On a System of Nonlinear Ordinary Differential Equations with an Irregular Type Singularity: A Degenerate Case

Jerome John Przybylski

Western Michigan University

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ON A SYSTEM
OF NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS
WITH AN IRREGULAR TYPE SINGULARITY:
A DEGENERATE CASE

by
Jerome John Przybylski

A Dissertation
Submitted to the
Faculty of The Graduate College
in partial fulfillment
of the
Degree of Doctor of Philosophy

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Kalamazoo, Michigan
August 1978

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Jerome John Przybylski
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ON A SYSTEM OF NONLINEAR ORDINARY
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WESTERN MICHIGAN UNIVERSITY, PH.D., 1978
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>CHAPTER</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td></td>
</tr>
<tr>
<td>INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>1. The Irregular Type Singularity</td>
<td>1</td>
</tr>
<tr>
<td>2. Notations and Definitions</td>
<td>5</td>
</tr>
<tr>
<td>II</td>
<td></td>
</tr>
<tr>
<td>MAIN RESULTS</td>
<td>11</td>
</tr>
<tr>
<td>3. A Formal Solution</td>
<td>11</td>
</tr>
<tr>
<td>4. A Special Existence Theorem</td>
<td>17</td>
</tr>
<tr>
<td>III</td>
<td></td>
</tr>
<tr>
<td>KNOWN EXISTENCE THEOREMS</td>
<td>21</td>
</tr>
<tr>
<td>5. Existence Theorem I</td>
<td>21</td>
</tr>
<tr>
<td>6. Existence Theorem II</td>
<td>22</td>
</tr>
<tr>
<td>7. Existence Theorem III</td>
<td>24</td>
</tr>
<tr>
<td>8. Existence Theorem IV</td>
<td>27</td>
</tr>
<tr>
<td>IV</td>
<td></td>
</tr>
<tr>
<td>THE FUNCTIONS $\tilde{V}(x)$ AND $W(x)$</td>
<td>30</td>
</tr>
<tr>
<td>9. The Function $\tilde{V}(x)$</td>
<td>30</td>
</tr>
<tr>
<td>10. The Function $W(x)$</td>
<td>30</td>
</tr>
<tr>
<td>V</td>
<td></td>
</tr>
<tr>
<td>PROOF OF THEOREM A</td>
<td>35</td>
</tr>
<tr>
<td>11. Equations That Determine $\varphi_{&lt;x;w,v&gt;}$ $\psi_{&lt;x;w,v&gt;}$</td>
<td>35</td>
</tr>
<tr>
<td>12. The Functions $\varphi_0(v)$ and $\psi_0(v)$</td>
<td>38</td>
</tr>
<tr>
<td>13. The Functions $\varphi_1(w,v)$ and $\psi_1(w,v)$</td>
<td>40</td>
</tr>
<tr>
<td>14. The Functions $\varphi_2(w,v)$ and $\psi_2(w,v)$</td>
<td>47</td>
</tr>
<tr>
<td>15. Equations That Determine the Series Part of the Formal Solution</td>
<td>51</td>
</tr>
</tbody>
</table>
# TABLE OF CONTENTS (Continued)

<table>
<thead>
<tr>
<th>CHAPTER</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>16. The Functions $r_0(x,w)$ and $s_0(x,w)$</td>
<td>56</td>
</tr>
<tr>
<td>17. The Functions $r_p(x,w)$ and $s_p(x,w)$</td>
<td>61</td>
</tr>
<tr>
<td>VI PROOF OF THEOREM B</td>
<td>64</td>
</tr>
<tr>
<td>18. Initial Reduction</td>
<td>64</td>
</tr>
<tr>
<td>19. A Stable Domain</td>
<td>68</td>
</tr>
<tr>
<td>20. The Functions $\omega(\tau)$ and $\chi(\tau)$</td>
<td>70</td>
</tr>
<tr>
<td>21. The Paths of Integration</td>
<td>72</td>
</tr>
<tr>
<td>22. The Behavior of $W(x)$</td>
<td>76</td>
</tr>
<tr>
<td>23. An Integral Inequality</td>
<td>79</td>
</tr>
<tr>
<td>24. Successive Approximations</td>
<td>83</td>
</tr>
<tr>
<td>VII EXAMPLES OF THEOREM A</td>
<td>86</td>
</tr>
<tr>
<td>BIBLIOGRAPHY</td>
<td>90</td>
</tr>
</tbody>
</table>
1. The Irregular Type Singularity

Consider the system of nonlinear ordinary differential equations of the form

\[
\begin{align*}
\left\{ \begin{array}{l}
x^{s+1} y' = F(x, y, z), \\
xz' = G(x, y, z), \quad \left( = \frac{d}{dx} \right)
\end{array} \right.
\end{align*}
\]

(1.1)

where:

i) \( x \) is an independent complex variable.

ii) \( y \) and \( z \) are \( m \)-column and \( n \)-column vectors respectively, \( (m, n \geq 1) \).

iii) \( F(x, y, z) \) and \( G(x, y, z) \) are \( m \)-column and \( n \)-column vectors respectively, whose components are holomorphic functions of \( (x, y, z) \) in a domain of the form

\[
|x| < a, \quad \|y\| < b, \quad \|z\| < b. \quad \left(\|y\| = \max_{i=1}^{m} |y_i|\right)
\]

iv) \( s \) is a positive integer.

v) \( F(0, 0, 0) = 0 \) and \( G(0, 0, 0) = 0 \).

Then the point \( x = 0 \) is called an irregular type singularity of the system (1.1). Equations of the form (1.1) have
played an important role in many diverse areas of mathematics. These include the study of the Briot-Bouquet type singularity in the analytic theory of differential equations on the complex plane and the study of the stability of the Falkner-Skan boundary layer problem in fluid dynamics.

The point $x = 0$ of the system of nonlinear differential equations of the form

$$(1.3) \quad xy' = H(x, y), \quad H(0, 0) = 0, \quad \left( \frac{d}{dx} \right)$$

is called a Briot-Bouquet type singularity. Here $x$ is a complex independent variable, $y$ is an $n$-column vector and $H(x, y)$ is an $n$-column vector whose components are holomorphic functions of $(x, y)$ in a domain of the form $|x| \leq a, \|y\| \leq b$. This type of singularity has been of interest to many mathematicians, since, in 1856, C.C.A. Briot and J. C. Bouquet [1], proved that:

If the system (1.3) has a formal solution of the form

$$(1.4) \quad y = x D_1 + x^2 D_2 + \cdots + x^\ell D_\ell + \cdots,$$

where the $D_\ell$ are $n$-column constant vectors, the power series has a positive radius of convergence.

Although, in general, it is not possible to find a solution for (1.3) as a power series in $x$, solutions, in
many cases, have been found which depend both on $x$ and a function which is the solution of a reduced or simplified equation. The method of constructing this solution usually depends on the value of $H_y(0,0)$, the Jacobian of $H(x,y)$ with respect to $y$ evaluated at $(0,0)$. When M. Iwano [12, 13] studied the multidimensional cases where $H_y(0,0) = 0$ or where $H_y(0,0)$ is of the form

$$H_y(0,0) = \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix},$$

for some invertible square matrix $K$ of size less than $n$, he found it necessary to solve equations of the form (1.1).

The Falkner-Skan problem [14, 15] can be characterized by the boundary value problem:

$$(1.6) \quad f'''' + ff'' + \lambda (k^2 - f'^2) = 0, \quad k > 0, \quad \left( = \frac{d}{dt} \right)$$

with the boundary conditions

$$(1.7) \quad f(0) = 0, \quad f'(0) = 0, \quad f'(\infty) = k.$$
(1.8) \( F_y(0, 0, 0) = \frac{1}{k}, \quad G_z(0, 0, 0) = -2\lambda \).

Hence, in order to solve equation (1.6) it is again necessary to study a system of the form (1.1).

Because of the importance of the system with an irregular type singularity, both M. Iwano [10, 11, 14] and P. Hsieh [2-5] have devoted considerable effort to studying this system. Their general approach has been to first find a "suitable" formal solution for the equation and then use the formal solution to generate an actual solution. Iwano uses a fixed-point approach developed by Hukuhara [9], while Hsieh uses a successive approximation approach.

In order to obtain these results, both authors found it necessary to assume that \( G_z(0, 0, 0) \) was an invertible matrix of a particular form. However, Iwano [12] has found an analytic expression for a solution of equations (1.3) when \( H_y(0, 0) = 0 \). Encouraged by his result, this paper begins the study of system (1.1) when \( G_z(0, 0, 0) = 0 \).

By placing certain further assumptions on \( G(x, y, z) \), it is possible to show that system (1.1) has a formal solution in powers of \( V(x) \), a solution of a reduced system, whose coefficients are holomorphic functions in \( x \) and a special parametric function \( W(x) \). It appears likely that this formal solution may eventually be used to generate an analytic expression for an actual solution of system (1.1).
This result is a partial solution to a problem proposed by Hsieh [6].

In constructing this formal solution, it becomes necessary to solve equations of the form

\[
\begin{align*}
x^{\sigma+1} y' &= A(x, W(x); y, z), \\
xz' &= B(x, W(x); y, z) \cdot \left( \frac{d}{dx} \right) 
\end{align*}
\]

(1.9)

A proof of the existence of a solution of this system will also be presented here.

The remainder of Chapter I explains definitions and notations. In Chapter II the main results will be stated. Chapter III contains four known existence theorems which play an important part in this work. The functions \( \tilde{V}(x) \) and \( W(x) \) are explained in Chapter IV. Chapter V contains the construction of the formal solution while Chapter VI contains the proof of the existence of a solution of equations (1.9). Two examples are given in Chapter VII.

2. Notations and Definitions

The symbol \( l_n \) denotes the \( n \times n \) identity matrix. For an \( n \)-column vector \( y \) with components \( \{y_j\} \), \( l_n(y) \) is the \( n \times n \) diagonal matrix with diagonal entries \( \{y_j\} \).

If \( y \) is an \( n \)-column vector with components \( \{y_j\}, [y] \)
is an n-column vector with components \( |y_j| \). For two n-column vectors \( y \) and \( \tilde{y} \) with entries \( \{y_j\} \) and \( \{\tilde{y}_j\} \) respectively, we write \( [y] \leq [\tilde{y}] \) if \( |y_j| \leq |\tilde{y}_j| \) for each index \( j \).

When \( y \) is an n-column vector with entries \( \{y_j\} \),

\[
(2.1) \quad \|y\| = \max_{j=1}^{n} |y_j| ,
\]

while

\[
(2.2) \quad \|y\|' = \min_{j=1}^{n} |y_j| .
\]

For the n-row vector \( p = (p_1, p_2, \ldots, p_n) \), whose components are all nonnegative integers,

\[
(2.3) \quad |p| = p_1 + p_2 + \cdots + p_n .
\]

If \( y \) is an n-column vector and \( p \) is a row vector, the symbol \( yp \) stand for the scalar expression

\[
(2.4) \quad yp = y_1^p_1 y_2^p_2 \cdots y_n^p_n .
\]

When \( y \) is an n-row vector and \( w \) is a scalar,

\[
(2.5) \quad wy = (w^y_1, w^y_2, \ldots, w^y_n) ,
\]

\[
(2.6) \quad \exp y = (\exp y_1, \exp y_2, \ldots, \exp y_n) ,
\]

\[
(2.7) \quad \Re y = (\Re y_1, \Re y_2, \ldots, \Re y_n) ,
\]
(2.8) \[ \text{Im } y = (\text{Im } y_1, \text{Im } y_2, \ldots, \text{Im } y_n) \].

If \( y \) is an \( n \)-column vector, \( \exp y \), \( \text{Re } y \), \( \text{Im } y \) are \( n \)-column vectors defined in a similar manner.

For the \( m \)-column vector \( y \) with elements \( \{y_j\} \) and an \( n \)-column vector function \( f(x, y) \) with elements \( \{f_j(x, y)\} \), the expression \( f_y(x, y) \) denotes the \( n \times m \) Jacobian matrix given by

(2.9) \[ f_y(x, y) = \begin{pmatrix} \frac{\partial}{\partial y_1} f(x, y) & \ldots & \frac{\partial}{\partial y_m} f(x, y) \end{pmatrix} \].

A function \( f(x) \) belongs to the class \( \mathcal{C}(\varnothing, \varnothing; a) \) if it is holomorphic and bounded for \( x \) in

(2.10) \[ 0 < |x| < a, \quad \varnothing < \text{arg } x < \varnothing, \]

and admits an asymptotic expansion in powers of \( x \) as \( x \) tends to 0 through sector (2.10).

A vector \( f(x, y) \) which is a holomorphic function of \( (x, y) \) in

(2.11) \[ 0 < |x| < c, \quad \varnothing < \text{arg } x < \varnothing, \quad ||y|| < d \]

is said to have Property-\( \mathcal{U} \) with respect to \( y \) in (2.11) if the components of \( f(x, y) \) admit uniformly convergent expansions in powers of \( y \) for \( (x, y) \) in (2.11) whose coefficients belong to the class \( \mathcal{C}(\varnothing, \varnothing; c) \).
The symbol \( f[x; y, z] \) denotes a polynomial in \( x \) of degree \( \sigma \) whose coefficients are functions of \( y \) and \( z \).

If the coefficients of the polynomial are holomorphic vector functions of \((y, z)\) for \(||y|| < \delta, ||z|| < \delta\), we shall say \( f[x; y, z] \) has Property-\( \sigma \) with respect to \( x \) in \( ||y|| < \delta, ||z|| < \delta \).

For a scalar \( w \) and an \( n \)-column vector \( v \), the symbol \( f<x; w, v> \) stands for a polynomial in \( x \) of degree \( \sigma \) having the form

\[
(2.12) \quad f<x; w, v> = f_0(v) + \sum_{i=1}^{\sigma} x^i f_i(w, v)
\]

If \( f_0(v) \) is a holomorphic function of \( v \) in \(||v|| < \delta\) and the \( f_i(w, v) \) have Property-\( \mathcal{U} \) with respect to \( v \) in

\[
(2.13) \quad 0 < |w| < c, \, \mathcal{P} < \arg w < \mathcal{P}, \, ||v|| < \delta,
\]

we shall say \( f<x; w, v> \) has Property-\( \sigma^* \) with respect to \( x \) in (2.13).

Now for a positive integer \( \sigma \) and nonzero complex constants \( \gamma_j, j = 1, 2, \ldots, m \), let

\[
\Omega_j(x) = -\frac{\gamma_j}{\sigma x^\sigma}.
\]

Then the sectors of the form

\[
(2.14) \quad \frac{1}{\sigma} \left( \arg \gamma_j - \frac{\pi}{2} + 2\pi n \right) < \arg x < \frac{1}{\sigma} \left( \arg \gamma_j + \frac{\pi}{2} + 2\pi n \right),
\]
(2.15) \[ \frac{1}{\sigma} (\arg \gamma_j + \frac{\pi}{2} + 2\pi h') < \arg x < \frac{1}{\sigma} (\arg \gamma_j + \frac{3\pi}{2} + 2\pi h') , \]

where \( h \) and \( h' \) are any integers, are said to be maximal negative regions of \( \Omega_j(x) \) and maximal positive regions of \( \Omega_j(x) \) respectively. As \( x \) approaches the origin in any subsector of (2.14) the function \( \exp(\Re \Omega_j(x)) \) tends to zero exponentially, while as \( x \) approaches the origin in any subsector of (2.15) the function \( \exp(\Re \Omega_j(x)) \) tends to infinity exponentially.

A sector \( \theta < \arg x < \bar{\theta} \) is said to have Property-\( J \) with respect to the monomials \( \{ \Omega_1(x), \ldots, \Omega_m(x) \} \) if the sector does not contain a maximal negative region of \( \Omega_j(x) \) for each index \( j \) and if in the sector there exists a direction for each index \( j \) such that as \( x \) approaches the origin along this direction, \( \exp(\Re \Omega_j(x)) \) tends to infinity exponentially. It should be noted that this definition for Property-\( J \) may differ from its definition by other authors. In particular, M. Hukuhara [8] assumes only that the sector \( \theta < \arg x < \bar{\theta} \) does not contain any maximal negative region of \( \Omega_j(x) \) for each index \( j \).

Moreover, in his case, the \( \Omega_j(x) \) may have different degrees.

It is possible to construct a sector that has Property-\( J \) with respect to any set of monomials \( \{ \Omega_1(x), \ldots, \Omega_m(x) \} \) and contains any fixed direction \( \arg x = \theta_0 \). In particular, the sectors \( S_j \) and \( S_j' \) of the form...
\[ S_j : \frac{1}{\sigma}(\arg \gamma_j - \frac{5\pi}{2} + 2\pi h_j) + \varepsilon_1 < \arg x < \frac{1}{\sigma}(\arg \gamma_j + \frac{\pi}{2} + 2\pi h_j) - \varepsilon_2, \]

\[ S_j' : \frac{1}{\sigma}(\arg \gamma_j - \frac{\pi}{2} + 2\pi h_j') + \varepsilon_1 < \arg x < \frac{1}{\sigma}(\arg \gamma_j + \frac{5\pi}{2} + 2\pi h_j') - \varepsilon_2, \]

where \( \varepsilon_1, \varepsilon_2 \) are constants satisfying \( 0 < \varepsilon_1, \varepsilon_2 < \frac{2\pi}{\sigma} \), \( \varepsilon_1 + \varepsilon_2 < \frac{3\pi}{\sigma} \), both have Property-\( \mathcal{J} \) with respect to \( \{ \Omega_j(x) \} \). If a direction \( \arg x = \Theta_o \) is given, we can choose \( h_j \) and \( h_j' \) so that \( \Theta_o \in S_j \) and \( \Theta_o \in S_j' \).

Put

\[ S = \bigcap_{j=1}^{m} S_j, \quad S' = \bigcap_{j=1}^{m} S_j'. \]

Then, both \( S \) and \( S' \) are non-empty and have Property-\( \mathcal{J} \) with respect to \( \{ \Omega_1(x), \ldots, \Omega_m(x) \} \). Also, since

\[ \max_{j=1}^{m}\{2\pi h_j + \arg \gamma_j\} - \min_{j=1}^{m}\{2\pi h_j + \arg \gamma_j\} < 2\pi, \]

and

\[ \max_{j=1}^{m}\{2\pi h_j' + \arg \gamma_j\} - \min_{j=1}^{m}\{2\pi h_j' + \arg \gamma_j\} < 2\pi, \]

we can choose \( \varepsilon_1 \) and \( \varepsilon_2 \) sufficiently small so the sectors \( S \) and \( S' \) have central angle greater than \( \pi/\sigma \).
CHAPTER II

MAIN RESULTS

3. A Formal Solution

Consider a system of nonlinear ordinary differential equations

\[
\begin{align*}
    x^{\sigma+1} y' &= F(x, y, z), \\
    x z' &= G(x, y, z), \quad (z = \frac{d}{dx})
\end{align*}
\]

in the degenerate case when \( G_z(0, 0, 0) = 0 \). In particular, let system (S) satisfy the following assumptions:

ASSUMPTION I. \( x \) is an independent complex variable, \( y \) and \( z \) are \( m \)- and \( n \)-column vectors respectively.

ASSUMPTION II. \( \sigma \) is a positive integer.

ASSUMPTION III. \( F(x, y, z) \) and \( G(x, y, z) \) are \( m \)- and \( n \)-column vectors respectively whose components are holomorphic and bounded functions of \((x, y, z)\) in

\[
|x| < a, \quad \|y\| < d, \quad \|z\| < d,
\]

for positive constants \( a \) and \( d \).

11

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ASSUMPTION IV. $F(0,0,0) = 0$ and $G(0,0,0) = 0$.

ASSUMPTION V. $F_y(0,0,0)$ is a nonsingular matrix in Jordan canonical form. That is,

\[
F_y(0,0,0) = \lambda_m(\mu) + E, \quad \det \lambda_m(\mu) \neq 0,
\]

where $E$ is a nilpotent matrix. For simplicity we will denote $F_y(0,0,0)$ by $\mathcal{J}$.

ASSUMPTION VI. $G_z(0,0,0) = 0$.

ASSUMPTION VII. Without loss of generality we can assume that

\[
\begin{align*}
(3.3) & \quad F_z(0,0,0) = 0, \quad F_x(0,0,0) = 0, \\
(3.4) & \quad G_y(0,0,0) = 0, \\
(3.5) & \quad G_x(0,0,0) = 0.
\end{align*}
\]

If (3.3) is not true, apply the linear transformation

\[
y = \hat{y} - \mathcal{J}^{-1} F_z(0,0,0) \hat{z} - x \mathcal{J}^{-1} F_x(0,0,0),
\]

\[
z = \hat{z}.
\]

In the event that (3.4) does not hold, make the change of variables

\[
y = \hat{y},
\]

\[
z = -x^\sigma G_y(0,0,0) \mathcal{J}^{-1} \hat{y} + \hat{z}.
\]
If (3.5) is false, let $a_k$ be the $k$th component of $G(0,0,0)$. Then apply the transformation

$$y = \hat{y}, z_h = \hat{z}_h \ (h \neq k),$$

$$z_k = x\alpha_k + \hat{z}_k.$$

It is difficult, if not impossible, to guarantee that a formal solution has any analytic properties unless more specific assumptions are applied to $G(x,y,z)$. Hence, similar to Iwano [12], we will additionally assume:

**ASSUMPTION VIII.** For the function $G(x,y,z)$,

$$G(0,y,z) = z^q \sum_n (\hat{g}(z)) z,$$

where $q$ is an $n$-row vector of nonnegative integers, not all zero, and the components of the $n$-column vector $\hat{g}(z)$ are holomorphic and bounded functions of $z$ in $||z|| < d$.

**ASSUMPTION IX.** $\hat{g}(0) = \beta$ where all components of $\frac{\beta}{q\beta}$ have positive real parts.

**ASSUMPTION X.** For all arrangements $(\ell,p)$ of $1+n$ nonnegative integers $\ell$ and $\{p_k\}$ such that $\ell + |p| \geq 2$ we have

$$\frac{\beta_i}{q\beta} \neq \ell + \frac{p\beta}{q\beta},$$

for each index $i$. 

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The formal solution will depend on the solutions, \( \tilde{V}(x) \) and \( W(x) \), of two reduced equations. The function \( \tilde{V}(x) \) is a solution of the equation,

\[ x \frac{dv}{dx} = v^q \ln(g(v))v, \]

where \( g(v) \) is a suitable polynomial. The function \( W(x) \) is a solution of

\[ x \frac{dw}{dx} = w^2 qg(\tilde{V}(x)). \]

The specific properties of \( \tilde{V}(x) \) and \( W(x) \) will be discussed in Chapter IV.

Now let

\[ \Omega_j(x) = \frac{-\mu_j}{\sigma x^j}, \]

for each \( j = 1, \ldots, m \). Then, under the above assumptions, the following theorem is true:

**Theorem A.** Let \( W(x) \) and \( \tilde{V}(x) \) be as defined above. Then the system \((S)\) has a formal solution of the form
\( y \sim \varphi(x; W(x), \tilde{V}(x)) \)

\[ + x^{\sigma+1} \sum_{|p|=0}^{\infty} \tilde{V}(x)^{p} r_p(x, W(x)), \]

(FS)

\( z \sim \psi(x; W(x), \tilde{V}(x)) \)

\[ + x^{\sigma+1} \sum_{|p|=0}^{\infty} \tilde{V}(x)^{p} s_p(x, W(x)), \]

with the following properties:

i) \( \varphi(x; w, v) \) and \( \psi(x; w, v) \) have Property-\( \sigma^* \) with respect to \( x \) for \( (w, v) \) in

(3.6) \( 0 < |w| < b', \; \tau < \arg w < \bar{\tau}, \; ||v|| < c' \).

for suitable positive constants \( b' \) and \( c' \). Here

(3.7)

\[ \tau = -\arg q\beta - \frac{3\pi}{2} + \epsilon, \]

\[ \bar{\tau} = -\arg q\beta + \frac{3\pi}{2} - \epsilon, \]

for a suitable positive constant \( \epsilon \). Also

(3.8)

\[ \frac{\partial}{\partial v} \psi_0(0) = l_n. \]
ii) The functions \( r_p(x, w) \) and \( s_p(x, w) \) are holomorphic and bounded functions of \( (x, w) \) in

\[
0 < |x| < a', \quad \theta < \arg x < \theta',
\]

(3.9)

\[
0 < |w| < b', \quad \Gamma < \arg w < \Gamma'.
\]

Here the sector \( \theta < \arg x < \theta' \) has Property-\( J \) with respect to \( \{ \Omega_1(x), \ldots, \Omega_m(x) \} \) and contains the positive real axis. Moreover, the functions \( r_p(x, w) \) and \( s_p(x, w) \) possess asymptotic expansions of the form

\[
r_p(x, w) \sim \sum_{\ell=0}^{\infty} x^\ell P_{\ell p}(w),
\]

(3.10)

\[
s_p(x, w) \sim \sum_{\ell=0}^{\infty} x^\ell Q_{\ell p}(w),
\]

(3.11)

as \( x \) tends to 0 in

\[
0 < |x| < a', \quad \theta < \arg x < \theta'.
\]

The functions \( P_{\ell p}(w) \) and \( Q_{\ell p}(w) \) are of the form

\[
P_{\ell p}(w) = \overline{P_{\ell p}(w, U(w))},
\]

(3.12)

\[
Q_{\ell p}(w) = \overline{Q_{\ell p}(w, U(w))},
\]
where \( U(w) = \frac{B}{n(w^\beta)} C_1 \), for an arbitrary constant vector \( C_1 \), and \( F_p(w, u), Q_p(w, u) \) both have Property-\( \mathfrak{U} \) with respect to \( u \) in

\[
0 < |w| < b', \hspace{1em} \Pi < \arg w < \Phi, \hspace{1em} ||u|| < e',
\]

for a positive constant \( e' \).

This theorem is a generalization of a result by Iwano [11]. A proof of Theorem A can be found in Chapter V.

4. A Special Existence Theorem

In order to construct the functions \( r_p(x, w) \) and \( s_p(x, w) \) in Theorem A it is necessary to study equations of the form

\[
\begin{cases}
  x^{\sigma+1} y' = A(x, W(x); y, z), \\
  xz' = B(x, W(x); y, z). \quad \left( \frac{d}{dx} \right)
\end{cases}
\]

Therefore, for the system (SS), let us assume that:

i) \( x \) is an independent complex variable, \( y \) and \( z \) are \( m- \) and \( n- \) column vectors respectively.

ii) \( \sigma \) is a positive integer.

iii) The function \( W(x) \) is as defined in Theorem A.

iv) \( A(x, w; y, z) \) and \( B(x, w; y, z) \) are \( m- \) and \( n- \) column vectors respectively whose components are
holomorphic and bounded functions of \((x, w; y, z)\) in

\[ 0 < |x| < a_0, \quad 0 < |w| < b_0, \]

\(0 < \arg x < \theta^*, \quad 0 < |w| < b_0,\)

(4.1)

\[ \Gamma < \arg w < \overline{\Gamma}, \quad ||y|| < d_0, ||z|| < d_0, \]

for positive constants \(a_0, b_0\) and \(d_0\). Here \(\Gamma\) and \(\overline{\Gamma}\) are the same as in (3.7). Furthermore, suppose

\(A(x, w; y, z)\) and \(B(x, w; y, z)\) have uniformly convergent expansions in powers of \(y\) and \(z\) in (4.1) of the form

\[ A(x, w; y, z) = \sum_{|\ell| + |k| = 0}^{\infty} y^\ell z^k A_{\ell k}(x, w), \]

(4.2)

\[ B(x, w; y, z) = \sum_{|\ell| + |k| = 0}^{\infty} y^\ell z^k B_{\ell k}(x, w). \]

The coefficients \(A_{\ell k}(x, w), B_{\ell k}(x, w)\) have asymptotic expansions as \(x\) tends to 0 in

\[ 0 < |x| < a_0, \quad 0 < \arg x < \overline{\theta}^*, \]

(4.3)

of the form
\[ A_{\ell k}(x, w) \sim \sum_{j=0}^{\infty} x^j A_{\ell kj}(w, U(w)) , \]

(4.4)

\[ B_{\ell k}(x, w) \sim \sum_{j=0}^{\infty} x^j B_{\ell kj}(w, U(w)) . \]

Here \( U(w) = l_n(w^{q_0}) C_1 \) for an arbitrary constant vector \( C_1 \), and \( A_{\ell kj}(w, u), B_{\ell kj}(w, u) \) have Property-\( \mathcal{U} \) with respect to \( u \) in

(4.5) \( 0 < |w| < b_0, \quad \overline{\Gamma} < \arg w < \Gamma, \quad ||u|| < e_0, \)

for a positive constant \( e_0 \).

v) \( A_y(0, 0; 0, 0) = l_m(\mu) + E \) where \( \det l_m(\mu) \neq 0 \) and \( E \) is a nilpotent matrix. Also suppose

\( A_Z(0, 0; 0, 0) = 0 \).

vi) Equations (5S) have a formal solution of the form

\[ y \sim \sum_{\ell=0}^{\infty} x^\ell P_\ell(w) = \sum_{\ell=0}^{\infty} x^\ell \overline{P_\ell}(w, U(w)) , \]

(4.6)

\[ y \sim \sum_{\ell=0}^{\infty} x^\ell Q_\ell(w) = \sum_{\ell=0}^{\infty} x^\ell \overline{Q_\ell}(w, U(w)) , \]

where \( \overline{P_\ell}(w, u) \) and \( \overline{Q_\ell}(w, u) \) have Property-\( \mathcal{U} \) with respect to \( u \) in (4.5). Furthermore suppose
Now let
\[ \Omega_j(x) = \frac{-u_j}{\sigma x^\sigma} \]
for \( j = 1, \ldots, m \). Then the following is true:

**Theorem B.** Let \( 0 < \arg x < \theta^* \) be a subsector of \( \theta^* < \arg x < \theta^* \) with Property-\( \mathcal{F} \) with respect to \( \{ \Omega_1(x), \ldots, \Omega_m(x) \} \). Then the system \((SS)\) has a unique solution of the form

\[ y = \phi(x, W(x)), \quad z = \psi(x, W(x)), \]

whenever the values of \((x, w)\) are in

\[ 0 < |x| < a_0', \quad \theta < \arg x < \theta^* , \]

or

\[ 0 < |w| < b_0', \quad \Gamma < \arg w < \Gamma^* , \]

for suitably chosen positive constants \( a_0' \) and \( b_0' \). Moreover, \( \phi(x, w) \) and \( \psi(x, w) \) possess asymptotic expansions of the form \((4.6)\) as \( x \) tends to \( 0 \) in

\[ 0 < |x| < a_0', \quad \theta < \arg x < \theta^* . \]

A proof of Theorem B can be found in Chapter VI.
CHAPTER III

KNOWN EXISTENCE THEOREMS

The following existence theorems are used extensively in this paper.

5. Existence Theorem I

Consider the system of differential equations

\[(5.1) \quad xy' = y q_1^n (\hat{g}(y)) y, \quad (\frac{\gamma}{dx})\]

where:

i) \(x\) is an independent complex variable and \(y\) is an \(n\)-column vector.

ii) \(q\) is an \(n\)-row vector of nonnegative integers, not all zero.

iii) The components of the \(n\)-column vector \(\hat{g}(y)\) are holomorphic functions of \(y\) in \(|\|y\|| < b\).

iv) \(\hat{g}(0) = \beta\) and all components of the \(n\)-column vector \(\frac{\beta}{q\beta}\) have positive real parts.

Then the following is true:

**Existence Theorem I.** There exists a transformation of the form

\[(5.2) \quad y = u + 1_n(u)P(u), \quad P(0) = 0,\]

21

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where \( P(u) \) is an \( n \)-column vector whose components are holomorphic functions of \( u \) in \( \|u\| \leq b_0 \), which transforms equation (5.1) into

\[
(5.3) \quad xu' = u^q_1 n (g(u))u .
\]

Here

\[
(5.4) \quad g(u) = \beta + \sum_{p \in \mathcal{P}} u^p \beta_p ,
\]

where \( \mathcal{P} \) is a finite set of \( n \)-row vectors \( p \) such that

\[
(5.5) \quad (p - q) \beta = 0 ,
\]

and the \( \beta_p \) are \( n \)-column constant vectors satisfying

\[
(5.6) \quad (\ell - q) \beta_p = 0
\]

for every \( \ell \in \mathcal{P} \).

This theorem is a special case of a theorem proved by M. Iwano [12].

6. Existence Theorem II

Consider the system of nonlinear ordinary differential equations of the form

\[
\begin{align*}
\tag{6.1}
x^{q+1} y' &= A(x, y, z) , \\
xz' &= B(x, y, z) , \quad \left( ' = \frac{d}{dx} \right)
\end{align*}
\]

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where:

i) \( \sigma \) is a positive integer.

ii) \( x \) is a complex independent variable, \( y \) and \( z \) are \( m \)- and \( n \)-column vectors respectively.

iii) \( A(x, y, z) \) and \( B(x, y, z) \) are \( m \)- and \( n \)-column vectors respectively whose components have Property-\( \mathcal{W} \) with respect to \( y \) and \( z \) in

\[
0 < |x| < c, \ \Theta < \arg x < \Theta, \ ||y|| < d, \ ||z|| < d.
\]

iv) For a nilpotent matrix \( D \),

\[
A_y(0, 0, 0) = l_m(y) + D,
\]

(6.3)

\[
A_z(0, 0, 0) = 0, \ \det l_m(y) \neq 0.
\]

v) Equations (6.1) possess a formal solution of the form

\[
y \sim \sum_{\ell=0}^{\infty} x^\ell f_\ell, \ z \sim \sum_{\ell=0}^{\infty} x^\ell g_\ell,
\]

(6.4)

where \( f_\ell \) and \( g_\ell \) are \( m \)- and \( n \)-column constant vectors respectively and

\[
||f_0|| < d, \ ||g_0|| < d.
\]

Then the following is true:
Existence Theorem II. Let

\[ \Omega_j(x) = \frac{-\gamma_j}{\sigma x^\gamma} \]

for \( j = 1, 2, \ldots, m \). If the sector \( \theta < \arg x < \bar{\theta} \) contains a subsector \( \theta^* < \arg x < \bar{\theta}^* \) which has

Property \( P \) with respect to \( \{ \Omega_1(x), \ldots, \Omega_m(x) \} \), then

equations (6.1) possess a unique solution \( \{ \phi(x), \psi(x) \} \)

which is holomorphic and bounded in

(6.5) \[ 0 < |x| < c_\sigma, \ \theta^* < \arg x < \bar{\theta}^* . \]

This solution admits asymptotic expansions of the form

(6.4) as \( x \) tends to 0 in the sector (6.5).

Proofs for this theorem can be found in Hsieh [2] and Iwano [11].

7. Existence Theorem III

Consider a system of nonlinear ordinary differential equations of the form

\[
\begin{align*}
    x^{\sigma+1} y' &= A(x, U(x); y, z), \\
    xz' &= B(x, U(x); y, z),
\end{align*}
\]

(7.1)

where \( U(x) = l_n(x^\mu) C \) for an arbitrary \( n \)-column constant

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vector C. Suppose that:

i) \( \sigma \) is a positive integer.

ii) \( x \) is an independent complex variable, \( y \) and \( z \) are \( m \)- and \( n \)-column vectors respectively.

iii) \( A(x, u; y, z) \) and \( B(x, u; y, z) \) are \( m \)- and \( n \)-column vector functions which admit uniformly convergent expansions in powers of \( y \) and \( z \) in a domain of the form

\[
0 < |x| < a, \quad \theta < \arg x < \overline{\theta},
\]

\[
(7.2)
\]

\[
||u|| < c, \quad ||z|| < d,
\]

whose coefficients are functions with Property-\( \mathcal{U} \) with respect to \( u \) in

\[
(7.3) \quad 0 < |x| < a, \quad \theta < \arg x < \overline{\theta}, \quad ||u|| < c.
\]

iv) For a nilpotent \( D \),

\[
A_y(0, 0; 0, 0) = l_m(y) + D,
\]

\[
(7.4)
\]

\[
A_z(0, 0; 0, 0) = 0, \quad \det l_m(y) \neq 0.
\]

v) Equations (7.1) have a formal solution of the form
\[
\sum_{|p|=0}^{\infty} U(x)^p f_p(x),
\]
(7.5)

\[
\sum_{|p|=0}^{\infty} U(x)^p g_p(x),
\]

where \( f_p(x) \) and \( g_p(x) \) are \( m \)- and \( n \)-column vector functions respectively which belong to the class \( C(\theta, \bar{\theta}; a) \) and

\[
|| f_0(x) || < d, \quad || g_0(x) || < d,
\]

when the values of \( x \) stay in

\[
0 < |x| < a, \quad \theta < \arg x < \bar{\theta}.
\]

Then the following is true:

**Existence Theorem III.** Let

\[
\Omega_j(x) = - \frac{\gamma_j}{\sigma x}
\]

for \( j = 1, 2, \ldots, m \). If the sector \( \theta < \arg x < \bar{\theta} \) contains a subsector \( \theta^* < \arg x < \bar{\theta}^* \) which has Property-J with respect to \( \{\Omega_1(x), \ldots, \Omega_m(x)\} \), then equations (7.1) have a solution of the form
(7.6) \[ y = \phi(x, U(x)), \quad z = \psi(x, U(x)), \]

whenever values of \( x \) and \( u = U(x) \) are in

(7.7) \[ 0 < |x| < a_0, \quad \theta^* < \arg x < \theta^*, \quad ||u|| < c_0. \]

The solution admits a uniformly convergent expansion of the form (7.5) and hence the functions \( \phi(x, u) \) and \( \psi(x, u) \) have Property-\( \Phi \) with respect to \( u \) in (7.7).

Proofs for this theorem can be found in Hsieh [2] and Iwano [11].

8. Existence Theorem IV

Consider the system of nonlinear ordinary differential equations of the form

\[
\begin{align*}
\sigma + 1 y' &= F(x, y, z), \\
xz' &= G(x, y, z), \quad (') = \frac{d}{dx}
\end{align*}
\]

(8.1)

where:

i) \( \sigma \) is a positive integer.

ii) \( x \) is an independent complex variable, \( y \) and \( z \) are \( m \)- and \( n \)-column vectors respectively.

iii) \( F(x, y, z) \) and \( G(x, y, z) \) are \( m \)- and \( n \)-column vectors respectively whose components are holomorphic and bounded functions of \((x, y, z)\) in

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$|x| < c, \; \|y\| < d, \; \|z\| < d.$

iv) $F(0, 0, 0) = 0$ and $G(0, 0, 0) = 0.$

v) $F_y(0, 0, 0)$ is nonsingular and in Jordan canonical form.

vi) $G_z(0, 0, 0) = l_n(\mu)$ where all components of have positive real parts.

vii) The kth component of $G_x(0, 0, 0)$ is zero if $\mu_k = 1.$

viii) For all vectors $(\ell, q)$ of $1 + n$ nonnegative integers such that $\ell + |q| \geq 2, \; \mu_k \neq \ell + q\mu$ for each index $k.$

Let the set of eigenvalues of $F_y(0, 0, 0)$ be denoted by \{\nu_1, \nu_2, \ldots, \nu_m\} and let

$$\Lambda_j(x) = \frac{-\nu_j}{\sigma x}$$

for each $j = 1, 2, \ldots, m.$ Then the following is true:

**Existence Theorem IV.** If $0 < \arg x < \overline{\Theta}$ is a sector with Property-\(\mathcal{T}\) with respect to \{\Lambda_1(x), \ldots, \Lambda_m(x)\} which contains the positive real axis, then equations (8.1) have a particular solution of the form

$$y = \Phi(x, l_n(x^\mu)C), \quad z = \Psi(x, l_n(x^\mu)C),$$
whenever values of $x$ and $\ln(x^\mu)$ belong to

(8.3) $0 < |x| < c_\circ$, $\emptyset < \arg x < \emptyset$, $\| \ln(x^\mu) \| < d_\circ$.

Here $\phi(x, u)$ and $\psi(x, u)$ have unique representations of the form

\[
\phi(x, u) = \varphi[x; u] + x^{\sigma+1} \phi^0(x, u),
\]

(8.4)

\[
\psi(x, u) = \psi[x; u] + x^{\sigma+1} \psi^0(x, u),
\]

where $\varphi[x; u]$ and $\psi[x; u]$ have Property-$\sigma$ with respect to $x$ in $\| u \| < d_\circ$, while $\phi^0(x, u)$ and $\psi^0(x, u)$ have Property-$\psi$ with respect to $u$ in

(8.5) $0 < |x| < c_\circ$, $\emptyset < \arg x < \emptyset$, $\| u \| < d_\circ$.

Also

(8.6) $\frac{\partial}{\partial u} \psi[0; 0] = \ln$

This theorem can be found in Iwano [11].
CHAPTER IV

THE FUNCTIONS \( \tilde{V}(x) \) AND \( W(x) \)

9. The Function \( \tilde{V}(x) \)

Let \( \tilde{V}(x) \) be a general solution of the ordinary differential equation

\[
(9.1) \quad xv' = v^q_{1n}(g(v))v. \quad \left( ' = \frac{d}{dx} \right)
\]

Here \( v \) is an \( n \)-column vector, \( q \) is an \( n \)-row vector of nonnegative integers, not all zero, and \( g(v) \) is an \( n \)-column vector function of the form

\[
(9.2) \quad g(v) = \beta + \sum_{p \in S} v^p \beta_p,
\]

where \( S \) is a finite set of \( n \)-row vectors \( p \) such that

\[
(9.3) \quad (p - q)\beta = 0,
\]

and the \( \beta_p \) are \( n \)-column constant vectors satisfying

\[
(9.4) \quad (\ell - q)\beta_p = 0, \quad \text{for any } \ell \in S
\]

10. The Function \( W(x) \)

Now let
(10.1) \[ x = X(w), \quad v = V(w) \]

be a holomorphic solution of the differential equations

\[
(10.2) \quad w^2 \frac{dx}{dw} = \frac{x}{qg(v)}, \quad w \frac{dv}{dw} = l \left( \frac{g(v)}{qg(v)} \right) v,
\]

such that \[ X(w_1) = x_1 \] and \[ V(w_1) = v_1, \] where \((x_1, w_1, v_1)\) is in a domain of the form

\[
|x| < a, \quad 0 < |w| < b, \quad \Gamma < \arg w < \bar{\Gamma}, \quad ||v|| < c.
\]

Then equations (10.1) define a solution \( v = \tilde{V}(x) \) of (9.1), if and only if \( V(w)^q = w \). However, \( V(w)^q - w \) is a solution of the linear differential equation

\[
w \frac{dv}{dw} = Y,
\]

and hence equations (10.1) define a solution to (9.1) if and only if \( (v_1)^q = w_1 \). If we denote the inverse of \( X(w) \) by \( W(x) \), then \( \tilde{V}(x) = V(W(x)) \).

Equations (10.2) may be simplified by noticing that \( \tilde{V}(x)^{q-l} \) is a constant for every \( l \in \mathbb{S} \). In fact, since \( \tilde{V}(x) \) is a solution of (9.1),
\[
\frac{dV(x)}{dx} = V(x) \ell - q (V(x)) \ell - q \frac{d}{dx} V(x)
\]

\[
= \tilde{V}(x) \ell (\ell - q) \tilde{q} (V(x))
\]

\[
= \tilde{V}(x) \ell \{ (\ell - q) \beta + \sum_{p \in \mathbb{S}} (\ell - q) \beta_p \tilde{V}(x)^p \}
\]

\[
= 0.
\]

Therefore the n-column vector

\[
\beta^* = \sum_{\ell \in \mathbb{S}} \tilde{V}(x) \ell - q \beta_\ell
\]

is constant. So a solution \( v = \tilde{V}(x) \) of (9.1) can be represented by a solution of the system

\[
(10.4) \quad w^2 \frac{dx}{dw} = \frac{w}{q \beta + q \beta^* w}, \quad w \frac{dv}{dw} = \frac{1}{n} \left( \frac{\beta + \beta^* w}{q \beta + q \beta^* w} \right) v,
\]

where \( x = x_1 \) and \( v = v_1 \) at \( w_1 \) and \( w_1 = (v_1)^q \).

Equations (10.4) can be solved by elementary means and yield a parametric representation of \( \tilde{V}(x), \tilde{V}(x_1) = v_1 \) of the form
where \( c \) and \( C \) are chosen so that

\[
X(w_1) = x_1, \quad V(w_1) = v_1, \quad Cq = 1.
\]

A representation of \( W(x) \), the inverse function of \( X(w) \), can be obtained through use of a transcendental function introduced by M. Hukuhara [7]. Let

\[
x = \exp \left( -\frac{q\theta^*}{(q\theta)^2} R \right), \quad w = \frac{q\theta}{q\theta^*} S.
\]

Then the first equation of (10.4) becomes

\[
(10.6) \quad \frac{dS}{dR} = 1 + \frac{1}{S}.
\]

A solution of (10.6), \( S = G(R) \), is defined implicitly by

\[
R = S - \log (S + 1).
\]

Although the function \( G(R) \) has two types of branches, if we pick the branch such that \( G(R) - R - \log R \) vanishes at \( R = \infty \), \( W(x) \) can be expressed as
(10.7) \[ W(x) = \frac{q^\beta}{q^{\beta*}} \left\{ \log \left( c - \frac{(q^\beta)^2}{q^{\beta*}} \log x \right) \right\}^{-1}, \]

and a general solution \( V(x) \) of (9.1) can be written as

(10.8) \[ \tilde{V}(x) = \ln \left( \frac{G(R + c)}{q^\beta} \right) \ln \left( \frac{q^\beta + \frac{q^\beta}{G(R + c)}}{q^{\beta*}} - \frac{q^\beta}{q^{\beta*}} \right), \]

where \( R = -\frac{(q^\beta)^2}{q^{\beta*}} \log x \). Here \( \tilde{c} \) and \( \tilde{c}^* \) must be chosen so that \( W(x_1) = w_1 \) and \( \tilde{V}(x_1) = v_1 \).
CHAPTER V

PROOF OF THEOREM A

11. Equations That Determine $\varphi<x;w,v>$ and $\psi<x;w,v>$

Let

$$\varphi<x;w,v> = \varphi_0(v) + \sum_{i=1}^{(1)} x^i \varphi_i(w,v), \tag{11.1}$$

$$\psi<x;w,v> = \psi_0(v) + \sum_{i=1}^{(1)} x^i \psi_i(w,v). \tag{11.2}$$

Since $F(x,y,z)$ and $G(x,y,z)$ are holomorphic at $(0,0,0)$, they have unique representations of the form

$$F(x,y,z) = F[x;y,z] + x^{\sigma+1} F^O(x,y,z), \tag{11.3}$$

$$G(x,y,z) = G[x;y,z] + x^{\sigma+1} G^O(x,y,z),$$

where $F[x;y,z]$ and $G[x;y,z]$ have Property-$\sigma$ with respect to $x$ for values of $(y,z)$ in

$$||y|| < d, \quad ||z|| < d, \tag{11.4}$$

and $F^O(x,y,z)$ and $G^O(x,y,z)$ have Property-$\delta$ with respect to $y$ and $z$ in
By virtue of assumptions V, VI and VIII, $F[x; y, z]$ and $G[x; y, z]$ have the following properties:

\[
F_y[0; 0, 0] = 3, \quad F_z[0; 0, 0] = 0, \\
G_y[0; 0, 0] = 0, \quad G_z[0; 0, 0] = 0.
\]

The equations that determine (11.1) are created by substituting the formal solution $(F S)$ into system (S) and isolating the coefficients of $x_i$ for $i = 0, 1, \ldots, \sigma$. By substituting into the right side of (S) and making use of (11.2) we have

\[
x^\sigma+1 y' = F[0; \varphi_0(v), \psi_0(v)] \\
+ x[K(v) \varphi_1(w, v) + L(v)\psi_1(w, v) + R_1(v)] \\
+ \cdots + x^\sigma[K(v) \varphi_\sigma(w, v) + L(v)\psi_\sigma(w, v) + R_\sigma(w, v)] \\
+ x^{\sigma+1}\{\cdots\},
\]

\[
xz' = G[0; \varphi_0(v), \psi_0(v)] \\
+ x[M(v) \varphi_1(w, v) + N(v)\psi_1(w, v) + S_1(v)] \\
+ \cdots + x^\sigma[M(v) \varphi_\sigma(w, v) + N(v)\psi_\sigma(w, v) + S_\sigma(w, v)] \\
+ x^{\sigma+1}\{\cdots\}.
\]

Here
\[ K(v) = F_y[0; \varphi_0(v), \psi_0(v)], \quad L(v) = F_z[0; \varphi_0(v), \psi_0(v)], \]
\[ M(v) = G_y[0; \varphi_0(v), \psi_0(v)], \quad N(v) = G_z[0; \varphi_0(v), \psi_0(v)]. \]

The functions \( \varphi_i \) are linear forms of the \( m \)-column vectors
\[ F_x[0; \varphi_0(v), \psi_0(v)], \ldots, \frac{\partial \varphi_i}{\partial x_i} F[0; \varphi(v), \psi(v)] \]
with polynomial coefficients in \( \varphi_k \) and \( \psi_k \) for \( 0 \leq k < i \). The functions \( g_i \) have a similar property.

In particular,
\[ R_1(v) = F_x[0; \varphi_0(v), \psi_0(v)], \]
\[ (11.7) \]
\[ S_1(v) = G_x[0; \varphi_0(v), \psi_0(v)]. \]

Now, by formally differentiating (FS) term by term, we have
\[ x^{\sigma+1} y' = x^\sigma \{ x \varphi_0'(v) \} + x^{\sigma+1} (...) 
\]
\[ (11.8) \]
\[ xz' = \{ x \psi_0'(v) \} + x(x \psi_1'(w,v) + \psi_1(w,v)) + \cdots + x^\sigma \{ x \psi_\sigma'(w,v) + \sigma \psi_\sigma(w,v) \} + x^{\sigma+1}(...). \]

Since \( v \) is a solution of \( xv' = v^\sigma 1_n(g(v))v \), and \( w \) is a solution of \( xw' = w^2 qg(v) \), the functions in the braces \{\cdots\} of statement (11.9) can be considered as functions of \( w \) and \( v \) alone.
From (11.6) and (11.9) the equations that determine
the functions $\varphi_i$ and $\psi_i$ are

(11.10) $F[0; \varphi_0, \psi_0] = 0, \quad x\psi_0' = G[0; \varphi_0, \psi_0],$

and, for $h = 1, 2, \ldots, \sigma$,

(11.11) $K(v)\varphi_h + L(v)\psi_h + \varphi_h = \frac{\partial}{\partial v} \varphi_{h-\sigma} - v^q \ln(g(v))v,$

(11.12) $x\psi_h' + h\psi_h = M(v)\varphi_h + N(v)\psi_h + S_h,$

where $\varphi_{h-\sigma} \equiv 0$ for $h - \sigma < 0$.

12. The Functions $\varphi_0(v)$ and $\psi_0(v)$

Since $F[0; 0, 0] = 0, \quad F_y[0; 0, 0] = 3$ and
$F_z[0; 0, 0] = 0$, the Implicit Function Theorem assures
the existence of a function $f(\psi_0)$ which is holomorphic
in a neighborhood at $\psi_0 = 0$ and has the following
properties:

(12.1) $\varphi_0 = f(\psi_0), \quad f(0) = 0, \quad f_{\psi_0}(0) = 0.$

To solve for $\psi_0$ notice that

$$G[0; \varphi_0, \psi_0] = G(0, \varphi_0, \psi_0) = \psi_0^q \ln(\hat{g}(\psi_0))\psi_0.$$ 

So the second equation of (11.10) becomes

(12.2) $x\psi_0' = \psi_0^q \ln(\hat{g}(\psi_0))\psi_0.$
This equation is solved by an application of Existence Theorem I in Chapter III. In particular, there exists a transformation

\[(12.3) \quad \psi_0(v) = v + \ln(v)P(v), P(0) = 0,\]

which transforms (12.2) into

\[(12.4) \quad x\nu' = vq \ln(g(v))v',\]

where \(g(v)\) is a special polynomial. Here \(P(V)\) is a holomorphic in some neighborhood of \(v = 0\).

Let \(\psi_0(v)\) be defined by (12.3) and

\[(12.5) \quad \phi_0(v) = f(\psi_0(v)).\]

Then, by picking a neighborhood of \(v = 0\) small enough, we have:

i) \(\phi_0(v)\) and \(\psi_0(v)\) are holomorphic in the neighborhood.

ii) \(\phi_0(v)\) and \(\psi_0(v)\) satisfy (11.10).

iii) \(\phi_0(0) = 0\) and \(\psi_0(0) = 0\).

iv) \(\frac{\partial}{\partial v} \phi_0(0) = 0\) and \(\frac{\partial}{\partial v} \psi_0(0) = 1_n\).

v) \(\|\phi_0(v)\| < d\) and \(\|\psi_0(v)\| < d\) for \(v\) in the neighborhood.
13. The Functions $\psi_1(w, v)$ and $\psi_1(w, v)$

Using (11.8), (11.11) and (11.12), the equations that determine $\psi_1(w, v)$ and $\psi_1(w, v)$ are

$$K(v)\psi_1 + L(v)\psi_1 + F_x[0; \varphi_0(v), \psi_0(v)] = 0,$$

(13.1)

$$x\psi_1' + \psi_1 = M(v)\varphi_1 + N(v)\psi_1 + G_x[0; \varphi_0(v), \psi_0(v)],$$

where

$$K(v) = F_y[0; \varphi_0(v), \psi_0(v)], \quad L(v) = F_z[0; \varphi_0(v), \psi_0(v)],$$

(13.2)

$$M(v) = G_y[0; \varphi_0(v), \psi_0(v)], \quad N(v) = G_z[0; \varphi_0(v), \psi_0(v)].$$

Since $\varphi_0(0) = 0$ and $\psi_0(0) = 0$, we have

$$K(0) = \bar{K}, \quad L(0) = 0,$$

(13.3)

$$M(0) = 0, \quad N(0) = 0,$$

and

$$F_x[0; \varphi_0(0), \psi_0(0)] = 0, \quad G_x[0; \varphi_0(0), \psi_0(0)] = 0.$$

(13.4)

Hence, in particular, $K^{-1}(v)$ exists in some neighborhood of $v = 0$. So the first equation of (13.1) can be written as
(13.5) \[ \varphi_1 = C(v) \psi_1 + D_1(v), \quad C(0) = 0, \quad D_1(0) = 0, \]

where \( C(v) \) and \( D(v) \) are holomorphic in some neighborhood of \( v = 0 \). Using (13.5), the second equation of (13.1) becomes

(13.6) \[ x\psi_1' = -\psi_1 + C(v)\psi_1 + D_1(v), \quad C(0) = 0, \quad D_1(0) = 0, \]

where \( \tilde{C}(v) \) and \( \tilde{D}_1(v) \) are also holomorphic in some neighborhood of \( v = 0 \).

Instead of solving equation (13.6) directly, we make use of the parametric representation \( \{X(w), V(w)\} \) for the function \( \tilde{V}(x) \) found in Chapter IV. To do so consider an equation of the form

(13.7) \[ w^2 \frac{dQ}{dw} = \frac{1}{qg(V(w))} h(X(w), w, V(w), Q). \]

where \( h(x, w, v, Q) \) is an n-column vector whose components are holomorphic and bounded functions of \((x, w, v, Q)\) in

\[
|x| < a^*, \quad 0 < |w| < b^*, \quad \underline{\gamma} < \text{arg } w < \overline{\gamma},
\]

\[
||v|| < c^*, \quad ||Q|| < d^*,
\]

for positive constants \( a^*, b^*, c^* \) and \( d^* \) where
\[
\Gamma = - \arg q - \frac{3\pi}{2} + \varepsilon , \\
\overline{\Gamma} = - \arg q + \frac{3\pi}{2} - \varepsilon ,
\]

for a sufficiently small positive constant \( \varepsilon \). Then the following is true:

**Lemma 1.** Suppose equation (13.7) has a holomorphic solution \( \zeta(X(w), w, V(w)) \) whenever values of \( (x, w, v) \) belong to the domain

\[
|x| < a^*, \quad 0 < |w| < b^*, \quad |v| < c^* ,
\]

(13.9)

\[
\Gamma < \arg w < \overline{\Gamma} , \quad ||v|| < c^* ,
\]

for suitably chosen positive constants \( a^* \), \( b^* \), and \( c^* \). Let \( w = W(x) \) be the inverse function of \( X(w) \) and \( \tilde{V}(x) = V(W(x)) \). Then \( \zeta(X, W(x), \tilde{V}(x)) \) is a solution of the equation

\[
(13.10) \quad x \frac{dQ}{dx} = h(x, W(x), \tilde{V}(x), Q) ,
\]

whenever the values of \( (x, w, v) \) belong to (13.9).

To prove this lemma notice that since \( \{X(w), V(w)\} \) is a solution of equations (10.2) and \( \zeta(X(w), w, V(w)) \) is a solution of equation (13.7), the function \( \zeta(x, w, v) \) is a solution of the partial differential equation.
whenever the values of \((x, w, v)\) belong to (13.9).

Now let \(\zeta = \zeta(x, W(x), V(x))\). Then by simple calculation

\[
x \frac{d\zeta}{dx} = x \frac{\partial \zeta}{\partial x} + x \frac{\partial \zeta}{\partial W(x)} \frac{dW(x)}{dx} + x \frac{\partial \zeta}{\partial V(x)} \frac{dV(x)}{dx}
\]

\[
= x \frac{\partial \zeta}{\partial x} + w(x) q(x(V(x))) \frac{\partial \zeta}{\partial W(x)}
\]

\[
+ \tilde{v}(x) \frac{\partial \zeta}{\partial V(x)} l_n(g(V(x))) V(x).
\]

But \(\tilde{v}(x) = V(W(x)) q = W(x)\) and by use of (13.11),

\[
x \frac{d\zeta}{dx} = h(x, W(x), V(x), \zeta),
\]

whenever the values of \(x, W(x)\) and \(V(x)\) belong to (13.9). This completes the proof of the lemma. A similar result can be found in Iwano [12].

Hence to solve equation (13.6), consider the equation

\[
(13.12) \quad w^2 \frac{d\psi_1}{dw} = \frac{1}{qg(V(w))} \left( -\psi_1 + C(V(w)) \psi_1 + D(V(w)) \right).
\]

Since \(V(w)\) is a general solution of the second equation of (10.2), equation (13.12) is equivalent to the system.
\[
\begin{aligned}
\frac{d^2 \psi_1}{d\theta^2} & = \frac{1}{q_\theta (v)} \left( - \psi_1 + \tilde{C}(v) \psi_1 + \tilde{D}_1(v) \right) = A(\psi_1, v), \\
\frac{dv}{d\theta} & = l_n \left( \frac{q(v)}{q_\theta (v)} \right) v = B(v) .
\end{aligned}
\] (13.13)

Notice that:

i) \( A(\psi_1, v) \) and \( B(v) \) are \( n \)-column vectors whose components are holomorphic functions of \( \psi_1 \) and \( v \) in a neighborhood of \( \psi_1 = 0, \ v = 0 \).

ii) \( A(0, 0) = 0 \) and \( B(0) = 0 \).

iii) \( A_{\psi_1}(0, 0) = \frac{-1}{q_\theta} l_n \).

iv) \( B_v(0) = l_n \left( \frac{\beta}{q_\theta} \right) \).

System (13.13) can be solved by an application of Existence Theorem IV in Chapter III. However, due to the form of \( B(v) \), the solution of the second equation in (13.13) has a simpler form than in Existence Theorem IV. In particular, we have:

Lemma 2. Let \( U(w) = l_n \left( \frac{\beta}{q_\theta} \right) C \), where \( C \) is an arbitrary constant \( n \)-column vector. Then the second equation of system (13.13) has a solution of the form \( V(w) = \Psi(U(w)) \), where \( \Psi(u) \) is holomorphic in a neighborhood of \( u = 0 \) and \( \Psi_u(0) = l_n \).
To prove this lemma first note the differential equation can be written in the form

\[(13.14) \quad \frac{dv}{dw} = l_n \left( \frac{\beta}{q^\delta} \right) v + \sum_{|\ell| \geq 2} v^\ell d^\ell. \]

We claim that this equation has a formal solution of the form

\[(13.15) \quad v = u + \sum_{|p| \geq 2} u^p c^p. \]

Indeed, differentiation of (13.15) yields

\[(13.16) \quad \frac{dw}{dw} = l_n \left( \frac{\beta}{q^\delta} \right) u + \sum_{|p| \geq 2} u^p \frac{p\delta}{q^\delta} c^p, \]

and substitution of (13.15) into (13.14) yields

\[(13.17) \quad \frac{dw}{dw} = l_n \left( \frac{\beta}{q^\delta} \right) u + \sum_{|p| \geq 2} u^p \left( l_n \left( \frac{\beta}{q^\delta} \right) c^p + K^p \right), \]

where the \(K^p\) depend on \(c^\ell\) for \(|\ell| < |p|\) only.

Thus, for each \(p\), \(c^p\) is determined by

\[(13.18) \quad \left\{ \frac{p\delta}{q^\delta} l_n - l_n \left( \frac{\beta}{q^\delta} \right) \right\} c^p = K^p. \]

By virtue of Assumption X, the matrix in the brackets is invertible and the \(c^p\)'s can be solved successively.

Since the formal solution (13.15) exists, the solution \(\Psi(u)\) can be guaranteed by Existence Theorem III in
Chapter II. Moreover, the series (13.15) has a positive radius of convergence. This completes the proof of the lemma.

Hence, by use of Lemma 2 and Existence Theorem IV, equations (13.13) have a solution of the form

\[ \{ \phi(w, U(w)), \psi(U(w)) \} \]

whenever the values of \( w \) and \( u = U(w) \) stay within

\[(13.19) \quad 0 < |w| < b_2, \quad \Gamma < \arg w < \Gamma, \quad ||u|| < c_2, \]

Here

\[(13.20) \quad \Gamma = - \arg \psi^3 - \frac{3\pi}{2} + \varepsilon, \quad \Gamma = - \arg \psi^3 + \frac{3\pi}{2} - \varepsilon, \]

and \( b_2, c_2 \) and \( \varepsilon \) are suitably chosen positive constants. Furthermore, the components of \( \phi(w, u) \) have Property-\( \Psi \) with respect to \( u \) in (13.19).

Since \( \psi'(0) = 1_n, \psi(u) \) is invertible in some neighborhood of \( u = 0 \). Let \( \psi^{-1}(v) \) be the inverse function. Then, for a suitably chosen positive constant \( c_2' \), we can assume that \( \psi^{-1}(v) \) is holomorphic in \( ||v|| < c_2' \) and there satisfies \( ||\psi^{-1}(v)|| < c_2 \).

Now define \( \psi_1(w, v) = \phi(w, \psi^{-1}(v)) \) and \( \varphi_1(w, v) = c(v)\psi_1(w, v) + D_1(v) \). Then, by using Lemma 1 and picking
small enough, \( \{ \varphi_1(W(x), \tilde{V}(x)) \}, \)
\( \psi_1(W(x), \tilde{V}(x)) \) is a solution of (13.1) whenever the
values of \((x, w, v)\) stay in
\[
|x| < a_1, \quad 0 < |w| < b_1, \quad |v| < c_1.
\]
Moreover, \( \varphi_1(w, v) \) and \( \psi_1(w, v) \) have Property-\( \mathcal{U} \) with
respect to \( v \) in
\[
0 < |w| < b_1, \quad \Gamma < \arg w < \pi, \quad |v| < c_1.
\]

14. The Functions \( \varphi_2(w, v) \) and \( \psi_2(w, v) \)

Since the determination of the functions \( \varphi_h(w, v) \)
and \( \psi_h(w, v) \) is the same for each \( h = 2, \ldots, \sigma \),
only the construction of \( \varphi_2(w, v) \) and \( \psi_2(w, v) \) will
be presented here. From (11.11) and (11.12), the
equations that determine \( \varphi_2(w, v) \) and \( \psi_2(w, v) \) are
of the form
\[
\varphi_2 = C(v)\psi_2 + D_2(w, v),
\]
\[
\psi_2' - 2\psi_2 + \tilde{C}(v)\psi_2 + \tilde{D}_2(w, v),
\]
where \( C(v) \) and \( \tilde{C}(v) \) are as described in (13.5) and
(13.6), and \( D_2(w, v) \) and \( \tilde{D}_2(w, v) \) have Property-\( \mathcal{U} \).
with respect to \( v \) in

\[(14.3) \quad 0 < |w| < b_1, \quad \Gamma < \arg w < \Gamma, \quad ||v|| < c_1.\]

As before, instead of solving (14.2) directly, we consider

\[(14.4) \quad w^2 \frac{d}{dw} \psi_2 = \frac{1}{qg(v)} (-2 \psi_2 + \tilde{C}(v) \psi_2 + \tilde{D}_2(w, v)),\]

\[(14.5) \quad w \frac{dv}{dw} = \ln \left( \frac{g(v)}{qg(v)} \right) v.\]

We already know equation (14.5) has a solution \( \psi(U(w)) \)

where

\[(14.6) \quad \psi(u) = u + \sum_{|p| \geq 2} u^p c_p\]

is convergent in some neighborhood of \( u = 0 \). A solution of (14.4) is developed in the following lemma.

**Lemma 3.** Equation (14.4) has a solution of the form

\[(14.7) \quad \psi_2(w, U(w)) = \sum_{|p|=0}^\infty U(w)^p I_p(w),\]

whenever the values of \( w \) and \( u = U(w) \) stay within

\[(14.8) \quad 0 < |w| < b_1, \quad \Gamma < \arg w < \Gamma, \quad ||u|| < c_1.\]
Moreover, $\psi_2(w, u)$ has Property-UN with respect to $u$ in (14.8).

To prove this lemma we will first show (14.5) has a formal solution of the form

\[
(14.9) \quad \psi_2 \sim \sum_{|p|=0}^{\infty} u^p I_p(w),
\]

where the $I_p(w)$ belong to the class $C(\Gamma, \bar{\Gamma}; b_1)$. By substituting (14.6) for $v$, equation (14.4) becomes

\[
(14.10) \quad w^2 \frac{d\psi_2}{dw} = \hat{C}(u)\psi_2 + \hat{D}_2(w, u), \quad \hat{C}(0) = \frac{-2}{q_\beta^2} \ln,
\]

where $\hat{C}(u)$ is holomorphic in $||u|| < c_1$ and $\hat{D}_2(w, u)$ has Property-UN with respect to $u$ in (14.8). Formally differentiating (14.9) we have

\[
(14.11) \quad w^2 \frac{d^2\psi_2}{dw^2} = w^2 \frac{dI_0}{dw} + \sum_{|p| \geq 1} u^p \left( w \frac{d}{dw} I_p + w^2 \frac{dI_p}{dw} \right)
\]

On the other hand, substituting (14.9) into the right-hand side of (14.10) we have

\[
(14.12) \quad w^2 \frac{d\psi_2}{dw} = \sum_{|p|=0}^{\infty} u^p \left( \frac{-2}{q_\beta^2} I_p + I_p(w) \right),
\]

where the $I_p(w)$ belong to the class $C(\Gamma, \bar{\Gamma}; b_1)$ and
depend only on $I_{\ell}(w)$ for $|\ell| < |p|$. Hence, the equations that determine (14.9) are

\begin{align}
(14.13) \quad w^2 \frac{d}{dw} I_0 &= \frac{-2}{q^\beta} I_0 + I_0(w), \\
(14.14) \quad w^2 \frac{d}{dw} I_p &= \frac{-2}{q^\beta} I_p - w \frac{D_p}{q^\beta} I_p + I_p(w).
\end{align}

It is easy to see that equation (14.13) has a formal solution of the form

\begin{equation}
(14.15) \quad I_0 \sim \sum_{\ell=0}^{\infty} w^\ell r_\ell
\end{equation}

where the $r_\ell$ are constant vectors. Hence, by use of Existence Theorem II, equation (14.13) has a solution $I_0(w)$ which admits an asymptotic expansion of the form (14.15). Since the right-hand side of (14.13) is holomorphic in $||I_0|| < \infty$, the solution of $I_0(w)$ belongs to the class $C(\overline{\Gamma}, \overline{\Gamma}; b_1)$. For $|p| > 1$, the $I_p(w)$ can be solved in a similar manner.

Given the formal solution (14.9), Existence Theorem III guarantees the existence of a solution $\psi_2(w, U(w))$ of (14.4) with the aforementioned properties. This completes the proof of the lemma.

Now by use of Lemma 1 and an argument similar to the one in the previous section, equations (14.1) and (14.2)
have a solution of the form \{ \varphi_2(W(x), V(x)), \psi_2(W(x), V(x)) \}
whenever \((w, v)\) are in
\[(14.16) \quad 0 < |w| < b_1, \quad \Gamma < \arg w < \Gamma, \quad ||v|| < c_1.
\]
Moreover, \(\varphi_2(w, v)\) and \(\psi_2(w, v)\) have Property-\(U\)
with respect to \(v\) in (14.16).

15. Equations That Determine the Series Part of the Formal Solution

Having determined \(\varphi(x; w, v)\) and \(\psi(x; w, v)\), the
construction of the series part of the formal solution (FS) is facilitated by the transformation
\[
y = \varphi(x; W(x), \tilde{V}(x)) + x^{\sigma+1} y,
\]
\[
z = \psi(x; W(x), \tilde{V}(x)) + x^{\sigma+1} z,
\]
By differentiating (15.1) we have
\[
x^{\sigma+1} y' = x^{\sigma+1} \varphi'_{x; w, v} + x^{\sigma+1} ((\sigma+1)x^\sigma y + x^{\sigma+1} y'),
\]
(15.2)
\[
x z' = x \psi'_{x; w, v} + x^{\sigma+1} ((\sigma+1)z + xz') .
\]
Making use of (11.2) the right-hand side of system (S) becomes

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\[ x^{\sigma+1} y' = F[x;\varphi<x;w,v> + x^{\sigma+1} y, \psi<x;w,v> + x^{\sigma+1} z] \]

(15.3)

\[ + x^{\sigma+1} F^0(x,\varphi<x;w,v> + x^{\sigma+1} y, \psi<x;w,v> + x^{\sigma+1} z) , \]

\[ xz' = G[x;\varphi<x;w,v> + x^{\sigma+1} y, \psi<x;w,v> + x^{\sigma+1} z] \]

(15.4)

\[ + x^{\sigma+1} G^0(x,\varphi<x;w,v> + x^{\sigma+1} y, \psi<x;w,v> + x^{\sigma+1} z) . \]

Hence the transformed system can be written as

\[ x^{\sigma+1} y' = -(\sigma+1)x^{\sigma} y \]

(15.5)

\[ + F^0(x,\varphi<x;w,v> + x^{\sigma+1} y, \psi<x;w,v> + x^{\sigma+1} z) \]

\[ - x^{-\sigma-1}\{x^{\sigma+1}\varphi',<x;w,v> \]

\[ - F[x;\varphi<x;w,v> + x^{\sigma+1} y, \psi<x;w,v> + x^{\sigma+1} z] \} , \]

(5.6)

\[ xz' = -(\sigma+1)z \]

\[ + G^0(x,\varphi<x;w,v> + x^{\sigma+1} y, \psi<x;w,v> + x^{\sigma+1} z) \]

\[ - x^{-\sigma-1}\{x^{\sigma+1}\psi',<x;w,v> \]

\[ - G[x;\varphi<x;w,v> + x^{\sigma+1} y, \psi<x;w,v> + x^{\sigma+1} z] \} . \]

By virtue of the nature of \( \varphi<x;w,v> \) and \( \psi<x;w,v> \), the expressions

\[ x^{\sigma+1} \varphi',<x;w,v> - F[x;\varphi<x;w,v> , \psi<x;w,v> ] , \]

\[ x \psi',<x;w,v> - G[x;\varphi<x;w,v> , \psi<x;w,v> ] . \]

both contain factors of \( x^{\sigma+1} \). Hence, the right-hand sides of (15.5) and (15.6) are holomorphic and bounded

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functions of \((x, w, v, Y, Z)\) in a domain of the form

\[
|x| < a_2, \quad 0 < |w| < b_1, \quad \Gamma < \arg w < \bar{\Gamma},
\]

(15.7)

\[
||v|| < c_2, \quad ||Y|| < d_2, \quad ||Z|| < d_2.
\]

By choosing \(a_2\) and \(c_2\) sufficiently small, \(d_2\) can be made arbitrarily large. For convenience (15.5) and (15.6) can be rewritten as

\[
\begin{cases}
\begin{align*}
x^{\sigma+1} y' &= A(x, W(x), \tilde{V}(x), Y, Z) \\
x \, Z' &= B(x, W(x), \tilde{V}(x), Y, Z).
\end{align*}
\end{cases}
\]

(15.8)

The Jacobians of the system have the form

\[
A_y(x, w, v, Y, Z) = -(\sigma+1)x^\sigma + x^{\sigma+1} F^0_y(x, y, z) + F_y[x; y, z],
\]

\[
A_z(x, w, v, Y, Z) = x^{\sigma+1} F^0_z(x, y, z) + F_z[x; y, z],
\]

(15.9)

\[
B_y(x, w, v, Y, Z) = x^{\sigma+1} G^0_y(x, y, z) + G_y[x; y, z],
\]

\[
B_z(x, w, v, Y, Z) = -(\sigma+1)x^n + x^{\sigma+1} G^0_z(x, y, z) + G_z[x; y, z],
\]

where \(y\) and \(z\) are given by (15.1). Hence
\[ A_y(0, w, 0, y, z) = 0, \quad \delta(0, w, 0, y, z) = 0, \]

(15.10)

\[ B_y(0, w, 0, y, z) = 0, \quad B_z(0, w, 0, y, z) = -(\sigma+1) \lambda_n, \]

and there exists functions \( A_0(w) \) and \( B_0(w) \) which belong to the class \( C(\Gamma, \Gamma'; b_1) \) such that

\[ A(0, w, 0, y, z) = A_0(w) + 3y, \]

(15.11)

\[ B(0, w, 0, y, z) = B_0(w) - (\sigma+1)z. \]

We will show system (15.8) has a formal solution of the form

\[ y \sim r_0(x, W(x)) + \sum_{|p|=1}^{\infty} \tilde{V}(x)^p r_p(x, W(x)), \]

(15.12)

\[ z \sim s_0(x, W(x)) + \sum_{|p|=1}^{\infty} \tilde{V}(x)^p s_p(x, W(x)). \]

Formally differentiating (15.12) we have

\[ x^{\sigma+1} y' \sim x^{\sigma+1} r_0' + \sum_{|p|=1}^{\infty} v^p[x^\sigma v^p g(v) r_p + x^{\sigma+1} r_p'], \]

(15.13)

\[ xz' \sim x s_0' + \sum_{|p|=1}^{\infty} v^p[v^p g(v) s_p + x s_p']. \]
On the other hand substituting (15.12) into the right-hand side of (15.8) we have

\[ x^{\sigma+1} y' = A(x, w, 0, r_0, s_0) \]

\[ + \sum_{|p|=1}^{\infty} v^p \{ K(x, w) r_p + L(x, w) s_p + \delta_p(x, w) \} , \]

(15.14)

\[ x z' = B(x, w, 0, r_0, s_0) \]

\[ + \sum_{|p|=1}^{\infty} v^p \{ M(x, w) r_p + N(x, w) s_p + \delta_p(x, w) \} , \]

(15.15)

where

\[ K(x, w) = A_\eta(x, w, 0, r_0(x, w), s_0(x, w)) , \]

\[ L(x, w) = A_\zeta(x, w, 0, r_0(x, w), s_0(x, w)) , \]

\[ M(x, w) = B_\eta(x, w, 0, r_0(x, w), s_0(x, w)) , \]

\[ N(x, w) = B_\zeta(x, w, 0, r_0(x, w), s_0(x, w)) . \]

So the equations that determine \( r_0(x, w) \) and \( s_0(x, w) \) are

\[ x^{\sigma+1} r_0' = A(x, w, 0, r_0, s_0) , \]

(15.16)

\[ x \ s_0' = B(x, w, 0, r_0, s_0) . \]
And the equations that determine $r_p(x, w)$ and $s_p(x, w)$ for $|p| > 1$, are

$$x^{\sigma+1} r_p' = K(x, w) r_p + L(x, w) s_p + \overline{r_p}(x, w),$$  \hspace{1cm} (15.17)

$$x s_p' = M(x, w) r_p + N(x, w) s_p + \overline{s_p}(x, w),$$

where $\overline{r_p}(x, w)$ and $\overline{s_p}(x, w)$ depend only on $r_\ell(x, w)$ and $s_\ell(x, w)$ for $|\ell| < |p|$. 

16. The Functions $r_0(x, w)$ and $s_0(x, w)$

We will first show equation (15.16) has a formal solution of the form

$$r_0(x, w) \sim P_0(w) + \sum_{\ell=1}^{\infty} x^\ell P_\ell(w),$$

(16.1)

$$s_0(x, w) \sim Q_0(w) + \sum_{\ell=1}^{\infty} x^\ell Q_\ell(w).$$

By formally differentiating (16.1) and using the fact that $x \frac{dw}{dx} = w^2qg(v)$ we have
\[ x^{\sigma+1} r_0' = w^2 q_g(v) \frac{d}{dw} \frac{P_0}{\sigma} x^\sigma \]

\[ + \sum_{\ell=1}^\infty x^{\ell+\sigma} \left\{ w^2 q_g(v) \frac{d}{dw} \frac{P_\ell}{\ell} + \ell P_\ell \right\} , \]

(16.2)

\[ x s_0' = w^2 q_g(v) \frac{d}{dw} Q_0 \]

\[ + \sum_{\ell=1}^\infty x^{\ell} \left\{ w^2 q_g(v) \frac{d}{dw} Q_\ell + \ell Q_\ell \right\} . \]

On the other hand substituting (16.1) into the right-hand side of (15.16) we have

\[ x^{\sigma+1} r_0' \sim A(0, w, 0, P_0, Q_0) \]

\[ + \sum_{\ell=1}^\infty x^{\ell} \left\{ \bar{R}(w) P_\ell + \bar{I}(w) Q_\ell + \bar{R}_\ell(w) \right\} , \]

(16.3)

\[ x s_0' \sim B(0, w, 0, P_0, Q_0) \]

\[ + \sum_{\ell=1}^\infty x^{\ell} \left\{ \bar{M}(w) P_\ell + \bar{N}(w) Q_\ell + \bar{S}_\ell(w) \right\} , \]

where

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\[ \Phi(w) = A_Y(0, w, 0, P_0(w), Q_0(w)), \]
\[ \Theta(w) = A_Z(0, w, 0, P_0(w), Q_0(w)), \]
\[ (16.4) \]
\[ \Psi(w) = B_Y(0, w, 0, P_0(w), Q_0(w)), \]
\[ \Omega(w) = B_z(0, w, 0, P_0(w), Q_0(w)), \]
and \( \Phi_\xi(w), \Phi_\eta(w) \) both belong to the class \( C(\Gamma, \Gamma; b_1) \).

With the use of (15.11) the equations that determine \( P_0(w) \) and \( Q_0(w) \) are of the form
\[ (16.5) \quad 0 = A_0(w) + \mathfrak{F}P_0, \]
\[ (16.6) \quad w^2 \frac{d}{dw} Q_0 = \frac{B_0(w) - (\sigma + 1) Q_0}{qg(v)}. \]

Since \( \mathfrak{F} \) is invertible, the function \( P_0(w) \) is determined by (16.5). As before, equations of the form (16.6) have a formal solution of the form
\[ (16.7) \quad Q_0 \sim \sum_{|p| = 0}^{\infty} U(w)^p I_p(w), \]
where the \( I_p(w) \) belong to the class \( C(\Gamma, \Gamma; b_1) \). Thus, Existence Theorem III guarantees the power series (16.7) will be uniformly convergent for \( (w, u) \) in
\[ (16.8) \quad 0 < |w| < b_1, \quad \Gamma < \arg w < \Gamma, \quad \|u\| < c_1. \]
Define $Q_0(w)$ by (16.7). Then equations (16.5) and (16.6) have a solution \( \{P_0(w), Q_0(w)\} \) for values of \( w \) in

\[
0 < |w| < b_1, \quad \Gamma < \text{arg } w < \pi.
\]

Moreover, since we can pick \( d_2 \) as large as we wish, we can assume \( ||P_0(w)|| < d_2 \) and \( ||Q_0(w)|| < d_2 \) in (16.9).

By virtue of (15.1) the equations that determine \( P_\ell(w) \) and \( Q_\ell(w) \) for \( \ell \geq 1 \) are

\[
w^2 qg(V(w)) \frac{d}{dw} P_{\ell-\sigma} + (\ell-\sigma) P_{\ell-\sigma}(w) = J P_{\ell} + \overline{R}_{\ell}(w),
\]

(16.10)

\[
w^2 \frac{d}{dw} Q_{\ell} = - (\ell+\sigma+1) Q_{\ell} + \overline{S}_{\ell}(w) qg(v),
\]

(16.11)

where \( P_{\ell-\sigma} \equiv 0 \) for \( \ell < \sigma \). Here \( \overline{R}_{\ell}(w) = \overline{R}_{\ell}(w, U(w)) \) and \( \overline{S}_{\ell}(w) = \overline{S}_{\ell}(w, U(w)) \) where both \( \overline{R}_{\ell}(w, u) \) and \( \overline{S}_{\ell}(w, u) \) have Property-\( \Psi \) with respect to \( u \) in (16.8).

The function \( P_{\ell}(w) \) is clearly determined by equations (16.10). Equation (16.11) has a formal solution of the form
where the $I_p(w)$ belong to the class $C(T, \bar{T}; b_1)$. Hence, Existence Theorem III guarantees the power series (16.12) will be uniformly convergent in (16.8). Therefore, equations (16.10) and (16.11) have a solution \{P_\ell(w), Q_\ell(w)\} for values of $w$ in (16.9).

By virtue of the existence of the formal solution (16.1), equation (15.16) can now be solved by use of Theorem B. In particular, let

(16.13) \[ \Omega_j(x) = \frac{-\mu_j}{\sigma^j} \]

for $j = 1, \ldots, m$. Then if $0 < \arg x < \theta$ is a sector with Property-J with respect to \{\Omega_1(x), \ldots, \Omega_m(x)\} that contains the positive real axis, there exists a solution \{r_0(x, W(x)), s_0(x, W(x))\} of (15.16) where $r_0(x, w)$ and $s_0(x, w)$ are holomorphic and bounded functions of $(x, w)$ in

(16.14) \[ 0 < |x| < a', \quad 0 < \arg x < \theta, \]

\[ 0 < |w| < b', \quad \Gamma < \arg w < \bar{T}, \]

which admit asymptotic expansions of the form (16.1) as
x tends to 0 in

\[(16.15)\quad 0 < |x| < a', \quad \theta < \arg x < \theta' .\]

17. The Functions \( r_p(x, w) \) and \( s_p(x, w) \)

We will first show equation (15.17) has a formal solution of the form

\[
\begin{align*}
 r_p(x, w) & \sim P_{p0}(w) + \sum_{\ell=1}^{\infty} w^{\ell} P_{p\ell}(w) , \\
 s_p(x, w) & \sim Q_{p0}(w) + \sum_{\ell=1}^{\infty} x^{\ell} Q_{p\ell}(w) .
\end{align*}
\]

(17.1)

Using (15.1), the equations that determine \( P_{p0}(w) \) and \( Q_{p0}(w) \) are

\[
(17.2) \quad 0 = \overline{A}_o(w, U(w)) + J P_{p0} ,
\]

\[
(17.3) \quad w^2 \frac{d}{dw} Q_{p0} = \frac{\overline{O}_o(w, U(w)) - (a+1) Q_{p0}}{qg(v)}
\]

where \( \overline{A}_o(w, u) \) and \( \overline{O}_o(w, u) \) have Property-\( \Psi \) with respect to \( u \) in

\[(17.4) \quad 0 < |w| < b', \quad \bar{\Gamma} < \arg w < \bar{\Gamma}, \quad ||u|| < e' ,\]
for a suitable positive constant e'. \( P_{p0}(w) \) is determined by (17.2). Clearly equation (17.3) has a formal solution of the form

\[
(17.5) \quad Q_{p0} = \sum_{|\ell|=0}^n U(w)^{\ell} I_p(w),
\]

where the \( I_p(w) \) belong to the class \( C(\Gamma, \Gamma'; b') \). Hence, by use of Existence Theorem III, the equations (17.2) and (17.3) have a solution \( \{ P_{p0}(w), Q_{p0}(w) \} \) for values of \( w \) in

\[
(17.6) \quad 0 < |w| < b', \quad \Gamma < \arg w < \Gamma .
\]

The equations that determine \( P_{p\ell}(w) \) and \( Q_{p\ell}(w) \) are of the form,

\[
(17.7) \quad \frac{\partial}{\partial w} Q_{p\ell}(w, U(w)) = \Phi_{p\ell}(w, U(w)) + \Psi_{p\ell}(w, U(w)),
\]

\[
(17.8) \quad w^2 \frac{d}{dw} Q_{p\ell} = \frac{-(\ell+\sigma+1)Q_{p\ell} + S_{p\ell}(w, U(w))}{qg(v)},
\]

where \( \Phi_{p\ell}(w, u), \Psi_{p\ell}(w, u) \) and \( S_{p\ell}(w, u) \) are all known functions having Property-\( \mathcal{U} \) with respect to \( u \) in (17.4). The function \( P_{p\ell}(w) \) is clearly determined by (17.7). Equation (17.8) has a formal solution of the form.

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where the $I_p(w)$ belong to the class $\mathcal{C}(\Gamma, \Gamma b')$.

Hence, by use of Existence Theorem III, equations (17.7) and (17.8) have a solution $\{P_{p\ell}(w), Q_{p\ell}(w)\}$ for values of $w$ in (17.6).

Finally, by virtue of the existence of the formal solution (17.1) and use of Theorem B, equation (15.17) has a solution $\{r_p(x, W(x)), s_p(x, W(x))\}$ where $r_p(x, w)$ and $s_p(x, w)$ are holomorphic and bounded functions of $x, w$ in (16.4). Moreover, this solution admits asymptotic expansions of the form (17.1) as $x$ tends to $0$ in (16.15).

Hence, the proof of Theorem A is complete.
18. Initial Reduction

To prove this theorem we first make a transformation of the form

\[
Y = \sum_{\ell=0}^{N-1} x^\ell P_\ell(W(x)) + Y,
\]
(18.1)

\[
Z = \sum_{\ell=0}^{N-1} x^\ell Q_\ell(W(x)) + Z.
\]

Then the transformed system can be written as

\[
x^{\sigma+1} Y' = l_m(u) Y + \hat{A}(x, W(x); Y, Z),
\]
(18.2)

\[
x Z' = \hat{B}(x, W(x); Y, Z),
\]

where \(\hat{A}(x, w; Y, Z)\) and \(\hat{B}(x, w; Y, Z)\) are holomorphic and bounded vector functions of \((x, w; Y, Z)\) in

\[
0 < |x| < a_N, \quad 0 < |w| < b_N,
\]
(18.3)

\[
0 < |x| < a_N, \quad 0 < |w| < b_N,
\]

\[
0 < |x| < a_N, \quad 0 < |w| < b_N.
\]

64
Here \( a_N, b_N \) and \( d_N \) are positive constants that depend on \( N, a, b \) and \( d \). Also, we have

\[
A_Y(0, 0; 0, 0) = E, \quad A_Z(0, 0; 0, 0) = 0,
\]

(18.4)

\[
A(0, 0; 0, 0) = 0, \quad B(0, 0; 0, 0) = 0.
\]

Therefore, there exist positive constants \( C, \overline{C} \) and \( D_N \) such that

\[
\| \hat{A}(x, w; Y, Z) \| \leq C (\| Y \| + \| Z \| ) + D_N |x|^N,
\]

(18.5)

\[
\| \hat{B}(x, w; Y, Z) \| \leq \overline{C} (\| Y \| + \| Z \| ) + D_N |x|^N,
\]

for (18.3). Moreover, the functions satisfy the Lipschitz conditions

\[
\| \hat{A}(x, w; Y_1, Z_1) - \hat{A}(x, w; Y_2, Z_2) \| \leq C (\| Y_1 - Y_2 \| + \| Z_1 - Z_2 \| ),
\]

(18.6)

\[
\| \hat{B}(x, w; Y_1, Z_1) - \hat{B}(x, w; Y_2, Z_2) \| \leq \overline{C} (\| Y_1 - Y_2 \| + \| Z_1 - Z_2 \| ),
\]

in (18.3). Here \( C \) and \( \overline{C} \) are independent of \( N \) but \( D_N \) depends on \( N \). Since \( E \) is a nilpotent matrix, we can assume without loss of generality that

\[
8C < \| \mu \| \sin 2\sigma \varepsilon.
\]

(18.7)

Also take \( N \) large enough so that

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(18.8) \[ 4\mathcal{C} < N. \]

Now let

(18.9) \[ Y = \lim_{\mathcal{m}}(e^{\Omega(x)}) n_N, \quad Z = \zeta_N. \]

The equations (18.2) become

\[
\begin{align*}
\eta_N' &= x^{-\sigma-1} \lim_{\mathcal{m}}(e^{-\Omega(x)}) \hat{A}(x, W(x)) \lim_{\mathcal{m}}(e^{\Omega(x)}) n_N, \zeta_N), \\
\zeta_N' &= x^{-1} \hat{B}(x, W(x)) \lim_{\mathcal{m}}(e^{\Omega(x)}) n_N, \zeta_N).
\end{align*}
\]

Then the proof of Theorem B reduces to proving:

**Proposition 1.** Given that (18.7) and (18.8) are true, there exists a unique solution \( \{\varphi_N(x, W(x)), \psi_N(x, W(x))\} \) of (18.10)\(_N\) such that for suitably chosen positive constants \( a_N', b_N' \) and \( k_N \) the functions \( \varphi_N(x, w) \) and \( \psi_N(x, w) \) are holomorphic and bounded functions of \( (x, w) \) in

\[
0 < |x| < a_N', \quad 0 < \arg x < \bar{\varnothing}, \quad 0 < |w| < b_N', \quad 0 < \arg w < \bar{\Gamma}.
\]

The solution also satisfies the inequalities
\[ [\varphi_N(x, W(x))] \leq K_N |x|^N [\exp(-\Re \Omega(x))] \]

(18.12)

\[ ||\psi_N(x, W(x))|| < K_N |x|^N, \]

for \((x, w)\) in (18.11). Moreover, a solution of \((18.10)^N\) satisfying

(18.13) \[ [\eta_N] = O(|x|^N) [\exp(-\Re \Omega(x))] \], \[ ||z_N|| = O(|x|^N), \]

is unique.

Theorem B is a direct consequence of Proposition 1. Indeed due to the transformations (18.1) and (18.9), the functions

\[ \sum_{\ell=0}^{N-1} x^\ell P_{\ell}(W(x)) + l_m \left( e^{\Omega(x)} \right) \varphi_N(x, W(x)) \]

(18.14)

\[ \sum_{\ell=0}^{N-1} x^\ell Q_\ell(W(x)) + \psi_N(x, W(x)) \]

are solutions of equations (SS) provided \((X, W(x))\) is in the domain \((18.11)^N\). Now let \(N'\) be an integer greater than \(N\). Then,
\[ l_m \left( e^{-\Omega(x)} \right)^{N'-1} \sum_{\ell=N}^{N'-1} x^\ell P_\ell(w(x)) + \varphi_N(x, W(x)), \]

is also a solution of \((18.10)_N\) which satisfies \((18.13)_N\) provided \((x, W(x))\) belong to the common part of the domains \((18.11)_N\) and \((18.11)_{N'}\). Hence, the solution \((18.15)\) must coincide with \(\{ \varphi_N(x, W(x)), \psi_N(x, W(x)) \}\). Therefore, a solution of \((SS)\) expressed by \((18.14)\) must be unique. Denote this solution by \(\{ \phi(x, W(x)), \psi(x, W(x)) \}\). Then by analytic continuation the functions \(\phi(x, w)\) and \(\psi(x, w)\) are defined in a domain of the form \((4.1)\) with \(a'_o = \sup a'_N,\)

\[ b'_o = \sup b'_N. \]

19. A Stable Domain

Due to the presence of the exponential, rather than proving the existence of a solution of \((18.10)_N\) in a domain of the form \((18.11)_N\), it is more convenient to use a domain of the form
0 < |x| < a_N^" \omega(\arg x), \quad 0 < \arg x < \theta

(19.1)_N

0 < |w| < b_N^" \chi(\arg x), \quad \Gamma < \arg w < \Gamma^*

where \( \omega(\tau) \) and \( \chi(\tau) \) are strictly positive and bounded functions of \( \tau \) for \( \theta < \tau < \theta^* \). Hence, instead of proving Proposition 1, we will prove:

**Proposition 2.** There exist functions \( \omega(\arg x) \) and \( \chi(\arg x) \) and positive constants \( a_N^", b_N^" \) and \( K_N \) such that \( (18.10)_N \) has a unique solution

\[
\{ \varphi_N(x, W(x)), \psi_N(x, W(x)) \}
\]

with the properties:

i) \( \varphi_N(x, w) \) and \( \psi_N(x, w) \) are holomorphic and bounded vector functions in \( (19.1)_N \).

ii) \( \varphi_N(x, W(x)) \) and \( \psi_N(x, W(x)) \) satisfy the inequalities

\[
[\varphi_N(x, W(x))] \leq K_N |x|^N \exp(-\Re \Omega(x)),
\]

(19.2)

\[
\| \psi_N(x, W(x)) \| \leq K_N |x|^N,
\]

for \((x, w)\) in \((19.1)_N\).

Since both \( \omega(\tau) \) and \( \chi(\tau) \) are positive and bounded, the domains \((18.11)_N\) and \((19.1)_N\) are equivalent in the sense that any point in \((18.11)_N\) is contained in \((19.1)_N\) for suitable choice of \( a_N^", b_N^" \) and vice versa.
Hence Proposition 1 is solved if we can prove Proposition 2.

20. The Functions $\omega(\tau)$ and $\chi(\tau)$

In order to determine $\omega(\tau)$ and $\chi(\tau)$ it is first necessary to define a function $A(\tau)$.

The directions $\arg x = \theta_j$ in the sector

$$\theta < \arg x < \overline{\theta}$$

such that $\Re \Omega_j(x) = 0$ for $\arg x = \theta_j$ are called singular directions of $\Omega_j(x)$. For indices $j$ such that $\Re \Omega_j(x)$ changes sign in (20.1), choose $\arg \mu_j$ so that at least one of the two singular directions

$$\theta_j^+ = \frac{1}{\sigma} (\arg \mu_j + \frac{\pi}{2})$$

$$\theta_j^- = \frac{1}{\sigma} (\arg \mu_j + \frac{3\pi}{2})$$

is contained in (20.1). By the assumption that (20.1) has Property-$\mathcal{J}$ with respect to $\{\Omega_1(x), \ldots, \Omega_m(x)\}$, we can classify the indices $j$ of the set $J = \{1, \ldots, m\}$ into four classes:
\[ J_0 = \{ j \mid \text{Re } \Omega_j(x) < 0 \text{ for } 0 < \arg x < \bar{\sigma} \}, \]
\[ J_1 = \{ j \mid \bar{\theta} < \theta_j^+ < \theta_j^- < \bar{\sigma} \}, \]
\[ J_2 = \{ j \mid \bar{\theta} < \theta_j^+ < \bar{\sigma} < \theta_j^- \}, \]
\[ J_3 = \{ j \mid \theta_j^+ < \bar{\theta} < \theta_j^- < \bar{\sigma} \}. \]

Some of these sets may be empty. In one case, when \( \bar{\sigma} - \bar{\theta} < \frac{\pi}{\sigma} \), \( J_1 \) is empty and when \( \bar{\sigma} - \bar{\theta} > \frac{\pi}{\sigma} \), \( J_0 \) is empty. Therefore \( \{1, \ldots, m\} = J_1 \cup J_2 \cup J_3 \) or \( \{1, \ldots, m\} = J_0 \cup J_2 \cup J_3 \).

Since the sector (20.1) has Property-\( \mathcal{J} \) with respect to \( \{ \Omega_1(x), \ldots, \Omega_m(x) \} \), the angles \( \theta \) and \( \bar{\theta} \) must satisfy, for sufficiently small \( \varepsilon \), the inequality

\[(20.2) \max_{j=1}^m \theta_j^+ - (\frac{\pi}{\sigma} + 6\varepsilon) \leq \theta \leq \min_{j=1}^m \theta_j^- + (\frac{\pi}{\sigma} - 6\varepsilon) \]

for all \( j \in J_1 \cup J_2 \cup J_3 \) or \( j \in J_2 \cup J_3 \). Put

\[(20.3) \theta_{k^+} = \max_{j \in J_k} \theta_j^+, \quad \theta_{k^-} = \min_{j \in J_k} \theta_j^- \]

where \( k = 1, 2, 3 \) or \( k = 2, 3 \). Then \( A(\tau) \) is defined by
The function $A(t)$ satisfies

$$2\sigma \varepsilon \leq A(t) \leq \pi - 2\sigma \varepsilon \quad \text{for} \quad \theta < t < \bar{\sigma}.$$

Then the functions $\omega(t)$ and $\chi(t)$ are defined by

$$\omega(t) = \exp \int_{\theta_0}^{t} \cot A(t) \, dt,$$

$$\chi(t) = \exp \left\{ \left| \frac{q\beta}{q\beta^*} \right| \left( \int_{\theta_0}^{t} \cot A(t) \, dt + \left| t - \theta_0 \right| \right) \right\},$$

where $\theta_0 = \theta$ if $\theta < t \leq \frac{1}{2}(\theta_3^- + \theta_2^+)$ and $\theta_0 = \bar{\sigma}$ if $\frac{1}{2}(\theta_3^- + \theta_2^+) < t \leq \bar{\sigma}$. Clearly $\omega(t)$ and $\chi(t)$ are positive and bounded.

21. The Paths of Integration

In order to prove Proposition 2, it is necessary to define special paths of integration and estimate the behavior of $W(x)$ on these paths. The needed results are stated in the following lemma.
Lemma 4. Let \((x_1, w_1)\) be an arbitrary point in a domain of the form

\[
\begin{align*}
0 < |x| < a_N^n \omega(\arg x), & \quad 0 < \arg x < \Theta, \\
0 < |w| < b_N^n \chi(\arg x), & \quad \Gamma < \arg w < \Pi.
\end{align*}
\]

Then there exists an \(m\)-vector path \(T_{x_1}\) with elements \(\{T_{jx_1}\}\) such that:

i) Each curve \(T_{jx_1}\) joins the point \(x_1\) with the origin and is contained in (21.1) except for the origin.

ii) If \(a_N^n\) is chosen sufficiently small, then as \(x\) moves on the curve \(T_{jx_1}\) we have

\[
0 < |W(x)| < 2b_N^n \chi(\arg x), \quad \Gamma < \arg(W(x)) < \Pi.
\]

Here \(a_N^n\) and \(b_N^n\) are chosen so that \(a_N^n \omega(\arg x) < a_N\) and \(2b_N^n \chi(\arg x) < b_N\) for \(0 < \arg x < \Theta\).

To prove Lemma 4 we first define an \(m\)-vector function \(a(\tau)\) with elements \(\{a_j(\tau)\}\) as follows:

If \(j \in J_0\),

\[
a_j(\tau) = \frac{\pi}{2}, \quad 0 < \tau < \Theta.
\]
If \( j \in J_1 \),

\[
(21.5) \quad a_j(\tau) = \begin{cases} 
\sigma(\tau - \theta_j - 2\varepsilon), & \theta_j + \frac{\pi}{2\sigma} - 2\varepsilon \leq \tau \leq \overline{\theta}, \\
\frac{\pi}{2} - \theta_j - \frac{\pi}{2\sigma} + 2\varepsilon < \tau < \theta_j + \frac{\pi}{2\sigma} - 2\varepsilon, \\
\sigma(\theta - \theta_j - 2\varepsilon) + \pi, & \theta \leq \tau \leq \theta_j + \frac{\pi}{2\sigma} + 2\varepsilon.
\end{cases}
\]

If \( j \in J_2 \),

\[
(21.6) \quad a_j(\tau) = \begin{cases} 
\frac{\pi}{2} - \theta_j + \frac{\pi}{2\sigma} + 2\varepsilon \leq \tau \leq \overline{\theta}, \\
\sigma(\tau - \theta_j - 2\varepsilon) + \pi, & \theta \leq \tau \leq \theta_j + \frac{\pi}{2\sigma} + 2\varepsilon.
\end{cases}
\]

If \( j \in J_3 \),

\[
(21.7) \quad a_j(\tau) = \begin{cases} 
\sigma(\tau - \theta_j - 2\varepsilon), & \theta_j + \frac{\pi}{2\sigma} - 2\varepsilon \leq \tau \leq \overline{\theta}, \\
\frac{\pi}{2}, & \theta \leq \tau \leq \theta_j + \frac{\pi}{2\sigma} - 2\varepsilon.
\end{cases}
\]

Hence, the functions \( a_j(\tau) \) satisfy

\[
(21.8) \quad 2\sigma \varepsilon \leq a_j(\tau) \leq \pi - 2\sigma \varepsilon \text{ for } \theta \leq \tau \leq \overline{\theta},
\]

and

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\[ a_j(t) > A(t), \quad \theta_j - 2\varepsilon \leq \tau \leq \theta, \quad (j \in J_1, J_3), \]

\[ a_j(t) \leq A(t), \quad 0 \leq \tau \leq \theta_j + 2\varepsilon, \quad (j \in J_1, J_2). \]

Therefore, we have

\[ \int_\theta^\tau \cot a_j(t) \, dt \leq \int_\theta^\tau \cot A(t) \, dt, \]

for \( 0 \leq \tau \leq \theta_j + 2\varepsilon \) and \( (j \in J_1, J_2) \) and for

\[ \theta_j - 2\varepsilon \leq \tau \leq \theta \) \) and \( (j \in J_1, J_3) \).

Let \((r, \theta)\) and \((\rho, \tau)\) be the polar coordinates for the point \(x_1\) and a variable point \(x\) on the curve \(T_{jx_1}\) respectively. Then the curve \(T_{jx_1}\) is defined as follows:

i) If \(0 < \theta_j + 2\varepsilon\) or \(\theta_j - 2\varepsilon < \theta\), the curve \(T_{jx_1}\) consists of a curvilinear part \(T_j':\)

\[ \rho = r \exp \int_\theta^\tau \cot a_j(t) \, dt, \]

\[ \theta \leq \tau \leq \theta_j + 2\varepsilon \quad \text{or} \quad \theta_j - 2\varepsilon \leq \tau \leq \theta, \]

and a rectilinear part \(T_j'':\)

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\[ 0 \leq \tau \leq r \exp \int_\theta^\tau \cot a_j(t) \, dt, \]

(21.12)

\[ \tau = \theta_j^+ + 2\varepsilon \quad \text{or} \quad \tau = \theta_j^- - 2\varepsilon. \]

ii) If \( \theta_j^+ + 2\varepsilon \leq \tau \leq \theta_j^- - 2\varepsilon \), the curve \( T_{jx_1} \)

consists only of a rectilinear part \( T_j'' \),

(21.13) \[ 0 \leq \tau \leq r, \quad \tau = \theta. \]

By virtue of (21.10), the curve \( T_{jx_1} \) is contained in the domain (21.1) except for the origin. This completes the proof of assertion (i) in Lemma 4.

22. The Behavior of \( W(x) \)

To prove assertion (ii) in Lemma 4 let \( (x_1, w_1) \) be an element of the domain (21.1), (21.2) and recall

(22.1) \[ W(x) = \frac{q^*_0}{q^*} \left\{ G(\bar{c} - \frac{(q^*_0)^2}{q^*} \log x) \right\}^{-1}, \]

where \( \bar{c} \) is chosen so that \( W(x_1) = w_1 \). Since \( G(X) \) satisfies the condition that \( G(X) - X - \log X \) vanishes at \( X = \infty \), there exists a positive constant \( L \) such that

(22.2) \[ \frac{1}{\sqrt{2}} |x| \leq |G(X)| \leq \sqrt{2} |x| \quad \text{for} \quad |x| \geq L. \]

Hence

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Here

\[ (22.4) \quad x = c \cdot \frac{(q^\beta)^2}{q^\beta} \cdot \log x. \]

Now let

\[ (22.5) \quad x_1 = c - \frac{(q^\beta)^2}{q^\beta} \cdot \log x_1, \]

so that

\[ (22.6) \quad x_1 = x + \frac{(q^\beta)^2}{q^\beta} \cdot \log \frac{x}{x_1}. \]

Hence,

\[ (22.7) \quad \left| \frac{W(x)}{W(x_1)} \right| \leq 2 \left| \frac{x_1}{x} \right| \leq 2 \left( 1 + \frac{1}{|x|} \right) \left| \frac{(q^\beta)^2}{q^\beta} \right| \left| \log \frac{x}{x_1} \right|. \]

Now pick a_N small enough so that \[|x| \geq 1 \] and \[|x| < L\]
for \[x\] in (21.1). Then we have

\[ (22.8) \quad \left| \frac{W(x)}{W(x_1)} \right| \leq 2 \left( 1 + \left| \frac{(q^\beta)^2}{q^\beta} \right| \left| \log \frac{x}{x_1} \right| \right). \]

But on the curvilinear part \(T_j\) of \(T_{jx_1}\) we have

\[ (22.9) \quad \log \frac{x}{x_1} = \int_{\theta}^{\tau} \cot a_j(t) dt + i(\tau - \theta), \]

and therefore, noting that the integral is positive,
(22.10) \[ |W(x)| \leq 2|w_1| \exp\left( \left| \frac{(q\beta)^2}{q\beta^*} \right| \left( \int_0^\tau \cot a_j(t) |t + \tau - \theta| \right) \right). \]

But since \( w_1 \) is in (21.2),

(22.11) \[ |W(x)| \leq 2 b_{N"}^" \chi(\arg x). \]

For the rectilinear part \( T_j" \), notice that \( X \) moves along a straight line parallel to \( \arg X = \arg X_1 \) and tends to \( \infty \) as \( x \) tends to \( 0 \). Thus \( |X| \leq |X_1| \) and

(22.12) \[ |W(x)| \leq 2|w_1| \leq 2b_{N"}^" \chi(\arg x). \]

From (22.10) and the fact that \( G(X) - X - \log X \to 0 \) as \( X \to \infty \) we have

\[ W(x) \sim \frac{-1}{q\beta \log x} \]

as \( x \to 0 \). Thus, for a positive constant \( \varepsilon \), if \( a_n" \) is sufficiently small,

\[ -\arg q\beta + \pi - \varepsilon < \arg W(x) < -\arg q\beta + \pi + \varepsilon, \]

for all \( x \) on \( T_{X_1} \). Thus,

(22.13) \[ \Gamma < \arg W(x) < \bar{\Gamma}, \]

for all \( x \) on \( T_{X_1} \).

This completes the proof of Lemma 4.
23. An Integral Inequality

The proof of Proposition 2 also requires an integral inequality that can be stated as follows:

**Lemma 5.** Let $x_1$ be an arbitrary point in the domain (21.1) and $s_j$ be the arc length of the curve $T_j x_1$ measured from the origin to the point $x$ on this curve. Then,

\[
(23.1) \quad \frac{d}{ds_j} e^{-\text{Re} \, \Omega_j(x)} \geq |x|^{-1} e^{-\text{Re} \, \Omega_j(x)} \| \mu \| \sin 2\sigma \epsilon
\]

and

\[
(23.2) \quad |x|^{-1} \frac{d|x|}{ds_j} \geq - |x|^{-1}
\]

hold as $x$ moves on $T_{xj_1}$. Moreover, if in addition to the conditions imposed by Lemma 4, $a_N''$ satisfies

\[
(23.3) \quad 2N(a_N'' \max \omega(\tau))^C \leq \| \mu \| \sin 2\sigma \epsilon,
\]

then

\[
(23.4) \quad \int_{x_1}^{x} |x|^{-\sigma - 1} e^{-\text{Re} \, \Omega_j(x)} |dx| \leq \frac{2}{\| \mu \| \sin 2\sigma \epsilon} |x_1|^N e^{-\text{Re} \, \Omega_j(x_1)}
\]
Here, the integration is carried along $T_{x_1}$. The maximum in (23.3) is taken over $\theta \leq \tau \leq \theta^\circ$.

Indeed, if (23.1) and (23.2) are true, then

$$\frac{d}{ds_j} \left( |x|^N e^{-\text{Re} \Omega_j(x)} \right) \geq \frac{1}{|x|^{N-\sigma-1}} e^{-\text{Re} \Omega_j(x)} (||\mu||' \sin 2\varepsilon - N|x|^\sigma).$$

Thus, if $a_N''$ satisfies (23.2), then

$$\frac{d}{ds_j} \left( |x|^N e^{-\text{Re} \Omega_j(x)} \right) \geq \frac{||\mu||' \sin 2\varepsilon}{2} \left( |x|^{N-\sigma-1} e^{-\text{Re} \Omega_j(x)} \right).$$

for $x$ on $T_{x_1}$ and (23.4) is true.

The proof of (23.1) and (23.2) can be found in Iwano [11]. In particular, let $x = \rho e^{i\tau}$. Then on $T_{x'}$, (23.7)

$$\frac{dx}{ds_j} = \exp \{ (a_j(\tau) + \tau) i \},$$

where the sign is - when $\theta \leq \tau \leq \theta_{j+} + 2\varepsilon$ and + when $\theta_{j-} - 2\varepsilon \leq \tau \leq \theta$. Hence, we have the equality
where the sign is + when \( \theta \leq \tau \leq \theta_j^+ - 2\epsilon \) and - when \( \theta_j^- - 2\epsilon \leq \tau \leq \theta \).

But

\[
a_j(\tau) - \sigma \tau + \arg \mu_j = \begin{cases} 
- \frac{3\pi}{2} + 2\sigma \epsilon, & \theta_j^- + \frac{\pi}{2\sigma} - 2\epsilon \leq \tau \leq \theta \\
\frac{\pi}{2} - 2\sigma \epsilon, & \theta \leq \tau \leq \theta_j^- - \frac{\pi}{2\sigma} + 2\epsilon
\end{cases}
\]

Also,

\[-\frac{3\pi}{2} + 2\sigma \epsilon < a_j(\tau) - \sigma \tau + \arg \mu_j < -\pi + 2\sigma \epsilon\]

for \( \theta_j^- - 2\epsilon < \tau \leq \theta_j^- + \frac{\pi}{2\sigma} - 2\epsilon \), and

\[-2\sigma \epsilon < a_j(\tau) - \sigma \tau + \arg \mu_j < \frac{\pi}{2} - 2\sigma \epsilon\]

for \( \theta_j^+ - \frac{\pi}{2\sigma} + 2\epsilon \leq \tau \leq \theta_j^+ + 2\epsilon \). Hence,

\[(23.9) \quad \cos(a_j(\tau) - \sigma \tau + \arg \mu_j) \leq -\sin 2\sigma \epsilon\]

for \( \theta_j^- - 2\epsilon < \tau \leq \theta \), and

\[(23.10) \quad \cos(a_j(\tau) - \sigma \tau + \arg \mu_j) \geq \sin 2\sigma \epsilon\]

for \( \theta \leq \tau \leq \theta_j^+ + 2\epsilon \). This proves inequality (23.1) on \( T_j' \).
On the rectilinear part $T_j''$, $x = \rho e^{i\theta}$ and $s_j = \rho$.

Thus

\[
\frac{d}{ds_j} e^{-\Re \Omega_j(x)} = -e^{-\Re \Omega_j(x)} \frac{d}{dp} \Re \Omega_j(x)
\]

(23.11)

\[
= -e^{-\Re \Omega_j(x)} \rho^{-\sigma-1} |u_j| \cos(\arg u_j - \sigma \theta)
\]

\[
\geq e^{-\Re \Omega_j(x)} \rho^{-\sigma-1} |u_j| \sin 2\sigma \varepsilon,
\]

because $\theta_{j+} + 2\varepsilon < \theta < \theta_{j-} - 2\varepsilon$. Therefore, (23.1) is true as $x$ moves on $T_j x_1$. To prove (23.2), notice $s_j$ is real, so that

\[
|x|^{-1} \frac{d|x|}{ds_j} = \frac{d}{ds_j} \log |x| = \frac{d}{ds} (\Re \log x)
\]

(23.12)

\[
= \Re \left( \frac{d}{ds_j} \log x \right) = \Re \left( x^{-1} \frac{dx}{ds_j} \right)
\]

\[
\geq - |x|^{-1}.
\]

This inequality is a consequence of (23.7) when $x$ is on $T_j'$, and of the fact that $|x| = s_j$ on $T_j''$.

Thus, the proof of Lemma 5 is complete.
24. Successive Approximations

Given Lemmas 4 and 5, the proof of Proposition 2 can be realized by use of successive approximations. In particular, corresponding to equations \((18.10)_N\), consider the system of integral equations

\[
\begin{align*}
\phi(x_1, w) &= \int_0^{x_1} x^{-\sigma-1} l_m \left( e^{-\Omega(x)} \right) \, dx, \\
\psi(x_1, w) &= \int_0^{x_1} e^{-\Omega(x)} \, dx.
\end{align*}
\]

(24.1)

Here \((x_1, w)\) is an arbitrary point in \((19.1)_N\) and \(W(x_1) = w\). The path of integration in the first equation is \(T_{jx_1}\), while the path in the second is \(\overline{jx_1}\), the line segment between the origin and \(x_1\).

The successive approximations for (24.1) are defined to be the sequence of functions \(\{\phi^{(m)}(x_1, w), \psi^{(m)}(x_1, w)\}\) given recursively by
(24.2) \[ \phi^{(0)}(x_1, w_1) = 0, \quad \psi^{(0)}(x_1, w_1) = 0, \]

\[ \phi^{(m+1)}(x_1, w_1) \]
\[ = \int_0^{x_1} x^{-\sigma - 1} \left( e^{-\Omega(x)} \right) \hat{A}(x; W(x); 1) \left( e^{\Omega(x)} \right) \phi^{(m)}(x, W(x)), \psi^{(m)}(x, W(x)) \, dx, \]

(24.3)

\[ \psi^{(m+1)}(x_1, w_1) \]
\[ = \int_0^{x_1} x^{-1} \left( e^{\Omega(x)} \right) \phi^{(m)}(x, W(x)), \psi^{(m)}(x, W(x)) \, dx, \]

(m = 0, 1, 2, ...).

Here the paths of integration are the same as in (24.1).

By use of Lemma 4 to ensure the analyticity of the integrands and Lemma 5 to ensure the convergence of the integrals, the successive approximations can be shown to converge to a solution of (24.1). The proof of the uniqueness of the solution is also aided by Lemma 5. This method is similar to Hsieh [2].
Thus, the proof of Proposition 2 and Theorem B is complete.
CHAPTER VII

EXAMPLES OF THEOREM A

The functions which form the coefficients of the formal solution in Theorem A are determined by the implicit function theorem and other existence theorems and hence cannot usually be given in simple, concrete form. However, in some cases it is possible to determine the coefficients $\varphi_0(v)$ and $\psi_0(v)$ and the functions $V(x)$ and $W(x)$. This will give some insight into the nature of the formal solution.

For the two examples given here we will assume $\sigma = 2$ and $y$ and $z$ are scalars. In this case, the formal solution will have the form

$$y \sim \varphi_0(v) + x\varphi_1(x,v) + x^2\varphi_2(x,v) + x^3\phi^0(x,w,v),$$

$$z \sim \psi_0(v) + x\psi_1(w,v) + x^2\psi_2(w,v) + x^3\psi^0(x,w,v),$$

where $\phi^0(x,w,v)$ and $\psi^0(x,w,v)$ are power series in $v$ with coefficient functions in $x$ and $w$. All quantities in (25.1) are one dimensional.

First consider the system

$$x^3 y' = (\lambda + x^2)y + z^2,$$

$$xz' = z^3 + xy,$$

(25.2)
where \( \lambda \neq 0 \). Then the equations that determine \( \psi_0(v) \) and \( \psi_0(v) \) are
\[
\lambda \psi_0 + \psi_0^2 = 0,
\]
(25.3)
\[
x \psi_0' = \psi_0^3.
\]
Since the reduced equation in this case is \( xv' = v^3 \), we can take \( \psi_0(v) = v \) and \( \varphi_0(v) = \frac{-v^2}{\lambda} \). The functions \( \varphi_1(w, v) \) and \( \varphi_2(w, v) \) are simple combinations of known functions but the construction of \( \psi_1(w, v) \) and \( \psi_2(w, v) \) will require the use of Existence Theorems III and IV. In particular the equation that determines \( \psi_1(w, v) \) is
\[
x \psi_1' = -\psi_1 + v^2 \psi_1 - \frac{v^2}{\lambda}.
\]
(25.4)
To solve this equation notice \( q(v) = 1, \beta = 1 \) and \( q = 2 \). Then the equations that determine the parametric representation of \( \tilde{V}(x) \) are
\[
w^2 \frac{dx}{dw} = \frac{x}{2}, \quad w \frac{dv}{dw} = \frac{v}{2}.
\]
So equation (25.4) is equivalent to the system
\[
\begin{align*}
  w^2 \frac{d\psi_1}{dw} &= \frac{1}{2}( -\psi_1 + v^2 \psi_1 - \frac{v^2}{\lambda} ) , \\
  w \frac{dv}{dw} &= \frac{v}{2} ,
\end{align*}
\]
(25.5)
which is solved by Existence Theorem IV. Note that in this case \( V(x) \) does not depend on Hukuhara's \( G(x) \) function. In fact, \( V(w) = w^{\frac{3}{2}} C \), and

\[
W(x) = \frac{1}{\bar{c} - 2 \log x}.
\]

Hence,

\[
V(x) = \left( \frac{1}{\bar{c} - 2 \log x} \right)^{\frac{1}{2}} C.
\]

Here \( \bar{c} \) and \( C \) are arbitrary constants.

Now consider the system

\[
x^3 y' = (\lambda + x^2)y + z^2,
\]

(25.7)

\[
xz' = z^2(1 + 2z) + xy,
\]

where \( \lambda \neq 0 \). Again \( \psi_0(v) = v \) and \( \omega_0(v) = -\frac{v^2}{\lambda} \) but the reduced equation in this case is \( xv' = v^2(1 + 2v) \).

so \( g(v) = 1 + 2v \), \( \beta = 1 \) and \( q = 1 \). Hence the parametric form of \( \tilde{V}(x) \) is now determined by

\[
(25.8) \quad w^2 \frac{dx}{dw} = \frac{x}{1 + 2v} \quad , \quad w \frac{dv}{dw} = v.
\]

So \( V(w) = wC \) but both \( W(x) \) and \( \tilde{V}(x) \) will depend on the \( G(x) \) function. In fact,
\[ W(x) = \frac{1}{2C} \left\{ G\left( \tilde{c} - \frac{\log x}{2C} \right) \right\}^{-1}, \]

and

\[ V(x) = \frac{1}{2} \left\{ G\left( \tilde{c} - \frac{\log x}{2C} \right) \right\}^{-1}. \]

Here \( C \) and \( \tilde{c} \) are arbitrary constants.


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12. Iwano, M., "A general solution of a system of non-linear ordinary differential equations \( xy' = f(x, y) \) in the case when \( f_y(0, 0) \) is the zero matrix." Annali di Matematica pura ed applicata (IV), LXXXIII (1969), 1-42.

