Partitions of the Vertex Set or Edge Set of a Graph

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PARTITIONS
OF THE VERTEX SET OR EDGE SET
OF A GRAPH

by
H. Joseph Straight

A Dissertation
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of the
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H. Joseph Straight
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For Wendy,
without whom there is nothing,
and for our parents.

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CHAPTER 1
DEFINITIONS, NOTATION, AND INTRODUCTION

In this chapter we give the basic definitions in graph theory that we will require throughout this discussion. With a few exceptions, our terminology follows that of Behzad and Chartrand [8] or White [56]. We also provide a background for the problems we shall encounter in Chapters 2 - 4 by discussing and presenting known results for several related problems.

A graph G consists of a pair of finite sets, V(G) and E(G), called the vertex set and edge set, respectively. Elements of the vertex set are called vertices, and the edge set is made up of two-element subsets of the vertex set; these are called edges. The cardinality of V(G) is called the order of G and the number of edges is the size of G. Two vertices of G are adjacent if they constitute an edge; in this case the vertices are incident with the edge, and conversely. For convenience of notation, the edge \( \{v_1, v_2\} \) will be denoted by \( v_1v_2 \) (or \( v_2v_1 \)). A pleasing aspect of graph theory is the pictorial nature of the subject. Given a graph G, we may draw a "picture" of G by associating with each vertex a point in the plane, and then joining two vertices with a simple curve, or line, if they are
adjacent. In Figure 1.1 we show the graph $G$ having $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $E(G) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_1, v_1v_5, v_2v_4, v_3v_6\}$. For our purposes, this "picture" is the graph $G$.

Given a graph $G$ with $V(G) = \{v_1, v_2, \ldots, v_n\}$, the adjacency matrix of $G$, denoted by $A(G)$, is the $n \times n$, symmetric, $(0,1)$ matrix $(a_{ij})$, where $a_{ij} = 1$ if $v_i v_j \in E(G)$ and $a_{ij} = 0$, otherwise. Since $A(G)$ is symmetric and each main diagonal entry is 0, all information describing the graph is contained in those entries which lie above the main diagonal. We shall sometimes focus our attention on this array of entries only, rather than on the whole of $A(G)$.

Two graphs, $G_1$ and $G_2$, are isomorphic if there exists a one-to-one mapping $\phi$ from $V(G_1)$ onto $V(G_2)$ such that $\phi$ preserves adjacency and nonadjacency, i.e., $uv \in E(G_1)$ if and only if $\phi(u)\phi(v) \in E(G_2)$. If $G_1$
and $G_2$ are isomorphic we write $G_1 = G_2$, and often say "$G_1$ equals $G_2" instead of "G_1 is isomorphic to G_2." The important point is that two isomorphic graphs are structurally the same, even though they may look or be labelled differently.

$H$ is said to be a subgraph of $G$, denoted by $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $U \subseteq V(G)$, the subgraph induced by $U$, denoted by $\langle U \rangle$, has vertex set $U$ and contains all edges of $G$ which join two vertices of $U$. Similarly, for $F \subseteq E(G)$, the subgraph induced by $F$ contains the edges of $F$ and all vertices incident with an edge of $F$. This will be denoted by $\langle F \rangle$. Also, for a proper subset $U$ of $V(G)$, $G - U$ denotes the graph obtained by deleting from $G$ the vertices of $U$ (and all edges thereby affected). For $F \subseteq E(G)$, $G - F$ is the graph with $V(G - F) = V(G)$ and $E(G - F) = E(G) - F$.

The complement, $\overline{G}$, of a graph $G$ has the same vertex set as that of $G$, and two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in $G$. For $H \subseteq G$, the relative complement of $H$ in $G$ has $V(G)$ as its vertex set and contains all edges of $G$ which are not in $H$. The union of two graphs $G_1$ and $G_2$ with $V(G_1) \cap V(G_2) = \emptyset$, denoted by $G_1 \cup G_2$, is defined by $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. 

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Also if $G_i, 1 \leq i \leq n$, are $n$ pairwise disjoint graphs, each isomorphic to $G$, then $nG$ denotes the graph $G_1 \cup G_2 \cup \ldots \cup G_n$.

Often a great deal of information concerning a graph can be discovered if we examine the degree of each of its vertices. If $v \in V(G)$, the degree of $v$, denoted by $\deg(v)$, is the number of vertices of $G$ to which $v$ is adjacent. We let $\Delta(G)$ and $\delta(G)$ denote, respectively, the maximum and minimum degrees among the vertices of $G$. A graph $G$ is said to be regular of degree $r$ or $r$-regular if each vertex of $G$ has degree $r$.

A path in a graph $G$ is a sequence of distinct vertices $v_0, v_1, \ldots, v_n$ such that $v_i, v_{i+1} \in E(G)$, for $0 \leq i \leq n - 1$. The length of this path is $n$, i.e., the length of a path is the number of edges it contains, or, one less than the number of vertices on the path. A cycle in a graph $G$ is simply a closed path, i.e., all vertices are distinct except the first and last. The length of a cycle is the number of vertices or edges on it. For example, in the graph of Figure 1.1, $v_1 v_5 v_4 v_2$ is a path of length 3 and $v_1 v_6 v_3 v_2 v_1$ is a cycle of length 4. A graph is said to be connected if for any two vertices $u$ and $v$ there is a path from $u$ to $v$. The distance from $u$ to $v$ in a connected graph $G$ is the minimum length among all paths from $u$ to $v$. The diameter of $G$ is the maximum distance in $G$. 

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We next wish to define the class of graphs known as trees and introduce a notation with which we can describe any tree. A tree is defined as a connected graph which contains no cycles. Alternately, it is well-known that a graph $G$ of order $p$ and size $q$ is a tree if and only if $G$ is connected and $p = q + 1$. An end vertex of a tree $T$ is a vertex having degree one in $T$.

The basic trees from which we can build and define all others are the stars. The star of size $n$, $S_n$, is the unique tree $T_n$ of size $n$ having $\Delta(T_n) = n$. (See Figure 1.3a.) Thus $S_n$ is also the unique tree of size $n$ having $n$ end vertices. The root of $S_n$ is the vertex of degree $n$. (In the case of $S_1$ we arbitrarily choose one of the two vertices to be the root.) We now define $S_n(S_m)$ to be the tree obtained by identifying the root of $S_m$ with an end vertex of $S_n$; the root of $S_n$ becomes the root of $S_n(S_m)$. Note that it does not matter at which end vertex the identification is made; all graphs obtained by this procedure are isomorphic. As an example, we show the tree $S_4(S_3)$ in Figure 1.2a. More generally, suppose $T_1, T_2, \ldots, T_m$ are trees which have been defined and suppose each has a specified root. We then define $S_n(T_1, T_2, \ldots, T_m), n \geq m$, as the tree obtained by identifying the roots of $T_1, T_2, \ldots, T_m$ each with a different end vertex of $S_n$. For the purpose of further
identifications, the root of $S_n$ becomes the root of this new tree. Again $S_n(T_1, T_2, \ldots, T_m)$ is well-defined up to isomorphism. For example, in Figure 1.2b we show the tree $S_4(S_3(S_1, S_1), S_3(S_2(S_1)), S_2)$.

![Diagram](a. S_4(S_3) b. S_4(S_3(S_1, S_1), S_3(S_2(S_1)), S_2)

FIGURE 1.2

The path $P_n$ is the unique tree of size $n$ that has exactly two end vertices. It follows that the $n - 1$ non-end vertices of $P_n$ all have degree 2. (See Figure 1.3b.) We may also inductively define $P_n$ in terms of our tree notation. We let

$$P_1 = S_1$$

and

$$P_n = S_1(P_{n-1}) \quad \text{for} \quad n \geq 2.$$  

Another class of trees we shall encounter, $L_n$, may be defined as follows. First of all,
$L_1 = S_1$ and $L_2 = S_2$

Then, for $n \geq 3$, $L_n = S_2(L_{n-2})$.

These trees are shown in Figures 1.3 c-d.

In a connected graph $G$, the **eccentricity** of a vertex $u$ is the maximum distance from $u$ to the other vertices. The **center** of $G$ is the set of vertices having minimum eccentricity. It is known that the center of a tree consists either of a single vertex or a pair of adjacent vertices. Given a tree, there are many descriptions for it, depending on which vertex one chooses to start with. However, it is not difficult to argue that if one wishes the shortest notation, in the sense of minimizing the number of right parentheses, then one should start...
with a central vertex.

If an edge is added to a tree of size at least two, we obtain a unicyclic graph, i.e., a connected graph containing exactly one cycle. The most important unicyclic graphs are the cycles. The cycle of length \( n \geq 3 \), \( C_n \), is obtained when the end vertices of \( P_{n-1} \) are joined by an edge.

A set of vertices [resp. edges] in a graph \( G \) is said to be an independent set of vertices [resp. edges] if no two elements of the set are adjacent. The complete \( n \)-partite graph \( K(p_1, p_2, \ldots, p_n) \) is formed by taking \( n \) independent sets of vertices, \( V_1, V_2, \ldots, V_n \), called partite sets, of respective sizes \( p_1, p_2, \ldots, p_n \), and joining with an edge any two vertices which belong to different partite sets. If \( p_1 = p_2 = \ldots = p_n = m \), we denote this graph by \( K_n(m) \). Finally, the complete graph, \( K_n \), is the graph of order \( n \) in which every pair of vertices is joined by an edge. Thus the size of \( K_n \) is the binomial coefficient \( \binom{n}{2} = \frac{n(n-1)}{2} \).

Throughout our discussion we let \( [x] \) denote the greatest integer less than or equal to the real number \( x \).

Often it is desirable to draw a graph in such a way that there are no extraneous crossings of edges, that is, the only place where two edges may meet is at a common
incident vertex. To accomplish this, we sometimes draw graphs on surfaces other than the sphere or plane. A surface is a connected, closed 2-manifold. A surface is orientable if it admits a 2-cell decomposition with coherent orientation; otherwise, it is nonorientable. It is well-known that the orientable surfaces are the spheres with $n$ handles, $S_n$, and that the nonorientable surfaces are spheres with $n$ crosscaps, $\tilde{S}_n$, $n = 0, 1, \ldots$. (The sphere is $S_0$ or $\mathcal{S}_0$.) A graph $G$ is said to imbed on a surface $S$ if it is possible to properly draw $G$ on $S$, in the above sense. Such a drawing is called imbedding of $G$ in (or on) $S$.

The genus, $\gamma(G)$, [resp. nonorientable genus, $\bar{\gamma}(G)$,] of a graph $G$ is the minimum $n$ such that $G$ imbeds in $S_n$ [resp. $\tilde{S}_n$]. Graphs which have genus 0 are called planar, and such graphs have been characterised by the famous result of Kuratowski (see, for example, [8], 96-98) as those graphs which do not contain subgraphs homeomorphic with $K_5$ or $K(3, 3)$. Graphs which have genus 1 are called toroidal. In Figure 1.4 we show an imbedding of the toroidal graph $K_6$ on $S_1$. We use the usual representation for the torus as a rectangle with opposite edges identified as indicated.

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In Chapters 2 and 3 of this discussion we are basically concerned with decomposing the edge set of a graph into trees of different sizes. In Chapter 2 we look at the particular case where the trees are all paths, while Chapter 3 focuses on the case where the graph is a complete graph. Extensive research in graph theory has been done on various problems concerning decomposing the edge set of a graph into subgraphs satisfying certain properties.

A problem similar to the one we consider in Chapter 3 was posed by Ringel [45] at a graph theory conference in Smolenice, Czechoslovakia in 1963, and reposed by Duke [20] in 1969. They asked if it is possible to decompose
$K_{2n+1}$ into $2n + 1$ copies of an arbitrary tree having $n$ edges. Rosa [46] modified the conjecture by showing how the decomposition could be "cyclic". He showed that the modified conjecture was equivalent to proving that for any tree $T$ of order $n$, there exists a bijection $f$ from $V(T)$ onto $\{1, 2, \ldots, n\}$ such that for each $i$, $1 \leq i \leq n - 1$, there exists $uv \in E(T)$ with $|f(u) - f(v)| = i$. Trees possessing this property are called "graceful", a term introduced by Golumb. Rosa, Golumb [31], Kotzig [37], Cahit and Cahit [16, 17], Stanton and Zarnke [49], Haggard and McWha [34], Gardner [30], and others have all solved special cases of the problem, and it is now known that "almost all" trees satisfy Ringel's Conjecture.

Another question that has been considered is that of factoring $K_n$ into disjoint copies of the star $S_m$, where $m$ divides $\binom{n}{2}$. Cain-Hogarth [18] showed that $K_{rm}$ and $K_{rm+1}$, $r > 1$, can be decomposed into copies of $S_m$ if and only if $r$ is even or $m$ is odd. Later, Yamamoto and company [57], showed that it is possible to factor $K_n$ into copies of $S_m$ whenever $m$ divides $\binom{n}{2}$ and $n \geq 2m$.

A graph $F$ has star number $m$ if every set of $m$ vertices of $F$ belongs to a subgraph that is a star. In [48], Sauer and Schaer show that if $n \geq \binom{m^2 + 1}{m}$, then...
$K_n$ is decomposable into $k$ factors, each having star number $m$. This bound for $n$ is improved in [25].

Much attention has been given to the problem of factoring the complete graph into factors with given diameters. Let $f_m(d)$ be the minimum number of vertices required for a complete graph to contain $m$ factors of diameter $d$. Sauer [47], Bosák, Erdös, and Rosa [12], and Bosák [11] give bounds in the case where $d = 2$. The general problem is considered by Palumbíný [42, 43], Bosák, Rosa, and Znám [13] and Tomová [53]. Also one may consult Baranovicová [7], Březina [14], and Palumbíný and Znám [44]. The case of isomorphic factors of a given diameter is discussed by Kotzig and Rosa [38] and Tomosta [52]. They define $G_m(d)$ to be the smallest $n$ such that $K_n$ can be decomposed into $m$ isomorphic factors with diameter $d$ and $H_m(d)$ to be the smallest $n$ such that for all admissible $N \geq H_m(d)$, $K_N$ can be decomposed into $m$ isomorphic factors with diameter $d$. It is conjectured that $G_m(d) = H_m(d)$. This conjecture is true for $m$ equal to 1 or 2 and some partial results have been obtained for $m \geq 3$.

An $r$-factorization of a graph $G$ is a decomposition of $G$ into spanning $r$-regular subgraphs, called $r$-factors, and $G$ is said to be $r$-factorable. It is well-known that $K_{2n}$ is 1-factorable and that $K_{2n+1}$ is 2-factorable. In [50] Sugiyama showed in general that $K_n$ is $r$-factorable.
for any \( r \) that has the same parity as and divides \( n - 1 \).

Other problems that have been looked at include decomposing \( K_n \) into regular bipartite factors - Kotzig [36], into copies of a small connected subgraph - Erdős and Schonheim [26], or into copies of a spanning tree - Beineke [9].

In Chapter 4 we study a parameter called the cochromatic number, which was introduced by Lesniak-Foster and Straight in [40]. This parameter involves partitioning the vertex set of a graph \( G \) into subsets which are independent in \( G \) or which induce a complete subgraph of \( G \). The chromatic number of a graph \( G \), denoted by \( \chi(G) \), is the fewest number of colors needed to color the vertices of \( G \) so that no two vertices which are adjacent are assigned the same color. It may also be defined as the fewest number of subsets into which \( V(G) \) can be partitioned so that each subset is independent in \( G \). Hence the cochromatic number is an extension of the chromatic number.

Considerable research concerning \( \chi(G) \) has involved finding bounds for this parameter. A question that has generated much interest not only in graph theory but throughout mathematics is the Heawood Map Coloring Problem. This question asks, "What is the maximum value of \( \chi(G) \) among all graphs \( G \) that imbed in \( S_n \), for \( n = 0, 1, 2, \ldots \)?" Let us denote this number by \( \chi(S_n) \).
Heawood showed in 1890 that

\[ \chi(S_n) \leq f(n) = \left\lceil \frac{7 + (1 + 48n)^{1/2}}{2} \right\rceil, \]

for \( n > 0 \). The proof that the converse inequality also holds was completed by Ringel and Youngs in 1968. They showed that \( \gamma(K_p) = \left\{ \frac{(p-4)(p-3)}{12} \right\}, \ p \geq 3, \) a result which implies that \( K_f(n) \) can be imbedded in \( S_n \). (Here \( [x] \) denotes the least integer greater than or equal to \( x \).) This result has quite a history and many people contributed to the solution; for a thorough discussion of it and also the problem of finding \( \chi(S_n) \) we refer the reader to [56]. Thus the case \( n = 0 \) was the only one left to solve. However, this problem was the famous Four-Color Problem, which has been around for about 100 years. Heawood knew that \( \chi(S_0) \leq 5, \) but it was conjectured that 4 colors would always suffice. Many people had tried to prove the Four-Color Conjecture and had failed, or could only manage to increase the number of vertices which a counterexample would require. Then, recently, Kenneth Appel and Wolfgang Haken [6] completed a remarkable proof of the conjecture, aided by a computer at the University of Illinois. Their method basically reduces the problem to one of considering over 1700 "reducible configurations", and then uses the computer to verify that 4-colorings always exist in every case. In Chapter 4 we shall find various bounds for the
cochromatic number of a graph, and discuss the analog to the Heawood Map Coloring Problem for this parameter.
CHAPTER 2

PATH - PERFECT GRAPHS

Recall the well-known formula for the sum of the first $n$ positive integers, namely,

$$1 + 2 + \ldots + n = \frac{n(n+1)}{2} = \binom{n+1}{2}.$$ 

If a graph has $\frac{n(n+1)}{2}$ edges for some positive integer $n$, it may be possible to decompose the graph into mutually edge-disjoint subgraphs, one for each size $i$, $1 \leq i \leq n$. In this chapter we consider this notion, where the subgraphs of the decomposition are all paths. Specifically, we define a graph $G$ having $\frac{n(n+1)}{2}$ edges to be path-perfect if $E(G)$ can be partitioned into $E_1 \cup E_2 \cup \ldots \cup E_n$ so that the subgraph of $G$ induced by $E_i$ is isomorphic to $P_i$, for $1 \leq i \leq n$. We shall show that several well-known classes of graphs are path-perfect, and also consider the problem of the existence of regular path-perfect graphs.

Also well-known are the formulas for the sum of the first $n$ odd or even positive integers:

$$1 + 3 + \ldots + (2n-1) = n^2$$
$$2 + 4 + \ldots + 2n = n(n+1).$$

In an analogous way, we say that a graph $G$ having $n^2$ [resp. $n(n+1)$] edges is odd [resp. even] path-perfect.
if $E(G)$ can be partitioned as $E_1 \cup E_2 \cup \ldots \cup E_n$ so that the subgraph of $G$ induced by $E_i$ is isomorphic to $P_{2i-1}$ [resp. $P_{2i}$]. Several classes of graphs possessing these properties are also exhibited.

Let us first consider the notion of a graph being path-perfect. As an example, consider the famous Petersen graph, which is shown in Figure 2.1. This graph has $1 + 2 + 3 + 4 + 5 = 15$ edges, and we show one possible decomposition of it into $P_1, P_2, P_3, P_4,$ and $P_5$.

\[ P_5: v_1 v_2 v_3 v_4 v_5 v_6 \quad P_1: v_7 v_8 \]
\[ P_4: v_3 v_7 v_6 v_{10} v_2 \quad P_2: v_4 v_9 v_{10} \]
\[ P_3: v_5 v_1 v_8 v_9 \]

**FIGURE 2.1**

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We shall say that a graph $G$ is potentially path-perfect if it has $\frac{n(n+1)}{2}$ edges for some $n$. It is easy to see that not every potentially path-perfect graph is path-perfect, for example, consider the graph $\frac{n(n+1)}{2} P_1$ for $n > 1$. However, suppose that a potentially path-perfect graph is connected. The graph $K_2$ is path-perfect, and both connected graphs on three edges, $K_3$ and $P_3$, are path-perfect. Also, there are 30 connected graphs on 6 edges (see [35], pp. 215, 217, 219, 220, 233) and all are path-perfect with the exception of $S_6$. This establishes another obvious necessary condition for a connected graph with $\frac{n(n+1)}{2}$ edges to be path-perfect; it must contain a path of length $n$. However, as our first result shows, even this condition is far from being sufficient.

**THEOREM 2.1** For each positive integer $n \geq 4$, there exists a connected graph $G_n$ with $\frac{n(n+1)}{2}$ edges which is not path-perfect, but whose edge set can be partitioned into $E_0 \cup E_1 \cup \ldots \cup E_{n-2}$ such that for $0 \leq i \leq n-3$, $<E_i> = P_{n-i}$, and $<E_{n-2}> = 3P_1$.

**Outline of proof.** The graph $G_4$ is shown in Figure 2.2a. It is easy to argue that $G_4$ is not path-perfect. A partition of this graph into $P_4$, $P_3$, and $3P_1$ is shown.

For $n \geq 5$, the graph $G_n$ is given in Figure 2.2b.
(In the case \( n = 5 \), \( v_{15} = v_{16} = \ldots = \frac{v_n(n+1)}{2} \).)

Suppose this graph is path-perfect and that a partition into \( P_1, P_2, \ldots, P_n \) is given. Without loss of generality we may assume that \( P_1 \) is none of the edges \( v_1v_7, v_2v_8, v_3v_9, v_7v_8 \), or \( v_8v_9 \). This will imply that these five edges must be used to form \( P_2 \) and \( P_3 \), otherwise, one of these edges will become "isolated", and so cannot be used in any of the longer paths. Next consider the edge \( v_5v_{14} \). If this edge is not \( P_1 \), then it belongs to some \( P_i \) for \( i > 3 \). This will imply that \( v_6v_{13} \) (or \( v_4v_{15} \)) is \( P_1 \), but then the edge \( v_{14}v_{15} \) (or \( v_{13}v_{14} \)) becomes isolated when we try to place into \( G_n \) the path \( P_j \), \( j \geq 5 \), to which \( v_{14}v_{15} \) (or \( v_6v_{13} \)) belongs. Thus \( v_5v_{14} \) is \( P_1 \). We next argue that \( P_4 \) must contain the edges \( v_6v_{13}, v_{13}v_{14}, v_{14}v_{15}, v_{15}v_4 \). But now the path of length 4 from \( v_9 \) to \( v_{13} \) is isolated, and we arrive at a contradiction. Therefore \( G_n \) is not path-perfect. However, letting

\[ 3P_1 = \langle v_8v_9, v_5v_{14}, v_7v_{\frac{n(n+1)}{2}} \rangle, \quad P_3 = \langle v_1v_7, v_7v_8, v_8v_2 \rangle, \]

\[ P_4 = \langle v_6v_{13}, v_{13}v_{14}, v_{14}v_{15}, v_{15}v_4 \rangle, \]

\[ P_5 = \langle v_3v_9, v_9v_{10}, v_{10}v_{11}, v_{11}v_{12}, v_{12}v_{13} \rangle, \]

and partitioning the path of length \( \frac{n(n+1)}{2} - 15 \) from \( v_{15} \) to \( \frac{v_n(n+1)}{2} \) into \( P_6, P_7, \ldots, P_n \), we obtain

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a partition of $G_n$ into $P_n, P_{n-1}, \ldots, P_3$ and $3P_1$. "

Theorem 2.1 is best possible, in the sense that if a graph having $\frac{n(n+1)}{2}$ edges contains edge-disjoint copies of $P_2, P_3, \ldots, P_{n-1}$ and $P_n$, then the remaining edge automatically equals $P_1$, and so the graph is path-perfect.

We have noted that the complete graphs $K_2$, $K_3$, and $K_4$ are path-perfect. In fact, since $K_n$ has $\frac{n(n-1)}{2} = 1 + 2 + \ldots + (n-1)$ edges, every complete graph is potentially path-perfect, with the "n" of the
Our first theorem shows that every (non-trivial) complete graph is path-perfect.

**THEOREM 2.2** For each positive integer \( n \geq 2 \), the complete graph \( K_n \) is path-perfect.

**Proof.** We have already noted that \( K_n \) has \( \frac{n(n-1)}{2} \) edges.

If \( n \) is odd, it is well-known that \( K_n \) can be factored into \( \frac{(n-1)}{2} \) hamiltonian cycles (see, for example, [35] p. 89). Partitioning the \( i^{th} \) cycle into \( P_i \) and \( P_{n-i} \), \( 1 \leq i \leq \frac{(n-1)}{2} \), yields paths of the desired lengths.

If \( n \) is even, Beineke [9] has shown that \( K_n \) can be factored into \( \frac{n}{2} \) hamiltonian paths. We may partition the \( i^{th} \) path into \( P_i \) and \( P_{n-i-1} \) for \( 1 \leq i \leq \frac{n}{2} - 1 \), and use the remaining path as \( P_{n-1} \).

The complete bipartite graphs \( K(s,t) \) form another important and often studied family of graphs. This graph has \( st \) edges and hence is potentially path-perfect if \( st = \frac{n(n+1)}{2} \) for some \( n \). We next consider the case where \( n = 2r \) or \( n = 2r - 1 \) and \( s = r \).

**THEOREM 2.3** Let \( r \) be a positive integer. Then the graphs \( K(r, 2r-1) \) and \( K(r, 2r+1) \) are path-perfect.
Proof. First consider $K(r, 2r-1)$. Let $U = \{u_1, u_2, \ldots, u_{2r-1}\}$ and $V = \{v_1, v_2, \ldots, v_r\}$ be the partite sets of this graph. To help the reader see the construction of the various paths, we form a $2r-1$ by $r$ matrix, called $S = (s_{ij})$, where $s_{ij} = m$ means that the edge $u_i v_j$ is included in the path of length $m$. (As examples, see Figures 2.3a and 2.3b.)

We construct the paths (matrix $S$) as follows. The path $P_{2r-1}$ equals $u_1 v_1 u_2 v_2 \cdots v_{r-1} u_r v_r$. (Or, equivalently, $s_{11} = s_{21} = s_{22} = \cdots = s_{r r-1} = s_{rr} = 2r-1$.) For $m \in \{2, 3, \ldots, r-1\}$, we let $P_{2m}$ equal $u_{2r-2m+1} v_1 u_{2r-2m+2} v_2 \cdots u_{2r-m} v_m u_{2r-m+1}$

and $P_{2m-1}$ equal $v_{r-m+1} u_{r+m+1} v_{r-m+2} \cdots u_{r+2m-1} v_r u_{r+2m}$. (Here the subscripts on the $u$'s must be taken modulo $n-1$.) Lastly, let $P_1$ equal $u_2 v_1$ and $P_2$ equal $u_{r+1} v_r u_{r+2}$.

Next consider $K(r, 2r+1)$, having partite sets $U = \{u_1, u_2, \ldots, u_{2r+1}\}$ and $V = \{v_1, v_2, \ldots, v_r\}$. As above we may construct a matrix, this time called $Q$; $Q$ will be $2r+1$ by $r$. Let the path of length $2r$ be given by $u_1 v_1 u_2 v_2 \cdots u_r v_r u_{r+1}$ i.e., $q_{11} = q_{21} = q_{22} = \cdots = q_{rr} = q_{r+1 r} = 2r$. Now note that the remaining parts of $Q$ can be combined into a $2r-1$ by $r$ matrix $Q'$. Hence the problem of finding
paths of lengths 1, 2, \ldots, 2r-1 in $Q'$ is solved by the construction of $S$. ■

In Figures 2.3a and 2.3b we show the matrix $S$ in the cases where $r$ equals 5 and 6, respectively. In Figure 2.3c is shown the matrix $Q$ for the case $r = 6$. Note that once the path of length 12 has been inserted, the construction of the rest of $Q$ is exactly the same as that of $S$ when $r = 6$.

\begin{align*}
9 & 7 & 7 & 5 & 5 & 11 & 9 & 9 & 7 & 7 & 5 & 12 & 9 & 9 & 7 & 7 & 5 \\
9 & 9 & 7 & 7 & 5 & 11 & 11 & 9 & 9 & 7 & 7 & 12 & 12 & 9 & 9 & 7 & 7 \\
8 & 9 & 9 & 7 & 7 & 10 & 11 & 11 & 9 & 9 & 7 & 7 & 11 & 12 & 12 & 9 & 9 \\
8 & 8 & 9 & 9 & 7 & 10 & 10 & 11 & 11 & 9 & 9 & 9 & 11 & 12 & 12 & 9 & 9 \\
6 & 8 & 8 & 9 & 9 & 8 & 10 & 10 & 11 & 11 & 9 & 10 & 11 & 12 & 12 & 9 & 9 \\
1 & 4 & 5 & 5 & 5 & 4 & 6 & 8 & 3 & 3 & 3 & 6 & 8 & 8 & 10 & 10 & 2 & 2 \\
1 & 4 & 5 & 5 & 5 & 4 & 6 & 6 & 8 & 3 & 3 & 6 & 6 & 8 & 8 & 10 & 10 & 2 & 2 \\
\end{align*}

a) $S$ when $r = 5$.

\begin{align*}
1 & 4 & 7 & 7 & 5 & 5 & 4 & 6 & 8 & 3 & 3 & 4 & 4 & 6 & 5 & 5 & 3 & 3 \\
\end{align*}

b) $S$ when $r = 6$.

\begin{align*}
1 & 4 & 7 & 7 & 5 & 5 & 4 & 4 & 6 & 5 & 5 & 3 & 3 \\
\end{align*}

c) $Q$ when $r = 6$.

\begin{figure}
\caption{Figure 2.3}
\end{figure}

Suppose that $K(s, t)$ is potentially path-perfect, where $st = \frac{n(n+1)}{2}$ and $s < t$. If $s$ is too small, namely,
if \( s < \left\lfloor \frac{n+1}{2} \right\rfloor \), then it is impossible for \( K(s, t) \) to be path-perfect. For then the longest path in \( K(s, t) \) has length \( 2s \), and \( 2s < n \). Theorem 2.3 tells us, however, that in the "worst" possible case, namely, when \( s = \left\lfloor \frac{n+1}{2} \right\rfloor \), that \( K(s, t) \) is path-perfect. We would also expect \( K(s, t) \) to be path-perfect when the sizes of the partite sets of differ by a smaller amount.

**CONJECTURE 2.1** If \( s, t, \) and \( n \) are positive integers such that \( 2st = n(n+1) \) and \( \left\lfloor \frac{n+1}{2} \right\rfloor \leq s \leq t \), then the complete bipartite graph \( K(s, t) \) is path-perfect.

The first instance where \( 2st = n(n+1) \) and \( \left\lfloor \frac{n+1}{2} \right\rfloor < s \leq t \) occurs when \( s = t = 6 \). In Figure 2.4 we give a matrix construction showing that \( K(6, 6) \) is path-perfect.

\[
\begin{array}{cccccc}
v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\
u_1 & 4 & 4 & 7 & 7 & 3 & 3 \\
u_2 & 6 & 4 & 8 & 7 & 7 & 3 \\
u_3 & 6 & 6 & 8 & 8 & 7 & 7 \\
u_4 & 5 & 6 & 6 & 8 & 8 & 7 \\
u_5 & 5 & 5 & 6 & 2 & 8 & 8 \\
u_6 & 4 & 5 & 5 & 2 & 1 & 8 \\
\end{array}
\]

**FIGURE 2.4**

Recall that an odd path-perfect graph must have size \( n^2 \) for some positive integer \( n \). An obvious graph
to investigate in this case is the complete bipartite graph $K(n, n)$. The next result is due to John Fink [27].

**THEOREM 2.4** For every positive integer $n$ the complete bipartite graph $K(n, n)$ is odd path-perfect.

**Proof.** Let $U = \{u_1, u_2, \ldots, u_n\}$ and $V = \{v_1, v_2, \ldots, v_n\}$ denote the partite sets of $K(n, n)$. Suppose as before that $R$ is an $n$ by $n$ matrix $(r_{ij})$, where $r_{ij} = 2m - 1$ means that $u_i v_j$ is included in the path of length $2m - 1$. Also in $R$ call consecutively, from lower left to upper right, the $2n - 1$ diagonals parallel to, and including, the main diagonal $D_1, D_2, \ldots, D_{2n-1}$ (e.g. $D_1$ is $r_{n1}$, $D_n$ is the main diagonal, $D_{2n-1}$ is $r_{1n}$). For notational convenience we shall use $D_i$ to refer also to the set of edges of $K(n, n)$ corresponding to the elements of $D_i$. Also define $D_0 = \emptyset$.

Observe that if $i$ is odd, the subgraph induced by $D_{i-1} \cup D_i$ is the path $P_{2i-1}$, and if $i$ is even, $D_{2n-i} \cup D_{2n-i+1}$ induces $P_{2i-1}$. Therefore, in our partition of $E(K(n, n))$, we define

$$E_i = \begin{cases} 
D_{i-1} \cup D_i, & \text{if } i \text{ is odd, } 1 \leq i \leq n. \\
D_{2n-i} \cup D_{2n-i+1}, & \text{if } i \text{ is even, } 2 \leq i \leq n.
\end{cases}$$

In Figure 2.5 we show the matrix $R$ for the cases $n$ equals 8 and $n$ equals 9.
FIGURE 2.5

The last definition concerning us in this chapter is that of a graph being even path-perfect. We next exhibit a well-known family of graphs possessing this property.

THEOREM 2.5 Let n be a positive integer. The complete bipartite graph $K(n, n+1)$ is even path-perfect if, and only if, $n$ does not equal 2.

Proof. Let $\{u_1, u_2, \ldots, u_{n+1}\}$ and $\{v_1, v_2, \ldots, v_n\}$ be the partite sets of $K(n, n+1)$.

First consider the case where $n$ is odd. We exhibit the paths as follows. For $m \in \{0, 1, \ldots, (n-1)/2\}$, $P_{2n-4m}$ equals

$u_1 v_{2m+1} u_2 v_{2m+2} \cdots u_{n-2m} v_n u_{n-2m+1}$.

For $m \in \{1, 2, \ldots, (n-1)/2\}$, $P_{2n-4m+2}$ equals

$u_{1+2m} v_1 u_{2+2m} v_2 \cdots u_{n+1} v_{n-2m+1} u_1$.

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Next suppose \( n \) is even. If \( n \) equals 2 it is easy to see that \( K(2, 3) \) is not even path-perfect, for the relative complement of \( P_2 \) in \( K(2, 3) \) is either a 4-cycle or contains a vertex of degree 3. If \( n \) is at least 4, we can construct the paths this way.

\[
P_{2n} = u_1 v_1 u_2 v_2 \cdots u_n v_n u_{n+1}.
\]

\[
P_2 = v_1 u_{n+1} v_{n-1}.
\]

\[
P_{2n-2} = u_1 v_n u_3 v_1 u_4 v_2 \cdots u_n v_{n-2} u_{n+1}.
\]

For \( m \in \{1, 2, \ldots, \frac{n}{2} - 1\} \), \( P_{2n-4m} \) equals

\[
v_{2m} u_1 v_{2m+1} u_2 \cdots v_{n-1} u_{n-2m} v_n.
\]

For \( m \in \{2, 3, \ldots, \frac{n}{2} - 1\} \), \( P_{2n-4m+2} \) equals

\[
u_{2m+1} v_1 u_{2m+2} v_2 \cdots v_{n-2m} u_{n+1} v_{n-2m+1}.
\]

To help visualize the constructions given in the above theorem, we could again construct a matrix; suppose we call it \( T \). Here \( t_{ij} = 2m \) means that \( u_i v_j \) should be included in \( P_{2m} \). In Figure 2.6 we show \( T \) when \( n \) equals 7 and when \( n \) equals 8. Again the reader is encouraged to construct \( T \) for several values of \( n \).

The girth of a graph is the length of the smallest cycle which it contains. An \( n \)-cage is a graph of minimum order which is 3-regular and has girth \( n \). The complete graph \( K_4 \) is the unique 3-cage and the Petersen graph is
known to be the only 5-cage. Note that we have shown that both of these graphs are path−perfect. Are any other cages path−perfect? In general, what can be said about the existence of regular path−perfect graphs?

If $G$ is an $r$-regular graph of order $p$ and size $n = \frac{n(n+1)}{2}$, then

$$pr = n(n+1).$$

Hence $r$ must divide $n(n+1)$. We first consider the case where $r$ is odd and $r$ divides $n$ or $r$ divides $n + 1$.

**THEOREM 2.6** Let $r$ be an odd positive integer. There exists an $r$-regular path−perfect graph of order $m(mr+1)$ if, and only if, $m$ equals 1. Also, there exists an $r$-regular path−perfect graph of order $m(mr−1)$ if, and only if, $m$ equals 2.
**Proof.** Let $r$ be odd and suppose $G$ is an $r$-regular path-perfect graph of order $p$ with $\frac{n(n+1)}{2}$ edges. Then $pr = n(n+1)$. Since each vertex of $G$ must be an end vertex of at least one path, $p$ is at most $2n$. We consider two cases.

**Case 1.** If $p$ equals $m(mr+1)$, then $n$ equals $mr$. Thus we have $m(mr+1) \leq 2mr$, which implies that $mr + 1 \leq 2r$, whence $m$ equals 1. Conversely, $K_{r+1}$ is an $r$-regular path-perfect graph of order $r + 1$.

**Case 2.** If $p$ equals $m(mr - 1)$, then $n$ equals $mr - 1$. Thus $m(mr - 1) \leq 2(mr - 1)$, whence $m \leq 2$. Also as $p$ is at least $r + 1$, $m$ is at least 2. Therefore, $m$ equals 2.

Conversely, we must exhibit an $r$-regular path-perfect graph of order $2(2r - 1)$ for each odd integer $r$. If $r$ is equal to 1 or 3, we can use as examples $K_2$ and the Petersen graph, respectively. For $r \geq 5$, we define a graph $H_r$ as follows. Let $V(H_r) = X \cup Y \cup U \cup V$, where

$$X = \{x_1, x_2, \ldots, x_r\}, \quad Y = \{y_1, y_2, \ldots, y_r\},$$

$$U = \{u_1, u_2, \ldots, u_{r-1}\} \text{ and } V = \{v_1, v_2, \ldots, v_{r-1}\}.$$

Form the complete bipartite graphs on $X$ and $Y$, and on $U$ and $V$. $H_r$ results by removing the edge $x_1y_r$ and adding the edges $x_1v_1, v_{r-1}y_r, u_1u_2, u_3u_4, \ldots, u_{r-2}u_{r-1}$ and $v_2v_3, v_4v_5, \ldots, v_{r-3}, v_{r-2}$. Clearly $H_r$ is $r$-regular.
and has order \(2(2r - 1)\). Since the subgraph of \(H_r\) induced by \(X \cup Y\) equals \(K(r, r)\) minus an edge, the odd length paths \(P_3, P_5, \ldots, P_{2r-1}\) are in \(H_r\). Also \(P_1 = v_{r-1}v_r\). Next we have that the subgraph of \(H_r\) induced by \(U \cup V\) contains \(K(r-1, r-1)\) as a subgraph. The edges of this \(K(r-1, r-1)\) can be partitioned into odd paths, say we call them \(P'_1, P'_3, \ldots, P'_{2r-3}\). Our plan is to add an edge of \(H_r\) to each of these paths to obtain the paths of even length.

Note that by the construction of Theorem 2.4 the following are true:

a) \(P'_{2r-3}\) starts with \(v_1\),

b) for \(i\) even, \(2 \leq i \leq r - 1\), \(P'_{2r-2i-1}\) starts with \(u_i\) and does not include \(u_{i-1}\),

c) for \(i\) odd, \(3 \leq i \leq r - 2\), \(P'_{2r-2i-1}\) starts with \(v_i\) and does not include \(v_{i-1}\).

Therefore, we may extend these paths as follows:

a) \(P'_{2r-3} + x_1v_1 = P_{2r-2}\),

b) for \(i\) even, \(2 \leq i \leq r - 1\),

\[P'_{2r-2i-1} + u_{i-1}u_i = P_{2r-2i}\]

c) for \(i\) odd, \(3 \leq i \leq r - 2\),

\[P'_{2r-2i-1} + v_{i-1}v_i = P_{2r-2i}\]

This shows that \(H_r\) is path-perfect. ■

The graph \(H_5\) is shown in Figure 2.7.
FIGURE 2.7

We have from Equation 2.1 that \( r \) must divide the product \( n(n+1) \). If \( r \) is a power of a prime integer, then \( r \) must divide either \( n \) or \( n+1 \). We thus obtain the following corollary.

**Corollary 2.1** If \( G \) is an \( r \)-regular path-perfect graph and \( r \) is a power of an odd prime number, then either \( G \) equals \( K_{r+1} \), or the order of \( G \) is \( 2(2r-1) \).

It follows from Corollary 2.1 that \( K_4 \) and the Petersen graph are the only path-perfect cages.

If \( r \) is even, Equation 2.1 can be written in the
form

\[ p \left( \frac{r}{2} \right) = \frac{n(n+1)}{2} . \]

Given an \( n \) and an even \( r \), with \( r \) dividing \( n(n+1) \), we would like to find an \( r \)-regular path-perfect graph on \( p = \frac{n(n+1)}{r} \) vertices. The problem of finding \( G \) is related to the following number theoretical problem.

Let \( N = \{1, 2, 3, \ldots\} \) and \( Z_n = \{1, 2, \ldots, n\} \). For \( S \) a finite subset of \( N \), let \( \Sigma S \) denote the sum of the elements of \( S \). Now \( \Sigma Z_n \) equals \( 1 + 2 + \ldots + n \), which equals \( \frac{n(n+1)}{2} \); suppose

\[ n \frac{n(n+1)}{2} = ts , \]

where \( t \) and \( s \) are integers and \( t \geq n \). We ask, do there exist \( s \) pairwise disjoint subsets of \( Z_n \), say \( T_1, T_2, \ldots, T_s \), such that \( \Sigma T_i = t, \ 1 \leq i \leq s \)?

As an example, let \( n = 20, \ t = 35, \) and \( s = 6 \).

Let \( T_1 = \{15, 20\}, \ T_2 = \{16, 19\}, \ T_3 = \{17, 18\}, \ T_4 = \{1, 4, 7, 10, 13\}, \ T_5 = \{2, 8, 11, 14\}, \) and \( T_6 = \{3, 5, 6, 9, 12\} \). Note that the \( T_i, \ 1 \leq i \leq 6 \), form a partition of \( \{1, 2, \ldots, 20\} \), and the sum of the elements belonging to any \( T_i \) equals 35.

**Lemma 2.1** Suppose \( n \frac{n(n+1)}{2} = ts \), where \( t \) and \( s \) are integers and \( t \geq n \). Suppose at least one of the following conditions holds:
i) $2s$ divides $n$ or $n + 1$,
ii) $t < 2n - 2s + 4$, or
iii) $s \in \{1, 2, 3, 4\}$.

Then $\mathbb{Z}_n$ can be partitioned into $T_1 \cup T_2 \cup \ldots \cup T_s$ so that $\sum T_i = t$, $1 \leq i \leq s$.

**Proof.**

i) First suppose $2s$ divides $n$, say $m$ times. Then for $1 \leq i \leq s$, let $T_i$ equal

$$\{im - m + 1, im - m + 2, \ldots, im, n - im + 1, \ldots, n - im + m\}.$$

We check that $\sum T_i = m(im - m) + \frac{m(m + 1)}{2} + m(n - im) + \frac{m(m + 1)}{2}$

$$= im^2 - m^2 + mn - im^2 + m^2 + m$$

$$= m(n + 1) = t.$$

Secondly, if $2s$ divides $n + 1$, $m$ times, we let $T_i$, $1 \leq i \leq s$, equal

$$\{im - m, im - m + 1, \ldots, im - 1, n - im + 1, n - im + 2, \ldots, n - im + m\}.$$

Then, $\sum T_i = m(im - m) + \frac{m(m - 1)}{2} + m(n - im) + \frac{m(m + 1)}{2}$

$$= im^2 - m^2 + m^2 + mn - im^2$$

$$= mn = t.$$

The partitions in this case are due to John Fink.

ii) Suppose $t < 2n - 2s + 4$. Here we may assume that $s \geq 2$, as the case $s = 1$ is trivial. Hence $t < 2n$, which implies that $t - n < n$. We also have that $t - n + s - 2 < n - s + 2$. Thus we let
\[ T_i = \{n-i+1, t-n+i-1\}, 1 \leq i \leq s-1, \text{ and} \]
\[ T_s = \{1, 2, \ldots, t-n-1, t-n+s-1, t-n+s, \ldots, n-s+1\}. \]
Clearly \( \sum T_i = t \) for \( i \) between 1 and \( s-1 \), and thus \( \sum T_s \)
must be \( t \) also.

\( \text{iii}) \) If \( s \in \{1, 2, 4\} \), then condition (i) is
satisfied, so suppose \( s = 3 \). Then \( t = n(n+1)/6 \), and so
either \( n \) or \( n+1 \) is divisible by 3. Also note that the
requirement \( t \geq n \) implies that \( n \geq 6 \).

**Case 1.** Suppose that 3 divides \( n \). Let \( A = \{1, 4, \ldots, n-2\} \),
\( B = \{2, 5, \ldots, n-1\} \), and \( C = \{3, 6, \ldots, n\} \). Note that \( \sum A = t - n/3 \), \( \sum B = t \), and \( \sum C = t + n/3 \). Hence if \( n/3 \in A \),
let
\[ T_1 = (A \cup \{n/3 + 1, n/3 - 1\}) - \{n/3\}, \]
\[ T_2 = (B \cup \{n/3, 3\}) - \{n/3 + 1, 2\}, \]
and
\[ T_3 = (C \cup \{2\}) - \{3, n/3 - 1\}. \]
Then \( \sum T_1 = \sum T_2 = \sum T_3 = t \). If \( n/3 \in B \), let
\[ T_1 = (A \cup \{n/3 + 1\}) - \{1\}, \]
\[ T_2 = B, \]
and
\[ T_3 = (C \cup \{1\}) - \{n/3 + 1\}. \]
Again \( \sum T_1 = \sum T_2 = \sum T_3 = t \). Finally, if \( n/3 \in C \), let
\[ T_1 = A \cup \{n/3\}, \]
\[ T_2 = B, \]
\[ T_3 = C - \{n/3\}. \]
Clearly \( \sum T_i = t, 1 \leq i \leq 3 \).
Case 2. Suppose 3 divides $n + 1$. Let
\[ A = \{1,4,\ldots,n-1\}, \quad B = \{2,5,\ldots,n\}, \quad \text{and} \quad C = \{3,6,\ldots,n-2\}. \]
Note that $\Sigma A = t$, $\Sigma B = t + (n+1)/3$, and $\Sigma C = t - (n+1)/3$.
Thus, if $(n+1)/3 \in A$, let $T_1 = A$, 
\[ T_2 = (B \cup \{3\}) - [(n+4)/3, 2], \]
and 
\[ T_3 = (C \cup [(n+4)/3, 2]) - \{3\}. \]
Then $\Sigma T_1 = \Sigma T_2 = \Sigma T_3 = t$. If $(n+1)/3 \in B$, we let 
\[ T_1 = A, \quad T_2 = B - [(n+1)/3] \quad \text{and} \quad T_3 = C \cup [(n+1)/3]. \]
Again $\Sigma T_1 = \Sigma T_2 = \Sigma T_3 = t$. Finally, if $(n+1)/3 \in C$, let 
\[ T_1 = (A \cup \{2, (n+1)/3\}) - [(n+4)/3, 1], \]
\[ T_2 = (B \cup \{1\}) - [(n-2)/3, 2], \]
and 
\[ T_3 = (C \cup [(n-2)/3, (n+4)/3]) - [(n+1)/3]. \]
Again it is easy to check that $\Sigma T_i = t$, $1 \leq i \leq 3$. ■

As an example of a case where condition (i) of the lemma applies, let $n = 18$, $s = 3$, and $t = 57$. Here $2s$ divides $n$, and applying Fink's construction we obtain the partition $T_1 = \{1,2,3,16,17,18\}$, $T_2 = \{4,5,6,13,14,15\}$, and $T_3 = \{7,8,9,10,11,12\}$. Since $s = 3$, we could also use the construction given for condition (iii); we would obtain $T_1 = \{1,4,6,7,10,13,16\}$, $T_2 = \{2,5,8,11,14,17\}$, and $T_3 = \{3,9,12,15,18\}$. When $n = 15$, $s = 6$, and $t = 20$, we have the case where
We then let \( T_1 = \{5,15\}, T_2 = \{6,14\}, T_3 = \{7,13\}, T_4 = \{8,12\}, T_5 = \{9,11\}, \) and \( T_6 = \{1,2,3,4,10\} \).

Let us now apply the lemma to the problem of finding \( r \)-regular path-perfect graphs when \( r \) is even. Note Equations 2.2 and 2.3 are exactly the same when \( t = p \) and \( s = r/2 \).

**THEOREM 2.7** Let \( p \) and \( n \) be positive integers and let \( r \) be an even positive integer such that \( p \geq r + 1 \) and \( pr = n(n+1) \). Then there exists an \( r \)-regular path-perfect graph \( G \) of order \( p \) and size \( \frac{n(n+1)}{2} \) provided that at least one of the following conditions is satisfied:

1) \( r \) divides \( n \) or \( r \) divides \( n + 1 \),
2) \( p < 2n - r + 4 \), or
3) \( r \leq 8 \).

**Proof.** Let \( p, n \) and \( r \) even be given with \( p \geq r + 1 \), \( pr = n(n+1) \), and suppose that at least one of conditions 1-3 holds. We wish to construct an \( r \)-regular path-perfect graph \( G \) of order \( p \).

If \( p \) is even we know that \( K_p \) can be factored into \( (p-2)/2 \) hamiltonian cycles plus a 1-factor. If \( p \) is odd, \( K_p \) can be factored into \( (p-1)/2 \) hamiltonian cycles. In either case we may use \( r/2 \) of these cycles, call them \( C_1, C_2, \ldots, C_{r/2} \), to form an \( r \)-regular, spanning sub-
graph of $K_p$. This subgraph will be $G$.

To show that $G$ is path-perfect, apply Lemma 2.1 with $s = r/2$ and $t = p$. For $1 \leq i \leq r/2$, the subset $T_i$ of $Z_n$ given by the lemma tells us how to partition $C_i$ into paths—the elements of $T_i$ will be the lengths of the paths.

For those readers who find the problem of partitioning $Z_n$ into $s$ subsets, each summing to $t$, interesting, we present one final result. This result states that if $t$ contains a factor between $n$ and $2n$, then there is a method of reducing the given problem to one with a smaller value for "$n". It follows that the only case preventing an inductive proof that $Z_n$ can be partitioned as questioned is where $t > 2n$ and $t$ has no factor $t_1$ with $n \leq t_1 \leq 2n$.

**Theorem 2.8** Let $n$, $t$ and $s$ be positive integers such that $2ts = n(n + 1)$ and $t \geq n$. Suppose $t$ has a factor $t_1$ such that $n \leq t_1 \leq 2n$. Then the problem of partitioning $Z_n$ into $s$ subsets, each of which sums to $t$, can be solved, or reduced to a problem of partitioning $Z_{t-n-1}$.

**Proof.** Suppose $t = t_1t_2$. If we can partition $Z_n$ into $t_2$'s subsets, each of which sums to $t_1$, then by combining these subsets $t_2$ at a time, we obtain a
partition into $s$ subsets, each summing to $t$. We thus assume that $t$ itself is between $n$ and $2n$.

If $t = n$, $t = 2n$, $s \leq 4$, or if $t < 2n - 2s + 4$, then we may find the desired partition by Lemma 2.1. So assume that $2n > t \geq 2n - 2s + 4$. We now consider two cases.

**Case 1.** There exists a positive integer $j$ such that $n - j = t - n + j$. This implies that $j = (2n - t)/2$, and, that $n - j = t/2$. Thus $t$ is even in this case.

We let $T_i$ equal $\{n-i+1, t-n+i-1\}$ for $1 \leq i \leq j$. Then $Z_n - (T_1 \cup T_2 \cup \ldots \cup T_j) = \{1, 2, \ldots, t-n-1, t/2\}$. Now

$$1 + 2 + \ldots + t-n-1 = \frac{(t-n-1)(t-n)}{2} = \frac{t}{2}(t-2n+2s-1)$$

and $t - 2n + 2s - 1$ is odd. Therefore, if $Z_{t-n-1}$ can be partitioned into $t - 2n + 2s - 1$ subsets, each summing to $t/2$, one of these subsets can be combined with $t/2$, and the rest taken two at a time, to obtain the desired partition of $Z_n$.

**Case 2.** There exists a positive integer $j$ such that $n - j = t - n + j - 1$. This implies that $n - j = (t+1)/2$, and thus $t$ is odd.

Let $T_i$ equal $\{n-i+1, t-n+i-1\}$ for $1 \leq i \leq j$. Then $Z_n - (T_1 \cup T_2 \cup \ldots \cup T_j) = \{1, 2, \ldots, t-n-1\}$. Now

$$1 + 2 + \ldots + (t-n-1) = \frac{t(t-1-2n+2s)}{2},$$

and therefore, if we can partition $Z_{t-n-1}$ into
(t - 1 - 2n + 2s)/2 subsets, each summing to t, we will have the desired partition of $\mathbb{Z}_n$. ■

In the last part of this chapter we consider two operations with graphs, the product and the join. We use these operations to find several examples of graphs which are path-perfect, and also discover many questions concerning graphs which potentially possess these properties.

If $G_1$ and $G_2$ are two graphs, their product, denoted by $G_1 \times G_2$, is defined by

$$V(G_1 \times G_2) = V(G_1) \times V(G_2)$$

and

$$E(G_1 \times G_2) = \{(u, v)(u', v') \mid u = u' \text{ and } vv' \in E(G_2) \text{ or } v = v' \text{ and } uu' \in E(G_1)\}.$$  

The way this graph is normally constructed is as follows. First of all, replace each vertex of $G_1$ with an entire copy of $G_2$. Then, if $uu'$ is an edge of $G_1$, each vertex of $G_2$ in the copy corresponding to $u$ is joined to its "twin" in the copy corresponding to $u'$. One should note that $G_1 \times G_2 = G_2 \times G_1$.

Using this method of construction one observes that if $G_i$ has $p_i$ vertices and $q_i$ edges, $i = 1, 2$, then $G_1 \times G_2$ has $p_1p_2$ vertices and $p_1q_2 + p_2q_1$ edges. We wish to find values for $p_1, q_1, p_2$ and $q_2$ which will make $G_1 \times G_2$ potentially path-perfect, odd path-perfect, or even path-perfect.
For example, if $G_1$ is a unicyclic graph on $n$ edges, $n \geq 3$, and $G_2$ is a tree on $n - 1$ edges, then $G_1 \times G_2$ has $n(n-1) + n^2 = n(2n-1) = \binom{2n}{2}$ edges, and hence is potentially path-perfect. Similarly, if $G_1$ is a unicyclic graph on $n$ edges and $G_2$ is a tree on $n$ edges, then $G_1 \times G_2$ has $\binom{2n+1}{2}$ edges, and is potentially path-perfect. Our next result concerns these two situations.

**THEOREM 2.9** Let $n$ be a positive integer, $n \geq 3$. Let $G_1$ be a unicyclic graph on $n$ edges, formed by adding an edge to $P_{n-1}$ which is incident to an end vertex of $P_{n-1}$, and let $G_2$ be either $P_{n-1}$ or $P_n$. Then $G_1 \times G_2$ is path-perfect.

**Proof.** Let $V(G_1) = \{u_1, u_2, \ldots, u_n\}$ where $u_1 u_2 \ldots u_n$ is a path of length $n - 1$ and $u_n u_i$ is the additional edge of $G_1$, $1 \leq i \leq n - 2$.

Firstly, if $G_2 = P_{n-1}$, let $V(G_2) = \{v_1, v_2, \ldots, v_n\}$ where $v_1 v_2 \ldots v_n$ is the path. For ease of notation let $(k, j)$ denote the vertex $(u_k, v_j)$ of $G_1 \times G_2$ as shown in Figure 2.8. We form the paths as follows. For $m \in \{1, 2, \ldots, n-1\}$,

$$P_{2n-2m+1} = (m, 1) (m+1, 1) (m+1, 2) (m+2, 2) \ldots (n, n-m) (n, n-m+1) (i, n-m+1)$$

and

$$P_{2n-2m} = (1, m) (1, m+1) (2, m+1) (2, m+2) \ldots (n-m, n-1) (n-m, n) (n-m+1, n).$$
Also \( P_1 = (n, 1) (i, 1) \)

Secondly, if \( G_2 = P_n \), label the path

\[ V_1 v_2 \ldots v_n v_{n+1} \].

Again let \((k, j) = (u_k, v_j)\). We let \( P_{2n} \) equal

\[ (1,1) (1,2) (2,2) (2,3) \ldots (n-1,n) (n,n) (n,n+1) (i,n+1). \]

For \( m \in \{1, 2, \ldots, n-1\} \), \( P_{2n-2m} = (1,m+1) (1,m+2) (2,m+2) (2,m+3) \ldots (n-m,n+1) (n-m+1,n+1). \)

Finally, for \( m \in \{1, 2, \ldots, n\} \), \( P_{2n-2m+1} \) is the same as in the case when \( G_1 = P_{n-1}. \)

![Diagram](https://example.com/diagram.png)

**FIGURE 2.8**
The following is an immediate corollary of Theorem 2.9.

**Corollary 2.2** The graphs $C_n \times P_{n-1}$ and $C_n \times P_n$ are path-perfect when $n \geq 3$.

We can also find several values for $p_1, q_1, p_2,$ and $q_2$ which will make $G_1 \times G_2$ potentially even path-perfect. Consider the following cases.

1) $G_1$ is a unicyclic graph on $n$ edges and $G_2$ is a unicyclic graph on $2n + 1$ edges.

2) $G_1$ is a tree on $n$ edges and $G_2$ is a unicyclic graph on $2n + 1$ edges.

3) $G_1$ is a unicyclic graph on $n$ edges and $G_2$ is a unicyclic graph on $2n - 1$ edges.

4) $G_1$ is a tree on $n + 1$ edges and $G_2$ is a unicyclic graph on $2n$ edges.

Let us focus on the cases where the unicyclic graphs of (1) and (3) are cycles.

**THEOREM 2.10** For $n \geq 3$, the graphs $C_n \times C_{2n+1}$ and $C_n \times C_{2n-1}$ are even path-perfect.

**Proof.** First consider $C_n \times C_{2n+1}$, where $C_n$ and $C_{2n+1}$ are labeled $u_1 u_2 \ldots u_n u_1$ and $v_1 v_2 \ldots v_{2n+1} v_1$, respectively. Let $(i, j)$ denote the vertex $(u_i, v_j)$ of $C_n \times C_{2n+1}$. We let $P_{4n}$ equal $(1,1)$ $(1,2)$ $(2,2)$ $(2,3)$ \ldots $(n,n+1)$ $(1,n+1)$ $(1,n+2)$ \ldots $(n,2n)$ $(n,2n+1)$ $(1,2n+1)$. For $m \in \{1,2,\ldots,n-1\}$ or $m \in \{0,1,\ldots,n-1\}$, resp.
\[ P_{4n-2m} = (1,m+1) (l,m+2) (2,m+2) (2,m+3) \ldots (n,n+m+1) \]
\[ (1,n+m+1) (l,n+m+2) \ldots (n-m,2n+l) (n-m+1,2n+l) \]
and
\[ P_{2n-2m} = (m+1,2n+1) (m+1,l) (m+2,1) (m+2,2) \ldots (n,n-m-1) \]
\[ (n,n-m) (l,n-m). \]

Secondly, for \( C_n \times C_{2n-1} \), label \( C_n \) as above and label \( C_{2n-1} \) \( v_1 v_2 \ldots v_{2n-1} v_1 \), and let \((i,j)\) denote \((u_i,v_j)\).  For \( m \in \{1,2,\ldots,n\} \) let
\[ P_{4n-2m} = (1,m) (2,m) (2,m+1) (3,m+1) \ldots (n,n+m-2) (n,n+m-1) \]
\[ (1,n+m-1) (l,n+m) \ldots (n-m,2n-1) (n-m+1,2n-1) \]
\[ (n-m+1,l). \]

Lastly, for \( m \in \{1,2,\ldots,n-1\} \), let
\[ P_{2n-2m} = (1+m,1) (2+m,1) (2+m,2) \ldots (n,n-m) (1,n-m) (l,n-m+1). \]

In Figures 2.9 a-b we show the construction of the various paths for \( C_3 \times C_7 \) and \( C_4 \times C_7 \).  Since \( C_n \times C_m \) has genus one, we show these graphs on the torus.

Recall that an odd path-perfect graph has \( m^2 \) edges for some \( m \).  If \( G_1 \) is a unicyclic graph on \( n \) vertices and \( G_2 \) is a unicyclic graph on \( 2n \) vertices, then \( G_1 \times G_2 \) has \( 4n^2 = (2n)^2 \) edges, and hence is potentially odd path-perfect.  We next consider the graph \( C_n \times C_{2n} \).

**Theorem 2.11**  For \( n \geq 3 \), the graph \( C_n \times C_{2n} \) is odd path-perfect.
a. $C_3 \times C_7$ is even path-perfect

b. $C_4 \times C_7$ is even path-perfect

FIGURE 2.9
Proof. Consider the construction given in the proof of Theorem 2.10 to show that \( C_n \times C_{2n+1} \) is even path-perfect. Note that if we suppress the vertices \((u_i, v_{2n+1})\), \(1 \leq i \leq n\), we obtain the graph \( C_n \times C_{2n} \) and, moreover, we shorten each of the paths \( P_2, P_4, \ldots, P_{4n} \) by one edge. This yields a decomposition of \( C_n \times C_{2n} \) into \( P_1, P_3, \ldots, P_{4n-1} \). 

Another question one might ask is what happens when we take the product of two path-perfect graphs \( G_1 \) and \( G_2 \)? If \( G_1 \) and \( G_2 \) are non-empty path-perfect graphs, and \( G_1 \times G_2 \) has \( \binom{n+1}{2} \) edges for some \( n \), is \( G_1 \times G_2 \) path-perfect?

We first note that we should require both \( G_1 \) and \( G_2 \) to be connected, so that \( G_1 \times G_2 \) will be. For consider the graph \( P_1 \times (P_1 \cup P_2 \cup P_3) = (P_1 \times P_1) \cup (P_1 \times P_2) \cup (P_1 \times P_3) \). This graph has 21 edges, and both factors are path-perfect. However, it is not path-perfect. For the path of length 6 must be placed in \( P_1 \times P_3 \), and the path of length 5 in \( P_1 \times P_2 \). But then the path of length 4 cannot be placed anywhere.

Having considered many of the cases where \( G_1 \times G_2 \) has small size, we make the following conjecture.
**CONJECTURE 2.2**  If $G_1$ and $G_2$ are non-empty, connected, path-perfect graphs and $G_1 \times G_2$ is potentially path-perfect, then $G_1 \times G_2$ is path-perfect.

The same question concerning operations with path-perfect graphs can be investigated for other operations. The **join** of two graphs $G_1$ and $G_2$, denoted by $G_1 \vee G_2$, is the graph defined by

$$V(G_1 \vee G_2) = V(G_1) \cup V(G_2), \quad \text{and}$$

$$E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{v_1v_2 \mid v_1 \in V(G_1), v_2 \in V(G_2)\}.$$  

Thus $G_1 \vee G_2$ can be formed from $G_1 \cup G_2$ by adding all edges joining a vertex of $G_1$ to a vertex of $G_2$. If $G_i$ has order $p_i$ and size $q_i$, $i = 1, 2$, then $G_1 \vee G_2$ has order $p_1 + p_2$ and size $q_1 + q_2 + p_1p_2$.

Consider now the join of the path-perfect graphs $P_1$ and $P_{21}$. $P_1 \vee P_{21}$ has $66 = \binom{12}{2}$ edges, but we can argue that this graph is not path-perfect. Briefly, note that the path of length $i$ would have to use a least $i - 4$ edges of $P_{21}$, for $i = 8, 9, 10, 11$. But this adds to 22 edges, one more than $P_{21}$ contains. Therefore, the join of two non-empty, connected path-perfect graphs may not be path-perfect, even if it has the right size.

We end Chapter 2 with a list of unsolved problems concerning path-perfect graphs, some of which we have already encountered.
Prove or disprove:

1. If \( r \) is odd, \( r \) not equal to the power of a prime, then there exists an \( r \)-regular path-perfect graph having order other than \( r + 1 \) or \( 2(2r - 1) \).

2. Suppose \( s, t, \) and \( n \) are positive integers such that \( 2st = n(n+1) \) and \( t \geq n \). Then \( \mathbb{Z}_n \) can be partitioned into \( s \) subsets, \( T_1, T_2, \ldots, T_s \), such that \( \sum T_i = t \) for each \( i, 1 \leq i \leq s \). It follows that if \( r \) is even, then there exists an \( r \)-regular path-perfect graph of order \( p \) whenever \( pr = n(n+1) \) for some \( n \).

3. If \( T_n \) is a tree on \( n \) edges, \( n \geq 2 \), then \( \overline{T_n} \) is path-perfect.

4. If \( G_1 \) is a unicyclic graph on \( n + 1 \) edges and \( G_2 \) is a tree on \( n \) edges, then \( G_1 \times G_2 \) is path-perfect.

5. If \( G_1 \) is a unicyclic graph on \( n \) edges and \( G_2 \) is a tree on \( n \) edges, then \( G_1 \times G_2 \) is path-perfect.

6. If \( G_1 \) is a unicyclic graph on \( n \) edges and \( G_2 \) is a unicyclic graph on \( 2n + 1 \) edges, then \( G_1 \times G_2 \) is even path-perfect.
7. If $G_1$ is a unicyclic graph on $n$ edges and $G_2$ is a unicyclic graph on $2n - 1$ edges, then $G_1 \times G_2$ is even path-perfect.

8. If $G_1$ is a unicyclic graph on $2n$ edges and $G_2$ is a tree on $n$ edges, then $G_1 \times G_2$ is even path-perfect.

9. If $G_1$ is a unicyclic graph on $2n$ edges and $G_2$ is a tree on $n + 1$ edges, then $G_1 \times G_2$ is even path-perfect.

10. If $G_1$ is a unicyclic graph on $n$ edges and $G_2$ is a unicyclic graph on $2n$ edges, then $G_1 \times G_2$ is odd path-perfect.

11. If $G_1 = nK_2$ and $G_2 = K_n$, then $G_1 \vee G_2$ is path-perfect.

12. If $G_1 = K_{2n}$ and $G_2 = C_n$, then $G_1 \vee G_2$ is path-perfect.

13. If $G_1 = K_n$ and $G_2 = K_n \cup K(m, m+1)$, $m \geq 1$, then $G_1 \vee G_2$ is even path-perfect. (If $m = 0$ we have $G_1 \vee G_2 = K(n, n+1)$, which we showed was even path-perfect unless $n = 2$.)

14. If $G_1 = K_n$ and $G_2 = K_n \cup K(m, m)$, then $G_1 \vee G_2$ is odd path-perfect. (If $m = 0$ we have
$G_1 \lor G_2 = K(n,n)$, which is odd path-perfect.)

15. If $G_1$ and $G_2$ are non-empty, connected path-perfect graphs and $G_1 \times G_2$ is potentially path-perfect, then $G_1 \times G_2$ is path-perfect.

16. Suppose $n$, $s$, and $t$ are positive integers such that $2st = n(n+1)$, $s \leq t$, and $s \geq n/2$. Then $K(s,t)$ is path-perfect.
CHAPTER 3
PACKING TREES OF DIFFERENT SIZES
INTO THE COMPLETE GRAPH

In this chapter we investigate the following conjecture, which we shall call the Tree Packing Conjecture, or TPC for short.

**CONJECTURE 3.1** Let \( n \) be a positive integer, \( n \geq 2 \). For each \( i, \ 1 \leq i \leq n - 1 \), let \( T_i \) be any tree on \( i \) edges. Then there exists a partition of the edge set of \( K_n \), say \( E_1 \cup E_2 \cup \ldots \cup E_{n-1} \), such that the subgraph of \( K_n \) induced by \( E_i \) equals \( T_i \), \( 1 \leq i \leq n - 1 \).

Such a partition of \( E(K_n) \) will be called a packing of the sequence of trees \( T_1, T_2, \ldots, T_{n-1} \) into \( K_n \).

Let us consider an example of this notion.

**EXAMPLE 3.1** Consider \( K_6 \) and the sequence of trees \( T_1, T_2, S_3, P_4, L_5 \). In Figure 3.1 we show one possible packing of this sequence into \( K_6 \).

![Figure 3.1](image_url)

**FIGURE 3.1**

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In [33], A. Győrfi and J. Lehel showed that $T_1', T_2', \ldots, T_{n-1}$ can be packed into $K_n$ if each $T_i$ is a path or a star. This result was independently discovered by S. Zaks and C. L. Liu [58], who gave an especially interesting proof. We shall next provide an extension of this result. Before proceeding to the theorem, the following comments are presented for the sake of clarity.

Suppose $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. Consider an array called $X_n$, having entries $x_{ij}$, for $1 \leq i < j \leq n$. Note that there is a one-to-one correspondence between the edges of $K_n$ and the entries of $X_n$. We wish to assign a number to each edge of $K_n$ telling us to which of the trees in the packing it belongs. We accomplish this by letting $x_{ij} = k$, if and only if the edge $v_i v_j$ belongs to $T_k$, for $1 \leq k \leq n-1$.

As examples, consider $K_6$ and the sequence of trees $T_1', T_2', S_3', P_4', L_5$ (as in Example 3.1), and also $K_7$ and the sequence $T_1', T_2', P_3', S_4', L_5', P_6'$. In Figure 3.2 we show the arrays $X_6$ and $X_7$ which provide the desired packings for these sequences. Note that in both cases $X_n$ can be partitioned into two sub-arrays, one consisting of those entries which are odd, and the other made up of the even entries. Note also that the manner in which this partition is made depends on the parity of $n$.

In general, let $O_n$ denote the sub-array of $X_n$ consisting of those entries $x_{ij}$, where $1 \leq i \leq \lfloor n/2 \rfloor$.
and $i < j < m-i$. Here $m$ equals $n$ or $n+1$ depending on whether $n$ is odd or even, respectively. Let $M_n$ denote the remaining entries of $X_n$. In the proof of the next theorem we shall need to show that the entries of $O_n$ [resp. $M_n$] can be used to obtain all the trees $T_i$ where $i$ is odd [resp. even]. We are now able to provide an extension of the Zaks and Liu result, using their proof technique.

**THEOREM 3.1** Let $n$ be a positive integer, $n \geq 2$. For each $i$, $1 \leq i \leq n-1$, let $T_i \in \{P_i, S_i, L_i\}$. Then there exists a packing of $T_1, T_2, \ldots, T_{n-1}$ into $K_n$.

**Proof.** Let $O_n$ and $M_n$ be defined as above. For convenience of notation let $(i, j)$ denote $x_{ij}$, and let this refer not only to the entry of $X_n$, but to the corresponding edge of $K_n$ as well. Let the sequence
T1, T2, ..., Tn-1 be given, with \( T_i \in \{ P_i, S_i, L_i \} \) for \( 1 \leq i \leq n - 1 \). We wish to prove that \( T_1, T_3, \ldots, T_{2\lfloor n/2 \rfloor - 1} \) can be packed into \( O_n \) and that \( T_2, T_4, \ldots, T_{2\lfloor (n-1)/2 \rfloor} \) can be packed into \( M_n \). We proceed by induction on \( n \).

Clearly the desired packing can be found if \( n = 2 \) or \( n = 3 \). Assume that a packing can be found for \( n = m \); we wish to show that such a packing can be found when \( n = m+1 \).

**Case 1.** The integer \( m \) is even. In this case \( O_{m+1} \) and \( O_m \) are "isomorphic" in the sense that the graphs induced by the corresponding sets of edges are isomorphic, and also in the sense that they have exactly the same size and shape as arrays. Hence, by the induction hypothesis, we know that \( T_1, T_3, \ldots, T_{m-1} \) can be packed into \( O_{m+1} \). Now consider \( T_m \). If \( T_m = S_m \), we let

\[
T_m = \langle \{(i, m+1) \mid 1 \leq i \leq m\} \rangle.
\]

If \( T_m = P_m \), let

\[
T_m = \langle \{(1, m+1), (2, m+1), (2, m), (3, m), \ldots, (m/2, m/2 + 3), (m/2, m/2 + 2), (m/2 + 1, m/2 + 2)\} \rangle.
\]

Or, if \( T_m = L_m \), let

\[
T_m = \langle \{(1, m+1), (2, m), (3, m-1), \ldots, (m/2, m/2 + 2) \}
U \{(m/2 + 1, m/2 + 2), (m/2 + 2, m/2 + 3), \ldots, (m, m + 1)\} \rangle.
\]
In Figure 3.3 we show $T_m$ in the three cases. Note that when the entries of $T_m$ are removed from $M_{m+1}$, that which remains is isomorphic, in the above sense, to $M_m$. Now, by the induction hypothesis, $T_2, T_4, \ldots, T_{m-2}$ can be packed into $M_m$, and the proof is completed in this case.

![Diagram of Figure 3.3](image)

**FIGURE 3.3**

**Case 2.** Suppose that $m$ is odd. This case is very similar to case 1. In this case $M_{m+1}$ is isomorphic to $M_m$, and we need to place $T_m$ into $O_{m+1}$ in such a way that the remaining part of $O_{m+1}$ is isomorphic to $O_m$. This can be accomplished by letting

$$T_m = \langle [(1, i) \mid 2 \leq i \leq m+1] \rangle, \text{ if } T_m = S_m,$$

$$T_m = \langle [(1, m+1), (1, m), (2, m), (2, m-1), \ldots, \rangle,$$
((m-1, m+5), (m-1, m+3), (m+1, m+3)), if \( T_m = P_m \), and

\[ T_m = \langle \{(1, 2), (2, 3), \ldots, \frac{(m-1, m+1)}{2} \} \]

\[ \cup \{(\frac{m+1}{2}, m+3), (\frac{m-1}{2}, m+5), \ldots, (1, m+1)\} \]

if \( T_m = L_m \). See Figure 3.4.

By induction, this completes the proof.

---

Since for \( i \leq 4 \), any tree \( T_i \in \{S_i, P_i, L_i\} \), it follows from Theorem 3.1 that the TPC holds for \( K_n \), \( n \leq 5 \).

In their paper, Gyárfás and Lehel also showed that if each \( T_i \), with at most two exceptions, is a star,
then there exists a packing of $T_1, T_2, \ldots, T_n$ into $K_n$. A star of size $q$ is the unique tree $T_q$ of size $q$ having $\Delta(T_q) = q$. There is also a unique tree of size $q$ having maximum degree $q - 1$, namely $S_{q-1}(S_1)$. Moreover, there are three non-isomorphic trees of size $q$ having $\Delta = q - 2$, namely, $S_{q-2}(S_2), S_{q-2}(S_1, S_1)$, and $S_{q-2}(S_1(S_1))$. These are shown in Figure 3.5.

**FIGURE 3.5**

We next present two results which deal with the maximum degrees of the $T_i$. The first of these results extends the aforementioned theorem of Gyárfás and Lehel.

**THEOREM 3.2** Let $n$ be a positive integer, $n \geq 2$. For each $i$, $1 \leq i \leq n - 1$, let $T_i$ be a tree of size $i$, and suppose that for each $i$, with at most
two exceptions, $\Delta(T_i) \geq i - 1$. Then $T_1, T_2, \ldots, T_{n-1}$ can be packed into $K_n$.

**Proof.** Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. We proceed by induction on $n$. By the previous result we have that the TPC holds for $K_m$, $m \leq 5$. Hence we assume that the theorem holds for $m < n$, where $n$ is some integer, $n \geq 6$. Let $T_1, T_2, \ldots, T_{n-1}$ be given satisfying the hypothesis of the theorem. We now consider several cases.

**Case 1.** Suppose $T_{n-1}$ equals $S_{n-1}$. By the induction hypothesis, $T_1, T_2, \ldots, T_{n-2}$ can be packed into $K_{n-1} = K_n - v_n$. Letting $S_{n-1}$ equal $\langle v_n, v_i | 1 \leq i \leq n-1 \rangle$, we obtain a packing of $T_1, T_2, \ldots, T_{n-1}$ into $K_n$.

**Case 2.** Let $T_{n-1}$ equal $S_{n-2}(S_1)$. Again use the induction hypothesis to pack $T_1, T_2, \ldots, T_{n-2}$ into $K_n - v_n$. Assume, without loss of generality, that $T_1 = \langle v_1, v_2 \rangle$. We let $T_{n-1}$ equal $\langle v_1, v_2, v_n, v_i | 1 \leq i \leq n-2 \rangle$ and let $T_1$ equal $\langle v_n, v_{n-1} \rangle$. This produces the desired packing of $T_1, T_2, \ldots, T_{n-1}$ into $K_n$.

We may now assume that $\Delta(T_{n-1}) < n-2$, and so, by the hypothesis of the theorem, $\Delta(T_i) \geq i - 1$ for $1 \leq i \leq n - 2$, with at most one exception.

**Case 3.** Assume that $T_{n-2}$ equals $S_{n-2}$. Denote by $T'_{n-2}$ a tree obtained from $T_{n-1}$ by deleting an end vertex. Then the sequence $T_1, T_2, \ldots, T_{n-3}, T'_{n-2}$ satisfies the hypothesis of the theorem, and so, by the
induction hypothesis, this sequence can be packed into $K_n - v_n$. Now one of the edges of $K_n$ incident with $v_n$ can be used to extend $T_{n-2}'$ to $T_{n-1}$; the other edges incident with $v_n$ induce $S_{n-2}$.

**Case 4.** Suppose that $T_{n-2}'$ equals $S_{n-3}(S_1)$. Let $T_{n-2}$ be as in Case 3. Pack $T_1, T_2, \ldots, T_{n-3}, T_{n-2}'$ into $K_n - v_n$ and assume, without loss of generality, that $T_1 = \langle v_1, v_2 \rangle$ and that the edge used to extend $T_{n-2}'$ to $T_{n-1}$ is either $v_2v_n$ or $v_3v_n$. We may then let $T_{n-2}$ equal $\langle v_1, v_2, v_1v_n \cup \{v_i^*v_i \mid 4 \leq i \leq n-1\} \rangle$ and we may take $T_1$ to be whichever of the edges $v_3v_n$ or $v_2v_n$ is not used to extend $T_{n-2}'$ to $T_{n-1}$. This provides the desired packing of $T_1, T_2, \ldots, T_{n-1}$ into $K_n$.

**Case 5.** Suppose that $\Delta(T_{n-2}) < n-3$. In this case we know that $\Delta(T_i) \geq i-1$ for $1 \leq i \leq n-3$. Let $T_{n-4}'$ and $T_{n-3}'$ be obtained from $T_{n-2}$ and $T_{n-1}$, respectively, by removing from each two end vertices which are at a distance three or more from each other. We use the induction hypothesis to pack $T_1, T_2, \ldots, T_{n-5}, T_{n-4}', T_{n-3}'$ into $K_{n-2} = K_n - v_{n-1} - v_n$. It is clear that $T_{n-4}'$ and $T_{n-3}'$ can be extended to $T_{n-2}$ and $T_{n-1}$, respectively, by using four edges, two incident with each of $v_{n-1}$ and $v_n$. We now need to show that the remaining edges incident with $v_{n-1}$ and $v_n$ can be used to form $T_{n-4}$ and $T_{n-3}$. Let these edges be partitioned as $E \cup F$, where $E$ equals the set of
those edges incident with \( v_n \) and \( F \) equals the set incident with \( v_{n-1} \). Note that each of \( E \) and \( F \) contains \( n-4 \) edges. We also have use of the edge \( v_{n-1}v_n \). We now consider several subcases.

**Subcase 5A.** If \( T_{n-3} \) equals \( S_{n-3} \) and \( T_{n-4} \) equals \( S_{n-4} \), then we may let \( T_{n-3} = \langle E \cup \{v_{n-1}v_n\} \rangle \) and \( T_{n-4} = \langle F \rangle \).

**Subcase 5B.** Suppose \( T_{n-3} \) equals \( S_{n-4}(S_1) \) and \( T_{n-4} \) equals \( S_{n-4} \). If \( n \geq 7 \), then there exists a vertex \( v_j \) such that \( v_nv_j \in E \) and \( v_{n-1}v_j \in F \). In this case we let \( T_{n-3} = \langle E \cup \{v_{n-1}v_j\} \rangle \) and \( T_{n-4} = \langle (F - \{v_{n-1}v_j\}) \cup \{v_{n-1}v_j\} \rangle \). However, if \( n \) equals 6, it follows that our sequence of trees is \( T_1, T_2, P_3, P_4, T_5 \), where \( T_5 \in \{P_5, L_5, S_3(S_1(S_1)), S_3(S_2)\} \). If \( T_5 \) equals \( P_5 \) or \( L_5 \), we can apply Theorem 3.1 to obtain a packing.

Finally, for the cases where \( T_5 \) equals \( S_3(S_1(S_1)) \) or \( S_3(S_2) \), we show packings below. (Recall that if the entry of the array corresponding to \( v_i \) and \( v_j \) is \( k \), then the edge \( v_iv_j \) is to be included in \( T_k, 1 \leq k \leq 5 \).)

\[
\begin{array}{cccccccc}
  v_2 & v_3 & v_4 & v_5 & v_6 \\
  v_1 & 3 & 1 & 3 & 5 & 5 \\
  v_2 & 5 & 5 & 5 & 4 \\
  v_3 & 3 & 4 & 4 \\
  v_4 & 4 & 2 \\
  v_5 & 2 \\
  T_1, T_2, P_3, P_4, S_3(S_1(S_1)) \\
\end{array}
\quad
\begin{array}{cccccccc}
  v_2 & v_3 & v_4 & v_5 & v_6 \\
  v_1 & 5 & 5 & 3 & 3 & 5 \\
  v_2 & 1 & 5 & 5 & 4 \\
  v_3 & 3 & 4 & 4 \\
  v_4 & 4 & 2 \\
  v_5 & 2 \\
  T_1, T_2, P_3, P_4, S_3(S_2) \\
\end{array}
\]

**FIGURE 3.6**

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To avoid Subcases 5A and 5B, we assume below that \( n \geq 7 \).

**Subcase 5C.** Suppose \( T_{n-3} = S_{n-3} \) and \( T_{n-4} = S_{n-5}(S_1) \). Assume, again without loss of generality, that \( T_1 \) equals \( \langle v_1, v_2 \rangle \). First of all, suppose that either \( v_1 \) or \( v_2 \) is incident with an edge of \( E \cup F \), say \( v_1 v_{n-1} \in F \). We then let \( T_{n-3} \) equal \( \langle E \cup \{v_n v_{n-1}\} \rangle \), \( T_{n-4} \) equal \( \langle (F \cup \{v_1 v_2\}) - \{v_{n-1} v_j\} \rangle \), and \( T_1 = \langle v_{n-1} v_j \rangle \), where \( v_{n-1} v_j \in F \) and \( j \neq 1 \). On the other hand, if neither \( v_1 \) nor \( v_2 \) is incident with an edge of \( E \cup F \), then \( v_n v_1, v_n v_2, v_{n-1} v_1, \) and \( v_{n-1} v_2 \) are the edges used to extend \( T_{n-4}' \) and \( T_{n-3}' \) to \( T_{n-2} \) and \( T_{n-1} \). Now consider \( T_2 \). If either edge of \( T_2 \) is incident with \( v_1 \) or \( v_2 \), then by exchanging an edge of \( T_2 \) with \( T_1 \), we could obtain the previous situation. Hence we may assume that \( T_2 \) equals \( \langle v_3 v_4, v_4 v_5 \rangle \). We then let \( T_{n-3} \) equal \( \langle E \cup \{v_n v_{n-1}\} \rangle \), \( T_{n-4} \) equal \( \langle (F \cup \{v_3 v_4\}) - \{v_{n-1} v_4\} \rangle \) and \( T_2 \) equal \( \langle v_{n-1} v_4, v_4 v_5 \rangle \).

**Subcase 5D.** Lastly we may assume that \( T_{n-3} \) equals \( S_{n-4}(S_1) \) and \( T_{n-4} \) equals \( S_{n-5}(S_1) \). Since \( n \geq 7 \) there exists a vertex, say \( v_{n-2} \), such that \( v_n v_{n-2} \in E \) and \( v_{n-1} v_{n-2} \in F \). Suppose there exists another vertex \( v_j \) with \( v_{n-1} v_j \in F \) and \( v_j v_n \not\in E \). We can then let \( T_{n-3} \) equal \( \langle (E \cup \{v_n v_{n-1}, v_{n-1} v_j\}) - \{v_n v_{n-2}\} \rangle \) and \( T_{n-4} \) equal \( \langle (F \cup \{v_n v_{n-2}\}) - \{v_{n-1} v_j\} \rangle \). However, if such a vertex \( v_j \) does not exist, then we may assume, as in subcase 5C, that \( v_n v_1, v_n v_2, v_{n-1} v_1, \) and \( v_{n-1} v_2 \).
are the edges used to extend \( T_{n-4} \) and \( T_{n-3} \). Again consider \( T_1 \), which we can assume equals \( \langle v_j v_{j+1} \rangle \), where \( j = 1, 2, \) or \( 3 \). If \( j = 2 \) we let \( T_{n-3} \) equal 
\( \langle (E \cup \{ v_n v_{n-1}, v_2 v_3 \}) - \{ v_n v_{n-2} \} \rangle \), \( T_{n-4} \) equal 
\( \langle (F \cup \{ v_n v_{n-2} \}) - \{ v_{n-1} v_3 \} \rangle \), and \( T_1 \) equal \( \langle v_{n-1} v_3 \rangle \).

If \( j = 3 \) let \( T_{n-3} \) equal \( \langle (E \cup \{ v_n v_{n-1}, v_3 v_4 \}) - \{ v_n v_4 \} \rangle \), \( T_{n-4} \) equal \( \langle (F \cup \{ v_n v_4 \}) - \{ v_{n-1} v_3 \} \rangle \), and \( T_1 \) equal \( \langle v_{n-1} v_3 \rangle \). However, if \( j = 1 \), we may assume, as in subcase 5C, that \( T_2 \) equals \( \langle v_3 v_4, v_4 v_5 \rangle \). In this case we may let \( T_{n-3} \) equal \( \langle (E \cup \{ v_n v_{n-1}, v_3 v_4 \}) - \{ v_n v_4 \} \rangle \), \( T_{n-4} \) equal \( \langle (F \cup \{ v_n v_4 \}) - \{ v_{n-1} v_5 \} \rangle \), and \( T_2 \) equal \( \langle v_4 v_5, v_{n-1} v_5 \rangle \).

Thus in all cases we have found the desired packing of \( T_1, T_2, \ldots, T_{n-1} \) into \( K_n \). By induction the proof is completed. ■

Since \( \Delta(T_i) \geq i - 1 \) whenever \( i \leq 3 \), it follows from Theorem 3.2 that the TPC is true for \( K_6 \). We next present a result which will help us settle the conjecture in the case \( n \) equals 7.

**Theorem 3.3.** Let \( n \) be a positive integer, \( n \geq 2 \). For each \( i, 1 \leq i \leq n - 1 \), let \( T_i \) be a tree of size \( i \) and suppose that for each \( i \), with at most one exception, \( \Delta(T_i) \geq i - 2 \). Then \( T_1, T_2, \ldots, T_{n-1} \) can be packed into \( K_n \).
Proof. We proceed by induction on \( n \). The result holds for \( n \leq 6 \) by the previous theorem. Thus we assume that the result holds for all integers greater than 1 and less than \( n \), where \( n \) is some integer, \( n \geq 7 \). Let \( T_1, T_2, \ldots, T_{n-1} \) be given satisfying the hypothesis of the theorem, let \( V(K_n) = \{v_1, v_2, \ldots, v_n\} \) and let \( E \) denote the set of edges incident with \( v_n \). We break the proof down into several cases.

Case 1. If \( T_{n-1} \) equals \( S_{n-1} \), we use the induction hypothesis to pack \( T_1, T_2, \ldots, T_{n-2} \) into \( K_{n-1} = K_n - v_n \) and let \( T_{n-1} \) equal \( \langle E \rangle \).

Case 2. Suppose \( T_{n-1} \) equals \( S_{n-2}(S_1) \). Again use the induction hypothesis to pack \( T_1, T_2, \ldots, T_{n-2} \) into \( K_n - v_n \) and assume, without loss of generality, that \( T_1 \) equals \( \langle v_1 v_2 \rangle \). Letting \( T_{n-1} \) equal

\[
\langle (E \cup \{v_1 v_2\}) - \{v_n v_2\} \rangle
\]

and \( T_1 \) equal \( \langle v_n v_2 \rangle \), we obtain a packing of \( T_1, T_2, \ldots, T_{n-1} \) into \( K_n \).

Case 3. Suppose \( T_{n-1} \) equals \( S_{n-3}(S_1(S_1)) \). We pack \( T_1, T_2, \ldots, T_{n-2} \) into \( K_n - v_n \) and assume, without loss of generality, that \( T_2 \) equals \( \langle v_1 v_2, v_2 v_3 \rangle \). We then let \( T_{n-1} \) equal \( \langle (E \cup \{v_1 v_2, v_2 v_3\}) - \{v_n v_2, v_n v_3\} \rangle \) and \( T_2 \) equal \( \langle v_n v_2, v_n v_3 \rangle \), to obtain the desired packing.

Case 4. If \( T_{n-1} \) equals \( S_{n-3}(S_2) \), pack \( T_1, T_2, \ldots, T_{n-2} \) into \( K_n - v_n \) as in Case 3. Let \( T_{n-1} \)
equal \((E \cup \{v_1v_2, v_2v_3\}) - \{v_nv_1, v_nv_3\}\) and \(T_2\) equal \(\langle v_nv_1, v_nv_3 \rangle\).

**Case 5.** Suppose \(T_{n-1}\) equals \(S_{n-3}(S_1, S_1)\). Again we pack \(T_1, T_2, \ldots, T_{n-2}\) into \(K_n - v_n\). The case where \(T_{n-2}\) equals \(S_{n-2}\) is handled in Case 6, thus we assume here that \(T_{n-2} \neq S_{n-2}\). First of all, suppose \(T_3 = P_3 = \langle v_1v_2, v_2v_3, v_3v_4 \rangle\). Then we let \(T_{n-1}\) equal \((E \cup \{v_1v_2, v_3v_4\}) - \{v_nv_2, v_nv_4\}\) and \(P_3\) equal \(\langle v_3v_2, v_2v_n, v_nv_4 \rangle\). Secondly, suppose \(T_3 = S_3 = \langle v_1v_2, v_1v_3, v_1v_4 \rangle\). Since \(T_{n-2} \neq S_{n-2}\), there exists an edge of \(T_{n-2}\) not incident with \(v_1\). Suppose this edge is \(v_iv_{i+1}\), \(i \geq 3\), where \(v_{i+1}\) is the end vertex.

We may then let \(T_{n-1}\) equal \((E \cup \{v_1v_2, v_i v_{i+1}\}) - \{v_nv_1, v_nv_i\}\), \(S_3\) equal \(\langle v_nv_1, v_3v_1, v_4v_1 \rangle\), and use \(v_nv_i\) as the new end edge of \(T_{n-2}\). We thus obtain a packing of \(T_1, T_2, \ldots, T_{n-1}\) into \(K_n\).

We henceforth assume that \(\Delta(T_{n-1}) \leq n - 4\) and so, by the hypothesis of the theorem, \(\Delta(T_i) \geq i - 2\) for \(1 \leq i \leq n - 2\). In the remaining cases, let \(T'_{n-2}\) be a tree obtained by deleting an end vertex from \(T_{n-1}\). We use the induction hypothesis to pack \(T_1', T_2', \ldots, T_{n-3}', T'_{n-2}\) into \(K_n - v_n\).

**Case 6.** If \(T_{n-2}\) equals \(S_{n-2}'\), we use one of the edges of \(E\) to extend \(T_{n-2}\) to \(T_{n-1}\) and the remaining edges of \(E\) will induce \(S_{n-2}\). To handle the situation where \(T_{n-1} = S_{n-3}(S_1, S_1)\), let \(T'_{n-2} = S_{n-3}(S_1)\).
Case 7. Suppose that $T_{n-2}$ equals $S_{n-3}(S_1)$. We pack $T_1, T_2, \ldots, T_{n-3}, T'_{n-2}$ into $K_n - v_n$ and assume, without loss of generality, that $T_1$ equals $\langle v_1v_2 \rangle$ and that we use the edge $v_nv_j$ to extend $T'_{n-2}$ to $T_{n-1}$, where $j$ equals 2 or 3. Letting $T_{n-2}$ equal
\[
\langle (E \cup \{v_1v_2\}) - \{v_nv_j, v_nv_k\}\rangle
\]
and $T_1$ equal $\langle v_nv_k \rangle$, where $k \in \{2, 3\}$ and $k \neq j$, we obtain a packing of $T_1, T_2, \ldots, T_{n-1}$ into $K_n$.

Case 8. Suppose that $T_{n-2}$ equals $S_{n-4}(S_1(S_1))$. Pack $T_1, T_2, \ldots, T_{n-3}, T'_{n-2}$ into $K_n - v_n$ as above. Assume that $T_2 = \langle v_1v_2, v_2v_3 \rangle$ and that $v_nv_j$ is the edge used to extend $T'_{n-2}$ to $T_{n-1}$, where $j \in \{2, 3, 4\}$. Let $k, m \in \{2, 3, 4\}$ where $m \neq k \neq j \neq m$. We let $T_{n-2}$ equal $\langle (E \cup \{v_1v_2, v_2v_3\}) - \{v_nv_j, v_nv_k, v_nv_m\}\rangle$ and $T_2$ equal $\langle v_nv_k, v_nv_m \rangle$, thus obtaining the desired packing.

Case 9. Suppose $T_{n-2}$ equals $S_{n-4}(S_2)$. Pack $T_1, T_2, \ldots, T_{n-3}, T'_{n-2}$ into $K_n - v_n$ and assume that $T_2$ equals $\langle v_1v_2, v_2v_3 \rangle$. We consider several subcases.

Subcase 9A. Suppose that $v_2v_n$ is not the edge used to extend $T'_{n-2}$ to $T_{n-1}$. We may then assume that $v_nv_j$ is the edge used, where $j$ equals 3 or 4. Let $k$ equal 3 or 4 and $k$ not equal $j$. We may then let $T_{n-2}$ equal $\langle (E \cup \{v_1v_2, v_2v_3\}) - \{v_nv_1, v_nv_j, v_nv_k\}\rangle$ and $T_2$ equal $\langle v_nv_1, v_nv_k \rangle$. 

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Assume therefore, in the remaining subcases, that $v_2v_n$ is the edge that must be used to extend $T_{n-2}$ to $T_{n-1}$. We wish to argue that one of Subcases 9B-9E holds.

**Subcase 9B.** Suppose the edge used for $T_1$ is incident with $v_1$ or $v_3$. Without loss of generality, we assume that $T_1 = \langle v_3v_j \rangle$, $j \neq 2$. If $j \neq 1$, let

$$T_{n-2} = \langle (E \cup \{v_2v_3, v_3v_j\}) - \{v_nv_1, v_nv_2, v_nv_j\} \rangle,$$

$$T_2 = \langle v_nv_1, v_1v_2 \rangle,$$

and $T_1 = \langle v_nv_j \rangle$. If $j = 1$, let

$$T_{n-2} = \langle (E \cup \{v_1v_2, v_1v_3\}) - \{v_nv_2, v_nv_3, v_nv_4\} \rangle,$$

$$T_2 = \langle v_nv_3, v_3v_2 \rangle,$$

and $T_1 = \langle v_nv_4 \rangle$. In either instance, we obtain a packing of $T_1, T_2, \ldots, T_{n-1}$ into $K_n$.

**Subcase 9C.** Suppose that $v_1$ or $v_3$ is an end vertex of $T_{n-3}$. We may assume that $v_3$ is the end vertex, and that $v_jv_3$ is the end edge of $T_{n-3}$, $j \neq 2$. If $j \neq 1$, we may use the same construction as in Subcase 9B, except that $v_nv_j$ will be used as the new end edge of $T_{n-3}$.

So suppose $j = 1$. Consider again $T_1$. We may assume that $T_1$ is not incident with $v_1$ or $v_3$, and so

$$T_1 = \langle v_4v_m \rangle$$

where $m$ does not equal 1 or 3. We can then let

$$T_{n-2} = \langle (E \cup \{v_2v_3, v_1v_3\}) - \{v_nv_1, v_nv_2, v_nv_4\} \rangle,$$

$$T_2 = \langle v_nv_4, v_4v_m \rangle,$$

and $T_1 = \langle v_1v_2 \rangle$, and let $v_1v_n$ be the new end edge of $T_{n-3}$. This gives a packing of $T_1, T_2, \ldots, T_{n-1}$ into $K_n$.

**Subcase 9D.** Suppose that $T_{n-3}$ contains a path of length two, say $v_i v_j v_k$, where $v_j$ has degree 2 in $T_{n-3}$, and neither $v_i$ nor $v_j$ is $v_2$. If $k \neq 2$,
then letting $T_{n-2}$ equal
\[
\langle (E \cup \{v_i v_j, v_j v_k\}) - \{v_n v_i, v_n v_k, v_n v_2\}\rangle
\]
and $v_i v_n v_k$ be the new path of length two in $T_{n-3}$, we obtain the desired packing. Thus we suppose that $v_i v_j v_2$ is the path of length two in $T_{n-3}$. If $v_2$ is an end vertex of $T_{n-3}$, we let $T_{n-2}$ equal
\[
\langle (E \cup \{v_i v_j, v_j v_2\}) - \{v_n v_i, v_n v_2, v_n v_t\}\rangle,
\]
and let $v_i v_n v_t$ be the new end path of length two for $T_{n-3}$, where $v_t$ is the vertex of $K_n - v_n$ not used in $T_{n-3}$. Finally, we may assume that $v_2$ has degree 2 in $T_{n-3}$ and let $v_m$ denote the end vertex of $T_{n-3}$ adjacent with $v_2$. In this case we let $T_{n-2}$ equal
\[
\langle (E \cup \{v_i v_j, v_j v_2\}) - \{v_n v_i, v_n v_2, v_n v_m\}\rangle,
\]
and let $v_i v_n v_m v_2$ be the new end path of length 3 for $T_{n-3}$.

**Subcase 9E.** Suppose $T_1 = \langle v_b v_c \rangle$, $v_b$ is an end vertex of $T_{n-3}$, with $v_a$ the vertex adjacent to it, and neither $v_a$ nor $v_b$ is $v_2$. Also let $v_d$ be another vertex of $K_n - v_n$, where $v_d = v_2$ if $v_c \neq v_2$. We may then let $T_{n-2}$ equal
\[
\langle (E \cup \{v_a v_c, v_b v_a\}) - \{v_n v_a, v_n v_c, v_n v_d\}\rangle
\]
use $v_n v_a$ as the new end edge of $T_{n-3}$, and let $T_1$ be either $v_n v_c$ or $v_n v_d$. This again provides a packing of $T_1, T_2, \ldots, T_{n-1}$ into $K_n$.

To complete the proof in Case 9, we claim that if $v_2 v_n$ is the edge used to extend $T_{n-2}$ to $T_{n-1}$ then one
of Subcases 9 B-E must hold. We now verify this claim for each of the five possible trees which $T_{n-3}$ may be. We may assume that $v_1$ is used in $T_{n-3}$, for otherwise $v_3$ is, since $T_{n-3}$ contains $n-2$ of the vertices of $K_n-v_n$.

Case 9'.1 Suppose $T_{n-3}$ equals $S_{n-3}$ as in Figure 3.7a. The vertex $v_1$ must be as shown or it will be an end vertex of $T_{n-3}$ (Subcase 9C). But then $v_3$ must be as shown, since $v_2$ cannot be adjacent to $v_1$ in $T_{n-3}$. Thus we cannot avoid Subcase 9C.

Case 9'.2 Suppose $T_{n-3}$ equals $S_{n-4}(S_1)$ as shown in Figure 3.7b. If we are to avoid Subcase 9C, then $v_1 \in \{v_i, v_j\}$. But $v_2 \not\in \{v_i, v_j\}$ and we have Subcase 9D.

Case 9'.3 Suppose $T_{n-3}$ equals $S_{n-5}(S_1(S_1))$ as in Figure 3.7c. Again to avoid Subcase 9C, $v_1 \in \{v_i, v_i, v_j\}$. For example, if $v_1$ equals $v_i$ then $v_2$ cannot equal $v_j$, and Subcase 9D applies. We reach the same conclusion if $v_1$ is $v_h$ or $v_j$.

Case 9'.4 Suppose $T_{n-3}$ equals $S_{n-5}(S_1, S_1)$ as in Figure 3.7d. To avoid Subcase 9C, $v_1 \in \{v_i, v_j, v_j\}$. In any event we may apply Subcase 9D.

Case 9'.5 Suppose $T_{n-3}$ equals $S_{n-5}(S_2)$. In order not to have Subcase 9C, the locations of $v_1$, $v_2$, and $v_3$ with respect to $T_{n-3}$ must be as shown in one of Figures 3.7 e-g. Because of Subcase 9B, $T_1$ cannot be incident with $v_1$ or $v_3$. But then it is easy to see that Subcase
$9E$ cannot be avoided.

This completes the proof in Case 9.

**FIGURE 3.7**

**Case 10.** In this last case $T_{n-2}$ equals $S_{n-4}(S_1, S_1)$.

Pack $T_1, T_2, \ldots, T_{n-3}, T_{n-2}$ into $K_{n-1}$ as before.

We again divide this case into several subcases.

**Subcase 10A.** Suppose that $T_3 = P_3 = \langle v_1v_2, v_2v_3, v_3v_4 \rangle$.

Assume, without loss of generality, that $v_nv_j$ is the
edge used to extend $T'_{n-2}$ to $T_{n-1}$, where $j \in \{3, 4, 5\}$. If $j$ equals 3 or 4 we may let $T_{n-2}$ equal
$$\langle (E \cup \{v_1 v_2, v_3 v_4\}) - \{v_n v_2, v_n v_j, v_n v_5\} \rangle$$
and $P_3$ equal $\langle v_2 v_3, v_n v_2, v_n v_5 \rangle$. Or, if $j$ equals 5, we then let $T_{n-2}$ equal
$$\langle (E \cup \{v_1 v_2, v_3 v_4\}) - \{v_n v_2, v_n v_4, v_n v_5\} \rangle$$
and $P_3$ equal $\langle v_2 v_3, v_n v_2, v_n v_4 \rangle$. In either case we obtain a packing of $T_1, T_2, \ldots, T_{n-1}$ into $K_n$.

We may assume from now on that $T_3 = S_3$.

**Subcase 10B.** Suppose $T_{n-3}$ equals $S_{n-3}$. Let $T'_{n-3}$ equal $S_{n-4}(S_1)$. Use the induction hypothesis to pack $T_1, T_2, \ldots, T_{n-4}, T'_{n-3}, T'_{n-2}$ into $K_n - v_n$. Since there exist at least two vertices from which $T'_{n-3}$ may be extended to $T_{n-2}$, we can take two distinct edges incident with $v_n$ to extend $T'_{n-3}$ and $T'_{n-2}$ to $T_{n-2}$ and $T_{n-1}$, respectively. The remaining edges incident with $v_n$ induce $S_{n-3}$.

Henceforth, we shall assume that $T_{n-3} \neq S_{n-3}$.

**Subcase 10C.** Suppose that $v_j v_n$ is the edge used to extend $T'_{n-2}$ to $T_{n-1}$. We may assume, without loss of generality, that $T_1 = \langle v_1 v_2 \rangle$ and that $j \neq 2$. Our plan is to find some $T_i, 2 \leq i \leq n - 3$, with an end edge $v_x v_y$ ($v_y$ being the end vertex) satisfying the following conditions:

1) $v_x v_y$ is independent of $v_1 v_2$, and
2) $j \notin \{x, y\}$. 

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For then we may use $v_x v_n$ as the new end edge of $T_1$, let $T_{n-2}$ equal $(E \cup \{v_1 v_2, v_x v_y\}) - \{v_n v_j, v_n v_x, v_n v_k\}$ and let $T_1$ equal $(v_n v_k)$, where $k \notin \{2, j, x, y\}$. If we find such a $T_1$, call this situation $\ast$. We will also be done if we find two distinct trees, $T_a$ and $T_b$, $2 \leq a < b \leq n - 3$, having end edges $v_x v_y$ and $v_w v_z$, respectively, such that $v_x v_y$ and $v_w v_z$ are independent and neither is incident with $v_j$. (Here $v_y$ and $v_z$ are the end vertices). For then we let $T_{n-2}$ equal
\[(E \cup \{v_x v_y, v_w v_z\}) - \{v_n v_x, v_n v_w, v_n v_j\}\]
and $v_n v_x$ and $v_n v_w$ be the new end edges of $T_a$ and $T_b$, respectively. Call this second situation $\ast\ast$. We now consider two cases depending on the value of $j$.

**Subcase 10C.1.** Suppose $j = 1$. Since $T_{n-3} \neq S_{n-3}$, we can find an end edge of $T_{n-3}$ not incident with $v_1$. This end edge must be incident with $v_2$ or we have $\ast$. We may assume this edge is $v_2 v_3$. Now consider $T_2$. Both edges of $T_2$ must be incident with $v_1$ or $v_2$ or $\ast$ happens, but if one of them is not incident with $v_2$, then there exists an edge of $T_2$ not incident with either $v_2$ or $v_3$. If we let this edge be $T_1$ and use $v_1 v_2$ as its replacement in $T_2$, we obtain situation $\ast$. It must therefore be the case that both edges of $T_2$ are incident with $v_2$. Similarly, one can argue that $v_2$ must be the degree 3 vertex of $T_3 = S_3$. Thus $n \geq 9$. 

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Let $v_k$ be the vertex of maximum degree in $T_{n-3}$. Since the degree of $v_2$ in $T_{n-3}$ can be at most $n - 9$ and the degree of $v_k$ in $T_{n-3}$ is at least $n - 5$, $k$ cannot equal 2. Hence there are at least three end edges of $T_{n-3}$ not incident with $v_2$. If any one of them is also not incident with $v_1$, we obtain situation $\ast$, so we may assume otherwise. Thus all three of these end edges must be incident with $v_1$, and it follows that $k = 1$.

Finally, consider $T_{n-4}$. Since $\Delta(T_{n-4}) \geq n - 6$ and $v_1$ has degree at most $n - 7$ in $T_{n-4}$, the vertex of maximum degree in $T_{n-4}$ is neither $v_1$ nor $v_2$. Therefore, there exists an end edge of $T_{n-4}$ independent of $v_1v_2$, giving us situation $\ast$.

Subcase 10C.2. Suppose $j \neq 1$. (Recall that we have previously assumed $j \neq 2$.) Since $T_{n-3} \neq S_{n-3}$, there exists an end edge of $T_{n-3}$ not incident with $v_j$. If this edge is also independent of $T_1 = \langle v_1v_2 \rangle$, we have situation $\ast$. Thus we may assume that this end edge of $T_{n-3}$ is $v_2v_3$. Now consider $T_2$. If both edges of $T_2$ are independent of $v_1v_2$, then we may assume, without loss of generality, that $T_2 = \langle v_mv_{m+1}, v_{m+1}v_{m+2} \rangle$, where $m \geq 3$ and $j \neq m$. We then let $T_{n-2}$ equal 
\[
\langle (E \cup \{v_1v_2, v_mv_{m+1}\}) - \{v_nv_2, v_nv_j, v_nv_k\} \rangle,
\]
$T_1$ equal $\langle v_nv_2 \rangle$, and $T_2$ equal $\langle v_{m+1}v_{m+2}, v_nv_k \rangle$. Here $k = m + 1$ if $j \neq m + 1$, otherwise $k = m + 2$. 

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Secondly, if exactly one of $v_1$ and $v_2$ is an end vertex of $T_2$, then the other edge of $T_2$ is independent of $T_1$. That is, unless $*$ occurs, one of its incident vertices is $v_j$. Letting this edge be $T_1$ and replacing it in $T_2$ by the edge $v_1v_2$, we can reduce this case to Subcase 10C.1. Also, if $T_2 = \langle v_1v_m, v_mv_2 \rangle$ for some $m > 3$, let $T_1 = \langle v_1v_m \rangle$ and $T_2 = \langle v_1v_2, v_mv_2 \rangle$. Then $v_2v_3$ is independent of $T_1$ and we have $*$. Lastly, then, either $v_1$ or $v_2$ has degree 2 in $T_2$. If $v_1$ has degree 2 in $T_2$, we again switch edges of $T_1$ and $T_2$ to obtain situation $*$. Thus we may assume that $T_2 = \langle v_2v_4, v_2v_5 \rangle$.

Next consider $T_3$, which equals $S_3$. First of all, if one of the vertices of $T_3$ is $v_j$, then by exchanging edges between $T_1$ and $T_2$ and $T_3$ we can obtain $v_j$ incident with $T_1$, which is just Subcase 10C.1. So assume that $v_j$ is incident with no edge of $T_3$. Then $v_2$ must be the vertex of degree 3 in $T_3$, for otherwise an edge of $T_3$ will be independent of $T_1$ or an edge of $T_2$, giving us situation $*$ or $**$. Hence we may assume that $T_3 = \langle v_2v_6, v_2v_7, v_2v_8 \rangle$. Note that we may use any of the edges $v_1v_2$ or $v_2v_m$, $4 \leq m \leq 8$, as $T_1$, and so in order to avoid being in Subcase 10C.1, $j \notin \{1, \ldots, 8\}$. Hence $n \geq 10$. 

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Now we need only find an edge of some $T_i$, $4 \leq i \leq n - 3$, not incident with either $v_j$ or $v_2$. Let $v_k$ be the vertex of maximum degree in $T_{n-3}$. Since the degree of $v_2$ in $T_{n-3}$ is at most $n - 9$, $k$ cannot equal 2. Hence there are at least four end edges of $T_{n-3}$ not incident with $v_2$. If any of these is also not incident with $v_j$, which must happen if any two are independent, we're done. In particular, then, we may assume $T_{n-3} \neq S_{n-5}(S_1, S_1)$ and $j = k$.

Finally, consider $T_{n-4}$. Since $\Delta(T_{n-4}) \geq n - 6$ and the degree of $v_j$ in $T_{n-4}$ is at most $n - 7$, the vertex of maximum degree in $T_{n-4}$ is neither $v_2$ nor $v_j$. Hence there exists an end edge of $T_{n-4}$ not incident with either $v_j$ or $v_2$, as we wished to show.

Therefore, we have shown in all cases that a packing of the given sequence $T_1, T_2, \ldots, T_{n-1}$ into $K_n$ can be found. By induction this completes the proof. ■

Corollary 3.1. The Tree Packing Conjecture holds for the complete graph $K_7$.

Proof. Let $T_1, T_2, \ldots, T_6$ be given.

If $T_4 \neq P_4$, then the condition $\Delta(T_i) \geq i - 1$ is satisfied for $i \leq 4$. Thus, Theorem 3.2 guarantees that a packing of $T_1, T_2, \ldots, T_6$ into $K_7$ exists. We assume, therefore, that $T_4$ equals $P_4$. 

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Next consider $T_5$. If $T_5 \neq P_5$, then the condition
\[ \Delta(T_i) \geq i - 2 \] holds whenever $i \leq 5$. Theorem 3.3
guarantees the existence of a packing of $T_1, T_2, \ldots, T_6$
into $K_7$ in this case. Thus we may also assume that
$T_5 = P_5$.

Now consider $T_6$. If $T_6 \in \{S_6, P_6, L_6\}$, we can
find a packing by Theorem 3.1. Also if $\Delta(T_6) \geq 4$,
then $\Delta(T_i) \geq i - 2$ for $i \neq 5$, and we may again apply
Theorem 3.3. Therefore, there are eight cases left to
consider: $T_1, T_2, T_3, P_4, P_5, T_6$, where $T_3 \in \{S_3, P_3\}$ and
$T_6 \in \{S_2(S_2, S_1(S_1)), S_2(S_2, S_2), S_3(S_1, S_1, S_1), S_3(S_1, S_1, S_1)\}$.

We exhibit packings for these eight sequences in Figures
3.8a-h. (Recall that when the entry of the table corres-
ponding to $v_i$ and $v_j$ is $m$, $1 \leq m \leq 6$, then the
edge $v_iv_j$ is to be included in $T_m$.

\[
\begin{array}{ccccccc}
  & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \\
v_1 & 3 & 3 & 3 & 5 & 5 & 6 \\
v_2 & 1 & 5 & 5 & 6 & 4 & \\
v_3 & & 5 & 4 & 6 & 2 & \\
v_4 & & & 6 & 6 & 2 & \\
v_5 & & & & 4 & 6 & \\
v_6 & & & & & 4 & \\
\end{array}
\]

a. $T_1, T_2, S_3, P_4, P_5, S_2(S_2, S_1(S_1))$

FIGURE 3.8
\begin{table}
\begin{tabular}{cccccccc}
\hline
 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \\
\hline
v_1 & 1 & 3 & 3 & 5 & 5 & 5 & 6 \\
v_2 & 3 & 5 & 5 & 6 & 4 &  \\
v_3 & 5 & 4 & 6 & 2 &  \\
v_4 & 6 & 6 & 2 &  \\
v_5 & 4 & 6 &  \\
v_6 &  \\n\hline
\end{tabular}
\end{table}

b. T_1, T_2, P_3, P_4, P_5, S_2(S_2, S_1(S_1))
\begin{table}
\begin{tabular}{cccccccc}
\hline
 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \\
\hline
v_1 & 3 & 3 & 3 & 5 & 5 & 5 & 6 \\
v_2 & 1 & 5 & 5 & 4 & 6 &  \\
v_3 & 5 & 4 & 6 & 6 &  \\
v_4 & 2 & 6 & 2 &  \\
v_5 & 6 & 4 &  \\
v_6 & 4 &  \\n\hline
\end{tabular}
\end{table}

c. T_1, T_2, S_3, P_4, P_5, S_2(S_2, S_2)
\begin{table}
\begin{tabular}{cccccccc}
\hline
 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \\
\hline
v_1 & 1 & 3 & 3 & 5 & 5 & 5 & 6 \\
v_2 & 3 & 5 & 5 & 4 & 6 &  \\
v_3 & 5 & 4 & 6 & 6 &  \\
v_4 & 2 & 6 & 2 &  \\
v_5 & 6 & 4 &  \\
v_6 & 4 &  \\n\hline
\end{tabular}
\end{table}

d. T_1, T_2, P_3, P_4, P_5, S_2(S_2, S_2)
\begin{table}
\begin{tabular}{cccccccc}
\hline
 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \\
\hline
v_1 &  \\
v_2 &  \\
v_3 &  \\
v_4 &  \\
v_5 &  \\
v_6 &  \\n\hline
\end{tabular}
\end{table}

FIGURE 3.8 (cont.)

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\[ \begin{array}{cccccc}
& v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \\
v_1 & 3 & 3 & 3 & 5 & 5 & 6 \\
v_2 & 1 & 5 & 5 & 6 & 6 & \\
v_3 & 5 & 6 & 4 & 6 & \\
v_4 & 4 & 6 & 4 & \\
v_5 & 2 & 2 & \\
v_6 & & & & 4 & \\
\end{array} \]

e. \(T_1, T_2, s_3, p_4, p_5, s_3(s_1, s_1(s_1))\)

\[ \begin{array}{cccccc}
& v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \\
v_1 & 1 & 3 & 3 & 5 & 5 & 6 \\
v_2 & 3 & 5 & 5 & 6 & 6 & \\
v_3 & 5 & 6 & 4 & 6 & \\
v_4 & 4 & 6 & 4 & \\
v_5 & 2 & 2 & \\
v_6 & & & & 4 & \\
\end{array} \]

f. \(T_1, T_2, s_3, p_4, p_5, s_3(s_1, s_1(s_1))\)

\[ \begin{array}{cccccc}
& v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \\
v_1 & 3 & 3 & 3 & 5 & 5 & 6 \\
v_2 & 1 & 5 & 5 & 6 & 4 & \\
v_3 & 5 & 6 & 4 & 2 & \\
v_4 & & 6 & 6 & 6 & \\
v_5 & & 4 & 4 & & \\
v_6 & & & 2 & & \\
\end{array} \]

g. \(T_1, T_2, s_3, p_4, p_5, s_3(s_1, s_1, s_1)\)

FIGURE 3.8 (cont.)
Note that for each of the packings we have thus far exhibited, namely, those of Theorem 3.1, Figure 3.6 and Figure 3.8, it is true that the trees of odd size can be packed into the subgraph of $K_n$ induced by the set of edges

$$\{v_i v_j \mid 1 \leq i < j \leq m - i\},$$

where $m$ equals $n$ or $n + 1$ depending on whether $n$ is odd or even, respectively. Can this always be done?

**CONJECTURE 3.2** Let $n$ be a positive integer, $n \geq 2$. For each $i$, $1 \leq i \leq n - 1$, let $T_i$ be a tree of size $i$. Also, let $m$ equal $n$ or $n + 1$ depending on whether $n$ is odd or even, respectively, and let

$$V(K_n) = \{v_1, v_2, \ldots, v_n\}.$$  If $T_1, T_2, \ldots, T_{n-1}$ can be packed into $K_n$, then there exists a packing in which each of the edges $v_i v_j$, for $1 \leq i < j \leq m - i$, belongs
to a tree of odd size.

The graph $K_n$ can be thought of as the complete n-partite graph with 1 vertex in each partite set. Thus the graph $K_n(m)$, the complete n-partite graph with $m$ vertices in each partite set, is a generalization of $K_n$. This graph has $\frac{m^2(n-1)}{2}$ edges. We may, therefore, generalize the TPC as follows:

**CONJECTURE 3.3** (Generalized Tree Packing Conjecture)

Let $n$ and $m$ be positive integers, $n \geq 2$. For each $i$ and $j$, $1 \leq i \leq n-1$, $1 \leq j \leq m^2$, let $T_{i,j}$ be any tree on $i$ edges. Then $E(K_n(m))$ may be partitioned as

$$E_1,1 \cup E_1,2 \cup \ldots \cup E_1,m^2 \cup$$
$$E_2,1 \cup E_2,2 \cup \ldots \cup E_2,m^2 \cup$$
$$\vdots$$
$$E_{n-1},1 \cup E_{n-1},2 \cup \ldots \cup E_{n-1},m^2,$$

such that the subgraph of $K_n(m)$ induced by $E_{i,j}$ equals $T_{i,j}$, $1 \leq i \leq n-1$, $1 \leq j \leq m^2$.

As our final result of this chapter we show that the Generalized Tree Packing Conjecture holds in the case where every tree is a star.
THEOREM 3.4 Let \( n \) and \( m \) be positive integers, \( n \geq 2 \). For \( 1 \leq i \leq n - 1 \) and \( 1 \leq j \leq m^2 \), let \( T_{i,j} \) equal \( S_i \). Then the \( T_{i,j} \)'s can be packed into \( K_n(m) \).

**Proof.** We proceed by induction on \( n \). If \( n \) equals 2 we have the graph \( K(m,m) \), which may certainly be decomposed into \( m^2 \) copies of \( S_1 \). So assume the result holds for \( K_{n-1}(m) \) and consider \( K_n(m) \), where \( n \geq 3 \). Let \( W \) be one partite set of \( K_n(m) \). Since \( K_n(m) - W \) equals \( K_{n-1}(m) \), we can pack \( T_{i,j}, 1 \leq i \leq n - 2, 1 \leq j \leq m^2 \), into \( K_n(m) - W \) by the induction hypothesis. Now let \( w \in W \). Since the degree of \( w \) in \( K_n(m) \) is \( m(n-1) \), the edges incident with \( w \) may be partitioned into \( m \) copies of \( S_{n-1} \). Now \( W \) contains \( m \) vertices, and so the edges of \( K_n(m) \) incident with a vertex of \( W \) may be partitioned into \( m^2 \) copies of \( S_{n-1} \), as we wished to show. By induction, the proof is completed. 

As a final word, note that the set of positive integers for which the TPC holds is either \( \{2, 3, \ldots\} \) or \( \{2, 3, \ldots, N\} \) for some \( N \geq 7 \). For suppose that we have a sequence \( T_1, T_2, \ldots, T_{n-1} \) which cannot be packed into \( K_n \). Then the sequence \( T_1, T_2, \ldots, T_{n-1}, S_n \) cannot be packed into \( K_{n+1} \) either.
CHAPTER 4

THE COCHROMATIC NUMBER OF A GRAPH

In chapters 2 and 3 of this discussion we were concerned with partitions of the edge set of a graph. In this chapter we focus our attention on the vertex set, and, in particular, on the problem of partitioning the vertex set of a graph $G$ into subsets which are independent in $G$ or which induce a complete subgraph of $G$.

A clique of a graph $G$ is a subset $U$ of $V(G)$ such that $\langle U \rangle$ is complete. Clearly, $U$ is a clique of $G$ if and only if $U$ is independent in $\overline{G}$. In [28], Foldés and Hammer define a graph $G$ to be polarized if $V(G)$ can be partitioned as $V_1 \cup V_2$, where each of $V_1$ and $V_2$ is independent in $G$ or a clique in $G$. We extend this concept by defining a parameter $z(G)$, called the cochromatic number of $G$, which equals the minimum number of subsets into which $V(G)$ can be partitioned so that each subset is independent in $G$ or a clique in $G$. Such a partition will be called a cocoloring of $G$. This parameter was first defined and studied by Lesniak-Foster and Straight in [40]. In this chapter we shall find some bounds for $z(G)$ and determine the cochromatic number for several classes of graphs. We also give some results concerning the problem of finding the maximum cochromatic number among all graphs of a particular
order or among all graphs which can be imbedded in $S_n$, for $n = 0, 1, 2, ...$. Let us begin by discussing some upper bounds for the cochromatic number.

We can immediately see that $z(G) = 1$ if and only if $G$ is an empty graph or a complete graph. Also, $z(G) = 2$ implies that $G$ is nonempty, noncomplete, and that $G$ is polarized in the sense of Foldés and Hammer. Graphs which are polarized but not bipartite are characterized as not containing the cycle $C_4$, its complement $\overline{C_4}$, or $C_5$ as an induced subgraph. This characterization, known to Fred Galvin, was rediscovered by Foldés and Hammer. It may also be deduced from a result of Gyárfás and Lehel [32].

It is evident that $z(G) = z(\overline{G})$. Since any two vertices of $G$ are either independent or form a clique, we obtain the upper bound $z(G) \leq [(p + 1)/2]$, where $p$ is the order of $G$. It is easy to see that this bound is attained if $p \leq 4$ and the graphs $C_5$ and $C_5 \cup K_1$ show that it is also attained when $p = 5$ and $p = 6$, respectively. The following argument shows that these are the only values of $p$ for which $z(G)$ may equal $[(p + 1)/2]$.

For positive integers $n$ and $m$, let the Ramsey number $r(n, m)$ denote the least positive integer $k$ such that any graph $G$ of order $k$ contains $K_n$ or $\overline{K_m}$ as an induced subgraph. The problem of determining $r(n, m)$ is well-known; in Figure 4.1 we show a table giving all
Ramsey numbers \( r(n, m) \) known to date. Note that 
\[ r(n, m) = r(m, n). \]

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**FIGURE 4.1**

Now let \( G \) be a graph of order \( p \geq 7 \). If \( p \) is odd, choose the maximum \( n \) such that \( r(n, n) \leq p \). Since \( p \geq 7 \), we have \( n \geq 3 \). Then

\[
z(G) \leq 1 + \left\lceil \frac{p - n + 1}{2} \right\rceil \leq 1 + \left\lceil \frac{p - 2}{2} \right\rceil = \frac{p - 1}{2}.
\]

If \( p \) is even, \( p \geq 10 \), choose the maximum \( n \) such that \( r(n, n) \leq p - 3 \). Then \( n \geq 3 \) and

\[
z(G) \leq 2 + \left\lceil \frac{p - (3 + n) + 1}{2} \right\rceil \leq 2 + \left\lceil \frac{p - 5}{2} \right\rceil = \frac{p - 2}{2}.
\]

Thus if \( p \geq 7 \) and \( p \neq 8 \), we have \( z(G) \leq \lceil (p + 1)/2 \rceil - 1 \).

We leave the case \( p = 8 \) as our first theorem.
THEOREM 4.1 If $G$ is a graph of order 8, then $z(G) \leq 3$.

Proof. As noted above, $z(G) \leq 4$. Since $r(3, 3) = 6$, either $G$ or $\overline{G}$ contains $K_3$ as a subgraph. Since $z(G) = z(\overline{G})$, we can assume without loss of generality that $K_3$ is a subgraph of $G$. Let $a, b, c$ denote the vertices of this subgraph and let $W$ denote the 5 remaining vertices of $G$.

Suppose $z(G) = 4$. Then $\langle W \rangle_G$ contains no induced $K_3$ or $\overline{K_3}$. Thus $\langle W \rangle_G$ is a 5-cycle, say $u, w, x, y, z, u$.

Case 1. The vertex $a$ is adjacent to a pair of adjacent vertices in $W$. Without loss of generality, assume $\langle a, u, w \rangle_G = K_3$. Then, as before, it must be that $\langle c, b, x, y, z \rangle_G = C_5$, where we may assume that $bx, cz \notin E(G)$. Now, $cu \notin E(G)$; for otherwise, $\langle c, u, z \rangle_G = K_3$, which implies that $\langle a, b, w, x, y \rangle_G = C_5$. However, since $x$ has degree 3 in $\langle a, b, w, x, y \rangle_G$ this is impossible. Similarly, it can be argued that $bw \notin E(G)$.

But then $\{c, u, y\}$ and $\{b, w, z\}$ are independent in $G$, which implies $z(G) \leq 3$. Thus Case 1 cannot occur.

Case 2. The vertex $a$ is adjacent to no pair of adjacent vertices in $W$. Then without loss of generality we may assume that $\{a, u, x\}$ is independent in $G$. Then $\langle a, u, x \rangle_G = K_3$. Since $z(\overline{G}) = 4$, we have that
\( \langle b, c, w, y, z \rangle_G = C_5 \). However, this is impossible since \( zw \) and \( wy \) are edges of \( \overline{G} \) while \( bc \) is not an edge of \( \overline{G} \). Thus Case 2 cannot occur.

Since neither Case 1 nor Case 2 can occur, \( z(G) \leq 3 \). ■

Recall that the chromatic number of \( G \), \( \chi(G) \), is defined to be the least number of subsets into which \( V(G) \) can be partitioned so that each subset is independent in \( G \). Note that any coloring of a graph \( G \) is a cocoloring of \( G \), and it follows from this and the fact that \( z(G) = z(\overline{G}) \), that \( z(G) \leq \min\{\chi(G), \chi(\overline{G})\} \). The next result gives a sufficient condition for this bound to be attained.

**THEOREM 4.2** If \( G \) is a graph of order \( p \geq 3 \) and not both \( K_3 \) and \( \overline{K_3} \) are induced subgraphs of \( G \), then \( z(G) = \min\{\chi(G), \chi(\overline{G})\} \).

**Proof.** Let \( G \) be a graph of order \( p \geq 3 \) and let \( n = z(G) \). Let \( V(G) = V_1 \cup V_2 \cup \ldots \cup V_n \) be a cocoloring of \( G \) using \( n \) colors. First suppose that \( K_3 \) is not a subgraph of \( G \). Since \( z(G) \leq \chi(G) \), it suffices to show that \( \chi(G) \leq n \).

If each \( V_i, 1 \leq i \leq n \), is independent in \( G \), then we have an \( n \)-coloring of \( G \), so that \( \chi(G) \leq n \). Thus we may assume that for some \( m, 1 \leq m \leq n \), we have that \( \langle V_i \rangle_G = K_2 \) if \( 1 \leq i \leq m \) and that \( V_i \) is in-
dependent in $G$ if $i > m$.

We first show that $m \leq 3$. Suppose, to the contrary, that $m > 4$. Then by Theorem 4.1

$$z(<V_1 \cup V_2 \cup V_3 \cup V_4>_G) \leq 3.$$ 

But then $z(G) \leq n - 1$, which is a contradiction. Thus $m \leq 3$.

If $m = 1$, let $V_1 = \{u, v\}$. Since $p \geq 3$, $V_1 \neq V(G)$. Therefore $V_2$ exists and is independent in $G$. Let $W_1 = \{x \in V_2 \mid xu \in E(G)\}$ and let $W_2 = V_2 - W_1$. Since $G$ does not contain $K_3$, $[v] \cup W_1$ and $[u] \cup W_2$ are independent in $G$. Thus $\chi(<V_1 \cup V_2>_G) \leq 2$, so that $\chi(G) \leq n$.

If $m = 2$, then since $G$ does not contain $K_3$, we have that $<V_1 \cup V_2>_G$ is a subgraph of $C_4$. Therefore $\chi(<V_1 \cup V_2>_G) \leq 2$, so that $\chi(G) \leq n$.

Finally, suppose $m = 3$. Then $<V_1 \cup V_2 \cup V_3>_G$ contains $C_5$ as a subgraph; otherwise, $<V_1 \cup V_2 \cup V_3>_G$ has no odd cycles and is therefore bipartite. But this implies that $\chi(<V_1 \cup V_2 \cup V_3>_G) \leq 2$, so that $z(G) \leq n - 1$, which is a contradiction. Since $G$ does not contain $K_3$, it is easily shown that $<V_1 \cup V_2 \cup V_3>_G$ is a subgraph of $H$, where $H$ consists of a 5-cycle $C$ together with a vertex which is adjacent to two nonadjacent vertices of $G$. Therefore $\chi(<V_1 \cup V_2 \cup V_3>_G) \leq 3$, so that $\chi(G) \leq n$. 

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Secondly, suppose $K_3$ is a subgraph of $G$ but that $G$ does not contain an induced $\overline{K}_3$. Then $\overline{G}$ does not contain $K_3$. Applying the above argument to $\overline{G}$ we find that $z(G) = z(\overline{G}) = \chi(\overline{G})$, which completes the proof. 

A somewhat surprising known result is that there exist triangle-free graphs with arbitrarily high chromatic numbers. A recursive construction for such graphs was first given by Blanches Descartes in 1954. An easier construction was given by Mycielski [41] in 1955. Applying this result together with Theorem 4.2 yields the following corollary.

**Corollary 4.1** For each positive integer $n$, there exists a graph $G_1$ with $z(G_1) = n$ which does not contain $K_3$. Also, there exists a graph $G_2$ with $z(G_2) = n$ which does not contain $\overline{K}_3$ as an induced subgraph.

We note that the converse of Theorem 4.2 is not true. For consider the graph $H = K_1 \cup K_2 \cup \ldots \cup K_n$, for $n \geq 3$. $\chi(H) = \chi(\overline{H}) = n$ (since $H$ is complete $n$-partite) and we shall show later that $z(H) = z(\overline{H}) = n$. Yet $H$ does contain both $K_3$ and $\overline{K}_3$ as induced subgraphs.

Since we have $z(G) \leq \min\{\chi(G), \chi(\overline{G})\}$, any result which gives an upper bound for $\chi(G)$ immediately yields an upper bound for $z(G)$. One of the best known of such results, due to Brooks [15], is given below. Recall that the symbols $\Delta(G)$ and $\delta(G)$ denote the maximum degree and
minimum degree, respectively, among the vertices of a graph $G$.

**Theorem 4.3** (Brooks) For any graph $G$,

$$\chi(G) \leq 1 + \Delta(G).$$

Furthermore, if $G$ is connected, then equality holds if and only if $G$ is a complete graph or an odd cycle.

Noting that $\Delta(G) = p - 1 - \delta(G)$, we obtain a corollary which gives an upper bound for $\chi(G)$.

**Corollary 4.2** For any graph $G$ of order $p$,

$$\chi(G) \leq \min(1 + \Delta(G), p - \delta(G)).$$

Furthermore, if $p \geq 5$ and both $G$ and $\overline{G}$ are connected, then equality holds if and only if $G$ or $\overline{G}$ is an odd cycle.

Many other upper bounds for the chromatic number are known. We next list several of the more well-known, in each case followed by a corollary giving the corresponding upper bound for the cochromatic number. We give brief proofs for those which do not follow immediately from the theorem.

**Theorem 4.4** (Szekeres and Wilf [51]) For any graph $G$,

$$\chi(G) \leq 1 + \max \delta(H),$$

where the maximum is taken over all induced subgraphs $H$ of $G$. 

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Corollary 4.3  For any graph G,
\[ z(G) \leq \max \min\{1 + \delta(H), |V(H)| - \Delta(H)\}, \]
where the maximum is taken over all induced subgraphs H of G.

Proof.  Applying Theorem 4.4 to \( \chi(\overline{G}) \) we have that
\[ \chi(\overline{G}) \leq 1 + \max \delta(\overline{H}), \]
where the maximum is taken over all induced subgraphs \( \overline{H} \) of \( \overline{G} \). But \( \delta(\overline{H}) = |V(H)| - 1 - \Delta(H) \), where H is the subgraph of G induced by \( V(\overline{H}) \). Thus
\[ 1 + \max \delta(\overline{H}) = \max\{|V(H)| - \Delta(H)\}. \]
The result follows from the fact that
\[ \min\{1 + \max \delta(H), \max\{|V(H)| - \Delta(H)\}\} = \]
\[ \max \min\{1 + \delta(H), |V(H)| - \Delta(H)\}. \]

Theorem 4.5 (Welsh and Powell [54])  If G has degree sequence \( d_1 \geq d_2 \geq \ldots \geq d_p \), then
\[ \chi(G) \leq \max \min\{d_i + 1, i\}. \]

Corollary 4.4  If G has degree sequence \( d_1 \geq d_2 \geq \ldots \geq d_p \), then
\[ z(G) \leq \max \min\{d_i + 1, i, p - d_i, p - i + 1\} \]

Proof.  If G has degree sequence \( d_1 \geq d_2 \geq \ldots \geq d_p \), then \( \overline{G} \) has degree sequence \( p - 1 - d_1 \leq p - 1 - d_2 \leq \ldots \leq p - 1 - d_p \). Applying the theorem to \( \overline{G} \) we thus obtain
\[ \chi(\overline{G}) \leq \max \min\{p - d_i, p - i + 1\}. \]
The result follows by interchanging the order in which we
take the maximum and minimum.

**THEOREM 4.6** (Wilf [55]) Let \( \varepsilon(G) \) denote the maximum eigenvalue of the adjacency matrix of \( G \). Then

\[
\chi(G) \leq 1 + \varepsilon(G).
\]

Moreover, equality holds iff \( G \) is a complete graph or an odd cycle.

**Corollary 4.5.** For any graph \( G \),

\[
z(G) \leq 1 + \min\{ \varepsilon(G) , \varepsilon(G) \}.
\]

Moreover, if \( p \geq 5 \) and both \( G \) and \( \overline{G} \) are connected, then equality holds if and only if \( G \) or \( \overline{G} \) is an odd cycle.

**THEOREM 4.7** (Wilf [55]) If \( G \) has order \( p \) and size \( q \) then

\[
\chi(G) \leq 1 + \left( \frac{2q(p-1)}{p} \right)^{1/2}.
\]

**Corollary 4.6** Suppose \( G \) has order \( p \) and size \( q \) and let \( \overline{q} = \frac{p(p-1)}{2} - q \). Then

\[
z(G) \leq 1 + \min\{ \left( \frac{2q(p-1)}{p} \right)^{1/2} , \left( \frac{2\overline{q}(p-1)}{p} \right)^{1/2} \}.
\]

**THEOREM 4.8** (Gallai [29]) Let \( m(G) \) denote the length of a longest path in \( G \). Then \( \chi(G) \leq m(G) + 1 \).

**Corollary 4.7** Let \( m(G) \) be defined as in Theorem 4.8. Then \( z(G) \leq 1 + \min\{ m(G) , m(\overline{G}) \} \).
THEOREM 4.9 (Erdős and Hajnal [24]) Let \( o(G) \) denote the length of the longest odd cycle in \( G \); if \( G \) has no odd cycles, define \( o(G) = 1 \). Then
\[
\chi(G) \leq 1 + o(G).
\]

Corollary 4.8 Let \( o(G) \) be defined as in Theorem 4.9. Then
\[
z(G) \leq 1 + \min\{o(G), o(\overline{G})\}.
\]

THEOREM 4.10 (see [35] p. 128) Let \( G \) be a graph of order \( p \) and let \( \beta(G) \) denote the maximum size of an independent set in \( G \). Then
\[
\chi(G) \leq p + 1 - \beta(G).
\]

Corollary 4.9 If \( G \) is a graph of order \( p \), then
\[
z(G) \leq p + 1 - \max\{\beta(G), \beta(\overline{G})\}.
\]

We next give a lower bound for \( z(G) \).

THEOREM 4.11 If \( G \) is a graph of order \( p \), then
\[
z(G) \geq p/\max\{\beta(G), \beta(\overline{G})\}.
\]

Proof. Since \( \beta(\overline{G}) \) is the maximum size of a clique in \( G \), \( \max\{\beta(G), \beta(\overline{G})\} \) is the maximum number of vertices which can belong to the same set in a cocoloring of \( G \). Hence it would be impossible to cocolor \( G \) with fewer than \( p/\max\{\beta(G), \beta(\overline{G})\} \) colors. \(\blacksquare\)

In our next result we give a formula for the cochromatic number of the complete \( n \)-partite graph \( K(p_1, p_2, \ldots, p_n) \). This result is also due to Lesniak-Foster

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and appears in [40]. We present here a new proof. Note that since $z(G) = z(G)$, this result also provides the cochromatic number for the graphs $K_{p_1} \cup K_{p_2} \cup \ldots \cup K_{p_n}$.

**THEOREM 4.12** Let $G = K(p_1, p_2, \ldots, p_n)$, where $p_1 \leq p_2 \leq \ldots \leq p_n$. Then $z(G) = \min_{1 \leq j \leq n} \{n, n-j+p_j\}$.

**Proof.** Denote the partite sets of $G$ by $S_1, S_2, \ldots, S_n$, where $|S_j| = p_j$ for $1 \leq j \leq n$. If $p_n = 1$ or $n = 1$, the result is immediate, so assume that $p_n \geq 2$ and $n \geq 2$.

Clearly $\chi(G) = n$ and $\chi(G) = p_n$. Furthermore, for $1 \leq j < n$, we can cocolor $G$ with $n-j+p_j$ colors by partitioning $\langle S_1 \cup S_2 \cup \ldots \cup S_j \rangle$ into $p_j$ cliques and $\langle S_{j+1} \cup \ldots \cup S_n \rangle$ into $n-j$ independent sets. Hence

$$z(G) \leq \min\{n, n-1+p_1, n-2+p_2, \ldots, p_n\}.$$ 

To prove the converse inequality, we proceed by induction on $n$. For $n = 2$ we have

$$2 = z(K(p_1, p_2)) \geq \min\{2, p_2\} = 2.$$

So assume the converse inequality holds for $n-1$ and consider $G = K(p_1, p_2, \ldots, p_n)$, where $n \geq 3$. Let $V_1 \cup V_2 \cup \ldots \cup V_z$ be a cocoloring of $G$. We must show that $z \geq \min\{n, n-1+p_1, n-2+p_2, \ldots, p_n\}$.

We may assume, without loss of generality, that every vertex of $S_n$ is colored the same or that each vertex of $S_n$ is assigned a different color. If each vertex of $S_n$...
is colored differently, then \( z \geq p_n \geq \min \{n, n-1+p_1, \ldots, p_n\} \),
as we wished to show. On the other hand, if every vertex
of \( S_n \) is colored the same, then we may assume that \( V_z = S_n \).
Thus \( V_1 \cup V_2 \cup \ldots \cup V_{z-1} \) is a cocoloring of
\( K(p_1, p_2, \ldots, p_{n-1}) \). Applying the induction hypothesis:

\[
z - 1 \geq \min \{n-1, n-2+p_1, n-3+p_2, \ldots, p_{n-1}\},
\]
or
\[
z \geq \min \{n, n-1+p_1, n-2+p_2, \ldots, 1+p_{n-1}\}
\]

\[
\geq \min \{n, n-1+p_1, n-2+p_2, \ldots, 1+p_{n-1}, p_n\}.
\]

**Corollary 4.10**

1. \( z(K_a(b)) = \min(a, b) \).
2. \( z(K(p_1, p_2, \ldots, p_n)) = n \) if and if \( p_j \geq j, 1 \leq j \leq n \).
3. \( z(K(p_1, p_2, \ldots, p_n)) = p_n \) if and only if \( n - p_n = \max_{1 \leq j \leq n} (j-p_j) \geq 0 \).

If \( G \) is a graph of order \( n \), what is the largest possible value of \( z(G) \)? To begin a study of this ques-
tion, we define

\[
z(n) = \max_{|V(G)|=n} z(G),
\]

where the maximum is taken over all graphs \( G \) of order \( n \).
Also define \( C(m) \) to be the least \( n \) such that there exists
a graph \( G \) of order \( n \) having \( z(G) = m \). \( C(m) \) and
\( z(n) \) are then related; \( C(m) \leq n < C(m+1) \) if and only
if \( z(n) = m \).

It is easy to see that \( C(1) = 1, C(2) = 3, \) and \( C(3) = 5 \). The following result summarizes what is known
THEOREM 4.13

(a) \( C(4) = 9 \)
(b) \( C(5) \in \{12, 13\} \)
(c) \( C(6) \in \{15, 16\} \)
(d) \( C(m) \leq C(m-1) + m \)
(e) \( C(m) \geq C(m-1) + k \),

where \( k \) is the largest positive integer satisfying \( C(m-1) + k \geq r(k, k) + 1 \).

Proof.  
(a) By Theorem 4.1, \( z(8) = 3 \), and thus \( C(4) \geq 9 \). To prove that \( C(4) \leq 9 \), observe that \( z(C_5 \cup K_4) = 4 \).

(b) Since \( z(8) = 3 \) and any graph of order 11 contains an induced \( K_3 \) or \( \overline{K}_3 \), \( z(11) \leq 4 \). Hence \( C(5) \geq 12 \).
Since \( r(4, 4) = 18 \), there exists a graph \( H \) of order 13 which does not contain \( K_4 \) or \( \overline{K}_4 \) as an induced subgraph. Thus \( \beta(H) = \beta(\overline{H}) = 3 \). Applying Theorem 4.11 we see that \( z(H) = 5 \). Thus \( C(5) \leq 13 \).

(c) The argument here is analogous to that of part (b). We note that \( z(14) = 5 \) and that there exists a graph on 16 vertices not containing \( K_4 \) or \( \overline{K}_4 \) as induced subgraphs.

(d) Let \( H \) be a graph of order \( C(m-1) \) such that \( z(H) = m - 1 \), and let \( G = H \cup K_m \). We wish to show that \( z(G) = m \). If each vertex of the copy of \( K_m \) is cocolored with a different color, then at least \( m \) colors are used.
to cocolor $G$. Hence we may assume that one color is used to cocolor $K_m$. But now $m - 1$ other colors must be used for $H$, and so $z(G) = m$. Since $G$ has order $C(m-1) + m$, $C(m) \leq C(m-1) + m$.

(e) Let $G$ be a graph of order $C(m-1) + k - 1$. We wish to show that $z(G) \leq m - 1$. Since $C(m-1) + k - 1 \geq r(k, k)$, $G$ contains an independent set or clique with at least $k$ elements. Upon removing these vertices from $G$, we obtain a graph $G'$ of order $C(m-1) - 1$. Therefore,

$$z(G) \leq z(G') + 1 \leq (m - 2) + 1 = m - 1,$$

as we wished to show. It follows that $C(m) \geq C(m-1) + k$.

The author has as yet been unsuccessful in proving that $z(12) = 4$, or in finding an example of a graph $G$ of order 12 having $z(G) = 5$. Note that such a graph $G$ may not contain an induced $K_4$ or $\overline{K}_4$, but must have $\chi(G) \geq 5$. The reader should also note that if $C(5) = 13$, then $C(6) = 16$ and $C(7) \geq 20$.

In Chapter 1 we mentioned the Heawood Map Coloring Problem, which is to determine the maximum chromatic number among all graphs $G$ which imbed in the surface $S_n$. The solution to this problem, recently completed by W. Haken and K. Appel when they solved the Four-Color Problem for the sphere, is known as the Heawood Map Coloring Problem.
Theorem.

**THEOREM 4.14** (Heawood, Ringel and Youngs, Haken and Appel, and others). Let \( n \geq 0 \). Then

\[
\chi(S_n) = \left\lceil \frac{7 + \sqrt{1 + 48n}}{2} \right\rceil.
\]

We next discuss the analog of this problem for the cochromatic number.

Define

\[
z(S_n) = \max_{G \rightarrow S_n} \chi(G),
\]

where the maximum is taken over all graphs \( G \) which imbed in \( S_n (G \rightarrow S_n) \). Since for any graph \( G \), \( z(G) \leq \chi(G) \), we have that \( z(S_n) \leq \chi(S_n) \). For \( n = 0 \), the graph \( C_5 \cup K_4 \) is planar and has \( z(C_5 \cup K_4) = 4 \). Therefore, \( z(S_0) = \chi(S_0) = 4 \). We now proceed to show that the sphere is the unique orientable surface for which \( z = \chi \). In what follows let \( f(n) = \left\lceil \frac{k + 48n}{48} \right\rceil \) and let \( [x] \) and \( [x]_o \) denote the least integer greater than or equal to \( x \) and the greatest odd integer less than or equal to \( x \), respectively.

**Lemma 4.1** \( \gamma(K_f(n)) = \left\{ \frac{k^2 - 1}{48} \right\} \), where \( k = [\sqrt{1 + 48n}]_o \).

**Proof.** It is well-known that the genus of \( K_n \) is given by

\[
\gamma(K_n) = \left\{ \frac{(n - 3)(n - 4)}{12} \right\}, \quad n \geq 3.
\]
Replacing \( n \) by \( f(n) \) we obtain

\[
\gamma(K_{f(n)}) = \left\lfloor \frac{\sqrt{1 + 48n + 1} + \sqrt{1 + 48n - 1}}{2} \right\rfloor.
\]

Let \( m = \lfloor \sqrt{1 + 48n} \rfloor \). Note that if \( m \) is odd,

\[
\left\lfloor \frac{\sqrt{1 + 48n + 1}}{2} \right\rfloor = \frac{m + 1}{2} \quad \text{and} \quad \left\lfloor \frac{\sqrt{1 + 48n - 1}}{2} \right\rfloor = \frac{m - 1}{2},
\]

and thus

\[
\gamma(K_{f(n)}) = \left\lfloor \frac{m^2 - 1}{48} \right\rfloor = \left\lfloor \frac{k^2 - 1}{48} \right\rfloor.
\]

If \( m \) is even,

\[
\left\lfloor \frac{\sqrt{1 + 48n + 1}}{2} \right\rfloor = \frac{m}{2} \quad \text{and} \quad \left\lfloor \frac{\sqrt{1 + 48n - 1}}{2} \right\rfloor = \frac{m - 2}{2}.
\]

Then

\[
\gamma(K_{f(n)}) = \left\lfloor \frac{m^2 - 2m}{48} \right\rfloor = \left\lfloor \frac{k^2 - 1}{48} \right\rfloor,
\]

since \( k = m - 1 \).

\[\text{THEOREM 4.15} \quad \text{Let } n \geq 1. \text{ Then } z(S_n) < \chi(S_n).\]

\[\text{Proof.} \quad \text{Let } G \text{ be a graph imbedded in } S_n, \quad n \geq 1, \text{ and suppose } \chi(G) = f(n). \text{ Dirac [22] showed that such a graph } G \text{ must contain } K_{f(n)} \text{ as a subgraph. Let } G' \text{ be the graph obtained from } G \text{ by removing the vertices of this } K_{f(n)}. \text{ Then}
\]

\[
z(G) \leq z(G') + 1 \leq \chi(G') + 1.
\]

We need to show that \( \chi(G') \leq f(n) - 2 \).

Suppose that \( \gamma(G') = m \). Then
\[ \chi(G') \leq \left\lceil \frac{7 + \sqrt{1 + 48m}}{2} \right\rceil, \]

and so we shall be done if

\[ f(m) = \left\lceil \frac{7 + \sqrt{1 + 48m}}{2} \right\rceil \leq f(n) - 2 = \left\lceil \frac{3 + \sqrt{1 + 48n}}{2} \right\rceil. \]

This last inequality is true provided that

\[ m \leq n - \frac{\sqrt{1 + 48n} - 2}{6}. \]

Now \( G' \cup K_f(n) \) is a subgraph of \( G \). Hence

\[ \gamma(G' \cup K_f(n)) \leq \gamma(G), \]

which implies that

\[ \gamma(G') \leq n - \gamma(K_f(n)). \]

By the lemma

\[ \gamma(K_f(n)) = \left\lceil \frac{k^2 - 1}{48} \right\rceil, \]

where \( k = \lceil \sqrt{1 + 48n} \rceil \).

Therefore,

\[ \gamma(G') \leq n - \frac{k^2 - 1}{48}. \]

Now \( k = \lceil \sqrt{1 + 48n} \rceil \) implies that \( k + 2 > \sqrt{1 + 48n} \), or that

\[ \frac{k^2 - 1}{48} > n - \frac{\sqrt{1 + 48n} - 1}{12}. \]

Hence

\[ n - \frac{k^2 - 1}{48} < \frac{\sqrt{1 + 48n} - 1}{12}. \]

Thus (4.1) holds and we are done if

\[ \frac{\sqrt{1 + 48n} - 1}{12} \leq n - \frac{\sqrt{1 + 48n} - 2}{6}. \]

But this last inequality holds if and only if

\[ 12n - 3\sqrt{1 + 48n} + 5 \geq 0. \]

Let \( x = \sqrt{1 + 48n} \). Then \( n = \frac{x^2 - 1}{48} \). Since \( n \geq 1 \), \( x \geq 7 \). So we wish to know when \( \frac{x^2 - 1}{48} - 3x + 5 \geq 0 \),
or when \( x^2 - 12x + 19 \geq 0 \). Note that this quadratic in \( x \) is positive whenever \( x \geq 11 \). Hence

\[ 12n - 3\sqrt{1+48n} + 5 \] is positive for \( n \geq 3 \).

For the cases \( n = 1 \) and \( n = 2 \), \( f(n) = 7 \) and \( 8 \), respectively, and \( \gamma(K_f(n)) = n \). It follows that in these cases \( \gamma(G') = 0 \), which implies that

\[ z(G) \leq 1 + z(G') \leq 5 < f(n) \].

As a result of Theorem 4.15 we have that \( z(S_1) \leq 6 \), \( z(S_2) \leq 7 \), \( z(S_3) \leq 8 \), and so on. We can obtain a lower bound for \( z(S_m) \) by using the graph

\[ G_n = K_1 \cup K_2 \cup \ldots \cup K_n \] . By Theorem 4.12, \( z(G_n) = n \).

Hence \( z(S_m) \geq n \) for any \( n \) for which \( G_n \) imbeds in \( S_m \).

It is known that \( \gamma(H_1 \cup H_2) = \gamma(H_1) + \gamma(H_2) \), and it follows that \( \gamma(G_n) = \sum_{i=3}^{n} \left\{ \frac{(i-3)(i-4)}{12} \right\} \) for \( n \geq 3 \). The following remark gives a closed formula for \( \gamma(G_n) \).

**Remark 4.1**

Let \( G_n = K_1 \cup K_2 \cup \ldots \cup K_n \), \( n \geq 1 \). Then

\[
\gamma(G_n) =
\begin{cases}
48m^3 - 36m^2 + 13m - 2, & \text{if } n = 12m \\
48m^3 - 24m^2 + 8m - 1, & \text{if } n = 12m + 1 \\
48m^3 - 12m^2 + 5m, & \text{if } n = 12m + 2 \\
48m^3 + 4m, & \text{if } n = 12m + 3 \\
48m^3 + 12m^2 + 5m, & \text{if } n = 12m + 4 \\
48m^3 + 24m^2 + 8m + 1, & \text{if } n = 12m + 5
\end{cases}
\]
\[
\begin{align*}
48m^3 + 36m^2 + 13m + 2, & \quad \text{if } n = 12m + 6 \\
48m^3 + 48m^2 + 20m + 3, & \quad \text{if } n = 12m + 7 \\
48m^3 + 60m^2 + 29m + 5, & \quad \text{if } n = 12m + 8 \\
48m^3 + 72m^2 + 40m + 8, & \quad \text{if } n = 12m + 9 \\
48m^3 + 84m^2 + 53m + 12, & \quad \text{if } n = 12m + 10 \\
48m^3 + 96m^2 + 68m + 17, & \quad \text{if } n = 12m + 11.
\end{align*}
\]

For example, \( \gamma(G_5) = 1 \), which implies that \( z(S_1) \) is 5 or 6. Similarly \( z(S_2) = 6 \) or 7, and \( z(S_3) = 7 \) or 8.

We note that if \( G \) is a toroidal graph with \( z(G) = 6 \), then \( \chi(G) = 6 \), but \( G \) cannot contain \( K_5 \). For if this 5-clique is removed, the resulting graph is planar, and can thus be cocolored with 4 colors. But then \( z(G) \leq 5 \). One might wonder if there even exists a 6-chromatic toroidal graph which does not contain \( K_5 \).

We show such a graph, called \( J \), due to Albertson and Hutchinson \([1],[4]\) in Figure 4.2. We might hope to combine \( J \) with a planar graph to obtain a toroidal graph having cochromatic number 6. However, \( J \) can be 5-cocolored using a clique of size 4 and four independent sets. Therefore, \( z(J \cup P) \leq 5 \), whenever \( P \) is planar. Also, Albertson and Hutchinson conjecture that \( J \) is the only 6-chromatic toroidal graph which is regular of degree six, and show that if \( G \) is a 6-regular toroidal graph with more than 25 vertices, then \( G \) can be five colored. We conjecture that \( z(S_1) = 5 \), and, moreover, that the
class of graphs $G_n$ will play a central role in the problem of determining $z(S_m)$, much as the complete graphs did for the Heawood problem.

**CONJECTURE 4.1** For $m \geq 0$,

$$z(S_m) = n,$$

where $n$ is the largest positive integer for which the graph $G_n = K_1 \cup K_2 \cup \ldots \cup K_n$ imbeds in $S_m$.

---

*FIGURE 4.2*
We conclude this chapter by listing several problems for future study concerning the cochromatic number.

1. Try to prove that \( z(12) = 4 \) or find a graph \( G \) of order 12 with \( z(G) = 5 \).

2. Prove that \( z(S_n) = 5 \) or find a toroidal graph \( G \) with \( z(G) = 6 \). Try to determine \( z(S_n) \) for values of \( n \) greater than 1.

3. Analogous to \( z(S_n) \), one can define \( z(S_n^2) \), \( n \geq 1 \), to be the maximum cochromatic number among all graphs \( G \) which imbed in \( S_n^2 \), the nonorientable surface of genus \( n \). It is known that

\[
\chi(S_n^2) = \left\lceil \frac{7 + \sqrt{1 + 24n}}{2} \right\rceil, \ n \neq 2,
\]

and

\[
\chi(S_2^2) = 6.
\]

Thus for \( S_1 \), the projective plane, and \( S_2 \), the Klein bottle, the chromatic number is 6. Since

\( \gamma(K_1 \cup K_2 \cup \ldots \cup K_5) = 1 \) and \( \gamma(K_1 \cup K_2 \cup \ldots \cup K_6) = 2 \),

we have that \( z(S_1^2) = 5 \) or 6 and \( z(S_2^2) = 6 \). Try to determine \( z(S_1^2) \) exactly and find \( z(S_n^2) \) for \( n > 2 \).

(Note: The author has very recently been able to prove that \( z(S_n^2) < \chi(S_n^2) \) for \( n \geq 4 \). Since \( \gamma(K_1 \cup K_2 \cup \ldots \cup K_7) = 4 \), it follows that \( z(S_4^2) = 7 \).)

4. For \( n \geq 2 \), define a graph \( G \) to be critically \( n\)-cochromatic if \( z(G) = n \) and \( z(G - v) = n-1 \) for all \( v \in V(G) \). Similarly, define \( G \) to be minimally \( n\)-cochromatic if
$z(G) = n$ and $z(G - e) = n-1$ for all $e \in E(G)$. Note that if $G$ is critically $n$-cochromatic then so is $\overline{G}$, while if $G$ is minimally $n$-cochromatic then $z(G + e) = z(G - e) = n-1$ for all $e \in E(G)$. It is not difficult to argue that every graph $G$ with $z(G) = n \geq 2$ contains a critically $n$-cochromatic subgraph. However, because the cochromatic number may increase when an edge is removed from $G$, is it true that any such $G$ contains a minimally $n$-cochromatic subgraph? Try to discover other properties of critically or minimally $n$-cochromatic graphs. (These concepts may be helpful in attempting to solve some of the above problems. For example, suppose there exists a graph $G$ of order 12 with $z(G) = 5$. Then $G$ is critically 5-cochromatic and we may also assume that $G$ is minimally 5-cochromatic.)

5. Define a graph $G$ to be $n$-antiminimal if $z(G - e) = z(G) + 1$ for all $e \in E(G)$. Try to characterize these graphs.

6. Define $C_1(m)$ as the least positive integer $q$ for which there exists a graph $G$ of size $q$ with $z(G) = m$. It is easy to see that $C_1(1) = C_1(2) = 1$ and that $C_1(3) = 4$. Find $C_1(m)$ for other values of $m$ and obtain some bounds on this number.
BIBLIOGRAPHY


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