Indegrees, Outdegrees, and the Hamiltonian Theme

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John Roberts
For John and Chris,
who have waited so often,
and for Delphia,
who was always there.
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CHAPTER I

PRELIMINARIES

In this chapter, we provide a brief historical sketch of the hamiltonian problem for the purpose of contrasting the accomplishments in graph theory as opposed to those achieved in directed graph theory. The second section provides an overview of the dissertation and the third section provides some definitions and conventions central to this presentation.

Section 1.1

Historical Precedence and The Hamiltonian Problem

In 1857, Sir William Rowan Hamilton reportedly introduced a "game" consisting primarily of a solid dodecahedron with each vertex representing an important city of the time. The object of the game was to produce a round trip of the cities by following the edges of the dodecahedron and passing through each vertex exactly once. This is equivalent to finding a cycle containing every vertex in the graph (or 1-skeleton) of the dodecahedron. In honor of Hamilton, graphs with this property are ordinarily referred to as hamiltonian graphs.

Although many sufficient conditions are available, there is no known characterization of hamiltonian graphs. In fact, it
wasn't until 1952 that Dirac provided the first nontrivial sufficient condition for a graph to be hamiltonian. This result was first extended by Ore in 1960 and subsequently by many others (in an apparently endless cascade of "best possible" results).

The first sufficient condition for hamiltonian digraphs was provided by Ghouila-Houri in 1960. Unaware of this result, Nash-Williams later provided a Dirac-type result for digraphs but withdrew it upon learning of the more general result by Ghouila-Houri. In 1970, Woodall presented a second sufficient condition for hamiltonian digraphs. As with (undirected) graphs, there is no known characterization of the hamiltonian property; in fact, it can be shown that a characterization of either hamiltonian digraphs or hamiltonian graphs will yield a characterization of the other class.

The results mentioned above are presented in Section 3.1 and 3.4 of Chapter III.

Section 1.2

An Overview

This dissertation considers the hamiltonian theme from the viewpoint of the indegrees and outdegrees associated with the vertices of a digraph. For example, Chapter II considers the sequences and sets of outdegrees associated with hamiltonian digraphs. In addition to showing that every finite sequence $x_1, x_2, \ldots, x_p$ of positive integers is the outdegree sequence of
some hamiltonian digraph if each $x_i \leq p-1$, we characterize the outdegree sequences of asymmetric hamiltonian digraphs. Also, it is shown that every finite nonempty set of positive integers is the outdegree set of some asymmetric hamiltonian digraph and, the minimum order of any such digraph is determined. The corresponding statements for indegree sequences and sets follow immediately because of the directional duality principle [12; p.38] in directed graph theory.

Chapter III considers the indegrees and the outdegrees of the vertices in a digraph together with an adjacency relation. This avenue of research leads to a sufficient condition for a digraph to be $k$-path hamiltonian. Besides yielding Woodall's hamiltonian sufficiency condition as a corollary, it provides a sufficient condition for a digraph to be $k$-path hamiltonian-connected (a generalization of a property introduced for graphs by Ore [21]) and at the same time generalizes many existing results in (undirected) graph theory.

Chapter IV considers the indegrees and outdegrees associated with the vertices of a digraph and the line digraph function. In [3], Chartrand showed that the $n$th iterated line graph (of a connected graph different from a path) is hamiltonian for all sufficiently large $n$. This is not the case for line digraphs; in fact, there are digraphs whose sequence of iterated line digraphs contain infinitely many hamiltonian and non-hamiltonian digraphs. By considering an analogous problem in eulerian digraphs, we produce a characterization of digraphs with the desired
hamiltonian properties. Also, using a characterization of k-trail eulerian digraphs and a result relating weak arc-connectivity in line digraphs to their degree structure, we are able to establish a sufficient condition for a sequence of iterated line digraphs to contain infinitely many k-path hamiltonian digraphs.

Section 1.3
Definitions and Notation

It is convenient at this time to present some definitions and notational conventions which will be used throughout this presentation. In addition to the following conventions, more specialized definitions will be introduced as needed. All other terms will be as defined in [11] and [12]. Occasionally, it will be convenient to refer to (undirected) graphs and, in this event, we adopt the conventions in [2].

The indegree, outdegree, and total degree of a vertex $v$ in a digraph $D$ are denoted by $id_v$, $od_v$, and $td_v$ respectively. The maximum indegree (respectively outdegree and total degree) among the vertices of $D$ is denoted by $i\Delta(D)$ (respectively $o\Delta(D)$ and $t\Delta(D)$). The corresponding minimum values are designated by $i\delta(D)$, $o\delta(D)$, and $t\delta(D)$ respectively.

The order and size of a digraph $D$ refer to the number of elements in $V(D)$ and $E(D)$ respectively. Clearly, the size of a digraph is the sum of the indegrees (outdegrees) of its vertices. The indegree set $I\deg(D)$ of $D$ is the set of indegrees associated
with the vertices of $D$. The outdegree set $\text{Odeg}(D)$ of $D$ is defined analogously.

By the terms walk, trail, path, circuit and cycles we mean respectively directed walk, directed trail, directed circuit and directed cycle; i.e., the arcs in each are similarly oriented.

A digraph $D$ is hamiltonian if it has a cycle containing every vertex of $D$ and every such cycle is a hamiltonian cycle. For a nonnegative integer $k$, a digraph is $k$-path hamiltonian if every path of length not exceeding $k$ is contained in a hamiltonian cycle of the digraph. Also, a digraph $D$ is hamiltonian-connected if for every two vertices $u$ and $v$ of $D$, there are $u$-$v$ and $v$-$u$ paths in $D$ each containing every vertex of $D$. Clearly, if a hamiltonian-connected digraph has order at least two, then it is hamiltonian.

A digraph $D$ is complete if for every two vertices of $D$, it contains an arc joining them and it is complete symmetric if for every two vertices $u$ and $v$ it contains both of the arcs $uv$ and $vu$. The complete symmetric digraph with $p$ vertices is denoted by $K_p$ while the (asymmetric) cycle with $p$ vertices is designated by $C_p$.

The disjoint union $D_1 \cup D_2$ of two disjoint digraphs is the digraph with vertex set $V(D_1) \cup V(D_2)$ and arc set $E(D_1) \cup E(D_2)$. The symmetric join $D_1 + D_2$ of two disjoint digraphs $D_1$ and $D_2$ is the digraph $D_1 \cup D_2$ together with the arcs $v_1v_2$ and $v_2v_1$ for each $v_1 \in V(D_1)$ and $v_2 \in V(D_2)$. If $S$ is a nonempty subset of the vertex set $V(D)$ of $D$, the
Induced subdigraph \( <S> \) is the digraph with vertex set \( S \) and arc set \( E(<S>) = \{uv \in E(D): u,v \in S\} \). If \( S \) is a proper subset of \( V(D) \), then \( D-V \) is the digraph \( <V(D) - S> \).

In Chapters II and III, the term digraph always refers to a simple directed graph (i.e., no parallel arcs and no loops) while in Chapter IV the definition of a digraph is broadened to include parallel arcs and multiple loops.
CHAPTER II

ON THE DEGREE SEQUENCES AND DEGREE SETS
OF HAMILTONIAN DIGRAPHS

In this chapter, we consider the outdegree sequences and outdegree sets associated with hamiltonian digraphs. Although we discuss digraphs in general, the primary results concern asymmetric hamiltonian digraphs. In addition to characterizing the outdegree sequences of asymmetric hamiltonian digraphs, we show that every finite nonempty set of positive integers is the outdegree set of some asymmetric hamiltonian digraph and determine the minimum order of any such digraph.

Section 2.1

Graphs and Their Degree Sets

The degree set $\text{Deg}(G)$ of a graph $G$ is the set of degrees associated with the vertices of $G$. Clearly, $\text{Deg}(G)$ can contain only nonnegative integers. Moreover, if $A$ is a finite nonempty set of nonnegative integers, then it is easy to show there is a graph $G$ for which $\text{Deg}(G) = A$. In [14], Kapoor, Polimeni, and Wall show that there is a connected graph $G$ with $\text{Deg}(G) = A$ if $A$ is nonempty and contains only positive integers. Also, they determined the minimum order of any such graph. Before presenting this result, we make the following definition. For a finite nonempty set $A$ of positive integers, let $\mu(A)$ denote the minimum
order among all connected graphs with degree set $A$. 

**Theorem 2.1** (Kapoor, Polimeni, Wall): If $A = \{a_1, a_2, \ldots, a_n\}$ is a set of positive integers, where $n \geq 1$ and $a_1 < a_2 < \ldots < a_n$, then $\mu(A) = a_n + 1$.

In the event that $G$ is a hamiltonian graph, then $\text{Deg}(G)$ is clearly nonempty and consists of integers, each of which is at least two. For a finite nonempty set $A$ of integers (all of which exceed one), let $\mu_H(A)$ denote the minimum order among all hamiltonian graphs with degree set $A$. In addition to proving that $\mu_H(A)$ exists for all such sets, Lesniak, Polimeni, and VanderJagt [17] showed the following:

**Theorem 2.2** (Lesniak, Polimeni, VanderJagt): If $A = \{a_1, a_2, \ldots, a_n\}$ is a set of integers satisfying $n \geq 1$ and $2 \leq a_1 < a_2 < \ldots < a_n$, then:

(i) $\mu_H(A) = a_1 + 1$ if $n = 1$;  
(ii) $\mu_H(A) = \begin{cases} 4 & \text{if } a_1 = 2 \text{ and } a_2 = 3 \\ a_2 + 2 & \text{if } a_1 = 2 \text{ and } a_2 \geq 4 \\ a_2 + 1 & \text{if } a_1 \geq 3 \end{cases}$ and $n=2$;  
(iii) $\mu_H(A) = a_3 + 1$ if $n = 3$; and  
(iv) $\mu_H(A) = a_4 + 1$ if $n = 4$ and $a_1 \geq 3$.

Also, in [17], they established the following more general result:

**Theorem 2.3** (Lesniak, Polimeni, VanderJagt): Let $A = \{a_1, a_2, \ldots, a_n\}$...
be a set of integers satisfying $n \geq 3$, $a_i \geq 2$, and $a_i + 2 \leq a_{i+1}$ for $1 \leq i \leq n-1$. Then, $\nu_H(A) = a_n + 1$ if either $n \geq 3$ and odd or $n \geq 6$ and even.

In spite of the numerous special cases considered, the general solution for $\nu_H(A)$ is still an open problem.

Section 2.2

Degree Sequences for Asymmetric Hamiltonian Digraphs

The outdegree sequence of a digraph $D$ with vertex set $V(D) = \{v_1, v_2, \ldots, v_p\}$ is the sequence $s_1, s_2, \ldots, s_p$ where $s_i = \text{od } v_i$ for $1 \leq i \leq p$. It is easily established that every such sequence of positive integers is the outdegree sequence of some hamiltonian digraph provided $p \geq 2$ and each $s_i \leq p - 1$.

For this, let $p \geq 2$ and $S: s_1, s_2, \ldots, s_p$ be a sequence of positive integers with $s_i \leq p - 1$ for $1 \leq i \leq p$. Let $D$ be the digraph with vertices $v_i$ and arcs $v_i v_j$ where $1 \leq i \leq p$ and $i + 1 \leq j \leq i + s_i$. Then, $D$ is a hamiltonian digraph with outdegree sequence $S$.

We now consider the problem for asymmetric digraphs. In particular, a sequence $S: s_1, s_2, \ldots, s_p$ of integers is hamiltonian graphical (or briefly, H-graphical) if there is an asymmetric hamiltonian digraph with outdegree sequence $S$. Clearly, if $D$ is an asymmetric hamiltonian digraph with order $p$, then the outdegree of each vertex in $D$ is at least 1 and does not exceed
p - 2. Hence, if $S: s_1, s_2, \ldots, s_p$ is an H-graphical sequence, then $1 \leq s_i \leq p - 2$ for $1 \leq i \leq p$.

**Theorem 2.4:** If $S: s_1, s_2, \ldots, s_p$ is a nondecreasing sequence of positive integers satisfying $s_p + s_s \leq p - 1$, then $S$ is H-graphical.

**Proof:** Let $D$ be the digraph with vertices $v_i$ and arcs $v_i v_j$ where $1 \leq i \leq p$ and $i + 1 \leq j \leq i + s_i$. Clearly, $D$ is a hamiltonian digraph with outdegree sequence $S$. If $D$ is asymmetric, then the theorem follows. Suppose that $D$ is not asymmetric. Then $D$ has arcs $v_i v_j$ and $v_j v_i$ for some $i$ and $j$ where $1 \leq i < j \leq p$. Since $1 \leq i < j$ and $v_i v_j \in E(D)$, it follows that $j - i \leq s_i$. Also, since $1 \leq i < j \leq p$ and $v_j v_i \in E(D)$, it follows that $p + i - j \leq s_j \leq s_p$ and that $1 \leq i \leq s_p$. But this implies that

$$p \leq s_j + s_i \leq s_p + s_s \leq p - 1$$

which is impossible. Hence, $D$ is an asymmetric hamiltonian digraph with outdegree sequence $S$ and the theorem is proved.

To illustrate the preceding results, let $S_1: s_1, s_2, \ldots, s_6$ be the sequence where $s_1 = s_2 = s_3 = 2$ and $s_4 = s_5 = s_6 = 3$. By the theorem, $S_1$ is H-graphical since $s_6 + s_s = 3 + 2 = 5$.

In particular, using the construction given in the proof of Theorem 2.4, we obtain the digraph $D_1$ in Figure 2.1. Although the condition given in the preceding theorem is sufficient, it is
not necessary. For example, let $S^*: s_1, s_2, \ldots, s_6$ be the sequence where $s_1 = s_2 = 1$, $s_3 = 2$, $s_4 = 3$, and $s_5 = s_6 = 4$. Since $s_6 + s_5 = 7 > 5$, this sequence does not satisfy the hypothesis of the theorem. However, the digraph $D_2$ in Figure 2.2 is an asymmetric hamiltonian digraph with outdegree sequence $S_2$.

**Corollary 2.5:** Let $S: s_1, s_2, \ldots, s_p$ be a sequence of positive integers where $s_i = s$ for $1 \leq i \leq p$. Then, $S$ is H-graphical if and only if $1 \leq s \leq (p - 1)/2$. 

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Proof: If \( 1 \leq s \leq \frac{(p - 1)}{2} \), then \( s_p + s_s = 2s \leq p - 1 \) and, by Theorem 2.4, the sequence \( S \) is H-graphical. If \( s \geq \frac{p}{2} \), then \( \sum_{i=1}^{p} s_i \geq \frac{p^2}{2} \) and \( S \) is not the outdegree sequence of any asymmetric digraph.

Corollary 2.6: If \( S: s_1, s_2, \ldots, s_p \) is a sequence of integers satisfying \( 2 \leq s_i = s \leq \frac{(p - 1)}{2} \) for \( 1 \leq i \leq p \), then \( S^* : s_1^*, s_2^*, \ldots, s_p^* \) is H-graphical where \( s_1^* = s^* \) for \( 1 \leq i \leq s \) and \( s_i^* = s_i - 1 \) for \( s + 1 \leq i \leq p - 1 \).

Proof: Let \( T: t_1, t_2, \ldots, t_{p-1} \) be a nondecreasing listing of the terms in \( S^* \). Then, \( t_i = s - 1 \) for \( 1 \leq i \leq p - s - 1 \) and \( t_i = s \) for \( p - s \leq i \leq p - 1 \). Since \( s \leq p - s - 1 \), it follows that \( s_p - 1 + s_s = 2s - 1 \leq p - 2 \). Hence, by Theorem 2.4, the sequence \( S^* \) is H-graphical.

A digraph \( D \) is \( v \)-hamiltonian if \( D-v \) is hamiltonian for some vertex \( v \) in \( D \).

Lemma 2.7: If \( p \geq 4 \) and \( S: s_1, s_2, \ldots, s_p \) is an H-graphical sequence in nondecreasing order, then there is an asymmetric digraph \( D \) with outdegree sequence \( S \) such that \( D \) is \( v_p \)-hamiltonian where \( v_p \in V(D) \) and \( od_{v_p} = s_p \).

Proof: Let \( D \) be an asymmetric hamiltonian digraph with \( V(D) = \{v_1, v_2, \ldots, v_p\} \) where \( od_{v_i} = s_i \) for \( 1 \leq i \leq p \). The digraph \( D \) has a hamiltonian cycle \( C \) containing the arcs \( v_a v_p \).
and $v_p v_b$ for some two vertices $v_a$ and $v_b$ in $D$. If $v_a v_b$ is an arc of $D$, then $D$ is $v_p$-hamiltonian since $D-v_p$ is hamiltonian. If $v_b v_a$ is an arc of $D$, then the digraph obtained by reversing the arcs on the cycle $v_a, v_p, v_b, v_a$ is an asymmetric $v_p$-hamiltonian digraph with outdegree sequence $S$. If neither $v_a v_b$ nor $v_b v_a$ is an arc of $D$, then the digraph obtained by replacing the arc $v_a v_p$ with the arc $v_a v_b$ is an asymmetric $v_p$-hamiltonian digraph with outdegree sequence $S$. In any event, there is an asymmetric digraph $D$ with outdegree sequence $S$ which is $v_p$-hamiltonian where $v_p \in V(D)$ and $od(v_p) = s_p$.

Before proceeding further, we illustrate the preceding result with the sequence $S: s_1, s_2, s_3, s_4$ where $s_1 = s_2 = s_3 = s_4 = 1$. By Theorem 2.4, this sequence is $H$-graphical. In fact, it is the outdegree sequence of the 4-cycle $C: v_1, v_2, v_3, v_4, v_1$ where $od(v_i) = s_i = 1$ for $1 \leq i \leq 4$. Clearly, $C$ is not $v_4$-hamiltonian. However, neither $v_3 v_1$ nor $v_1 v_3$ are arcs of $C$. Let $D$ be the digraph obtained by replacing the arc $v_3 v_4$ in $C$ with the arc $v_3 v_1$. Then $D$ consists of the cycle $v_1, v_2, v_3, v_1$ and the vertex 4 together with the arc $v_4 v_1$. Thus, $D$ is an asymmetric $v_4$-hamiltonian digraph with outdegree sequence $S$.

It was observed above that the sequence $S: 1, 1, 1, 1$ is the outdegree sequence of the 4-cycle. In general, if $p \geq 3$ and $S: s_1, s_2, \ldots, s_p$ is the sequence for which $s_i = 1$ for $1 \leq i \leq p$, then $S$ is the outdegree sequence of the $p$-cycle $C_p$ which is both asymmetric and hamiltonian and, therefore, $S$ is $H$-graphical.
Theorem 2.8: Let $p \geq 4$ and let $S = s_1, s_2, \ldots, s_p$ be a non-decreasing sequence of positive integers with $2 \leq s_p \leq p - 2$. Let $J$ denote the least integer such that $J \geq s_p$ and $s_{J+1} \geq 2$. Then, $S$ is H-graphical if and only if $S^* = s_1^*, s_2^*, \ldots, s_{p-1}^*$ is H-graphical where $s_i^* = s_i$ for $1 \leq i \leq J$ and $s_i^* = s_i - 1$ for $J + 1 \leq i \leq p - 1$.

Proof: Suppose that the sequence $S^*$ is H-graphical and let $D_1$ be an asymmetric hamiltonian digraph with $V(D_1) = \{v_1, v_2, \ldots, v_{p-1}\}$ and $od(v_i) = s_i^*$ for $1 \leq i \leq p - 1$. Let $C$ be any hamiltonian cycle of $D_1$ and let $v_p$ be a vertex not in $D_1$.

Suppose that $J \leq p - 2$. Then $C$ has an arc $v_a v_b$ where $1 \leq b \leq J$ and $J + 1 \leq a \leq p - 1$. By joining every vertex $v_i$ (where $J + 1 \leq i \leq p - 1$) to $v_p$ and by joining $v_p$ to $v_b$ and to $s_{p-1}$ other vertices $v_i$ (where $1 \leq i \leq J$), we obtain an asymmetric hamiltonian digraph with outdegree sequence $S$.

On the other hand, if $J = p - 1$, then $s_{p-1} = 1$; otherwise, $s_{p-1} \geq 2$ and $s_p \leq p - 2$ implies that $J \leq p - 2$. Since $S$ is a nondecreasing sequence, it follows that $s_1 = s_2 = \cdots = s_{p-1} = 1$. Then, $s_p + s_p \leq p - 1$ since $s_p \leq p - 2$ and $s_p = 1$. Hence, by Theorem 2.4, the sequence $S$ is H-graphical. In either case, the sequence $S$ is H-graphical if $S^*$ is H-graphical.

Conversely, suppose that the sequence $S$ is H-graphical and consider $S^*$. Let $T = t_1, t_2, \ldots, t_{p-1}$ be the terms of $S^*$ listed in nondecreasing order. If $t_{p-1} + t_{p-1} \leq p - 2$, then it follows
from Theorem 2.4 that the sequence $T$ and, therefore, the sequence $S^*$ are both H-graphical.

We now consider the case where $t_{p-1} + t_{t_{p-1}} \geq p - 1$. If $J = p - 1$ then $s_1 = s_2 = \cdots = s_{p-1} = 1$. If $J \leq p - 2$, then $s_{p-1} < p - 2$ and $s_i < p - 3$ for $1 \leq i \leq p - 2$ since no asymmetric digraph of order $p$ can have more than two vertices of outdegree $p - 2$. This implies that $s_i \leq p - 3$ for $1 \leq i \leq p - 1$. Consequently, $t_{p-1} \leq p - 3$ and, therefore, $t_{t_{p-1}} \geq 2$. Observe that $t_i \leq s_i$ for $1 \leq i \leq p - 1$. Then, $t_{p-1} \leq s_p$ implies that $2 \leq t_{t_{p-1}} \leq s_p$ and, therefore, $J = s_p$.

By Lemma 2.7, there exist asymmetric $v_p$-hamiltonian digraphs with vertex set $V = \{v_1, v_2, \ldots, v_p\}$ where $od v_i = s_i$ for $1 \leq i \leq p$.

Among all such asymmetric $v_p$-hamiltonian digraphs, let $D$ be one which minimizes $N_D = \sum_{v \in N(v_p)} i$ where $N(v_p)$ is the set of od $v_p = s_p = J$ vertices adjacent from $v_p$.

If $N_D = \sum_{i=1}^{J} i = \binom{J+1}{2}$, then $v_p$ is adjacent to the vertices $v_i$ where $1 \leq i \leq J = s_p$. Hence, if $v_i v_p$ is an arc of $D$, then $J + 1 \leq i \leq p - 1$. Then, $D - v_p$ is an asymmetric hamiltonian digraph with od $v_i = t_i$ where $t_i = s_i^*$ if $1 \leq i \leq J$ and $s_i^* \leq t_i \leq s_i^* + 1$ for $J + 1 \leq i \leq p - 1$. Let $C$ be any hamiltonian cycle of $D - v_p$. For each $v_i$ where $J + 1 \leq i \leq p - 1$, if $t_i = s_i^* + 1 \geq 2$, then we can choose an arc $e_i$ not in the hamiltonian cycle $C$. Let $D_1$ be the digraph obtained by removing
these arcs from $D-v_p$. Then, $D_1$ is an asymmetric hamiltonian digraph with outdegree $v_i = s_i^*$ for $1 \leq i \leq p - 1$ and, therefore, the sequence $S^*$ is H-graphical.

On the other hand, suppose that $N_D > \left( \frac{j+1}{2} \right)$. Then $v_p$ is not adjacent to some vertex $v_i$ where $1 \leq i \leq J$. Let $m$ be the least integer for which $v_p v_m$ is not an arc of $D$. Also, let $j \geq m$ be the least integer such that $v_p v_{j+1}$ is an arc of $D$; since $m \leq J = s_p = \text{od } v_p$, the integer $j$ does exist. Observe that $s_j < s_{j+1}$; otherwise, we could relabel $v_j$ and $v_{j+1}$ and, thereby reduce the value of $N_D$. Also, $v_j v_p$ is an arc of $D$; for if $v_j v_p \notin E(D)$, then we could replace the arc $v_p v_{j+1}$ by the arc $v_p v_j$ and produce an asymmetric $v_p$-hamiltonian digraph $F$ with outdegree sequence $S$ and $N_F < N_D$, which is impossible.

Furthermore, if $v$ is any vertex of $D$ different from $v_j$, then either $v v_j \in E(D)$ or $v_j v \in E(D)$. Suppose that this is not the case. Then, there is a vertex $v$ such that neither $v v_j$ nor $v_j v$ is an arc of $D$. Since neither $v_j v_p$ nor $v_p v_{j+1}$ belong to any hamiltonian cycle of $D-v_p$, we can replace them with the arcs $v_j v$ and $v_p v_j$ and obtain an asymmetric $v_p$-hamiltonian digraph $F$ with outdegree sequence $S$ and $N_F < N_D$. Since this is impossible, it follows that $v_j$ is adjacent either to or from every other vertex of $D$. Let $C$ be any hamiltonian cycle of $D$. We consider two cases, depending on which of the arcs $v_{j+1} v_j$ and $v_j v_{j+1}$ are in $D$.

Case 1: Assume that $v_{j+1} v_j$ is an arc of $D$. If $v_{j+1} v_j$ is not
an arc of $C$, then $C$ contains no arcs of the cycle $C': v_{j+1}, v_j, v_{j+1}$ and we can replace the arcs of $C'$ with the arcs $v_j, v_{j+1}, v_{j+1}, v_j$ and obtain an asymmetric $v_p$-hamiltonian digraph $F$ with outdegree sequence $S$ and $N_F < N_D$. Since this is impossible, the arc $v_{j+1}, v_j$ must belong to the cycle $C$. Since $od v_{j+1} = 1 + od v_j$ and since $v_p \in N(v_j) - N(v_{j+1})$, there is a vertex $v \in N(v_{j+1}) - N(v_j)$. Then, $vv_j \notin E(D)$ and $D$ contains the cycle $C': v_j, v_p, v_{j+1}, v, v_j$ which has no arcs belonging to $C$. If we replace the cycle $C'$ by the cycle $v_j, v, v_{j+1}, v_p, v_j$, then we obtain an asymmetric $v_p$-hamiltonian digraph $F$ with outdegree sequence $S$ and $N_F < N_D$. Again, this is a contradiction and, consequently, Case 1 cannot occur.

Case 2: Assume that $v_j, v_{j+1}$ is an arc of $D$. Since $od v_{j+1} = 1 + od v_j$ and since both $v_{j+1}$ and $v_p$ are in $N(v_j) - N(v_{j+1})$, it follows that $N(v_{j+1}) - N(v_j)$ has at least three vertices. Then, there is a vertex $v \in N(v_{j+1}) - N(v_j)$ such that neither $v_j$ nor $v_{j+1}v$ is an arc of $C$. However, $vv_j \notin E(D)$ and, therefore, $D$ contains the cycle $C': v_j, v_p, v_{j+1}, v, v_j$ which has no arcs of the hamiltonian cycle $C$ in $D - v_p$. Replacing $C'$ by the cycle $v_j, v, v_{j+1}, v_p, v_j$ yields an asymmetric $v_p$-hamiltonian digraph $F$ with outdegree sequence $S$ and $N_F < N_D$. Since this is a contradiction, Case 2 cannot occur. Consequently, the assumption that $N_D > \binom{j+1}{2}$ must be false and the theorem follows.

In the following, we illustrate an application of Theorem 2.8. For a sequence $S$, not necessarily in nondecreasing order, we write
S → T if T is the sequence obtained by listing the terms of S in nondecreasing order. Let $S_0: s_1, s_2, \ldots, s_7$ be the sequence for which $s_1 = s_2 = 1, s_3 = s_4 = 3, s_5 = s_6 = 4,$ and $s_7 = 5.\) Observe that $s_7 + s_5 = 9 > 6.$ Hence, Theorem 2.4 does not apply. However, Theorem 2.8 does apply. Successive applications of Theorem 2.8 yields the following sequences:

- $S_0: 1, 1, 3, 3, 4, 4, 5$
- $S_0^*: 1, 1, 3, 3, 4, 3$ → $S_1: 1, 1, 3, 3, 4$
- $S_1^*: 1, 1, 3, 2$ → $S_2: 1, 1, 2, 3, 3$
- $S_2^*: 1, 1, 2$ → $S_3: 1, 1, 2, 2$
- $S_3^*: 1, 1, 1$

where $S_i$ is H-graphical if and only if $S_i^*$ is H-graphical for $0 \leq i \leq 3.$ Also, $S_i^*$ is H-graphical if and only if $S_{i+1}$ is H-graphical. Since $S_3^*: 1, 1, 1$ is the outdegree sequence of the 3-cycle, it follows that $S_0$ is the outdegree sequence of an asymmetric hamiltonian digraph.

Section 2.3

Degree Sets for Hamiltonian Digraphs

Let $A$ be a finite nonempty set of positive integers and let $\Delta_A$ and $\delta_A$ respectively denote the maximum and minimum elements of $A.$ Then, there is a hamiltonian digraph $D$ with order $\Delta_A + 1$ and outdegree set $\operatorname{Odeg}(D) = A.$ In particular, if $A = \{a_1, a_2, \ldots, a_n\}$ where $a_1 < a_2 < \cdots < a_n,$ then the digraph with vertices $v_i$ and arcs $v_iv_j$ where $1 \leq i \leq a_n + 1$ and
\[
i + 1 \leq j \leq i + a_{n-i+1} \quad \text{if} \quad 1 \leq i \leq n \quad \text{and} \\
i + 1 \leq j \leq i + a_1 \quad \text{if} \quad n + 1 \leq i \leq a_n + 1
\]
is a Hamiltonian digraph with order \( a_n + 1 = \Delta_A + 1 \) and outdegree set \( A \). Moreover, every digraph with outdegree set \( A \) must have at least \( \Delta_A + 1 \) vertices. Consequently, if \( n(A) \) denotes the minimum order among all Hamiltonian digraphs with outdegree set \( A \), then \( n(A) = \Delta_A + 1 \) for every finite nonempty set \( A \) of positive integers.

We now consider the analogous problem for asymmetric digraphs. In particular, if \( A \) is a finite nonempty set of positive integers, then \( n_H(A) \) will denote the minimum order among asymmetric Hamiltonian digraphs with outdegree set \( A \). The following result establishes the existence of \( n_H(A) \) for every such set \( A \).

**Theorem 2.9:** If \( A \) is a finite nonempty set of positive integers, then \( n_H(A) \leq \Delta_A + \delta_A + 1 \).

**Proof:** Let \( A = \{a_1, a_2, \ldots, a_n\} \) where \( a_1 < a_2 < \cdots < a_n \). Let \( m = a_n + a_1 + 1 \) and let \( D \) be the digraph with vertices \( v_i \) and arcs \( v_i v_j \) where \( 1 \leq i \leq m \) and
\[
i + 1 \leq j \leq i + a_{n-i+1} \quad \text{for} \quad 1 \leq i \leq n \quad \text{and} \\
i + 1 \leq j \leq i + a_1 \quad \text{for} \quad n + 1 \leq i \leq m.
\]
Clearly, \( D \) is a Hamiltonian digraph with outdegree set \( A \) and order \( m \). Also, if \( 1 \leq i \leq n \), then \( v_i \) can be adjacent only to vertices \( v_j \) where \( i + 1 \leq j \leq i + a_{n-i+1} \leq 1 + a_n \). If \( n + 1 \leq i \leq 1 + a_n \), then \( v_i \) can only be adjacent to vertices \( v_j \) where \( i + 1 \leq j \leq a_n + a_1 + 1 \). Finally, if \( a_n + 2 \leq i \leq a_n + a_1 + 1 \),

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then $v_i$ can only be adjacent to vertices where $i + 1 \leq j \leq a_n + a_1 + 1$ or $1 \leq j \leq a_1$. Thus, $D$ is asymmetric and the theorem follows.

The preceding result is sharp in the sense that there do exist finite nonempty sets $A$ for which $\eta_H(A) = \Delta_A + \delta_A + 1$. In fact, this is illustrated by the following two results.

**Corollary 2.10**: If $A$ consists only of the positive integer $n$, then $\eta_H(A) = 2n + 1$.

**Proof**: Since $\Delta_A = \delta_A = n$, Theorem 2.9 implies that $\eta_H(A) \leq 2n + 1$. Suppose that $D$ is an asymmetric hamiltonian digraph with order $p \leq 2n$ and outdegree set $A$. Then, each vertex $v$ in $D$ has $\text{id}_v \leq p - n - 1$ and this implies that

$$pn = \sum_{v \in V(D)} \text{od}_v = |E(D)| = \sum_{v \in V(D)} \text{id}_v \leq p(p - n - 1) < pn.$$

Since this is impossible, it follows that $\eta_H(A) = 2n + 1$.

**Corollary 2.11**: If $A$ is a finite nonempty set of positive integers and $\delta_A = 1$, then $\eta_H(A) = \Delta_A + 2$.

**Proof**: Since $\delta_A = 1$, it follows from Theorem 2.9 that $\eta_H(A) \leq \Delta_A + 2$. If $D$ is any asymmetric hamiltonian digraph with outdegree set $A$, then $D$ has a vertex $v$ such that $\text{od}_v = \Delta_A$. Also, $\text{id}_v \geq 1$ since $D$ is hamiltonian. Since $D$ is asymmetric, the vertices adjacent to $v$ are distinct from the $\Delta_A$ vertices adjacent from $v$. Hence, $D$ has at least $\Delta_A + 2$ vertices.
In spite of the sharpness associated with Theorem 2.9, \( \eta_H(A) \) may be significantly smaller than \( \Delta_A + \delta_A + 1 \). For example, let \( r \geq 2 \) and consider the asymmetric Hamiltonian digraph with vertices \( v_i \) and arcs \( v_i v_j \) where \( 1 \leq i \leq 2r + 1 \) and \( 1 + i \leq j \leq i + r \).

![Figure 2.3: An asymmetric Hamiltonian digraph with outdegree set \( \{2,3,4\} \).](image)

From this digraph and a vertex \( v_{2r+2} \) not in it, we construct a new asymmetric Hamiltonian digraph \( D(r) \) by joining \( v_{2r+1} \) to \( v_{2r+2} \) and by joining \( v_{2r+2} \) to \( v_i \) where \( 1 \leq i \leq 2r \). (See Figure 2.3 for the case \( r = 2 \).) The outdegree set of \( D(r) \) is \( A_r = \{r,r+1,2r\} \). Since any asymmetric Hamiltonian digraph with a vertex of outdegree \( 2r \) must have order at least \( 2r + 2 \) and since \( D(r) \) is such a digraph with order \( 2r + 2 \), it follows that

\[
\eta_H(A_r) = 2r + 2 < 3r + 1 = \Delta_A + \delta_A + 1
\]

whenever \( r \geq 2 \). Moreover, by an appropriate choice of \( r \), this
difference may be made arbitrarily large.

**Lemma 2.12:** If $A$ is a finite nonempty set of positive integers, then there is an asymmetric hamiltonian digraph $D$ with $\text{Odeg}(D) = A$ and order $p$ for each $p \geq \eta_H(A)$.

**Proof:** Since Theorem 2.9 establishes the result when $p = \eta_H(A)$, it suffices to show that given an asymmetric hamiltonian digraph with $\text{Odeg}(D) = A$ and order $p \geq \eta_H(A)$, then there is an asymmetric hamiltonian digraph $D_1$ with $\text{Odeg}(D) = A$ and order $p + 1$.

So, suppose that $D$ is such a digraph with order $p$ for some $p \geq \eta_H(A)$. Then, $D$ has an arc $uv$ belonging to some hamiltonian cycle $C$ of $D$. If $w$ is a vertex not in $D$, then the digraph $D_1$ obtained by replacing the path $u,v$ by the path $u,w,v$ and joining $w$ to any $\Delta_A - 1$ other vertices in $V(D) - \{u,v\}$ is an asymmetric hamiltonian digraph with $\text{Odeg}(D_1) = A$ and order $p + 1$. Hence, the lemma follows.  

The following result provides a partial evaluation of $\eta_H(A)$.

**Lemma 2.13:** If $A$ is a finite nonempty set of positive integers with at least two elements and $B = A - \{\Delta_A\}$, then:

(i) $\eta_H(A) = \Delta_A + 2$ if $\Delta_A \geq \eta_H(B) - 1$ and

(ii) $\eta_H(A) \leq \eta_H(B) + 1$ if $\Delta_A < \eta_H(B) - 1$.

**Proof:** Clearly, $\eta_H(A) \geq \Delta_A + 2$. By Lemma 2.12, there is an asymmetric hamiltonian digraph $D$ with $\text{Odeg}(D) = B$ and order $\Delta_A + 1$. Let $uw$ be an arc of a hamiltonian cycle in $D$ and let
v be a vertex not in D. The digraph D' obtained by replacing the arc uv by the arcs uv and vw and the joining of v to any other \( \Delta_A - 1 \) vertices in \( V(D) - \{u,w\} \) is an asymmetric hamiltonian digraph with \( \text{Odeg}(D') = A \) and order \( \Delta_A + 2 \). Thus, (i) is proved.

To show that (ii) holds, let D be an asymmetric hamiltonian digraph with \( \text{Odeg}(D) = B \) and order \( n_H(B) \). Again, by repeating the construction employed above, we obtain an asymmetric digraph \( D_1 \) with \( \text{Odeg}(D_1) = A \) and order \( n_H(B) + 1 \). Thus, \( n_H(A) \leq n_H(B) + 1 \).

To present the primary result of this section, it is convenient to introduce the following notion. For a finite nonempty set \( A \) of nonnegative integers, let \( \mu(A) \) denote the minimum order among all asymmetric digraphs with outdegree set \( A \). In addition to proving the existence of \( \mu(A) \) for all such sets \( A \), the following was proved in \([5]\).

**Theorem 2.14 (Chartrand, Lesniak, Roberts):** Let \( A = \{a_1, a_2, \ldots, a_n\} \) be a set of \( n \geq 2 \) integers with \( 0 \leq a_1 < a_2 < \cdots < a_n \) and let \( t \geq 2 \) be the least integer for which \( (n + t - 2)a_1 + \left(\frac{t}{2}\right) \geq \sum_{i=2}^{n} a_i \). Then

\[
\mu(A) = \begin{cases} 
    a_n + 1 & \text{if } a_n \geq \mu(A - \{a_n\}) \\
    2a_1 + t & \text{if } a_n < \mu(A - \{a_n\}) 
\end{cases}
\]

The following result proved instrumental in establishing the preceding theorem. Since it will also play an integral role in proving the primary result of this section, we recreate the proof.
of it.

Lemma 2.15 (Chartrand, Lesniak, Roberts): If $A$ is a set of $n$ nonnegative integers, $n \geq 2$, then $\mu(A) \geq 2\delta_A + t$ where $t \geq 2$ is the least integer such that

$$(n + t - 1)\delta_A + \left(\frac{t}{2}\right) \geq \sum_{i \in A} i.$$ 

Proof: Let $D$ be an asymmetric digraph with $Odeg(D) = A$ and having order $p = \mu(A)$. Since the size of $D$, which is equal to the sum of the outdegrees of the vertices of $D$, cannot exceed $p(p-1)/2$ in an asymmetric digraph, it follows that

$$(p - n)\delta_A + \sum_{i \in A} i \leq p(p - 1)/2. \quad (2.1)$$

and this implies that

$$0 < 2 \sum_{i \in A} (i - \delta_A) \leq p(2\delta_A - 1).$$

since $|A| = n \geq 2$. Thus, $p - 2\delta_A - 1 \geq 1$ and this implies that $\mu(A) = p = 2\delta_A + t^*$ for some $t^* \geq 2$. Also, substituting $p = 2\delta_A + t^*$ into (1.1) yields, upon simplification,

$$(n + t^* - 1)\delta_A + \left(\frac{t^*}{2}\right) \geq \sum_{i \in A} i.$$

Consequently, $\mu(A) \geq 2\delta_A + t$ where $t \geq 2$ is the least integer for which $(n + t - 1)\delta_A + \left(\frac{t}{2}\right) \geq \sum_{i \in A} i$.

In order to present the primary result of this section, it is also convenient to introduce the following notation. An asymmetric digraph $D$ is bipartite if $V(D)$ can be partitioned into sets $U$ and $V$ (called partite sets) where each arc of $D$ joins a vertex of $U$ and a vertex of $V$. Two sequences $s_1, s_2, \ldots, s_m$
and $t_1, t_2, \ldots, t_n$ of nonnegative integers are **asymmetrically constructable** if there is an asymmetric bipartite digraph with partite sets $U = \{u_1, u_2, \ldots, u_m\}$ and $V = \{v_1, v_2, \ldots, v_n\}$ where $od u_i = s_i$ for $1 \leq i \leq m$ and $od v_j = t_j$ for $1 \leq j \leq n$. The following, which was also demonstrated in [5], characterizes those sequences which are asymmetrically constructable.

**Lemma 2.16:** The sequences $S: s_1 \leq s_2 \leq \cdots \leq s_m$ and $T: t_1, t_2, \ldots, t_n$ are asymmetrically constructable if and only if $m \geq t_1$ and the sequences $S^*: s_1^*, s_2^*, \ldots, s_{k+1}^*, s_{k+1}^* - 1, \ldots, s_m - 1$ and $T^*: t_2, t_3, \ldots, t_n$ (where $s_{k+1}^*$ is the first nonzero term for which $k \geq t_1$) are asymmetrically constructable.

**Proof:** Let $S^*$ and $T^*$ be asymmetrically constructable sequences. Then, there is an asymmetric bipartite digraph $D'$ with bipartite sets $U' = \{u_1, u_2, \ldots, u_m\}$ and $V' = \{v_2, v_3, \ldots, v_n\}$ where $od u_i = s_i$ for $1 \leq i \leq k$, $od u_i = s_i - 1$ for $k + 1 \leq i \leq m$, and $od v_i = t_i$ for $2 \leq i \leq n$. Let $v_1$ be a vertex not in $D'$. Let $D$ be the digraph obtained from $D'$ and $v_1$ by joining $v_1$ to $u_i$ for $1 \leq i \leq t_1$ and from $u_i$ for $k + 1 \leq i \leq m$.

Since $od u_i = s_i$ for $1 \leq i \leq m$ and $od v_i = t_i$ for $1 \leq i \leq n$, it follows that $S$ and $T$ are asymmetrically constructable.

Conversely, suppose that $S$ and $T$ are asymmetrically constructable. Clearly, $m \geq t_1$. Among all asymmetric bipartite digraphs with bipartite sets $U = \{u_1, u_2, \ldots, u_m\}$ and $V = \{v_1, v_2, \ldots, v_n\}$, where $od u_i = s_i$ for $1 \leq i \leq m$ and $od v_i = t_i$.
for $1 \leq i \leq n$, let $D$ be one for which $N = \sum_{u_i \in N(v_1)} i$ is a minimum.

If $t_1 = 0$, then we construct a new digraph $D'$ from $D$ in the following manner: for $1 \leq i \leq m$ and $s_i \geq 1$, if $u_i v_1 \notin E(D)$, then select any arc incident from $u_i$ and replace it with $u_i v_1$. Then, in $D'-v_1$, we have that $od u_i = s_i$ for $1 \leq i \leq k$, $od u_i = s_i - 1$ for $k + 1 \leq i \leq m$, and $od v_1 = t_i$ for $2 \leq i \leq n$; i.e., $S^*$ and $T^*$ are asymmetrically constructable.

So, suppose that $t_1 \geq 1$ and let $j_0$ be the least positive integer such that $v_1 u_{j_0} \notin E(D)$. If $j_0 = t_1 + 1$, then $v_1 u_{j_0} \notin E(D)$ for $t_1 + 1 \leq i \leq m$. Consequently, for $i \geq t_1 + 1$ where $s_i \geq 1$ and $u_i v_1 \notin E(D)$, we can choose an arc incident from $u_i$ and replace it with the arc $u_i v_1$. This yields a digraph $D'$ with the same partition sets and outdegree sequences. Also, in $D'-v_n$, we have that $od u_i = s_i$ for $1 \leq i \leq k$, $od u_i = s_i - 1$ for $k + 1 \leq i \leq m$, and $od v_1 = t_i$ for $2 \leq i \leq n$.

Now suppose that $j_0 \neq t_1 + 1$. Then, $j_0 \leq t_1$. Let $j_1 > j_0$ be the least positive integer such that $v_1 u_{j_1} \in E(D)$. This implies that $u_{j_0} v_1 \in E(D)$; if not, then we could replace the arc $v_1 u_{j_1}$ by $v_1 u_{j_0}$ and achieve a smaller value for $N = \sum_{u_i \in N(v_1)} i$, which is impossible. Since $od u_{j_1} \geq od u_{j_0}$ and since $v_1 \in N(u_{j_0}) - N(u_{j_1})$, there is a vertex $v_k \in N(u_{j_1}) - N(u_{j_0})$.

If $v_k u_{j_0} \in E(D)$, then $v_1, u_{j_1}, v_k, u_{j_0}, v_1$ is a cycle in $D$. 

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By replacing this cycle with the cycle $v_1, u_{j_0}, v_k, u_{j_1}, v_1$ we produce an asymmetric bipartite digraph $D_1$ with the same partite sets and outdegree sequences $S$ and $T$. However, in $D_1$ the value of $N_{D_1} = \sum_{i \in N(v_1)} i$ is smaller than $N_D$. Since this is impossible, $v_k u_{j_0} \notin E(D)$.

Since neither $v_k u_{j_0}$ nor $u_{j_0} v_k$ are arcs of $D$, we can replace the arcs $u_{j_0} v_1$ and $v_1 u_{j_1}$ with the arcs $v_1 u_{j_0}$ and $u_{j_0} v_k$. This yields an asymmetric bipartite digraph $D_1$ with the same partite sets and outdegree sequences $S$ and $T$. However, $N_{D_1} < N_D$ and this is impossible. Consequently, $j_0 = t_0 + 1$ and the result follows.

We illustrate the preceding result by exhibiting two sequences which are asymmetrically constructable. For example, let $S: 1,2,3,3,3$ and $T: 2,2,1,3$. Observe that if $S$ and $T$ are the outdegree sequences associated with the partite sets $U = \{u_1, u_2, u_3, u_4, u_5\}$ and $V = \{v_1, v_2, v_3, v_4\}$ of some asymmetric bipartite digraph $D$, then the sum of the terms in $S$ and $T$ correspond to the sum of the outdegrees in $D$ (and, therefore, to the size of $D$). Consequently, this sum cannot exceed $|U| \cdot |V| = 20$ which is the maximum size of $D$. Since this sum, in fact, equals 20, we proceed to sequentially apply the theorem to the sequences $S$ and $T$. Our first application to $S: 1,2,3,3,3$ and $T: 2,2,1,3$

yields the sequences
\[ S_1 = S^* : 1,2,2,2,2 \text{ and } T_1 = T^* : 2,1,3; \]
and a second application yields
\[ S_1^* : 1,2,1,1,1 \text{ and } T_2 = T^* : 1,3. \]
Upon rearranging the terms of \( S_1^* \) we obtain
\[ S_2 : 1,1,1,1,2 \text{ and } T_2 : 1,3; \]
and a third application of the theorem yields
\[ S_2^* : 1,0,0,0,1 \text{ and } T_2^* : 3. \]
A rearrangement of \( S_2^* \) gives the sequences
\[ S_3 : 0,0,0,1,1 \text{ and } T_3 : 3. \]
which are the outdegree sequences of the asymmetric bipartite digraph with partite sets \( U = \{u_1, u_2, u_3, u_4, u_5\} \) and \( V = \{v\} \) with arcs \( vu_1, vu_2, vu_3, u_4 v, u_5 v \). Thus, by Lemma 2.16, the sequences \( S \) and \( T \) are asymmetrically constructable. In fact, \( S \) and \( T \) are the outdegree sequences of the asymmetric bipartite digraph \( D \) shown in Figure 2.4.

![D:](image)

**Figure 2.4:** An asymmetric bipartite digraph.

We also observe that not all sequences \( S: s_1, s_2, \ldots, s_m \) and \( T: t_1, t_2, \ldots, t_n \) are asymmetrically constructable. In fact, although it is necessary that \( 0 \leq s_i \leq n \) (for \( 1 \leq i \leq m \)),
$0 \leq t_i \leq m$ (for $1 \leq i \leq n$), and $\sum_{i=1}^{m} s_i + \sum_{i=1}^{n} t_i \leq mn$, not even these requirements are sufficient. For example, consider the sequences $S: 0,3,3,3$ and $T: 0,2,2,3$. Repeated application (and possibly some rearrangements) of Lemma 2.16 yields the following:

- $S: 0,3,3,3$ and $T: 0,2,2,3$;
- $S_1: 0,2,2,2$ and $T_1: 2,2,3$;
- $S_2: 0,1,1,2$ and $T_2: 2,3$; and
- $S_3: 0,0,1,1$ and $T_3: 3$.

Since $S_3$ and $T_3$ cannot be the outdegree sequences associated with the partite sets of any asymmetric bipartite digraphs, the sequences $S$ and $T$ are not asymmetrically constructable.

We may now present the primary result of this section, the proof of which parallels the discussion in [5] of Theorem 2.14.

**Theorem 2.17:** If $A$ is a set of $n$ positive integers, $n \geq 2$, and $B = A - \{\Delta_A\}$, then

$$n_H(A) = \begin{cases} \Delta_A + 2 & \text{if } \Delta_A \geq n_H(B) - 1 \\ 2\Delta_A + t & \text{if } \Delta_A < n_H(B) - 1 \end{cases}$$

where $t \geq 2$ is the smallest integer satisfying

$$(*) \quad (n + t - 1)\Delta_A + \left(\begin{array}{c} t \\ 2 \end{array}\right) \geq \sum_{i \in A} i.$$

**Proof:** If $\Delta_A \geq n_H(B) - 1$, then it follows from Lemma 2.13(i) that $n_H(A) = \Delta_A + 2$. Consequently, it suffices to consider the case where $\Delta_A < n_H(B) - 1$. For convenience, let $A = \{a_1, a_2, \ldots, a_n\}$ where $a_1 < a_2 < \cdots < a_n$ and let $A_i = \{a_1, a_2, \ldots, a_i\}$ for
\[ 1 \leq i \leq n. \] Thus, \[ \Delta_A = a_n \] and \[ \delta_A = a_1. \] We now proceed by induction to show if \[ a_n < \eta_H(A_{n-1}) - 1 \] then \[ \eta(A) = 2a_1 + t \] where \[ t \geq 2 \] is the smallest integer satisfying (*). In view of Lemma 2.15, it suffices to show that \[ \eta_H(A) \leq 2a_1 + t. \]

Let \[ n = 2. \] Since \[ a_2 < \eta_H(A_1) - 1 = 2a_1, \] it follows that \[ t = 2 \] is the smallest integer exceeding one which satisfies (*). By Lemma 2.15, \[ \eta_H(A_2) \geq 2a_1 + 2. \] Also, by Lemma 2.13(i), \[ \eta_H(A) \leq \eta_H(A_1) + 1 = 2a_1 + 2 \] since \[ \eta_H(A_1) = 2a_1 + 1. \] Thus, \[ \eta_H(A) = 2a_1 + t \] if \[ n = 2. \] We now assume that the theorem holds for all sets of order not exceeding \[ n - 1 \] for some \[ n - 1 \geq 2. \]

Since \[ a_n < \eta_H(A_{n-1}) - 1, \] it follows that \[ a_{n-1} < \eta_H(A_{n-2}) - 1; \] if not, then \[ a_n + 2 \leq \eta_H(A_{n-1}) = a_{n-1} + 2 \] which is impossible. Let \[ t \geq 2 \] be the smallest integer satisfying (*) and note that \[ ((n-1) + t - 1)a_1 + \binom{t}{2} > \{ i \in A_{n-1} \}. \] Furthermore, \[ t - 1 \] will not suffice; for otherwise, \[ t - 1 \] would have sufficed in (*). Thus, by the induction hypothesis, \[ \eta_H(A_{n-1}) = 2a_1 + t. \] Consequently, \[ a_n \leq 2a_1 + t - 2 \] since \[ a_n < \eta_H(A_{n-1}) - 1. \]

We showed above that \[ a_n < \eta_H(A_{n-1}) - 1 \] implies that \[ a_{n-1} < \eta_H(A_{n-2}) - 1. \] In a similar manner, it follows that \[ a_{n-i+1} < \eta_H(A_{n-i}) - 1 \] implies \[ a_{n-i} < \eta_H(A_{n-i-1}) - 1 \] if \[ 1 \leq i \leq m - 2. \] By Lemma 2.13(ii), it follows that \[ \eta_H(A_{n-i+1}) \leq \eta_H(A_{n-i}) + 1 \] for \[ 1 \leq i \leq n. \] Since \[ \eta_H(A_1) = 2a_1 + 1, \] we have that \[ \eta_H(A) \leq 2a_1 + n. \] Also \[ t \leq n \] since \[ 2a_1 + t \leq \eta_H(A) \leq 2a_1 + n. \] We now consider two cases.

Case 1: Suppose that \[ t \] and \[ n \] are of the same parity. First
observe that \( a_n + 2 \leq 2a_1 + t \) implies
\[
a_i - (i - 2) \leq 2a_1 + t - n
\]
for \( 2 \leq i \leq n \) since \( a_i + (n - i) \leq a_n \). Also, \( 2a_1 + t - n \geq 2 \)
since \( a_n \geq n \) and \( a_n^2 + 2 \leq 2a_1 + t \). We first show that the sequences
\[
S_0: s_1^0, s_2^0, \ldots, s_m^0
\]
\[
T_0: a_2, a_3 - 1, \ldots, a_n - (n - 2)
\]
are asymmetrically constructable where \( m = 2a_1 + t - n + 1 \) and
\( s_i^0 = (n - t)/2 \geq 0 \) for \( 1 \leq i \leq m \). Since \( a_i - (i - 2) \leq m \) for \( 2 \leq i \leq n \), we can consider the following sequences:
\[
S_1: s_1^1 < s_2^1 < \cdots < s_m^1
\]
\[
T_1: a_3 - 1, a_4 - 2, \ldots, a_n - (n - 2)
\]
\[
S_2: s_1^2 < s_2^2 < \cdots < s_m^2
\]
\[
T_2: a_4 - 2, a_5 - 3, \ldots, a_n - (n - 2)
\]
\[
\vdots
\]
\[
S_{n-2}: s_1^{n-2} < s_2^{n-2} < \cdots < s_m^{n-2}
\]
\[
T_{n-2}: a_n - (n - 2)
\]
where the sequence \( S_j \) is obtained from \( S_{j-1} \) by Lemma 2.16
(with a possible rearrangement) for \( 1 \leq i \leq n - 2 \).
Since \( s_i^0 = (n - t)/2 \) for \( 1 \leq i \leq m \), each sequence \( S_j \)
consists of either one integer or two consecutive integers. Hence,
if some \( S_j \) contains at least \( a_{j+2} - j \) zero terms, then \( S_{j+1} \)
consists entirely of zero terms and the sequences \( S_{j+1} \) and \( T_{j+1} \)
are asymmetrically constructable and, therefore, so are the
sequences $S_0$ and $T_0$. Suppose no sequence $S_j$ (for $0 \leq j \leq n-3$) contains $a_{j+2} - j$ zero terms. This implies that

$$
\sum_{i=1}^{m} s_{i}^{1} = \sum_{i=1}^{m} s_{i}^{0} - (2a_{1} + t - n + 1) - a_{2}
$$

$$
\sum_{i=1}^{m} s_{i}^{2} = \sum_{i=1}^{m} s_{i}^{1} - (2a_{1} + t - n + 1) - (a_{3} - 1)
$$

$$
\vdots
$$

$$
\sum_{i=1}^{m} s_{i}^{j} = \sum_{i=1}^{m} s_{i}^{j-1} - (2a_{1} + t - n + 1) - (a_{j+1} - (j-1))
$$

$$
\vdots
$$

$$
\sum_{i=1}^{m} s_{i}^{n-2} = \sum_{i=1}^{m} s_{i}^{n-3} - (2a_{1} + t - n + 1) - (a_{n-1} - (n-3))
$$

which in turn implies that

$$
\sum_{i=1}^{m} s_{i}^{n-2} = \sum_{i=1}^{m} s_{i}^{0} - (n-2)(2a_{1} + t - n + 1) + \sum_{i=2}^{n-1} a_{i} - \sum_{i=2}^{n-1} (i-2) .
$$

$$
= \left( \frac{n-t}{2} \right) (2a_{1} + t - n + 1) - (n-2)(2a_{1} + t - n + 1) +
$$

$$
\sum_{i=2}^{n-1} a_{i} - \frac{(n-3)(n-2)}{2}
$$

$$
= -(n+t-3)a_{1} + \left[ \frac{t}{2} \right] + (t-1) + \sum_{i=1}^{n-1} a_{i}
$$

Recall that $(n+t-1)a_{1} + \left[ \frac{t}{2} \right] \geq \sum_{i=1}^{m} a_{i}$, then

$$
2a_{1} + t - n + 1 - \sum_{i=1}^{m} s_{i}^{n-2} = (n+t-1)a_{1} + \left[ \frac{t}{2} \right] - \sum_{i=1}^{n-1} a_{i} - (n-2)
$$

$$
\geq a_{n} - (n-2) ;
$$

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i.e., if $S_j$ has fewer than $a_{j+2} - j$ zero terms (for $0 \leq j \leq n-3$), then $S_{n-2}$ has at least $a_n - (n - 2)$ zero terms. Consequently, $S_0$ and $T_0$ are asymmetrically constructible.

Let $D_1$ be an asymmetric bipartite digraph with partite sets $U = \{u_1, u_2, \ldots, u_m\}$ and $V = \{v_1, v_2, \ldots, v_n\}$ where $\text{od } u_i = (n-t)/2$ for $1 \leq i \leq m$ and $\text{od } v_j = a_j - (j - 2)$ for $2 \leq j \leq n$. Let $T_{n-1}$ be the transitive tournament of order $n - 1$ with vertex set $V$ where $v_i v_j \in E(T_{n-1})$ if $2 \leq j < i \leq n$. Then, in the digraph $D_1 \cup T_{n-1}$, $\text{od } v_i = a_i$ for $2 \leq i \leq n$. Also, $T_{n-1}$ has the hamiltonian path $P: v_n, v_{n-1}, \ldots, v_2$. In the bipartite digraph $D_1$,

$$\text{od } v_n = a_n - (n - 2) \leq 2a_1 + t - n = |U| - 1.$$ 

Thus, there is a vertex $u_1$ in $U$ to which $v_n$ is not adjacent. Also, $v_2$ is adjacent to a vertex $u_2 \in U - \{u_1\}$ since $\text{od } v_2 = a_2 \geq 2$ in $D_1$. Furthermore, since $a_1 - (n - t)/2 \geq 1$ and $n_H(\{a_1 - (n - t)/2\}) = |U|$, it follows that we can construct an asymmetric hamiltonian digraph $D_2$ using the vertices of $U$ and such that $\text{Odeg}(D_2) = \{a_1 - (n - t)/2\}$ and such that the arc $u_1 u_2$ lies on a hamiltonian cycle $C$ of $D_2$. Let $D = D_1 \cup D_2 \cup T_{n-1}$.

If $u_1 v_n \in E(D)$, then $D$ has a hamiltonian cycle consisting of the $v_n - v_2$ path $P$ in $T_{n-1}$ and $u_2 - u_1$ path in $D_2$ together with the arcs $u_1 v_n$ and $v_2 u_2$. If $u_1 v_n \notin E(D)$, then we can replace the arc $u_1 u_2$ with the arc $u_1 v_n$ and the resulting digraph is hamiltonian. In either case, $D$ is an asymmetric hamiltonian digraph with $\text{Odeg}(D) = A$ and order $2a_1 + t$ where $t \geq 2$ is the smallest integer satisfying (*).
Case 2: Suppose that $t$ and $n$ are of opposite parity. Because $a_n + 2 \leq 2a_1 + t$, 
$$a_i - (i - 1) \leq 2a_1 + t - n - 1 < 2a_1 + t - n$$
for $1 \leq i \leq n$. Also, it follows from $n + 2 \leq a_n + 2 \leq 2a_1 + t$ that $2a_1 + t - n \geq 3$ since $t$ and $n$ are of different parity.

We now show that the sequences
$$S_0: s_0^0, s_2^0, \ldots, s_m^0$$
and
$$T_0: a_1, a_2, a_3, -2, \ldots, a_n - (n-1),$$
where $m = 2a_1 + t - n$ and $s_i^0 = (n - t + 1)/2 \geq 1$ for $1 \leq i \leq m$, are asymmetrically constructable. As before, we can consider the sequences
$$S_j: s_j^1, s_2^1, \ldots, s_m^1$$
$$T_j: a_{j+1}, a_j - 1, a_j + 2, \ldots, a_n - (n-1),$$
where $0 \leq j \leq n - 1$ and $S_j$ is obtained (as before) from $S_{j-1}$ by Lemma 2.16. As in the preceding case, if $S_j$ (where $0 \leq j \leq n-1$) contains at least $a_{j+1} - j$ zero terms, then $S$ and $T$ are asymmetrically constructable. So, suppose this is not the case for any $S_j$ where $0 \leq j \leq n - 2$. Then, it follows that
$$\sum_{i=1}^{m} s_i^{n-1} = \sum_{i=1}^{m} s_i^0 - (n-1)(2a_1 + t - n) + \sum_{i=1}^{n-1} a_i - \binom{n-1}{2}$$
from which we obtain
$$\sum_{i=1}^{m} s_i^{n-1} = \left(\frac{n-t+1}{2}\right)(2a_1 + t - n) - (n-1)(2a_1 + t - n) + \sum_{i=1}^{n-1} a_i - \binom{n-1}{2}$$
$$= -(n+t-3)a_1 - \binom{t}{2} + (t-1) + \sum_{i=1}^{n-1} a_i.$$
Since \((n+t-1)a_1 + \binom{t}{2} \geq \sum_{i=1}^{n} a_i\), it follows that

\[
2a_1 + t - n - \sum_{i=1}^{m} s_i^{n-1} = a_1(n + t - 1) + \binom{t}{2} - \sum_{i=1}^{n-1} a_i - (n - 1) \geq a_n - (n - 1).
\]

Thus, if each \(S_j\) (where \(0 \leq i \leq n - 1\)) has fewer than \(a_j - (j-1)\) zero terms, then \(S_n\) has at least \(a_n - (n - 1)\) zero terms. Consequently, the sequences \(S_0\) and \(T_0\) are asymmetrically constructable.

We now proceed (as in the previous case) to construct an asymmetric hamiltonian digraph \(D\) with \(\text{Odeg}(D) = A\) and having order \(2a_1 + t\). Let \(D_1\) be an asymmetric bipartite digraph with partite sets \(U = \{u_1, u_2, \ldots, u_m\}\) and \(V = \{v_1, v_2, \ldots, v_n\}\) where \(\text{od } u_i = (n - t + 1)/2\) and \(\text{od } v_j = a_j -(j-1)\) for \(1 \leq i \leq m\) and \(1 \leq j \leq n\). Let \(T_n\) be the transitive tournament with \(V(T_n) = V\) and arcs \(v_i v_j\) for \(1 \leq j < i \leq n\). Then, \(T_n\) has a hamiltonian path \(P: v_n, v_{n-1}, \ldots, v_1\). Also, in \(D_1 \cup T_n\), \(\text{od } v_i = a_i\) for \(1 \leq i \leq n\).

We now construct a third digraph \(D_2\) where \(V(D_2) = U\). In the bipartite digraph \(D_1\), observe that

\[
\text{od } v_n = a_n - (n - 1) \leq 2a_1 + t - n - 1 = |U| - 1.
\]

Thus, there is a vertex \(u_1 \in U\) to which \(v_n\) is not adjacent. Also, in \(D_1\), \(\text{od } v_1 = a_1 \geq 2\); this follows from Theorem 2.9 since we have assumed that \(\Delta_A < n_H(A - \{A_1\}) - 1\). Thus, \(v_1\) is adjacent to a vertex \(u_2\) in \(U\) different from \(u_1\). Since \(a_1 - (n - t + 1)/2 \geq 1\) and \(n_H(a_1 - (n - t + 1)/2) = |U|\), we
can construct an asymmetric hamiltonian digraph $D_2$ with $V(D_2) = U$ and such that the arc $u_1u_2$ is contained in a hamiltonian cycle $C$ of $D_2$. Also, observe that $\text{od} u_i = a_i$ in $D_1 \cup D_2$. Let $D = D_1 \cup D_2 \cup T_n$. If $u_1v_n \in E(D)$, then (as in the preceding case) the digraph $D$ is hamiltonian. If $u_1v_n \notin E(D)$, then replace the arc $u_1u_2$ by $u_1v_n$ and the resulting digraph $D'$ is an asymmetric hamiltonian digraph with $\text{Odeg}(D') = A$ and having order $2a_1 + t$ where $t$ is the smallest integer satisfying (*). #

We illustrate the preceding result by showing that $\eta_H((5,9,10)) = 13$; i.e., the minimum order among all asymmetric hamiltonian digraphs having outdegree set $\{5,9,10\}$ is thirteen.

Corollary 2.10 yields $\eta_H((5)) = 11$. Since $9 < \eta_H((5)) - 1$, Theorem 2.17 implies that $\eta_H((5,9)) = 10 + t$ where $t \geq 2$ is the least integer satisfying $5(t + 1) + \left\lceil \frac{t}{2} \right\rceil \geq 14$. Because the inequality is satisfied if $t = 2$, it follows that $\eta_H((5,9)) = 12$.

Since $10 < \eta_H((5,9)) - 1$, we again apply Theorem 2.17. In this case, $\eta_H((5,9,10)) = 10 + t$ where $t \geq 2$ is the least integer satisfying $5(t + 2) + \left\lceil \frac{t}{2} \right\rceil \geq 24$. Since the inequality is satisfied if $t = 3$ but not if $t = 2$, it follows that $\eta_H((5,9,10)) = 13$.

As a final illustration of the preceding theorem, we show that $\eta_H((5,6,7,8)) = 12$. Clearly, $\eta_H((5)) \leq \eta_H((5,6,7))$. Since $8 < \eta_H((5)) - 1$, it follows that $8 < \eta_H((5,6,7)) - 1$. Thus, Theorem 2.17 implies that $\eta_H((5,6,7,8)) = 10 + t$ where $t \geq 2$ is the least integer satisfying $5(t + 3) + \left\lceil \frac{t}{2} \right\rceil \geq 26$. Since the inequality is satisfied by $t = 2$, we have $\eta_H((5,6,7,8)) = 12$. 

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CHAPTER III

ORE-TYPE DIGRAPHS

In this chapter we present a classification scheme for directed graphs, discuss some structural characteristics of the digraphs in each class, and provide a sufficiency condition for the k-path hamiltonian property which generalizes many results in directed and undirected graph theory.

Section 3.1

Some Preliminary Remarks

The existence of hamiltonian digraphs with a relatively small number of arcs is readily apparent. For example, if $p \geq 2$ then the directed cycle $C_p$ has $p$ vertices and $p$ arcs. Similarly, if $p \geq 3$, then the symmetric cycle $S_p$ has $p$ vertices and $2p$ arcs. On the other hand, a digraph can have a rather substantial number of arcs and still have no hamiltonian cycle. For example, a transitive tournament with $p$ vertices has no cycles even though it has $p(p - 1)/2$ arcs. Also, if $p \geq 2$, then the complete symmetric digraph $K_{p-1}$ together with an isolated vertex has $p$ vertices and $(p - 2)(p - 3)$ arcs, but no hamiltonian cycle.

Clearly, $K_p$ is hamiltonian if $p \geq 2$. Thus, if the indegree and outdegree of each vertex of a digraph $D$ are sufficiently large (with respect to the order of $D$), then $D$ must be hamiltonian. Nash-Williams [19] verified this when he proved the
Theorem 3.1 (Nash-Williams): A digraph $D$ with order $p \geq 2$ is hamiltonian if the indegree and outdegree of each vertex is at least $p/2$.

The lower bound "$p/2$" in Theorem 3.1 is sharp in the sense that the conclusion no longer follows if "$p/2$" is replaced by "$(p-1)/2$". For example, let $K_1 + 2K_n$ denote the symmetric join of $K_1$ with two disjoint copies of $K_n$. Although each vertex has indegree and outdegree at least $n = (p-1)/2$, the graph $K_1 + 2K_n$ has no hamiltonian cycle. Moreover, this class of examples is by no means unique in this respect.

In spite of the sharpness associated with the above theorem, two strengthenings of this result are known. The first one is by Ghouila-Houri [8] and it requires that a digraph be strong and that the total degree of each vertex be at least as large as the order of the digraph. The second generalization is by Woodall [24] and it involves both the order of the digraph and the adjacency relations among the vertices. We now present formal statements of both of these results.

Theorem 3.2 (Ghouila-Houri): A strong digraph $D$ of order $p \geq 2$ is hamiltonian if

$$\text{id}_v + \text{od}_v \geq p$$

for each vertex $v$ of $D$.
**Theorem 3.3** (Woodall): A nontrivial digraph \( D \) of order \( p \) is hamiltonian if

\[
id u + \text{od } v \geq p
\]

whenever \( u \) and \( v \) are distinct vertices and the arc \( uv \) is not present in \( D \).

Since any digraph which satisfies the hypothesis of Theorem 3.1 must be strong, it must also satisfy the hypothesis of Theorem 3.2. Likewise, any digraph which satisfies the hypothesis of Theorem 3.1 obviously satisfies the requirements of Theorem 3.3. Consequently, Theorem 3.1 is a corollary to both Theorems 3.2 and 3.3.

To see that Theorems 3.2 and 3.3 are independent of each other, we consider two classes of digraphs. Let \( m \) and \( n \) be integers satisfying \( m > n^2 \geq 2 \) and let

\[
D = K_2 + (K_{m-1} \cup K_{n-1})
\]

be the symmetric join of \( K_2 \) with the disjoint union of \( K_{m-1} \) and \( K_{n-1} \). Then \( D \) has order \( p = m + n > 2n \). It also has vertices having total degree \( 2n \) which is less than \( p \). Consequently, Theorem 3.2 does not apply. However, \( D \) does satisfy the hypothesis of Theorem 3.3 and, therefore, it is hamiltonian. Thus, Theorem 3.3 is not implied by Theorem 3.2.

On the other hand, let \( p \geq 3 \) and let \( D \) be the digraph having vertices \( v_1, v_2, \ldots, v_p \) and arcs \( v_1 v_j \) and \( v_k v_{jk} \) where \( 2 \leq j \leq p \) and \( 1 \leq k \leq p \). The digraph \( D \) does not satisfy the hypothesis of Theorem 3.3; for example, \( v_2 v_3 \notin E(D) \) and \( \text{od } v_2 + \text{id } v_3 = p - 1 \). Hence, Theorem 3.3 does not apply. However,
D is strong and each vertex has total degree $p$. Thus, $D$ satisfies the hypothesis of Theorem 3.2 and therefore, it is hamiltonian. Consequently, Theorem 3.2 and 3.3 are independent of each other.

While hamiltonian sufficiency conditions have been developed for certain specific classes of digraphs (such as tournaments and line digraphs), the three preceding results are the only ones which consider digraphs in general. Of these three theorems, it is the hypothesis of Theorem 3.3 which we will explore in greater detail for the remainder of this chapter. In doing so, we will generalize not only Woodall's result, but also many other results in (undirected) graph theory as well.

Section 3.2

Ore-type Digraphs Introduced

Let $k$ be an integer. A digraph $D$ having order $p$ is of Ore-type $(k)$ if

$$\od{u} + \id{v} \geq p + k$$

whenever $u$ and $v$ are distinct vertices for which $uv$ is not an arc of $D$. Every digraph $D$ is of Ore-type $(k)$ for some $k$; for example, if $D$ has order $p$, then $D$ is of Ore-type $(-p)$. Also, if $D$ is of Ore-type $(k)$, then $D$ is of Ore-type $(j)$ for every integer $j \leq k$. The Ore index $\Theta(D)$ of a digraph $D$ is the supremum of all integers $k$ for which $D$ is of Ore-type $(k)$. If $D$ is of Ore-type $(k)$ for every integer $k$, then we write $\Theta(D) = +\infty$. For example, the directed cycle $C_p$ has Ore index
$\Theta(C_p) = 2 - p$ if $p \geq 3$ while the complete symmetric bipartite digraph $K_{n,n}$ has $\Theta(K_{n,n}) = 0$ if $n \geq 2$. Since the complete symmetric digraphs $K_p$ are of Ore-type $(k)$ for every integer $k$, $\Theta(K_p) = +\infty$. Moreover, the complete symmetric digraphs are unique in this respect.

Let $D$ be a digraph with order $p$ which is different from $K_p$. Let $E(\overline{D})$ be the arc set of the complement $\overline{D}$ of $D$ and define

$$k = \min_{uv \in E(\overline{D})} (\text{od}_u + \text{id}_v - p).$$

Since $E(\overline{D}) \neq \emptyset$, $k$ exists. Also, $D$ is of Ore-type $(k)$ but is not of Ore-type $(k + 1)$. Hence, $\Theta(D) = k$. Moreover, it is easily seen that

$$-p \leq \Theta(D) \leq p - 4$$

(3.1)

if $\Theta(D) < +\infty$. Furthermore, these bounds are sharp since $\Theta(D) = -p$ if $D$ is the totally disconnected digraph $\overline{K_p}$ (where $p \geq 2$) while $\Theta(D) = p - 4$ if $D$ is the digraph obtained by removing an arc from $K_p$. Because of (3.1), if $D$ is a digraph for which $\Theta(D) < +\infty$, then $D$ has order $p \geq 4 + \Theta(D)$. Consequently, if $D$ is of Ore-type $(k)$ with order $p$ and $D \neq K_p$, then $p \geq k + 4$.

The following lemma compares the Ore-type of a digraph and a vertex deleted subgraph.

**Lemma 3.4:** If a nontrivial digraph $D$ is of Ore-type $(k)$, then $D-v$ is of Ore-type $(k-1)$ for every vertex $v$.

**Proof:** Let $D$ have order $p \geq 2$ and let $v$ be any vertex of $D$. 

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Then, $od_H^x \geq od_D^x - 1$ and $id_H^x \geq id_D^x - 1$ for every vertex $x$ in $H = D-v$. For every pair $x$ and $y$ of vertices in $H = D-v$, $xy$ is an arc of $H = D-v$ if and only if it is also an arc of $D$. Thus,

$$od_H^x + id_H^y \geq (p - 1) + (k - 1)$$

whenever $x \neq y$ and $xy$ is not an arc of $H = D-v$. Since $D-v$ has order $p - 1$, it is of Ore-type $(k - 1)$ and the lemma follows.

In a strong digraph $D$, the distance $d(u,v)$ is the length of any shortest $u$-$v$ path. The diameter $\text{diam } D$ of a strong digraph $D$ is the maximum among the distances $d(u,v)$ over all pairs of vertices $u$ and $v$ in $D$.

Lemma 3.5: If a digraph $D$ is of Ore-type $(-1)$, then $\text{diam } D \leq 2$.

Proof: Let $D$ have order $p$. If $p = 1$, then $\text{diam } D = 0$. So, suppose $p \geq 2$ and let $u$ and $v$ be vertices for which $d(u,v) = \text{diam } D$. If $uv$ is an arc of $D$, then $\text{diam } D = 1$. If $uv$ is not an arc of $D$, then

$$od_u + id_v \geq p - 1$$

which implies that $D$ has a third vertex $w$ such that $uw$ and $wv$ are arcs of $D$. Then, $\text{diam } D = 2$. In any event, $\text{diam } D \leq 2$.

For $n \geq 0$, a strong digraph $D$ is strongly $n$-connected if the removal of fewer than $n$ vertices always results in a strong digraph. For example, the complete symmetric digraph $K_p$ is strongly $(p-1)$-connected. Similarly, if $p \geq 3$, then the cycle $C_p$
and the symmetric cycle $S_p$ are strongly 1-connected and 2-connected respectively.

**Theorem 3.6**: If a digraph $D$ of order $p$ has Ore-type $(k)$, $k \geq -1$, and $D \neq K_p$, then $D$ is strongly $(k+2)$-connected.

**Proof**: Since $D \neq K_p$, $p \geq k + 4$. Let $V_1$ be any set of $k + 1$ or fewer vertices. By Lemma 3.4, $D-V_1$ is of Ore-type $(k - n)$ where $n$ is the order of $V_1$. But $k - n \geq -1$ and, by Lemma 3.5, $D-V_1$ is strongly connected. #

The preceding result is sharp in the sense that for each $k \geq -2$, there is an infinite class of digraphs of Ore-type $(k)$ which are not strongly $(k+3)$-connected. In particular, let $m$ and $n$ be any positive integers. Let $D = \overline{K}_m \cup K_n$ if $k = -2$ and let $D = \overline{K}_{k+2} + (\overline{K}_m \cup K_n)$ if $k \geq -1$. Then $D$ is of Ore-type $(k)$ but is not strongly $(k+3)$-connected.

We next consider the degree structure of Ore-type digraphs. Recall that $i\delta(D)$, $o\delta(D)$, and $t\delta(D)$ respectively denote the minimum among the indegrees, outdegrees, and total degrees of the vertices of $D$.

**Theorem 3.7**: Let $D$ be a digraph with $\theta(D) = k < +\infty$. Then:

1. $i\delta(D) \geq k + 2$;
2. $o\delta(D) \geq k + 2$; and
3. $t\delta(D) \geq 2k + 4$.

**Proof**: Let $D$ have order $p$. Then $p \geq k + 4$ and $o\delta(D) \leq p - 2$
since $D \neq K_p$. Let $v$ be a vertex for which $id v = i \delta(D)$. Then $D$ has a vertex $u$, $u \neq v$, such that $uv$ is not an arc of $D$. Hence,

$$i \delta(D) = id v \geq p + k - od u \geq k + 2.$$ 

A similar argument shows that $o \delta(D) \geq k + 2$. Since 

$$td u = id u + od u \geq i \delta(D) + o \delta(D)$$

for each vertex $u$, $t \delta(D) \geq 2k + 4$.

The bounds given in Theorem 3.7 are sharp in that for each $k \geq -2$, there is an infinite class of digraphs each of which have Ore index $k$ and attain the specific lower bounds. For example, let $m$ be any positive integer and let $D(k,m) = K_1 \cup K_m$ if $k = -2$ and $D(k,m) = K_{k+2} \cup (K_1 \cup K_m)$ if $k \geq -1$. Then 

$$i \delta(D(k,m)) = o \delta(D(k,m)) = k + 2$$

and $t \delta(D(k,m)) = 2k + 4$.

The maxima among the indegrees, the outdegrees, and the total degrees of vertices of $D$ are denoted by $i \Delta(D)$, $o \Delta(D)$, and $t \Delta(D)$ respectively.

**Theorem 3.8:** Let $D$ be a graph with order $p$ and $\theta(D) = k < +\infty$. Then:

1. $i \Delta(D) \geq (p + k)/2$;
2. $o \Delta(D) \geq (p + k)/2$; and
3. $t \Delta(D) \geq (p + k)$.

**Proof:** Suppose that $i \Delta(D) < (p + k)/2$. Then $D$ has fewer than $p(p + k)/2$ arcs. Let $u$ be any vertex of $D$. If $od u = p - 1$, then $od u \geq (p + k)/2$. If $od u < p - 1$ then $D$ has a vertex $v$
such that $v \neq u$ and $uv$ is not an arc of $D$. Thus,
\[
\text{od } u \geq p + k - \text{id } v \geq (p + k)/2
\]
since $\text{id } v = i\Delta(D)$. In any event, each vertex has outdegree at least $(p + k)/2$. Hence, $D$ has at least $p(p + k)/2$ arcs and this is a contradiction. Thus, $i\Delta(D) \geq (p + k)/2$. In a similar fashion, $o\Delta(D) \geq (p + k)/2$.

To show that $t\Delta(D) \geq p + k$ it suffices to exhibit a vertex $v$ with $td v \geq p + k$. Without loss of generality, we may suppose that $o\delta(D) \leq i\delta(D)$. Let $u$ be a vertex such that $od u = o\delta(D)$. Since $D \neq K_p$, $D$ has a vertex $v$, $v \neq u$, such that $uv$ is not an arc of $D$. Hence,
\[
td v = od v + id v \geq od u + id v \geq p + k
\]
and the lemma follows.

To demonstrate the sharpness associated with the previous result, it is convenient to present some additional notation. A spanning subdigraph $H$ of a digraph $D$ is a $k$-factor of $D$ if each vertex has indegree and outdegree $k$ in $H$. Also, a digraph $D$ is $k$-factorable if it can be represented as the union of arc-disjoint $k$-factors. For example, if $p \geq 3$, then the symmetric cycle $S_p$ is 1-factorable since $C_p$ is a 1-factor and $S_p$ can be represented as the arc-disjoint union of two copies of $C_p$.

Implementing two techniques of Harary [11, pp. 85, 89], the complete symmetric digraph $K_p$ can be 1-factored for $p \geq 2$. So, let $p \geq 2$ and let
\[
F(p,1), F(p,2), \ldots, F(p,p-1)
\]
be a 1-factorizing of $K_p$. Let $F(p,0)$ be the totally disconnected digraph $\overline{K}_p$ and define $f(p,j)$, $0 \leq j < p - 1$, to be the union of $F(p,0), F(p,1), \ldots, F(p,j)$. Then $f(p,j)$ has Ore index $k = 2j - p$ and $i\Delta(f(p,j)) = o\Delta(f(p,j)) = (p + k)/2$. Hence, the bounds on the maximum indegree and maximum outdegree are sharp. For this class of digraphs, we also have that $t\Delta(f(p,j)) = p + k = 2j$. Hence, the lower bound given in (3) of Theorem 3.8 cannot be substantially improved, if at all. For a real number $r$ let $\{r\}$ denote the smallest integer which is not less than $r$. We offer the following.

**Conjecture:** If a digraph $D$ has order $p$ and $\Theta(D) = k < +\infty$, then $t\Delta(D) \geq 2\{(p + k)/2\}$.

Section 3.3

The Hamiltonian Nature of Ore-type Digraphs

Let $k$ be a nonnegative integer and recall that a digraph $D$ is $k$-path hamiltonian if every path of length not exceeding $k$ is contained in a hamiltonian cycle of $D$. In the following, we develop a sufficient condition for a digraph to be $k$-path hamiltonian. But first it is convenient to establish two lemmas.

**Lemma 3.9:** Let $k$ be a nonnegative integer and let $D$ be a digraph of order $p$ and of Ore index $k$. Then, every path in $D$ of length not exceeding $k$ is contained in a cycle of length at least $k + 2$.
**Proof:** Since $D$ is of finite Ore index $k$, it follows that $p \geq k + 4$. Let $P: v_1, v_2, \ldots, v_{n+1}$ be any path in $D$ of length $n$, $0 \leq n \leq k$. We first extend $P$ to a path $P_1$ of length $k + 1$ in the following manner. By Lemma 3.7, $\od v_{n+1} \geq \od(D) \geq k + 2$. Since $n \leq k$, the vertex $v_{n+1}$ is adjacent to at least two vertices not belonging to $P$. Let $v_{n+2}$ be such a vertex and extend $P$ to the new path $P_1: v_1, v_2, \ldots, v_{n+1}, v_{n+2}$ which has length $n + 1$. If $n + 1 < k + 1$, then the procedure may be repeated until a path $P_1: v_1, v_2, \ldots, v_{k+2}$ of length $k + 1$ is obtained.

If $v_{k+2}v_1$ is an arc of $D$, then

$$C: v_1, v_2, \ldots, v_{k+2}, v_1$$

is a cycle of length $k + 2$ containing $P$. So, suppose that $v_{k+2}v_1$ is not an arc of $D$. Since $D$ has Ore index $k$,

$$\od v_{k+2} + \id v_1 \geq p + k$$

Suppose that every vertex $w$, for which $v_{k+2}w$ and $wv_1$ are both arcs of $D$, lies on $P_1$. Then

$$\od v_{k+2} + \id v_1 \geq p + k - 2$$

and this is a contradiction. Hence, $D$ has a vertex $w$ not in $P_1$ such that $v_{k+2}w$ and $wv_1$ are arcs of $D$. Then, $D$ has the cycle

$$C: v_1, v_2, \ldots, v_{k+2}, w, v_1$$

of length $k + 3$ which contains $P$. Hence, the lemma follows.

Let $D$ be a nontrivial digraph and let $V$ be a proper subset of $V(D)$. Then $D - V$ is the subdigraph of $D$ obtained from $D$.
by removing the vertices in $V$.

**Lemma 3.10:** Let $k$ be a nonnegative integer and let $D$ be a digraph having Ore index $k$. Let $P$ be any path of length $k$ and let $C$ be any longest cycle of $D$ containing $P$. If $V(C) \neq V(D)$, then $D - V(C)$ is of Ore-type $(0)$.

**Proof:** Let $D$ be a digraph of order $p$ satisfying the hypothesis of the lemma. Then $D \neq K_p$ and $p \geq k + 4$ since $D$ has Ore index $k$. By Lemma 3.9, $D$ has paths of length $k$ and each path of length $k$ is contained in a cycle of length at least $k + 2$. So, suppose that $D$ has a path $P: v_1, v_2, \ldots, v_{k+1}$ which is contained in no hamiltonian cycle of $D$. Let $C: v_1, v_2, \ldots, v_m, v_{m+1} = v_1$ be a longest cycle containing $P$. Then $k + 2 \leq m < p$. Since $V(C) \neq V(D)$, we let $D_1 = D - V(C)$.

For a vertex $y$ in $D_1$, let $n(y, C)$ and $n(C, y)$ denote respectively the number of vertices in $C$ dominated by $y$ and the number of vertices in $C$ dominating $y$. Let $P: y_1, y_2, \ldots, y_a$ be any (possibly trivial) path in $D_1$ and consider the sum $n(C, y_1) + n(y_a, C)$. For $k + 1 \leq i \leq m$, at most one of the arcs $v_{i-1}v_i$ and $y_av_{i+1}$ can be in $D$; otherwise, $D$ has a longer cycle containing $P$. Thus, $n(C, y_1) + n(y_a, C) \leq m + k$.

For each vertex $v$ of $D_1$ we define the following three sets:

1. $A_v = \{u \in V(D_1): vu \in E(D_1)\}$;
2. $R_v = \{u \in V(D_1): D_1$ has a $u-v$ path\}; and
3. $N_v = V(D_1) - (A_v \cup R_v)$.
For each vertex \( v \) of \( D_1 \), \( v \in R_v \) since \( D_1 \) has the trivial \( v-v \) path \( v \).

Suppose that \( N_v \neq \emptyset \) for some vertex \( v \) of \( D_1 \). Let \( x \in N_v \) and consider the partition \( A_v - R_v \), \( A_v \cap R_v \), \( R_v - (A_v \cup \{v\}) \), \( \{v\} \), \( N_v - \{x\} \), and \( \{x\} \) of \( V(D_1) \). Let these sets have cardinalities \( a_1, a_2, a_3, l, b, \) and \( 1 \) respectively. Then

\[
p - m = a_1 + a_2 + a_3 + b + 2 \quad (3.2)
\]
since \( D_1 \) has order \( p - m \). Also, \( x \in N_v \) implies that \( D_1 \) (and therefore \( D \)) contains neither the arc \( xv \) nor \( vx \). Hence,

\[
\text{od}_Dv + \text{id}_Dv \geq p + k \quad (3.3)
\]
and

\[
\text{od}_Dv + \text{id}_Dx \geq p + k . \quad (3.4)
\]

Since \( v \) is adjacent from no vertex in \( (A_v - R_v) \cup N_v \),

\[
\text{id}_Dv \leq a_2 + a_3 + n(v,C) .
\]

Also, since \( x \) is adjacent to no vertex in \( R_v \),

\[
\text{od}_Dx \leq a_1 + b + n(x,C) .
\]

These two observations together with (3.2) and (3.3) imply that

\[
m + k + 2 \leq n(C,v) + n(x,C) . \quad (3.5)
\]

Since \( v \) is adjacent to no vertex in either of the sets

\[ R_v - (A_v \cup \{v\}) \] and \( N_v \),

\[
\text{od}_Dv \leq a_1 + a_2 + n(v,C) .
\]

Because of (3.5), \( D_1 \) has no \( v-x \) path. Hence, \( x \) is adjacent from no vertex of \( A_v \). Thus,

\[
\text{id}_Dx \leq a_3 + b + n(C,x) .
\]

The preceding two observations together with (3.2) and (3.4) imply that
From (3.5) and (3.6), we observe that either
\[ n(C,v) + n(v,C) \geq m + k + 2 \]
or
\[ n(C,x) + n(x,C) \geq m + k + 2. \]
However, either situation implies that \( D \) has a longer cycle containing \( P \) and this is a contradiction. Thus, \( N_{v} = \emptyset \) for each vertex \( v \) of \( D_{1} \).

To see that \( D_{1} \) is of Ore-type (0), let \( u \) and \( v \) be any two vertices of \( D_{1} \) such that \( uv \) is not an arc of \( D_{1} \). Then, \( uv \) is not an arc of \( D \), which implies that
\[ p + k \leq od_{u} + id_{v} = od_{D_{1}}u + id_{D_{1}}v + n(u,C) + n(C,v). \]
Since \( v \notin A_{u} \), it follows that \( v \in R_{u} \) and, therefore, \( D_{1} \) has a \( v-u \) path. Hence, \( n(C,v) + n(u,C) \leq m + k \), and this implies that
\[ od_{D_{1}}u + id_{D_{1}}v \geq p - m. \]
Since \( D_{1} \) is of order \( p - m \), the subdigraph \( D_{1} \) is of Ore-type (0) and the lemma follows.

We are now prepared to present the primary result of this chapter.

**Theorem 3.11:** Let \( k \) be a nonnegative integer. If \( D \) is a non-trivial digraph of Ore-type (\( k \)), then \( D \) is \( k \)-path hamiltonian.

**Proof:** Let \( D \) have order \( p \geq 2 \) and suppose that \( D \) has a path \( P \) of length not exceeding \( k \) which is contained in no hamiltonian cycle. Then \( D \neq K_{p} \) and \( p \geq k + 4 \). By Lemma 3.9, either the path \( P \) is of length \( k \) or it is contained in a path \( T \) of length \( k \). Since \( T \) can lie on no hamiltonian cycle of \( D \), we
may assume, without loss of generality, that \( P \) has length \( k \).

Let \( C: v_1, v_2, \ldots, v_m, v_{m+1} = v_1 \) be a longest cycle containing \( P: v_1, v_2, \ldots, v_{k+1} \). By Lemma 3.9, \( k + 2 \leq m < p \). Since \( V(C) \neq V(D) \), Lemma 3.10 infers that \( D_1 = D - V(C) \) is strongly connected.

We next verify that the subpath \( P_1: v_{k+1}, v_{k+2}, \ldots, v_m, v_{m+1} = v_1 \) of \( C \) has at least three vertices which dominate vertices in \( D_1 \). Assume that this is not the case. Then each vertex in \( D_1 \) is dominated by at most \( k + 1 \) vertices of \( C \). Let \( x \) be any vertex of \( D_1 \) and observe that \( \text{id}_{D_1}x \leq p + k - m \). Since \( P_1 \) has length \( m - k + 1 \geq 3 \), it has a vertex \( v_i \) which dominates no vertex of \( D_1 \) and this implies that \( \text{od}_{D_1} v_i \leq m - 1 \). Since \( v_i x \) is not an arc of \( D \), it follows that

\[
p + k \leq \text{od}_{D_1} v_i + \text{id}_{D_1}x \leq p + k - 1
\]

which is a contradiction. Thus, \( P_1 \) has at least three vertices which dominate vertices in \( D_1 \).

A similar argument also shows that \( P_1 \) has at least three vertices which are dominated by vertices of \( D_1 \). It now suffices to consider two cases.

Case 1: Suppose that \( P_1 \) has vertices \( v_a \) and \( v_b \), where \( k + 1 \leq a < b \leq m + 1 \), such that \( v_a \) is adjacent to some vertex \( u_a \) of \( D_1 \) while \( v_b \) is adjacent from some vertex \( u_b \) of \( D_1 \). Since \( D_1 \) is strongly connected, it has a \( u_a - u_b \) path \( P_2 \). Consequently, \( v_a \) and \( v_b \) cannot be consecutive, for otherwise, \( D_1 \) would have a longer cycle containing \( P \). It may also be
assumed that no vertex $v_i$ (for $a < i < b$) is adjacent to or from any vertex of $D$; otherwise, $v_i$ could play the role of a "new" $v_a$ or $v_b$, depending upon the direction of the arc, and the preceding arguments would still apply.

Also, $D$ cannot contain both of the arcs $v_i v_{a+1}$ and $v_{b-1} v_{i+1}$ whenever $i$ is such that either $k + 1 \leq i < a$ or $b \leq i \leq m$; otherwise, $D$ has the $u_b - u_a$ path $u_b, v_b, v_{b+1}, \ldots, v_i, v_{a+1}, v_{a+2}, \ldots, v_{b-1}, v_{i+1}, v_{i+2}, \ldots, v_a, u_a$ which, together with the $u_a - u_b$ path $P_2$ in $D_1$, is a longer cycle containing $P$. Consequently,

$$1d v_{a+1} + od v_{b-1} \leq m + k - a + b - 2.$$  

Since the arcs $u_b v_{a+1}$ and $v_{b-1} u_a$ are not in $D$,

$$od u_b + id u_a \geq 2p + 2k - id v_{a+1} - od v_{b-1} \geq 2p + k - m + a - b + 2.$$  

In $D_1$ the outdegree of $u_b$ plus the indegree of $u_a$ is at most $2p - 2m - 2$. Hence, the number $n(u_b, C)$ of vertices in $C$ dominated by $u_b$ and the number $n(C, u_a)$ of vertices in $C$ dominating $u_a$ satisfy

$$n(C, u_a) + n(u_b, C) \geq m + k + a - b + 4. \quad (3.7)$$

We now count the arcs from vertices in $C$ to $u_a$ and from $u_b$ to vertices in $C$ by a second method in order to obtain an upper bound for $n(C, u_a) + n(u_b, C)$. For $1 \leq i \leq m$ consider the arcs $v_i u_a$ and $u_b v_{i+1}$. If $1 \leq i \leq k$, then all such arcs may occur for a total of $2k$ arcs. If $k + 1 \leq i < a$, then at most one such arc may occur for each choice of $i$, for a total of $a - k - 1$ arcs. If $a \leq i < b$, then only the arcs $v_a u_a$ and $u_b v_b$ may occur. If $b \leq i \leq m$, then at most one of the arcs $v_i u_a$ and
$u_b v_{i+1}$ may occur, for a total of $m - b + 1$ arcs. Hence,

$$n(C, u_a) + n(u_b, C) \leq m + k + a - b + 2$$

which contradicts (3.7). Therefore, Case 1 cannot occur.

Case 1 considered the situation where $P_1: v_{k+1}, v_{k+2}, \ldots, v_m, v_{m+1} = v_1$ had two vertices $v_a$ and $v_b$ (where $k + 1 \leq a < b \leq m + 1$) such that $v_a$ dominated a vertex of $D_1$ while $v_b$ was dominated by a vertex of $D_1$. Since Case 1 cannot occur, it suffices to consider the following case.

**Case 2:** If $v_a$ and $v_b$ are any two vertices of $P_1$ such that $v_b$ dominates vertices of $D_1$ and $v_a$ is dominated by vertices of $D_1$, then $a < b$. Let $v_a$ be the first vertex of $P_1: v_{k+1}, v_{k+2}, \ldots, v_m, v_{m+1} = v_1$ which is both different from $v_{k+1}$ and is adjacent from some vertex $u_a$ in $D_1$. Also, let $v_b$ be the last vertex of $P_1$ which is both different from $v_{m+1} = v_1$ and is adjacent to some vertex $u_b$ in $D_1$. Since $P_1$ has at least three vertices which are dominated by vertices in $D_1$ and at least three vertices which dominate vertices of $D_1$, such choices are possible and, in fact, $u_a \neq u_b$. Thus,

$$k + 1 < a < b < m + 1.$$  

In the following, let $P_2$ be any $u_b-u_a$ path.

Due to the hypothesis of Case 2, the vertex $v_{i-1}$ is adjacent to no vertex of $D_1$ and $v_{i+1}$ is adjacent from no vertex of $D_1$.

For $a < i \leq b$, $D$ cannot have both of the arcs $v_{i-1}v_{i+1}$ and $v_iv_{i+1}$; otherwise, $D$ has the $u_a-u_b$ path $u_a, v_a, v_{a+1}, \ldots, v_i, v_{i+1}, v_{i+2}, \ldots, v_b, u_b$ which together with the
u_b-u_a \text{ path } P_2 \text{ is a cycle longer than } C \text{ which also contains } P, \text{ and this cannot happen. Thus,}
\od u_{a-1} + \id v_{b+1} \leq 2m + a - b - 2.

Since D contains neither the arc \( u_a v_{b+1} \) nor the arc \( v_{a-1} u_b \),
\od u_a + \id u_b \geq 2p + 2k - \od v_{b+1} - \od v_{a-1} \geq 2p+2k-2m-a+b+2. \text{ In } D, \text{ the vertex } u_a \text{ is adjacent to at most } p - m - 1 \text{ vertices}
\text{ whereas } u_b \text{ is adjacent from at most } p - m - 1 \text{ vertices. Hence,}
\od n(C, u_b) + \id n(u_a, C) \geq 2k + b - a + 4. \quad (3.8)

As in Case 1, the sum \( \od n(C, u_b) + \id n(u_a, C) \) is now counted in a second manner. Consider the arcs \( v_i u_b \) and \( u_{a} v_{i+1} \) for \( 1 \leq i \leq m \).
If \( 1 \leq i \leq k \), then both arcs may be present for a total of 2k arcs. If \( k + 1 \leq i < a \), then only the arc \( u_a v_a \) may be present.
If \( a \leq i < b \), then at most one of the two arcs \( v_i u_b \) and \( u_{a} v_{i+1} \)
may be present for a total of \( b - a \) arcs. If \( b \leq i \leq m \), then
only the arc \( v_b u_b \) is present. Hence,
\od n(C, u_b) + \id n(u_a, C) \leq 2k + b - a + 2
which contradicts (3.8). Hence, Case 2 cannot occur and the theorem follows.

Theorem 3.11 is best possible in the sense that for each non-negative integer \( k \) there exist an infinite class of digraphs, each of Ore-type \((k - 1)\) which are not \( k \)-path hamiltonian. For example, let \( m \) and \( n \) be positive integers and let \( k \) be any nonnegative integer. Then, the symmetric digraph \( K_{k+1} + (K_m \cup K_n) \) is of Ore-type \((k - 1)\) but it is not \( k \)-path hamiltonian. In particular, if \( P \) is any spanning path of the subdigraph \( K_{k+1} \), then it is a
path of length \( k \) which is contained in no hamiltonian cycle of \( K_{k+1} + (K_m \cup K_n) \).

**Corollary 3.12:** Let \( k \) be a nonnegative integer. If a digraph \( D \) of order \( p \) has \( q \) arcs, \( q \geq p^2 - 2p + 2 + k \), then \( D \) is \( k \)-path hamiltonian.

**Proof:** Suppose the assertion is false. Let \( k \) be the smallest nonnegative integer for which a counterexample exists and let \( D \) be a counterexample of minimum order. Let \( D \) have \( p \) vertices and \( q \) arcs, then \( q \geq p^2 - 2p + 2 + k > 0 \) and, therefore, \( D \) is nontrivial. Since \( D \) is not \( k \)-path hamiltonian, it is not of Ore-type \((k)\). Hence, there exist distinct vertices \( u \) and \( v \) in \( D \) such that \( uv \) is not an arc of \( D \) and \( od\ u + od\ v < p + k \). Let

\[
N_v^* = \{ x \in V(D) : x = v \text{ or } xv \in E(D) \}
\]

and let

\[
S = V(D) - (N_v^* \cup \{ u \}) .
\]

Then \( N_v^* \) has \( 1 + id\ v \) vertices each having outdegree at most \( p - 1 \). Also, \( S \) has \( p - 2 - id\ v \) vertices each having out-degree at most \( p - 2 \). Since the number \( q \) of arcs in \( D \) equals the sum of the outdegrees of its vertices,

\[
q \leq od\ u + (1 + id\ v)(p - 1) + (p - 2 - id\ v)(p - 2) = od\ u + id\ v + p^2 - 3p + 3.
\]

Since \( od\ u + id\ v < p + k \) and \( q \geq p^2 - 2p + 2 + k \),

\[
p^2 - 2p + 2 + k \leq q \leq p^2 - 2p + 2 + k
\]

and equality must hold throughout. In particular, \( od\ u + id\ v = p + k - 1 \), each vertex \( N_v^* \) has outdegree \( p - 1 \), and each vertex
in $S$ has outdegree $p - 2$. This implies that $\od v \geq k + 1$, $\id v \geq k + 1$, and $N^*_v = N^*_v - \{v\}$ is nonempty. It also implies that $p \geq k + 3$ and that $D - \{u, v\}$ is a complete subdigraph of order $p - 2$.

We first consider the case $k = 0$. If $u$ is adjacent to any vertex $x$ of $S$, then we can easily extend the path $u, x$ to a hamiltonian $u-v$ path of $D$, which together with the arc $vu$ would be a hamiltonian cycle of $D$. If $u$ is adjacent to no vertex of $S$, or if $S = \emptyset$, then $u$ is adjacent to some vertex of $N^*_v$. Hence, there is a spanning $u-v$ path in the induced subdigraph $<N^*_v \cup \{u, v\}>$ and a spanning $v-u$ path in $<S \cup \{u, v\}>$. The union of these two paths is a hamiltonian cycle of $D$. Hence, if $k = 0$, then $D$ is 0-path hamiltonian and this is a contradiction. Thus, $k > 0$.

By the initial arguments, each vertex has outdegree at least $k + 1$. Thus, every path of length not exceeding $k$ is contained in a path of length $k$. Therefore, since $D$ is not $k$-path hamiltonian, $D$ has a path $P: v_1, v_2, \ldots, v_{k+1}$ of length $k$ which is contained in no hamiltonian cycle of $D$. It now suffices to consider two cases.

**Case 1:** Suppose that $\od v_{k+1} = p - 1$. Then $D - v_{k+1}$ has $p - 1$ vertices and $q'$ arcs where $q \geq (p-1)^2 - 2(p-1) + 2 + (k-1)$ arcs. Since $k - 1 \geq 0$, the induction hypothesis holds and $D - v_{k+1}$ has a hamiltonian cycle containing the path $P - v_{k+1}$ of length $k - 1$. Since $\od v_{k+1} = p - 1$ and since $v_kv_{k+1}$ is an arc of $D$, this
cycle can be extended to a Hamiltonian cycle of $D$. Since this is a contradiction, Case 1 cannot occur.

**Case 2:** Suppose that $\od v_{k+1} \leq p - 2$. Since $\od v_{k+1} \geq k + 1$, there is a vertex $v_{k+2}$ in $D - V(P)$ such that $v_{k+1}v_{k+2}$ is an arc of $D$. Let $D_1 = D - v_{k+1}$ if $v_kv_{k+2}$ is an arc of $D$ and let $D_1 = D - v_{k+1} + v_kv_{k+2}$ if $v_kv_{k+2}$ is not an arc of $D$. Then $D_1$ is a digraph of order $p - 1$ and $q'$ arcs where $q' \geq (p - 1)^2 - 2(p - 1) + 2 + k$. By the inductive hypothesis, $D_1$ is $k$-path Hamiltonian. Hence, $D_1$ has a Hamiltonian cycle containing the path $v_1, v_2, \ldots, v_k, v_{k+2}$. Since $D$ has both of the arcs $v_kv_{k+1}$ and $v_{k+1}v_{k+2}$, this cycle can be used to construct a Hamiltonian cycle of $D$ which contains $P$. Since this is a contradiction, Case 2 cannot occur and the result follows.

We now demonstrate the sharpness of the bound of $q$ in the preceding result. Let $k$ be any nonnegative integer and let $p$ be any integer satisfying $p \geq k + 2$. Let $D(p,k)$ be the digraph having the vertex set $\{v_1, v_2, \ldots, v_p\}$, where $\langle v_1, v_2, \ldots, v_{p-1}\rangle = K_{p-1}$, and the additional arcs $v_iv_p$ for $1 \leq i \leq p - 1$ and $v_pv_i$ for $1 \leq i \leq k$. Then, $D$ has $p^2 - 2p + 1 + k$ arcs. If $k = 0$, then $\od v_p = 0$ and, therefore, $D(p,k)$ is not Hamiltonian. If $k \geq 1$, then $D(p,k)$ has no Hamiltonian cycle containing the path $v_1, v_2, \ldots, v_k, v_p$ of length $k$. Hence, $D(p,k)$ is a digraph of order $p \geq k + 2$ having $q = p^2 - 2p + 1 + k$ arcs which is not Hamiltonian. Thus, the bound on "$q$" is sharp.
Section 3.4

Some Sufficiency Conditions
For Other Hamiltonian Properties

In the following, we adopt the conventions of Behzad and Chartrand [2] for (undirected) graphs. The k-path hamiltonian and k-path traceable properties in graphs are analogous to those in digraphs. A graph $G$ of order $p$ is of Ore-type $(k)$ if

$$\text{deg } u + \text{deg } v \geq p + k$$

for every two nonadjacent vertices $u$ and $v$ of $G$.

Let $G$ be a graph. The symmetric digraph $D(G)$ is the digraph obtained from $G$ by replacing each edge by the corresponding pair of symmetric directed arcs. Clearly, $G$ is k-path traceable if and only if the digraph $D(G)$ is k-path traceable. Also, if $G$ is of order $p \geq 3$, then $G$ is k-path hamiltonian if and only if $D(G)$ is k-path hamiltonian. Since $G$ is of Ore-type $(k)$ if and only if $D(G)$ is of Ore-type $(k)$, Theorem 3.11 provides a method for establishing a variety of results in graph theory. The first of these is the following result which is due to Dirac [6].

**Corollary 3.13 (Dirac):** Let $G$ be a graph of order $p \geq 3$. If $\text{deg } u \geq p/2$ for each vertex $u$, then $G$ is hamiltonian.

**Proof:** Let $D = D(G)$. Since

$$\text{id}_D u = \text{od}_D u = \text{deg}_G u \geq p/2$$

for every vertex $u$ in $V(D) = V(G)$, the nontrivial digraph $D(G)$ has Ore-type $(0)$ and, by Theorem 3.11, it is hamiltonian. Since $G$
is of order \( p \geq 3 \), \( G \) is hamiltonian.

The preceding result was generalized in two ways. For example, Nash-Williams [19] provided the following extension to digraphs which are not necessarily symmetric.

**Corollary 3.14 (Nash-Williams):** A digraph \( D \) of order \( p \geq 2 \) is hamiltonian if \( \text{id} \ u \geq p/2 \) and \( \text{od} \ u \geq p/2 \) for every vertex \( u \) of \( D \).

Since any digraph satisfying the hypothesis must be a non-trivial digraph of Ore-type (0), the conclusion follows immediately from Theorem 3.11. The following generalization by Ore [20] strengthens Dirac's results substantially.

**Corollary 3.15 (Ore):** If a graph \( G \) is of order \( p \geq 3 \) and \( \text{deg} \ u + \text{deg} \ v \geq p \) for every two nonadjacent vertices \( u \) and \( v \), then \( G \) is hamiltonian.

**Proof:** The graph \( G \) and, therefore, the digraph \( D(G) \) are of Ore-type (0). By Theorem 3.11, the nontrivial digraph \( D(G) \) is hamiltonian. Thus, \( G \) is hamiltonian since it has order \( p \geq 3 \). #

The preceding results by Nash-Williams and Ore were, in turn, generalized by Woodall [24].

**Corollary 3.16 (Woodall):** Let \( D \) be a digraph of order \( p \geq 2 \). If \( \text{id} \ u + \text{od} \ v \geq p \)
whenever $u$ and $v$ are distinct vertices for which $uv$ is not an arc of $D$, then $D$ is hamiltonian.

**Proof:** Since the hypothesis implies that $D$ is a nontrivial digraph of Ore-type $(0)$, it follows by Theorem 3.11 that $D$ is hamiltonian.

Ore's result was also generalized by Kronk [16] when he provided the following condition for a digraph to be $k$-path hamiltonian.

**Corollary 3.17 (Kronk):** Let $k$ be a nonnegative integer and let $G$ be a graph of order $p \geq 3$. If

$$\text{deg } u + \text{deg } v \geq p + k$$

whenever $u$ and $v$ are distinct nonadjacent vertices, then $G$ is $k$-path hamiltonian.

**Proof:** By Theorem 3.11, the nontrivial digraph $D(G)$ is $k$-path hamiltonian. Since $G$ is of order $p \geq 3$, the graph $G$ is also $k$-path hamiltonian.

In the preceding, we have been concerned with paths contained in hamiltonian cycles. A second path containment problem concerns the $k$-path traceable properties in graphs and digraphs.

**Corollary 3.18:** Let $k$ be a nonnegative integer. If a digraph $D$ is of Ore-type $(k - 1)$, then $D$ is $k$-path traceable.

**Proof:** Let $P$ be any path in $D$ of length not exceeding $k$. Let $v$ be a new vertex not already in $D$. Then, the symmetric join
D + v is a nontrivial digraph of Ore-type (k). Hence, P lies on a hamiltonian cycle C of D + v. Thus, P lies on the hamiltonian path C - v of D and the corollary follows. #

The preceding is an extension of the following result by Kapoor and Theckedath [15] for graphs.

**Corollary 3.19 (Kapoor and Theckedath):** Let k be a nonnegative integer. A graph G of order p is k-path traceable if

\[
\deg u + \deg v \geq p + k - 1
\]

for every two nonadjacent vertices u and v of G.

A third type of path problem was suggested by Ore [21] when he showed that every Ore-type (1) graph is hamiltonian-connected. Although this result could easily be established at this point, a more general result is just as easily obtained. To present this result, it is convenient to introduce some additional notation. Let P_1 and P_2 be disjoint paths in a digraph D. Then a P_1-P_2 path in D is a path beginning with the initial vertex of P_1, ending with the terminal vertex of P_2, and containing the paths P_1 and P_2. A digraph D is k-path hamiltonian-connected if given any two disjoint paths P_1 and P_2 of lengths \( \ell_1 \) and \( \ell_2 \), respectively where \( \ell_1 + \ell_2 \leq k \), then D has a hamiltonian P_1-P_2 path. We now offer the following result.

**Corollary 3.20:** Let k be a nonnegative integer. If a digraph D is of Ore-type (k + 1), then D is k-path hamiltonian-connected.
Proof: If $D = K_1$, then the conclusion follows vacuously. So, suppose that $D$ is nontrivial. Let $P_1$ and $P_2$ be any two disjoint paths having lengths $\varepsilon_1$ and $\varepsilon_2$ where $\varepsilon_1 + \varepsilon_2 \leq k$. Let $u$ be the terminal vertex of $P_2$ and $v$ be the initial vertex of $P_1$. Let $D_1 = D$ if $uv$ is an arc of $D$ and $D_1 = D + uv$ if $uv$ is not an arc of $D$. Then, $D_1$ is a nontrivial digraph of Ore-type $(k + 1)$ and, therefore, it is $(k + 1)$-path hamiltonian. Hence, the $P_2$-$P_1$ path $P$ containing the arc $uv$ has length $\varepsilon_1 + \varepsilon_2 + 1 \leq k + 1$ and lies on some hamiltonian cycle $C$ of $D_1$. Thus, $D$ has the hamiltonian $P_1$-$P_2$ path $C$-$uv$ and the corollary follows.

For graphs, the definition of $k$-path hamiltonian-connected is analogous to that given for digraphs. Clearly, a graph $G$ is $k$-path hamiltonian-connected if and only if $D(G)$ is $k$-path hamiltonian-connected. Hence, the following is an immediate consequence of Corollary 3.20.

Corollary 3.21: Let $k$ be a nonnegative integer. Let $G$ be a graph of order $p$ such that

$$\deg u + \deg v \geq p + k + 1$$

for every two nonadjacent vertices $u$ and $v$. Then $G$ is $k$-path hamiltonian-connected.

Since $0$-path hamiltonian-connectedness (in graphs and digraphs) corresponds to hamiltonian-connectedness, we have the following special cases of the preceding results; the last of which is the
result by Ore [21] mentioned previously.

**Corollary 3.22:** If a digraph $D$ is of Ore-type (1) then $D$ is hamiltonian-connected.

**Corollary 3.23 (Ore):** A graph $G$ of order $p$ is hamiltonian-connected if

$$\deg u + \deg v \geq p + 1$$

for every two nonadjacent vertices $u$ and $v$ of $G$. 

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CHAPTER IV

ITERATED LINE DIGRAPHS

In this chapter, the sequence $\lambda(D)$ of iterated line digraphs derivable from a nonempty digraph $D$ is considered and a characterization is provided of those connected digraphs for which this sequence contains infinitely many hamiltonian digraphs. Alternate (and more usable) characterizations are developed, stronger hamiltonian properties are investigated, a comparison with line graphs is made, and some applications are discussed.

Section 4.1

Pseudodigraphs and Some Alternate Notation

In general, the digraphs in this chapter are pseudodigraphs (i.e., multiple arcs and/or loops are permitted). A digraph is simple if it has neither multiple arcs nor loops. If $e$ is an arc of a digraph, then $i(e)$ and $r(e)$ respectively denote the initial and terminal vertices of $e$. For a nontrivial walk, we will often be interested in the arcs used rather than the vertices visited. Consequently, our usual representation (a sequence of vertices) is not adequate for this task. However, a nontrivial walk may also be represented by a sequence $e_1, e_2, \ldots, e_n$ of arcs where $e_i$ is adjacent to $e_{i+1}$ for $1 \leq i \leq n-1$. Since circuits, cycles, paths and trails are walks, this alternate
representation carries forward in the obvious manner. For conve-
nience and clarity, the choice of a particular representation
scheme will be determined by context.

A digraph \( D \) is said to be a **complete symmetric digraph** if
every two vertices of \( D \) are joined by a symmetric pair of arcs.
However, \( K_n \) will denote the unique simple complete symmetric
digraph with \( n \) vertices. Also, \( K_n^{(m)} \) will denote the digraph
obtained by replacing each arc of \( K_n \) by \( m \) copies of the arc
(hence, \( K_n^{(1)} = K_n \)) and \( PK_n^{(m)} \) will denote the digraph \( K_n^{(m)} \)
together with \( m \) loops at each vertex (if \( m = 1 \), we write \( PK_n \)
instead of \( PK_n^{(1)} \)).

**Section 4.2**

**The Line Digraph**

The **line digraph** \( L(D) \) of a nonempty digraph \( D \) has as its
vertex set \( V(L(D)) = \{ \hat{e} : e \in E(D) \} \) where \( \hat{e} \) is adjacent in
\( L(D) \) to \( \hat{f} \) if and only if the arc \( e \) is adjacent in \( D \) to \( f \).
For example, the line digraph of a cycle is a cycle of the same
length while the line digraph of a nontrivial path is a path of
length one less. A more illustrative example is provided by the
digraph \( D_1 \) (cf. Figure 4.1) consisting of two vertices, two
parallel arcs \( f \) and \( g \) which join one vertex to the other, and
a loop at the initial vertex of the arcs \( f \) and \( g \). The line di-
graph \( L(D_1) \) illustrates the fact that while a digraph may have
parallel arcs, its line digraph never does. Also, \( L(D) \) has loops
if and only if $D$ has loops; however, $L(D)$ never has parallel loops.

An arbitrary digraph $F$ is called a line digraph if it is (isomorphic to) the digraph $L(D)$ for some digraph $D$. Two of the more elegant characterizations of line digraphs were provided by Heuchenne [13] and Richards [23]. Before presenting these results, it is convenient to make the following definition. The adjacency matrix $A(D) = (a_{ij})$ of a digraph $D$ with $V(D) = \{v_1, v_2, \ldots, v_p\}$ is the $p \times p$ matrix where $a_{ij}$ is the number of arcs from $v_i$ to $v_j$.

Theorem 4.1 (Heuchenne): A digraph $D$ is a line digraph if and only if $D$ contains the arc $vy$ whenever it contains the arcs $ux$, $uy$ and $vx$.

Theorem 4.2 (Richards): A digraph $D$ is a line digraph if and only if every two rows of $A(D)$ are either identical or orthogonal.

If a line digraph $L(D)$ is nonempty, then the line digraph $L^2(D) = L(L(D))$ exists. In general, for $n \geq 1$, if $L^n(D)$ is nonempty, then $L^{n+1}(D) = L(L^n(D))$ exists. Consequently, with

\[D_1: \quad \begin{array}{ccc}
  e & f & \circ \\
  \circ & g & \\
\end{array} \quad \text{Figure 4.1: A digraph and its line digraph}
\]

\[L(D): \quad \begin{array}{ccc}
  \hat{e} & \hat{f} & \circ \\
  \circ & \circ & \circ \\
  \hat{g} & \circ & \circ \\
\end{array} \]
each nonempty digraph \( D \) there is associated a sequence
\[ \lambda(D): L(D), L^2(D), L^3(D), \ldots \]
of iterated line digraphs whose \( n \)th term is \( L^n(D) \). For example, the sequence \( \lambda(C) \) for a cycle \( C \) contains \( L^n(C) \) for all \( n \geq 1 \) while \( \lambda(P): L(P), L^2(P), \ldots, L^m(P) \)
for a path \( P \) of length \( m \geq 1 \).

Associated with each digraph \( L^n(D) \) in \( \lambda(D) \) there is a
natural labelling of its vertices and arcs which is induced by \( D \).
For example, if there is an arc in \( L(D) \) from \( \hat{e} \) to \( \hat{f} \), then it
is labelled \( \hat{e}\hat{f} \). Since \( L(D) \) has no parallel arcs, each arc
receives a distinct label. Also, the arc \( \hat{e}\hat{f} \) is in \( L(D) \) if and
only if the walk \( e,f \) of length two is in \( D \). Because of the
1-1 correspondence between \( E(L(D)) \) and \( V(L^2(D)) \), the arc
labelling of \( L(D) \) becomes the vertex labelling of \( L^2(D) \). Also,
there is an arc in \( L^2(D) \) from \( \hat{e}\hat{f} \) to \( \hat{g}\hat{h} \) if and only if \( \hat{f} = \hat{g} \);
if such is the case, then the arc receives the unique label \( \hat{e}\hat{f}\hat{g}\hat{h} \).
As above, the arc \( \hat{e}\hat{f}\hat{g} \) is in \( L^2(D) \) if and only if \( e,f,g \) is a
walk in \( D \). This process generalizes in the following manner. In
\( L^n(D) \), \( n \geq 2 \), there is an arc from the vertex \( \hat{f}_1\hat{f}_2\cdots\hat{f}_n \)
to the vertex \( \hat{g}_1\hat{g}_2\cdots\hat{g}_n \) if and only if \( \hat{f}_{i+1} = \hat{g}_i \) for \( 1 \leq i \leq n - 1 \)
and it is labelled \( \hat{f}_1\hat{f}_2\cdots\hat{f}_n\hat{g}_n \). Then, the arc \( \hat{f}_1\hat{f}_2\cdots\hat{f}_{n+1} \)
is in \( L^n(D) \) if and only if \( f_1, f_2, \ldots, f_{n+1} \) is a walk in \( D \). This
labelling of the digraphs in \( \lambda(D) \) is called the induced labelling
and we will presume this labelling for any digraph in \( \lambda(D) \).

Let \( D \) be a nonempty digraph with \( V(D) = \{v_1, v_2, \ldots, v_p\} \). A
vertex \( \hat{e}_1\hat{e}_2\cdots\hat{e}_n \) of \( L^n(D) \) in \( \lambda(D) \) is of type-\( (v_i, v_j) \) if
\( e_1, e_2, \ldots, e_n \) is a \( v_i-v_j \) walk in \( D \). Because of the 1-1
correspondence between the vertices of $L^n(D)$ and the walks in $D$ of length $n$, each vertex of $L^n(D)$ is of type-$(v_i,v_j)$ for some unique pair $v_i$ and $v_j$ of vertices in $D$. The following result makes use of a well-known property of the adjacency matrix.

**Theorem 4.3:** Let $D$ be a nonempty digraph with $V(D) = \{v_1,v_2,\ldots,v_p\}$. If $L^n(D)$ is in $\lambda(D)$ and $A^n = (a_{ij}^{(n)})$, where $A = A(D)$, then $L^n(D)$ has:

1. $a_{ij}^{(n)}$ vertices of type-$(v_i,v_j)$ for $1 \leq i, j \leq p$;
2. order $\sum_{1 \leq i,j \leq p} a_{ij}^{(n)}$.

**Proof:** Since $a_{ij}^{(n)}$ is the number of $v_i$-$v_j$ walks in $D$ of length $n$, (1) follows. Since each vertex in $L^n(D)$ is of type-$(v_i-v_j)$ for some pair $v_i$ and $v_j$ of vertices in $D$, (2) follows immediately from (1). \#

**Theorem 4.4:** For a nonempty digraph $D$, the sequence $\lambda(D)$ is infinite if and only if $D$ has a cycle.

**Proof:** Let $D$ have order $p$ and suppose that $\lambda(D)$ is infinite. Since $L^p(D)$ is in $\lambda(D)$, the digraph $D$ has a walk of length $p$ and, therefore, must contain a cycle. Conversely, if $D$ has a cycle, then it has walks of length $n$ for every $n \geq 1$. Thus, $L^n(D)$ exists for all $n \geq 1$ and $\lambda(D)$ is infinite. \#

From the induced labelling of digraphs in $\lambda(D)$ we can directly deduce the degree structure of any digraph $L^n(D)$ in $\lambda(D)$. 

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Theorem 4.5: Let $D$ be a nonempty digraph. If the vertex $e_1e_2\cdots e_n$ of $L^n(D)$ in $\lambda(D)$ is of type-$(u,v)$, then $id(e_1e_2\cdots e_n) = id u$ and $od(e_1e_2\cdots e_n) = od v$.

Proof: For $n \geq 1$, this is immediate. So, suppose that $n \geq 2$ and observe that $e_1e_2\cdots e_n$ is adjacent only from vertices in $L^n(D)$ of the form $e_0e_1\cdots e_{n-1}$. Since $D$ has exactly $id u$ such walks, $id(e_1e_2\cdots e_n) = id u$. Similarly, the outdegree of $e_1e_2\cdots e_n$ is $od v$ and the result follows.

Section 4.3

Connectedness and Connectivity
in Iterated Line Digraphs

It can easily be shown that the line digraph of a strongly connected nonempty digraph is strongly connected (and nonempty). However, strong connectedness is unique in this respect. For example, the transitive tournament $TT_p$ of order $p \geq 3$ is strictly unilateral (i.e., unilaterally connected but not strongly connected) and has a disconnected line digraph. For a second and more illustrative example, let $D$ (cf. Figure 4.2) be the digraph consisting of the path $v_1v_2\cdots v_6$ and the additional arcs $v_1v_5$ and $v_4v_2$. Then, $\lambda(D)$ is periodic and, for $i \geq 1$, consists of the strictly weak digraph $L(D) = L^{3i+1}(D)$, the disconnected digraph $L^2(D) = L^{3i+2}(D)$, and the strictly unilateral digraph $D = L^{3i}(D)$. Consequently, strong connectedness is the only type of connectedness which is invariant with respect to line digraphs.
Figure 4.2: A digraph and its line digraphs.

Clearly, if \( \lambda(D) \) contains at least one strong nonempty digraph \( L^n(D) \), then \( \lambda(D) \) contains infinitely many such digraphs since \( L^m(D) \) is strong and nonempty for all \( m \geq n \). Moreover, it can easily be shown that \( \lambda(D) \) contains a strong digraph if and only if either \( D \) is strong or \( D \) is the disjoint union of a strong nonempty digraph and an acyclic digraph.

The strong connectivity \( \kappa^s(D) \) of a digraph \( D \) is the minimum number of vertices whose removal results in a digraph which is either trivial or not strongly connected. The unilateral connectivity \( \kappa^u(D) \) and weak connectivity \( \kappa^w(D) \) are defined analogously.

Clearly, \( \kappa^s(D) \leq \kappa^u(D) \leq \kappa^w(D) \) for every digraph \( D \). The strong arc-connectivity \( \kappa^s_1(D) \) is the minimum number of arcs whose removal results in a digraph which is either trivial or not strongly connected. The unilateral arc-connectivity \( \kappa^u_1(D) \) and weak arc-connectivity \( \kappa^w_1(D) \) are defined analogously and \( \kappa^s_1(D) \leq \kappa^u_1(D) \leq \kappa^w_1(D) \) for every digraph \( D \).

**Theorem 4.6**: For every digraph \( D \),

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(1) \( \kappa^S(D) \leq \kappa^L_1(D) \leq \min \{i\delta(D), o\delta(D)\} \), and
(2) \( \kappa^W(D) \leq \kappa^W_1(D) \leq t\delta(D) \).

**Proof:** We prove only (1) since (2) follows in a similar manner. Since the conclusion is obvious if \( D \) is complete symmetric, we assume that \( D \) is not complete symmetric. Without loss of generality, let \( o\delta(D) \leq i\delta(D) \) and let \( u \) be a vertex of \( D \) such that \( od_u = o\delta(D) \). Let \( E \) be the set of arcs incident from \( u \), then \( D - E \) is not strongly connected. Hence, \( \kappa^S(D) \leq |E| \leq o\delta(D) = \min \{i\delta(D), o\delta(D)\} \).

We now verify that \( \kappa^S(D) \leq \kappa^S_1(D) \). Let \( E \) be a smallest set of arcs such that \( D - E \) is not strongly connected. If \( E = \emptyset \), then \( \kappa^S(D) = \kappa^S_1(D) = 0 \). If \( E = \{uv\} \), then either \( D - u \) or \( D - v \) is not strongly connected since \( D \) is not complete. So suppose that \( |E| \geq 2 \) and let \( uv \) be an arc of \( E \). Let \( E' \) be a subset of \( E \) containing no arcs from \( u \) to \( v \) or from \( v \) to \( u \). For each arc in \( E' \), select a vertex of \( D \) different from \( u \) and \( v \) and let \( V \) be the set of these vertices. If \( V(D) - V = \{u,v\} \), then \( D - (V - \{u\}) \) is trivial and \( \kappa^S(D) \leq |E| = \kappa^S_1(D) \). If \( V(D) - V \neq \{u,v\} \), then either \( D - (V \cup \{u\}) \) or \( D - (V \cup \{v\}) \) is not strongly connected and, therefore, \( \kappa^S(D) \leq |E| = \kappa^S_1(D) \).

The following result relates strong arc-connectivity in nonempty digraphs to strong connectivity in their line digraphs.

**Theorem 4.7:** If \( D \) is a nonempty digraph, then \( \kappa^S_1(D) \leq \kappa^S(L(D)) \).
Proof: If \( L(D) \) is complete, then \( D \) is \( K_1 \) together with some loops (and possibly some isolated vertices) and the result follows. Hence, we suppose that \( L(D) \) is not complete and let \( \hat{E} \) be a smallest subset of \( V(L(D)) \) such that \( L(D) - \hat{E} \) is not strong. Let \( E \) be the arcs in \( D \) corresponding to the vertices in \( \hat{E} \). Since \( L(D) - \hat{E} \) is not strong, it has vertices \( \hat{e} \) and \( \hat{f} \) for which no \( \hat{e}-\hat{f} \) path exists. Recall that \( \tau(e) \) is the terminal vertex of the arc \( e \) (corresponding to \( \hat{e} \)) and \( \iota(f) \) is the initial vertex of \( f \) (corresponding to \( \hat{f} \)). Then, \( D - E \) has no \( \tau(e) - \iota(f) \) path. Hence, \( \kappa^S_1(D) \leq |E| = |\hat{E}| = \kappa^S(L(D)) \).

Corollary 4.8: If \( D \) is a nonempty digraph, then \( \kappa^S(D) \leq \kappa^S(L(D)) \) and \( \kappa^S_1(D) \leq \kappa^S_1(L(D)) \).

Proof: From Theorems 4.6 and 4.7 we have that \( \kappa^S(D) \leq \kappa^S_1(D) \leq \kappa^S(L(D)) \leq \kappa^S_1(L(D)) \) and the result follows.

In general, there are no corresponding results for unilateral or weak connectedness. For example, let \( p \geq 4 \) and let \( TT_p \) denote the transitive tournament on \( p \) vertices. Then \( \kappa^U(L(TT_p)) = \kappa^W(L(TT_p)) = 0 < 1 = \kappa^U_1(TT_p) < \kappa^W_1(TT_p) \). However, if a line digraph \( F \) is strongly connected, then its weak arc-connectivity satisfies the following relation.

Theorem 4.9: Let \( F \) be a strongly connected nontrivial line digraph. Then, \( \kappa^W_1(F) \geq \begin{cases} 2 & \text{if } \delta(F) = 1, \\ 2\delta(F) - 2 & \text{if } \delta(F) \geq 2. \end{cases} \)
**Proof:** If $\delta(F) = 1$, then the result follows since $\kappa_{1}^{W}(D) \geq 2$ for every strong nontrivial digraph $D$. So, suppose that $\delta(F) \geq 2$ and let $E$ be a smallest subset of $E(F)$ such that $F - E$ is disconnected. Clearly, $F - E$ has exactly two weak components, say $F_1$ and $F_2$, and no arc of $E$ joins two vertices in the same weak component of $F - E$. Also, since $F$ is strong, $E$ has arcs from vertices in $F_1$ to vertices in $F_2$ and from vertices in $F_2$ to vertices in $F_1$. Since $F$ is a line digraph, it has no multiple arcs. For a vertex $v$ let $N(v)$ denote the vertices to which $v$ is adjacent and let $N^*(v)$ denote the vertices from which $v$ is adjacent. Note that $v \in N(v) \cap N^*(v)$ if and only if there is a loop at $v$.

Suppose that $E$ has an arc from $u_1$ in $F_1$ to $v_2$ in $F_2$ such that $N(u_1) \cap V(F_1) \neq \emptyset$ and $N^*(v_2) \cap V(F_2) \neq \emptyset$. Let $v_1 \in N(u_1) \cap V(F_1)$ and $u_2 \in N^*(v_2) \cap V(F_2)$. Since $N(u_1) \cap N(u_2) \neq \emptyset$ Heuchenne's condition (Theorem 4.1) implies that $N(u_1) = N(u_2)$. Likewise, $N^*(v_1) \cap N^*(v_2) \neq \emptyset$ implies that $N^*(v_1) = N^*(v_2)$. Again by Heuchenne's condition every vertex in $N^*(v_1)$ is adjacent to every vertex in $N(u_1)$. Let the sets $N(u_1) \cap V(F_1)$ and $N^*(v_1) \cap V(F_1)$ have cardinalities $n_i$ and $m_i$ respectively for $i = 1, 2$. Then $E$ will have to contain the $n_1m_2$ arcs from vertices in $N^*(v_1) \cap V(F_2)$ to those in $N(u_1) \cap V(F_1)$ and the $n_2m_1$ arcs from vertices $N^*(v_1) \cap V(F_1)$ to those in $N(u_1) \cap V(F_2)$. Hence, $E$ has cardinality at least $n_1m_2 + n_2m_1$. However, $\text{od } u_1 = \text{od } u_2 = n_1 + n_2 \geq \delta(F)$ and $\text{id } v_1 = \text{id } v_2 = m_1 + m_2 \geq \delta(F)$. Thus,
\begin{align*}
\kappa_1^w(F) &= |E| \geq n_1m_2 + n_2m_1 = (n_1 - 1)(m_2 - 1) + (n_2 - 1)(m_1 - 1) + \\
n_1 + n_2 + m_1 + m_2 - 2 &\geq 2\delta(F) - 2 \text{ since } n_i \geq 1 \text{ and } m_i \geq 1 \text{ for } i = 1,2.
\end{align*}

If the preceding case does not happen, then for each arc \(v_iv_j\) in \(E\) from \(v_i\) in \(F_i\) to \(v_j\) in \(F_j\), where \(\{i,j\} = \{1,2\}\), either \(N(v_i) \cap N(F_i) = \emptyset\) or \(N(v_j) \cap V(F_j) = \emptyset\); i.e., \(E\) contains all arcs incident from \(v_i\) or incident to \(v_j\). Since \(E\) contains arcs from vertices in \(F_1\) to vertices in \(F_2\) and from vertices in \(F_2\) to vertices in \(F_1\), the set \(E\) contains at least \(2\delta(F)\) arcs. In any event, \(\kappa_1^w(F) = |E| \geq 2\delta(F) - 2\).

To see that the preceding bound is sharp, let \(m \geq 2\) and let \(PK_m\) denote the complete pseudosymmetric digraph of order \(m\). Then \(PK_m\) is a line digraph (in fact, of the trivial digraph having \(m\) arcs) and has \(\kappa_1^w(PK_m) = 2\delta(PK_m) - 2 = 2m - 2\).

Section 4.4

Line Graphs and a Comparison

It is convenient at this point to discuss the undirected analogue to line digraphs. In particular, the line graph \(L(G)\) of a nonempty graph \(G\) has as its vertex set \(V(L(G)) = \{\hat{e} : e \in E(G)\}\) where two vertices are adjacent in \(L(G)\) if and only if the corresponding edges are adjacent in \(G\). As with the line digraph, the line graph of a cycle is a cycle of the same length and the line graph of a path of length \(m \geq 1\) is a path of length \(m - 1\). However, unlike the line digraph, the line
graph of an acyclic graph may possess a cycle (for example, the graph $G$ of Figure 4.3). In fact, if a graph has a vertex of degree at least three, then its line graph will contain a cycle.

The sequence $\lambda(G)$ of iterated line graphs of a nonempty graph $G$ is the sequence $L(G), L^2(G), \cdots$ where $L^{n+1}(G) = L(L^n(G))$ is in $\lambda(G)$ provided $L^n(G)$ exists and it is nonempty. Clearly, if $H$ is a nonempty subgraph of a graph $G$, then $L(H)$ is a subgraph of $L(G)$. With this observation, it can easily be shown that $\lambda(G)$ is infinite if and only if $G$ has either a cycle or a vertex of degree at least three.

In the directed case, the degree structure of digraphs in $\lambda(D)$ is always bounded. In particular, if $\Delta(D)$ denotes the maximum among the indegrees and the outdegrees of the vertices in $D$ and if $D$ is nonempty, then Theorem 4.5 implies that $\Delta(L^n(D)) \leq \Delta(D)$ for every digraph $L^n(D)$ in $\lambda(D)$. This is not the case for line graphs. For example, if a graph $G$ is regular of degree $d \geq 3$, then $L(G)$ is regular of degree $2d - 2 > d$. Consequently, if $G$ has a vertex of degree at least four, then $L(G)$ has a subgraph which is regular of degree at least three and this implies that $\Delta(L^n(G)) \to \infty$ as $n \to \infty$. In fact, for a nonempty graph $G$, it can be shown that $\Delta(L^n(G))$ is bounded as $n \to \infty$ if and only if
\( \Delta(G) \leq 3 \) and every vertex in \( G \) of degree three is adjacent with only vertices of degree one.

Because of the symmetrical adjacency relation, connectedness in graphs corresponds (in a sense) to strong connectedness in digraphs. This analogy extends to the interrelation between connectedness in graphs and their line graphs. For example, the line graph of a nontrivial connected graph is always connected. The connectivity \( \kappa(G) \) and edge-connectivity \( \kappa_1(G) \) of a graph \( G \) are analogous to their directed counter-parts and are related by \( \kappa(G) \leq \kappa_1(G) \leq \delta(G) \) where \( \delta(G) \) is the minimum among the degrees of the vertices of \( G \). Also, if \( G \) has at least two edges, then (in a manner similar to the proof of Theorem 4.7) it can easily be established that \( \kappa(G) \leq \kappa_1(G) \leq \kappa(L(G)) \leq \kappa_1(L(G)) \). In fact, Chartrand [4] has shown that \( \kappa(L^n(G)) \to \infty \) as \( n \to \infty \) whenever \( G \) is connected and has either a vertex of degree at least four or a vertex of degree three adjacent to a vertex of degree at least two. This cannot happen in iterated line digraphs since \( \kappa(L^n(D)) \leq \kappa_1(L^n(D)) \leq \Delta(L^n(D)) \leq \Delta(D) \) for all digraphs in \( \lambda(D) \). On the other hand, there is no analogue to Theorem 4.9 which relates the edge-connectivity in line graphs to the degree structure of the line graph. In fact, it is possible to find line graphs with arbitrarily large minimum degree and very small edge-connectivity. For example, let \( G_n \) (the graph \( G_3 \) is shown in Figure 4.4) be the graph consisting of two disjoint copies of a complete graph on \( n \) vertices (\( n \geq 3 \)) joined by a path of length two. Then, \( \delta(L(G_n)) = n - 1 \) but \( \kappa_1(L(G_n)) = 1 \). Moreover, their construction can be
generalized so as to produce any minimum degree (at least two) and any positive edge-connectivity smaller than the minimum degree.

A connected nonempty graph is eulerian if it has a circuit which contains all of its edges. Eulerian graphs have been characterized by Euler [7] as those connected nonempty graphs in which every vertex has even degree. If $G$ is an eulerian graph and $uv$ is an edge of $G$, then $uv$ is adjacent with $\deg u - \deg v - 2$ edges in $G$ (where $\deg x$ denotes the degree of $x$). Consequently, $L(G)$ is connected and contains only vertices of even degree and, therefore, it is eulerian. Hence, if $H$ is any nonempty graph and $\lambda(H)$ contains one eulerian graph, then $L^n(H)$ is eulerian for all sufficiently large $n$. In order to determine if $\lambda(H)$ contains any eulerian graph, it suffices to check only a finite number of graphs. In particular, Chartrand [3] has shown for a connected graph $G$ that $\lambda(G)$ contains an eulerian graph if and only if $G$, $L(G)$ or $L^2(G)$ is eulerian.

Clearly, if a graph $G$ has an eulerian circuit $e_1, e_2, \ldots, e_m$ then $L(G)$ has a hamiltonian cycle $\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_m, \hat{e}_1$. Consequently, the line graph of an eulerian graph is always hamiltonian. In fact, if $G$ has at least three edges and either a vertex or a circuit incident with all edges of $G$, then it can easily be shown that

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L(G) is hamiltonian. Thus, if G is hamiltonian, then so is L(G). Consequently, if λ(G) contains a hamiltonian graph L^n(G), then λ(G) contains infinitely many hamiltonian graphs; in particular, L^m(G) is hamiltonian for all m ≥ n. Moreover, Chartrand [3] has shown for any connected graph G, say of order p ≥ 3, which is not a path, that L^{p-3}(G) is hamiltonian. Consequently, if G is a connected graph which is not a path, then λ(G) contains infinitely many hamiltonian graphs since once the hamiltonian property is encountered in some line graph of λ(G), it is inherent in all subsequent line graphs.

Section 4.5

Hamiltonian and Eulerian Periods in Iterated Line Digraphs

We now investigate the properties associated with any digraph D for which λ(D) contains infinitely many hamiltonian digraphs. First, if λ(D) contains a hamiltonian digraph, then it can easily be shown that either D is strongly connected or else it is the disjoint union of an acyclic digraph and a strongly connected nonempty digraph. Since any acyclic weakly connected component will eventually cease to contribute to the iterated line digraphs, it suffices to consider only the sequences λ(D) where D is a strongly connected nonempty digraph.

Let D₁ (shown in Figure 3.5) be the digraph consisting of two vertices joined by a symmetric pair of arcs and a loop at one of the vertices. Clearly, D₁ and L(D₁) are hamiltonian while

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L₂(D₁) is not hamiltonian. Consequently, the occurrence of a hamiltonian digraph in λ(D) does not imply that all subsequent line digraphs are hamiltonian. We now demonstrate a necessary and sufficient condition for a line digraph to be hamiltonian.

A digraph D is eulerian if it has a circuit containing each vertex and arc of D and any such circuit is called an eulerian circuit of D. If a digraph D has an eulerian circuit C: e₁, e₂, ..., eₙ, then L(C): ê₁, ê₂, ..., êₙ, ê₁ is a hamiltonian cycle of L(D). Conversely, if D is a strongly connected digraph for which L(D) has a hamiltonian cycle C: ê₁, ê₂, ..., êₙ, ê₁, then e₁, e₂, ..., eₙ is a circuit in D containing all arcs (and therefore all vertices) of D; i.e., D is eulerian. Hence a strongly connected digraph has a hamiltonian line digraph if and only if it is eulerian. Since Lⁿ⁺¹(D) = L(Lⁿ(D)), we have that λ(D) has infinitely many hamiltonian digraphs if and only if it contains infinitely many eulerian digraphs.

Eulerian digraphs have been characterized as those strongly connected nonempty digraphs which are locally regular (i.e., id x = od x for each vertex x), If a digraph D is regular of degree d ≥ 1, then so is Lⁿ(D) for all n ≥ 1. Hence, if D is a
A strongly connected digraph which is regular of degree \( d \geq 1 \), then \( \lambda^n(D) \) is hamiltonian and eulerian for all \( n \geq 1 \) and \( \lambda(D) \) is an infinite sequence containing only hamiltonian digraphs. Conversely, if \( D \) is a strongly connected nonempty digraph and \( \lambda(D) \) contains only hamiltonian digraphs, then \( D \) and every digraph in \( \lambda(D) \) is eulerian (and, in particular, locally regular). Let \( u \) be any vertex of \( D \). If \( v \) is any other vertex of \( D \), then \( D \) has a \( u-v \) path of length \( d(u,v) = n \) and \( \lambda^n(D) \) has a vertex of type- \((u,v)\). Since \( \lambda^n(D) \) is locally regular, \( \text{id}_u = \text{od}_v = \text{id}_v \) and this implies that \( D \) is regular. Hence, since the situation is clear for regular digraphs, it suffices to consider nonregular digraphs.

For a strongly connected nonempty digraph \( D \), the eulerian digraphs in \( \lambda(D) \) are simply those which are locally regular. Recall that each vertex \( e_1 e_2 \cdots e_n \) in \( \lambda^n(D) \) has type- \((i(e_1), t(e_n))\), where \( i(e) \) and \( t(e) \) denote respectively the initial and terminal vertices of the arc \( e \), and has \( \text{id}(e_1 e_2 \cdots e_n) = \text{id}(e_1) \) and \( \text{od}(e_1 e_2 \cdots e_n) = \text{od}(e_n) \). Hence, it suffices to determine what vertex types occur in \( \lambda^n(D) \) for each \( n \geq 1 \).

Let \( V(D) = \{v_1, v_2, \ldots, v_p\} \). The boolean matrix \( B(D) = (b_{ij}) \) of \( D \) is the \( p \times p \) \((0,1)\)-matrix where \( b_{ij} = 1 \) if and only if there is an arc from \( v_i \) to \( v_j \). The \( n^{th} \) power \( B^n(D) \) of \( B(D) \) is computed by standard matrix multiplication with the coefficient arithmetic being performed in the boolean ring \( \{0,1\} \); i.e., \( 0 + x = x, 1 + x = 1, 0 \cdot x = 0 \), and \( 1 \cdot x = x \) for all \( x \in \{0,1\} \). The \( n^{th} \) power \( A^n(D) = (a_{ij}^n) \) of the adjacency
matrix $A(D)$ and $B^n(D) = (b_{ij}^{(n)})$ are related in that $b_{ij}^{(n)} = 1$ if and only if $a_{ij}^{(n)} \neq 0$; i.e., $b_{ij}^{(n)} = 1$ if and only if $D$ has a $v_i$-$v_j$ walk of length $n$. Because of the correspondence between the $v_i$-$v_j$ walks in $D$ of length $n \geq 1$ and the vertices of type-$(v_i, v_j)$ in $L^n(D)$, the matrix $B^n(D)$ describes the vertex types in $L^n(D)$. Consequently, $L^n(D)$ is locally regular if and only if $b_{ij}^{(n)} = 1$ implies id $v_i = \text{od } v_j$. Since the sequence $B(D), B^2(D), B^3(D), \ldots$ can have only finitely many distinct terms, there is a largest $n$ such that $B(D), B^2(D), \ldots, B^n(D)$ are all distinct. Also, there is a smallest $m$, $1 \leq m \leq n$, such that $B^m(D) = B^{n+1}(D)$ and this implies that $B^{m+j}(D) = B^{n+1+j}(D)$ for all $j \geq 0$. Consequently, $\lambda(D)$ has infinitely many eulerian digraphs if and only if $L_j(D)$ is locally regular for some $j$ where $n - m + 1 \leq j \leq n$. Also, if such a digraph does exist in $\lambda(D)$, then they must occur periodically throughout $\lambda(D)$ and $D$ is said to have an eulerian period of length $m$.

We now consider a previous example, in particular, the digraph $D_1$ of Figure 4.5. Since

$$B(D_1) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$B^n(D_1) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

for all $n \geq 2$, the digraph $L^n(D_1)$ has a vertex of type-$(v_1, v_2)$ for each $n \geq 1$. Since $2 = \text{id } v_1 \neq \text{od } v_2 = 1$, the line digraph $L^n(D)$ is noneulerian for every $n \geq 1$. Thus, $\lambda(D)$ has only one hamiltonian digraph, in particular, the digraph $L(D)$.

We now exhibit a nonregular digraph $D$ for which $\lambda(D)$
contains infinitely many hamiltonian and nonhamiltonian digraphs.
In particular, let $D$ consist of the vertices $v_1$ and $v_2$
together with two arcs from $v_1$ to $v_2$ and one arc from $v_2$ to
$v_1$. Then
\[
B^{2k-1}(D) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
\]
and
\[
B^{2k}(D) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]
for all $k \geq 1$. Consequently, $L^{2k-1}(D)$ has only vertices of
type-$(v_1,v_2)$ and type-$(v_2,v_1)$ and, therefore, it is eulerian.
Hence, $L^{2k}(D)$ is hamiltonian for all $k \geq 1$. Similarly, $L^{2k}(D)$
has only vertices of type-$(v_1,v_2)$ and type-$(v_2,v_2)$ and, therefore,
it is not eulerian. Hence, $L^{2k+1}(D)$ is not hamiltonian for all
$k \geq 1$. Thus, $\lambda(D)$ contains infinitely many hamiltonian and
nonhamiltonian digraphs.

For a digraph $D$ let $\text{Odeg}(D)$ and $\text{Ideg}(D)$ respectively
denote the set of outdegrees and indegrees associated with the
vertices of $D$. Also, let $\text{Deg}(D)$ be the union of $\text{Odeg}(D)$ and
$\text{Ideg}(D)$.

**Theorem 4.10:** If $D$ is a strongly connected digraph with an
eulerian period, then:

1. $\text{Deg}(D) = \text{Odeg}(D) = \text{Ideg}(D)$ and
2. for each $d \in \text{Deg}(D)$ and for each cycle $C$ in $D$,
   \[ \text{idu} = \text{odv} = d \]
   for some pair of vertices $u$ and $v$
in $C$.

**Proof:** Since $D$ has an eulerian period, $L^n(D)$ is eulerian for
some $n \geq 1$. Let $d \in \text{Ideg}(D)$ and let $u \in V(D)$ be such that $id u = d$. Since $D$ is strongly connected and nonempty, $D$ has a $u$-$v$ walk of length $n$ for some vertex $v$. Then $L^n(D)$ has a vertex of type-$(u,v)$ and, since $L^n(D)$ is locally regular, $id u = od v = d$. Thus, $\text{Ideg}(D)$ is a subset of $\text{Odeg}(D)$.

Similarly, $\text{Odeg}(D)$ is a subset of $\text{Ideg}(D)$ and (1) follows.

Let $d \in \text{Deg}(D)$ and let $C: v_1,v_2,\ldots,v_k,v_1$ be any cycle of $D$. Then $D$ has a vertex $u$ such that $id u = d$. Also, $L^n(D)$ is eulerian for some $n \geq d(u,v_1)$ where $d(u,v_1)$ is the distance from $u$ to $v_1$. Let $(i)_k$ denote the smallest positive integer congruent to $i$ modulo $k$. Then, $D$ has a $u$-$v_{(i)_k}$ walk of length $d(u,v_1) - 1 + i$ for each $i \geq 1$; in particular, follow any shortest $u$-$v_1$ path and then continue to follow $C$ until a walk of the desired length is obtained. Let $i = n + 1 - d(u,v_1)$, then $L^n(D)$ has a vertex of type-$(u,v_{(i)_k})$. Since $L^n(D)$ is eulerian, $od v_{(i)_k} = id u = d$. In a similar fashion, $C$ has a vertex $v$ such that $id v = d$ and the theorem follows.

The length of a walk $W$ is denoted by $\ell(W)$ and the greatest common divisor of a set $N$ of positive integers is denoted by $\gcd N$. Let $\text{Circ}(D)$ be the set of all nontrivial closed walks in the digraph $D$. If $D$ has a cycle, then $g = \gcd(\ell(W) : W \in \text{Circ}(D))$ exists and $\text{Circ}(D)$ has walks $W_1,W_2,\ldots,W_n$ (for some $n$) such that $g = \sum_{i=1}^n k_i \ell(W_i)$ for some integers $k_1,k_2,\ldots,k_n$.

**Lemma 4.11:** Let $a_1,a_2,\ldots,a_n$ be positive integers. Then, there exist integers $k_1,k_2,\ldots,k_n$ such that
\[ g = \gcd(a_1, a_2, \ldots, a_n) = \sum_{i=1}^{n} k_i a_i \text{ and } k_i \geq 1 \text{ for } 2 \leq i \leq n. \]

**Proof:** Clearly, there exist integers \( b_1, b_2, \ldots, b_n \) such that
\[ g = \sum_{i=1}^{n} b_i a_i. \]
For \( i \geq 2 \), if \( b_i \geq 1 \), let \( k_i' = k_i'' = 0 \).
If \( b_i \leq 0 \), then choose any integer \( c_i \) such that \( c_i a_1 + b_i > 0 \)
and let \( k_i' = c_i a_1 \) and \( k_i'' = -c_i a_i \). Then,
\[ \sum_{i=1}^{n} k_i a_i + \sum_{i=1}^{n} k_i'' a_i = 0. \]
Let \( k_1 = b_1 + \sum_{i=1}^{n} k_i' \) and for \( j \geq 2 \) let \( k_j = b_j + k_j' \). Then \( k_j \geq 1 \) for \( j \geq 2 \) and \( g = \sum_{i=1}^{n} k_i a_i \).

**Theorem 4.12:** Let \( D \) be a strongly connected nonregular digraph
with an eulerian period. Then, \( g = \gcd(\varepsilon(W) : W \in \text{Circ}(D)) \geq 2. \)

**Proof:** Suppose that \( g = 1 \). By Theorem 4.10, the digraph \( D \) has
no loops. Let \( \{W_1', W_2', \ldots, W_n'\} \) be a smallest subset of \( \text{Circ}(D) \)
for which \( \gcd(\varepsilon(W_i') : 1 \leq i \leq n) = \gcd(\varepsilon(W) : W \in \text{Circ}(D)) \). Then
\( n \geq 2 \); for otherwise, \( n = 1 \) and \( 1 = g = \varepsilon(W_1') \geq 2 \). There exist
integers \( k_1, k_2, \ldots, k_n \) such that \( 1 = \sum_{i=1}^{n} k_i \varepsilon(W_i') \) and \( k_i \geq 1 \)
for \( i \geq 2 \). For \( 1 \leq i \leq n \), let \( W_i \) be \( W_i' \) repeated \( |k_i| \)
times. Then, \( 1 = \varepsilon(W_1) + \sum_{i=2}^{n} \varepsilon(W_i) \); i.e., \( \gcd(\varepsilon(W_1), \varepsilon(W_i)) = 1. \)

Let \( \varepsilon = \varepsilon(W_1) \) and let \( W_1 : v_1, v_2, \ldots, v_r, v_1 \). For \( i \geq 2 \)
let \( u_i \) be the initial vertex of \( W_i \) and let \( U_i \) be a \( u_i - u_{i+1} \)
path where \( u_{n+1} = v_1 \). For \( j \geq 0 \), let \( W(j) \) denote the \( u_2 - v_1 \)
walk obtained by repeating \( W_i \) a total of \( j \) times followed by
\( U_i \) for \( i = 2, 3, \ldots, n \). If \( \varepsilon(W(0)) = k \), then \( \varepsilon(W(j)) = k + jL \)
where \( L = \sum_{i=2}^{n} \varepsilon(W_i) \).
We now show that $D$ has a $u_2$-$v$ walk of length $k + rL$ for every vertex $v$ in $W_1$. Let $P(j)$ be the $v_1$-$V(j+1)$, walk of length $j$ which follows $W_1$. For $0 \leq j < r$, the walk $W(j)$ followed by $P(rL-jL)$ is a $u_2$-$V(rL-jL)$, walk of length $k + rL$. If $(rL-j_1L) \equiv (rL-j_2L) \mod r$ for some $j_1$ and $j_2$, $0 \leq j_1 \leq j_2 < r$, then $j_1 = j_2$ since gcd$(r,L) = 1$. Hence, there exists at least one $u_2$-$v$ walk of length $k + rL$ in $D$ for each vertex $v$ in $W_1$. Hence, for each vertex $v$ in $W_1$, there is a $u_2$-$v$ walk in $D$ of length $n$ for every $n \geq k + rL$. Since $D$ has an eulerian period, there is a smallest $N \geq k + rL$ for which $L^N(D)$ is eulerian. Since $L^N(D)$ has vertices of type-$(u_2,v)$ for every vertex $v$ in $W_1$, id $u_2 = od v$. From Theorem 4.10, we have that $Deg(D) = \{od v : v \in V(W_1)\} = \{id u_2\}$. Since $D$ is not regular, this is a contradiction and the theorem follows.

Let $Cyc(D)$ denote the set of cycles in $D$, then $Cyc(D)$ is a subset of Circ($D$). Hence, if $D$ is not acyclic, then $g_1 = \gcd\{\ell(W) : W \in Circ(D)\}$ divides $g_2 = \gcd\{\ell(W) : W \in Cyc(D)\}$. On the other hand, each closed walk $W$ in Circ$(D)$ has a cycle decomposition which accounts for arc and vertex multiplicities in $W$. Hence, $g_2$ divides $g_1$ since $g_2$ divides $\ell(W)$ for each $W$ in Circ$(D)$. Thus, $\gcd\{\ell(W) : W \in Circ(D)\} = \gcd\{\ell(W) : W \in Cyc(D)\}$. However, it is not necessary to enumerate all cycles of $D$ in order to determine $\gcd\{\ell(W) : W \in Cyc(D)\}$. For $i \geq 1$, let $J_i(D) = i$ if the diagonal of the boolean matrix $B^i(D)$ has a nonzero entry and $J_i(D) = 0$ if the diagonal contains only zero entries.

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Theorem 4.13: Let $D$ be a digraph with $V(D) = \{v_1, v_2, \ldots, v_n\}$.
If $D$ has cycles, then $\gcd(\ell(W) : W \in \text{Cyc}(D)) = \gcd(\ell_i(D) : 1 \leq i \leq n)$.

Proof: Since $\text{Cyc}(D) \subseteq \text{Circ}_n(D) = \{W \in \text{Circ}(D) : \ell(W) \leq n\} \subseteq \text{Circ}(D)$ implies that $g = \gcd(\ell(W) : W \in \text{Circ}(D)) \leq \gcd(\ell(W) : W \in \text{Circ}_n(D)) \leq \gcd(\ell(W) : W \in \text{Cyc}(D))$, the result follows.

Section 4.6

A Characterization of Digraphs 
Having an Eulerian Period

Let $S: s_1, s_2, \ldots, s_k$ be a sequence of positive integers and recall that $d_F(u, v)$ is the length of a shortest $u-v$ path in the digraph $F$. If $u$ is any vertex of a cycle $C$ in a digraph $D$, then $(C, u, a, b)_S$ is a cyclic $S$-score assignment of $C$ if the length $k$ of $S$ divides the length $\ell(C)$ of $C$ and $d_C(u, v) = j$ implies that $id_v = s(a+j)_k$ and $od_v = s(b+j)_k$ for every vertex $v$ of $C$. Clearly, if $(C, u, a, b)_S$ is a cyclic score assignment of $C$ and $d_C(u, v) = j$, then so is $(C, v, a+j, b+j)_S$. Two such cyclic $S$-score assignments are considered equivalent. A strongly connected digraph $D$ is $S$-score cyclic if each cycle of $D$ can be given a cyclic $S$-score assignment. Also, a digraph is score cyclic if it is $S$-score cyclic for some finite sequence $S$ of positive integers.

For example, let $S: s_1, s_2$ be the sequence where $s_1 = 1$ and $s_2 = 2$. The digraph $D_n$ $(n \geq 1)$ of Figure 4.6 consists of the two cycles $v_1, v_2, \ldots, v_n, v_1$ and $v_{4n}, v_{4n-1}, \ldots, v_{2n+1}, v_{4n}$ together with the additional arcs $v_{2i}v_{4n-2i+1}$ for $1 \leq i \leq 2n$. Let $V_1$ be the...
vertices of $D_n$ with odd index and $V_2$ be those with even index. Since vertices of $V_1$ are adjacent only to vertices of $V_2$ and vice versa, the digraph $D_n$ contains only cycles of even length. Hence, the length of $S$ divides the length of every cycle in $D_n$.

If $C$ is any cycle of $D_n$, then it contains some vertex $u$ in $V_1$. Consequently, $(C,u,2,1)_S$ is a cyclic $S$-score assignment of $C$ since $d_C(u,v) = j$ implies that $id v = s(j+2)$ and $od v = s(j+1)$ for each vertex $v$ in $C$. Likewise, each cycle of $D_n$ can be given a cyclic $S$-score assignment and, therefore, $D_n$ is $S$-score cyclic. In a similar fashion, one can show that every strongly connected $r$-regular digraph ($r \geq 1$) is $(r)$-score cyclic and that the digraph $m \cdot C_n$ (consisting of $m \geq 2$ copies of the $n$-cycle $C_n$, $n \geq 2$, which are pairwise disjoint except for one vertex which is common to every cycle) is $(m,a_2,a_3,\ldots,a_n)$-score cyclic where $a_2 = a_3 = \ldots = a_n = 1$.

An $S$-score cyclic digraph $D$ is {fixed $S$-score cyclic} if given any cyclic $S$-score assignment $(C,v,a,b)_S$ and any cycle $C'$ also containing $v$, then $(C',v,a,b)_S$ is a cyclic $S$-score assignment of $C'$. A digraph is {fixed score cyclic} if it is fixed $S$-score cyclic.
cyclic for some sequence $S$. It can easily be shown that each of the three preceding examples are fixed score cyclic. To see that not all score cyclic digraphs are fixed score cyclic, we consider the digraph $F$ in Figure 3.7. Let $S: s_1, s_2, s_3, s_4$ where $s_1 = s_2 = 1$ and $s_3 = s_4 = 2$. It can easily be verified that $F$ is $S$-score cyclic. Also, except for a cyclic permutation, $S$ is the only sequence for which $F$ is score cyclic; this follows since $F$ has a 4-cycle containing two consecutive vertices of indegree 1 and two consecutive vertices of indegree 2. Let $C_1$ be the 4-cycle of $F$ containing both $u$ and $w$ and let $C_2$ be the 8-cycle of $F$. Then, $(C_1, u, 4, 3)_S$ is a cyclic $S$-score assignment $C_1$ since:

1. $d_{C_1}(u, u) = 0$ and $\text{id } u = s_{(4+0)} = 2 = s_{(3+0)} = \text{od } u$,

2. $d_{C_1}(u, v) = 1$ and $\text{id } v = s_{(4+1)} = 1$ and $\text{od } v = s_{(3+1)} = 2$,
(3) \( d_{c_1}(u,w) = 2 \) and \( id\ w = s_{4+2} = 1 \), \( s_{3+2} = \) od\ w , and

(4) \( d_{c_1}(u,x) = 3 \) and \( id\ x = s_{4+3} = 2 \) and \( o d\ x = s_{(3+3)4} = 1 \).

However, \( C_2 \) also contains \( u \) and \( (C_2,u,4,3)_S \) is not a cyclic S-score assignment of \( C_2 \) since \( d_{c_2}(u,y) = 1 \) and \( o d\ y = 1 \neq s_{(3+1)4} = 2 \). Consequently, \( F \) is score cyclic but not fixed score cyclic.

**Theorem 4.14:** A strongly connected nonempty digraph \( D \) has an eulerian period if and only if it is fixed score cyclic.

**Proof:** We first suppose that \( D \) is fixed score cyclic for some sequence \( S: s_1, s_2, \ldots, s_k \). Since \( \lambda(D) \) is an infinite sequence of strongly connected nonempty digraphs, it suffices to show that it contains infinitely many locally regular digraphs. Let \( u \) be an arbitrary but fixed vertex of \( D \). Then \( u \) is contained in a cycle \( C \) and \( C \) has a cyclic S-score assignment \( (C,u,a,b)_S \). Also, there exists a smallest positive integer \( j \) such that \( (b + j)_k = (a)_k \).

Let \( n \geq 0 \) and consider any walk \( W \) in \( D \) of length \( nk + j \). Then, \( D \) has a path \( P \), say of length \( m \), from \( u \) to the initial vertex of \( W \). Let \( W^*: e_1, e_2, \ldots, e_{nk+j+m} \) be the walk consisting of \( P \) followed by \( W \) where \( e_i \) is an arc from \( v_i \) to \( v_{i+1} \); note that \( u = v_1 \), the initial vertex of \( W \) is \( v_{m+1} \) and the terminal vertex of \( W \) is \( v_{nk+j+m+1} \). Since \( D \) is strongly connected, \( D \) contains a cycle \( C \) through \( u \) in \( C \). Consequently, \( F \) is score cyclic but not fixed score cyclic.
connected, each arc $e_i$ lies on some cycle $C_i$. Since both $C$ and $C_i$ contain $u = v_1$, the cycle $C_1$ has a cyclic $S$-score assignment $(C_1, v_1, a, b)_S$ which is equivalent to $(C_1, v_2, a+1, b+1)_S$.

In general, for $1 \leq i \leq nk+j+m$, the cycle $C_i$ has a cyclic $S$-score assignment $(C_i, v_i, a+i-1, b+i-1)_S$ which is equivalent to $(C_i, v_{i+1}, a+i, b+i)_S$. Hence, $id_{m+1} = s(a+m)_k$ and $od_{v_{(nk+j+m+1)}} = s(b+nk+j+m) = s(a+m)_k$ since $(b+j)_k = (a)_k$.

Because of the correspondence between walks in $D$ of length $nk+j$ and vertices in $L^{nk+j}(D)$ and since $W$ was an arbitrary walk of length $nk+j$, the digraph $L^{nk+j}(D)$ is locally regular for every $n \geq 0$. Hence, $D$ has an eulerian period.

Conversely, suppose that $D$ has an eulerian period. Since $D$ is score cyclic if it is regular, we consider the case where $D$ is not regular. Then $D$ has at least two cycles. Let $g = \gcd\{\varepsilon(C) : C \in \text{Cyc}(D)\}$. We will first show that for each cycle $C$ in $D$ there is an integer $a$ (possibly dependent on $C$) such that for every pair $u$ and $v$ of vertices of $C$:

1. $od_u = od_v$ if $d_C(u, v) \equiv 0 \mod g$ and
2. $id_u = od_v$ if $d_C(u, v) \equiv a \mod g$.

Let $C_1: u_1, u_2, \ldots, u_m, u_1$ be any cycle of $D$ and let $C_2, C_3, \ldots, C_n$ be the remaining cycles. By Lemma 4.11, there exist integers $k_1, k_2, \ldots, k_n$ such that $\sum_{i=1}^{n} k_i \varepsilon(C_i) = g$ and $k_i \geq 1$ for $2 \leq i \leq n$. Then, $\gcd(m, L) = g$ where $L = \sum_{i=2}^{n} k_i \varepsilon(C_i)$. If the terminal vertex of a walk $W_1$ is the same as the initial vertex of a walk $W_2$, then $W_1, W_2$ will denote the walk consisting of $W_1$ followed by $W_2$. Also, if $k \geq 0$, then $kC_i$ is the walk obtained
by repeating the cycle $C_1$ a total of $k$ times. We now construct three principal walks in $D$.

For $1 \leq i \leq n$, let $W_i$ be a $v_i - v_{i+1}$ path in $D$ where $v_i$ is the initial vertex of $C_1$ and $v_{n+1} = v_1 = u$. For $j \geq 0$, let $W(j) : W_1, jk_2, C_2, W_2, jk_3, C_3, \ldots, jk_n, C_n, W_n$. Then $W(j)$ is a closed $v_1 - v_1$ walk of length $x(W(j)) = x_0 + jL$ where $x_0 = x(W(0)) = x(W_1, W_2, \ldots, W_n)$. Let $P_1(j)$ be the walk of length $j$ which ends at $u_1 = v_1$ and follows $C_1$; then the initial vertex of $P_1(j)$ is $u_{(1-j)}$. Finally, let $P_2(j)$ be the walk of length $j$ which begins at $u_1 = v_1$; then, the terminal vertex of $P_2(j)$ is $u_{(1+j)}$.

The least common multiple of $m$ and $L$ is $L_* = mL/g$. Since $D$ has an eulerian period there exists $J \geq 0$ such that

$L_0 + L_* + J$

$L(D)$ is locally regular. As such, the indegree of the initial vertex of every walk of length $x_0 + L_* + J$ is the same as the outdegree of the terminal vertex. For $0 \leq g_1 < g$, let

$W(g_1, j_1, j_2) : P_1(g_1), W(j_1), P_2(j_2L + J - g_1)$

where $0 \leq j_1 < m/g$ and $j_1 + j_2 = m/g$. Then $W(g_1, j_1, j_2)$ is a walk of length

$x(W(g_1, j_1, j_2)) = g_1 + (x_0 + j_1L) + (j_2L + J - g_1) = x_0 + L_* + J$.

If $u_{(1 + AL + J - g_1)} = u_{(1 + BL + J - g_1)}$ for some $A$ and $B$ where

$1 \leq A \leq B < m/g$, then $(1 + AL + J - g_1) = (1 + BL + J - g_1)$

which implies that $(B - A)L \equiv 0 \mod m$. But, this implies that $(B - A)L/g \equiv 0 \mod m/g$ which in turn implies that $B - A \equiv 0$.
mod \(m/g\) since \(L/g\) and \(m/g\) are relatively prime. Hence, \(B = A\)
and this implies that
\[
V(u(1-g_1)_m) = \{u(1+2_2+J-g_1)_m : 1 \leq j \leq m/g\}
\]
has \(m/g\) distinct vertices all of which have outdegree equal to
the indegree of \(u(1-g_1)_m\). Also, \(d_{C_1}(u,v) \equiv 0 \mod g\) for every
pair \(u\) and \(v\) in \(V(u(1-g_1)_m)\); i.e., the vertices of \(V(u(1-g_1)_m)\)
are equally spaced about \(C_1\) and therefore,
\[
V(u(1-g_1)_m) = \{u(1+kg+J-g_1)_m : 1 \leq k \leq m/g\}.
\]
Also, the sets \(V(u(1-g_1)_m)\) for \(0 \leq g_1 < g\) form a partition of
\(V(C_1)\). Hence, \(od\ u = od\ v\) whenever \(d_{C_1}(u,v) \equiv 0 \mod g\) for
every pair \(u\) and \(v\) in \(V(C_1)\); i.e., (1) is proved.

For \(1 \leq i \leq m\), let \(W(i)\) be the walk of length \(\kappa_0 + L\times + J\)
in \(C_1\) which begins at the vertex \(u_i\). Then, the terminal vertex
of \(W_i\) is \(u(i+\kappa_0+L\times+J)_m\) and \(id\ u_i = od\ u(i+\kappa_0+L\times+J)_m\) since
\(L_0+L\times+J\) (D) is eulerian. Consequently, for \(1 \leq i \leq m\), \(id\ u_i = od\ u(i+a)_m\) where \(a = (\kappa_0 + L\times + J)_m\). Since \(od\ u = id\ v\) whenever \(d_{C_1}(u,v) \equiv 0 \mod g\), \(id\ u = od\ v\) whenever \(d(u,v) \equiv a \mod m\) and (2) is proved.

Hence, for each cycle \(C\) of \(D\) there is a sequence
\(S: s_1, s_2, \ldots, s_g\) (possibly dependent on \(C\)) for which \(C\) has a
cyclic \(S\)-score assignment. We now show that \(D\) is fixed score
cyclic. To do this, it suffices to show that if any cycle \(C\) has
a cyclic \(S\)-score assignment then so does every other cycle of \(D\)
and that if \((C,v,a,b)\) is a cyclic S-score assignment of \(C\) and if \(C'\) is any other cycle of \(D\) containing \(v\), then \((C',v,a,b)\) is also a cyclic S-score assignment. However, since \(D\) is strongly connected, it suffices to verify that given any two cycles \(C_1\) and \(C_2\) with a common vertex, say \(u\), then \(id_{v_1} = id_{v_2}\) and \(od_{v_1} = od_{v_2}\) whenever \(d_{C_1}(u,v_1) = d_{C_2}(u,v_2) \mod g\) for \(v_1\) in \(V(C_1)\) and \(v_2\) in \(V(C_2)\). Since the indegrees and outdegrees occur cyclically around both \(C_1\) and \(C_2\), it suffices to consider only those \(v_i\) in \(V(C_i)\) where \(d_{C_i}(u,v_i) < g\) for \(i = 1,2\).

Since \(D\) has an eulerian period, there is a \(J \geq \max\{\varepsilon(C_1), \varepsilon(C_2)\}\) for which \(L^j(D)\) is eulerian. Let \(W(j)\) be the walk of length \(j\) in \(C_1\) which terminates at \(u\). For \(0 \leq j < g\), let \(P_i(j)\) be the path of length \(j\) in \(C_i\) which begins at \(u\) and ends at \(v_i\), for \(i = 1,2\). Then, for \(0 \leq j < g\), the walks \(W(j), P_1(j)\) and \(W(j), P_2(j)\) have a common initial vertex and both have length \(J\) and therefore, \(od_{v_1,j} = od_{v_2,j}\); i.e., \(od_{v_1} = od_{v_2}\) if \(d_{C_1}(u,v_1) = d_{C_2}(u,v_2) < g\).

We next observe that \(D\) has an eulerian period if and only if its converse \(F\) (obtained by reversing the direction of all arcs of \(D\)) has an eulerian period. In fact, \(L^j(F)\) is eulerian since \(L^j(D)\) is eulerian. Let \(F_i\) be the converse of \(C_i\) for \(i = 1,2\). By the preceding arguments, \(od_{F_1} v_1 = od_{F_2} v_2\) if \(d_{F_1}(u,v_1) = d_{F_2}(u,v_2) < g\). However, this implies that \(id_{F_1} v_1 = id_{F_2} v_1 = od_{F_2} v_2 = id_{F_2} v_2\) whenever \(d_{C_1}(v_1,u) = d_{C_2}(v_2,u) < g\). Since the indegrees occur cyclically around both \(C_1\) and \(C_2\) with period \(g\),
this implies that \(\text{id}_D v_1 = \text{id}_D v_2\) whenever \(d_{C_1}(u,v_1) = d_{C_2}(u,v_2) < g\). Hence, \(D\) is fixed S-score cyclic and the result follows.

### Section 4.7

**Fixed Score Cyclic Digraphs**

Although it is not particularly difficult to decide whether digraphs of relatively small order are fixed score cyclic (such as those encountered in the preceding section) this is not the case in general. For example, short of enumerating all cycles and (perhaps) exhausting all possible sequences, there is no way of determining from the definition whether or not an arbitrary strongly connected nonempty digraph is fixed score cyclic. In the following discussion we investigate the structure of such digraphs and develop a variety of characterizations.

**Theorem 4.15:** A strongly connected nonempty digraph \(D\) is fixed S-score cyclic if and only if for some sequence \(S: s_1, s_2, \ldots, s_n\) there is a mapping \(\alpha: V(D) \to ((i,j): 1 \leq i, j \leq k}\) such that for every \(u, v \in V(D)\):

1. \(\alpha(u) = (a, b)\) implies \(\text{id} u = s_a\) and \(\text{od} u = s_b\); and
2. \(\alpha(v) = ((a+1)_k, (b+1)_k)\) whenever \(\alpha(u) = (a, b)\) and \(D\) has an arc from \(u\) to \(v\).

**Proof:** Let \(S: s_1, s_2, \ldots, s_k\) be a sequence for which there exists a mapping \(\alpha\) satisfying (1) and (2). Clearly, \(k\) divides the length of every cycle in \(D\). Also, if \(v\) is a vertex of some
cycle $C$ and $\alpha(v) = (a, b)$, then $(C, v, a, b)_S$ is a cyclic $S$-score assignment of $C$. Hence, $D$ is $S$-score cyclic. Let $C_1$ and $C_2$ be any two cycles of $D$ having a common vertex $u$ and let $(C_1, u, a_1, b_1)_S$ be any cyclic $S$-score assignment of $C_1$. If $\alpha(u) = (a_1, b_1)$, then $(C_2, u, a_1, b_1)_S$ is a cyclic $S$-score assignment of $C_2$. Then, suppose that $\alpha(u) = (a, b) \neq (a_1, b_1)$. Since $(C_1, u, a, b)_S$ is also a cyclic $S$-score assignment of $C_1$, $s(a + i)_k = s(a_1 + i)_k$ and $s(b + i)_k = s(b_1 + i)_k$ for $1 \leq i \leq k$. Thus, $(C_2, u, a_1, b_1)_S$ is a cyclic $S$-score assignment of $C_2$. Hence, $D$ is fixed $S$-score cyclic.

Conversely, suppose that $D$ is fixed score cyclic for some sequence $S: s_1, s_2, \ldots, s_k$. Let $u$ be an arbitrary but fixed vertex of $D$. Then, $u$ lies on a cycle $C$ having a cyclic $S$-score assignment $(C, u, a, b)_S$. For $v \in V(D)$ let $\alpha(v) = ((a + j)_k, (b + j)_k)$ if $D$ has a $u-v$ walk of length $j$. Let $W_1$ and $W_2$ be any two $u-v$ walks of $D$. Then, $D$ has a $v-u$ path $P$. Since $k$ divides $\lambda(C)$ for every cycle $C$ in $D$, it follows that $k$ divides $\lambda(W)$ for every closed walk in $D$. Hence, $\lambda(W_1, P) = \lambda(W_1) + \lambda(P) \equiv 0 \mod k$ and $\lambda(W_2, P) = \lambda(W_2) + \lambda(P) \equiv 0 \mod k$. Thus, $\lambda(W_1) \equiv \lambda(W_2) \mod k$; i.e., $\alpha$ is well defined. Also, if there is a $u-v$ walk of length $j$ and $C'$ is any cycle containing $v$, then $(C', v, (a + j)_k, (b + j)_k)_S$ is clearly a cyclic $S$-score assignment of $C'$. Hence, (1) and (2) hold and the result follows.

The following result significantly restricts the number of sequences one must check in order to determine whether a given
digraph is fixed score cyclic.

**Corollary 4.16:** A strongly connected nonempty digraph $D$ is fixed
S-score cyclic if and only if for some sequence $S: s_1, s_2, \cdots, s_g$
where $g = \gcd(C(C): C \in \text{Cyc}(D))$ there is a mapping $\beta: V(D) \to$
{$(i,j): 1 \leq i, j \leq g$} such that for every $u, v \in V(D):$

1. $\beta(u) = (a,b)$ implies $id(u) = s_a$ and $od(u) = s_b$; and
2. $\beta(v) = ((a+1)g, (b+1)g)$ whenever $\beta(u) = (a,b)$ and there
   is an arc from $u$ to $v$.

**Proof:** If the sequence $S$ and the mapping $\beta$ do exist, then $D$
is fixed S-score cyclic by Theorem 4.15. Conversely, suppose that
$D$ is fixed score cyclic. By Theorem 4.15, there is a sequence
$T: t_1, t_2, \cdots, t_k$ and a mapping $\alpha$ satisfying (1) and (2) of Theorem
4.15. Since the result follows if $k = g$, we assume $k \neq g$.
Since $k$ divides the length of every cycle in $D$, it also divides
g. Let $S: s_1, s_2, \cdots, s_g$ where $s_i = t_{(i)k}$ for $1 \leq i \leq g$. Let
$u$ be any vertex of $D$, then $\alpha(u) = (a,b)$ for some $a$ and $b$ in
{$1, 2, \cdots, k$}. For $v \in V(D)$ define $\beta(v) = ((a+j)g, (b+j)g)$ if $D$
has a u-v walk of length $j$. As before, $\beta$ is well-defined.
Also, since $k$ divides $g$ we have that $((n+j)g)_k = (n+j)_k$ for all
integers $n$ and $j$. Consequently, $s_{(n+j)g} = t_{(n+j)k}$ and
therefore, (1) and (2) hold.

#

For a vertex $u$ in a digraph $D$ and for $1 \leq i \leq n$ let
$N_i(u; n) = \{v \in V(D): d(u, v) \equiv i \mod n\}$. We now present the final
characterization of fixed score cyclic digraphs. However, since a
strongly connected digraph containing loops is fixed score cyclic if and only if it is regular, it suffices to consider only loop-free digraphs.

Theorem 4.17: Let \( u \) be any vertex of a strongly connected non-regular digraph \( D \). Also, let \( g = \gcd(\lambda(C): C \in \text{Cyc}(D)) \). Then, \( D \) is fixed score cyclic if and only if: for \( 1 \leq i \leq g \),

1. the induced subdigraph \( \langle N_i(u;g) \rangle \) is empty;
2. there exist \( s_i, t_i \) such that \( w \in N_i(u;g) \) implies \( \text{id} v = s_i \) and \( \text{od} v = t_i \); and
3. there is an integer \( d \) such that \( s_i = t_{(i+d)g} \).

Proof: Suppose that \( D \) is fixed score cyclic. By Corollary 4.16, there is a sequence \( S: s_1, s_2, \ldots, s_g \) and a mapping \( \alpha: V(D) \to \{(i,j): 1 \leq i, j \leq g\} \) such that for every \( v, w \in V(D) \):

1. \( \alpha(v) = (a,b) \) implies \( \text{id} v = s_a \) and \( \text{od} v = s_b \); and
2. \( \alpha(w) = ((a+1)g, (b+1)g) \) whenever \( \alpha(v) = (a,b) \) and there is an arc from \( v \) to \( w \).

Let \( \alpha(u) = (a,b) \). Then, for \( 1 \leq i \leq g \), \( N_i(u;g) = \{ v \in V(U): d(u,v) \equiv i \mod g \} = \{ v \in V(D): \alpha(v) = ((a+i)g, (b+i)g) \} \).

Consequently, (1) - (3) hold.

Conversely, suppose that (1) - (3) hold. Suppose that there is an arc from a vertex \( v \) in \( N_i(u;g) \) to a vertex \( w \) in \( N_j(u;g) \) where \( (j-i)_k \neq 1 \). Then, \( D \) has a shortest \( u-w \) path \( P_1 \) (where \( \lambda(P_1) \equiv j \mod g \)) and a \( u-w \) walk \( W_1 \) (where \( \lambda(W_1) \equiv (i+1) \mod g \)) consisting of a shortest \( u-v \) path \( P_2 \) and the path \( v,w \). If \( W \) is any \( w-u \) path, then \( P_1,W \) and \( W_1,W \) are closed.
u-u walks. Thus, \( \lambda(P_1, W) \equiv \lambda(W_1, W) \mod g \). But this implies that \( \lambda(P_1) \equiv \lambda(W_1) \mod g \) and this is a contradiction. Thus, for \( 1 \leq i \leq g \), each vertex in \( N_i(u; g) \) is adjacent only to vertices in \( N_{(i+1)_k}(u; g) \). Let \( S: s_1, s_2, \ldots, s_g \) where \( s_1, s_2, \ldots, s_g \) are given by (2). For \( 1 \leq i \leq g \) and \( v \in N_i(u; g) \) let \( \alpha(v) = (i, (i+d)_g) \) where \( d \) is given by (3). Then, \( S \) and \( \alpha \) satisfy Corollary 4.16 and \( D \) is fixed S-score cyclic.

For positive integers \( n_1, n_2, \ldots, n_k \) let \( C(n_1, n_2, \ldots, n_k) \) be the digraph with vertices \( v_1, v_2, \ldots, v_n \) and \( n_i \) arcs from \( v_i \) to \( v_{(i+1)_k} \) for \( 1 \leq i \leq k \).

**Corollary 4.18:** If \( S: s_1, s_2, \ldots, s_k \) is a sequence of positive integers, then there is a digraph \( D \) which is fixed S-score cyclic. The fixed S-score cyclic digraph of minimum order is \( C(s_1, s_2, \ldots, s_k) \).

**Proof:** Since any fixed S-score cyclic digraph must have at least \( k \) vertices, the digraph \( C(s_1, s_2, \ldots, s_k) \) is clearly a fixed S-score cyclic digraph of minimum order. Moreover, the characterization in Theorem 4.17 shows that it is unique.

**Corollary 4.19:** If \( S: s_1, s_2, \ldots, s_k \) is a sequence of positive integers, then the fixed S-score cyclic digraph of minimum order and having no multiple arcs is \( L(C(s_1, s_2, \ldots, s_k)) \).

**Proof:** Let \( D \) be any fixed S-score cyclic digraph satisfying the hypothesis. By Theorem 4.17, there is a partition \( V_1, V_2, \ldots, V_k \) of \( V(D) \) such that for \( 1 \leq i \leq k \)
(1) every vertex of $V_i$ is adjacent only to vertices in $V(i+1)_k$;
(2) there exist $s_i, t_i$ such that $u \in V_i$ implies $id u = s_i$ and $od u = t_i$; and
(3) there is an integer $d$ such that $s_i = t_{(i+d)}_k$.

Since $D$ has no multiple arcs, (1) - (3) imply that $|V(i+1)_k| \geq s_i$ for $1 \leq i \leq k$. Hence, $D$ has order at least $\sum_{i=1}^{k} s_i$. Since $L(C(s_1, s_2, \ldots, s_k))$ is a fixed $S$-score cyclic digraph without multiple arcs having order $\sum_{i=1}^{k} s_i$, the digraph $D$ has order $\sum_{i=1}^{k} s_i$; i.e., $|V(i+1)_k| = s_i$ for $1 \leq i \leq k$. But, this implies that each vertex of $V_i$ is adjacent to every vertex of $V(i+1)_k$ for $1 \leq i \leq k$. Hence, $D = L(C(s_1, s_2, \ldots, s_k))$.

Section 4.8

Hamiltonian Properties
in Iterated Line Digraphs

A digraph $D$ has a **hamiltonian period** if there exist positive integers $m$ and $n$ such that the line digraph $L^{m+kn}(D)$ is hamiltonian for all nonnegative integers $k$. In the preceding sections we showed that $\lambda(D)$ contains infinitely many hamiltonian line digraphs if and only if the nonempty digraph $D$ has a hamiltonian period. Also, we established an equivalence between digraphs with a hamiltonian period and those with an eulerian period and, for strongly connected digraphs, extended this equivalence to fixed score cyclic digraphs. Consequently, the following is an immediate consequence of Theorem 4.14.
Theorem 4.20: A strongly connected nonempty digraph has a hamiltonian period if and only if it is fixed score cyclic.

Since a digraph, which is not strongly connected, has a hamiltonian period if and only if it is the disjoint union of an acyclic digraph (which eventually ceases to contribute to the iterated line digraphs) and a strongly connected digraph with a hamiltonian period, no generality will be lost if we restrict our attention to only strongly connected digraphs. Moreover, this observation holds for any property which implies that a digraph has a hamiltonian period. Consequently, in this section we restrict our attention to strongly connected digraphs.

Clearly, every strongly connected nonempty regular digraph has a hamiltonian period. Also, if a strongly connected digraph with loops has a hamiltonian period, then it must be regular. If a digraph is not regular, then Theorems 4.17 and 4.20 provide another characterization of digraphs with a hamiltonian period. For this result, recall that given a vertex \( u \) in a digraph \( D \) and a positive integer \( n \), then \( N_i(u;n) = \{v \in V(D): d(u,v) \equiv i \mod n\} \).

Theorem 4.21: Let \( u \) be any vertex of a strongly connected nonempty digraph \( D \) which is not regular. Also, let \( g = \gcd\{\chi(C): C \in \text{Cyc}(D)\} \). Then, \( D \) has a hamiltonian period if and only if for \( 1 \leq i \leq g \):

1. the induced subdigraph \( N_i(u;g) \) is empty;
2. there exist integers \( s_i, t_i \) such that \( \text{id } v = s_i \) and \( \text{od } v = t_i \) for all \( v \in N_i(u;g) \); and
(3) there exists an integer $d$ such that $s_i = t(i+d)_{g}$.

We next consider hamiltonian-connectedness in iterated line digraphs. However we first present the following result.

**Lemma 4.22:** Let $D$ be a strongly connected nonempty digraph. Then, $L(D)$ is hamiltonian-connected if and only if either $D$ is $K_1$ with loops or $D = K_2$.

**Proof:** The line digraph $L(D)$ is a complete symmetric digraph if and only if either $D$ is $K_1$ with loops or $D = K_2$. Since all complete symmetric digraphs are hamiltonian-connected, it suffices to consider the case where $L(D)$ is not a complete symmetric digraph. Then, $L(D)$ has two vertices $\hat{e}$ and $\hat{f}$ where $\hat{e}$ is not adjacent to $\hat{f}$. If $L(D)$ has no hamiltonian $\hat{e}$-$\hat{f}$ path, then $L(D)$ is not hamiltonian-connected. On the other hand, if $L(D)$ has a hamiltonian $\hat{e}$-$\hat{f}$ path $\hat{e} = \hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n = \hat{f}$ then $D$ has an open eulerian trail $e = e_1, e_2, \ldots, e_n = f$ and therefore, $D$ is not eulerian. This implies that $L(D)$ is not hamiltonian and therefore, it is not hamiltonian-connected. In any event, if $L(D)$ is not a complete symmetric digraph, then $L(D)$ is not hamiltonian-connected and the result follows.

**Theorem 4.23:** If $D$ is a strongly connected digraph and $\lambda(D)$ contains infinitely many hamiltonian-connected digraphs, then $D$ is $K_1$ with one loop or $D = K_2$.

**Proof:** Since $\lambda(D)$ contains infinitely many hamiltonian-connected
digraphs, let $L^n(D)$ be such a digraph where $n \geq 2$. Then, $L^{n-1}(D)$ is either $K_1$ with loops or $K_2$. Since $n - 1 \geq 1$, the digraph $L^{n-1}(D)$ has no multiple loops. Hence, $L^{n-1}(D)$ is either $K_1$ with one loop or $K_2$. In either case, this implies that $D = L^{n-1}(D)$ and the theorem is proved.

In order to discuss $k$-path hamiltonian line digraphs, $k \geq 0$, it is convenient to make the following definition. An eulerian digraph is $k$-trail eulerian if every trail (no repeated arcs) of length $k$ can be extended to an eulerian circuit of the digraph. Clearly, every eulerian digraph is 1-trail eulerian. Let $k \geq 1$ and suppose that an eulerian digraph $D$ is $k$-trail eulerian. If $P: \hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n$ is any path in $L(D)$ of length not exceeding $k - 1$, then $e_1, e_2, \ldots, e_n$ is a trail in $D$ of length not exceeding $k$. Consequently, $D$ has an eulerian circuit $e_1, e_2, \ldots, e_q$ and $L(D)$ has a hamiltonian cycle $\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_q, \hat{e}_1$ containing $\hat{P}$; i.e., $L(D)$ is $(k-1)$-path hamiltonian. The converse is established similarly and we have for $k \geq 1$, an eulerian digraph $D$ is $k$-trail eulerian if and only if $L(D)$ is $(k-1)$-path hamiltonian. The following result characterizes $k$-trail eulerian digraphs.

**Lemma 4.24:** For any positive integer $k$, an eulerian digraph $D$ is $k$-trail eulerian if and only if for every open trail $T$ and every circuit $C$, neither of length exceeding $K$:

1. $D - E(T)$ has one nontrivial component; and
2. $D - E(C)$ is either weakly connected or empty.

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Proof: Let $D$ be $k$-trail eulerian and let $T$ be any open trail of length not exceeding $k$. Then, $D - E(T)$ has a nontrivial trail $T'$ such that $T$ followed by $T'$ is an eulerian circuit of $D$. Since $T'$ exhausts all arcs of the nonempty digraph $D - E(T)$ and since $<E(T')>$ is connected, condition (1) must hold.

To show that (2) must also hold, let us suppose that $D$ has a circuit $C$ of length not exceeding $k$ for which $D - E(C)$ is nonempty and disconnected. Since $D$ is strongly connected, this implies that $D - E(C)$ has a nonempty weakly connected component $D_1$ and a vertex $v$ belonging to $C$ but not to $D_1$. Without loss of generality, we may assume that $C$ is a $v$-$v$ circuit. Since there is no path in $D - E(C)$ from $v$ to any vertex of $D_1$, the $v$-$v$ circuit $C$ cannot be extended to a circuit which includes the arcs of $D_1$. However, this is a contradiction since $D$ is $k$-trail eulerian and $C$ is a circuit of length not exceeding $k$. Consequently, (2) must hold.

Conversely, suppose that conditions (1) and (2) hold for some eulerian digraph $D$. Let $T$ be any open trail of length not exceeding $k$, then $D - E(T)$ has exactly one nontrivial weakly connected component $D$. Moreover, $D_1$ has an open eulerian trail $T'$ whose initial and terminal vertices are respectively the terminal and initial vertices of $T$. Thus, $T$ followed by $T'$ is an eulerian circuit of $D$ containing $T$.

Also, if $C_1$ is any circuit (say, a $v$-$v$ circuit) of length not exceeding $k$ and if $D - E(C_1)$ is nonempty, then $D - E(C_1)$
is weakly connected and locally regular. Hence, it has an eulerian v-v circuit $C_2$ which can be used to extend $C_1$ to an eulerian circuit $C$ of $D$. Thus, $D$ has an eulerian circuit containing $C_1$ and the lemma is proved. 

Recall that $\delta(D)$ denotes the minimum among the indegrees and the outdegrees of the vertices of $D$. If $n \geq 3$ and $D$ consists of two disjoint copies of the complete symmetric digraph $K_n$ joined by a pair of symmetric arcs, then $\delta(D) = n - 1 \geq 2$ and $D$ is an eulerian digraph which is not 2-trail eulerian. At the other extreme, every cycle $C$ is $k$-trail eulerian for all $k \geq 0$ even though $\delta(C) = 1$.

**Theorem 4.25:** If $D$ is an eulerian line digraph, then $D$ is $k$-trail eulerian where $k = \max\{1, 2\delta(D) - 3\}$.

**Proof:** By Theorem 4.9, the digraph $D - E(T)$ is weakly connected for any (open or closed) trail $T$ of length not exceeding $k$. Consequently, by Lemma 4.24, the digraph $D$ is $k$-trail eulerian.

The preceding result is sharp in the sense that there exist eulerian line digraphs which are $k$-trail eulerian ($k$ as defined in Theorem 4.25) which are not $(k+1)$-trail eulerian. For example, recall that $PK_n$ is the complete symmetric digraph $K_n$ ($n \geq 2$) with one loop at each vertex. Then, $PK_n$ is the eulerian line digraph of the digraph having one vertex and $n$ loops. Then, $k = \max\{1, 2\delta(PK_n) - 3\} = 2n - 3$ and $PK_n$ is $(2n-3)$-trail eulerian. To see that $PK_n$ is not $(2n-2)$-trail eulerian, let

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$V(PK_n) = \{v_1,v_2,\ldots,v_n\}$ and let $C$ be the circuit $v_1,v_2,v_1,v_3,v_1,\ldots,v_1,v_n,v_1$. Then $C$ has length $2n-2$ and $PK_n - E(C)$ is disconnected and nonempty. Thus, by Lemma 4.24, the digraph $PK_n$ is not $(2n-2)$-trail eulerian if $n \geq 2$.

Lemma 4.26: If $D$ is an eulerian line digraph, then $L(D)$ is $k$-path hamiltonian where $k = \max\{0,2\delta(D)-4\}$.

Proof: Since $L(D)$ is $k$-path hamiltonian if and only if $D$ is $(k+1)$-trail eulerian, the result follows immediately from Theorem 4.25.

Moreover, since $L(D)$ is $k$-path hamiltonian if and only if $D$ is $(k+1)$-trail eulerian (where $D$ is strongly connected), the sharpness of Theorem 4.25 carries over to the preceding result. Also, Lemma 4.26 possesses particular significance for digraphs which have a hamiltonian period.

Theorem 4.27: If a strongly connected digraph $D$ has a hamiltonian period, then every hamiltonian digraph in $\lambda(D)$, with the possible exception of $L(D)$, is $k$-path hamiltonian where $k = \max\{0,2\delta(D)-4\}$.

Proof: If $n \geq 2$ and $L(D)$ is hamiltonian, then by Lemma 4.26 the digraph $L(D)$ is $k$-path hamiltonian where $k = \max\{0,2\delta(L(D))-4\}$. From Theorem 4.5, it follows that $\delta(L(D)) = \delta(D)$. Hence, the assertion is proved.

In spite of the sharpness associated with Lemma 4.26, this result (and its consequences) can be improved provided the digraphs...
in question have no loops.

**Theorem 4.28:** If $D$ is an eulerian line digraph without loops, then $D$ is $k$-trail eulerian where $k = 2\delta(D) - 1$.

**Proof:** Since the result is obvious if $\delta(D) = 1$, we assume that $\delta(D) \geq 2$. Since $D$ is a line digraph, it has no multiple arcs. Also, $D$ has no transitive triples since the Heuchenne condition would then imply that $D$ has loops, which it does not. Now suppose $D$ has a trail $T$ of length $n \leq 2\delta(D) - 1$ for which $D - E(T)$ is disconnected. Without loss of generality we may assume that $T$ is a shortest such trail. Then, $D - E(T)$ has two components $D_1$ and $D_2$, one of which must contain the initial vertex of $T$; let it be $D_1$. Since $D$ is eulerian and $D - E(T)$ is disconnected, the component $D_2$ is eulerian and the terminal vertex of $T$ is in $D_1$ (regardless of whether $T$ is open or closed). Then, $T$ contains an arc $uv$ where $u$ is in $D_1$ and $v$ is in $D_2$. Let $N(u)$ be the vertices of $D$ adjacent from $u$ and $N^*(v)$ be those adjacent to $v$. Then, $N(u) \cap N^*(v) = \emptyset$ since $D$ has no transitive triples. Also, $|N(u)| \geq 6(D)$ and $|N^*(v)| \geq 6(D)$.

Let $T$ contain a total of $(od u) - m$ arcs incident from $u$. Then it must also contain $(od u) - m - 1$ arcs incident to $u$. Likewise, suppose that $T$ contains $(id v) - n$ arcs incident to $v$, then it contains $(id v) - n$ arcs incident from $v$ since $v$ is an intermediate vertex of $T$. Let
\[ N_1(u) = \{ w \in N(u): uw \not\in E(T) \} \] and
\[ N_1^\ast(v) = \{ w \in N^\ast(v): wv \not\in E(T) \} . \]
Then \(|N_1(u)| = m\) and \(|N_1^\ast(v)| = n\). Since \(T\) has length \(n_0 \leq 2\delta(D) - 1\), and since \(id v + od v \geq 2\delta(D)\), the trail \(T\) cannot exhaust all arcs incident to or from \(v\); i.e., \(|N_1^\ast(v)| = n \geq 1\).

Suppose that \(N_1(u) \neq \emptyset\). From the Heuchenne condition, \(xy\) is an arc of \(D\) for every \(x \in N_1^\ast(v)\) and \(y \in N_1(u)\). Since \(D - E(T)\) is disconnected, \(T\) must also contain these \(mn\) arcs. This implies that \(T\) contains at least \(2(od u) - 2m - 1\) arcs incident to or from \(u\), \(2(id u) - 2n\) arcs incident to or from \(v\), and the \(mn\) arcs from vertices in \(N_1^\ast(v)\) to vertices in \(N_1(u)\). Thus, \(mn + 4\delta(D) - 2m - 2n - 1 \leq mn + 2(od u) - 2m - 1 + 2(id v) - 2n \leq n_0 \leq 2\delta(D) - 1\).

This implies that \((m - 2)(n - 2) + 2\delta(D) \leq 4\). Since \(\delta(D) \geq 2\), not both \(m \geq 3\) and \(n \geq 3\) can occur. If \(n = 1\), then \(m \geq 2\delta(D) - 2 \geq 2\) and \(T\) contains at least \(2\delta(D) - 2\) arcs incident from the single vertex of \(N_1^\ast(v)\) together with the arc \(uv\). Since \(T\) has length \(n_0 \leq 2\delta(D) - 1\), this is impossible and \(n \geq 2\).

Likewise, \(m \geq 2\) and we have that \((m - 2)(n - 2) = 0\) and \(\delta(D) = 2\). Thus, \(T\) is a trail of length not exceeding three. Since \(N_1^\ast(v) \neq \emptyset\), \(T\) must exhaust all arcs incident from \(u\). Thus, \(T: u, v, u, w\) for some vertex \(w\) in \(D_1\). However, every vertex in \(N_1^\ast(v)\) is also adjacent in \(D - E(T)\) to \(w\) and this is a contradiction. Thus, our assumption that \(N_1(u) \neq \emptyset\) was false.

Since \(N_1(u) = \emptyset\), this implies that \(T\) contains \(od u\) arcs incident from \(u\) and at least \((od u) - 1\) arcs incident to \(u\). Since \(T\) has length not exceeding \(2\delta(D) - 1 \leq 2(od u) - 1\), every
arc of $T$ is either incident to $v$ or incident from $v$. Also, $D - E(T)$ has one arc $wu$ incident to $u$. Since $D$ has no multiple arcs, only one arc of $T$ is incident from $v$. Thus, $D - E(T)$ has an arc $vy$ since $\delta(D) \geq 2$. Hence, $D - E(T)$ has the arc $wy$ (by the Heuchenne condition) and therefore, $D - E(T)$ has a $u$-$v$ semiwalk. Since $u$ and $v$ lie in different weakly connected components, this is a contradiction. Hence, if $T$ is any trail in $D$ of length not exceeding $2\delta(D) - 1$, then $D - E(T)$ is connected. Thus, by Lemma 4.24, the digraph $D$ is $(2\delta(D) - 1)$-trail eulerian and the result follows.

The preceding result is sharp in the sense that for each $n \geq 2$ there is an eulerian line digraph $D$ without loops for which $\delta(D) = n$ and $D$ is not $2n$-trail eulerian. For example, the complete symmetric bipartite digraph $K(n,n)$ is the eulerian line digraph of the multidigraph having two vertices and $n$ arcs from each vertex to the other vertex. To see that $K(n,n)$ is not $2n$-trail eulerian, if $n \geq 2$, let $v$ be any vertex of $K(n,n)$ and let $e_1, e_2, \ldots, e_n$ be the arcs incident from $v$ and, for $1 \leq i \leq n$, let $e_i^*$ be the converse of $e_i$. Then, $C: e_1, e_1^*, e_2, \ldots, e_n, e_n^*$ is a circuit of length $2n$ containing all arcs incident to or from $v$. Then, $K(n,n) - E(C)$ is disconnected and by Lemma 4.24, the digraph $K(n,n)$ is not $2n$-trail eulerian. For digraphs having no loops, we now present the results which respectively correspond to Lemma 4.26 and Theorem 4.27.

**Lemma 4.29:** If $D$ is an eulerian line digraph without loops, then
L(D) is k-path hamiltonian where \( k = 2\delta(D) - 1 \).

**Theorem 4.30:** If \( D \) is a strongly connected loop-free digraph with a hamiltonian period, then every hamiltonian digraph in \( \lambda(D) \), with the possible exception of \( L(D) \), is k-path hamiltonian where \( k = 2\delta(D) - 1 \).

**Section 4.9**

**de Bruijn Sequences and Some Applications**

In this section, which is expository in nature, we discuss a class of sequences which have come to be called de Bruijn sequences. In particular, we present a brief history of their development and some of their (industrial) applications. For this purpose, let \( \Sigma_k = \{e_1, e_2, \ldots, e_k\} \) be a set of \( k \geq 1 \) symbols and let \( \Sigma_k^n \) be the set of sequences \( e_{i_1} e_{i_2} \cdots e_{i_n} \) of length \( n \) with terms in \( \Sigma_k \).

Also, let \( D_k \) denote the complete digraph of order one with loops \( e_1, e_2, \ldots, e_k \). Since \( D_k \) is strongly connected and regular of degree \( k \geq 1 \), the digraph \( L^n(D) \) is both hamiltonian and eulerian for all \( n \geq 1 \). Also, there is an obvious 1-1 correspondence between the vertices of \( L^n(D) \) and the sequences in \( \Sigma_k^n \); in particular, \( e_{i_1} e_{i_2} \cdots e_{i_n} \in V(L^n(D)) \) corresponds to \( e_{i_1} e_{i_2} \cdots e_{i_n} \in \Sigma_k^n \).

The first occurrence in the literature of these sequences is found in a problem proposed by Martin [18]. This problem considers the existence of a sequence \( S \) with terms in \( \Sigma_k \) having the property that \( S \) contains each sequence in \( \Sigma_k^n \) exactly once as a
subsequence of $n$ consecutive symbols. For example, if $k = 2$ and $n = 3$, then the sequence

$$e_1 e_2 e_1 e_2 e_1 e_2$$

has this property since $\sum_2^3 = \{e_1 e_1 e_1, e_1 e_1 e_2, e_1 e_2 e_1, e_2 e_1 e_2, e_1 e_2 e_2, e_2 e_2 e_1, e_2 e_1 e_2\}$. In [18], Martin demonstrated the existence of such sequences for arbitrary choices of $k \geq 1$ and $n \geq 1$ using an algorithmic technique. However, the existence of such sequences follows quite easily from the theory of line digraphs.

Since $L^n(D_k)$ is hamiltonian for each $n \geq 1$, it has a hamiltonian path $P: x_1, x_2, \ldots, x_N$ where $N = V(L^n(D_k)) = k^n$ and where each $x_i$ is of the form $e_{i_1} e_{i_2} \cdots e_{i_n}$. Since $x_i$ is adjacent to $x_j$ if and only if $e_{i_{m+1}} = e_{j_m}$ for $1 \leq m \leq n - 1$, it follows that the sequence

$$e_1, e_2, \ldots, e_n, 1 e_N, 2 \cdots e_N, n$$

has the property that each vertex of $L^n(D_k)$ appears exactly once as a subsequence of $n$ consecutive symbols. Because of the 1-1 correspondence between the vertices of $L^n(D_k)$ and the sequences in $\sum_k^n$, it follows that each sequence in $\sum_k^n$ appears once as a subsequence of $n$ consecutive symbols in the sequence

$$e_1, e_2, \ldots, e_N, 1 e_N, 2 \cdots e_N, n.$$ 

Hence, the existence of such sequences for arbitrary $k \geq 1$ and $n \geq 1$ is shown.

An equivalent problem regarding "normal recurring decimals" was considered by Good [10] and Rees [22] in adjacent articles. For this purpose, let $f(x,s,m)$ be the number of times the sequence $s$ (of length $n$) appears in the first $m$ terms of the decimal number
x where $0 < x < 1$. If $f(x,s,m)/m \to 10^{-n}$, then s has normal frequency in x. If all sequences of length k have normal frequency in x, then x is n-normal. The existence of n-normal recurring decimals for arbitrary $n \geq 1$ was shown by Good [10] using directed graph theory and by Rees [22] using finite field theory. The method employed by Good consisted of constructing a digraph (in particular, $L^{n-1}(D_k)$) and showing that this digraph was eulerian. An eulerian circuit in this digraph was then used to construct an n-normal recurring decimal. We proceed with an equivalent development in the following.

Since the digraph $L^n(D_k)$ is hamiltonian (for each $n \geq 1$) it has a hamiltonian cycle $C: x_1, x_2, \ldots, x_N, x_1$ where $N = |V(L^n(D_k))| = k^n$ and where each $x_i$ is of the form $e_{i_1}e_{i_2}\ldots e_{i_n}$. As in the preceding discussion, $x_i$ is adjacent to $x_j$ if and only if $e_{i_{m+1}} = e_{j_m}$ for $1 \leq m \leq n - 1$. Consequently, each vertex of $L^n(D_k)$ appears exactly once among any N consecutive terms of the infinite sequence

$e_1, e_2, e_3, \ldots, e_N, e_1, e_2, e_3, \ldots$

Consequently, each sequence in $\sum_k^n$ appears exactly once among any $N = k^n$ consecutive terms in the decimal

$x = e_1, e_2, e_3, \ldots, e_N, e_1, e_2, \ldots$

which is expressed in the base k. Thus, $f(x,s,m)/m \to k^{-n}$ for each sequence in $\sum_k^n$. Hence, x (in base k) is n-normal. Consequently, n-normal recurring decimals (in an arbitrary base $k \geq 1$) exist for every choice of $n \geq 1$.

If we consider the first $N$ terms of the infinite sequence
to occur sequentially and equally spaced about a circle, we obtain an oriented circular array with the property that each vertex of \( L^k(D_k) \) occurs exactly once as a sequence of \( n \) consecutive terms. Such oriented circular arrays are called de Bruijn sequences and are denoted simply as \( e_1, e_2, \ldots, e_N, 1 \). Clearly, there is a 1-1 correspondence between the de Bruijn sequences of length \( N = k^n \) and the Hamiltonian cycles in \( L^n(D_k) \), where two Hamiltonian cycles are equivalent if one corresponds to a cyclic permutation of the second.

Also, because of the 1-1 correspondence between the vertices in \( L^n(D_k) \) and the sequences in \( \Sigma^n_k \), we see that each de Bruijn sequence gives rise to a unique oriented circular array of terms in \( \Sigma^n_k \) with the property that each sequence in \( \Sigma^n_k \) occurs exactly once as \( n \) consecutive terms.

In [1], de Bruijn considered these sequences for the case \( k = 2 \) and showed that the number of such (de Bruijn) sequences is \( 2^f(n) \) where \( f(n) = 2^{n-1} - n \); i.e., the number \( |L^n(D_2)|_h \) of Hamiltonian cycles in \( L^n(D_2) \) is given by

\[
|L^n(D_2)|_h = 2^{2^{n-1} - n}.
\]

This result was later generalized by van Aardenne-Ehrenfest and de Bruijn [1] when they showed that: if \( D \) is a strongly connected \( k \)-regular digraph with \( |D|_e \) distinct Eulerian circuits, then \( L(D) \) has

\[
|L(D)|_e = k^{-1}(k!)^N(k^{-1})|D|_e
\]

distinct Eulerian circuits where \( N \) is the order of \( D \). Since \( D_k \) has \( (k - 1)! \) distinct Eulerian circuits, this implies that...
$L^n(D_k)$ has
\[ |L^n(D_k)|_e = \frac{(k!)^k}{(k^n+1)} \]
distinct eulerian circuits. Because of the 1-1 correspondence between eulerian circuits in a digraph $D$ and the hamiltonian cycles in $L(D)$, we have that $L^n(D_k)$ has
\[ |L^n(D_k)|_h = \frac{(k!)^{n-1}}{(k^n)} \]
hamiltonian cycles and, consequently, there is a like number of distinct de Bruijn sequences. Also, it follows from Lemma 4.26, that $L^n(D_k)$ is $(2k-4)$-path hamiltonian for each $n \geq 1$ provided $k \geq 2$. Consequently, for any $N = 2k - 3$ distinct "consecutive" terms $e_1 e_2 \cdots e_n$, $e_1 e_2 \cdots e_{n+1}$, $\cdots$, $e_1 e_2 \cdots e_{N+n-1}$ in $\Sigma^n_k$, it is possible to find a de Bruijn sequence which (in generating $\Sigma^n_k$) will list these $N$ sequences of $\Sigma^n_k$ in a consecutive manner.

There is a less restrictive form of de Bruijn sequences. In particular, every cycle of length $m \geq n$ in $L^n(D_k)$ gives rise to an oriented circular array of symbols in $\Sigma^n_k$ with the property that no sequence in $\Sigma^n_k$ occurs more than once as $n$ consecutive terms. These oriented circular arrays are the output of rather simple electronic devices called shift registers and, consequently, they are called shift register sequences.

A shift register of degree $n$ consists of $n$ consecutive (binary) storage positions $P_1, P_2, \cdots, P_n$, a transfer operation which outputs the contents of $P_n$ and transfers the contents of $P_i$ to $P_{i+1}$ for $1 \leq i \leq n - 1$, and a logical feedback function.
(defined on the contents of the cells $P_1, P_2, \ldots, P_n$) which prohibits the register from emptying. A complete development of shift registers is provided by Golomb in [9]; however, we consider only some of their properties and uses.

Among their more important features is the inherent stability of the signal. In particular, they are self-correcting with respect to sporadic electronic noise. Also, it has been shown that any periodicity (upto $2^n$ in the case of binary shift registers of degree $n$) can be attained. They can be equipped with an "output logic" which will change the period as desired. Besides the preceding assets, they are described as relatively easy to manufacture.

Shift registers can be used in a variety of applications; for example: encipherment, address coding, error correcting codes, signal devices operating with extreme background noise, and random bit generators for Monte Carlo programs.


