8-1975

Self-Dual Embeddings of Graphs

Saul Stahl

Western Michigan University

Follow this and additional works at: http://scholarworks.wmich.edu/dissertations

Part of the Mathematics Commons

Recommended Citation

http://scholarworks.wmich.edu/dissertations/2942

This Dissertation-Open Access is brought to you for free and open access by the Graduate College at ScholarWorks at WMU. It has been accepted for inclusion in Dissertations by an authorized administrator of ScholarWorks at WMU. For more information, please contact maira.bundza@wmich.edu.
SELF-DUAL EMBEDDINGS OF GRAPHS

by

Saul Stahl

A Dissertation
Submitted to the
Faculty of The Graduate College
in partial fulfillment
of the
Degree of Doctor of Philosophy

Western Michigan University
Kalamazoo, Michigan
August 1975
ACKNOWLEDGEMENTS

I wish to thank Professor Arthur T. White, my thesis adviser for his guidance and encouragement. I am also grateful to Professor Alden Wright of Western Michigan University and Professor Paul Kainen of Case Western Reserve University for their careful reading of the manuscript and their many helpful suggestions. My thanks also to Professors Joseph T. Buckley and Gary Chartrand for serving on my committee as well as for many interesting and enlightening conversations.

Saul Stahl
For all those who have taught me mathematics.
INFORMATION TO USERS

This material was produced from a microfilm copy of the original document. While the most advanced technological means to photograph and reproduce this document have been used, the quality is heavily dependent upon the quality of the original submitted.

The following explanation of techniques is provided to help you understand markings or patterns which may appear on this reproduction.

1. The sign or "target" for pages apparently lacking from the document photographed is "Missing Page(s)". If it was possible to obtain the missing page(s) or section, they are spliced into the film along with adjacent pages. This may have necessitated cutting thru an image and duplicating adjacent pages to insure you complete continuity.

2. When an image on the film is obliterated with a large round black mark, it is an indication that the photographer suspected that the copy may have moved during exposure and thus cause a blurred image. You will find a good image of the page in the adjacent frame.

3. When a map, drawing or chart, etc., was part of the material being photographed the photographer followed a definite method in "sectioning" the material. It is customary to begin photoing at the upper left hand corner of a large sheet and to continue photoing from left to right in equal sections with a small overlap. If necessary, sectioning is continued again — beginning below the first row and continuing on until complete.

4. The majority of users indicate that the textual content is of greatest value, however, a somewhat higher quality reproduction could be made from "photographs" if essential to the understanding of the dissertation. Silver prints of "photographs" may be ordered at additional charge by writing the Order Department, giving the catalog number, title, author and specific pages you wish reproduced.

5. PLEASE NOTE: Some pages may have indistinct print. Filmed as received.

Xerox University Microfilms
300 North Zeob Road
Ann Arbor, Michigan 48106

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
75-28,992

STAHL, Saul, 1942-
SELF-DUAL EMBEDDINGS OF GRAPHS.

Western Michigan University, Ph.D., 1975
Mathematics

Xerox University Microfilms, Ann Arbor, Michigan 48106

THIS DISSERTATION HAS BEEN MICROFILM ED EXACTLY AS RECEIVED.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>CHAPTER</th>
<th>TOPICAL PRELIMINARIES</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>TOPOLOGICAL PRELIMINARIES</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Surfaces</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Fundamental Groups</td>
<td>6</td>
</tr>
<tr>
<td>II</td>
<td>THE MAIN THEOREM</td>
<td>13</td>
</tr>
<tr>
<td></td>
<td>Graph Theoretical Terminology</td>
<td>13</td>
</tr>
<tr>
<td></td>
<td>The Main Theorem</td>
<td>23</td>
</tr>
<tr>
<td>III</td>
<td>APPLICATIONS OF ORIENTABLE EMBEDDINGS</td>
<td>38</td>
</tr>
<tr>
<td></td>
<td>Orientable Surfaces</td>
<td>38</td>
</tr>
<tr>
<td></td>
<td>Self-dual Embeddings of Abelian Groups</td>
<td>40</td>
</tr>
<tr>
<td></td>
<td>A Self-dual Embedding of $\pi(S_n)$</td>
<td>50</td>
</tr>
<tr>
<td></td>
<td>Self-dual Embeddings of $K_r(s)$</td>
<td>55</td>
</tr>
<tr>
<td>IV</td>
<td>APPLICATIONS TO NON-ORIENTABLE EMBEDDINGS</td>
<td>59</td>
</tr>
<tr>
<td></td>
<td>Non-orientable surfaces</td>
<td>59</td>
</tr>
<tr>
<td></td>
<td>Applications</td>
<td>69</td>
</tr>
<tr>
<td>V</td>
<td>GENERALIZED EMBEDDING SCHEMES</td>
<td>85</td>
</tr>
<tr>
<td></td>
<td>Background</td>
<td>85</td>
</tr>
<tr>
<td></td>
<td>Generalized Embedding Schemes</td>
<td>94</td>
</tr>
<tr>
<td></td>
<td>The Classification of Embedding Schemes</td>
<td>107</td>
</tr>
<tr>
<td></td>
<td>The Nature of $\lambda$</td>
<td>119</td>
</tr>
<tr>
<td>VI</td>
<td>COVERING PSEUDOGRAPHS</td>
<td>134</td>
</tr>
<tr>
<td></td>
<td>The Construction</td>
<td>134</td>
</tr>
<tr>
<td></td>
<td>Self-dual Embeddings</td>
<td>154</td>
</tr>
</tbody>
</table>

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
CHAPTER I

TOPOLOGICAL PRELIMINARIES

The purpose of this chapter is twofold. It is here we supply the topological terminology and summarize the covering space theory required by the theorem of the next chapter. For proofs of the topological theorems stated here the reader is referred to [1] and [25]. Proofs of the covering projection theorems can be found in [33] and [25]. No essentially new theorems are stated here. The only "novelty" offered is the definition of singular surfaces and the calculation of their fundamental groups.

Section I.1

Surfaces

A surface is a connected Hausdorff topological space in which every point has a neighborhood homeomorphic to the Euclidean plane $\mathbb{R}^2$. A closed surface is a compact surface. It is well known (see [11], for example) that every closed surface can be presented as a collection of plane polygons, the totality of whose sides are identified in pairs. This identification process will be more explicitly described as soon as some terminology has been developed.

1
A bordered surface $S$ is a connected Hausdorff space in which every point has a neighborhood homeomorphic to an open subset of the Euclidean half-plane $\{(x,y) \in \mathbb{R}^2 \mid y \geq 0\}$. The set of points of $S$ which do not possess a neighborhood homeomorphic to an open subset of $\mathbb{R}^2$ is called the border or boundary of $S$ and is denoted by $\partial(S)$. Note that our definition agrees with that of [25], while disagreeing with [1], in that surfaces are bordered surfaces; in other words, surfaces are bordered surfaces for which $\partial(S) = \emptyset$. Bordered surfaces are obtained when a collection of open disks, with disjoint closures, are excised from a surface. It can be shown (see, for example, [25]) that every compact bordered surface has a presentation as a finite collection of polygons only some of whose sides are identified in pairs.

It will prove convenient to have yet another variation on the concept of a surface available. We first set $\mathbb{R}^2/1 = \mathbb{R}^2$ and define, for any positive integer $n \geq 2$, $\mathbb{R}^2/n$ to be that subset of the plane, parametrized by polar coordinates, described by

$$\{(r,\theta) \in \mathbb{R}^2 \mid r \geq 0 \text{ and } 2k\pi \leq (n-1)\theta \leq (2k+1)\pi \text{ for some } k=0,1,\ldots,n-2\}.$$
For example, $\mathbb{R}^2/2$ is the aforementioned Euclidean half-plane and the unshaded area in figure 1.1 describes $\mathbb{R}^2/4$. A singular surface is a connected Hausdorff space in which every point has a neighborhood homeomorphic to an open subset of $\mathbb{R}^2/n$ for some positive integer $n$. The value of the integer $n$ may vary from point to point.

Figure 1.1

For example, a singular surface is obtained when a collection of disjoint open disks is excised from a surface, the closure of any two of which may intersect at a finite number of points. Another method for obtaining singular surfaces is to identify a finite number of boundary points of some bordered surface. A point on a singular surface is a singular point if it possesses no neighborhood homeomorphic with an open subset of the Euclidean half-plane.

In particular, let $M$ denote a plane polygon with...
As the succession of its sides together with a rule which specifies some pairs of edges and some vertices which are to be identified (see Figure 1.2). Suppose further that as a result of these identifications all the vertices of \( \gamma_s \) have been identified into a single point \( P \). The resulting singular surface is called a unisingular surface. It has at most one singular point, and in that case the singular point is \( P \). We refer to \((P, \gamma_1, \gamma_2, \ldots, \gamma_s)\) as a presentation for this surface (the identification rule is left implicit). It seems quite likely that any compact singular surface with at most one singular point is in fact a unisingular surface, but this question will not be addressed here. Figure I.3a is a presentation for the modified torus of Figure I.3b. Here as elsewhere in this treatise, the shaded part of a diagram denotes a "hole" in the surface.
We summarize this section by listing the variety of spaces defined in order of their generality, from the most specialized to the most general:

closed surface
surface
bordered surface
unisingular surface
singular surface.
Section I.2

Fundamental Groups

Given a topological space \( X \), a path is a continuous map \( \omega : [0,1] \to X \) where \([0,1]\) is the closed unit interval. A path \( \omega \) is said to be simple if it is injective. If \( \omega(0) = P \) and \( \omega(1) = Q \), we say that \( \omega \) is a \( P-Q \) path. A path is closed if \( \omega(0) = \omega(1) \). A simple closed path is a closed path which is injective on the open interval \((0,1)\). Given a path \( \omega \), its inverse, denoted by \( \omega^{-1} \), is the path defined by

\[
\omega^{-1}(t) = \omega(1-t) \quad 0 \leq t \leq 1.
\]

Strictly speaking, paths are maps; nevertheless we shall frequently identify the path \( \omega \) with the point set \( \omega([0,1]) \subseteq X \). Given two paths \( \omega_1 \) and \( \omega_2 \) such that \( \omega_1(1) = \omega_2(0) \), their product \( \omega_1 \cdot \omega_2 \) is the path defined by

\[
(\omega_1 \cdot \omega_2)(t) = \begin{cases} 
\omega_1(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\
\omega_2(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1.
\end{cases}
\]

The two points \( \omega(0) \) and \( \omega(1) \) are the endpoints of \( \omega \). The homotopy class of \( \omega \), relative to its endpoints, is denoted by \([\omega]\), and the fundamental group of \( X \),
based at some point $x$, is denoted by $\pi(X,x)$. The product of two elements of $\pi(X,x)$ is denoted by their juxtaposition; thus,

$$[w_1][w_2] = [w_1 \cdot w_2].$$

A map of pairs $f:(X,A) \to (Y,B)$ is defined when $A \subset X$, $B \subset Y$, $f:X \to Y$, and $f(A) \subset B$. In particular, a continuous map $f:(X,x) \to (Y,y)$ induces a natural group homomorphism $f^*:\pi(X,x) \to \pi(Y,y)$.

The sides of the polygons used in the presentation of a unisingular surface $S$ are all simple paths or simple closed paths of $S$. Let $\gamma_1, \ldots, \gamma_s$ denote the totality of sides in such a presentation. An identification process may be defined as an ordered pair $(\phi, \epsilon)$ of maps where

$$\phi:[i]_i=1^s \to [i]_i=1^s \quad \text{and} \quad \epsilon:[i]_i=1^s \to [-1,1]$$

such that

a) $\phi$ is an involution (i.e., $\phi^2$ is the identity),

b) if $\phi(i) = i$ then $\epsilon(i) = 1$,

c) $\gamma_i$ is identified with $\gamma^{\epsilon(i)}_{\phi(i)}$; in other words, the surface is the quotient of the disjoint union of the given polygons by the relations

$$\gamma_i(t) = \gamma^{\epsilon(i)}_{\phi(i)}(t) \text{ for } 0 \leq t \leq 1 \text{ and } i = 1, 2, \ldots, s.$$
It will be necessary to have a presentation for the fundamental group of a unisingular surface. The reader is referred to [24] for the definition of the algebraic terms used here. There is nothing surprising about the following theorem and its proof is a straightforward application of theorems and methods used in [25] to derive the fundamental groups of closed surfaces.

1.2.1 Theorem: If a unisingular surface $S$ has presentation $(P, \gamma_1, \ldots, \gamma_s)$ with the identification rule $\gamma_i = \gamma^e(i)$, then $\pi(S, P)$ has the presentation

$$\langle g_1^e, \ldots, g_s^e; g_1g_2\cdots g_s, g_i = g^e_{\phi(i)} \quad i = 1, 2, \ldots, s \rangle.$$

Proof: Referring to the presentation of $S$ given in Figure 1.4, define

$$V = S - \bigcup_{i=1}^{s} \{\gamma_i\} \quad \text{and} \quad U = S - \{Q\}.$$

In terms of the polygonal presentation, the set $V$ is the interior of the defining polygon. Hence it is homeomorphic to the Euclidean plane. The set $U \cup V$, which is critical to this proof, is obtained by deleting the point $Q$ from $V$. Thus, $U \cup V$ is a homeomorph of the punctured Euclidean plane.
Figure 1.4

Since \( \alpha = \delta^{-1} \gamma_1 \cdot \ldots \cdot \gamma_s \delta \) is a deformation retract of \( U \), it follows that \( \pi(U,P_0) \cong \pi(\alpha,P_0) \) (see Theorem 4.2 in Chapter 2 of [25]). However, \( \pi(\alpha,P_0) \) is a free group with the presentation

\[
\langle g_1, \ldots, g_s \mid g_i = g_s^{\epsilon(i)} \phi(i), \ i = 1,2,\ldots,s \rangle
\]

under the defining map \( g_i \rightarrow [\delta^{-1} \gamma_i \delta] \). Moreover, the sets \( U, V, \) and \( U \cap V \) are all arcwise connected open sets and \( V \) is simply connected. Hence, the Van Kampen Theorem (see Theorem 4.1 of Chapter 4 of [25]) is applicable. Thus, if \( j:U \cap V \rightarrow V \) is the inclusion map, and \( N \) is the smallest normal subgroup of \( \pi(U,P_0) \) which contains \( \#(\pi(U \cap V,P_0)) \), then
\[ \pi(S, P_0) = \frac{\pi(U, P_0)}{N}. \]

But \( \pi(U \cap V, P_0) \) is the infinite cyclic group generated by \( \zeta \) and hence, since \( \zeta \) is homotopic in \( U \) to \( \zeta^{-1}, \gamma_1, \ldots, \gamma_2, \zeta \),

\[ j_\#: \left[ \zeta \right] = [\zeta^{-1}, \gamma_1, \ldots, \gamma_s, t] \in \pi(U, P_0). \]

Thus, a presentation for \( \pi(S, P_0) \) is obtained by adding the relator \( g_1 g_2 \ldots g_s \) to (1). Consequently, \( \pi(S, P) \) has the required presentation under the defining map \( g_i \rightarrow [\gamma_i] \).

q.e.d.

Section 1.3

Covering Projections

A continuous map \( p:\tilde{X} \rightarrow X \) is a covering projection if every point \( x \) of \( X \) has a neighborhood \( U \) such that \( p \) maps each component of \( p^{-1}(U) \) homeomorphically onto \( U \). In all our applications here, \( X \) will be a singular surface. Hence, if \( \tilde{X} \) is connected and \( p:\tilde{X} \rightarrow X \) is a covering projection, then \( \tilde{X} \) is also a singular surface. If \( Y \) is a subset of \( X \) and \( \tilde{Y} \) is a subset of \( \tilde{X} \) such that \( p \) maps \( \tilde{Y} \) homeomorphically onto \( Y \), we say that \( Y \) lifts to \( \tilde{Y} \), or that \( \tilde{Y} \) is a lift of \( Y \). The following theorems are classical results.
of covering projection theory, whose statements have been slightly modified to reflect the restriction that \( X \) is a singular surface.

1.3.1 Theorem (unique path lifting): Given a covering projection \( p: (\tilde{X}, \tilde{x}) \to (X, x) \) and a path \( \omega \) such that \( \omega(0) = x \), then there exists a unique path \( \tilde{\omega} \) in \( \tilde{X} \), which is a lift of \( \omega \), such that \( \tilde{\omega}(0) = \tilde{x} \).

**Proof:** See [25] Chapter 5 Lemma 3.1.

1.3.2 Theorem: Given a covering projection \( p: (\tilde{X}, \tilde{x}) \to (X, x) \), the induced group homomorphism \( p^\#: \pi(\tilde{X}, \tilde{x}) \to \pi(X, x) \) is injective.

**Proof:** See [25] Chapter 5 Theorem 4.1.

1.3.3 Theorem: Given a subgroup \( K \) of \( \pi(X, x) \), there exists a covering projection \( p: (\tilde{X}, \tilde{x}) \to (X, x) \) such that \( p^\#(\pi(\tilde{X}, \tilde{x})) = K \).

**Proof:** See [25] Chapter 5 Theorem 10.2.

1.3.4 Theorem: Given a covering projection \( p: (\tilde{X}, \tilde{x}) \to (X, x) \), then every simply connected subset of \( X \) lifts to a simply connected subset of \( \tilde{X} \).

**Proof:** See [25] Chapter 5 Theorem 5.1.
1.3.5 Theorem: Given a covering projection $p: (\tilde{X}, \tilde{x}) \to (X, x)$, then for any $y$ in $X$, the number of points in $p^{-1}(y)$ equals the index of $p_{\#}(\pi(\tilde{X}, \tilde{x}))$ in $\pi(X,x)$.

CHAPTER II

THE MAIN THEOREM

The first half of this chapter sets up the graph theoretical terminology required by the statement of the theorem in the second half. In general the definitions given here agree with those in [4] and [19], except that graphs may be infinite and duality is given an unconventional twist.

Section II.1

Graph Theoretical Terminology

A pseudograph $G$ consists of a pair of sets $V(G)$ and $E(G)$ called, respectively, the vertex and edge sets of $G$. An edge is either a singleton or a pair of elements of $V(G)$. Collectively, the vertices and edges of $G$ are referred to as the elements of $G$. Note that we allow the possibility that two distinct edges should be equal as subsets of $V(G)$; such edges are called multi-edges of $G$. Edges which are singletons are called loops. In general an edge which consists of a pair $\{u,v\}$ will be denoted by $uv$. We shall have occasion to refer to the directed edges (or arcs) of $G$; these are the ordered pairs.
(u,v) where u and v are vertices of G and uv is an edge of G. Given an arc (u,v), u is its initial vertex and v is its terminal vertex. By saying that the arc d is at v we mean that v is the terminal vertex of d. The set of directed edges of G is denoted by D(G) and the above directed edge (u,v) will also be denoted by uv, as the context will make it clear, explicitly, if necessary, whether or not we have a directed edge in mind.

Two vertices of a pseudograph are adjacent if they constitute an edge of G. An edge is incident to a vertex, and vice versa, if the edge contains the vertex. An alternating sequence of vertices and edges, which both starts and ends with a vertex, is a walk of G if every consecutive pair of elements are incident. If the first and last vertices of a walk are identical, we say that the walk is closed. A closed walk in which no edge is repeated is a circuit. A pseudograph is said to be connected if every two vertices are contained in some walk. The set of all the walks of the pseudograph G is denoted by W(G).

Two pseudographs G and H are isomorphic if there exists a map f: V(G)∪E(G) → V(H)∪E(H) such that
a) $f$ induces a bijection between $V(G)$ and $V(H)$,
b) $f$ induces a bijection between $E(G)$ and $E(H)$,
c) $f$ preserves both adjacency and incidence.

It is clear that isomorphism is an equivalence relation and we consider isomorphic pseudographs to be equal.

A pseudograph is finite or infinite according as $E(G) \cup V(G)$ is finite or infinite. A multigraph is a pseudograph which has no loops, and a graph is a multigraph which has no multi-edges. In particular, a graph in which every two vertices are adjacent is called a complete graph and is denoted by $K_m$ where $m$ is the cardinality of $V(K_m)$. This notion is generalized in the following manner. If $V(G)$ can be partitioned into subsets $V_1, V_2, \ldots, V_n$ such that $uv$ is an edge of $G$ iff $u$ and $v$ belong to different $V_i$, we say that $G$ is complete $n$-partite and write $G = K(m_1, m_2, \ldots, m_n)$ where $m_i = |V_i|$. When $m_1 = m_2 = \ldots = m_n = m$, we write $K(m, m, \ldots, m) = K_n(m)$.

A voltage pseudograph is a triple $(G, \rho, \Gamma)$ where $G$ is a pseudograph and $\rho$ is a map of $D(G)$ into the group $\Gamma$ which satisfies the condition

$$\rho(uv) = [\rho(vu)]^{-1}.$$
Such a voltage assignment can be unambiguously extended to $W(G)$ by associating with every walk $u_1'u_1u_2'u_2u_3'$,..., $u_k'u_ku_k'u_k'$ of $G$ the element $\rho(u_1'u_2')\rho(u_2'u_3')...\rho(u_{k-1}'u_k')$ of $\Gamma$. We say that $\rho$ is full if $\rho(W(G)) = \Gamma$.

It is clear that if $\rho$ is full then $\rho(D(G))$ generates $\Gamma$.

A singular element of a group is an element of order at most two. Let $\Delta$ be a set of generators of the group $\Gamma$. A Cayley color multigraph of $\Gamma$ relative to $\Delta$ is a triple $(G,\lambda,\Gamma)$ where $G$ is a multigraph, $\Gamma$ is a group, and $\lambda$ is a map $\lambda:V(G) \cup D(G) \to \Gamma$ which satisfies the following conditions:

a) $\lambda|V(G)$ is a bijection of $V(G)$ onto $\Gamma$,
b) $\lambda|D(G)$ defines a full voltage pseudograph,
c) if $uv \in D(G)$ then $\lambda(u)\lambda(uv) = \lambda(v)$,
d) let $u$ be a vertex of $G$; then
d1) if $g$ is a non-singular element of $\Gamma$ then there exists a unique arc $d$ at $u$ such that $\lambda(d) = g$,
d2) if $g$ is a non-trivial singular element of $\Gamma$ then there are exactly two arcs $d, d'$ at $u$ such that $\lambda(d) = \lambda(d') = g$,
d3) $1_{\Gamma}, \not\in \lambda(D(G))$.

Given $\Gamma$ and $\Delta$, all the Cayley color multigraphs of $\Gamma$ relative to $\Delta$ are isomorphic and are collectively denoted...
by $G^m_{\Delta}(\Gamma)$. Figure II.1 contains a drawing of $G^m_{\{2,3\}}(Z_6)$.

![Diagram of G^m_{\{2,3\}}(Z_6)](image)

Figure II.1

A solid line is assigned voltage 2 in the direction of the arrowhead. Dotted lines are assigned voltage 3 in either direction. The vertex assignment is indicated in the figure.

If in the definition of Cayley color multigraphs condition d2 is replaced by

d2') if $g$ is a non-trivial singular element of $\Gamma$ then there exists exactly one arc $d$ at $u$ such that $\lambda(d) = g$,

we obtain a structure called the Cayley color graph of $\Gamma$ relative to $\Delta$ - denoted by $G^m_{\Delta}(\Gamma)$. Figure II.2 contains a drawing of $G^m_{\{2,3\}}(Z_6)$. The assignments in $\Gamma$ here
are the same as those in Figure II.1. For a more detailed examination of Cayley color graphs the reader is referred to [36] and [18]. In particular we shall make use of the fact that condition b for Cayley color multigraphs is equivalent to

\[ b') \lambda |D(G) \text{ is a voltage multigraph and } G \text{ is connected.} \]

It is convenient to define a Cayley graph as the graph obtained by ignoring the colors and the directions of the edges of a Cayley color graph.

Given a pseudograph \( G \) and a surface \( S \), an embedding \( i \) of \( G \) on \( S \) is an indexed collection

\[ i = \{ i_\mu | \mu \in V(G) \cup D(G) \} \]

such that

\[ \text{Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.} \]
a) if \( u \) and \( v \) are vertices of \( G \) then \( i_u \) and \( i_v \) are points of \( S \) and \( i_u = i_v \) iff \( u = v \),

b) if \( e = uv \) is an edge of \( G \) which is not a loop, then \( i_e \) is a simple path joining \( i_u \) and \( i_v \),

c) if \( e \) is a loop at \( v \) then \( i_v \) is a simple closed path at \( i_v \),

d) if \( e \) and \( f \) are distinct edges of \( G \) and \( x \in i_e \cap i_f \), then \( x \) is an endpoint of both \( i_e \) and \( i_f \),

e) if \( v \) is an isolated vertex of \( G \) and \( e \) is an edge of \( G \), then \( i_v \neq i_e \),

f) \( \bigcup_{v \in V(G)} i_v \bigcup \bigcup_{e \in E(G)} i_e \) is a closed subset of \( S \).

Even though \( i \) is not a map we shall denote the set described in condition \( f \) above by \( i(G) \). Since \( S \) is locally connected it follows that the complement of \( i(G) \) in \( S \) is the disjoint union of connected open sets, each of which is called a region of \( G \) on \( S \). A region which is homeomorphic to the Euclidean plane is called a 2-cell. If each region of an embedding is a 2-cell we refer to it as a 2-cell embedding.

We shall in general identify \( G \) with \( i(G) \) and consider walks in \( G \) as paths on \( S \). This identification enables us to characterize some of the edges of \( G \).

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
An edge is a **boundary edge** if it is entirely contained in $\partial(S)$. An **interior** edge is an edge which intersects $\partial(S)$ only at its endpoints, if at all. Such an edge has two sides in an obvious intuitive sense, although it is quite possible that the same region lies on both sides of an interior edge. A boundary edge, on the other hand, has only one side.

Given an embedding of a pseudograph $G$ on the surface $S$, a **dual of $G$ on $S$** is denoted by $G^*$ and constructed by selecting as vertices one point from each region of $G$ on $S$. If $uv$ is an interior edge of $G$ then the vertices (or vertex) of $G^*$ corresponding to the regions (or region) on the two sides of $uv$ are joined by a simple path (or closed path) which meets $G$ at exactly one point on $uv$. This edge is referred to as the dual edge to $uv$ and is denoted by $(uv)^*$. If $\gamma$ is a loop of $G$ which is also a boundary edge (a **boundary loop**), then its dual is described in Figure II.3. We start with a vertex $x$ of $G^*$ which lies in the only region $R$ of $G$ whose boundary contains $\gamma$, and join $x$ by a path to a point $y$ on the loop $\gamma$. Then proceed along the loop, in the sense of $\gamma^{-1}$, past the vertex $v$ of $\gamma$ to a point $z$ which precedes $\gamma$. Finally join $z$ to $x$ by a simple path whose interior lies in $R$. The resulting loop at $x$, denoted by $\gamma^*$,
is the dual edge of $\gamma$.

Figure II.3

We define duals for one additional class of edges of $G$. Suppose we have a boundary curve of $S$ which is the union of two multi-edges of $G$, $uv$ and $vu$ (see Figure II.4). Their duals are $xy$ and $yx$ as described in the same figure. Note that here we have a pair of edges of $G^*$ dual to a pair of edges of $G$, while it is not claimed that $uv$ itself is dual to either $xy$ or $yx$.

Let $(G, p_0, \Gamma)$ be a voltage pseudograph and suppose that $G$ is embedded in some surface $S$. Suppose further

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
that \( \rho_0(w) = l_T \) whenever \( w \) is a walk which consists of the complete oriented boundary of some region of \( G \) on \( S \). We say then that the Kirchoff voltage law (KVL) holds around each region of \( G \). This concept, which originates in [14], will be discussed in greater detail in Chapter VI.

It is clear that if \( G \) is embedded on \( S \) then the dual \( G^* \) of \( G \) on \( S \) is unique up to an isomorphism. If it happens that \( G^* \) and \( G \) are themselves isomorphic, we say that \( G \) has a self-dual embedding on \( S \). The theorem of the next section provides a method for generat-
ing a large class of self-dual embeddings.

Section II.2

The Main Theorem

II.2.1 Theorem: Let $S$ be a unisingular surface with the presentation $(P, \gamma_1, \ldots, \gamma_s)$. Let $G$ be the pseudograph with one vertex $P$ defined by the sides of this presentation and let $H$ be the graph dual to $G$ on $S$. Suppose $(G, \rho_0, \Gamma)$ is a full voltage graph such that $l_{\Gamma} \notin \rho_0(D(G))$ and KVL holds around the single region $G$ determines on $S$. Then $\rho_0$ induces a surjective homomorphism $\rho : \pi(S, P) \rightarrow \Gamma$ and if $p : (\tilde{S}, \tilde{P}) \rightarrow (S, P)$ is the covering projection which satisfies

$$p^*(\pi(S, P)) = \text{Ker} \rho \subset \pi(S, P),$$

then $p^{-1}(G)$ and $p^{-1}(H)$ are dual Cayley color multi-graphs for $\Gamma$ embedded in $\tilde{S}$.

More precisely, for each $i = 1, 2, \ldots, s$ define $W_i$ to be the following closed path in $G$:

a) if $\gamma_i$ is a boundary loop then

$$W_i = \gamma_1 \gamma_2 \cdots \gamma_{i-1} \gamma_{i+1} \cdots \gamma_s$$

b) if $\gamma_i = \gamma_j$ $(i < j)$ then

$$W_i = W_j = \gamma_1 \gamma_2 \cdots \gamma_i \gamma_{i+1} \cdots \gamma_s$$
c) if $\gamma_i = \gamma_j^{-1}$ (i<j) then

$$W_i = W_j = \gamma_{i-1}'...\gamma_{j+1}'...\gamma_s.$$ 

Now set $\Delta = \{\rho_0(\gamma_i)\}$ and $\Delta^* = \{\rho(W_i)\}$ for $i = 1, 2, ..., s$. Then

$$\rho^(-1)(\Delta) = G^m(\Gamma) \text{ and } \rho^(-1)(\Delta^*) = G^m(\Delta^*)(\Gamma).$$

Moreover, if the voltage assignment on $G$ is such that $\rho_0(\gamma)$ is singular iff $\gamma$ is a boundary loop of $G$, then these embeddings of $G^m(\Gamma)$ and $G^m(\Delta^*)(\Gamma)$ on $\overline{S}$ can be modified so as to produce embeddings of $G(\Gamma)$ and $G(\Delta^*)(\Gamma)$ which are dual to each other on some closed surface $\overline{S}$.

**Proof:** According to Theorem 1.2.1 $\pi(S,P)$ has, by a slight abuse of notation, the presentation

$$\langle[\gamma_1],[\gamma_2],...,[\gamma_t];[\gamma_1][\gamma_2]...[\gamma_t]\rangle.$$ 

Here $[\gamma_{ij}]_{j=1}^t$ is obtained from the original set $[\gamma_i]_{i=1}^s$ by deleting every $\gamma_j$ such that $\gamma_j = \gamma_i^{-1}$ for some $i<j$. On the other hand, since the voltage assignment is such that KVL holds around the region which $G$ determines on $S$, it follows that $\rho_0(\gamma_1)\rho_0(\gamma_2)...\rho_0(\gamma_s) = 1_{\Gamma}.$ Hence the association to each generator $\gamma_i$ in $\pi(S,P)$ of the element $\rho_0(\gamma_i)$ in $\Gamma$ extends to a group homomorphism.
\[ p : \pi(S,P) \to \Gamma. \]

Since \((G,\rho_0)\) is a full voltage pseudograph it follows that \(p\) is in fact surjective. Let \(K\) denote the kernel of \(p\); then \(K\) is a subgroup of \(\pi(S,P)\) and by Theorem I.3.3 there exists a covering projection \(p:(\tilde{S},\tilde{P}) \to (S,P)\) such that

\[ p_\#(\pi(\tilde{S},\tilde{P})) = K. \]

It follows from the path lifting properties of covering projections that \(p^{-1}(G)\) and \(p^{-1}(H)\) are indeed dual pseudographs on \(\tilde{S}\).

The reader is reminded here that in the proof of Theorem I.3.3 the points of \(\tilde{S}\) are in fact equivalence classes of pairs \((x,[\omega])\) with \(x\) a point of \(S\) and \(\omega\) a \(P\)-\(x\) path in \(S\); the equivalence is defined by

\[ (x_1,[\omega_1]) = (x_2,[\omega_2]) \iff x_1 = x_2 \quad \text{and} \quad [\omega_1,\omega_2^{-1}] \in K. \]

With this notation the given covering projection satisfies

\[ p((x,[\omega])) = x. \]

In particular, \((x,[\omega])\) is a vertex of \(p^{-1}(G)\) iff it is a point of \(p^{-1}(P)\) and this happens iff \(x = P\) and \(\omega\) is a closed path at \(P\). This "coordinate system" on \(S\) is used to show that \(p^{-1}(G)\) and \(p^{-1}(H)\) are Cayley color multigraphs.

We first define a map \(\lambda:V(p^{-1}(G)) \to \Gamma\) by setting

\[ \lambda(u) = \rho([\omega]). \]
whenever $u = (P, [w])$. To see that $\lambda$ is indeed well-defined, note that if $(P, [w_1]) = (P, [w_2])$ then $[w_1, w_2^{-1}] \in K$ and so $\rho([w_1]) = \rho([w_2])$. A reversal of this reasoning shows that $\lambda$ is an injection. It is in fact also surjective because $\rho$ itself is surjective.

Next, let $u_1 u_2$ be a directed edge of $p^{-1}(G)$ with $u_i = (P, [w_i])$ for $i = 1, 2$. Since every edge of $p^{-1}(G)$ is the lift of some edge of $G$, the edge $u_1 u_2$ is the lift of some $\gamma \in \{\gamma_i\}$. We note here that the $\gamma_i$ and the $\gamma_i^\ast$ should be considered as paths, i.e., as having a sense. Unless specifically indicated, this sense may be assigned in an arbitrary fashion. It is implicit in the construction used to prove Theorem 1.3.3 that $[w_2] [w_1, \gamma]^{-1} \in K$.

Now extend $\lambda$ to $D(p^{-1}(G))$ by defining

$$\lambda(u_1 u_2) = \rho([\gamma]).$$

We first show that $\lambda$ is still well defined. Let $\overline{w}_1$ and $\overline{w}_2$ be another pair of closed paths at $P$ such that $(P, [w_i]) = (P, [\overline{w}_i])$ for $i = 1, 2$, and suppose $\gamma$ is another edge in $\{\gamma_i\}$ such that $[\overline{w}_2][\overline{w}_1, \gamma]^{-1} \in K$. Then, since $K$ is the kernel of $\rho$, we have

$$\rho([\gamma]) = \rho([w_1^{-1}, w_2]) = \rho([w_1^{-1}]) \rho([w_2]) = \rho([w_1^{-1}]) \rho([\overline{w}_2]) = \rho([w_2]) = \lambda(u_1 u_2) = \rho([\gamma]).$$
\[= \rho([w_1^{-1}, \mu_2]) = \rho([\gamma]).\]

Persisting with the same notation, we see that

\[\gamma(u_1)\gamma(u_1u_2) = \rho([w_1])\rho([w_1^{-1}, \mu_2]) = \rho([w_1, w_1^{-1}, \mu_2]) = \rho([\mu_2]) = \lambda(u_2).\]

Finally, the unique lifting properties of \( p \) together with the voltage assignment of \( \rho_0 \) on \( G \) guarantee that at every vertex of \( p^{-1}(G) \), the non-singular generators of \( \Gamma \) are assigned to one arc, and the non-trivial singular generators to two arcs at that vertex. Thus, if we set \( \Delta = \{\rho_0(\gamma_i)\}_{i=1}^s \) then \( p^{-1}(G) = G_\Delta^m(\Gamma) \).

We now turn our attention to \( H \) which is the dual of \( G \) on \( S \). Let \( Q \) denote the unique vertex of \( H \) and let \( \delta \) be a \( Q-P \) path in \( S \), situated, as indicated in Figure II.5, between \( \gamma_s \) and \( \gamma_1 \). The vertex set of \( p^{-1}(H) \) is \( p^{-1}(Q) \) and its edge set consists of the collection of all the components of the \( p^{-1}(\gamma_i^*) \) (1 \( \leq i \leq s \)) . Each \( \gamma_i^* \) is of course a path dual (as edges) to \( \gamma_i \). Every point of \( p^{-1}(Q) \) has coordinates \((Q, [w])\) where \( w \) is a \( P-Q \) path. We use this to define a map \( f_\delta : p^{-1}(Q) \rightarrow p^{-1}(P) \) by setting

\[f_\delta((Q, [w])) = (P, [w, \delta]).\]
The map $f_\delta$ is well defined because if $(Q, [\omega_1]) = (Q, [\omega_2])$ then $[\omega_1, \omega_2^{-1}] \in K$ and $[\omega_1 \cdot \delta \cdot (\omega_2 \cdot \delta)^{-1}] \in K$, and so $(P, [\omega_1 \cdot \delta]) = (P, [\omega_2 \cdot \delta])$. The proof that $f_\delta$ is a bijection is straightforward and hence omitted.

In order to show that $p^{-1}(H)$ is a Cayley color multigraph we define a map $\mu : \mathcal{V}(p^{-1}(H)) \to \Gamma$ by setting

$$\mu((Q, [\omega])) = \lambda(f_\delta((Q, [\omega]))) = \rho([\omega, \delta]).$$

Since $f_\delta$ and $\lambda|_{p^{-1}(P)}$ are bijections, so is $\mu$ a bijection. Now let $v_1v_2$ be an arbitrary directed edge of $p^{-1}(H)$ with $v_i = (Q, [\omega_i])$ for $i = 1, 2$. Then $v_1v_2$ is the lift of some edge $\gamma^*$ of $H$. Hence $[\omega_2] \cdot [\omega_1 \cdot \gamma^*]^{-1} \in K$ and we define:

$$\mu(v_1v_2) = \rho([\delta^{-1}, \gamma^*, \delta]).$$

Thus,

$$\mu(v_1)\mu(v_1v_2) = \rho([\omega_1, \delta])\rho([\delta^{-1}, \gamma^*, \delta]) = \rho([\omega_1 \cdot \delta \cdot \delta^{-1}, \gamma^*, \delta]) = \rho([\omega_2, \delta]) = \mu(v_2).$$

It is clear that if $d_1$ and $d_2$ are any two lifts of $\gamma^*$ (i.e., components of $p^{-1}(\gamma^*_1)$) then $\mu(d_1) = \mu(d_2)$. Thus we may unambiguously define $\mu(\gamma^*_1) = \mu(d)$ where $d$ is any lift of $\gamma^*_1$. Set $\Delta^* = \{\mu(\gamma^*_i)\}^S_{i=1}$. It is obvious from the construction that $p^{-1}(H)$ is a connected multigraph.
and hence it now follows that $\Delta^*$ generates $\Gamma$. Again the unique path lifting properties of covering projections ensure that $p^{-1}(H)$ is indeed the Cayley color multi-

graph of $\Gamma$ relative to the set $\Delta^*$.

We next examine the elements of $\Delta^*$. A typical element of $\Delta^*$ has the form $p((\delta_1^*, \gamma^*, \delta))$ and we distinguish three possible cases.

a) $\gamma_i$ is a boundary edge of $G$ on $S$. Because of the nature of duals to edges which lie entirely on the boundary of $S$, the path $\delta_1^*, \gamma_i^*, \delta$ is described by the sequence of points $P, Q, T_1, P_1 = P_2 = P, T_2, Q, P$ as indicated by the arrows in Figure II.6. In fact, if we let $QT_i$ and $T_iP_i$ denote the obvious directed paths in this figure, then

$$[\delta_1^*, \gamma_i^*, \delta] = [\delta_1^*, QT_i, T_1P_1, P_2T_2, T_2Q, \delta) =$$

$$= [\delta_1^*, QT_i, T_1P_1][P_2T_2, T_2Q, \delta] = [\gamma_1^*, \gamma_2^*, \ldots, \gamma_{i-1}^*, \gamma_i^+, \ldots, \gamma_s^*]$$

(The reader is reminded that the definition of $W_i$ is given in the statement of the theorem).

b) $\gamma_i = \gamma_j$ $(1 < j)$. Here, in Figure II.7, and using notation similar to that used in case a, we note that $T_1 = T_2$
Figure II.6

and $T_1P_1 = (P_2T_2)^{-1}$ and so,

$[\delta^{-1} \cdot \gamma_1^* \cdot \delta] = [\delta^{-1} \cdot \gamma_j^* \cdot \delta] = [\delta^{-1} \cdot QT_1 \cdot T_2Q \cdot \delta] = $

$= [\delta^{-1} \cdot QT_1 \cdot T_1P_1 \cdot P_2T_2 \cdot T_2Q \cdot \delta] = [\delta^{-1} \cdot QT_1 \cdot T_1P_1] [P_2T_2 \cdot T_2Q \cdot \delta]$

$= [\gamma_1 \cdot \gamma_2 \cdot \ldots \cdot \gamma_j] \cdot [\gamma_{j+1} \cdot \ldots \cdot \gamma_s] = [W_1] = [W_j]$

(\text{c) } \gamma_i = \gamma_{i-1} (i<j) \). It is Figure II.8 that is relevant here and we note that in this case again $T_1 = T_2$ and $T_1P_1 = (P_2T_2)^{-1}$. Hence

$[\delta^{-1} \cdot \gamma_1^* \cdot \delta] = [\delta^{-1} \cdot \gamma_j^* \cdot \delta] = [\delta^{-1} \cdot QT_1 \cdot T_2Q \cdot \delta] =$

$= [\delta^{-1} \cdot QT_1 \cdot T_1P_1] [P_2T_2 \cdot T_2Q \cdot \delta] = [\gamma_1 \cdot \ldots \cdot \gamma_{i-1}] [\gamma_{j+1} \cdot \ldots \cdot \gamma_s]$

$= [W_1] = [W_j]$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Thus we have derived expressions for the elements of $\Delta^*$ in terms of the generator set $\Delta$. These will prove extremely useful later when we turn to down-to-earth calculations.
We have reached the third and final part of the theorem. The above derived surface and embeddings will now be modified to produce 2-cell embeddings of \( G_\Delta(\Gamma) \) and \( G_{\Delta^*}(\Gamma) \) in some closed surface \( \overline{S} \). First, however, the boundaries of \( \tilde{S} \) must be examined. We claim that each component of the boundary of \( \tilde{S} \) consists of exactly two lifts of the same boundary loop of \( S \). For suppose Figure II.9a describes a boundary loop \( \gamma_i \) of \( S \) (the shaded area denotes the hole) with voltage \( \rho_0(\gamma_i) = g \in \Gamma \). When this is lifted to a point \( \tilde{x} \) of \( p^{-1}(p) \) on \( \tilde{S} \), we obtain the configuration described in Figure II.9b. Since \( \gamma_i \) is a boundary curve, the element \( g \) has order 2 and hence

\[
\lambda(\tilde{x}_1) = \lambda(\tilde{x})\lambda(\tilde{x}x_1) = \lambda(\tilde{x})g = \lambda(\tilde{x})g^{-1} = \lambda(\tilde{x})\lambda(\tilde{x}x_2) = \lambda(\tilde{x}_2)
\]

\[
\therefore \quad \tilde{x}_1 = \tilde{x}_2,
\]

Hence it turns out that Figure II.9a lifts in fact to Figure II.9c. By identifying the paths \( \tilde{x}x_1 \) and \( \tilde{x}x_2 \) (each in the direction from \( \tilde{x} \) to \( \tilde{x}_1 \)) we simultaneously eliminate a hole from \( \tilde{S} \) and a pair of multi-edges of \( G^m_\Delta(\Gamma) \). Since Figure II.9 accounts for every hole of \( \tilde{S} \) and every pair of multiple edges of \( G^m_\Delta(\Gamma) \), it follows that the application of this process to all its boundaries will modify \( \tilde{S} \) into a closed surface \( \overline{S} \). Moreover,
the process will concurrently modify the embedding of $G_{\Delta}^{m}(\Gamma)$ on $\bar{S}$ into an embedding of $G_{\Delta}^{m}(\Gamma)$ on $S$.

We now consider the effect that this process has on $G_{\Delta}^{m}(\Gamma)$. Returning to our boundary loop $\gamma_i$ (see calculation a above) we note that $\mu(\gamma_i^+) = \rho([W_i])$ and hence
\[(\mu(\gamma_i^*))^2 = \rho([W_i^*W_i]) = \]
\[= \rho([\gamma_1^*\ldots\gamma_{i-1}^*\gamma_{i+1}^*\ldots\gamma_s^*\gamma_1^*\ldots\gamma_{i-1}^*\gamma_{i+1}^*\ldots\gamma_s^*]).\]

However, \([\gamma_1^*\gamma_2^*\ldots\gamma_s^*] = l_\pi(S,P),\) and hence
\[\{\gamma_{i+1}^*\ldots\gamma_s^*\gamma_1^*\ldots\gamma_{i-1}^*\} = [\nu_i]^{-1} .\]

Also, \(\rho([\nu_i]^{-1}) = \rho([\gamma_i^*])\) because \(\rho([\gamma_i^*]) = g\) has order 2. Hence,
\[(\mu(\gamma_i^*))^2 = \]
\[\rho([\gamma_1^*\ldots\gamma_{i-1}^*])\rho([\gamma_{i+1}^*\ldots\gamma_s^*\gamma_1^*\ldots\gamma_{i-1}^*])\rho([\gamma_{i+1}^*\ldots\gamma_s^*])\]
\[= \rho([\gamma_1^*\ldots\gamma_{i-1}^*])\rho([\gamma_i^*])\rho([\gamma_{i+1}^*\ldots\gamma_s^*]) = \rho([\gamma_1^*\ldots\gamma_s^*])\]
\[= \rho(l_\pi(S,P)) = 1_\Gamma .\]

On the other hand \(\mu(\gamma_i^*) = 1_\Gamma,\) for otherwise we would have
\[1_\Gamma = \mu(\gamma_i^*) = \rho([W_i^*]) = \rho([\gamma_1^*\ldots\gamma_{i-1}^*\gamma_{i+1}^*\ldots\gamma_s^*])\]
\[= \rho([\gamma_1^*])\rho([\gamma_2^*])\ldots\rho([\gamma_{i-1}^*])\rho([\gamma_{i+1}^*])\ldots\rho([\gamma_s^*]) .\]

However, since KVL holds we also know that
\[1_\Gamma = \rho([\gamma_1^*])\rho([\gamma_2^*])\ldots\rho([\gamma_s^*]) ,\]
and this implies that \(\rho([\gamma_i^*]) = 1_\Gamma,\) which we know not
to be the case. Hence $l_\Gamma \neq \mu(\nu_1^\ast)$ and so $\mu(\nu_1^\ast)$ has order 2 in $\Gamma$.

It can be shown in a similar manner that if $\mu(\nu_1^\ast)$ has order 2 then so does $\lambda(\nu_1)$. Thus multi-edges of $G_{\Lambda}^m(\Gamma)$ occur only as duals to multi-edges of $G_{\Delta}^m(\Gamma)$ which in turn arise only from boundaries of $S$. This implies that Figure II.10b, which is the lift to $\tilde{S}$ of Figure II.10a, describes a typical pair of multi-edges in $\tilde{S}$. In this figure $P_1(Q_1)$ is a lift of $P(Q)$ for $i = 1, 2$. When, in the above described process, the boundaries of $\tilde{S}$ are sealed up, the resulting configuration is shown in Figure II.10c. It is now clear that all that need be done to turn $G_{\Delta}^m(\Gamma)$ into $G_{\Lambda}^m(\Gamma)$ is to replace the two components of $\pi^{-1}(\nu_1^\ast)$ joining $Q_1$ and $Q_2$ by a single path joining the same points and dual to $P_1P_2$ (Figure II.10d). As it is clear from the construction that $G_{\Delta}(\Gamma)$ and $G_{\Lambda}(\Gamma)$ are embedded as dual graphs in $S$, the proof of the theorem is concluded.

q.e.d.
Figure II.10

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
CHAPTER III

APPLICATIONS TO ORIENTABLE EMBEDDINGS

A discussion of orientable surfaces is followed by the construction of self-dual embeddings of graphs on such surfaces. Embeddings of groups on surfaces are defined and it is shown that all finitely generated abelian groups, with two exceptions, and the fundamental groups of closed orientable surfaces have self-dual embeddings on orientable surfaces. A sufficient condition is given for some graphs $K_r(s)$ to have orientable self-dual embeddings.

Section III.1

Orientable Surfaces

Let $\{M_{\lambda}\}, \lambda \in \Lambda$, be a collection of plane polygons presenting a bordered surface $S$ and let $o$ be a function which assigns to each $\lambda$ the closed path consisting of the perimeter of $M_{\lambda}$, taken in one of its two possible senses. Suppose $o$ can be defined so that for any identified pair of sides $\gamma_1$ and $\gamma_2$, lying on the perimeters of $M_{\lambda_1}$ and $M_{\lambda_2}$ respectively, $o(\lambda_1)$ and $o(\lambda_2)$ traverse the resulting path on $S$ in opposite senses. We say then that the surface $S$ is orientable.
It is an easy consequence of Theorem 1.3.4 that if $\tilde{X}$ is a covering space of an orientable surface $X$, then $\tilde{X}$ is also orientable. It is a classical result (see [25] or [11]) that every closed orientable surface either has a presentation of the form shown in Figure III.1 with $n \geq 1$, or else it is homeomorphic to the sphere $S_0$. Such a surface $S_n$ is homeomorphic to a sphere with $n$ handles attached ($n \geq 0$), and we refer to $n$ as the genus of $S_n$. The genus of a closed orientable surface is a topological invariant.

![Figure III.1](Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.)
The unisingular surface obtained from $S_n$ by the excision of $k$ open disks, as indicated in Figure III.2, is denoted by $S(n, k)$. Note that $S(0, 0)$ is not described in either part of this figure.

Section III.2

Self-dual Embeddings of Abelian Groups

By an embedding of a group $\Gamma$ is meant an embedding of the Cayley graph $G_\Delta(\Gamma)$ relative to some generator

![Diagram](image-url)

Figure III.2

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
set \( \Delta \) of \( \Gamma \). This section's theorem generalizes and answers a question in [37] regarding the characterization of finite abelian groups which possess self-dual embeddings. First, we present a lemma which shows how Theorem II.2.1 can be used to construct self-dual embeddings of Cayley graphs on orientable surfaces. Note that abelian groups are written additively.

III.2.1 Lemma: Let \( \Gamma \) be a finitely generated abelian group with subsets \( \Delta_1 \) and \( \Delta_2 \), not containing 0, such that

a) \( \Delta_1 \) has an even number of elements, all non-singular,

b) \( \Delta_1 \cap \Delta_1^{-1} = \emptyset \),

c) \( \Delta_2 \) contains only singular elements of \( \Gamma \),

d) \( \sum_{g \in \Delta_2} g = 0 \),

d) \( \Delta_1 \cup \Delta_2 \) generates \( \Gamma \).

Then \( G_\Delta(\Gamma) \) has a self-dual embedding on an orientable surface. If \( \Gamma \) is finite, then the derived surface is closed.

Proof: Set \( n = \frac{1}{2}|\Delta_1| \) and \( k = |\Delta_2| \); we may write
\[ \Delta_1 = \{a_i, b_i\}_{i=1}^n \quad \text{and} \quad \Delta_2 = \{d_j\}_{j=1}^k. \] We select 
\( S = S(n, k) \) and define \( G \) to be the pseudograph whose
only vertex is \( P \) and whose edges are the paths 
\[ \{a_i, b_i, d_j \mid 1 \leq i \leq n, 1 \leq j \leq k \} \] (see Figure III.2).
The voltage assignment \( \rho_0 : D(G) \to \Gamma \) is defined by
\[
\rho_0(a_i) = a_i \quad i = 1, 2, \ldots, n \\
\rho_0(b_i) = b_i \quad i = 1, 2, \ldots, n \\
\rho_0(d_j) = d_j \quad j = 1, 2, \ldots, k.
\]
Condition \( d \) ensures that KVL holds, since the voltage around the single region of \( G \) on \( S \) is
\[
\sum_{i=1}^n (a_i + b_i - a_i - b_i) + \sum_{j=1}^k d_j = 0 + 0 = 0.
\]
Let \( H \) be a dual of \( G \) on \( S \). Clearly the hypotheses
of Theorem II.2.1 are fully satisfied and hence
\( p^{-1}(G) = G^*_{\Delta} (\Gamma) \) and \( p^{-1}(H) = G^*_{\Delta^*} (\Gamma) \), where
\( \Delta = \Delta_1 \cup \Delta_2 \). Moreover, these Cayley graphs are duals of
each other.

It now only remains to show that \( G_{\Delta} (\Gamma) \) and \( G_{\Delta^*} (\Gamma) \)
are in fact isomorphic graphs. We recall that
\( \Delta^* = \{\rho_0(W_\lambda)\}_{\lambda=1}^S \), where the \( W_\lambda \) corresponds to a boundary loop \( \xi_j \) of \( G \), then
\[ W_\lambda = \delta_1 \delta_2 \cdots \delta_{j-1} \delta_{j+1} \cdots \delta_K \alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1} \cdots \alpha_n \beta_n \alpha_n^{-1} \beta_n^{-1} \cdots \]

Hence,

\[ \rho_0(W_\lambda) = \sum_{\nu \neq j} d_\nu + \sum_{i=1}^n (a_i + b_i - a_i - b_i) = -d_j = d_j. \]

If \( W_\lambda \) corresponds to \( \alpha_i \), then

\[ W_\lambda = \delta_1 \cdots \delta_K \alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1} \cdots \alpha_{i-1} \beta_{i-1} \alpha_{i-1}^{-1} \beta_{i-1}^{-1} \cdots \alpha_{i+1} \beta_{i+1} \alpha_{i+1}^{-1} \beta_{i+1}^{-1} \cdots \alpha_n \beta_n \alpha_n^{-1} \beta_n^{-1} \cdots \]

Hence,

\[ \rho_0(W_\lambda) = \sum_{j=1}^K d_j + \sum_{\nu \neq i} (a_\nu + b_\nu - a_\nu - b_\nu) - b_i = -b_i. \]

Similarly, if \( W_\lambda \) corresponds to \( \beta_i \), then \( \rho_0(W_\lambda) = a_i \).

Hence it follows that

\[ \Delta^* \cup (\Delta^*)^{-1} = \Delta \cup \Delta^{-1} \]

and consequently \( G_\Delta(\Gamma) = G_{\Delta^*}(\Gamma) \).

To prove the orientability of \( \tilde{S} \) we note that each region of \( G_\Delta(\Gamma) \) is a lift of the unique region of \( G \) on \( S \). Thus, we assign to the region of \( G \) the clockwise orientation in Figure III.2 and lift that orientation.
to each region of $G_{\Delta}(\Gamma)$. We show that these orientations satisfy the necessary conditions for $\tilde{S}$ to be orientable. Accordingly, let $R_1$ and $R_2$ be two regions which abut along an edge on $G_{\Delta}(\Gamma)$. Suppose, for example, that they abut, as in Figure III.3, along an arc $uv$ which is a lift of $\alpha_1$. Note that in this figure $R_1$ has the clockwise orientation and we will show that $R_2$ is necessarily also clockwise oriented. For suppose not, then it follows that the arc $vw$ is a lift of $\beta_1$. Since the arc $vx$ is also a lift of $\beta_1$ it follows that we have derived a contradiction to the unique path lifting property. Hence $R_2$ is "correctly" oriented. The other possibilities for the abutment of $R_1$ and $R_2$, not described in Figure III.3, are settled by similar arguments.

Finally, we assume that $\Gamma$ is finite and prove that $\tilde{S}$ is compact. Since $\tilde{S}$ is a quotient space of $\hat{S}$, it suffices to show that $\hat{S}$ is compact. Now, since $\hat{S}$ is a compact singular surface it is easily seen that $\hat{S}$ possesses a cover $(F_i)_{i=1}^q$ consisting of compact and simply connected subsets. It follows from Theorems I.3.4 and I.3.5 that for each $i$, $p^{-1}(F_i)$ is the disjoint union of $|\Gamma|$ copies of $F_i$ ($p$ is the covering projection $p: \tilde{S} \rightarrow \hat{S}$). Since $\Gamma$ is finite
we conclude that each \( p^{-1}(F_i) \) is compact and hence so is \( \tilde{S} = \bigcup_{i=1}^{q.e.d.} p^{-1}(F_i) \).

Some notation must be standardized before we state the next theorem. The infinite cyclic abelian group and the finite cyclic abelian group of order \( m \) are denoted by \( \mathbb{Z} \) and \( \mathbb{Z}_m \) respectively. The direct sum of two groups \( A \) and \( B \) is denoted by \( A \oplus B \), and \( A^m \) denoted the direct sum of \( m \) copies of \( A \).

III.2.2 Theorem: A finitely generated abelian group has an orientable self-dual embedding if and only if its order is neither 2 nor 3; if the group is
finite, then the embedding is on a closed surface.

**Proof:** The only connected Cayley graphs of $Z_2$ and $Z_3$ are $K_2$ and $K_3$ respectively. It is easily seen that neither of these can have a self-dual embedding. The trivial group $Z_1$ has the obvious self-dual embedding as $K_1$ on the sphere. Now let $\Gamma$ be any finitely generated abelian group of order at least four. By the fundamental theorem of abelian groups there exist non-negative integers $m$ and $r$, and positive integers $m_1, \ldots, m_r$, with $m_i | m_{i+1}$ for $i = 1, 2, \ldots, r-1$ such that

$$\Gamma = Z_{m_1} \oplus Z_{m_2} \oplus \cdots \oplus Z_{m_r} \oplus Z^m.$$ 

Let $s + 1$ be the first integer such that $m_{s+1} > 2$; in other words $2 = m_1 = m_2 = \ldots = m_s$ and $m_{s+1} > 2$. Clearly $0 \leq s \leq r$. Let $g_i$ be a generator of $Z_{m_i}$ and let $(h_1, \ldots, h_m)$ be a basis for $Z^m$. The proof now splits into several cases, in each of which the sets $\Delta_1$ and $\Delta_2$ are defined so as to satisfy conditions a - e of Lemma III.2.1. The proof will then be concluded. As the details are quite easy their verification is mostly omitted.
Case 1: the group $\Gamma$ is cyclic. Assume first that $\Gamma$

![Figure III.4](image)

has order at least 5. We set $A_1 = \{f, f+f\}$ and $A_2 = \emptyset$, where $f$ is any generator of $\Gamma$. Figure III.4 exhibits this embedding for $\Gamma = Z_8$ and $f = 1$ on the torus $S_1$. If $\Gamma$ has order 4, it must be $Z_4$ since $\Gamma$ is cyclic. However, $G_{1,2}(Z_4) = K_4$ and it is well known that $K_4$ does have a self-dual embedding on the sphere (Figure III.5).

Case 2: The group $\Gamma$ is not cyclic and $s \neq 1$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Case 2a: $m + r - s$ is even. Here
\[ \Delta_1 = \{ g_{s+1}, g_{s+2}, \ldots, g_r, h_1, \ldots, h_m \} \]
\[ \Delta_2 = \{ g_1, g_2, \ldots, g_s, g_1 + g_2 + \ldots + g_s \}. \]

Case 2b: $m + r - s$ is odd and exceeds 1. Here
\[ \Delta_1 = \{ g_{s+1}, \ldots, g_r, h_1, \ldots, h_m, g_{s+1} + \ldots + g_r + h_1 + \ldots + h_m \} \]
\[ \Delta_2 = \{ g_1, \ldots, g_s, g_1 + g_2 + \ldots + g_s \}. \]

Case 2c: $m + r - s = 1$. If $r - s = 0$ and $m = 1$, set
\[ \Delta_1 = \{ h_1, h_1 + h_1 \} \]
If \( m = 0 \) and \( r - s = 1 \) then either \( s = 0 \) and \( \Gamma \) is cyclic (see case 1) or else \( s \neq 0 \), in which case we set

\[
\Delta_1 = \{ g_r, g_r + g_1 \}
\]

\[
\Delta_2 = \{ g_1, \ldots, g_s, g_1 + g_2 + \ldots + g_s \}.
\]

Case 3: \( \Gamma \) is not cyclic and \( s = 1 \).

Case 3a: \( m + r \) is even. Set \( f = g_2 \) if \( r \geq 2 \) and \( f = h_1 \) otherwise. Then

\[
\Delta_1 = \{ g_1 + f, h_1, \ldots, h_m, g_2, \ldots, g_r \}
\]

\[
\Delta_2 = \emptyset.
\]

Case 3b: \( m + r \) is odd. Since \( \Gamma \) is not cyclic, \( m + r \geq 3 \). Let \( f_1 \) and \( f_2 \) be arbitrary but distinct elements of

\[
\{ g_2, \ldots, g_r, h_1, \ldots, h_m \}.
\]

Then

\[
\Delta_1 = \{ g_1 + f_1, g_1 + f_2, g_2, g_3, \ldots, g_r, h_1, \ldots, h_m \}
\]

\[
\Delta_2 = \emptyset.
\]

q.e.d.
Section III.3

A Self-dual Embedding of \( \pi(S_n) \)

Let \( S_n \), the closed orientable surface of genus \( n \), be presented as in Figure III.1. Coxeter and Moser (in [6] pp 26-27) have exhibited two dual embeddings of \( \pi(S_n) \) on the universal covering space of \( S_n \). Since \( n \geq 1 \), this is of course the (hyperbolic) plane, as is indicated in [23]. We show here that these embeddings are in fact of isomorphic graphs and hence the graphs are self-dual. First, a lemma of a purely algebraic nature is proved; here again, we simplify the notation by using the same letters to denote the abstract symbols of a group presentation and the group element they correspond to.

III.3.1 Lemma: In the group \( \langle a_1, \ldots, a_n, b_1, \ldots, b_n; \prod_{i=1}^{n} a_i b_i a_i^{-1} b_i^{-1} \rangle \) set:

\[
c_i = a_i b_i a_i^{-1} b_i^{-1},
\]

\[
A_i = c_1 c_2 \ldots c_{i-1} b_i^{-1} c_{i+1} \ldots c_n, \tag{1}
\]

\[
B_i = c_1 c_2 \ldots c_{i-1} a_i c_{i+1} \ldots c_n. \tag{2}
\]
Then the map \[ \begin{cases} \phi(a_i) = B_i^{-1} \\ \phi(b_i) = A_i^{-1} \end{cases} \]
extends to an automorphism of \( \Gamma \) of order 2 (an involution).

**Proof:** Define \( C_i = B_i^{-1} A_i B_i A_i^{-1} \).

We first prove, by induction on \( k \), that

\[ c_1 c_2 \cdots c_k = C_k C_{k-1} \cdots C_1. \]  
(3)

For suppose \( k = 1 \). Then,

\[ C_1 = B_1^{-1} A_1 B_1 A_1^{-1} = (c_n^{-1} \cdots c_2^{-1} a_1^{-1})(b_1^{-1} c_2 \cdots c_n)(a_1 c_2 \cdots c_n)(c_n^{-1} \cdots c_2^{-1} b_1) = (c_n^{-1} \cdots c_2^{-1})a_1^{-1} b_1^{-1}(c_2 \cdots c_n)a_1 b_1. \]

However, \( c_1 c_2 \cdots c_n \) is a relator for \( \Gamma \) and hence,

\[ c_i = c_{i-1}^{-1} c_{i-2}^{-1} \cdots c_1^{-1} c_n^{-1} \cdots c_i+1 = (c_{i+1} \cdots c_n c_1 \cdots c_{i-1})^{-1}. \]

This fact will be implicitly used in many of the ensuing arguments. For example, it now follows that

\[ C_1 = c_1 a_1^{-1} b_1^{-1} c_1^{-1} a_1 b_1 = c_1 a_1^{-1} b_1^{-1} (b_1 a_1 b_1 a_1^{-1}) a_1 b_1 = c_1. \]

Assume now that the (3) has been established for \( k = p-1 \).

Then,
This completes the proof of (3). In particular, for 
k = n, identity (3) yields:

\[ l_r = C_n \cdots C_1 = \prod_{i=1}^{n} B_{n-i+1}^{-1} A_{n-i+1}^{-1} A_{n-i+1}, \] (4)

and hence \( \phi \) extends to a homomorphism of \( \Gamma \) into itself. Now,

\[ c_i \cdots c_n = (c_i \cdots c_{i-1})^{-1} = (c_{i-1} \cdots c_1)^{-1} = c_n \cdots c_1 . \] (5)

Moreover, it follows from (1) and (2) that

\[ a_i = (c_1 \cdots c_{i-1})^{-1} B_i (c_{i+1} \cdots c_n)^{-1} = \]

\[ (c_{i-1} \cdots c_1)^{-1} B_i (c_n \cdots c_{i+1})^{-1} = \]

\[ c_{i-1} c_{i-1} \cdots c_{i+1} c_{i+1} \cdots c_n^{-1} , \] (6)

and
Finally, it is easily verified that

$$\xi(c_i) = c_{n-i+1}.$$  \hfill (8)

Now, applying (5) - (8), for all $i = 1,2,...,n$, we see that

$$\xi^2(a_{n-i+1}) = \xi(\xi(a_{n-i+1})) = \xi(b_{i}) =$$

$$\xi(c_n \cdots c_{i+1} a_i c_{i-1} \cdots c_1) =$$

$$c_n \cdots c_{i+1} a_i c_{i-1} \cdots c_1 = a_{n-i+1}.$$

Similarly, $\xi^2(b_{n-i+1}) = b_{n-i+1}$ and hence $\xi$ is an involution of $\Gamma$. 

q.e.d.

**III.3.2 Theorem:** The fundamental group $\pi(S_n)$, $n \geq 1$, has a self-dual embedding in the plane.

**Proof:** Let $S_n$ be presented as in Figure III.1, and let $G$ be the pseudograph with the single vertex $P$ and edge set $[\alpha_i, \beta_i]_{i=1}^n$. We set $\Gamma = \pi(S_n)$ and define the voltage pseudograph $(G, \rho_0, \Gamma)$
\[ \rho_0(\alpha_i) = a_i = [\alpha_i] \in \pi(S_n) \]
\[ \rho_0(\beta_i) = b_i = [\beta_i] \in \pi(S_n). \]

The hypotheses of Theorem II.2.1 are clearly satisfied. Here,

\[ \Delta = \{a_i, b_i\}_{i=1}^n. \]

To calculate \( \Delta^* \) we set \( c_i = a_i b_i a_i^{-1} b_i^{-1} \) and then arguments similar to those used in the proof of Lemma III.2.1 show that

\[ \Delta^* = \{A_i, B_i\}_{i=1}^n \]

where

\[ A_i = c_1 \cdots c_{i-1} c_i^{-1} c_i a_i c_i+1 \cdots c_n \]
\[ B_i = c_1 \cdots c_{i-1} a_i c_i+1 \cdots c_n. \]

Since every group involution is an automorphism, it follows from Lemma III.3.1 and Lemma 5.3 of [37] that \( G_\Delta(\Gamma) \) and \( G_{\Delta^*}(\Gamma) \) are isomorphic graphs. Hence the derived embedding of \( \pi(S_n) \) is indeed self-dual. Since the map \( \rho: \pi(S_n) \to \Gamma \) induced by the voltage assignment \( \rho_0 \) is clearly injective, it has a trivial kernel and hence the derived surface \( \tilde{S} \) is the universal covering.
surface of $S^2$. In other words, $\sim$ is the plane.

q.e.d.

For example, if $n = 1$, the fundamental group $\pi(S^1)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ and it is embedded in the plane as the rectangular lattice $\{(x,y) | \text{either } x \text{ or } y \text{ is an integer}\}$. For the sake of completeness we note that $\pi(S^0)$, the only group not covered by the above theorem, is trivial and hence has an obvious self-dual embedding on the sphere.

Section. III.4

Self-dual Embeddings of $K_r(s)$

Various self-dual embeddings of complete n-partite graphs have been constructed by Heftter [22], Pengelley [26], and White [37]. In particular, it was shown in the latter that if $s(r-1)$ is divisible by 4, then $K_r(s)$ has an orientable self-dual embedding. We now demonstrate another sufficient condition.

III.4.1 Theorem: If both $s-1$ and $r$ are divisible by 4, then $K_r(s)$ has a self-dual embedding on an orientable closed surface.
**Proof:** Since $4 | r$ we may write $r = 2^x y$ where $x$ and $y$ are positive integers such that $x \geq 2$ and $y$ is odd. We set $\Gamma_1 = \mathbb{Z}_{sy}$, $\Gamma_2 = \mathbb{Z}_x$ and $\Gamma = \Gamma_1 \oplus \Gamma_2$, and select

$$\Delta_2 = \Gamma_2 - \{0\}.$$ 

Let $y\Gamma_1$ denote the unique subgroup of $\Gamma_1$ of order $s$. Then the set $T = \Gamma - y\Gamma_1 - \Gamma_2$ contains no singular elements and $T = T^{-1}$. Hence there exists a subset $\Delta_1 \subseteq T$ such that

$$\Delta_1 \cap \Delta_1^{-1} = \emptyset \quad \text{and} \quad \Delta_1 \cup \Delta_1^{-1} = T.$$ 

Moreover, the cardinality of $\Delta_1$ is $\frac{1}{2}(sr - s - (2^x - 1))$, which is even because $x \geq 2$, $4 | r$ and $4 | (s - 1)$.

Thus, if we set $\Delta = \Delta_1 \cup \Delta_2$, then by Lemma III.2.1 $G_\Delta(\Gamma)$ has a self-dual orientable surface. However, $\Gamma - (\Delta \cup \Delta^{-1}) = y\Gamma_1$, which is a subgroup of order $s$. Hence, $G_\Delta(\Gamma) = K_r(s)$.

q.e.d.

**III.4.2 Corollary:** The complete graph $K_m$ has a self-dual embedding on an orientable closed surface if and only if $m \equiv 0, 1 \pmod{4}$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Proof: The necessity follows from the Euler-Poincaré formula. Since $K_m = K_{m(1)}$, the case $m \equiv 1 \pmod{4}$ follows from White's condition and the case $m \equiv 0 \pmod{4}$ follows from the above theorem.

q.e.d.

For example, if $K(s) = K_4(1) = K_4$, we have $\Gamma = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $\Delta_1 = \emptyset$, $\Delta_2 = \Gamma - \{0\}$. Using the notation of section III.1, we have $S = S(0,3)$ (see Figure III.6). Since the boundaries of the regions of the derived embedding carry the same voltage as the boundary of the single region of $G$ on $S$, it follows that the regions of the derived embedding are the triangles of

a = (0,1)

b = (1,1)

Figure III.6
Figure III.7b: the generator assignment to edges is described in Figure III.7a. When the triangles are pasted together we obtain the well known self-dual embedding of $K_4$ in the plane, which was already exhibited in Figure III.5.
CHAPTER IV

APPLICATIONS TO NON-ORIENTABLE EMBEDDINGS

A criterion is established for deciding whether the derived embedding of Theorem II.2.1 is non-orientable. It is then applied to derive non-orientable self-dual embeddings of abelian groups and complete n-partite graphs.

Section IV.1

Non-orientable Surfaces

A bordered surface which is not orientable is said to be non-orientable. Examples of such surfaces are the Moebius strip (with or without its boundary) and the projective plane. It is a classical result of topology that any non-orientable closed surface has a presentation as shown in Figure IV.1a. This surface is denoted by $S_n$ (n ≥ 1). It is homeomorphic to a sphere with n cross-caps, a cross-cap resulting from the excision of an open disk and the identification of diametrically opposite points on the resulting boundary. The integer n is a topological invariant of $S_n$ and is referred to as the genus of $S_n$. The unisingular surface resulting

59
from the excision from $\tilde{S}_n$ of $k$ open disks whose boundaries intersect in a single point (see Figure IV.1b), is denoted by $\tilde{S}(n,k)$. These surfaces, much like the orientable $S(n,k)$, can be used to generate self-dual embeddings of Cayley graphs. Notice, however, that Lemma IV.1.1 makes no assertion as to whether or not the derived embedding is orientable. It will be shown in a later chapter that the derived surface may be either orientable or non-orientable.

IV.1.1 Lemma: Let $\Gamma$ be a finitely generated group with finite subsets $\Delta_1 = \{a_i\}_{i=1}^n$, $\Delta_2 = \{d_j\}_{j=1}^k$, not containing $1_{\Gamma}$, such that
a) all the elements of $\Delta_1$ are non-singular,
b) $\Delta_1 \cap \Delta_1^{-1} = \emptyset$
c) all the elements of $\Delta_2$ are singular,
d) $\left( \prod_{i=1}^{n} a_i^2 \right) \left( \prod_{j=1}^{k} d_j \right) = 1$.
e) $\Delta = \Delta_1 \cup \Delta_2$ generates $\Gamma$.

Then $G_\Delta(\Gamma)$ has a self-dual embedding on a closed surface. If $\Gamma$ is finite, then the derived surface is closed.

**Proof:** Suppose $\Delta_1 = \{a_i\}_{i=1}^{n}$ and $\Delta_2 = \{d_j\}_{j=1}^{k}$. We select $S = S(n,k)$ and define $G$ to be the pseudo-graph whose only vertex is $P$ and whose edges are the paths $\alpha_i, \delta_j, 1 \leq i \leq n, 1 \leq j \leq k$, as in Figure IV.1b. The voltage assignment $\rho_0 : D(G) \to \Gamma$ is defined by

$$\rho_0(\alpha_i) = a_i \quad i = 1,2,\ldots,n$$

$$\rho_0(\delta_j) = d_j \quad j = 1,2,\ldots,k.$$

Condition d ensures that KVL holds, since the voltage around the single region of $G$ in $S$ is

$$\left( \prod_{i=1}^{n} a_i^2 \right) \left( \prod_{j=1}^{k} d_j \right) = 1\Gamma.$$

Let $H$ be a dual of $G$ on $S$. Clearly the hypotheses of Theorem II.2.1 are fully satisfied and hence
Moreover, these two graphs are dual on $S$. Thus, it only remains to show that $G_{\Delta}(\Gamma)$ and $G_{\Delta^*}(\Gamma)$ are in fact isomorphic graphs. Again we compute $\Delta^* = \{ \rho_0(W_\lambda) \}_{\lambda=1}^S$. If $W_\lambda$ corresponds to the boundary loop $\delta_j$ of $G$, then

$$W_\lambda = \delta_1 \delta_2 \cdots \delta_{j-1} \delta_j \delta_{j+1} \cdots \delta_k \alpha_1 \alpha_1' \alpha_2 \alpha_2' \cdots \alpha_n \alpha_n'.$$

Hence,

$$\rho_0(W_\lambda) = \left( \prod_{\nu \neq j} d_{\nu} \left( \prod_{i=1}^{n} a_i^2 \right) \right) = d_j^{-1} = d_j.$$

Similarly, if $W_\lambda$ corresponds to some $\alpha_i$, then

$$W_\lambda = \delta_1 \cdots \delta_k \alpha_1 \alpha_1' \alpha_2 \alpha_2' \cdots \alpha_{i-1} \alpha_{i-1}' \alpha_i.'$$

Thus,

$$\rho_0(W_\lambda) = \left( \prod_{j=1}^{k} d_j \left( \prod_{i=1}^{i-1} a_i^2 \right) a_i \left( \prod_{\nu=i+1}^{n} a_{\nu}^2 \right) \right) = a_i^{-1}.$$

Thus, $\Delta^* = \Delta^{-1} = \{ g^{-1} | g \in \Delta_1 \cup \Delta_2 \}$. Hence, by Lemma 5.3 of [37], $G_{\Delta}(\Gamma)$ and $G_{\Delta^*}(\Gamma)$ are isomorphic (the identity map is the one required by this lemma).

q.e.d.
Before we derive a sufficient condition for the derived embedding to be non-orientable, an example is presented to show that it is possible for the derived self-dual graph to be orientably embedded. First we establish the following lemma, which is an analog of Lemma III.3.1.

**IV.1.2 Lemma:** If the group \( \Gamma \) has the presentation

\[
\langle a_1, a_2, \ldots, a_n, a_1^2a_2^2\ldots a_n^2 \rangle
\]

Then \( \Gamma \) has an involution \( \Phi \) such that

\[
\Phi(a_i) = a_1^2a_2^2\ldots a_{i-1}^2a_i^2a_{i+1}^2\ldots a_n^2.
\]

**Proof:** As the details resemble those of Lemma III.3.1, they are omitted and only a sketch of the proof is given. One sets

\[A_i = a_1^2a_2^2\ldots a_{i-1}^2a_i^2a_{i+1}^2\ldots a_n^2 \quad \text{for} \quad i = 1, 2, \ldots, n\]

and proves the following identities:

\[
A_1^2A_2^2\ldots A_i^2 = a_{i+1}^2\ldots a_n^2
\]

\[
A_1^2A_2^2\ldots A_n^2 = 1_\Gamma.
\]

Hence a homomorphism \( \Phi \) carrying \( a_i \) into \( A_i \) exists. It is then easily shown that \( \Phi \) is an involution by
proving that

$$\phi(a_i) = a_i.$$  

q.e.d.

IV.1.3 Theorem: The fundamental group $\pi(S_n)$ has a self-dual embedding in the plane.

Proof: Let $\tilde{S}_n$ be presented as in Figure IV.1a and let $G_n$ be the pseudograph with vertex $P$ and edge set $\{\alpha_i\}_{i=1}^n$. We set $\Gamma = \pi(\tilde{S}_n)$ and define the voltage pseudograph $(G, \rho_0, \Gamma)$ by

$$\rho_0(\alpha_i) = a_i = [\alpha_i] \in \pi(\tilde{S}_n).$$

Setting $\Delta = \{a_i\}_{i=1}^n$, we see that the hypotheses of Theorem II.2.1 are satisfied. Also, $\Delta^* = \{A_i\}_{i=1}^n$ with the $A_i$ defined as in the previous lemma. By the same lemma we have $G_\Delta(\Gamma) \cong G_{\Delta^*}(\Gamma)$ and, as in the proof of Theorem III.3.2, the derived surface is the universal covering space of $\tilde{S}_n$, which is the plane.

q.e.d.

A lemma is now proved, providing a sufficient condition for the derived surface $\tilde{S}$ to be non-orientable when the base surface $S$ is not orientable. A circuit
is odd or even according to the number of edges it contains.

**IV.1.4 Lemma**: Under the full hypothesis of Theorem II.2.1, suppose that \( S = \tilde{S}(n,k) \) with the presentation of Figure IV.1b. Let \( F \) be the pseudograph whose only vertex is \( P \) and whose edges are the loops \( (\alpha_i^n)_{i=1}^n \). Then \( \tilde{S} \) is non-orientable if \( \rho^{-1}(F) \) contains at least one odd circuit.

**Proof**: Let \( F^* \) denote the dual of \( F \) on \( S \). Then \( F^* \) has the single vertex \( Q \) and edges \( (\alpha_i^*)_{i=1}^n \). Now, if \( \lambda \) and \( \mu \) are the functions which define the Cayley structure on \( G^\Delta(\Gamma) = \rho^{-1}(G) \) and \( G^\Delta^*(\Gamma) = \rho^{-1}(H) \) respectively, then \( \lambda(\alpha_i^*) = a_i \) and \( \mu(\alpha_i^*) = a_i^{-1} \) (this was shown in the proof of Lemma IV.1.1). Hence the identity map used in the proof of the same lemma also induces an isomorphism \( \rho^{-1}(F) \cong \rho^{-1}(F^*) \). Since \( \rho^{-1}(F) \) contains an odd circuit, so does \( \rho^{-1}(F^*) \). In addition, since \( \rho^{-1}(F^*) \) is a Cayley graph, it follows that every component of \( \rho^{-1}(F^*) \) contains an odd circuit.

Since every region of \( G^\Delta(\Gamma) \) contains a vertex of \( G^\Delta^*(\Gamma) \), it follows that the corresponding element of \( \Gamma \) can be used to label both the vertex of \( G^\Delta^*(\Gamma) \) and the region of \( G^\Delta(\Gamma) \). Moreover, since the boundary...
of every region \( g \) on \( \tilde{S} \) consists of lifts of the \( \alpha_i \) and the \( \delta_j \) on \( S \), we can label these lifts as
\[
\{(g,a_i^1,v),(g,d_j)|1\leq i\leq n, 1\leq j\leq k, v = 1,2\},
\]
where the path \( (g,a_i^1) \) is the first, and \( (g,a_i^2) \) is the second, lift of \( \alpha_i \) along the oriented boundary of \( g \); similarly, \( (g,d_j) \) is the lift of \( \delta_j \). These labels are not all distinct, since the same path may occur on the boundary of two regions. In fact, referring to Figure IV.2, suppose \( xy = (g,a_i^1,1) \) is the lift, along

![Diagram](image)

Figure IV.2

the boundary of the region \( g \) on \( \tilde{S} \), of the edge \( \alpha_i \) of \( G \) on \( S \). It might help clarify matters if we observe here that \( yz = (g,a_i^2,2), y = xa_i, \) and \( z = ya_i \). As pointed out above, we are using \( g \) and \( h \) to denote both vertices and regions. Now the edge \( gh \) is the dual of \( xy \) and hence \( h = g\alpha_i^{-1} \). Hence we also have \( xy = (g\alpha_i^{-1}, a_i^1, p) \), with the precise value
of $p$ remaining to be ascertained. However, if $p$ were 1 it would follow that $yu = (g_{a_i}^{-1}, a_i, 2)$, thus creating two lifts of $\alpha_i$ at $y$; namely, $yu$ and $yz$. Note that $u$ and $z$ must be distinct points of $\tilde{S}$. For otherwise, every vertex of $p^{-1}(G)$ would be incident to exactly two edges. The pseudograph $G$ would then consist of the single interior loop $\alpha_1$. It would follow from the KVL that $[\rho_0(\alpha_i)]^2 = 1$, contradicting the fact that only boundary loops are assigned singular voltage. Thus we have two distinct lifts of $\alpha_i$ at $y$, contradicting the unique path lifting property on $\tilde{S}$. Consequently the assumption $p = 1$ was invalid and so we conclude that $p = 2$ and hence $xy = (g_{a_i}^{-1}, a_i, 2)$. Thus we have, on $\tilde{S}$,

$$(g, a_i, 1) = (g_{a_i}^{-1}, a_i, 2) \text{ for all } g \in \Gamma, \quad i = 1, 2, \ldots, n.$$ 

As for the $1$-cells of type $(g, d_j)$ we need only note that they are identified only with cells of the same type.

Assume now that $o$ is an orientation of the regions of $p^{-1}(G)$ which makes $\tilde{S}$ into an orientable surface: we shall derive a contradiction. For each region $g$ of $p^{-1}(G)$ define $\epsilon(g)$ to be 1 or -1 according as
\( o(g) \) is the lift of the boundary of the region of \( G \) on \( S \), or of its inverse. It follows from the above discussion that if \( g \) and \( h \) are abutting regions of \( p^{-1}(G) \) on \( \bar{S} \), and if \( gh^{-1} \) is a non-singular element of \( \Gamma \), then \( g \) and \( h \) must abut either as indicated in Figure IV.3, or else, we can still use the same figure but must interchange the roles of \( g \) and \( h \).

In either case we have \( \varepsilon(g) = -\varepsilon(h) \). However, the given odd circuit in \( p^{-1}(F^*) \) consists of a sequence \( g_1, g_2, \ldots, g_m \) of elements of \( \Gamma \) every two consecutive elements of which satisfy the same conditions as \( g \) and \( h \) above. Since the circuit is odd it follows that \( \varepsilon(g_1) = -\varepsilon(g_1) \), which is clearly absurd.
Hence $S$ must be non-orientable.  

q.e.d.

IV.2 Applications

We are now in position to construct some actual embeddings. The constructions call for some involved calculations in finite cyclic (abelian) groups. The reader is forewarned about the following notational ambiguities in these calculations. If $\Gamma = \mathbb{Z}_n$ and $r$ is an integer, $0 \leq r < n$, the symbol $r$ will also be used to denote the corresponding element of $\mathbb{Z}_n$. Conversely, we may start with a group element $b$ and subsequently treat it as an integer; when that happens, the integer $b$ is the smallest non-negative integer corresponding to that group element. In particular, when these symbols occur in expressions which involve inequalities, it is to be understood that they represent integers.

If $r$ is an element of $\mathbb{Z}_n$ and $m$ is a positive integer, then $\frac{r}{m}$ denotes the smallest positive integer $x$, if any, such that $mx \equiv r \pmod{n}$. Let $D$ be a set of non-singular elements of $\mathbb{Z}_n - \{0\}$ such that if $a$ is in $D$, so is $-a$. By $\frac{1}{2}D$ is meant the subset of $D$ such that $a$ is in $\frac{1}{2}D$ iff $a$ is in $D$ and
a < -a. For example, if \( \Gamma = \mathbb{Z}_9 \) and \( D = \{1, 2, 4, 5, 7, 8\} \),
then \( \frac{1}{2} D = \{1, 2, 4\} \).

The following theorem shows that at least 80% of all the complete regular \( n \)-partite graphs possess non-orientable self-dual embeddings.

**IV.2.1 Theorem:** The graph \( K_r(s) \), with \( r > 1 \), has a self-dual non-orientable embedding on a closed surface if any one of the following conditions holds:

a) \( s \equiv 0 \pmod{4} \) and \( r > 2 \);
b) \( s \equiv 2 \pmod{4} \) and \( r \not\equiv 2 \pmod{4} \);
c) \( s = 1 \) and \( r \equiv 0 \pmod{4}, r \not\equiv 4 \).
d) \( s \equiv 1 \pmod{2}, r \not\equiv 2 \pmod{4}, r \) is not a power of 2, and \( (r, s) \not= (3, 1), (5, 1), \) or \( (3, 3) \);

**Proof:** The cases are handled separately. In each case a finite abelian group \( \Gamma \) is defined and subsets \( \Delta_1 \) and \( \Delta_2 \) are defined which satisfy the hypotheses of Lemma IV.1.1. It will be clear that in each case \( \Gamma - (\Delta_1 \cup \Delta_1^{-1} \cup \Delta_2) \) is a subgroup of \( \Gamma \) of order \( s \) and hence \( G_{\Delta_1 \cup \Delta_2} (\Gamma) = K_r(s) \). Also, in each case except \( d \) the circuit with vertices labelled 0,1,2 is the one required for Lemma IV.1.4 to be applicable.
Thus all the embeddings which arise from the application of Lemma IV.1.1 are in fact non-orientable (the appropriate circuit for case d will be indicated later). Finally, the compactness of the surface $\bar{S}$ can be established by an argument identical with the one used in the proof of Lemma III.2.1.

a) $s \equiv 0 \pmod{4}$ and $r > 2$. We set $\Gamma = \mathbb{Z}_{rs}$ and note that since $s$ is even, the set $\Gamma - r\Gamma$ contains no singular elements. Hence $\Delta_1 = \frac{1}{2}(\Gamma - r\Gamma)$ is well defined. If we also set $\Delta_2 = \emptyset$, hypotheses a, b, c, and e of Lemma IV.1.1 are clearly satisfied. As for hypothesis d, we carry out the following calculations:

$$2 \sum_{g \in \Delta_1} g + \sum_{h \in \Delta_2} h = 2[(1+2+\ldots+(\frac{s}{2}-1)) - r(1+2+\ldots+(\frac{s}{2}-1))] + 0 =$$

$$= 2 \left( \frac{1}{2} \frac{rs-2}{2} - \frac{1}{2} \frac{s-2}{2} \right) = \frac{rs(r-2)}{4} - \frac{rs(s-2)}{4} =$$

$$= \frac{r^2 s^2}{4} - \frac{2rs}{4} = \frac{2rs - rs^2 + 2rs}{4} = \frac{s}{4} rs(r-1) \equiv 0 \pmod{rs}.$$

b) $s \equiv 2 \pmod{4}$ and $r \not\equiv 2 \pmod{4}$. Here we again select $\Gamma = \mathbb{Z}_{rs}$, $\Delta_2 = \emptyset$, and we note that since $s$ is even the set $\Gamma - r\Gamma$ contains no singular elements.
To define $\Delta_1$ we must consider two subcases.

b1) $r$ is odd. Here $\Delta_1$ is constructed as in case a, namely, $\Delta_1 = \frac{1}{2}(\Gamma - r\Gamma)$. Since both $s$ and $r - 1$ are now even, we have

$$2 \sum_{g \in \Delta_1} g = rs \frac{s}{2} \frac{r-1}{2} \equiv 0 \pmod{rs}.$$  

b2) $r$ is even. Since $r \not\equiv 2 \pmod{4}$ it follows that 4 divides $r$ and hence 8 divides $rs$. We now set $x_1 = \frac{rs}{8}$ and define

$$\Delta_1 = \left\{ \frac{1}{2}(\Gamma - r\Gamma) - \{x_1]\} \cup \{rs - x_1\}\right\}, \quad \Delta_2 = \emptyset.$$  

In other words, the element $x_1$ is deleted from $\frac{1}{2}(\Gamma - r\Gamma)$ and replaced by its additive inverse. Note that the hypothesis $s \equiv 2 \pmod{4}$ guarantees that $x_1$ is in $\Gamma - r\Gamma$. Now, however,

$$2 \sum_{g \in \Delta_1} g = \frac{rs^2}{4} \frac{(r-1)}{4} - \frac{rs}{4} + 2rs - \frac{rs}{4} =$$

$$\frac{s}{4} \cdot rs - \frac{s+2}{4} \cdot rs + 2rs \equiv 0 \pmod{rs}.$$

c) $s = 1$ and $r \equiv 0 \pmod{4}, r \neq 4$. There exists an integer $x$ such that $r = 4x$. We set $\Gamma = Z_{4x} = Z_{rs}$, $\Delta_1 = \{1,2,\ldots,2x-1\}$, and $\Delta_2 = \{2x\}$. Then,
\[ 2 \sum_{g \in \Delta_1} g + \sum_{g \in \Delta_2} h = 2(1 + 2 + \ldots + (2x-1)) + 2x = 4x^2 = 0 \pmod{4x}. \]

d) \( s \) is odd and \( r \neq 2 \pmod{4} \). There exist a non-negative integer \( x \neq 1 \) and a positive odd integer \( y \neq 1 \) such that \( r = 2^x y \). Define \( \Gamma_1 = \mathbb{Z}_{2y} \), \( \Gamma_2 = \mathbb{Z}_2^x \), and \( \Gamma = \Gamma_1 \oplus \Gamma_2 \). We set \( \Delta_2 = \Gamma_2 - \{0\} \) and note that \( \Delta_2 \) consists of all the non-trivial singular elements of \( \Gamma \). Moreover, since \( x \neq 1 \), it follows that

\[ \sum_{h \in \Delta_2} h = 0. \]

Preparatory to defining \( \Delta_1 \), we set

\[ \Delta_3 = \{(a,b) \in \Gamma - \{(0,0)\} | a \in \frac{1}{2}(\Gamma_1 - y\Gamma_1) \text{ or } b \in \frac{1}{2}(\Gamma_1 - y\Gamma_1) \text{ and } b \neq 0\}. \]

Then,

\[ \sum_{g \in \Delta_3} g = \sum_{a \in \frac{1}{2}(\Gamma_1 - y\Gamma_1)} \sum_{b \in \Gamma_2} (a,b) + \sum_{a \in \frac{1}{2}(\Gamma_1 - y\Gamma_1)} \sum_{b \in \Delta_2} (a,b) = \sum_{a \in \frac{1}{2}(\Gamma_1 - y\Gamma_1)} (2^x a, 0) + \sum_{a \in \frac{1}{2}(\Gamma_1 - y\Gamma_1)} ((2^x - 1)a, 0) = a \in \frac{1}{2}(\Gamma_1 - y\Gamma_1) \]

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Thus, as was the case in b2, we may again need to replace some of the elements of $\Delta_3$ by their additive inverses so as to force the KVL to hold. This will be done by specifying some elements $x_1, x_2, \ldots, x_k$ of $\frac{1}{2}(\gamma_1 - \gamma_1')$ and replacing each $(x_i, b)$ in $\Delta_3$ by $(-x_i, b)$. That is, we will define

$$D = \{(a, b) \in \Delta_3 | a = x_i \text{ for some } i = 1, 2, 3, \ldots, k\}$$

and

$$\Delta_1 = (\Delta_3 - D) \cup D^{-1}.$$ 

Then,

$$\sum_{g \in \Delta_1} g = (W, 0),$$

where

$$W = 2\left[ \sum_{g \in \Delta_3} g + ksy - 2 \cdot 2^x(x_1 + x_2 + \ldots + x_k) \right] =$$

$$2\left[ 2^x \frac{s^2y^2 - 1}{8} - y \frac{s^2 - 1}{8} + ksy - 2 \cdot 2^x(x_1 + x_2 + \ldots + x_k) \right].$$
In order for the KVL to hold we must have \( W \equiv 0 \) (mod \( sy \)). Since \( sy \) is odd, this condition can be simplified to:

\[
2^x(s^2y^2-1) - y(s^2-1) - 2^{x+4}(x_1 + x_2 + \ldots + x_k) \equiv 0 \pmod{sy}
\]
or

\[
-2^x + y - 2^{x+4}(x_1 + x_2 + \ldots + x_k) \equiv 0 \pmod{sy}
\]
or

\[
x_1 + x_2 + \ldots + x_k \equiv \frac{\frac{\nu}{2} - 1}{16} \pmod{sy}.
\]

Thus it suffices to show that equation (2) always has a solution in distinct \( x_i \in \frac{1}{2} (\Gamma_i - y\Gamma_i) \). Before we proceed to split the proof into a variety of subcases we note that in this case the vertices of the odd circuit required by Lemma IV.1.4 are labelled \((0,0), (1,0), \) and \((2,0)\). We now establish the existence of the above solution.

\[
\frac{\frac{\nu}{2} - 1}{16} \leq \frac{sy - 1}{2}.
\]

Here \( k = 1 \) and \( x_1 = \frac{\frac{\nu}{2} - 1}{16} \).

It is easy to see that in fact \( x_1 \in \frac{1}{2} (\Gamma_1 - y\Gamma_1) \).

\[
\frac{\frac{\nu}{2} - 1}{16} \leq \frac{sy + 1}{2}.
\]

We now regard \( \frac{2^x}{16} \) as a
positive integer and consider two subcases, according to the parity of this integer.

\[ \frac{2^x - 1}{16} = 2p, \] where \( p \) is an integer such that

\[ \frac{\sqrt{y} + 1}{4} \leq p \leq \frac{\sqrt{y} - 1}{2}. \] Again we have two cases:

**d211)** \( p < \frac{\sqrt{y} - 1}{2} \). We first show that \( p \neq 1 \). For suppose otherwise. Then the condition \( \frac{\sqrt{y} + 1}{4} \leq p \) implies that \( s = 1 \) and \( y = 3 \). This, however, contradicts the condition \( p < \frac{\sqrt{y} - 1}{2} \). Hence, \( p \neq 1 \).

**d2111)** \( y \) divides neither \( p - 1 \) nor \( p + 1 \). Here we set \( x_1 = p - 1 \) and \( x_2 = p + 1 \).

**d2112)** \( y \) divides \( p - 1 \). If \( p \neq 2, 4 \) we set \( x_1 = 2, x_2 = p - 2, \) and \( x_3 = p \). The possibility \( p = 2 \) is disposed of by observing that since \( y \) divides \( p - 1 \) it would follow that \( y = 1 \), contrary to the definition of \( y \). Finally, if \( p = 4 \), we set \( x_1 = 1, x_2 = 2, \) and \( x_3 = 5 \).

**d2113)** \( y \) divides \( p + 1 \). If \( p \neq 2 \) we may set \( x_1 = 1, x_2 = p - 1, \) and \( x_3 = p \). If \( p = 2 \), we note that since \( y \) here divides \( p + 1 \), it
must be that $y = 3$. From the condition 
\[
\frac{sv + 1}{4} \leq p
\]
it now follows that $s = 1$ or $2$.
Since $s$ is an odd integer, only $s = 1$ is possible. Now, if $x = 0$ we have $K_r(s) = K_3(1)$ which is one of the exceptional cases. If $x > 0$ then $x \geq 2$, $r \equiv 0 \pmod{4}$, and hence this graph is covered by case $c$ above.

\textbf{d212)} $p = \frac{sv - 1}{2}$. If $p = 1$ then $s = 1$ and $y = 3$, and so the same reasoning that was used to dispose of case d2113 for $p = 2$ can be applied here.
Hence we now assume that $p \geq 2$. It is also clear here that $y$ does not divide $p$.

\textbf{d2121)} $y$ does not divide $p - 1$. If $p \neq 2$ set $x_1 = 1$, $x_2 = p - 1$, and $x_3 = p$. If $p = 2$ then we deduce from the condition $p = \frac{sv - 1}{2}$ that $s = 1$ and $y = 5$. If $x = 0$ then $K_r(s) = K_5(1)$ which is one of the exceptional cases. If $x > 0$ then $x \geq 2$ and so again this eventuality is covered by case $c$.

\textbf{d2122)} $y$ divides $p - 1$. Hence $y$ does not divide $p - 2$. If $p \neq 2, 4$ we set $x_1 = 2$, $x_2 = p - 2$, $x_3 = p$. From the fact that $y$ divides $p - 1$ it follows that $p \neq 2$. The possibility $p = 4$
calls for some more subtle considerations. Since $y$ divides $p - 1$ it follows that $y = 3$. Solving the equation $p = \frac{sy - 1}{2}$ we conclude that $s = 3$. Suppose now that $\frac{y}{2^x} = a$; we recall that this means that

$$a \cdot 2^x \equiv y \pmod{sy}. \quad (3)$$

Since $\frac{a - 1}{16} = 2p$ it follows that $a - 1 \equiv 32p \pmod{sy}$. Substituting known values we obtain

$$a - 1 \equiv 128 \pmod{9}$$

or

$$a \equiv 3 \pmod{9}. \quad (5)$$

When this value of $a$ is substituted in $(3)$ we obtain an equation in $x$

$$3 \cdot 2^x \equiv 3 \pmod{9}. \quad (6)$$

It is an easy exercise to conclude that $x$ must be an even integer. Moreover, if $x = 0$, then $K_r(s) = K_3(3)$ which is one of the exceptional cases. Hence we may assume that $x$ is positive. In general, the desired solution to equation (2) on page 73 does not exist in
this case and we therefore return to (1) on page 72. Upon the substitution of known values this equation becomes
\[ \sum g = \left( \frac{2^x 3^2 - 3^2 - 1}{8} - 3 \left( \frac{3^2 - 1}{8} \right) , 0 \right) = \]
\[ \sum g \in \Delta_3 \]
\[ (10 \cdot 2^x - 3 , 0) . \]

Rather than replace whole classes of elements of \( \Delta_3 \) we shall now seek to replace individual elements. Thus, we will find elements \( u_1 \) and/or \( u_2 \) such that their replacement in \( \Delta_3 \) by their inverses in \( \Gamma \) yields the desired set \( \Delta_1 \). Note that since \( x \geq 2 \), \( \Gamma_2 \) contains at least two distinct elements \( a \) and \( b \) which are not trivial. Since \( s_y = 9 \) and \( 2^x \equiv 1, 4, \) or \( 7 \) (mod 9) according as \( x = 9, 2, \) or \( 4 \) (mod 6), the proof again splits into three subcases.

\[ d21221 \) \( x \equiv 0 \) (mod 6) . Hence \( 2^x \equiv 1 \) (mod 9) and \( 10 \cdot 2^x - 3 \equiv (4,b) . \)
Then,
\[ \sum g = (10 \cdot 2^x - 3 , 0) - (4,a) + (5,a) - (4,b) + (5,b) \]
\[ g \in \Delta_1 \]
\[ = (-2 - 4 + 5 - 4 + 5, -a + a - b + b) = (0,0) . \]

\[ d21222 \) \( x \equiv 2 \) (mod 6) . Hence \( 2^x \equiv 4 \) (mod 9) and
10 \cdot 2^x - 3 \equiv 1 \pmod{9} \text{ and we set } u_1 = (1,a) \text{ and } u_2 = (4,a). \text{ Then}
\[ g = \sum_{g \in A_1} (10 \cdot 2^x - 3, 0) - (1,a) + (8,a) - (4,a) + (5,a) = (1-1+8-4+5,-a+a-a+a) = (9,0) = (0,0). \]

\[ d21 \quad x \equiv 4 \pmod{6}. \text{ Hence } 2^x \equiv 7 \pmod{9}, \text{ and } 10 \cdot 2^x - 3 \equiv 4 \pmod{9}. \text{ We set } u_1 = (2,a). \text{ Then,}
\[ g = \sum_{g \in A_1} (10 \cdot 2^x - 3, 0) - (2,a) + (7,a) = (4 - 2 + 7, -a + a) = (9,0) = (0,0). \]

\[ d22 \quad \frac{2^x - 1}{16} = 2p + 1 \text{ for some integer } p \text{ such that } \frac{sV - 1}{4} \leq p \leq \frac{sV - 3}{2}. \text{ If } p = 1 \text{ it then follows that } s = 1 \text{ and } y = 5. \text{ Hence again } K_r(s) \text{ is either the exceptional graph } K_5(1) \text{ or else it is covered by case } c. \text{ Thus, we may assume that } p \geq 2. \]

\[ d221 \quad y \text{ divides neither } p \text{ nor } p + 1. \text{ Here we set } x_1 = p \text{ and } x_2 = p + 1. \]

\[ d222 \quad y \text{ divides } p. \text{ Then clearly } p \neq 2 \text{ and we may set } x_1 = 1, \ x_2 = p - 1, \text{ and } x_3 = p + 1. \]
d223) \( y \) divides \( p + 1 \). Thus \( y \) divides neither \( p \) nor \( p - 1 \). We now show that \( p \neq 2, 3 \). Since \( y \) is odd and divides \( p + 1 \) it follows that \( p \neq 3 \). If \( p = 2 \), then \( y = 3 \), and from the inequalities \( \frac{sv - 1}{4} \leq p \leq \frac{sv - 3}{2} \) we conclude that \( s = 3 \). Again we must examine the number

\[ \frac{Y}{2^x} - 1 \]

If \( \frac{Y}{2^x} = a \) then, since \( y = 3 \) and \( s = 3 \), we conclude that

\[ 3 = 2^x a \pmod{9}. \]  \( (4) \)

Also, since \( \frac{a - 1}{16} = 2p + 1 \) we conclude that

\[ a - 1 \equiv 16 \cdot 5 \pmod{9} . \]

\[ \therefore \quad a \equiv 0 \pmod{9} \]

which clearly contradicts \( (4) \) above. Hence \( p \neq 2 \). Now, since \( p \neq 2, 3 \), we may set \( x_1 = 2, x_2 = p - 1, \) and \( x_3 = p \).

This concludes the proof of all the subcases of part d.

q.e.d.

As for the exceptional cases \( K_3(1), K_5(1), \) and \( K_3(3) \) listed in hypothesis d, it is not known whether
the last one possesses self-dual non-orientable embeddings. However, \( K_3(1) = K_3 \) obviously does not possess such an embedding. The same can be proved for \( K_5(1) = K_5 \). To see this note that such an embedding would necessarily consist of five quadrilateral regions, with some of the vertices labelled as in Figure IV.4.

![Figure IV.4](image)

From here, however, it is easy to exhaust all the possible labellings of the other points in the diagram and to show that any such embedding must necessarily be orientable.

**IV.2.2 Theorem:** A finitely generated abelian group has a self-dual non-orientable embedding if and only if it has order at least six.
Proof: Let $\Gamma$ be a finitely generated abelian group of order at least 6. If $\Gamma$ is finite and $|\Gamma| \not\equiv 2 \pmod{4}$, we set $\Delta = \Gamma - \{0\}$ and note that $G(\Delta) = \Delta$. However, it follows from cases c and d of Theorem IV.2.1 that this graph does possess a self-dual non-orientable embedding. If $|\Gamma| \equiv 2 \pmod{4}$, then $\Gamma$ can be factored as $\Gamma = \Gamma_0 \oplus \Gamma_1 \oplus \cdots \oplus \Gamma_n$ where each $\Gamma_i$ is a cyclic subgroup of $\Gamma$, $|\Gamma_0| = 2$, and $|\Gamma_i|$ is the power of an odd prime, for each $i = 1, 2, \ldots, n$. Since $|\Gamma| \geq 6$ it follows that $n \geq 1$. For each $i$ let $\epsilon_i$ be a generator of $\Gamma_i$. Set $\Delta_1 = \{\epsilon_0^2, \epsilon_1, \epsilon_2, \ldots, \epsilon_n, -\epsilon_1, -\epsilon_2, \ldots, -\epsilon_n\}$, $\Delta_2 = \emptyset$. It is easily verified that these $\Delta_i$ satisfy the hypotheses of Lemma IV.1.1. Moreover, the circuit whose vertices are labelled $0, \epsilon_1, \epsilon_2, \ldots, (|\Gamma_n|-1)\epsilon_n$ clearly has odd length and so Lemma IV.1.4 is applicable as well. Thus we have produced self-dual non-orientable embeddings for all finite abelian groups of order at least six.

Now suppose $\Gamma$ is infinite. Hence there exist integers $1 \leq m \leq n$, and cyclic subgroups $\Gamma_i$ $(1 \leq i \leq n)$
such that

\[ \Gamma = \Gamma_1 \oplus \ldots \oplus \Gamma_n, \quad |\Gamma_i| \leq |\Gamma_{i+1}| \quad \text{for } i = 1, 2, \ldots, n-1. \]

and \[ |\Gamma_i| = 2 \text{ if and only if } 1 \leq i < m. \]

In particular, \( \Gamma_n \) is infinite cyclic. For each \( i \) let \( \epsilon_i \) be a generator for \( \Gamma_i \). Now, if \( m \neq 2 \), set

\[ \Delta_1 = \{ \epsilon_1, \ldots, \epsilon_{m-1}, \ldots, \epsilon_n, 2\epsilon_n, \epsilon_{m-1}, \ldots, \epsilon_{n-1}, -3\epsilon_n \} \]

\[ \Delta_2 = \{ \epsilon_1, \ldots, \epsilon_{m-1}, -\epsilon_1 - \epsilon_2 - \cdots - \epsilon_{m-1} \}. \]

If \( m = 2 \) set

\[ \Delta_1 = \{ \epsilon_2, \epsilon_1, \ldots, \epsilon_{n-1}, 2\epsilon_n, -\epsilon_2 - \epsilon_3 - \cdots - \epsilon_{n-1} - 3\epsilon_n \} \]

\[ \Delta_2 = \emptyset. \]

It is again easily verified that in either case the hypotheses of Lemma IV.1.1 are satisfied. Moreover, the circuit required for Lemma IV.1.4 to be applicable is the one whose vertices are labelled \( 0, \epsilon_n, 2\epsilon_n \). This concludes the proof in one direction.

The exceptional groups \( Z_i, 1 \leq i \leq 5 \), have Cayley graphs isomorphic to \( K_i, 1 \leq i \leq 5 \), and to the 4-cycle and the 5-cycle. None of these graphs, however, possesses self-dual non-orientable embeddings.

q.e.d.
In this chapter tools are developed which will enable us to present an alternate, and more conventional, description of the self-dual embeddings derived in the previous chapters. A short history of the subject of embedding schemes is given in Section 1. This is followed by the definition of generalized embedding schemes and an analysis of the components of these schemes. Criteria are given for determining when two such schemes define the same embedding and when such an embedding is orientable. Finally, these schemes are used to prove an interpolation type theorem for non-orientable graph embeddings.

Section V.1

Background

Given a pseudograph \( G \), a rotation system is a collection \( P = \{P_v \mid v \in V(G)\} \) where each \( P_v \) is a cyclic permutation of the arcs at \( v \). Each of the permutations \( P_v \) is referred to as the (local) rotation at \( v \). It is well known that every such system \( P \) of

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
rotations determines an orientable embedding of $G$ in the following manner. The system $P$ determines a permutation $K$ of the arcs of $G$ by

$$K(uv) = [P_v(uv)]^{-1} \text{ for all arcs } uv \text{ of } G.$$ 

Now let $(e_1^1, ..., e_{n_1}^1)(e_2^2, ..., e_{n_2}^2)...(e_m^m, ..., e_{n_m}^m)$ be a decomposition of $K$ into the product of disjoint cycles and let the plane polygon $\pi^i$ have the oriented boundary $e_1^i - e_2^i - ... - e_{n_i}^i - e_1^i$ for each $i = 1, 2, ..., m$.

The application of the classical pasting process to this collection $\{\pi^i\}$ of polygons yields an orientable surface $S$; the boundaries of these polygons, considered as a subspace of $S$, constitute an embedding of $G$. Moreover, the surface $S$ can be so oriented that for each vertex $v$ of $G$ the cyclic arrangement this orientation induces on the arcs at $v$ coincides with $P_v$. Finally, given any embedding of $G$ on an orientable surface, it is possible to construct a rotation system on $G$ which determines the given embedding. For these reasons rotation systems are also referred to as (embedding) schemes.

Although these schemes are frequently called "Edmonds schemes", their history goes back into the nineteenth
century. In a survey type paper published in 1888 Walther Dyck ([8]) comments that any polygonal presentation of a surface can be described by means of a table which lists the adjacencies among these polygons. Such a table is clearly equivalent to a system of rotations for the dual of the pseudograph which consists of the boundaries of the polygons. Heffter ([21]) made use of such tables in 1891 to describe some orientable embeddings of graphs. These embedding schemes were of course heavily used during the fifties and sixties by Gerhard Ringel in his work on the genus of $K_n$ and of $K_{m,n}$. In 1960 J. Edmonds announced a theorem in the Notices [9] which formalizes the notion of an embedding scheme, but never published any details. His formalization is in essence the one presented above. Youngs, in his classic paper [38] of 1963, supplied these details in the case where $G$ is a graph and the embedding is orientable. Although no general treatment of the orientable embedding schemes for pseudographs has been published, they are widely accepted and used.

In the above cited paper Heffter comments that the existence of "doppleflächen" is a twofold orientable cover of a non-orientable surface. That these covers do exist seems to be part of the folklore of algebraic
topology, and their existence is usually demonstrated by algebraic means. We indicate here a different proof which also makes obvious some of the geometric properties of the covering projection. Every closed orientable surface $S$ can be presented as in Figure V.1 so that $0$ is its center of symmetry. Now identify pairs of points which are symmetrical about $0$, thus producing a new quotient surface $\tilde{S}$. If the surface $S$ is sliced in two by a plane $\pi$ through $0$, then $\tilde{S}$ can be considered as the quotient space resulting when we take one of the "halves" of $S$ and identify only symmetric points on the intersection of the surface $S$ with the plane $\pi$.

When $S = S_{2n+1}$ we obtain a Klein bottle with $n$ handles attached; hence here $\tilde{S} = \tilde{S}_{2n+2}$ ($n \geq 0$). When $S = S_{2n}$, the derived surface is the projective plane with $n$ handles attached and so $\tilde{S} = \tilde{S}_{2n+1}$ ($n \geq 0$). The reader is asked to note that the identification process never conforms with the orientation $S$ receives from its embedding in $R^3$. In other words, let $\gamma$ be the boundary of a "small" disk on $S$ with the counterclockwise sense. Then, if $\gamma$ is the symmetric image of $\gamma$ (relative to $0$) on $S$, it has the clockwise sense.

Clearly now, any embedding of a pseudograph $G$ on the non-orientable surface $\tilde{S}$ lifts to an embedding of
Figure V.1
some associated pseudograph $G'$ on the orientable surface $S$. What Heffter had in mind was that in order to determine the non-orientable embeddings of $G$, one would study the orientable embeddings of $G'$. History did not bear Heffter's program out. Ringel produced his non-orientable genus embeddings for $K_n$ in the early fifties (see [27] and [28]), long before he and Youngs completed the solution of the orientable version of this problem. Then, in 1965 [32], Ringel exhibited non-orientable genus embeddings of the complete bipartite graph $K_{m,n}$. Thereafter, however, non-orientable embeddings have been shunned, on the whole, possibly due to the lack of Edmonds' type schemes for describing these embeddings. In each case where such embeddings were produced, ad-hoc arguments were used to prove non-orientability. In 1971 Gary Haggard, in his correspondence with J. Gross and A. T. White, disclosed a system of barred rotations which he used to describe embeddings. He also provided a criterion for determining just when these barred rotations do define non-orientable embeddings. His barred edges correspond to pairs of adjacent vertices which, in Ringel's terminology, satisfy rule $R$ but not rule $R^*$ (see [28] pp. 66-67). Haggard never published these results and has been communicating them on a personal basis only.
Soon after the completion of the first draft of this dissertation, the author met with Seth Alpert and Gerhard Ringel at the 81st annual conference of the American Mathematical Society which convened in Washington D. C. in January 1975. It was discovered that both had done work ([2],[31]) which parallels the content of Chapters V and VI of this thesis. In particular, both, being unaware of Haggard's work, developed embedding algorithms which are in essence the same as the one described here. Alpert, because of his interest in both voltage and current graphs and maps, also considered "face schemes", to describe the dual of a given embedding. Both their methods and final results, however, are quite different from those presented here. Also, neither had considered any of the classification and perturbation aspects of the theory, which constitute the major portion of this chapter. Alpert subsequently used his results to derive some genus formulas for octahedral graphs whereas Chapter VI is concerned mainly with complete tripartite graphs. Ringel's main result was that every (irreducible) graph has a non-orientable embedding with only one region. A weaker version of this was contained in the first draft. It has now been modified to its present form, which is actually stronger than Ringel's theorem, since the latter follows
as a corollary.

It is Heffter's idea of using "doppleflächen" that underlies the generalized schemes developed in the ensuing sections of this chapter. If the pseudograph $G$ is embedded on the non-orientable surface $\tilde{S}$, then each vertex $v$ of $G$ lifts to two vertices $(v,0)$ and $(v,1)$ on $S$. The cyclic arrangement of the arcs at $v$ induces opposite rotations of the arcs at $(v,0)$ and $(v,1)$. We do not always consider all pseudographs, however. It so happens that the statements of some of the theorems and their proofs are considerably simplified if attention is restricted to pseudographs in which every vertex has degree at least 3. Such pseudographs are said to be (homeomorphically) irreducible; it is convenient to consider the trivial pseudograph (with one vertex and no edges) as irreducible. It is clear that if $G$ is not irreducible then the contraction of $G$ (see [3]) along certain edges results in an irreducible pseudograph $G'$. It is just as clear that any embedding of $G'$ can be converted into an embedding of $G$ on the same surface in a very natural manner. Hence, nothing significant is lost by the restriction of the theory to irreducible pseudographs.

In conclusion, two alternative methods for describing
rotation systems are defined. Given a pseudograph $G$ and a rotation system $P$ on $G$, a planar description of $P$ is a drawing (not necessarily an embedding) of $G$ in the plane, such that the counterclockwise reading of the arcs at each vertex $v$ coincides with $P_v$.

For example, Figure V.2 is a planar description of the following rotation system $P$:

$$P_u = (vu, xu, wu) \quad P_v = (wv, xv, uv)$$

$$P_w = (uw, xw, vw) \quad P_x = (ux, vx, wx)$$

![Figure V.2](image-url)
The second alternative applies only when $G$ is a graph. In that case, we may, and usually do, dispense with the arcs in writing out the $P_v$, and replace them by their initial vertices without any risk of ambiguity. Thus the above rotation system $P$ should be written as

$$P_u = (v,x,w) \quad P_v = (w,x,u)$$

$$P_w = (u,x,v) \quad P_x = (u,v,w).$$

Section V.2

Generalized Embedding Schemes

We commence by describing a graph construction which has already been used to great advantage in other contexts by Gross and Alpert in [3] and [14], and by Biggs in [5]. Here it is used to "double" the given pseudograph.

Let $(G,\lambda)$ be a voltage pseudograph with assignments in the group $\Gamma$. We now construct a pseudograph $G_x^T$, whose vertex set is the cartesian product $V(G) \times \Gamma$. Given two vertices $(u,g)$ and $(v,h)$ of $G_x^T$, the ordered pair $[(u,g),(v,h)]$ is an arc of $G_x^T$ if and only if $uv$ is an arc of $G$ and $g\lambda(uv) = h$. The edges of $G_x^T$ are defined by identifying each arc with its inverse and ignoring their orientations. This
construct will be discussed in greater detail in the following chapter. For our purposes here it suffices to consider the case $\Gamma = \mathbb{Z}_2$. The reader may find an example of this construction in Figure V.3. We adopt here the convention that an unmarked arc is assigned the voltage 0 while an arc marked with the symbol "\~{}" is assigned the voltage 1.

If we regard pseudographs as topological spaces (with edges and loops homeomorphic to closed intervals

![Diagram](image-url)
and circles respectively) then \( G \times \lambda Z_2 \) is a twofold covering space of \( G \). This fact, though obvious, is crucial to the development of this subject. Note that \( G \times \lambda Z_2 \) need not be connected. In particular, if \( G \) is connected and \( \lambda \) assigns 0 to every arc of \( G \), then \( G \times \lambda Z_2 \) consists of two components each of which is isomorphic to \( G \). This observation generalizes easily to the following lemma.

**V.2.1 Lemma:** If \( G \) is a connected pseudograph and \( G \times \lambda Z_2 \) is disconnected, then \( G \times \lambda Z_2 \) has two components each of which is isomorphic to \( G \).

**Proof:** We regard both \( G \) and \( G \times \lambda Z_2 \) as topological spaces and denote by \( p:G \times \lambda Z_2 \rightarrow G \) the natural projection which carries \((u,i)\) into \( u \) for every \((u,i) \in V(G \times \lambda Z_2)\).

Fix a vertex \( u \) of \( G \). Any other vertex \( v \) of \( G \) can be joined to \( u \) by a path \( P \) in \( G \). Let \( H_i \) be the component of \( G \times \lambda Z_2 \) which contains \((u,i)\) for \( i = 1,2 \).

Then \( H_i \) contains a lift of \( P \) and hence each \( H_i \) contains one of the vertices \((v,0)\) and \((v,1)\). It is now clear that each \( H_i \) is isomorphic to \( G \) and that there are no other components of \( G \times \lambda Z_2 \).

q.e.d.
If the edge $f$ of $G \times \mathbb{Z}_2$ is defined by the edge $e$ of $G$, we speak of $f$ as a *lift* of $e$ and write $f = \tilde{e}^j$, where $j = 0,1$. Unless it is needed, the superscript $j$ of $\tilde{e}^j$ will be suppressed.

A *generalized embedding scheme* for the pseudograph $G$ is a pair $(P, \lambda)$ where $P$ is a rotation system on $G$ and $\lambda : \Delta(G) \to \mathbb{Z}_2$ defines a voltage pseudograph on $G$.

Our strategy for constructing a corresponding embedding for $G$ is to use $(P, \lambda)$ to define a rotation system $P^\lambda$ for $G \times \mathbb{Z}_2$. This rotation system $P^\lambda$ determines an orientable 2-cell embedding of $G \times \mathbb{Z}_2$ which is in fact a twofold cover of the desired embedding of $G$.

Let $(P, \lambda)$ be an embedding scheme for $G$. The rotation system $P^\lambda$ is defined on $G \times \mathbb{Z}_2$ by "lifting" $P_v$ to $(v,0)$ and lifting $P_v^{-1}$ to $(v,1)$ for each vertex $v$ of $G$. Thus, if $d$, $e$, and $f$ are consecutive arcs at $v$, i.e., $P_v(d) = e$ and $P_v(e) = f$, let $\tilde{d}^i$, $\tilde{e}^i$, and $\tilde{f}^i$ denote their lifts at $(v,i)$ for $i = 0,1$. Then,

\[
\begin{align*}
P^\lambda_{(v,0)}(\tilde{e}^0) &= \tilde{f}^0 \\
P^\lambda_{(v,1)}(\tilde{e}^1) &= \tilde{d}^1 \\
P^\lambda_{(v,0)}(\tilde{d}^0) &= \tilde{e}^0 \\
P^\lambda_{(v,1)}(\tilde{f}^1) &= \tilde{e}^1.
\end{align*}
\]

(1)

Figure V.4 contains illustrations, by means of planar...
We define \( \hat{0} \) and \( \hat{1} \) as 1 and 0 respectively. Similarly, if \( \tilde{e} \) is a lift of an arc \( e \) of \( G \) to \( G \times \mathbb{Z}_2 \), then \( \hat{e} \) denotes the other lift of \( e \). Thus, if the arc \( e = uv \) lifts to \( \tilde{e} = (u, i)(v, j) \), then \( \hat{e} = (u, \hat{i})(v, \hat{j}) \).

**V.2.2 Lemma:** Let \( (P, \lambda) \) be an embedding scheme for the irreducible pseudograph \( G \), and suppose \( \tilde{e}_1 \tilde{e}_2 \ldots \tilde{e}_s \tilde{e}_1 \) is the boundary of a region \( R \) in the 2-cell orientable embedding of \( G \times \mathbb{Z}_2 \) determined by \( (P, \lambda) \). Then there exists a unique region \( \hat{R} \neq R \) with
Moreover, \( \hat{R} = R \).

**Proof:** Let \( \tilde{e}_n \) and \( \tilde{e}_{n+1} \) be two consecutive arcs on the boundary of \( R \), with their vertices labelled as in Figure V.5. Let \( R \) denote the region whose boundary contains the arc \( (\tilde{e}_{n+1})^{-1} \). It follows from the definition (1) that \( P_{(v,j)}((\tilde{e}_{n+1})^{-1}) = (\tilde{e}_n) \). A simple application of the principle of induction allows us to conclude that the boundary of \( \hat{R} \) is indeed described by (2).

It is obvious that \( R \) is unique and \( \hat{R} = R \). It remains to verify that \( \hat{R} \neq R \). Suppose to the contrary that \( \hat{R} = R \); then portions of the boundary of \( R \) are described by (3) below.

---

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
If we apply arguments similar to those used in the first part of the proof to alternative (3), it is immediately seen that, depending on the parity of the number of arcs between \( e_{n+1} \) and \( (e_{n+1})^{-1} \), either a or b of Figure V.6 must describe the boundary of \( R \) between these two arcs. However, diagram a implies that \( e_x \) and \( e_x \) have the same terminal vertex, while diagram b says that \( e_x = (e_x)^{-1} \). Both of these implications,
however, are absurd. Hence (3) cannot describe the boundary of $R$. q.e.d.

We now proceed to describe a 2-cell embedding of an irreducible pseudograph $G$. Let $\rho$ be a collection of regions in the embedding of the above lemma, such that for each region $R$, one of the pair $\{R, \hat{R}\}$ is in $\rho$, but not both. Let $\tilde{e}_1 - \tilde{e}_2 - \ldots - \tilde{e}_n - \tilde{e}_1$ denote the boundary of a region $R$, where the $\tilde{e}_i$ in $G$. Then, let $p(R)$ denote a plane polygon whose oriented boundary is labeled $e_1 - e_2 - \ldots - e_n - e_1$. It follows from Lemma V.2.2 that each edge $e$ of $G$ occurs twice as the side of some $p(R)$. In other words, either there are two regions in $\rho$ on each of whose boundaries $e$ occurs once, or else there is a single region in $\rho$ on whose boundary $e$ occurs twice. For, if $\tilde{e}$ is a lift of $e$ then $\tilde{e}$ and $\tilde{e}^{-1}$ lie on the boundary of two regions $R_1$ and $R_2$ (possibly $R_1 = R_2$) of $G \times Z_2$. But then $\hat{R}_1$ and $\hat{R}_2$ both contain $\tilde{e}^{-1}$ and $\tilde{e}$ respectively on their boundaries. Since $\rho$ contains exactly one of the pair $\{R_1, \hat{R}_1\}$ for each $i = 1, 2$, the above assertion is valid. It is clear that an application of the classical edge identification process to the collection $p(\rho) = \{p(R) \mid R \in \rho\}$ yields a 2-cell embedding of $G$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
The above can be summarized as follows: the embedding scheme \((P, \lambda)\) determines an embedding of \(G \times \mathbb{Z}_2\) on an orientable surface; the map \(R \mapsto \hat{R}, \tilde{e} \mapsto \hat{e}\) determines a period 2 homeomorphism of this surface so that it covers a possibly non-orientable surface which contains \(G\).

It has been shown that any embedding scheme \((P, \lambda)\) determines a 2-cell embedding of \(G\). We now show that given an embedding of \(G\) on \(S\), a scheme \((P, \lambda)\) exists which induces that very embedding. First we define what it means for two embeddings to be the same. In general, we say that two embeddings of a pseudograph \(G\) are the same if there exists a one-to-one correspondence between their regions such that the unoriented boundaries of corresponding regions are identical (as closed walks of \(G\)). It should be noted that under this definition there are two rotation systems which define a given orientable embedding. The two are related in that they are inverses of each other and the resulting embeddings are mirror images of each other and hence the same.

If \(S\) is orientable, the rotation system \(P\) is the one which determines that orientable embedding and \(\lambda\) is the constant map which assigns \(0 \in \mathbb{Z}_2\) to every arc of \(G\). This, in effect, is the Edmonds-Youngs Theorem of [38].
Here $G \times _{\lambda} \mathbb{Z}_2$ is disconnected and hence the embedding given by Lemma V.2.2 consists of two orientable embeddings of isomorphic copies of $G$, both of which are identical with the given orientable embedding.

If $S$ is not orientable, let $p: \tilde{S} \to S$ be the twofold orientable covering projection of $S$. For every vertex $v$ of $G$ on $S$, assign the labels $(v,0)$ and $(v,1)$ arbitrarily to the two points in $p^{-1}(v)$. If $e = uv$ is an arc of $G$, then a lift $\tilde{e}$ of $e$ has endpoints $(u,i)$ and $(v,j)$. Define $\lambda(e) = 1$ if $i \neq j$ and $\lambda(e) = 0$ otherwise. Clearly, then, $p^{-1}(G)$ is an embedding of $G \times _{\lambda} \mathbb{Z}_2$ on $\tilde{S}$. Since $\tilde{S}$ is orientable there exists a rotation system $\mathcal{P} = \{P_{(v,i)} | (v,i) \in V(G \times _{\lambda} \mathbb{Z}_2)\}$ which induces this embedding. For each vertex $v$ of $G$ we define $P_v$ by setting $P_v(e) = f$ whenever $\tilde{P}_{(v,0)}(\tilde{e}) = \tilde{f}$. We claim that $(P,\lambda)$ induces the given embedding of $G$ on $S$. For, if we start with the scheme $(P,\lambda)$ and construct the auxiliary embedding of $G \times _{\lambda} \mathbb{Z}_2$, then we obtain exactly the above embedding of $G \times _{\lambda} \mathbb{Z}_2$ on $\tilde{S}$.

Hence, if we start with a region of $G$ under $(P,\lambda)$, it lifts to two regions of $G \times _{\lambda} \mathbb{Z}_2$ on $\tilde{S}$ which again project to a single region of $G$ on $S$. It is easy to see that this defines the desired one-to-one correspondence between the regions of the two embeddings of $G$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Hence \((P,\lambda)\) does indeed define the given embedding of 
\(G\) on \(S\). The above information is summarized in the 
following theorem.

**V.2.3 Theorem:** If \(G\) is a pseudograph, then every 
embedding scheme \((P,\lambda)\) determines a 2-cell embedding 
of \(G\) and every 2-cell embedding of \(G\) is determined 
by some such scheme.

**Example:** Set \(G = K_4\) and let \((P,\lambda)\) have the planar 
description given in Figure V.7 Since \(G \times \lambda Z_2\) is a 
graph, the local rotations can be described by their 
action on the vertices of \(G \times \lambda Z_2\). We now list these 
rotations.

\[
\begin{align*}
P(u,0) &= ((v,0), (w,1), (w,0)) \\
P(u,1) &= ((x,1), (w,0), (v,1)) \\
P(v,0) &= ((x,0), (w,0), (u,0)) \\
P(v,1) &= ((u,1), (w,1), (x,1))
\end{align*}
\]

Figure V.7

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
\[ P(w,0) = ((u,1), (v,0), (x,0)) \quad P(w,1) = ((x,1), (v,1), (u,0)) \]
\[ P(x,0) = ((u,0), (w,0), (v,0)) \quad P(x,1) = ((v,1), (w,1), (u,1)). \]

Next, we compute the regions of the embedding of \( G_{x,Z_2} \). Their boundaries are denoted by listing their vertices rather than their edges.

\[ R_1: (u,0)-(v,0)-(x,0)-(u,0)-(v,0) \]
\[ R_2: (v,0)-(u,0)-(w,1)-(x,1)-(u,1)-(w,0)-(v,0)-(u,0) \]
\[ R_3: (u,1)-(v,1)-(w,1)-(u,0)-(x,0)-(w,0)-(u,1)-(v,1) \]
\[ R_4: (v,1)-(u,1)-(x,1)-(v,1)-(u,1) \]
\[ R_5: (v,0)-(w,0)-(x,0)-(v,0)-(w,0) \]
\[ R_6: (w,1)-(v,1)-(x,1)-(w,1)-(v,1). \]

It is easy to see that \( \hat{R}_1 = R_4 \), \( \hat{R}_2 = R_3 \), and \( \hat{R}_5 = R_6 \).

Hence Figure V.8 is a polygonal presentation of our embedding of \( K_4 \).

![Figure V.8](image-url)
The above computations can be considerably simplified by suppressing the second coordinates. This is done as follows. We write

\[ P_u = (v, w, x) \quad P_v = (x, w, u) \]
\[ P_w = (u, v, x) \quad P_x = (u, w, v) . \]

The computation of the regions proceeds as in the orientable case, with one difference. Sometimes \( P_y^{-1}(z) \) is used instead of \( P_y(z) \). Specifically, suppose that we have already computed the portion \( v_1 - v_2 - \ldots - v_n \) of the boundary of some region \( R \). Then we set

\[ v_{n+1} = P_v^\epsilon(v_{n-1}) , \quad \text{where} \]

\[ \epsilon = \begin{cases} 1 \\ -1 \end{cases} \quad \text{if} \quad \sum_{i=1}^{n-1} \lambda(v_i v_{i+1}) = \begin{cases} 0 \\ 1 \end{cases} . \quad (4) \]

This version was first suggested by A. T. White in the aforementioned correspondence with Gross and Haggard.

We now apply this method to the above embedding. If \( \lambda(yz) = 1 \) we write \( y \sim z \) in describing the boundary of the region; otherwise, \( u - v \) is used.

\[ R_1: \quad u - v - x - u - v \]

\[ R_2: \quad v - u \sim w - x - u \sim w - v - u \]
The above could have been further abbreviated by noting that there is no need for writing $\hat{R}$ down once $R$ has been computed, as it will provide no new information. The converse problem, that of finding an embedding scheme which determines a given embedding, will be addressed in Section 4 of this chapter.

Section V.3

The Classification of Embedding Schemes

Consider the embedding schemes whose planar descriptions are given in Figure V.9. In computing their regions we note that for both schemes the regions consist of the four triangles in Figure V.10. Hence both schemes define the same planar self-dual embedding of $K_4$. This example gives rise to two questions which will be settled in this section. When do two schemes define the same embedding and when does an embedding scheme
determine an orientable embedding? Fortunately, the criteria established for making these decisions are easy to apply.

Figure V.9

Figure V.10
In the ensuing discussion, the pseudograph $G$ is fixed and $\pi$ denotes the set of all the generalized embedding schemes of $G$. If $v$ is a vertex of $G$, then $\sigma_v$, the switch at $v$, is a permutation of $\pi$ whose action is defined as follows: for any embedding scheme $(P, \lambda)$ of $G$, $(P, \lambda)\sigma_v$ is the scheme $(Q, \mu)$ with $Q$ defined by

\[
Q_u = P_u \text{ if } u \neq v \\
Q_v = P_v^{-1},
\]

also, $\mu(e) \neq \lambda(e)$ if $v$ is exactly one of the end-points of $e$, and $\mu(e) = \lambda(e)$ otherwise. For example, if $(P, \lambda)$ is the scheme of Figure V.11a, then part b of the same figure describes $(P, \lambda)\sigma_v$.

![Figure V.11](image_url)
The switch group $\Sigma$ is defined as that subgroup of $S$, the symmetric group of all the permutations of $\pi$, which is generated by the set $\{\sigma_v | v \in V(G)\}$. The following theorem shows that the switch group in itself contains very little information regarding $G$ and its embedding schemes.

V.3.1 Theorem: If $G$ is a connected pseudograph of order $n$, and $\Sigma$ is its switch group, then

$$\Sigma \cong \mathbb{Z}_2^{n-1}.$$  

Proof: Since $G$ is connected, every vertex $v$ is the endpoint of some edge which is not a loop, and hence $\sigma_v$ is not trivial. It is clear therefore that $\sigma_v$ has order 2. Moreover, if $u$ is any other vertex of $G$ then $\sigma_u \sigma_v = \sigma_v \sigma_u$. Hence an arbitrary element of $\Sigma$ can be written in the form $\sigma = \sigma_{v_1} \sigma_{v_2} \cdots \sigma_{v_m}$ where $v_1, v_2, \ldots, v_m$ is a list of $m$ distinct vertices of $G$. But then, suppose we have

$$\sigma_{v_1} \sigma_{v_2} \cdots \sigma_{v_m} = 1_{\Sigma} \quad \text{with} \quad m \geq 1. \quad (5)$$

We show that $v_1, v_2, \ldots, v_m$ is necessarily a list of all the vertices of $G$. Suppose not; then there exist adjacent vertices $u$ and $v$ of $G$ such that $u = v_i$
for some $1 \leq i \leq m$ while $v \neq v_j$ for all $1 \leq j \leq m$.

But then, if $(P, \lambda)$ is any embedding scheme of $G$ and $(Q, \mu) = (P, \lambda) \sigma_1 \sigma_2 \cdots \sigma_m$, it is clear that $\lambda(uv) \neq \mu(uv)$.

This, however, contradicts (5) above. We now know that $\Sigma$ is an abelian group which has the presentation

$$\langle a_1, a_2, \ldots, a_n \mid a_1^2 a_2^2 \cdots a_n^2, a_1^{-1} a_2^{-1} \cdots a_n^{-1}, 1 \leq i, j \leq n \rangle.$$  

Hence, $\Sigma \cong \mathbb{Z}_2^{n-1}$.

q.e.d.

Implicit in the above proof is the fact that if $\sigma$ is any element of the switch group $\Sigma$, then there exists a unique set $U = \{ u_1 \}_{i=1}^S$ with $u_1, u_2, \ldots, u_S \in V(G)$, such that $\sigma = \sigma_{u_1} \sigma_{u_2} \cdots \sigma_{u_S}$. We say that $\sigma$ is the switch at $U$, and write $\sigma = \sigma_U$. As the switch group depends only on the order and number of components of $G$, it is clear that something else must contain the information we seek. In fact, the following theorem shows that this information is to be found in the action of $\Sigma$ on $\pi$.

V.3.2 Theorem: Two embedding schemes of an irreducible pseudograph $G$ determine the same embedding if and only if they are in the same orbit of $\pi$ under the action of $\Sigma$.
Proof: The theorem may be restated as follows. If $(P, \lambda)$ and $(Q, \mu)$ are schemes for $G$, then they determine the same embedding if and only if there exists a set $U \subseteq V(G)$ such that $(P, \lambda)_{\sigma_U} = (Q, \mu)$. We commence by showing that schemes in the same orbit determine the same embedding.

Let $v$ be an arbitrary vertex of $G$ and $(P, \lambda)$ a scheme for $G$. Suppose $(P, \lambda)_{\sigma_v} = (Q, \mu)$. Define the map $\varphi: V(GxZ_2) \to V(GxZ_2)$ by

\begin{align*}
\varphi((u,i)) &= (u,i) \text{ if } u \neq v \text{ and } i = 0,1 \\
\varphi((v,i)) &= (v,\hat{i}) \text{ for } i = 0,1.
\end{align*}

It is clear, because of the way $\sigma_v$ acts, that $\varphi$ can be extended to an isomorphism of $GxZ_2$ and $GxZ_2$. Moreover,

\begin{align*}
Q_u &= P_u \text{ if } u \neq v, \text{ and } \\
Q_v &= P_v^{-1}.
\end{align*}

Hence, in the auxiliary embeddings of $GxZ_2$ and $GxZ_2$,

\begin{equation}
Q_{\varphi}((u,i)) = P(u,i) \text{ for all } (u,i) \in V(G)xZ_2.
\end{equation}

This means that the isomorphism $\varphi$ "conforms" with these
embeddings. However, the boundaries of the regions of the two embeddings of $G$ defined by $(P,\lambda)$ and $(Q,\mu)$ are determined by the sequence of the first coordinates along the boundaries of the lifting regions in the auxiliary embeddings of $G \times_{\lambda} Z_2$ and $G \times_{\mu} Z_2$. In view of (5), (6), and (7) above, these sequences are identical. Thus $(P,\lambda)$ and $(P,\lambda)\sigma_v$ determine the same embedding of $G$. Since $[\sigma_v|v\in V(G)]$ generates the switch group $\Sigma$, the proof of the sufficiency is concluded.

Conversely, suppose $(P,\lambda)$ and $(Q,\mu)$ determine the same embedding of $G$. For each $v \in V(G)$ both $P_v$ and $Q_v$ describe one of two possible cyclic permutations induced on the arcs at $v$ by the embedding. Hence $P_v = Q_v^\epsilon(v)$ where $\epsilon(v) = \pm 1$. Set $U = \{v \in V(G)|\epsilon(v) = -1\}$ and define $(Q',\mu) = (P,\lambda)\sigma_U$. Because of the manner in which the elements of $U$ were selected, we know that $Q'_v = Q_v$ for every vertex $v$ in $G$. By the first part of this theorem, the schemes $(Q',\mu)$ and $(P,\lambda)$ define the same embedding of $G$, and hence so do $(Q',\mu)$ and $(Q,\mu)$. Since we already know that $Q' = Q$, the proof will be concluded when it is demonstrated that $\mu' = \mu$. Suppose to the contrary that $\mu' \neq \mu$. Then, due to the symmetry of the situation, we may assume that for some edge $e = uv$, we have $\mu'(e) = 0$ and $\mu(e) = 1$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Suppose now that Figure V.12 describes $Q_u$ and $Q_v$ near the edge $e$. (The insertion of small circular arcs between edges denotes that they are consecutive in the appropriate rotation and the orientation of these circular arcs indicates the action of that rotation.)

The reader is reminded that the rotation $Q(u,0)$ is a lift of $Q_u$, while $Q(u,1)$ is a lift of $Q_u^{-1}$. Thus, in working out the regions of the auxiliary orientable embeddings of $G \times Z_2$ and $G \times Z_2^\mu$, we obtain Figure V.13. Now, however, it follows that the embedding determined by $(Q',\mu')$ has a region, namely $p(R_1)$, whose boundary contains the portion $\ldots - f - e - g^{-1} - \ldots$. An examination of the embedding of $G \times Z_2^\mu$ reveals that this is inconsistent with the requirement that each vertex have degree at least 3 (the theorem clearly holds for
the two exceptional irreducible pseudographs which have minimum degree less than 3). Thus, we must have \( \mu' = \mu \) and hence

\[
(P, \lambda)_{\sigma_U} = (Q', \mu') = (Q, \mu)
\]

Figure V.13

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
It is now very easy to determine whether two schemes \((P, \lambda)\) and \((Q, \mu)\) define the same embedding of \(G\).

First we must have \(F_v = Q_v^{\varepsilon(v)}\) with \(\varepsilon(v) = +1\) for all \(v\) in \(V(G)\); in addition, \((P, \lambda)_U\) must equal \((Q, \mu)_U\), where \(U = \{v \in V(G) \mid \varepsilon(v) = 1\}\). For example, the two embedding schemes of Figure V.14 do determine the same embedding of \(K_4\). Here we have \(U = \{v, x\}\).

![Diagram](image)

**Figure V.14**
Corollary: The scheme \((P,\lambda)\) determines an orientable embedding of the irreducible pseudograph \(G\) if and only if \(G_{\lambda}Z_2\) is disconnected.

Proof: If \(G_{\lambda}Z_2\) is disconnected, then it has two components, each of which is isomorphic to \(G\). In fact, this isomorphism extends to a homeomorphism of each component of the auxiliary embedding of \(G_{\lambda}Z_2\) and the embedding of \(G\) defined by \((P,\lambda)\). Since the auxiliary embedding is orientable, so is the embedding of \(G\).

Conversely, suppose \((P,\lambda)\) defines an orientable embedding of \(G\). Then this embedding can also be defined by some scheme \((Q,0)\) where \(0\) is the map assigning the identity element of \(Z_2\) to every arc of \(G\). Hence, there exists a subset \(U\) of \(V(G)\) such that \((P,\lambda)\sigma_U = (Q,0)\). However, it follows from (5) on p. 112 that \(G_{\lambda}Z_2\) and \(G_{0}Z_2\) are isomorphic. Since \(G_{0}Z_2\) is clearly disconnected, so is \(G_{\lambda}Z_2\).

q.e.d.

This corollary leads to the following theorem which presents a very easy to use criterion for determining the orientability character of an embedding scheme. Special cases of this rule have been used by Ringel and Youngs. An edge on which \(\lambda\) does not vanish corresponds to an
edge whose endpoints satisfy rule \( R \) in the terminology of [39]. The following theorem is thus a direct generalization of the triangular rule given on p.21 of [39].

**V.3.4 Theorem:** If \( G \) is an irreducible pseudograph then the embedding scheme \((P, \lambda)\) defines an orientable embedding of \( G \) if and only if every cycle of \( G \) contains an even number of arcs \( e \) for which \( \lambda(e) = 1 \).

**Proof:** Define

\[
D_\lambda = \{ e \in D(G) \mid \lambda(e) = 1 \}.
\]

Now, it follows from the above corollary that the embedding is non-orientable if and only if \( G \_{\lambda} \mathbb{Z}_2 \) is connected. This connectedness, however, is equivalent to the requirement that for some vertex \( v \) of \( G \), the vertices \((v,0)\) and \((v,1)\) are joined by a path of \( G \_{\lambda} \mathbb{Z}_2 \). Such a path projects onto a closed walk of \( G \). Since the second coordinates of \((v,0)\) and \((v,1)\) are different, such a closed walk must contain an odd number of arcs of \( D_\lambda \). Moreover, as every closed walk is the union of edge-disjoint cycles, it follows that \( G \) contains a closed walk with an odd number of arcs of \( D_\lambda \) if and only if \( G \) contains a cycle with the same property. Thus we have
shown that the embedding of $G$ defined by $(P, \lambda)$ is non-orientable if and only if $G$ has a cycle which contains an odd number of arcs of $D_\lambda$. This is clearly equivalent to the statement of the theorem.

q.e.d.

For an example the reader is referred to Figure V.3.1. It is important to note here that whether an embedding scheme $(P, \lambda)$ defines an orientable embedding depends only on $\lambda$ and has nothing whatsoever to do with the rotation system $P$.

There is an interesting coincidence here, which the author finds somewhat intriguing. Given an embedding scheme $(P, \lambda)$, let us replace the elements 0 and 1 of $\mathbb{Z}_2$ by the symbols "+" and "-" respectively. The pseudograph $G$ together with the function $\lambda$ then define a signed graph in the sense of [20]. The above theorem then says that $(P, \lambda)$ defines an orientable embedding if and only if the associated signed pseudograph is balanced.

Section V.4

The Nature of $\lambda$

Given an embedding scheme $(P, \lambda)$ for $G$, we refer to

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
the arc set \( D_\lambda = \{ e \in D(G) \mid \lambda(e) = 1 \} \) as the support of \( \lambda \). We now proceed to give a geometric interpretation of \( D_\lambda \); we shall also consider the result of modifying \( D_\lambda \) by a single edge.

Let \( i:G \to S \) be an embedding of the irreducible pseudograph \( G \) on the surface \( S \), and let \( o \) be an orientation of its regions. We say that the two regions on whose boundary an arc \( e \) occurs are **coherently oriented** if \( e^{-1} \) occurs on the oriented boundary of one and \( e \) occurs on the other. Otherwise the regions are said to be **non-coherently oriented**. The **non-coherence set** \( N(i,o) \) of the given orientation \( o \) of the embedding \( i:G \to S \) is the set of all those edges of \( G \) whose two sides are non-coherently oriented. Let \( i*:G* \to S \) be the embedding dual to the given embedding and for any set \( C \) of edges of \( G \) let \( C* \) denote the set of their duals in \( G* \).

**V.4.1 Theorem:** Let \( G \) be an irreducible pseudograph and suppose the embedding \( i:G \to S \) is defined by the scheme \((P,\lambda)\) with support \( D_\lambda \). If \( i*:G* \to S \) is the dual embedding, then there exists an orientation \( o* \) of the regions of \( G* \) such that

\[
N(i*,o*) = D_\lambda^*.
\]

Conversely, given a pair of dual embeddings \( i:G \to S \) and \( i*:G* \to S \),
Given \( i \) and \( i^* \), let \( V \leftrightarrow R \) denote the dual correspondence between the vertices of \( G \) and the regions of \( G^* \). Again, given the scheme \((P, \lambda)\) of the first half of the theorem, let \((v, i) \leftrightarrow R_{(v, i)} \) denote the analogous correspondence for \( G^* \). Since \( G^* \) is orientably embedded we may assume that its regions have all been coherently oriented. Now define the orientation \( o^* \) on the regions of \( G^* \) by assigning to \( R \) the orientation of \( R_{(v, 0)} \). Fix \( v \) and let \( e = uv \) be an arc at \( v \). If \( e \) is not in \( D^* \lambda \) (that is, \( \lambda(e) = 0 \)), then \( e \) lifts to \( \tilde{e} = (u, 0)(v, 0) \) in \( G^* \). Hence the regions \( R_{(u, 0)} \) and \( R_{(v, 0)} \) abut along \( (\tilde{e})^* \). Since they are coherently oriented along \( (\tilde{e})^* \) it follows that \( R_u \) and \( R_v \) are also coherently oriented along \( e^* \). Thus, \( e^* \notin N(i^*, o^*) \).

On the other hand, if \( e \) is in \( D^* \lambda \), then \( e \) lifts to \( \tilde{e} = (u, 0)(v, 1) \). In this case it is the regions \( R_{(u, 0)} \) and \( R_{(v, 1)} \) that abut along \( (\tilde{e})^* \). Since the boundary of \( R_{(v, 1)} \) is the "reverse" of that of \( R_{(v, 0)} \), it
follows that $R_u$ and $R_v$ are non-coherently oriented along $e^*$. Thus, $e^* \in N(i^*,o^*)$. Hence it has been shown that

$$D^*_\lambda = N(i^*,o^*) .$$

Conversely, suppose that $i$, $i^*$, and $o^*$ are given. For every arc $e$ of $G$ define

$$1 \text{ if } e^* \notin N(i^*,o^*)$$

$$0 \text{ if } e^* \in N(i^*,o^*) .$$

Finally, for every $v \in V(G)$ define $P_v$ as the rotation induced by the orientation $o^*(R_v)$, and set $P = \{ P_v \in v \mid V(G) \}$. It now remains to show that $(P, \lambda)$ defines the given embedding $i:G \rightarrow S$. To see this suppose that in computing the boundary of some region $R$ defined by $(P, \lambda)$ we have arrived at the section $\ldots - d - e$. Now there exists a region $R'$ of $i:G \rightarrow S$ which contains $\ldots - d - e - \ldots$ as a portion of its boundary. Figures V.15a and V.15b describe the local picture on $S$ when $\lambda(e) = 0$ and 1 respectively. We now refer the reader to the interpretation of $\lambda$ given on page 106. It is clear from (4) that the known portion of the boundary of $R$ can in either case be extended to $\ldots - d - e - f$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
It is now clear that the boundaries of $R$ and $R'$ are identical as closed walks of $G$. Thus $(P,X)$ does indeed determine the embedding $i:G \to S$.

q.e.d.

The above theorem will prove useful later in extending the Gross/Alpert construction of [14] to non-orientable surfaces. We now turn our attention to the question of what happens when the support of $\lambda$ is modified by the addition or deletion of a single edge. If $(P,\lambda)$ and $(Q,\mu)$ are embedding schemes for a pseudograph $G$, we say that $(Q,\mu)$ is a perturbation of $(P,\lambda)$ if $P = Q$ and there exists an edge $e$ of $G$ such that

- $\lambda(e) = 0$
- $\lambda(e) = 1$

Figure V.15
\[ u(d) = \lambda(d) \text{ iff } d \neq e \text{ for all } d \in E(G). \]

We say that \((Q, \mu)\) is the perturbation of \((P, \lambda)\) at \(e\).

**V.4.2 Theorem**: Suppose the embedding scheme \((P, \lambda)\) of the irreducible pseudograph \(G\) is perturbed at the edge \(e\) to produce \((P, \mu)\). Then all the regions determined by \((P, \mu)\) which do not contain \(e\) on their boundary are also regions of \((P, \mu)\). However,

1. **if** \(e\) is on the boundary of two distinct regions of \((P, \lambda)\), **they are coalesced into a single region of** \((P, \mu)\) **which is non-coherently self-abutting along** \(e\);
2. **if** a region \(R\) of \((P, \lambda)\) is coherently self-abutting along \(e\), **it is twisted into another coherently self-abutting region of** \((P, \mu)\);
3. **if** a region \(R\) of \((P, \lambda)\) is non-coherently self-abutting along \(e\), **it splits into two regions of** \((Q, \mu)\).**

**Proof**: The first part of the theorem is clear and requires no further comment. The remaining cases are treated individually.
a) Suppose $e$ is on the boundary of the regions $R_1$ and $R_2$ of Figure V.16a, where $R_1 \neq R_2$. Now trace out the boundary of the region which contains the arc $e$ in the embedding defined by $(P, \mu)$. We obtain:

$$e_1 - \cdots - e_m - e - f_n^{-1} - \cdots - f_n^{-1} - \cdots - f_1^{-1} - e - e_1.$$
b) Let $R$ be the self-abutting region of Figure V.4.2b with boundary
\[ e_1 - \ldots e_m - e^{-1} f_1 - \ldots - f_n e - e_1. \]
The resulting boundary in $(\mathcal{P},\mu)$ is
\[ e_1 - \ldots - e_m - e^{-1} f_n^{-1} - \ldots - f_1^{-1} e - e_1. \]
Note that in V.16b, if some $e_i = f_j$ then if self-abuttment of $R$ along $e_i = f_j$ used to be coherent, it is now non-coherent and vice versa.

c) Here $R$ is the non-coherently self-abutting region described in Figure V.16c with boundary
\[ e_1 - \ldots - e_m - e f_n e^{-1} - \ldots - f_1 e - e_1. \]
The resulting regions in $(\mathcal{P},\mu)$ have boundaries
\[ e_1 - \ldots e_m e - e_1 \quad \text{and} \quad f_1 - \ldots f_n e^{-1} f_1. \]
q.e.d.

The reader may recognize the above theorem as the extension of a device used by Youngs on pp. 15-16 or [39]. There, the placement of a cross-cap on an edge is used to coalesce the regions on both sides of that edge (as occurs in part a of the above theorem). In fact, the following...
corollary states explicitly a result which is probably widely known but to the best of the author's knowledge has yet to be stated in its full generality. First, a definition must be given; if \( G \) is a pseudograph, then \( Y(G) \) (or \( \tilde{Y}(G) \)) is the smallest integer \( n \) such that \( G \) has an embedding on \( S_n \) (or \( \tilde{S}_n \)). The parameters \( Y(G) \) and \( \tilde{Y}(G) \) are respectively the orientable and non-orientable genus of \( G \).

**V.4.3 Theorem:** For any connected pseudograph \( G \),

\[
\tilde{Y}(G) \leq 2Y(G) + 1.
\]

**Proof:** It clearly suffices to prove this when \( G \) is irreducible. Suppose now that \( G \) is orientably embedded on the surface \( S_n \), where \( n = Y(G) \), with \( r \) regions. Then, by the Euler-Poincaré formula,

\[
|V(G)| - |E(G)| + r = 2 - 2n.
\]

Let \((P,0)\) be any scheme which determines this embedding of \( G \), and let \((P,\mu)\) be the perturbation of \((P,0)\) at some edge \( e \) of \( G \). If \( G \) contains no cycles, then it must be the trivial pseudograph (because \( G \) is irreducible). In that case the statement of the theorem needs no further verification. Assume now that \( G \) does contain cycles and that the above edge \( e \) in fact lies
on some such cycle. It then follows from Theorem V.3.4 that \((P,\mu)\) determines a non-orientable embedding of \(G\).

Let \(r'\) denote the number of regions in this embedding of \(G\). In view of Theorem V.4.2 we conclude that \(r' \geq r - 1\). If \((P,\mu)\) embeds \(G\) on \(\mathbb{S}_m\), then we again conclude from the Euler-Poincaré formula that

\[
|V(G)| - |E(G)| + r' = 2 - m.
\]

Since \(r' \geq r - 1\) it follows that \(m \leq 2n + 1\) and hence

\[
\tilde{\gamma}(G) \leq m \leq 2n + 1 = 2\gamma(G) + 1.
\]

q.e.d.

It is a matter of routine calculations to verify that the 4-cube \(Q_4\) cannot have a quadrilateral embedding on the Klein bottle \(\mathbb{S}_2\). Hence, \(\tilde{\gamma}(Q_4) > 2\). However, since \(\gamma(Q_4) = 1\), it follows from Theorem V.4.3 that \(\tilde{\gamma}(Q_4) = 3\). This example also demonstrates the sharpness of the inequality of this theorem.

The following theorem is an analog of a theorem that Duke proved in [7] for orientable embeddings.

**V.4.4 Theorem:** Let \(r(G)\) denote the largest number of regions in any embedding of the pseudograph \(G\) on any closed surface. If \(G\) is connected then it possesses
non-orientable 2-cell embeddings with \( r \) regions (on closed surfaces) for any integer \( r \) such that \( 1 \leq r < r(G) \).

Proof: Let \((P,\lambda)\) be an embedding scheme which determines an embedding of \( G \) with \( r(G) \) regions. If \( r(G) \neq 1 \) then there clearly must exist two distinct regions which abut along some edge \( e \) of \( G \). Let \((P,\mu)\) be the scheme obtained by perturbing \((P,\lambda)\) at \( e \). It follows from Theorem V.4.2 that the new scheme \((P,\mu)\) determines an embedding of \( G \) with \( r(G) - 1 \) regions. Moreover, it follows from the same theorem that the new coalesced region is non-coherently self-abutting. Consequently, the embedding defined by \((P,\mu)\) is necessarily non-orientable. If \( r(G) - 1 = 1 \), then we are done. Else, the above process is repeated as often as is necessary.

q.e.d.

The above considerations regarding \( Q_4 \) show that sometimes it is not possible to construct a non-orientable embedding of \( G \) with \( r(G) \) regions. Hence, in that sense, the above theorem is the best possible. The following corollary is just a rephrasing of this theorem.
V.4.5 Corollary: Suppose the connected pseudograph $G$ has $p$ vertices and $q$ edges. Then $G$ has a 2-cell embedding on the non-orientable surface $S_k$ for all integers $k$ such that

$$\gamma(G) \leq k \leq q - p + 1.$$ 

Proof: According to Theorem V.4.4 $G$ has a non-orientable embedding with exactly one region. It follows from the Euler-Poincaré formula that such an embedding must be on the surface $S_{q-p+1}$. The proof of the corollary now follows easily from the previous theorem.

q.e.d.

We note here that Gerhard Ringel announced recently in [31] that every connected graph possesses a non-orientable embedding with exactly one region.

It will prove to be convenient to have yet another method for presenting embeddings which are not necessarily orientable. Using the notation introduced in the proof of Theorem V.4.1, we note that if $v$ is a vertex of $G$ then the specification of an orientation of $R_v$ is equivalent to a choice of one of the rotations $P_v$ and $P_v^{-1}$. Suppose now that we have the surface $S$ presented as the single polygon of Figure V.17, with the proper
identifications on the sides. How does one construct a scheme $(P, \lambda)$ which determines the embedding of the graph $G$ given in this figure? We first assign to each vertex the counter clockwise orientation it inherits from this drawing. In this case, $G = K_5$ and the rotations
It now remains to specify $\lambda$. An edge of $G$ is assigned the value 0 or 1 according as the number of cross-caps it crosses is even or odd. A cross-cap, here, is a side of the polygon along which $S$ is non-coherently self abutting. Thus in this special case $\lambda(13) = \lambda(35) = 1$ and $\lambda$ is zero elsewhere. The rationale behind this assignment is that each time an edge crosses a cross-cap, the region which was to the right side of the edge is now on the left and vice versa, and clearly it is the position of the region near the vertex $v$ that determines whether we are to use $P^v$ or $P^{-1}_v$. The regions of the above embedding have boundaries:

\[
\begin{align*}
A: & \quad 1 - 5 - 4 - 3 - 2 - 1 - 5 \\
B: & \quad 5 - 1 \sim 3 \sim 5 - 2 - 4 - 1 - 2 - 5 - 1 \\
C: & \quad 3 - 4 - 2 - 3 \sim 5 - 4 - 1 \sim 3 - 4 .
\end{align*}
\]

We note that the above technique assumes that no vertices are located on the periphery. Another example will be discussed now which shows how Theorem V.4.1 can
be used in the general case. Let the rectangle of Figure V.18a represent an embedding of a bouquet on two loops on the Klein bottle. Part b of the same figure contains an orientation of the single region $R$ of the dual of this bouquet. It is clear that $R$ is non-coherently self-adjacent only along $e^*$. Hence, if $(P, \lambda)$ is the embedding scheme which describes V.4.4a, then $\lambda(e) = 1$ and $\lambda(f) = 0$. The rotation system at $P$ is obtained by piecing together the corners of the rectangle. Thus $P_v = (e,f^{-1}, e^{-1}, f)$.
CHAPTER VI

COVERING PSEUDOGRAPHS

A method used by Gross and Alpert ([12, 14, 15, 16]) to construct orientable embeddings of pseudographs is extended to handle non-orientable embeddings as well. The extension is then used to construct a non-orientable genus embedding of $K(n,n,n-2)$ (a new result), among other graphs, and to characterize the self-dual embeddings constructed in Chapters III and IV.

This extension is known to Gross and Tucker (see [17]). The proof, however, has not been spelled out by them. The passage from current graphs to voltage graphs, which is repeatedly quoted to justify the voltage graph construction, does not apply to general surfaces. The reason for this is that if a current graph is embedded in a non-orientable surface, then there is no "natural" way for converting its dual into a voltage graph.

Section VI.1

The Construction

A covering pseudograph of $G$ is a pseudograph
$G \times \varphi \Gamma$ where $(G, \varphi)$ is a voltage pseudograph with values in $\Gamma$. It is easy to see that when graphs are regarded as topological spaces, then $G \times \varphi \Gamma$ is in fact a covering space of $G$. Moreover, the authors of [17] assert that every regular covering projection of $G$ can be obtained in this manner.

Given a voltage pseudograph $(G, \varphi)$ with values in the group $\Gamma$ and a closed walk $c: e_1, e_2, \ldots, e_n$ at a vertex $v$ of $G$, we define $\varphi(c) = \prod_{i=1}^{n} \varphi(e_i)$. The local group at $v$, denoted by $\Gamma_v$, is defined for all $v \in V(G)$ as

$$\Gamma_v = \{ \varphi(c) \mid c \text{ is a closed walk at } v \}.$$

It is easily verified that $\Gamma_v$ is in fact a subgroup of $\Gamma$. Moreover, if $u$ and $v$ are two vertices that belong to the same component of $G$, then $\Gamma_u$ and $\Gamma_v$ are conjugate subgroups of $\Gamma$; for if $c$ is a $u$-$v$ walk, then $\Gamma_v = [\varphi(c)]^{-1}\Gamma_u[\varphi(c)]$. Thus the index of $\Gamma_v$ in $\Gamma$ is independent of $v$ if the pseudograph $G$ is connected. The following theorem, which relates the index of $\Gamma_v$ to the components of $G \times \varphi \Gamma$, is the voltage version of a theorem originally proved for current.

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
As the proof of the original version is easily modified to apply to voltage pseudographs, no details are given here.

VI.1.1 Theorem: Given a connected voltage pseudograph $(G,\varphi)$, with values in $\Gamma$, the number of components of the covering graph $G \times \varphi \Gamma$ equals the index of $\Gamma_v$ in $\Gamma$ for any vertex $v$ of $G$.

In a series of papers ([3, 12, 14, 15, 16, 17]) J. Gross et al. have shown that many interesting embeddings can be constructed by "lifting" embeddings of pseudographs to their covering pseudographs. More precisely, let $(G,\varphi)$ be a voltage pseudograph with values in $\Gamma$. If $e$ is an arc of $G$ at $v$, then for any $g \in \Gamma$ we denote the lift of $e$ at $(v,g)$ by $e^g$. For any rotation system $P$ of $G$ we define the lift $P^\varphi$ of $P$ to $G \times \varphi \Gamma$ by specifying that if $P_v(e) = f$, then

$$P^\varphi_{(v,g)}(e^g) = f^g.$$

The relationship between the embeddings defined by $P^\varphi$ and $P$ is an example of a branched covering projection. For our purposes here it is sufficient to say that the map $p:S \rightarrow S$ is a branched covering projection if there exists a discrete set $B$ of points of $S$ such that the restriction

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
\[ p: \tilde{S} \to p^{-1}(B) \to S - B \]

is a covering projection. The points of \( B \) are the branch points. If \( b \) is a branch point, then for some sufficiently small open neighborhood \( U \) of \( b \), the restricted map \( p: \tilde{U} \to U - \{b\} \) is \( n \)-fold, where \( n \) is some cardinal and \( \tilde{U} \) is a component of \( p^{-1}(U - \{b\}) \) in \( \tilde{S} \). We refer to \( n \) as the multiplicity of branching at \( b \). For example, the map \( z \to z^3 \) defines a branched covering projection of the extended complex plane onto itself with the branch points 0 and \( \infty \); the multiplicity of branching is 3 at both branch points. For more details the reader is referred to [1] and [10]. The following notation will prove helpful in trying to describe the location of branch points. If \( R \) is a region of the embedding of \( G \) on \( S \) induced by the rotation system \( P \), and \( \varphi \) is a voltage assignment from \( G \) to \( \Gamma \), then \( |R|_\varphi \) is the order of \( \varphi(c) \) in \( \Gamma \), where \( c \) is the closed walk in \( G \) consisting of the boundary of \( R \). It is easily verified that \( |R|_\varphi \) is independent of the specific orientation of \( R \) and of the initial vertex of \( c \). The following theorem summarizes information in [3, 15, 16] and shows that the regions of \( G \times \Sigma_T \) are in fact easily computed.
VI.1.2 Theorem (Gross & Alpert): Let \((G,\varphi,\Gamma)\) be a voltage pseudograph with a rotation system \(P\), and let \(P^\varphi\) be the lift of \(P\) to \(G \times^\varphi \Gamma\). Let \(P\) and \(P^\varphi\) determine embeddings of \(G\) and \(G \times^\varphi \Gamma\) on \(S\) and \(S^\varphi\) respectively. Assume further that \(|R|_\varphi\) always has finite order. Then there exists a branched covering projection \(p:S^\varphi \to S\) such that

\begin{enumerate}
  \item[a)] \(p^{-1}(G) = G \times^\varphi \Gamma\);
  \item[b)] if \(b\) is a branch point of \(p\) of multiplicity \(n\), then \(b\) is in the interior of a region \(R\) such that \(|R|_\varphi = n\);
  \item[c)] if \(R\) is a region of \(G\) which is a \(k\)-gon, the \(p^{-1}(R)\) has \(\frac{|\Gamma|}{|R|_\varphi}\) components, each of which is a \(k|R|_\varphi\)-gon.
\end{enumerate}

We now show that this construction can be extended to generalized embedding schemes as well. Again \((G,\varphi,\Gamma)\) is a voltage pseudograph with the generalized embedding scheme \((P,\lambda)\). Let \(P^\varphi\) be the lift of \(P\) to \(G \times^\varphi \Gamma\). In addition, define \(\lambda^\varphi:D(G \times^\varphi \Gamma) \to Z_2\) by setting \(\lambda^\varphi(\hat{e}) = \lambda(e)\) for any lift \(\hat{e}\) of an arc \(e\) of \(G\). We define \((P^\varphi,\lambda^\varphi)\) as the lift of \((P,\lambda)\) to \(G \times^\varphi 1\).
VI.1.3 Theorem: Let \((G, \varphi, \Gamma)\) be an irreducible voltage pseudograph with the generalized embedding scheme \((P, \lambda)\), and let \((P^\varphi, \lambda^\varphi)\) denote the lift of \((P, \lambda)\) to \(G \times \varphi \Gamma\).

If \((P, \lambda)\) and \((P^\varphi, \lambda^\varphi)\) determine embeddings of \(G\) and \(G \times \varphi \Gamma\) on \(S\) and \(S^\varphi\) respectively, then there exists a branched covering projection \(p : S^\varphi \rightarrow S\) such that

a) \(p^{-1}(G) = G \times \varphi \Gamma\);

b) If \(b\) is a branch point of multiplicity \(n\) then \(b\) is in the interior of a region \(R\) such that \(|R|_\varphi = n\);

c) if \(R\) is a region of \(G\) which is a \(k\)-gon, then \(p^{-1}(R)\) has \(|R|/|R|_\varphi\) components each of which is a \(k\)-\(|R|_\varphi\)-gon.

Proof: Let \(p_1 : \tilde{S} \rightarrow S\) and \(p_2 : \tilde{S}^\varphi \rightarrow S^\varphi\) be twofold orientable covering projections. Then there exist lifted orientable embeddings of \(G \times \lambda Z_2\) and \((G \times \varphi \Gamma) \times \lambda \varphi Z_2\) on \(\tilde{S}\) and \(\tilde{S}^\varphi\) respectively. A voltage assignment \(\tilde{\varphi} : D(G \times \lambda Z_2) \rightarrow \Gamma\) is defined by setting \(\tilde{\varphi}(\tilde{e}) = \varphi(e)\) whenever \(\tilde{e}\) is the lift of an arc \(e\) of \(G\). Let \(\tilde{p} : \tilde{S} \rightarrow \tilde{S}\) be the branched covering projection whose existence is guaranteed by Theorem VI.1.2 (here the voltage pseudograph is \((G \times \lambda Z_2, \tilde{\varphi}, \Gamma)\)). Thus we have an
orientable embedding of $G_1 = (G \times \lambda Z_2) \times_{\varphi} \Gamma$ on $\bar{S}$ and an orientable embedding of $G_2 = (G \times \varphi \Gamma) \times_{\lambda \varphi} Z_2$ on $\bar{S}^\varphi$.

However, these two pseudographs have an obvious isomorphism $\Psi$ which carries a vertex $((v,i),g)$ of $G_1$ into the vertex $((v,g),i)$ of $G_2$ (see Figure VI.1). This isomorphism, moreover, conforms with their embeddings on $\bar{S}$ and $\bar{S}^\varphi$. To see this we note that both embeddings are orientable and hence it suffices to show that $\Psi$ preserves the rotations. Both rotation systems for $G_1$ and $G_2$, however, are lifted from $G$ as described in Figure VI.1, and it is clear from this

![Diagram](image-url)

Figure VI.2
figure that \( \varphi \) does indeed preserve rotations. Note that the values of \( \lambda \) are not inserted into this figure. Hence \( \varphi \) may be considered as a homeomorphism from \( \widetilde{S} \) onto \( \widetilde{S}^\varphi \). We combine all the above maps into Figure VI.2, where the maps are maps of pairs \((A,B)\) with \( A \supset B \), and define \( p = p_1 \circ \widetilde{\varphi} \circ \varphi^{-1} \circ p_2^{-1} \). Note that if \( R \) is any region of \( G \times \varphi \Gamma \) on \( S^\varphi \), then \( p_2^{-1}(R) \) consists of two regions which are again identified by \( p_1 \circ \widetilde{\varphi} \circ \varphi^{-1} \). We illustrate with an example. Suppose \( G = K_4 \) with \( \lambda : D(G) \to \mathbb{Z}_2 \) and \( \varphi : D(G) \to \mathbb{Z}_4 \) as indicated in Figure VI.2. To avoid confusion we use the symbols

![Diagram](image)

**Figure VI.3**

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
"+" and "-" to denote the elements of $\mathbb{Z}_2$ with the first denoting the identity element. Now the region $R$ of $G_{x^r}$ with boundary

$$(u,0) \sim (v,1) \sim (w,3) \sim (u,2) \sim (v,3) \sim (w,1) \sim (u,0)$$

lifts to two regions on $\tilde{S}^\varphi$ with boundaries

$$((u,0),+) - ((v,1),-) - ((w,3),+) - ((u,2),+) - ((v,3),-) - ((w,1),+)$$

and

$$((u,0),-) - ((v,1),+) - ((w,3),-) - ((u,2),-) - ((v,3),+) - ((w,1),-).$$

The map $\tilde{s}^{-1}$ maps these regions to the following regions of $\tilde{S}$:

$$((u,+),0) - ((v,-),1) - ((w,-),3) - ((u,-),2) - ((v,+),3) - ((w,-),1).$$

and

$$((u,-),0) - ((v,+),1) - ((w,-),3) - ((u,-),2) - ((v,+),3) - ((w,-),1).$$

Next, $\tilde{p}$ maps these regions onto the regions of $\tilde{S}$

$$(u,+)-(v,-)-(w,+)) \quad \text{and} \quad (u,-)-(v,+)-(w,-).$$

Finally, the effect of $p_1$ is to map both of these regions onto the region $u-v-w$ of $G$ on $S$.

Thus the map $p$ is well defined. That $p$ does indeed possess properties a, b, and c follows from
the fact that \( \tilde{p} \) possesses the analogous properties for the orientable case. This too is well illustrated by the above example.

q.e.d.

The surface \( S^\varphi \) need not be non-orientable (see Example VI.1.5 below) a definition we give a rule for determining the orientability character of \( S^\varphi \). Given a voltage graph \( (G, \eta, \Gamma) \), we say that the closed walk \( c \) of \( G \) is \( \eta \)-trivial if \( \eta(c) \) is the identity element of \( \Gamma \).

VI.1.4 Theorem: Under the hypotheses of Theorem VI.1.3, the derived surface \( S^\varphi \) is orientable if and only if every \( \varphi \)-trivial closed walk in \( G \) is also \( \lambda \)-trivial.

Proof: We know from Theorem V.3.4 that \( S^\varphi \) is orientable if and only if every closed walk of \( Gx^\Gamma \) is \( \varphi \)-trivial. Now, suppose \( S^\varphi \) is orientable, and let \( c \) be a \( \varphi \)-trivial closed walk of \( G \). Then \( c \) lifts to a circuit \( c^\varphi \) of the same length as \( c \), in \( Gx^\varphi \Gamma \). Since \( S^\varphi \) is orientable it follows that \( c^\varphi \) must be \( \lambda \)-trivial. Consequently, \( c \) itself must be \( \lambda \)-trivial. The converse is proved in a similar manner.

q.e.d.
VI.1.5 Example: Figure VI.4 exhibits an embedding of a pseudograph $G$ with one vertex and $m/2$ loops ($m = 2 \pmod{4}$) in the projective plane (diametrically opposite points on the circumference of the circle are identified). It follows from the discussion on pp. 131-132 that $\lambda = 1$ on every edge of $G$. Now define a voltage $\varphi$ of $1 \in \mathbb{Z}_n^*$ (n even) on each arc of $G$ in the direction indicated by the arrowhead in the figure. Here $G_{x,\varphi}^n$ is a graph with the vertices $\{(v,i) | i = 0,1,\ldots,n-1\}$ in which $(v,i)$ and $(v,i+1)$ are joined by $m/2$ edges for each $i = 0,1,\ldots,n-1$. For each region $R$ of $G$, 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
we have $|R|_\varphi = n/2$. Hence the regions of $G^{x}_\varphi n$ are all $n$-gons (by Theorem VI.1.3). In fact, it is easily verified that the sequence of vertices along the boundary of each region of $G^{x}_\varphi n$ is $(v,0) - (v,1) - ... - (v,n-1)$. Thus the boundary of each region of $G^{x}_\varphi n$ is a hamiltonian cycle in the sense that it contains each vertex of $G^{x}_\varphi n$ exactly once. There are $(m/2)2 = m$ such regions in the derived embedding of $G^{x}_\varphi n$. Now place a new vertex in the interior of each such region, join it by non-intersecting edges to all the vertices on its boundary, and delete all the original edges of $G^{x}_\varphi n$. The result is a quadrilateral embedding of $K(m,n)$. This device originates in [13]. Now, because $n$ is even and each arc of $G$ carries a voltage of $\pm 1 \in Z^n$, and moreover, for each arc of $G \lambda = 1 \in Z_2$, it is clear that every $\varphi$-trivial closed walk of $G$ is also $\lambda$-trivial. Thus we have obtained an orientable quadrilateral embedding of $K(m,n)$, where $n$ is even and $m = 2 \pmod 4$. This, of course, is not new. The first such embedding was given in [30]. The next example, since it displays a heretofore unknown embedding, is presented as a theorem.

**VI.1.6 Theorem:** $\bar{\gamma}(k(n,n,n-2)) = (n-2)^2$ for $n \geq 3$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Proof: Let $G$ be the multigraph which consists of $n$ edges joining two vertices. Suppose $G$ is embedded in the projective plane, as described in Figure VI.5 (with diametrically opposed points on the circumference identified). This embedding clearly consists of $n - 2$

![Figure VI.5]

2-gons and one quadrilateral. Let $\varphi : D(G) \rightarrow \mathbb{Z}_n$ be the voltage assignment given in the figure. It is clear that $G \times \mathbb{Z}_n = K(n,n)$, and that for any region $R$ of $G$ we
have $|R| = 1$ or $n$ according as $R$ is a quadrilateral or a 2-gon. Thus the lifted embedding of $G^\varphi \times Z_n^\varphi$ on $S^\varphi$ consists of $n$ quadrilaterals and $n - 2$ 2n-gons (the 2n-gons lift the 2-gons of $G$). It is easily verified that the boundary of each 2n-gon is in fact a hamiltonian cycle of $G^\varphi \times Z_n^\varphi$; thus by placing a new vertex inside each 2n-gon and joining it to all the vertices of $G^\varphi \times Z_n^\varphi$ we obtain an embedding of $K(n,n,n-2)$ on $S^\varphi$. To see that $S^\varphi$ is in fact non-orientable, observe that if $n$ is odd then the closed walk which consists of $n$ repetitions of the cycle $u \overset{1}{\rightarrow} v \overset{0}{\leftarrow} u$ is $\varphi$ trivial but not $\lambda$-trivial; if $n$ is even then the circuit

$$u \overset{\frac{n}{2}-1}{\rightarrow} v \overset{n-1}{\leftarrow} u \overset{\frac{n}{2}}{\rightarrow} v \overset{0}{\leftarrow} u$$

is also $\varphi$-trivial but not $\lambda$-trivial. Thus, in either case $S^\varphi$ is non-orientable. We apply the Euler-Poincaré formula to derive the genus of $S^\varphi$. The graph $G^\varphi \times Z_n^\varphi$ has $2n$ vertices, $n^2$ edges, and $2n - 2$ regions on $S^\varphi$. Hence

$$2n - n^2 + (2n-2) = 2 - \tilde{\gamma}(S^\varphi)$$

or,

$$\tilde{\gamma}(S^\varphi) = (n-2)^2.$$
On the other hand, it follows from Lemma 8.2 of [34] that any embedding of $K(n,n,n-2)$ contains at most $2n(n-2)$ triangular regions. Since this is exactly the number of triangular regions in the above embedding of $k(n,n,n-2)$ on $S^\phi$, and, since all the remaining regions are quadrilateral, it follows that $\gamma(K(n,n,n-2)) = (n-2)^2$. q.e.d.

A slight modification of the above construction yields a genus formula for $K(n,n,n-4)$ when $n$ is even.

**VI,1.7 Theorem:** $\gamma(K(n,n,n-4)) = n^2 - 5n + 6$ for $n \geq 4$ and $n$ even.

**Proof:** We use the same embedding as in the previous theorem but we change the voltage assignment $\phi$ as indicated in Figure VI.6 (here $m = n/2$). Each of the $n-4$ 2-gons but $R_2$ and $R_3$ lifts to a single $2n$-gon whose boundary is a hamiltonian cycle of $G_\phi^xZ_n$. Since $|R_2|_\phi = |R_3|_\phi = 2$, $R_2$ and $R_3$ (as well as $R_1$) lift to quadrilaterals on $S^\phi$. This embedding is again easily modified into an embedding of $K(n,n,n-4)$ with $2n(n-4)$ triangular and $n + n/2 + n/2 = 2n$ quadrilateral regions. The rest of the proof will be omitted here as well as in subsequent theorems, since
it does not differ materially from the conclusion of the proof of the previous theorem.

q.e.d.

We digress here to construct several orientable genus embeddings. All but one of these in fact give rise to new genus formulae.
Figure VI.7 exhibits a plane embedding of the above multigraph $G$ together with a voltage assignment $\phi: \Delta(G) \to \mathbb{Z}_n$. Here $G \times_{\phi} \mathbb{Z}_n = K(n,n)$ and the derived embedding has $n$ regions, each of which is bounded by a Hamiltonian cycle. Again, this embedding is easily modified to produce a triangular embedding $K(n,n,n)$. Such embeddings have of course been previously constructed in [29] and [35]. A slight modification of this construction yields the following theorem.

**VI.1.8 Theorem:** \( \gamma(K(n,n,n-2)) = \frac{(n-2)^2}{2} \) if $n$ is even and $n \geq 2$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Proof: We set \( m = n/2 \) and modify the previous voltage assignment as indicated in Figure VI.8. All the 2-gons except \( R_1 \) and \( R_2 \) lift to 2n-gons of \( G_\varphi Z_n \) with hamiltonian boundaries. On the other hand, \( |R_1|_\varphi = |R_2|_\varphi = 2 \) and so they lift to \( m \) quadrilaterals each. Thus the derived embedding of \( G_\varphi Z_n = K(n,n) \) can be modified into an embedding of \( K(n,n,n-2) \) with \( (n-2)2n \) triangular and \( n \) quadrilateral regions. That this embedding is orientable follows from the fact that we started out with an orientable embedding. Again an application of Lemma 8.2 of [34] shows that this is in fact a genus embedding.

\( \text{q.e.d.} \)

Figure VI.8
VI.1.9 Theorem: \( \gamma(K(2n,2n,\ldots,n)) = \frac{(3n - 2)(n - 1)}{2} \) for \( n \geq 1 \).

Proof: If \( n \) is odd then it is possible to list the elements of \( Z_{2n} \) as:

\[ 0, 1, n+1, n+2, 2, 3, n+3, n+4, \ldots, n-3, n-2, 2n-2, 2n-1, n-1, n. \] \( (8) \)

Again we use the multigraph \( G \) with \( 2n \) edges joining 2 vertices and embed it in the plane. We assign the elements of \( Z_{2n} \) to the arcs of \( G \) starting with the first arc on the left, and proceeding to the right, and making the assignments in the order indicated by the above sequence. Figure VI.9 exhibits this assignment for the

![Figure VI.9](image-url)
case $n = 7$. It is easily verified that for $n$ of the 2-gons we have $|R|_\phi = 2n$, while for the other $n$ 2-gons we have $|R|_\phi = 2$. Hence $G^x_\phi Z_{2n}$ has $n^2$ quadrilaterals and $n$ regions with hamiltonian boundaries. This is easily transformed into an embedding of $K(2n, 2n, n)$ with $n^2$ quadrilaterals and $4n^2$ triangles. A straightforward computation shows that the derived surface has the required genus.

If $n$ is even we replace sequence (8) above by

$$0, 1, n+1, n+2, 2, 3, n+3, n+4, \ldots \quad n-4, n-3, 2n-3, 2n-2, n-2.$$  

From this point the proof proceeds as above.

q.e.d.

Section VI.2

Self-dual Embeddings

We now show that the self-dual embeddings constructed in Chapters III and IV have a rather simple description in terms of lifted embedding schemes.

The terminology and notation of Lemma III.2.1 is used in discussing the orientable case, and it is assumed that $G_\Delta(\Gamma)$ is self-dually embedded on $\bar{S}$. For each vertex $v$ of $G_\Delta(\Gamma)$ define $\Delta_v$ to be the set of all the
elements of $\Gamma$ associated with the arcs at $v$. Since $\Delta(\Gamma)$ is a Cayley graph it is clear that $\Delta_v$ is in fact independent of the particular vertex $v$. Let $P$ be the rotation system which describes this orientable embedding of $\Delta(\Gamma)$ on $\bar{S}$. It is possible to regard $P_v$ as a cyclic permutation of $\Delta_v$. Since each $P_v$ is a lift of the single rotation which defines the embedding of $G$ on $S$, it follows that $P_v$ is also independent of $v$. Thus the rotation $P$ is easily determined by lifting the rotation on $G$ to $\Delta(\Gamma) = G \times \rho_0 \Gamma$.

In the non-orientable case attention must be focused on the voltage assignment $\lambda$. It will be convenient to describe a scheme $(\tilde{T}, \tilde{\lambda})$ for the dual $\Delta^*(\Gamma)$ of $\Delta(\Gamma)$. The terminology and notation used here are those of the proof of Lemma IV.1.1. In view of the identification on pp. 34-37 and Theorem V.4.2 it is clear that for any edge $e$ of $H$ (the dual of $G$), $\tilde{\lambda}(e) = 0$ if and only if $e$ is the dual of a boundary loop of $G$ on $S$; in other words, if and only if $e$ has singular voltage. Now, if $(T, \lambda)$ is any scheme describing the embedding of $H$ on $S$, then we lift $T$ to $\tilde{T}$ on $\Delta^*(\Gamma) = H \times \rho_0 \Gamma$. We now use Theorem V.4.2 to determine the corresponding $\tilde{\lambda}$. The rotation system $\tilde{T}$ on $\Delta^*(\Gamma) = [\Delta(\Gamma)]^*$ defines an orientation of the regions of
$G_\Delta(\Gamma)$ on $\tilde{S}$. In fact, this orientation is explicitly defined by stating that it simply lifts the orientation that $T$ determines for the unique region of $G$ on $S$. However, it is clear from Figure II.9 on p.34 and the argument on p.65-68 that the non-coherence set for this orientation consists of all the edges of the Cayley graph $G_\Delta^*(\Gamma)$ with which a non-singular element of $\Gamma$ is associated. Thus, if $\tilde{\lambda}$ denotes the lift of $\lambda$ to the edges of $G_\Delta^*(\Gamma)$, then the scheme $(\tilde{T},\tilde{\lambda})$ does indeed describe the embedding of $G_\Delta^*(\Gamma)$ on $\tilde{S}$. This scheme $(\tilde{T},\tilde{\lambda})$, however, is exactly that constructed in Theorem VI.1.3.

Thus we see that the self-dual embeddings constructed in Chapters III and IV are special cases of Theorem VI.1.3. This theorem, however, provides no obvious way for demonstrating the self-duality. Such self-duality was demonstrated by A. T. White in [37] using methods reminiscent of the Poincaré duality. This method, however, does not seem to generalize to the non-orientable case. Finally, we note that in view of Theorem VI.1.4 and the above discussion, it is possible to strengthen Lemma IV.1.4 as follows.

VI.2.1 Theorem: Under the hypotheses of Lemma IV.1.4, $\overline{S}$ is non-orientable if $p^{-1}(F)$ contains at least one
odd cycle.

**Proof:** Let \( (\tilde{T}, \tilde{\lambda}) \) be as above; then \( p^{-1}(F^*) \), where \( F^* \) is the dual of \( F \), consists of all those edges of \( G_{\Delta^*}(\Gamma) \) which correspond to non-singular elements of \( \Gamma \); i.e., all the edges \( e \) for which \( \lambda(e) = 1 \). Hence, in view of Theorem V.3.4, \( \overline{S} \) is non-orientable if \( p^{-1}(F^*) \) contains an odd cycle. As was shown in the proof of Lemma IV.1.4, \( p^{-1}(F^*) \cong p^{-1}(F) \), and this concludes the proof.

q.e.d.
Bibliograph


Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.


32. Ringel, G., "Der vollstandige paare Graph auf nicht-orientierbaren Flachen." Journal Fur Mathematik. 88-93.


