7

Let $T$ be a tree that contains no $S(K(1, 3))$. Then for each vertex $v$, there exists a hamiltonian path in $T^2$ with initial vertex $v$ and whose terminal vertex is an end-vertex of $T$.

Proof: By Theorem 4C, $T$ minus its end-vertices is a path, say $P:v_1, v_2, \ldots, v_k$. For each vertex $v_t$ ($t = 1, 2, \ldots, k$), let

$$W_t = \{x| \text{deg } x = 1 \text{ and } xv_t \in E(T)\}.$$

Case 1: Suppose $v = v_t$, $1 \leq t \leq k$. We construct a hamiltonian path in $T^2$ by traversing in order $v_t$, the vertices of $W_{t-1}$, $v_{t-2}$, $W_{t-3}$, and so on until either $v_1$ or the vertices of $W_1$ have been traversed, say $v_1$. Then traverse the vertices of $W_1, v_2, W_3$, and so on until the vertices of $W_s$ have been traversed. (Otherwise, traverse $v_1, W_2, v_3$, etc.) In either case, we now traverse vertices in the order $v_{t+1}, W_{t+1}, v_{t+2}, W_{t+2}, \ldots, v_k, W_k$. If $W_k \neq \emptyset$, we have ended with a vertex of degree 1. If $W_k = \emptyset$, then deg $v_k = 1$ and $v_k$ terminates the path.

Case 2: Suppose $v \in W_t$, $1 \leq t \leq k$. Then traverse the remaining vertices of $W_t$, $v_{t-1}$, $W_{t-2}$, and so on as before. Using an approach similar to Case 1, we can then construct a path that ends with a vertex of $W_k$, if $W_k \neq \emptyset$, or at $v_k$, otherwise. 

Suppose the tree $T$ contains a trunk $T$. By an end branch of $T$ we mean a branch $B$ at an end-vertex of $T$ such that $V(B) \cap V(T) = \emptyset$. 

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Theorem 4.8

Let $T$ be a tree that contains 3-roots but no $i$-roots, $i \geq 4$, and suppose $T^2$ is traceable. Then $T^2$ is traceable from the vertex $v$ if and only if $v$ lies in an end branch of $T$.

Proof: Let $T$ be a tree that contains 3-roots but no $i$-roots, $i \geq 4$. Further, suppose $T^2$ is traceable. Then $T$ contains a trunk $\mathcal{J}$. If $\mathcal{J}$ consists of a single 3-root $r$, then either $T$ is star-like or each nontrivial branch that is not a path contains no $S(K(1, 3))$. If $T$ is star-like, the result follows from Lemma 4.4. Otherwise, let $v$ be in a branch $B$ at $r$. By Lemma 4.7, there is a hamiltonian path in $B^2$ with initial vertex $v$ and terminal vertex $x$, where $d(x, r) \leq 2$. We can then traverse $r$ and all trivial branches at $r$. Now, by the proof techniques of Theorem 4.6, we know we can traverse the vertices of the two remaining nontrivial branches. Thus, $T^2$ is traceable from $v$. Note that, by Lemma 4.3, $T^2$ is not traceable from $r$. Thus, the result holds.

Next, suppose $\mathcal{J}$ contains a nontrivial trunk $v_1, v_2, \ldots, v_l$ ($l \geq 2$). If $v$ does not lie in an end branch of $T$, but $v$ is traceable from $v$, then the branches at one end-vertex of $\mathcal{J}$ must be traversed prior to those at the other. But this implies that a hamiltonian path in $S(K(1, 3))$ can begin and end in the same branch, contradicting Lemma 4.1. Thus, $T^2$ is traceable from $v$ if and only if $v$ lies in an end branch of $T$.■
Corollary 4.9

Let $T$ be a tree such that $T^2$ is traceable. Then $T^2$ is traceable from a vertex $v$ if and only if $v$ lies in an end branch of $T$ and $v$ is not adjacent to a 4-root.

Proof: If $T$ is star-like, we apply Lemma 4.4. If $T$ contains only 3-roots, then the result follows by Theorem 4.8. If $T$ contains exactly one $i$-root $r$, $i \geq 3$, and $r$ is a 4-root, then an argument analogous to that in Theorem 4.8 can be used; however, as in Lemma 4.4, $v$ cannot be adjacent to $r$. For the general case, an analogous proof technique again applies, noting that we must begin in an end branch of $T$ and that if $v$ is the initial vertex of an end branch at a 4-root, we cannot traverse all branches at that root. ■
CHAPTER V

A GENERALIZATION OF HOMOGENEously TRACEABLE GRAPHS

Section 5.1

Initially k-Path Traceable Graphs

In Chapter II we mentioned that homogeneously traceable graphs could be generalized in a variety of ways. In this chapter we introduce another generalization and consider some properties of these graphs.

A graph $G$ is called $k$-path traceable ($k \geq 0$) if each path in $G$ of length at most $k$ is contained on a hamiltonian path. Roberts [20] and Kronk [17], among others, have obtained sufficient conditions for a graph to be $k$-path traceable. A slightly more restrictive definition yields a new class of graphs.

A graph $G$ is initially $k$-path traceable ($k \geq 0$) if each path in $G$ of length at most $k$, with initial vertex $v$ say, can be extended to a hamiltonian path of $G$ having initial vertex $v$.

A graph is initially 0-path traceable if and only if it is homogeneously traceable. It is also readily seen that every initially $k$-path traceable graph is $k$-path traceable.

Both Roberts [20] and Kronk [17] noticed that if $G$ is a graph such that $\deg u + \deg v \geq p + k$ for all nonadjacent vertices $u$ and $v$, then $G$ is $(k+1)$-path traceable. Further, this condition

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also implies that $G$ is $k$-hamiltonian (that is, for every subset $S$ of $V(G)$ with $|S| \leq k$, $G - S$ is hamiltonian), which, in turn, implies that $G$ is initially $k$-path traceable.

The following example shows that initially $k$-path traceable graphs form a proper subclass of the $k$-path traceable graphs.

**Example 5.1**

Let the graph $G$ be formed by taking a copy of $K_n$ ($n > k + 2$ and $k \geq 0$) and a vertex $v$, then joining $v$ to $k + 2$ distinct vertices of $K_n$. The only nonadjacent vertex pairs are $v$ and a vertex $w$ of $K_n$ not in $N(v)$. Since $\text{deg} \; v = k + 2$ and $\text{deg} \; w = p - 2$ (where $p = n + 1$), we have that $\text{deg} \; v + \text{deg} \; w = p + 1$. This implies that $G$ is a $(k + 1)$-path traceable graph. However, $G$ is not initially $(k + 1)$-path traceable, since the path $w_1, w_2, \ldots, w_{k+2}$, of length $k + 1$ [where $N(v) = \{w_1, w_2, \ldots, w_{k+2}\}$], cannot be extended to a hamiltonian path.

The next result provides a sufficient condition for a graph to be initially $k$-path traceable.

**Theorem 5.1**

Let $G$ be a graph of order $p \geq 6$. If there exists a partition of $V(G)$ into two subsets $S_0$ and $S_1$ such that

(a) $\langle S_i \rangle$ is $k$-hamiltonian ($i = 0, 1$) and

(b) each vertex of $S_i$ is adjacent to at least $k + 1$ vertices of $S_{i+1}$ (subscripts modulo 2), then $G$ is initially $k$-path traceable.
Proof: Let $G$ be a graph of order $p \geq 6$ and suppose there exists a partition of $V(G)$ into two subsets $S_0$ and $S_1$ that satisfy conditions (a) and (b) of the hypothesis. Further, let $P$ be an arbitrary path of length at most $k$. If $V(P) \subseteq V(S_i)$ ($i = 0$ or $1$), the result is clear. Thus, suppose $V(P) \not\subseteq V(S_i)$, that is, $V(P)$ contains vertices of both $S_0$ and $S_1$.

Without loss of generality, suppose $P: v_0, v_1, \ldots, v_k$ and that $v_0 \in S_0$. Let $H_0 = S_0 \cap V(P)$ and $H_1 = S_1 \cap V(P)$. Note that $|H_0| \leq k$ and $|H_1| \leq k$.

Suppose $v_k \in S_0$. Since $\langle S_0 \rangle$ is $k$-hamiltonian, $\langle S_0 - H_0 \rangle$ contains a hamiltonian cycle $C: \omega_1, \omega_2, \ldots, \omega_k, \omega_1$ and $v_k$ must be adjacent to a vertex on this cycle; say $v_k$ is adjacent to $\omega_1$. Further, note that since $\omega_k$ is adjacent to at least $k + 1$ vertices of $S_1$; $\omega_k$ is adjacent to a vertex of $S_1 - H_1$. Since $\langle S_1 - H_1 \rangle$ is hamiltonian, let $C_1: x_1, x_2, \ldots, x_n, x_1$ be a hamiltonian cycle of $\langle S_1 - H_1 \rangle$. Further, without loss of generality, assume that $\omega_k$ is adjacent to $x_1$. Then $v_0, v_1, \ldots, v_k, \omega_1, \omega_2, \ldots, \omega_k, x_1, x_2, \ldots, x_n$ is a hamiltonian path in $G$ with initial subpath $v_0, v_1, \ldots, v_k$.

An analogous argument holds if $v_k \in S_1$. Thus, $G$ is initially $k$-path traceable.\[\Box\]

We note that if $k = 0$, Theorem 5.1 yields a slightly weaker version of Corollary 2.9.

Theorem 5.2

Let $G$ be a graph of order $p \geq 6$. If for each path $P$ of length at most $k$, there exists a partition of $V(G)$ into two subsets $S_0$ and $S_1$
such that (i) $\langle S_0 \rangle$ has a hamiltonian cycle containing $P$, (ii) each vertex of $S_0$ is adjacent to a vertex of $S_1$, and (iii) $\langle S_1 \rangle$ is homogeneously traceable. Then $G$ is initially $k$-path traceable.

**Proof:** Let $P:v_0, v_1, \ldots, v_m$, where $0 \leq m \leq k$, be an arbitrary path of length at most $k$ in the graph $G$ and let $S_0$ and $S_1$ be a partition of $V(G)$ satisfying the hypothesis of the theorem. Then a hamiltonian path with initial subpath $P$ is determined by following the hamiltonian cycle $C:v_0, v_1, \ldots, v_m, x_1, x_2, \ldots, x_n, v_0$ of $\langle S_0 \rangle$, until $x_n$ is traversed, then traversing any edge $x_ny_1$, where $y_1 \in S_1$. Now complete the hamiltonian path in $G$ by traversing the remainder of the path $P^*:y_1, y_2, \ldots, y_k$, the hamiltonian path in $\langle S_1 \rangle$. Since $P$ was arbitrary, $G$ is initially $k$-path traceable.

Let $k(G)$ denote the number of components of the graph $G$. Skupień [22] noticed that if $G$ is homogeneously traceable and $S$ is a cutset of $G$, then $k(G - S) \leq |S|$. Since every initially $k$-path traceable graph ($k \geq 0$) is homogeneously traceable, we have the following.

**Lemma 5.3**

If $G$ is initially $k$-path traceable ($k \geq 0$) and $X$ is a cutset for $G$, then $k(G - X) \leq |X|$.
Section 5.2

Initially $k$-Path Traceable Nonhamiltonian Graphs

Since homogeneously traceable nonhamiltonian graphs exist, initially $k$-path traceable nonhamiltonian graphs exist, at least when $k = 0$. In this section we consider initially $k$-path traceable nonhamiltonian graphs for values of $k \geq 1$.

We observe that every hypohamiltonian graph is initially 1-path traceable nonhamiltonian. Further, it is straightforward to show that the Petersen Graph is initially 3-path traceable and nonhamiltonian. Thus, infinitely many initially $k$-path traceable nonhamiltonian graphs exist for $k = 1$ and some exist for values of $k > 1$. It is not known in general for what values of $k > 1$ initially $k$-path traceable and nonhamiltonian graphs exist.

We now examine some properties of initially $k$-path traceable and nonhamiltonian graphs.

Theorem 5.4

If $G$ is an initially $k$-path traceable nonhamiltonian graph ($k \geq 1$) of order $p \geq 4$, then $G$ is 3-connected.

Proof: Let $G$ be an initially $k$-path traceable ($k \geq 1$) nonhamiltonian graph of order $p \geq 4$. Then, since $G$ is homogeneously traceable, $G$ is 2-connected. Suppose, in fact, that $\kappa(G) = 2$ and that $\{u, v\}$ is a cutset for $G$. 

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By Lemma 5.3, $G - \{u, v\}$ has exactly two components. Let $C_1$ and $C_2$ be the components of $G - \{u, v\}$. Further, suppose $\omega_1$ and $\omega_2$ are adjacent to $u$, where $\omega_1 \in C_1$ and $\omega_2 \in C_2$.

Since $G$ is initially 1-path traceable, $\omega_1 u$ is the initial edge of some hamiltonian path $P$ in $G$. But then there exists a $u - v$ subpath $P_1$ of $P$ such that $V(P_1) = C_1 \cup \{u, v\}$. Similarly, $\omega_2 u$ is the initial edge of some hamiltonian path, so that there exists a $u - v$ path $P_2$ where $V(P_2) = C_2 \cup \{u, v\}$. But then, by following $P_1$ and $P_2$, we obtain a hamiltonian cycle of $G$. Thus, $G$ is 3-connected.

**Corollary 5.5**

If $G$ is an initially $k$-path traceable nonhamiltonian $(p, q)$ graph ($k \geq 1$), then

(a) $\delta(G) \geq 3$ and

(b) $|q| \geq \left\lfloor \frac{3p}{2} \right\rfloor$.

Note that the bound obtained in Corollary 5.5(b) is achieved when $p = 10$ since the Petersen Graph is initially 3-path traceable and nonhamiltonian and contains exactly $\left\lfloor \frac{3 \cdot 10}{2} \right\rfloor = 15$ edges.

In [4] it was shown that if $G$ is a homogeneously traceable nonhamiltonian graph of order $p$, then $\Delta(G) \leq p - 4$. We now extend this theorem for initially $k$-path traceable nonhamiltonian graphs when $k \geq 1$.

**Theorem 5.6**

If $G$ is an initially $k$-path traceable ($k \geq 1$) nonhamiltonian graph of order $p \geq 9$, then $\Delta(G) \leq p - 6$. 

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Proof: Let $G$ be an initially $k$-path traceable ($k \geq 1$) nonhamiltonian graph of order $p \geq 9$. Since $G$ is homogeneously traceable and non-hamiltonian, $\Delta(G) \leq p - 4$.

Suppose $\Delta(G) = p - 4$ and $\deg v_1 = p - 4$. Further, let $P: v_1, v_2, \ldots, v_p$ be a hamiltonian path in $G$. Since $G$ is nonhamiltonian, $v_p$ is not adjacent to $v_1$. Let $v_i$ and $v_j$ be the other two vertices not adjacent to $v_1$ and suppose $i < j < p$.

**Case 1:** Suppose $i = p - 2$. Then $j = p - 1$ and by Corollary 5.5(a) $\delta(G) \geq 3$. Thus, $\deg v_p \geq 3$ and there exists some $k$, $2 \leq k \leq p - 3$, such that $v_i v_{k+1}$ and $v_k v_p$ are edges of $G$. But then we see that $v_1, v_2, \ldots, v_k, v_p, v_{p-1}, \ldots, v_{k+1}, v_1$ is a hamiltonian cycle in $G$, contradicting the assumption that $G$ is nonhamiltonian.

**Case 2:** Suppose $3 \leq i < p - 2$ and $j = p - 1$. Again, since $\deg v_p \geq 3$, we see that unless $v_p$ is adjacent to exactly $v_{p-1}$, $v_{p-2}$, and $v_{i-1}$, $G$ is hamiltonian. But since $\deg v_{p-1} \geq 3$, there exists a vertex $v_t (t \neq p - 2$ and $t \neq p)$ adjacent to $v_{p-1}$. If $t < i - 1$, then $v_p, v_{p-1}, v_t, v_{i-1}, \ldots, v_1, v_{i+1}, v_{i+2}, \ldots, v_{p-2}, v_p$ is a hamiltonian cycle in $G$. While, if $i \leq t \leq p - 3$, then $v_{t+1}, v_{t+2}, \ldots, v_{p-2}, v_p, v_{p-1}, v_t, v_{i-1}, \ldots, v_1, v_{t+1}$ is a hamiltonian cycle in $G$.

Finally, suppose $t = i - 1$. Since $\deg v \geq 3$, there exists a vertex $v_w (w \neq i + 1$ and $w \neq i - 1)$ adjacent to $v_i$. If $w < i - 1$, then $v_i, v_w, v_{w-1}, \ldots, v_1, v_{w+1}, \ldots, v_{i-1}, v_p, v_{p-1}, v_p, v_{p-2}, \ldots, v_i$ is a hamiltonian cycle in $G$ (see Figure 5.1).

If $i + 1 < w < p - 1$, then the hamiltonian cycle $v_w, v_i, v_{i+1}, \ldots, v_{w-1}, v_1, v_2, \ldots, v_{i-1}, v_{p-1}, v_p, v_{p-2}, v_{p-3}, \ldots, v_w$ exists in $G$ (see Figure 5.2).
If \( w = p \), then \( v_p v_i \) and \( v_1 v_{i+1} \) are edges of \( G \) and \( G \) is again hamiltonian. Thus, all possibilities lead to a contradiction.

**Case 3:** Suppose \( j = i + 1 < p - 1 \). Unless \( v_p \) is adjacent to precisely \( v_{p-1}, v_i, \) and \( v_{i-1} \), we find a vertex \( v_k \) such that \( v_i v_k \) and \( v_{k-1} v_p \) are edges of \( G \); which implies \( G \) is hamiltonian. Otherwise, we then see that \( v_i, v_{i+1}, \ldots, v_{p-1}, v_1, v_2, \ldots, v_{i-1}, v_p, v_i \) is a hamiltonian cycle in \( G \), and again we have a contradiction.

**Case 4:** Suppose \( 3 < i < j < p - 1 \). Then \( \deg v_p = 3 \) and \( v_p \) is adjacent to \( v_{p-1}, v_{i-1}, \) and \( v_{j-1} \), or \( G \) is hamiltonian. Since \( \deg v_i \geq 3 \), there is a vertex \( v_w \) (\( w \neq i - 1 \) or \( w \neq i + 1 \)) that is
adjacent to $v_i$. If $w < i - 1$, then $v_i, v_{i-1}, v_{i-2}, \ldots, v_1, v_{w+1}, v_{w+2}, \ldots, v_{i-1}, v_w, v_{i-1}, \ldots, v_i$ is a Hamiltonian cycle. If $i + 1 < w \leq j$ (or $j + 1 < w \leq p$), then $v_i, v_w, v_{i+1}, \ldots, v_p, v_{i-1}, v_{i-2}, \ldots, v_1, v_{w-1}, v_{w-2}, \ldots, v_i$ is a Hamiltonian cycle.

If $w = j + 1$, then there exists a vertex $v_z$ ($z \neq j + 1$) that is adjacent to $v_j$. If $1 \leq l < i - 1$, then $v_j, v_k, v_{k+1}, \ldots, v_1, v_{i-1}, v_p, v_{i-1}, \ldots, v_j$ is a Hamiltonian cycle. If $i + 1 < l \leq p$, then $v_j, v_z, v_{j+1}, \ldots, v_p, v_{j-1}, \ldots, v_1, v_{i-1}, \ldots, v_j$ is Hamiltonian. On the other hand, if $k = i - 1$, then $v_j, v_{i-1}, \ldots, v_1, v_{j+2}, \ldots, v_p, v_{j-1}, v_{j-2}, \ldots, v_i, v_{j+1}, v_j$ is a Hamiltonian cycle of $G$, and we have reached a contradiction.

Thus, all cases have led to a contradiction so that $\Delta(G) < p - 4$.

Next, suppose that $\deg v_1 = p - 5$ and again let $P:v_1, v_2, \ldots, v_p$ be a Hamiltonian path in $G$. Let $v_i, v_j, v_k$, and $v_p$ be the vertices not adjacent to $v_1$, where $1 < i < j < k < p$. We again examine the possible location of these vertices.

Case A: Suppose $i = p - 3$ so that $j = p - 2$ and $k = p - 1$. Then if $v_p$ is adjacent to any vertex $v_m$, $2 \leq m < p - 4$, $G$ contains a Hamiltonian cycle. Thus, $3 \leq \deg v_p \leq 4$, and $v_p$ is adjacent to a subset of $\{v_{p-4}, v_{p-3}, v_{p-2}, v_{p-1}\}$.

If $v_p v_{p-4} \in E(G)$, then $v_{p-3}$ is adjacent to no $v_y$, $2 \leq y < p - 4$, or a Hamiltonian cycle exists in $G$, namely $v_{p-3}, v_y, v_{y-1}, \ldots, v_1, v_{y+1}, \ldots, v_{p-4}, v_p, v_{p-1}, v_{p-2}, v_{p-3}$. Since $\deg v_{p-3} \geq 3$, at least one of $v_p$ or $v_{p-1}$ is adjacent to $v_{p-3}$.

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If $v_p v_{p-3} \in E(G)$, then $v_{p-2}$ is adjacent to no $v_w$, $2 \leq w < p - 4$, or again a hamiltonian cycle exists. But then since $\kappa(G) \geq 3$, we see that $v_{p-1} v_w \in E(G)$ for some $w < p - 4$, or else \{v_{p-4}, v_{p-3}\} would be a cutset for $G$. However, then

$v_{p-2}, v_{p-1}, v_w, v_{w-1}, \ldots, v_1, v_{p-4}, v_p, v_{p-3}, v_{p-2}$

is a hamiltonian cycle.

Thus, $v_p v_{p-3} \notin E(G)$, so that $v_p v_{p-2}$ must be an edge of $G$, since $\deg v_p \geq 3$. We know $v_{p-3}$ must also be adjacent to $v_{p-1}$ and since $\kappa(G) \geq 3$, $v_{p-1}$ must be adjacent to some $v_w$ where $2 \leq w < p - 4$. But then

$v_{p-2}, v_p, v_{p-4}, \ldots, v_{w+1}, v_1, v_2, \ldots, v_w, v_{p-1}, v_{p-3}, v_{p-2}$

is a hamiltonian cycle.

Finally, if $v_p v_{p-4} \notin E(G)$, then both $v_p v_{p-3}$ and $v_p v_{p-2}$ must be edges of $G$. Again, at least one of $v_{p-3}, v_{p-2},$ or $v_{p-1}$ is adjacent to a vertex $v_w$, $1 \leq w < p - 4$. If either $v_{p-1}$ or $v_{p-2}$ is adjacent to such a vertex, $G$ is hamiltonian. If $v_{p-3} v_w \in E(G)$, then $\kappa(G) < 3$ unless one of $v_p, v_{p-1},$ or $v_{p-2}$ is adjacent to a $v_w$, $1 \leq w < p - 4$. This contradicts our assumption, so in all cases $G$ is hamiltonian.

**Case B:** Suppose $i < p - 3$ and $j = p - 2$ (thus $k = p - 1$).

If $v_i$ is adjacent to any vertex other than $v_{i-1}, v_p - 2, v_{p-2}$, then the arguments of Case A apply. Thus, $\mathcal{N}(v_i) = \{v_{i-1}, v_{p-2}, v_{p-1}\}$. Since $\deg v_i \geq 3$, there exists a vertex $v_w$ ($w \neq i - 1$ and $w \neq i + 1$) adjacent to $v_i$. If $1 \leq w \leq i - 1$ or $i + 1 \leq w \leq p - 2$, we find hamiltonian cycles as before. If $w = p - 1$, then since $\deg v_{p-2} \geq 3$, there exists a vertex $v_r$ ($r \neq p - 3$ and $r \neq p - 1$) such that
Let \( v_{p-2}v_r \in E(G) \). If \( r = p \), then
\[
v_{p-3}, v_{p-2}, \ldots, v_{i+1}, v_1, v_2, \ldots, v_i, v_{p-1}, v_p, v_{p-2}, v_{p-3}
\]
is a hamiltonian cycle. If \( i \leq r < p - 3 \), then
\[
v_r, v_{r-1}, \ldots, v_1, v_{r+1}, \ldots, v_{i-1}, v_p, v_{p-1}, v_i, v_{i+1}, \ldots, v_{p-2}, v_r
\]
is a hamiltonian cycle and all possibilities lead to a contradiction.

**Case C:** Suppose \( i < j - 1 < p - 3 \) and \( k = p - 1 \). Since \( \deg v_p \geq 3 \), \( v_p \) is adjacent to at least one of \( v_{i-1} \) or \( v_{j-1} \). If \( v_pv_{i-1} \in E(G) \), then since \( \deg v_i \geq 3 \), there exists a vertex \( v_w \) (\( w \neq i - 1 \) and \( w \neq i + 1 \)) adjacent to \( v_i \). The techniques of the last cases show all possibilities for \( w \) lead to a hamiltonian cycle. In exactly the same manner, we obtain a hamiltonian cycle if \( v_pv_{j-1} \in E(G) \).

**Case D:** Suppose \( i + 1 = j < p - 2 \) and \( k = p - 1 \). Again, \( v_p \) can be adjacent only to \( v_{p-1} \) and \( v_{p-2} \), as well as at least one of \( v_{i-1} \) or \( v_i \). Suppose \( v_pv_i \in E(G) \). Since \( \deg v_j \geq 3 \), there exists a vertex \( v_w \) (\( w \neq i - 1 \) or \( w \neq i + 1 \)) such that \( v_wv_j \in E(G) \). If \( 1 \leq w < i - 1 \) or \( i + 1 < w \leq p \), \( G \) is hamiltonian, since
\[
(3) \quad v_j, v_w, v_{w-1}, \ldots, v_1, v_{w+1}, v_{w+2}, \ldots, v_{i-1}, v_p, v_{p-1}, \ldots, v_j
\]
and
\[
(4) \quad v_j, v_w, v_{w+1}, \ldots, v_p, v_{i-1}, v_{i-2}, \ldots, v_1, v_{w-1}, \ldots, v_j
\]
are hamiltonian cycles. If \( w = i - 1 \), then
\[
v_j, v_{i-1}, v_{i-2}, \ldots, v_1, v_{j+1}, \ldots, v_p, v_i, v_j
\]
is a hamiltonian cycle.

Suppose \( v_{i}v_{i-1} \in E(G) \). There exists a vertex \( v_r \) adjacent to \( v_i \) (\( r \neq i - 1 \) or \( r \neq i + 1 \)). If \( 1 \leq r < i - 1 \) or \( j + 1 < r \leq p - 1 \),
we obtain hamiltonian cycles similar to (3) and (4). If \( r = p \), \( G \) is hamiltonian by previous work, so suppose \( r = j + 1 \). We again consider a vertex \( v_r \) adjacent to \( v_j \). The techniques of the previous cases again show that each possible value for \( t \) leads to a hamiltonian cycle.

**Case E:** Suppose \( i + 1 = j < k - 1 < p - 2 \). Then \( v_p \) is adjacent to at least one of \( v_{i-1} \) or \( v_{k-1} \). Suppose \( v_p v_{k-1} \in E(G) \). Then since \( \deg v_k \geq 3 \), there is a vertex \( v_w \) (\( w \neq k - 1 \) and \( w \neq k + 1 \)) adjacent to \( v_k \). If \( 1 \leq w < i - 1 \), we obtain the hamiltonian cycle

\[
v_k, v_w, v_{w-1}, \ldots, v_1, v_{w+1}, \ldots, v_k, v_{k-1}, v_p, v_{p-1}, \ldots, v_k
\]

If \( j \leq w < k - 1 \), then a similar hamiltonian cycle is obtained. If \( k + 1 < w \leq p \), then

\[
v_k, v_w, v_{w+1}, \ldots, v_p, v_{k-1}, \ldots, v_1, v_{w-1}, \ldots, v_k
\]

is a hamiltonian cycle.

Thus, suppose \( w = i \). Then let \( v_r \) be adjacent to \( v_j \) (\( r \neq j - 1 \) and \( r \neq j + 1 \)). If \( 1 \leq r < i - 1 \), then

\[
v_v, v_r, v_{r-1}, \ldots, v_1, v_{r+1}, \ldots, v_i, v_k, v_{k+1}, \\
\ldots, v_p, v_{k-1}, v_{k-2}, \ldots, v_j
\]

is a hamiltonian cycle. If \( r = i - 1 \), then

\[
v_v, v_k, \ldots, v_p, v_{k-1}, \ldots, v_{j+1}, v_1, v_2, \ldots, v_{i-1}, v_j, v_i
\]

is a hamiltonian cycle. If \( j + 1 < r < k - 1 \), then

\[
v_v, v_r, v_{r+1}, \ldots, v_{k-1}, v_p, \ldots, v_k, v_{i}, v_{i-1}, \\
\ldots, v_1, v_{r-1}, \ldots, v_j
\]
is a hamiltonian cycle. If \( r = k - 1 \), then

\[ v_i, v_{k-1}, v_p, v_{p-1}, \ldots, v_k, v_i, v_{i-1}, \ldots, v_1, v_{k-2}, \ldots, v_j \]

is a hamiltonian cycle. If \( r = k \), then

\[ v_i, v_k, v_{k+1}, \ldots, v_p, v_{k-1}, \ldots, v_{j+1}, v_1, v_2, \ldots, v_j \]

is a hamiltonian cycle. If \( k + 1 \leq r < p - 1 \), then

\[ v_1, v_2, \ldots, v_i, v_k, v_{k+1}, \ldots, v_p, v_j, v_{j+1}, \ldots, v_{k-1}, v_p, v_{p-1}, \ldots, v_{r+1}, v_1 \]

is a hamiltonian cycle. If \( r = p - 1 \), then since \( v_p \) must be adjacent to one of \( v_{i-1} \) or \( v_i \), we again find that \( G \) is hamiltonian.

If \( v_p v_{i-1} \in E(G) \), then a similar analysis of the adjacencies of \( v_i, v_j, \) and \( v_k \) shows \( G \) is hamiltonian.

**Case F:** Suppose \( i + 1 = j \) and \( j + 1 = k < p - 1 \). Then \( v \) is adjacent to at least two of \( v_{i-1}, v_i, \) and \( v_j \). An analysis similar to the previous cases again shows a hamiltonian cycle must exist.

**Case G:** Suppose \( i < j - 1 \) and \( j + 1 = k \). This case is similar to Case D.

**Case H:** Suppose \( i < j - 1 < k - 2 \). Then \( v_p \) must be adjacent to at least two of \( v_{i-1}, v_{j-1}, \) and \( v_{k-1} \). If \( v_p v_{i-1} \in E(G) \), then since \( \deg v_i \geq 3 \), there exists a vertex \( v_w \) (\( w \neq i - 1 \) and \( w \neq i + 1 \)) such that \( v_w \) is adjacent to \( v_i \). If \( 1 \leq w < i - 1 \), or if \( i + 1 < w \leq j \), or if \( j + 1 < w \leq k \), or if \( k + 1 < w \leq p \), then \( G \) is found to be hamiltonian, as before. If \( v_p v_{j-1} \in E(G) \), then since \( \deg v_j \geq 3 \), there exists a vertex \( v_r \) adjacent to \( v_j \) (\( r \neq j - 1 \) and \( r \neq j + 1 \)).
If \( r = i - 1 \), then note that \( v_2 \) must be adjacent to a third vertex \( v_3 \). If \( 1 \leq s < i - 1 \), or if \( i + 1 < s \leq j - 1 \), then \( G \) is hamiltonian. Hence, all cases are concluded and \( \Delta(G) \leq p - 6 \).

It is not known in general if any initially \( k \)-path traceable and nonhamiltonian graphs \( (k \geq 1) \) actually contain vertices of degree \( p - 6 \).

**Corollary 5.7**

If \( G \) is initially \( k \)-path traceable and nonhamiltonian \( (k \geq 1) \), then \( G = K_2 \) or \( |V(G)| \geq 10 \).

**Proof:** If \( G \) is initially \( k \)-path traceable and nonhamiltonian, then, in particular, \( G \) is homogeneously traceable nonhamiltonian. Then, \( G = K_1, G = K_2, \) or \( |V(G)| \geq 9 \). But, since \( k \geq 1, G \neq K_1 \).

Now suppose \( |V(G)| = 9 \). Then, by Theorem 5.6, \( \Delta(G) \leq 9 - 6 = 3 \). By Corollary 5.4, \( \delta(G) \geq 3 \). Thus, \( G \) is 3-regular and has order 9, which is impossible. Hence, \( |V(G)| \geq 10 \).

We conclude by noting that except for \( K_2 \), the Petersen Graph has the minimum order for an initially \( k \)-path traceable graph \( (k \geq 1) \) as well as the minimum size.

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