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On the Theory of Hamiltonian Graphs

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ON THE THEORY OF HAMILTONIAN GRAPHS

by

Linda M. Lesniak

A Dissertation
Submitted to the
Faculty of The Graduate College
in partial fulfillment
of the
Degree of Doctor of Philosophy

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Linda M. Lesniak
For my family,
whose love and encouragement kept me smiling,
and for Joanna,
who understands.
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CHAPTER I

PRELIMINARIES

In this initial chapter, we present some background on the problem we are about to investigate and outline some of the topics treated in the chapters which follow. We also introduce some of the basic definitions pertinent to this dissertation and establish some notation.

Section 1.1

Introduction

Over a century ago, Sir William Hamilton introduced a simple game involving the existence of a cycle passing through all the vertices of a dodecahedron, which in recent years has led to the development of one of the most interesting areas of graph theory. A graph \( G \) that possesses a cycle containing all of the vertices of \( G \) is called a hamiltonian graph and such a cycle is referred to as a hamiltonian cycle of \( G \). The basic unsolved problem of determining which graphs are hamiltonian has been the impetus for a great amount of research in many diverse directions. The concept of a hamiltonian-connected graph was introduced by Ore in 1963. A path of a graph \( G \) containing every vertex of \( G \) is called a hamiltonian path.
and $G$ is said to be hamiltonian-connected if for each pair of distinct vertices $u$ and $v$ of $G$, there exists a hamiltonian $u-v$ path. If $G$ is a hamiltonian-connected graph with at least three vertices, then each of its edges belongs to some hamiltonian cycle of $G$. A graph with this property that each of its edges belongs to a hamiltonian cycle is called strongly hamiltonian and such graphs have been the objects of study. A variety of other avenues of investigation have stemmed from these basic definitions of hamiltonian graph, strongly hamiltonian graph, and hamiltonian-connected graph.

The primary purpose of this dissertation is to study hamiltonian properties related to elementary transformations in a graph, namely the deletion or addition of vertices or edges. The effect of such changes will be investigated for non-hamiltonian graphs as well as for hamiltonian graphs.

In Chapter II, we consider vertex-deleted subgraphs of a graph. An upper bound is obtained on the number of edges in a hamiltonian graph $G$ with the property that the removal of any vertex from $G$ results in a non-hamiltonian graph and a characterization is given for graphs attaining this upper bound in the case that the number of vertices is even. At the other end of the spectrum are the $n$-hamiltonian graphs which are those graphs having the property that the removal of any $n$ or fewer vertices
results in a hamiltonian graph. The definition of "n-hamiltonian" is extended to include negative values of n and then sufficient conditions are presented for n-hamiltonian graphs for an arbitrary integer n. Finally, graphs which are in some sense "nearly n-hamiltonian", for positive values of n, are classified.

Among the results of Chapter III are several conditions sufficient to insure that the removal of an arbitrary set of n edges from a graph results in a graph possessing one or more of the properties of being hamiltonian, strongly hamiltonian, and hamiltonian-connected, i.e. n-edge hamiltonian, strongly n-edge hamiltonian, and n-edge hamiltonian-connected, respectively. The property of being strongly hamiltonian is unlike that of being hamiltonian or hamiltonian-connected in the following sense. If G is hamiltonian (hamiltonian-connected), then the addition of any edge to G results in a graph which is also hamiltonian (hamiltonian-connected). However, although the addition of an edge to a graph which is not strongly hamiltonian may result in a strongly hamiltonian graph, it is the case that the addition of an edge to a strongly hamiltonian graph may destroy this property. Therefore, Chapter III also includes results which stem from this observation, some of which indicate new relationships between hamiltonian-connected and strongly hamiltonian graphs.
In recent years, the relationship between the line graph function and the properties of being hamiltonian and hamiltonian-connected has been investigated. Chapter IV basically deals with three aspects of hamiltonian theory arising from the consideration of the line graph function. First, graphs whose line graphs are strongly hamiltonian or 1-edge hamiltonian are characterized in terms of the existence of certain circuits in the graphs and the line graph function is shown to preserve these properties. We then prove with two results that, with suitable restrictions on a graph $G$, the second iterated line graph of $G$ is $n$-hamiltonian, where $n$ is an arbitrary nonnegative integer. Finally, as a corollary of these results it is shown that if $G$ is a connected graph which is neither a path, a cycle, nor the graph $K(1,3)$ and $n$ is a positive integer, then there exists a nonnegative integer $k$ such that the $m$-th iterated line graph of $G$ is $n$-hamiltonian for all $m \geq k$. Furthermore, for graphs $G$ belonging to several classes of graphs, we determine the least integer $k$ such that the $m$-th iterated line graph of $G$ is 1-hamiltonian for all $m \geq k$.

If $G$ is a strongly hamiltonian graph, then for all vertices $u$ and $v$ of $G$ for which the distance in $G$ between $u$ and $v$, denoted $d_G(u,v)$, does not exceed one, there exists a hamiltonian cycle $C$ of $G$ such that $d_C(u,v) = d_G(u,v)$. This property can be extended by
considering those graphs G and integers n satisfying
0 \leq n \leq \text{diam } G \text{ such that for all vertices } u \text{ and } v \text{ of } G \text{ for which } d_G(u,v) \leq n, \text{ there is a hamiltonian cycle } C \text{ of } G \text{ where } d_C(u,v) = d_G(u,v). \text{ Such graphs are }

studied in Chapter V and are called n-distant hamiltonian graphs. Among the results presented are a sufficient condition for a graph G to be diam G-distant hamiltonian, a theorem characterizing (p/2)-distant hamiltonian graphs for graphs of even order p, and several theorems pertaining to n-distant hamiltonian line graphs.

Section 1.2

Basic Definitions and Notation

In this section we present some basic definitions and notation that will be used throughout the dissertation. Additional more specialized definitions will be given later as required. Definitions of all other terms will be consistent with [2].

The order of a graph G is the number of elements in the vertex set of G and the size of G is the number of elements in the edge set of G. If v is a vertex of G, then the degree of v in G is denoted by deg_G(v), or simply deg v if the graph G is clear from context. A nondecreasing sequence d_1, d_2, \ldots, d_p of nonnegative integers is the degree sequence of a graph G having

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order $p$ if the vertices of $G$ can be labeled $v_1, v_2, \ldots, v_p$ so that $\deg v_i = d_i$, for $i = 1, 2, \ldots, p$. In this case, $d_1$ is called the minimum degree of $G$ and is denoted by $\delta(G)$; the maximum degree of $G$, namely $d_p$, is denoted by $\Delta(G)$.

If $S$ is a proper subset of the vertex set of a graph $G$, then $G - S$ denotes the (vertex-deleted) subgraph of $G$ with vertex set $V(G) - S$ and whose edges are all those edges of $G$ incident with no vertex in $S$. In the case that $S$ contains a single vertex $u$, we often employ the notation $G - u$. Similarly, if $F$ is a subset of $E(G)$, then $G - F$ denotes the (edge-deleted) subgraph of $G$ having vertex set $V(G)$ and edge set $E(G) - F$. If $F$ contains a single edge $e$, it is customary to write $G - e$.

If $u$ and $v$ are (not necessarily distinct) vertices of a graph $G$, then a $u - v$ walk in $G$ is an alternating sequence of vertices and edges of $G$ beginning with $u$, ending with $v$, and such that each edge is incident with the vertices immediately preceding and succeeding it. A $u - v$ walk is open if $u \neq v$ and closed if $u = v$. A $u - v$ trail is a $u - v$ walk in which no edge is repeated and a $u - v$ path is a $u - v$ walk in which no vertex is repeated. A nontrivial closed trail of $G$ is called a circuit of $G$ and a cycle is a circuit in which the intermediate vertices are not
repeated. A path (cycle) is said to have length \( t \) if the path (cycle) has \( t \) edges. In a connected graph \( G \), the distance between vertices \( u \) and \( v \) of \( G \), denoted \( d_G(u,v) \), is the length of a shortest \( u - v \) path in \( G \). The diameter of a connected graph \( G \), denoted \( \text{diam} \ G \), is defined by \( \text{diam} \ G = \max_{u,v \in V(G)} d_G(u,v) \).

An eulerian trail of a connected graph \( G \) is an open trail of \( G \) containing all the edges of \( G \), while an eulerian circuit of \( G \) is a circuit of \( G \) containing all the edges of \( G \). A graph possessing an eulerian circuit is called an eulerian graph. The following well-known characterizations of eulerian graphs and graphs with eulerian trails are frequently used in Chapters IV and V.

A. Let \( G \) be a nontrivial connected graph. Then \( G \) is eulerian if and only if every vertex of \( G \) is even.

B. Let \( G \) be a nontrivial connected graph. Then \( G \) contains an eulerian trail if and only if \( G \) has exactly two odd vertices.

The connectivity of a graph \( G \), denoted \( \kappa(G) \), is the minimum number of vertices whose removal from \( G \) results in a disconnected or trivial graph. The graph \( G \) is said to be \( n \)-connected, \( n \geq 1 \), if \( \kappa(G) \geq n \). The edge-connectivity of \( G \), denoted \( \kappa_1(G) \), is the minimum number of edges whose removal from \( G \) results in a
disconnected or trivial graph. A bridge \( e = u_1u_2 \) of a
graph \( G \) is a nonterminal bridge if \( \deg u_i \geq 2 \), for
\( i = 1,2 \).

A graph \( G \) is called complete \( n \)-partite, \( n \geq 2 \),
if it is possible to partition \( V(G) \) into \( n \) subsets
\( V_1, V_2, \ldots, V_n \), called partite sets, such that
\( uv \in E(G) \) if and only if there are distinct integers \( i \) and \( j \)
with \( u \in V_i \) and \( v \in V_j \). For \( n = 2 \) and \( n = 3 \),
such graphs are referred to as complete bipartite and com-
plete tripartite, respectively. If \( |V_i| = p_i \) for each
\( i \), where \( 1 \leq i \leq n \), then the complete \( n \)-partite graph
is denoted by \( K(p_1, p_2, \ldots, p_n) \). Let \( G \) and \( H \) be dis-
joint graphs. The union of \( G \) and \( H \), denoted \( G \cup H \),
is the graph for which \( V(G \cup H) = V(G) \cup V(H) \) and
\( E(G \cup H) = E(G) \cup E(H) \). The join of \( G \) and \( H \), de-
noted \( G + H \), is the graph for which \( V(G + H) =
V(G) \cup V(H) \) and \( E(G + H) = E(G) \cup E(H) \cup X \), where
\( X = \{uv | u \in V(G) \text{ and } v \in V(H)\} \). The union and join of
a finite number of pairwise disjoint graphs are defined
analogously. The union of \( n \) graphs, each of which is
isomorphic to a graph \( G \), is sometimes denoted by \( nG \).

Finally, we write \( G \cong H \) to denote that \( G \) and \( H \)
are isomorphic graphs. If there exists a label-preserving
isomorphism between two labeled graphs \( G \) and \( H \), then
\( G \) and \( H \) are identical and we write \( G \equiv H \).

As a matter of convenience for the reader, the
symbol ⊙ is used to designate the end of a proof.
CHAPTER II

VERTEX-RELATED HAMILTONIAN PROPERTIES

We are now prepared to begin our study of hamiltonian concepts. In this chapter we investigate hamiltonian properties and vertex-deleted subgraphs of certain hamiltonian graphs as well as of certain non-hamiltonian graphs.

Section 2.1

Critically Hamiltonian Graphs

Although every hamiltonian graph $G$ contains a spanning cycle, this need not be the case in $G - v$ for $v \in V(G)$. In fact, any cycle illustrates that there exist hamiltonian graphs for which no vertex-deleted subgraph contains a hamiltonian cycle. A graph $G$ is called critically hamiltonian if $G$ is hamiltonian but for each vertex $v$ of $G$, the graph $G - v$ is not hamiltonian. Thus every cycle is a critically hamiltonian graph. It is known (see [5]) that if a graph $G$ of order $p \geq 4$ has size at least $(p^2 - 3p + 8)/2$, then $G$ is hamiltonian and $G - v$ is hamiltonian for each vertex $v$ of $G$. Therefore the size of a critically hamiltonian graph of order $p \geq 4$ is clearly less than
Theorems 2-1 and 2-3 of this section will show that this bound can be substantially improved.

In order to present these theorems, we first prove the following lemma.

**Lemma 2-1.** If $G$ is a non-hamiltonian graph of order $p \geq 3$ which contains a hamiltonian path $v_1, v_2, \ldots, v_p$, then $\deg v_1 + \deg v_p \leq p - 1$. Moreover, if $\deg v_1 + \deg v_p = p - 1$, then the following must hold:

1. If $v_k v_p \notin E(G)$, $2 \leq k \leq p$, then $v_{k-1} v_p \in E(G)$.
2. If $p$ is even and $\deg v_1 \geq \deg v_p$, then $v_1 v_{t-1}$ and $v_1 v_{t+1}$ are edges of $G$ for some $t$ satisfying $2 \leq t \leq p - 2$.

**Proof.** That $\deg v_1 + \deg v_p \leq p - 1$ follows from a result in [20, p. 55]. Suppose $\deg v_1 + \deg v_p = p - 1$. Define the sets $V_1$ and $V_2$, where

$$V_1 = \{v_i \mid 1 \leq i \leq p - 1 \text{ and } v_1 v_{i+1} \in E(G)\}$$

and

$$V_2 = \{v_i \mid 1 \leq i \leq p - 1 \text{ and } v_1 v_{i+1} \notin E(G)\}.$$ 

Then $V_1 \cup V_2 = \{v_1, v_2, \ldots, v_{p-1}\}$ and $V_1 \cap V_2 = \emptyset$, implying that $|V_1| + |V_2| = p - 1$. Moreover, $|V_1| = \deg v_1$. Since $\deg v_1 + \deg v_p = p - 1$, we conclude that $\deg v_p = |V_2|$. Now, $v_p$ is adjacent to no vertex.
of $V_1$; for otherwise, there is an integer $i$
$(1 \leq i \leq p - 1)$ such that $v_i v_{i+1}, v_p v_i \in E(G)$. But
then $G$ contains the spanning cycle

$$v_1, v_2, \ldots, v_i, v_p, v_{p-1}, \ldots, v_{i+1}, v_1$$

which is a contradiction. Therefore $v_p$ is adjacent
only to vertices of $V_2$. Since $\deg v_p = |V_2|$, the
vertex $v_p$ is adjacent to every vertex of $V_2$. Thus if $v_k v_k \notin E(G)$, $2 \leq k \leq p$, we have that $v_{k-1} \in V_2$ so
that $v_{k-1} v_p \in E(G)$.

In order to prove part (ii) we note that the result
holds for $p = 4$. Thus we assume that $p$ is even and
$p \geq 6$. Since $\deg v_1 \geq \deg v_p$ and $\deg v_1 + \deg v_p = p - 1$, we have that $\deg v_1 \geq (p - 1)/2$, which im-
plies that $\deg v_1 \geq p/2$. Therefore $v_1$ is adjacent to
at least $p/2$ vertices in the set $\{v_2, v_3, \ldots, v_{p-1}\}$
since $v_1 v_p \notin E(G)$. But $|\{v_2, v_3, \ldots, v_{p-1}\}| = p - 2$, so that $v_1$ is adjacent to both $v_l$ and $v_{l+1}$ for some
$l$ satisfying $2 \leq l \leq p - 2$.

The first theorem of this section gives an upper
bound on the size of a critically hamiltonian graph of
even order in terms of the order of the graph. The cor-
ollary which follows is an immediate consequence of the
method of proof employed in the theorem.
Theorem 2-1. If $G$ is a critically hamiltonian graph of even order $p \geq 4$, then $|E(G)| \leq p^2/4$.

Proof. Let $C: u_1'u_2'\ldots'u_p'u_1'$ be a hamiltonian cycle of $G$. For each $i$, with $1 \leq i \leq p$, we wish to consider the sum $\deg_G u_i + \deg_G u_{i+2}$ (all subscripts expressed modulo $p$). Since $G$ is critically hamiltonian, $G - u_{i+1}$ is not hamiltonian. However, $G - u_{i+1}$ contains a hamiltonian $u_i - u_{i+2}$ path. Using Lemma 2-1 we obtain the inequality

$$\deg_{G-u_{i+1}} u_i + \deg_{G-u_{i+1}} u_{i+2} \leq (p - 1) - 1 = p - 2.$$ 

Therefore, $\deg_G u_i + \deg_G u_{i+2} \leq p$. We thus conclude that

$$p/2 \sum_{i=1}^{p/2} (\deg_G u_{2i-1} + \deg_G u_{2i+1}) \leq (p/2)p$$

and

$$p/2 \sum_{i=1}^{p/2} (\deg_G u_{2i} + \deg_G u_{2i+2}) \leq (p/2)p.$$ 

By adding corresponding sides of these inequalities we obtain

$$2 \sum_{i=1}^{p} \deg_G u_i \leq p^2.$$ 

Since $2 \sum_{i=1}^{p} \deg_G u_i = 4|E(G)|$, the proof is complete. $\diamondsuit$
Corollary 2-1-1. If $G$ is a critically hamiltonian graph of even order $p \geq 4$ and size $p^2/4$ and $C: u_1, u_2, \ldots, u_p, u_1$ is a hamiltonian cycle of $G$, then $\deg u_i + \deg u_{i+2} = p$ (all subscripts expressed modulo $p$) for $1 \leq i \leq p$.

Since, for even $p \geq 4$, the graph $K(p/2, p/2)$ is a critically hamiltonian graph of order $p$ and size $p^2/4$, the bound given in Theorem 2-1 is best possible. As the next theorem indicates, $K(p/2, p/2)$ is the unique critically hamiltonian graph which attains this bound.

Theorem 2-2. A graph $G$ of order $p \geq 4$ and size $p^2/4$ is critically hamiltonian if and only if $G$ is isomorphic to $K(p/2, p/2)$.

Proof. We first observe that if a graph has order $p$ and size $p^2/4$, then $p$ is even. As noted above, the graph $K(p/2, p/2)$ is a critically hamiltonian graph of order $p$ and size $p^2/4$, for even $p \geq 4$.

Conversely, let $G$ be a critically hamiltonian graph of order $p \geq 4$ and size $p^2/4$. If $p = 4$, then $G$ is isomorphic to $K(2, 2)$. So we may assume that $p \geq 6$.

Let $C: z_1, z_2, \ldots, z_p, z_1$ be a hamiltonian cycle of $G$ and let $i$ be an arbitrary integer satisfying $1 \leq i \leq p$. We will show that $\{w \in V(G) | z_i w \in E(G)\} = \{w \in V(G) | z_{i+2} w \in E(G)\} = \{z_{i+3}, z_{i+5}, \ldots, z_{i-1}, z_{i+1}\}$, where all
Subscripts are expressed modulo $p$. Relabel the vertices of $G$ in the following fashion:

$$u_1 = z_{i+2}, u_2 = z_{i+3}, u_3 = z_{i+4}, \ldots, u_{p-1} = z_i, u_p = z_{i+1}.$$ 

Then $u_1, u_2, \ldots, u_p, u_1$ is a Hamiltonian cycle of $G$. We wish to show that \[ \{w \in V(G) | u_{p-1}w \in E(G)\} = \{w \in V(G) | u_1w \in E(G)\} = \{u_2, u_4, u_6, \ldots, u_p\}. \]

We note that $u_1u_p, u_{p-1}u_p \in E(G)$.

Since $G$ is critically Hamiltonian, $G - u_p$ is not Hamiltonian. However, $G - u_p$ contains a Hamiltonian $u_1 - u_{p-1}$ path. Applying Lemma 2-1, we obtain the inequality $\deg_G u_1 + \deg_G u_{p-1} \leq (p - 1) - 1$. However, we must have $\deg_G u_1 + \deg_G u_{p-1} = (p - 1) - 1$; for otherwise, $\deg_G u_1 + \deg_G u_{p-1} < p$, which contradicts Corollary 2-1. Since $G$ is critically Hamiltonian,

$$u_1u_3, u_{p-3}u_{p-1} \notin E(G).$$

Therefore $u_1u_3, u_{p-3}u_{p-1} \notin E(G - u_p)$. By two applications of Lemma 2-1 (i), we have that $u_2u_{p-1}, u_1u_{p-2} \in E(G - u_p)$. Thus $u_2u_{p-1}, u_1u_{p-2} \in E(G)$.

We now observe that if $u_1u_\ell \in E(G)$ for some $\ell$ satisfying $2 \leq \ell \leq p - 2$, then $u_1u_{\ell+1} \notin E(G)$; for otherwise, $u_2, u_3, \ldots, u_\ell, u_1u_{\ell+1}, \ldots, u_{p-1}u_2$ is a Hamiltonian cycle of $G - u_p$, which contradicts the fact that $G$ is critically Hamiltonian. Similarly, if $u_{p-1}u_\ell \in E(G)$ for some $\ell$ satisfying $2 \leq \ell \leq p - 2$, then

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For otherwise, \( u_p, u_{p-1}, \ldots, u_1 \) is a Hamiltonian cycle of \( G - u_p \).

Since \( G \) is critically Hamiltonian, \( u_1 \) is adjacent to exactly \( \deg_G u_1 - 1 \) vertices in the set \( \{u_2, u_3, \ldots, u_{p-2}\} \) and \( u_{p-1} \) is adjacent to exactly \( \deg_G u_{p-1} - 1 \) vertices in the set \( \{u_2, u_3, \ldots, u_{p-2}\} \).

Using the observations made above together with the fact that \( \deg_G u_1 + \deg_G u_{p-1} = p \), we conclude that \( \deg_G u_1 = \deg_G u_{p-1} = \frac{p}{2} \) and

\[
\{w \in V(G) | u_{p-1}w \in E(G)\} = \{w \in V(G) | u_1w \in E(G)\} = \{u_2, u_4, u_6, \ldots, u_p\}. \]

We now turn our attention to critically Hamiltonian graphs of odd order. Initially, we note that the proof of Theorem 2-1 essentially does not depend upon \( p \) being even. Thus for critically Hamiltonian graphs \( G \) of odd order \( p \geq 5 \), the inequality \(|E(G)| \leq \frac{p^2}{4}\) holds. The following theorem shows that this bound can be improved.

**Theorem 2-3.** If \( G \) is a critically Hamiltonian graph of odd order \( p \geq 5 \), then \(|E(G)| \leq \lfloor p(p - 1)/4 \rfloor \).

**Proof.** Since the only critically Hamiltonian graph of order five is the 5-cycle \( C_5 \) and \(|E(C_5)| = 5 = \lfloor 5(5 - 1)/4 \rfloor \), we will assume \( p \geq 7 \). Let \( C: z_1, z_2, \ldots, z_p, z_1 \) be a Hamiltonian cycle of \( G \) and let \( i \) be an
arbitrary integer satisfying $1 \leq i \leq p$. We consider the sum $\deg_G z^*_i + \deg_G z^*_i + 2$ (all subscripts expressed modulo $p$). Without loss of generality, we may assume that $\deg_G z^*_i + 2 \geq \deg_G z^*_i$. Relabel the vertices of $G$ in the following fashion:

$$u_1 = z^*_i + 2, \quad u_2 = z^*_i + 3, \quad u_3 = z^*_i + 4, \ldots, \quad u_{p-1} = z^*_i, \quad u_p = z^*_i + 1.$$ 

Then $u_1, u_2, \ldots, u_p, u_1$ is a Hamiltonian cycle of $G$ and $\deg_G u_1 \geq \deg_G u_{p-1}$. We wish to show that $\deg_G u_1 + \deg_G u_{p-1} \leq p - 1$.

Since $G$ is critically Hamiltonian, $G - u_p$ is not Hamiltonian. However, $G - u_p$ contains a Hamiltonian $u_1 - u_{p-1}$ path. Using Lemma 2-1, we obtain the inequality $\deg_{G-u_p} u_1 + \deg_{G-u_p} u_{p-1} \leq (p - 1) - 1 = p - 2$. Thus $\deg_G u_1 + \deg_G u_{p-1} \leq p$. Suppose $\deg_G u_1 + \deg_G u_{p-1} = p$. Then $\deg_{G-u_p} u_1 + \deg_{G-u_p} u_{p-1} = (p - 2) = (p - 1) - 1$ and $\deg_{G-u_p} u_1 \geq \deg_{G-u_p} u_{p-1}$. Since $|V(G - u_p)|$ is even, we may apply parts (i) and (ii) of Lemma 2-1. By part (ii), we have that $u_1 u_{t}, u_1 u_{t+1} \notin E(G - u_p)$ for some $t$ satisfying $2 \leq t \leq (p - 1) - 2$, so that $u_1 u_{t}, u_1 u_{t+1} \notin E(G)$. Since $G$ is critically Hamiltonian, $u_1 u_3 \notin E(G)$. Thus $u_1 u_{t}, u_1 u_{t+1} \notin E(G - u_p)$ for some $t$ satisfying $4 \leq t \leq p - 3$. Now, since $u_1 u_3 \notin E(G)$, $u_1$ and $u_3$ are not adjacent in $G - u_p$. Therefore, by part (i), $u_2$ and $u_{p-1}$ are adjacent in $G - u_p$. 

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G - u_p. But then u_2, u_3, \ldots, u_t, u_1, u_{t+1}, \ldots, u_{p-1}, u_2 is a spanning cycle of G - u_p, which is a contradiction. Therefore we must have \( \deg_G u_1 + \deg_G u_{p-1} \leq p - 1 \).

In terms of the original labeling of the vertices of G, we have the inequality \( \deg_G z_{i+2} + \deg_G z_i \leq p - 1 \). Since \( i \) was arbitrary,

\[
\sum_{i=1}^{p} (\deg_G z_{2i-1} + \deg_G z_{2i+1}) \leq p(p - 1).
\]

Equivalently, \( 2 \sum_{i=1}^{p} \deg_G z_i \leq p(p - 1) \). Since \( 2 \sum_{i=1}^{p} \deg_G z_i = 4|E(G)| \), the proof is complete. \( \diamond \)

We close the discussions of this section by remarking that the bound given in Theorem 2-3 is best possible for \( p = 5 \) and \( p = 7 \). However, for larger values of odd \( p \), this does not seem to be the case.

Section 2.2

\( n \)-Hamiltonian Graphs

In Section 2.1 we considered the subclass of Hamiltonian graphs with the property that the removal of any vertex of a graph in this subclass resulted in a non-Hamiltonian graph. It is also natural to consider the related
class of hamiltonian graphs with the property that the removal of any vertex of one of these graphs results in a hamiltonian graph. A generalization of this concept was, in fact, suggested by Hamilton himself (see [1, p. 265]) and n-hamiltonian graphs were first studied by Chartrand, Kapoor, and Lick [5].

A graph $G$ of order $p \geq 3$ is called $n$-hamiltonian, $0 \leq n \leq p - 3$, if the removal of any $m$ vertices, $0 \leq m \leq n$, results in a hamiltonian graph. The $0$-hamiltonian graphs are then simply the hamiltonian graphs and the $(p-3)$-hamiltonian graphs of order $p \geq 3$ are the complete graphs. Several sufficient conditions for $n$-hamiltonian graphs are known, the most recent being due to Chvátal [8].

**Theorem of Chvátal.** If the degree sequence $d_1, d_2, \ldots, d_p$ of a graph $G$ satisfies $0 \leq n \leq p - 3$ and

$$d_k \leq k + m < (p + m)/2 = d_{p-m-k} > p - k \quad (0 \leq m \leq n)$$

then $G$ is $n$-hamiltonian.

As the following lemma indicates, the hypothesis of Chvátal's theorem can be simplified.

**Lemma 2-2.** If the degree sequence $d_1, d_2, \ldots, d_p$ of a graph $G$ satisfies $0 \leq n \leq p - 3$ and
\[ d_k \leq k + n < (p + n)/2 = d_{p-n-k} \geq p - k, \]

then \( G \) is \( n \)-hamiltonian.

**Proof.** We first note that \( \delta(G) \geq n + 2 \); for otherwise, there is a vertex \( u \) of \( G \) such that \( \deg u \leq 1 + n \).

Thus \( d_1 \leq 1 + n \). Since \( n < p - 2 \), we have that \( 1 + n < (p + n)/2 \). Therefore, by hypothesis, \( d_{p-n-1} \geq p - 1 \). This implies that the set \( W \) of all vertices of degree \( p - 1 \) has order at least \( n + 2 \). Since \( \deg u \leq 1 + n < p - 1 \), the vertex \( u \) is not in \( W \) and so \( u \) is adjacent to every vertex in \( W \). But then \( \deg u \geq n + 2 \), which presents a contradiction.

We now show that the degree sequence of \( G \) satisfies the hypothesis of Chvátal's theorem. This will imply that \( G \) is \( n \)-hamiltonian.

Suppose \( d_k \leq k + m < (p + m)/2 \), where \( 0 \leq m \leq n \).

Let \( t = k + m - n \), so that \( t \geq d_k - n \geq 2 \) and \( t \leq k \).

Then

\[ d_t \leq d_k \leq k + m = t + n \quad \text{and} \]

\[ t + n = k + m < (p + m)/2 < (p + n)/2. \]

Combining these inequalities we obtain

\[ d_t \leq t + n < (p + n)/2. \]

Thus, by hypothesis, \( d_{p-n-t} \geq p - t \). Since
\[ t + n = k + m \text{ and } t \leq k, \] we conclude that \[ d_{p-m-k} \geq p - k. \]

If a graph \( G \) is \( n \)-hamiltonian, \( n \geq 1 \), then \( G \) has the properties that \( G \) is also \((n-1)\)-hamiltonian and for each vertex \( v \) of \( G \), the graph \( G - v \) is \((n-1)\)-hamiltonian. Moreover, the graph \( G + w \), obtained from \( G \) by adding a vertex \( w \) which is adjacent to all vertices of \( G \), is \((n+1)\)-hamiltonian. A graph \( G \) of order \( p \geq 3 \) will be called \( n \)-hamiltonian, \(-p \leq n \leq -1\), if there exist \(-n\) or fewer pairwise disjoint paths in \( G \) which collectively span \( G \). According to this definition, the \((-1)\)-hamiltonian graphs are precisely those graphs containing a hamiltonian path, which is perhaps a natural "step down" from 0-hamiltonian graphs. It is readily apparent that if \( G \) is an \( n \)-hamiltonian graph of order \( p \) and \(-p + 1 \leq n \leq 0\), then \( G \) is also \((n-1)\)-hamiltonian. If \(-p + 2 \leq n \leq 0\), then the graph \( G - v \) is \((n-1)\)-hamiltonian for each vertex \( v \) of \( G \) and, finally, if \(-p \leq n \leq 0\), then the graph \( G + w \) is \((n+1)\)-hamiltonian.

The first theorem of this section presents a sufficient condition for \( n \)-hamiltonian graphs in the case that \( n \) is negative and is an analogue to Lemma 2-2.

**Theorem 2-4.** If the degree sequence \( d_1, d_2, \ldots, d_p \) of a graph \( G \) satisfies \(-p \leq n \leq -1\) \((p \geq 3)\) and

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\[ d_k \leq k + n < (p + n)/2 = d_{p-n-k} \geq p - k, \]

then \( G \) is \( n \)-hamiltonian.

**Proof.** Let \( H \) be the graph obtained from \( G \) by adding 
\(-n\) mutually adjacent vertices, each of which is adjacent to all vertices of \( G \). Then \( H \) has order \( p' = p - n \) and the degree sequence of \( H \) satisfies \( d_i' = d_i - n \), for \( 1 \leq i \leq p \), and \( d_i' = p' - 1 \), for \( p + 1 \leq i \leq p' \).

We wish to apply Chvátal's theorem to the degree sequence of \( H \) to show that \( H \) is hamiltonian.

Suppose \( d_k' \leq k < p'/2 \). Since \( d_i' = p' - 1 > p'/2 \) for \( p + 1 \leq i \leq p' \), we must have that \( 1 \leq k \leq p \).

Thus \( d_k' = d_k - n \) and, since \( p' = p - n \), we have that
\[ d_k - n \leq k < (p - n)/2 \]
so that
\[ d_k \leq k + n < (p + n)/2. \]

Thus by hypothesis, \( d_{p-n-k} \geq p - k \). Therefore,
\[ d_{p-n-k}' = d_{p-n-k} - n \geq (p - k) - n = p' - k. \]

Replacing \( p - n - k \) with \( p' - k \) we obtain \( d_{p'-k}' \geq p' - k \). Thus we may apply Chvátal's theorem to conclude that \( H \) is hamiltonian and so contains a hamiltonian cycle. However, this implies the existence of \(-n\) or fewer pairwise disjoint paths in \( G \) which span \( G \). \( \Diamond \)
Corollary 2-4-1. If the degree sequence $d_1, d_2, \ldots, d_p$ of a graph $G$ satisfies $-p \leq n \leq p - 3$ ($p \geq 3$) and

$$d_k \leq k + n < (p + n)/2 = d_{p-n-k} \geq p - k,$$

then $G$ is $n$-hamiltonian.

The most easily applied sufficient condition for a graph $G$ of order $p \geq 3$ to be $n$-hamiltonian, $n \geq 0$, is that every vertex of $G$ has degree at least $(p + n)/2$ (see [5]). Consider a graph $G$ with at least three vertices such that $\delta(G) \geq (p + n)/2$, with $-p \leq n \leq -1$. Then $G$ vacuously satisfies the hypothesis of Theorem 2-4 so that $G$ is $n$-hamiltonian. We therefore have a second corollary to Theorem 2-4.

Corollary 2-4-2. Let $G$ be a graph of order $p \geq 3$ and let $-p \leq n \leq p - 3$. If $\delta(G) \geq (p + n)/2$, then $G$ is $n$-hamiltonian.

We now show that if $n$ and $p$ are integers of opposite parity satisfying $-p + 3 \leq n \leq -2$, then there is a graph $G$ of order $p$ such that $G$ is $n$-hamiltonian by Theorem 2-4 but $G$ is not $(n+1)$-hamiltonian. Thus in a certain sense, Theorem 2-4 cannot be improved. Let $G$ consist of $-n$ components, $-n - 1$ of which are trivial. To determine the other component, begin with a copy of $K_{(p+n+1)/2} + \overline{K}_{(p+n+1)/2}$ where $K_m$, in general, denotes...
the complete graph of order \( m \) and \( \overline{K_m} \) its complement.

Now delete a set of \( (p + n + 1)/2 \) mutually nonadjacent edges. Then the degree sequence of \( G \) satisfies \( d_k = 0 \), for \( 1 \leq k \leq -n - 1 \), \( d_k = (p + n - 1)/2 \), for \( -n \leq k \leq (p - n - 1)/2 \), and \( d_k = p + n - 1 \), for \( k \geq (p - n + 1)/2 \). We first observe that \( k + n < (p + n)/2 \) if and only if \( k \leq (p - n - 1)/2 \). Now, for \( 1 \leq k \leq -n - 1 \), we have \( d_k = 0 > k + n \) and for \( -n \leq k \leq (p - n - 3)/2 \), we have \( d_k = (p + n - 1)/2 > k + n \).

However for \( k = (p - n - 1)/2 \), we have \( d_k = (p + n - 1)/2 = k + n \). Moreover, \( d_{p-n-k} = d_{(p-n+1)/2} = p + n - 1 \geq (p + n + 1)/2 = p - k \). Therefore \( G \) is \( n \)-hamiltonian by Theorem 2-4. Clearly \( G \) is not \((n+1)\)-hamiltonian.

The maximum order among the induced regular subgraphs of degree zero of a graph \( G \) is referred to as the independence number of \( G \) and is denoted by \( \beta(G) \). In [9], Chvátal and Erdös presented the following sufficient condition, involving the connectivity \( \kappa(G) \) and the independence number \( \beta(G) \) of a graph \( G \), for \( G \) to be hamiltonian.

**Theorem of Chvátal and Erdös.** Let \( G \) be a graph of order \( p \geq 3 \). If \( \kappa(G) \geq \beta(G) \), then \( G \) is hamiltonian.

We now proceed to present three results, the last of which is a generalization of the above theorem to \( n \)-hamiltonian
graphs for all possible values of \( n \).

**Theorem 2-5.** Let \( G \) be a graph of order \( p \geq 3 \) and let \( 0 \leq n \leq p - 3 \). If \( \kappa(G) \geq \beta(G) + n \), then \( G \) is \( n \)-hamiltonian.

**Proof.** By the theorem of Chvátal and Erdös, \( G \) is hamiltonian. Thus we may assume that \( n \geq 1 \) and it suffices to show that if \( S \) is a set of vertices of \( G \) such that \( 1 \leq |S| \leq n \), then \( G - S \) is hamiltonian. Clearly, \( \kappa(G - S) \geq \kappa(G) - n \) and \( \beta(G - S) \leq \beta(G) \). Thus \( \kappa(G - S) \geq \kappa(G) - n \geq \beta(G) \geq \beta(G - S) \). Another application of the theorem of Chvátal and Erdös yields the desired result. \( \Box \)

We remark that the condition \( \kappa(G) \geq \beta(G) + n \) in Theorem 2-5 cannot be replaced by

\[(1) \quad \kappa(G) \geq \beta(G) + (n - 1), \]

as the following examples indicate. For \( n = 0 \), the graph \( K(m,m+1) \), with \( m \geq 1 \), satisfies (1) but is not hamiltonian. Next, for \( n = 1 \), the graph \( K(m,m) \), with \( m \geq 2 \), satisfies (1) but is not 1-hamiltonian. Finally, for \( n \geq 2 \), the complete tripartite graph \( K(n-1,m-n+1,m-n+1) \), with \( m \geq 2n - 2 \), satisfies (1) but is not \( n \)-hamiltonian.

A technique similar to that used in the proof of
Theorem 2-4 can be used to obtain a result analogous to Theorem 2-5 for negative values of $n$.

**Theorem 2-6.** Let $G$ be a graph of order $p \geq 3$ and let $-p \leq n \leq -1$. If $\kappa(G) \geq \beta(G) + n$, then $G$ is $n$-hamiltonian.

**Proof.** Let $H$ be that graph obtained from $G$ by adding $-n$ mutually adjacent vertices, each of which is adjacent to all vertices of $G$. Then $\kappa(H) = \kappa(G) + |n|$ and $\beta(H) = \beta(G)$. Therefore $\kappa(H) = \kappa(G) - n \geq \beta(G) = \beta(H)$.

By the aforementioned result of Chvátal and Erdős, $H$ contains a hamiltonian cycle. This implies the existence of $-n$ or fewer pairwise disjoint paths in $G$ which span $G$, i.e. $G$ is $n$-hamiltonian. 

We have seen that for $n \geq 0$, the condition $\kappa(G) \geq \beta(G) + (n - 1)$ does not imply that $G$ is $n$-hamiltonian. The situation is analogous for negative values of $n$.

For example, if $n \leq -1$ and $G$ is a graph isomorphic to $K(m,m+|n|+1)$, with $m \geq 2$, then $\kappa(G) = m$ and $\beta(G) = m + |n| + 1$. Therefore $\kappa(G) = \beta(G) + (n - 1)$. However, $G$ is not $n$-hamiltonian.

Combining the results of Theorems 2-5 and 2-6, we obtain a corollary.

**Corollary 2-6-1.** Let $G$ be a graph of order $p \geq 3$ and let $-p \leq n \leq p - 3$. If $\kappa(G) \geq \beta(G) + n$, then $G$
is n-hamiltonian.

Section 2.3

Maximally Non-n-Hamiltonian Graphs

It is a trivial observation that if a graph $G$ is n-hamiltonian, then every supergraph of $G$ of order $|V(G)|$ is also n-hamiltonian. In Section 2.3 we consider those graphs which are not n-hamiltonian but have the property that every proper supergraph of the same order is n-hamiltonian. We note that $G$ is such a graph if and only if $G$ is not n-hamiltonian but the addition of any edge to $G$ results in an n-hamiltonian graph. Therefore, we make the following definition. If $G$ is any graph which is not n-hamiltonian and has the property that $G + uv$ is n-hamiltonian for each pair $u,v$ of nonadjacent vertices of $G$, then $G$ is a maximally non-n-hamiltonian graph. For $n = 0$, we will say that $G$ is maximally non-hamiltonian. The purpose of this section is to classify maximally non-n-hamiltonian graphs for positive values of $n$. We will first give some preliminary definitions and results.

In the course of the development of hamiltonian theory, one particular class of non-hamiltonian graphs has been singled out for investigation. A graph $G$ is hypo-hamiltonian if it is not hamiltonian but $G - v$ is hamiltonian for every vertex $v$ of $G$. Herz, Gaudin, and

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Rossi [14] showed that hypohamiltonian graphs exist and that the Petersen graph is the smallest (in terms of order) hypohamiltonian graph. We will call a graph \( G \) hypohamiltonian, \( n \geq 0 \), if \( G \) is not \( n \)-hamiltonian but \( G - v \) is \( n \)-hamiltonian for every vertex \( v \) of \( G \). Then the hypo-0-hamiltonian graphs are the hypohamiltonian graphs. The question of the existence of hypo-\( n \)-hamiltonian graphs for positive values of \( n \) immediately arises. At the present time, no hypo-\( n \)-hamiltonian graph, \( n \geq 1 \), is known to exist. However, the purpose of the definition of hypo-\( n \)-hamiltonian graphs will become apparent later in this section. Lemma 2-3 gives one necessary condition for a graph to be hypo-\( n \)-hamiltonian.

**Lemma 2-3.** If a graph \( G \) is hypo-\( n \)-hamiltonian, \( n \geq 0 \), then \( G \) is not hamiltonian.

**Proof.** If \( G \) is hypo-0-hamiltonian, then by definition \( G \) is not hamiltonian. So we may assume that \( n \geq 1 \).

Suppose, to the contrary, that \( G \) is hamiltonian. Since \( G \) is not \( n \)-hamiltonian, there exists a subset \( W \) of \( V(G) \), with \( 1 \leq |W| \leq n \), such that \( G - W \) is not hamiltonian. Let \( w \) be a vertex in \( W \). Since \( G \) is hypo-\( n \)-hamiltonian, the graph \( G - w \) is \( n \)-hamiltonian. Thus \( (G - w) - (W - \{w\}) \) is hamiltonian. But \( (G - w) - (W - \{w\}) = G - W \). Therefore \( G - W \) is hamiltonian, which presents a contradiction. \( \Box \)
It is an immediate consequence of Lemma 2-3 that if a graph $G$ is hypo-$n$-hamiltonian, then $G$ is hypo-$m$-hamiltonian for $0 \leq m \leq n$.

We now begin the process of developing a classification scheme for maximally non-$n$-hamiltonian graphs, $n \geq 1$. A maximally non-$n$-hamiltonian graph, $n \geq 1$, will be called a maximally non-$n$-hamiltonian graph of type I if it is not hamiltonian and will be called a maximally non-$n$-hamiltonian graph of type II, otherwise. Theorems 2-7 and 2-8 provide characterizations of maximally non-$n$-hamiltonian graphs of types I and II, respectively. At this point we recall two observations made in Section 2.2. If $G$ is $n$-hamiltonian, then $G - v$ is $(n-1)$-hamiltonian for every vertex $v$ of $G$. Moreover, the graph $G + w$, obtained from $G$ by adding a vertex $w$ which is adjacent to all vertices of $G$, is $(n+1)$-hamiltonian.

**Theorem 2-7.** A graph $G$ is maximally non-$n$-hamiltonian of type I if and only if $G$ is maximally non-hamiltonian and $G$ is hypo-$(n-1)$-hamiltonian ($n \geq 1$).

**Proof.** Let $G$ be a graph that is both maximally non-hamiltonian and hypo-$(n-1)$-hamiltonian. Since $G$ is not hamiltonian, $G$ is not $n$-hamiltonian. Let $v$ and $w$ be nonadjacent vertices of $G$. We wish to show that $G + vw$ is $n$-hamiltonian. Since $G$ is maximally non-hamiltonian, $G + vw$ is hamiltonian. Thus it suffices to show that the
removal of \( k \) vertices from \( G + vw \), with \( 1 \leq k \leq n \), results in a hamiltonian graph. Let \( Z \subseteq V(G + vw) \) with \( |Z| = k \) and let \( z \) be a vertex in the set \( Z \). Since \( G \) is hypo-(n-1)-hamiltonian, \( G - z \) is (n-1)-hamiltonian, implying that \( G - Z \) is hamiltonian. Since \( G - Z \) is a spanning subgraph of \((G + vw) - Z\), we conclude that \((G + vw) - Z\) is hamiltonian. Thus \( G \) is a maximally non-n-hamiltonian graph (of type I).

In order to verify the converse, we let \( G \) be a maximally non-n-hamiltonian graph of type I. Clearly \( G \) is maximally non-hamiltonian. We first show that \( G \) contains no vertex of degree \(|V(G)| - 1\). Suppose, to the contrary, that \( G \) contains a vertex \( z \) such that \( \deg_G z = |V(G)| - 1 \). Let \( u \) and \( w \) be nonadjacent vertices of \( G \). Since \( G \) is maximally non-n-hamiltonian and \( n \geq 1 \), the graph \((G + uw) - z\) contains a hamiltonian cycle. Moreover, since \( \deg_G z = |V(G)| - 1 \), we see that \( u \neq z \) and \( w \neq z \). Thus \((G + uw) - z \equiv (G - z) + uw\). Therefore \( G - z \) contains a hamiltonian path, implying that \( G \) contains a hamiltonian cycle (since \( \deg_G z = |V(G)| - 1 \)). This presents a contradiction so that \( G \) contains no vertex of degree \(|V(G)| - 1\). We now show that \( G \) is hypo-(n-1)-hamiltonian. Suppose, to the contrary, that \( G \) is not hypo-(n-1)-hamiltonian. Since \( G \) is not hamiltonian, this implies the existence of a vertex \( v \) of \( G \) such that \( G - v \) is not
(n-1)-hamiltonian. However, $\text{deg}_G v < |V(G)| - 1$ so that there is a vertex $u$ of $G$ such that $u$ and $v$ are nonadjacent vertices of $G$. But then $G + uv$ is n-hamiltonian, implying that $(G + uv) - v \equiv G - v$ is (n-1)-hamiltonian. We thus reach a contradiction, so that $G$ is hypo-(n-1)-hamiltonian. 

It was seen in the proof of Theorem 2-7 that if $G$ is a maximally non-n-hamiltonian graph of type I, $n \geq 1$, then $\Delta(G) < |V(G)| - 1$. We restate this as Corollary 2-7-1.

**Corollary 2-7-1.** If $G$ is a maximally non-n-hamiltonian graph of type I, $n \geq 1$, then $\Delta(G) < |V(G)| - 1$.

We now proceed to Theorem 2-8 for a characterization of maximally non-n-hamiltonian graphs of type II, $n \geq 1$.

**Theorem 2-8.** A graph $G$ is maximally non-n-hamiltonian of type II if and only if $G \cong M + z$ (for a vertex $z$), where $M$ is a maximally non-(n-1)-hamiltonian graph ($n \geq 1$).

**Proof.** Let $G$ be a maximally non-n-hamiltonian graph of type II. Since $G$ is hamiltonian and $G$ is not n-hamiltonian, there is a set $Z$ of vertices of $G$, with $1 \leq |Z| \leq n$, such that $G - Z$ is not hamiltonian. Let $z$ be a vertex in the set $Z$. We must have that
\( \deg_G z = |V(G)| - 1 \); for otherwise, there is a vertex \( u \) of \( G \) such that \( u \) and \( z \) are nonadjacent vertices of \( G \). Then \( G + uz \) is \( n \)-hamiltonian, implying that \( (G + uz) - Z \) is hamiltonian or, equivalently, that \( G - Z \) is hamiltonian, which presents a contradiction. We wish to show that \( G - z \) is maximally non-(\( n-1 \))-hamiltonian.

First, \( G - z \) is not (\( n-1 \))-hamiltonian; for otherwise, \( (G - z) + z \equiv G \) is \( n \)-hamiltonian, contrary to the hypothesis. Let \( w \) and \( v \) be nonadjacent vertices of \( G - z \). We now show that \( (G - z) + vw \) is (\( n-1 \))-hamiltonian. Since \( v \) and \( w \) are nonadjacent vertices of \( G - z \), they are also nonadjacent vertices of \( G \). Thus \( G + vw \) is \( n \)-hamiltonian, implying that \( (G + vw) - z \) is (\( n-1 \))-hamiltonian. However, since \( v \neq z \) and \( w \neq z \), we have that \( (G + vw) - z \equiv (G - z) + vw \).

If we let \( M \) be the graph \( G - z \), then \( M \) is maximally non-(\( n-1 \))-hamiltonian and \( G \equiv M + z \).

In order to verify the converse, we let \( M \) be a maximally non-(\( n-1 \))-hamiltonian graph and consider the graph \( M + z \). We wish to show that \( M + z \) is a maximally non-\( n \)-hamiltonian graph of type II. Since \( M \) is maximally non-(\( n-1 \))-hamiltonian and \( n - 1 \geq 0 \), the graph \( M \) contains a hamiltonian path so that \( M + z \) contains a hamiltonian cycle. Since \( M \) is not (\( n-1 \))-hamiltonian, \( M + z \) is clearly not \( n \)-hamiltonian. Let \( v \) and \( w \) be nonadjacent vertices of \( M + z \). We wish to show
that \((M + z) + vw\) is \(n\)-hamiltonian. We note that \(v \neq z\) and \(w \neq z\). Thus \(v\) and \(w\) are nonadjacent vertices of \(M\) and \((M + z) + vw = (M + vw) + z\). Since \(M\) is maximally non-(\(n-1\))-hamiltonian, \(M + vw\) is \((n-1)\)-hamiltonian so that \((M + vw) + z\) is \(n\)-hamiltonian. 

Since maximally non-\(m\)-hamiltonian graphs obviously exist for \(m \geq 0\), maximally non-\(n\)-hamiltonian graphs of type II exist for all positive values of \(n\). The situation is not clear for maximally non-\(n\)-hamiltonian graphs of type I. In the case \(n = 1\), the Petersen graph is an example of a maximally non-\(n\)-hamiltonian graph of type I since it is both maximally non-hamiltonian and hypohamiltonian. However, for \(n \geq 2\), the question of the existence of maximally non-\(n\)-hamiltonian graphs of type I remains open since an answer depends upon the existence of hypo-(\(n-1\))-hamiltonian graphs, where \(n - 1 \geq 1\).

As a first step in classifying maximally non-\(n\)-hamiltonian graphs for positive integers \(n\), we have divided all such graphs into two sets. We now consider maximally non-\(n\)-hamiltonian graphs of type II in detail. If \(G\) is a maximally non-\(n\)-hamiltonian graph of type II then since \(G\) is hamiltonian, there is a unique integer \(k\) satisfying \(0 \leq k \leq n - 1\) such that \(G\) is \(k\)-hamiltonian but \(G\) is not \((k+1)\)-hamiltonian. In this case, we will say that \(G\) is a maximally non-\(n\)-hamiltonian graph of type II\(_k\).
Theorem 2-9, which is actually a corollary of Theorem 2-8, indicates the relationship between maximally non-n-hamiltonian graphs of type $II_k$ and maximally non-$(n-1)$-hamiltonian graphs of type $II_{k-1}$ when $1 \leq k \leq n - 1$.

Theorem 2-9. Let $k$ and $n$ be integers satisfying $1 \leq k \leq n - 1$. Then $G$ is a maximally non-n-hamiltonian graph of type $II_k$ if and only if $G \cong M + z$ (for a vertex $z$), where $M$ is a maximally non-$(n-1)$-hamiltonian graph of type $II_{k-1}$.

Proof. Let $G$ be a maximally non-n-hamiltonian graph of type $II_k$. By Theorem 2-8, $G \cong M + z$, where $M$ is a maximally non-$(n-1)$-hamiltonian graph. Since $G$ is $k$-hamiltonian, $G - z (\cong M)$ is $(k-1)$-hamiltonian, where $k - 1 \geq 0$. Moreover, $M$ is not $k$-hamiltonian; for otherwise, $M + z (\cong G)$ is $(k+1)$-hamiltonian, which is a contradiction.

The converse follows from similar arguments. $\Diamond$

Theorem 2-9 provides one means of describing maximally non-n-hamiltonian graphs of type $II_k$ for values of $k$ and $n$ satisfying $1 \leq k \leq n - 1$. We next turn our attention to maximally non-n-hamiltonian graphs of type $II_0$ for values of $n \geq 2$.

Theorem 2-10. Let $n \geq 2$. Then $G$ is a maximally non-n-hamiltonian graph of type $II_0$ if and only if $G \cong M + z$
(for a vertex $z$), where $M$ is a maximally non-$(n-1)$-hamiltonian graph of type I.

**Proof.** Let $G$ be a maximally non-$n$-hamiltonian graph of type $\Pi_{10}^n$, $n \geq 2$. By Theorem 2-8, $G \cong M + z$, where $M$ is a maximally non-$(n-1)$-hamiltonian graph and $n - 1 \geq 1$. Clearly $M$ is not hamiltonian; for otherwise, $M + z$ ($\cong G$) is 1-hamiltonian, presenting a contradiction. Thus $M$ is a maximally non-$(n-1)$-hamiltonian graph of type I.

In order to verify the converse, we let $M$ be a maximally non-$(n-1)$-hamiltonian graph of type I and consider the graph $M + z$. By Theorem 2-8, $M + z$ is maximally non-$n$-hamiltonian. We wish to show that $M + z$ is hamiltonian but not 1-hamiltonian. Since $(M + z) - z \cong M$ and $M$ is not hamiltonian, it suffices to show that $M + z$ is hamiltonian. Since $M$ is maximally non-hamiltonian, $M$ contains a hamiltonian path so that $M + z$ contains a hamiltonian cycle. \(\diamondsuit\)

Using Theorems 2-7, 2-8, 2-9, and 2-10 we can now characterize maximally non-$n$-hamiltonian graphs of type I $(n \geq 1)$ and describe all maximally non-$n$-hamiltonian graphs $(n \geq 1)$ in terms of maximally non-hamiltonian graphs.

**A.** A graph $G$ is maximally non-$n$-hamiltonian of type I if and only if $G$ is maximally non-hamiltonian and $G$
is hypo-(n-1)-hamiltonian \( (n \geqslant 1) \).

B. Suppose \( 0 \leq k \leq n - 2 \). Then \( G \) is a maximally non-
\( n \)-hamiltonian graph of type II \(_k\) if and only if \( G \cong M + K_{k+1} \), where \( M \) is a maximally non-(n-k-1)-
hamiltonian graph of type I.

C. A graph \( G \) is maximally non-\( n \)-hamiltonian of type
II\(_{n-1}^\_\) if and only if \( G \cong M + K_n \), where \( M \) is a
maximally non-hamiltonian graph.

We note that if \( G \) is a maximally non-\( n \)-hamiltonian
graph of type II\(_k\) that is in category B, then \( G \) contains
exactly \( k + 1 \) vertices of degree \( |V(G)| - 1 \), since by
Corollary 2-7-1, the maximum degree of a maximally non-\( m \-
hamiltonian graph \( M \) of type I is less than \( |V(M)| - 1 \). If \( G \) falls into category C, then \( G \) contains at least
\( k + 1 = n \) vertices of degree \( |V(G)| - 1 \). However, \( G \)
may contain more than \( k + 1 \) vertices of degree \( |V(G)| - 1 \).
For example, let \( G \) be a graph isomorphic to \( K_{n+2} + \overline{K}_3 \),
for some \( n \geqslant 1 \). Then \( G \) is clearly isomorphic to
\( (K_2 + \overline{K}_3) + K_n \), where \( K_2 + \overline{K}_3 \) is a maximally non-
hamiltonian graph. Thus \( G \) is a maximally non-\( n \)-hamil-
tonian graph of type II\(_{n-1}^\_\). We note that \( G \) contains
\( n + 2 \) vertices of degree \( |V(G)| - 1 \).

In closing this discussion we remark that if it is
the case that hypo-\( n \)-hamiltonian graphs do not exist for

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values of $n \geq 1$, then the classification of maximally non-$n$-hamiltonian graphs ($n \geq 1$) can be made much more specific. In this case we can classify maximally non-$n$-hamiltonian graphs as follows.

A graph $G$ is maximally non-$1$-hamiltonian if and only if either

(i) $G$ is maximally non-hamiltonian and $G$ is hypohamiltonian, or

(ii) $G \cong M + K_1$, where $M$ is maximally non-hamiltonian.

A graph $G$ is maximally non-$n$-hamiltonian ($n \geq 2$) if and only if either

(i) $G \cong M + K_{n-1}^{
-1}$, where $M$ is maximally non-hamiltonian and $M$ is hypohamiltonian (i.e., $G$ is maximally non-$n$-hamiltonian of type $II_{n-2}$), or

(ii) $G \cong M + K_n^{
}$, where $M$ is maximally non-hamiltonian (i.e., $G$ is maximally non-$n$-hamiltonian of type $II_{n-1}$).
CHAPTER III

EDGE-RELATED HAMILTONIAN PROPERTIES

Next we consider graphs which retain various hamiltonian properties upon the removal of arbitrary sets of edges. We also discuss the effect (with respect to some of these same properties) produced by the addition of edges to a graph.

Section 3.1

n-Edge Hamiltonian Graphs

It was discovered by Dirac [12] in 1952 that if a graph \( G \) of order \( p \geq 3 \) has no vertices of degree less than \( p/2 \), then \( G \) is hamiltonian. Evidently, this was the first known sufficient condition for a graph to be hamiltonian. Since the complete bipartite graph \( K(s, s+1) \), \( s \geq 1 \), of order \( p = 2s + 1 \) has minimum degree \( s = (p - 1)/2 \) but is not hamiltonian, the simple minimum degree requirement of \( p/2 \) cannot be lowered. Therefore, as further sufficiency conditions (in terms of degrees) for hamiltonian graphs were developed, they took the form of allowing some vertices of small degree as long as other conditions were satisfied. For example, Pósa [22] showed that if a graph contains "only a few" vertices of small degree, then the graph is hamiltonian.
THEOREM OF PÓSA. Let $G$ be a graph of order $p \geq 3$ such that for every integer $j$ with $1 \leq j < p/2$, the number of vertices of degree not exceeding $j$ is less than $j$. Then $G$ is hamiltonian.

The first theorem of Section 3.1 shows that although we cannot improve Dirac's result in the sense of replacing "$\delta(G) \geq p/2$" with a lower minimum degree requirement, we can draw stronger conclusions about a graph of order $p \geq 7$ with minimum degree at least $p/2$. A preliminary definition is necessary. A graph $G$ of order $p \geq 3$ will be called $n$-edge hamiltonian, $0 \leq n \leq p - 3$, if the removal of any $n$ edges from $G$ results in a hamiltonian graph. We observe that if $G$ is $n$-edge hamiltonian, then $G$ is $m$-edge hamiltonian, $0 \leq m \leq n$.

**Theorem 3-1.** Let $p$ and $n$ be nonnegative integers satisfying $p \geq 4n + 3$. If $G$ is a graph of order $p$ such that $\delta(G) \geq p/2$, then $G$ is $n$-edge hamiltonian.

**Proof.** The case $n = 0$ is Dirac's theorem. Thus we may assume that $n \geq 1$. Let $E$ be an arbitrary set of $n$ edges of $G$. We wish to show that $G - E$ is hamiltonian. It suffices to show that if $j$ is an integer with $1 \leq j < p/2$, then the number of vertices of $G - E$ of degree not exceeding $j$ is less than $j$. Let $k$ denote the number of vertices of $G - E$ of degree not exceeding $j$. Then $k$ is less than $j$. But then $k$ is less than $j$. Therefore, $G - E$ is hamiltonian.
If \( k = 0 \), then \( k < j \). Thus we may assume that \( k \geq 1 \) and we label the vertices of \( G \) as \( v_1, v_2, \ldots, v_p \) so that for \( 1 \leq i \leq k \) we have \( p/2 - n \leq \deg_{G-E} v_i \leq j \). Then

\[
\sum_{i=1}^{k} (\deg_G v_i - \deg_{G-E} v_i) \leq 2n.
\]

Consider a function \( f \) defined for \( x \geq 1 \) by

\[
f(x) = \frac{4n}{x} + 2x - C,
\]

where \( C \) is an arbitrary constant. We observe that since \( f'(x) = \frac{-4n}{x^2} + 2 \), the function \( f \) is monotonically increasing for \( x \geq \sqrt{2n} \). By hypothesis, \( n < \frac{(p - 2)}{4} \) so that

\[
p - 2n > p - \frac{(p - 2)}{2} = \frac{(p + 2)}{2}.
\]

Therefore, \( \frac{p - 2n}{2} > \frac{(p + 2)}{4} = \frac{(p - 2)}{4} + 1 > n + 1 \). Since \( n + 1 > \sqrt{2n} \), we conclude that the function \( f \) is monotonically increasing for \( x \geq \frac{(p - 2n)}{2} \).

We now proceed in cases.

**Case 1.** Suppose \( p \) is even. Then necessarily,

\[
\frac{(p - 2n)}{2} \leq j \leq \frac{(p - 2)}{2}.
\]

Since \( \deg_G v_i \geq \frac{p}{2} \) and \( \deg_{G-E} v_i \leq j \), for \( 1 \leq i \leq k \), we have

\[
\deg_G v_i - \deg_{G-E} v_i \geq \frac{p}{2} - j \quad (1 \leq i \leq k).
\]

Thus,

\[
k(\frac{p}{2} - j) \leq \sum_{i=1}^{k} (\deg_G v_i - \deg_{G-E} v_i) \leq 2n
\]
so that

$$k \leq \frac{2n}{p/2 - j} = \frac{4n}{p - 2j}.$$ 

If \( p > 4n/j + 2j \), then \( p - 2j > 4n/j \) which implies that \( \frac{4n}{p - 2j} < j \). Therefore Case 1 will be complete once we have shown that \( p > 4n/j + 2j \).

Since \( (p - 2n)/2 \leq j \leq (p - 2)/2 \), by an earlier observation involving the monotonicity of \( f \), it remains only to show that

$$p > \frac{4n}{(p - 2)/2} + (p - 2).$$

However, since \( p > 4n + 2 \), we have \( (p - 2)/8n > 1/2 \), so that

$$8n/(p - 2) + p < 2 + p,$$

which implies that

$$p > \frac{4n}{(p - 2)/2} + (p - 2).$$

**Case 2.** Suppose \( p \) is odd. Then necessarily, \( \delta(G) \geq (p + 1)/2 \) and \( (p - 2n + 1)/2 \leq j \leq (p - 1)/2 \). Since \( \deg_{G_v_i} \geq (p + 1)/2 \) and \( \deg_{G-E_v_i} \leq j \), for \( 1 \leq i \leq k \), we have

$$\deg_{G_v_i} - \deg_{G-E_v_i} \geq (p + 1)/2 - j \ (1 \leq i \leq k).$$

Thus,

$$k((p + 1)/2 - j) \leq \sum_{i=1}^{k} (\deg_{G_v_i} - \deg_{G-E_v_i}) \leq 2n.$$
so that

\[ k \leq \frac{2n}{(p + 1)/2 - j} = \frac{4n}{p + 1 - 2j}. \]

If \( p > 4n/j + 2j - 1 \), then \( p + 1 - 2j > 4n/j \) which implies that \( \frac{4n}{p + 1 - 2j} < j \). Therefore Case 2 will be complete once we have shown that \( p > 4n/j + 2j - 1 \).

Since \( (p - 2n + 1)/2 \leq j \leq (p - 1)/2 \), by an earlier observation involving the monotonicity of \( f \), it remains only to show that

\[ p > \frac{4n}{(p - 1)/2} + (p - 1) - 1. \]

However, since \( p > 4n + 1 \), we have \( (p - 1)/8n > 1/2 \), so that

\[ 8n/(p - 1) + p < 2 + p, \]

which implies that

\[ p > \frac{4n}{(p - 1)/2} + (p - 1) - 1. \]

Let \( Q \) be the collection of all graphs with minimum degree at least half the order of the graph. Then Theorem 3-1 says that given any nonnegative integer \( n \), there is a number \( a(n) \), depending on \( n \), such that every graph in \( Q \) of order \( p \geq a(n) \) is \( n \)-edge hamiltonian. We cannot extend \( Q \) to \( Q^* \), by including hamiltonian graphs of odd order \( p \) with minimum degree \( (p - 1)/2 \), and obtain a similar result. For any positive
value of $n$, there exist graphs in $Q^*$ of arbitrarily large order that are not $n$-edge hamiltonian. This fact is illustrated by the graph of odd order $t$ obtained from $K([t/2],[t/2])$ by adding an edge between two vertices of degree $[t/2]$. Moreover, we cannot improve Theorem 3-1 by lowering $a(n)$ to $4n + 2$. For any positive value of $n$, the graph $\overline{K}_{2n} + ((n+1)K_2)$ has order $4n + 2$ and minimum degree $2n + 1 = (4n + 2)/2$. However, it is not $n$-edge hamiltonian.

According to Theorem 3-1, if $G$ is a graph of order $p \geq 3$ such that $\delta(G) \geq p/2$, then $G$ is $m$-edge hamiltonian for all $m$ satisfying $0 \leq m \leq [(p - 3)/4]$. We now consider sufficient conditions for a graph $G$ of order $p$ to be $n$-edge hamiltonian for arbitrary values of $n$ satisfying $0 \leq n \leq p - 3$. In [19], Ore showed that if $G$ is a graph of order $p \geq 3$ such that $\deg u + \deg v \geq p$ for all nonadjacent vertices $u$ and $v$ of $G$, then $G$ is hamiltonian, i.e. $G$ is 0-edge hamiltonian. In Theorem 3-2 we extend Ore's result to show that if $G$ is a graph of order $p \geq 3$ such that $\deg u + \deg v \geq p + n$ for all nonadjacent vertices $u$ and $v$ of $G$ ($0 \leq n \leq p - 3$), then $G$ is $n$-edge hamiltonian. We remark that, in general, this theorem does not follow as a corollary to Ore's theorem. For example, if $n$ and $p$ are integers satisfying $1 \leq n \leq p - 6$, consider the graph $G$ of order $p$ isomorphic to $K_{n+2} + (K_2 \cup K_{p-n-4})$. 

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Then \( G \) satisfies the hypothesis of Theorem 3-2. However, let \( u \) and \( v \) be adjacent vertices of \( G \) with \( \deg_G u = n + 3 = \deg_G v \). If \( E \) is any set of \( n \) edges of \( G \) such that each edge of \( E \) is incident with \( u \) and \( uv \in E \), then \( G - E \) does not satisfy the hypothesis of Ore's theorem since \( \deg_{G-E} u + \deg_{G-E} v = 3 + (n + 2) < p \).

Before proving Theorem 3-2, we make a preliminary observation. Suppose \( G \) is a graph of order \( p \geq 4 \) and \( F = \{e_1, e_2, \ldots, e_n\} \) is a set of \( n \) edges of \( G \), where \( 1 \leq n \leq p - 3 \). Let \( H \) be the graph defined by \( V(H) = V(G) \) and \( E(H) = F \). Then \( |E(H)| = \binom{p}{2} - n \geq \binom{p}{2} - (p - 3) = \frac{p^2 - 3p + 6}{2} \). It is easily verified that this bound on \( |E(H)| \) implies that \( \deg_H u + \deg_H v \geq p \) for all nonadjacent vertices \( u \) and \( v \) of \( H \). Therefore \( H \) is hamiltonian and so contains a hamiltonian cycle \( C: w_1, w_2, \ldots, w_p, w_{p+1} = w_1 \). Then the cyclic arrangement \( S: w_1, w_2, \ldots, w_p, w_{p+1} = w_1 \) of the vertices of \( G - F \) has the property that for \( 1 \leq j \leq n \), the two vertices incident with \( e_j \) in \( G \) are not consecutive vertices of \( S \).

**Theorem 3-2.** Let \( G \) be a graph of order \( p \geq 3 \) and let \( 0 \leq n \leq p - 3 \). If \( \deg u + \deg v \geq p + n \) for all pairs \( u, v \) of nonadjacent vertices, then \( G \) is \( n \)-edge hamiltonian.
Proof. For \( n = 0 \), the result holds by Ore's theorem. Thus we may assume that \( n \geq 1 \) so that \( p \geq 4 \). Let \( F = \{ e_1, e_2, \ldots, e_n \} \) be an arbitrary set of \( n \) edges of \( G \). We wish to show that \( G - F \) is hamiltonian. Assume, to the contrary, that \( G - F \) is not hamiltonian.

As noted above, there exists a cyclic arrangement \( S: w_1, w_2, \ldots, w_p, w_{p+1} = w_1 \) of the vertices of \( G - F \) which satisfies the following property \( P \).

(P). For \( 1 \leq j \leq n \), the two vertices incident with \( e_j \) in \( G \) are not consecutive vertices of the arrangement.

We will say there is a gap in \( S \) between \( w_t \) and \( w_{t+1} \) \( (1 \leq t \leq p) \) if \( w_t w_{t+1} \notin E(G - F) \).

Among all cyclic arrangements of the vertices of \( G - F \) which satisfy property \( P \), let

\[
S': v_1, v_2, \ldots, v_p, v_{p+1} = v_1\]

be a cyclic arrangement with a minimum number of gaps. Since \( G - F \) is not hamiltonian, we may assume, without loss of generality, that \( v_1 v_2 \notin E(G - F) \). This implies that \( v_1 v_2 \notin E(G) \); for otherwise, \( v_1 v_2 \in F \), which contradicts the manner in which \( S' \) was chosen.

If \( v_1 v_i \in E(G - F), 2 \leq i \leq p \), then \( v_2 v_{i+1} \notin E(G - F) \); for otherwise,

\[
T: v_1, v_i, v_{i-1}, \ldots, v_2, v_{i+1}, v_{i+2}, \ldots, v_p, v_{p+1} = v_1\]

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is a cyclic arrangement of the vertices of $G - F$ which satisfies property $P$ and which has at least one less gap than $S'$. Thus

$$\deg_{G-F}v_2 \leq (p - 1) - \deg_{G-F}v_1$$
or, equivalently,

$$\deg_{G-F}v_1 + \deg_{G-F}v_2 \leq p - 1.$$

Since $v_1v_2 \notin E(G)$,

$$\deg_Gv_1 + \deg_Gv_2 \leq (\deg_{G-F}v_1 + \deg_{G-F}v_2) + n \leq p + n - 1.$$ 

This contradicts the hypothesis of the theorem, so that $G - F$ is hamiltonian. \(\Box\)

**Corollary 3-2-1.** Let $G$ be a graph of order $p \geq 3$ and let $0 \leq n \leq p - 3$. If $\delta(G) \geq (p + n)/2$, then $G$ is $n$-edge hamiltonian.

The bound given in Theorem 3-2 is best possible. For integers $p$ and $n$ satisfying $0 \leq n \leq p - 3$, consider the graph $G$ of order $p$ obtained from the complete graph $K_{p-1}$ by adding a new vertex that is adjacent to exactly $n + 1$ vertices of $K_{p-1}$. Then the sum of the degrees of each pair of nonadjacent vertices of $G$ is $p + n - 1$; however, $G$ is not $n$-edge hamiltonian. We make one further observation. According to Theorem 3-1, every graph of order $p \geq 3$ that is hamiltonian by Dirac's theorem is, in fact, $m$-edge hamiltonian, where $m$ is an increasing function of $p$. The graph $G$ described
above for \( n = 1 \) indicates that an analogous result cannot be expected with respect to Ore's theorem.

The condition given in Theorem 3-2 for a graph to be \( n \)-edge Hamiltonian allows a graph \( G \) of order \( p \) to have a vertex \( u \) of degree less than \((p + n)/2\) as long as each vertex of \( G \) which is not adjacent to \( u \) has degree at least \( p + n - \deg u \). The following theorem gives a sufficient condition for a graph of order \( p \) to be \( n \)-edge Hamiltonian in terms of the total number of vertices of degree not exceeding an integer \( \ell \) \((n + 1 \leq \ell < (p + n)/2)\) and the number of vertices of degree at least \( p + n - \ell \). The special case \( n = 0 \) is a result due to Chvátal [8].

**Theorem 3-3.** If the degree sequence \( d_1, d_2, \ldots, d_p \) of a graph \( G \) satisfies \( 0 \leq n \leq p - 3 \) and

\[
d_k \leq k + n < (p + n)/2 = d_{p-n-k} \geq p - k,
\]

then \( G \) is \( n \)-edge Hamiltonian.

**Proof.** Assume that the theorem is false. Hence there exists a graph \( G \) which satisfies the hypothesis of the theorem but contains a set \( F \) of \( n \) edges whose removal from \( G \) results in a non-Hamiltonian graph. Since for \( p \geq 3 \), the graph \( K_p \) is \((p-3)\)-edge Hamiltonian (by Corollary 3-2-1), we may assume, without loss of generality, that the addition of any edge \( f \) to \( G \) creates a graph \( G + f \) with the property that \((G + f) - F \) is
Among all pairs of nonadjacent vertices of $G$, let $u_1$ and $u_p$ be two nonadjacent vertices such that $\deg_{G}u_1 + \deg_{G}u_p$ is maximum. Suppose $\deg_{G}u_1 \leq \deg_{G}u_p$. As we have already noted, the graph $(G + u_1u_p) - F$ is hamiltonian. Since $G - F$ is not hamiltonian, this implies that $u_1$ and $u_p$ are the endvertices of a path $P: u_1, u_2, \ldots, u_p$ in $G - F$ containing every vertex of $G - F$.

Now, if $u_1u_i \in E(G - F)$, $2 \leq i \leq p$, then $u_{i-1}u_p \notin E(G - F)$; for otherwise,

$$u_1, u_2, \ldots, u_{i-1}, u_p, u_{i-1}, \ldots, u_i, u_1$$

is a hamiltonian cycle of $G - F$. Thus

$$\deg_{G-F}u_p \leq (p - 1) - \deg_{G-F}u_1 \quad \text{or, equivalently,}$$

$$\deg_{G-F}u_1 + \deg_{G-F}u_p \leq p - 1.$$ 

Since $u_1u_p \notin E(G)$, we have

$$\deg_{G}u_1 + \deg_{G}u_p < p + n.$$ 

Therefore, $\deg_{G}u_1 < (p + n)/2$. Let $\deg_{G}u_1 = k + n$. Since $G$ is $n$-hamiltonian (cf. Lemma 2-2), $\delta(G) \geq 2 + n$. Thus $k \geq 2$.

Suppose $u_1$ is incident with $t$ edges of $F$ for some $t$ satisfying $0 \leq t \leq n$. Then $u_p$ is incident
with at most $n - l$ edges of $F$. Since there are at least $\deg_{G-F}u_1$ vertices which are not adjacent to $u_p$ in $G - F$, there are at least $\deg_{G-F}u_1 - (n - l)$ vertices which are not adjacent to $u_p$ in $G$. But $\deg_{G-F}u_1 = \deg_Gu_1 - l$. Thus there are at least $(\deg_Gu_1 - l) - (n - l) = \deg_Gu_1 - n = k$ vertices which are not adjacent to $u_p$ in $G$. From the manner in which $u_1$ and $u_p$ were chosen, each of these $k$ vertices has degree in $G$ not exceeding $\deg_Gu_1$. Since $\deg_Gu_1 = k + n$, we have

$$d_k \leq k + n < (p + n)/2.$$ 

Thus by hypothesis, $d_{p-n-k} \geq p - k$. This implies that there are at least $k + n + 1$ vertices whose degrees in $G$ are at least $p - k$. Let $W$ be the set of all vertices of $G$ whose degrees in $G$ are at least $p - k$. Since $k + n < (p + n)/2$, we have that $2k + 2n < p + n$, so that $k + n < p - k$. Thus $\deg_Gu_1 < p - k$ and $u_1 \notin W$. Since $\deg_Gu_1 = k + n$ and $|W| \geq k + n + 1$, there is a vertex $w \in W$ which is not adjacent to $u_1$ in $G$. But then $\deg_Gu_1 + \deg_Gw \geq (k + n) + (p - k) > \deg_Gu_1 + \deg_Gu_p$. This presents a contradiction, so that $G$ is $n$-edge hamiltonian. \(\Box\)

We now show that if $n$ and $p$ are integers satisfying $2 \leq n \leq p - 5$, then there is a graph $G$ of order
p such that G is n-edge hamiltonian by Theorem 3-3 but G is not (n+1)-edge hamiltonian. We construct G by beginning with a copy of $K_{n-1} + (K_2 \cup K_{p-n-1})$. Let $u_1, u_2, v_1, v_2$ be four distinct vertices of degree $p - 3$ in this graph and let $w_1$ and $w_2$ be the two vertices of degree $n$. We obtain G by adding the edges $w_1u_1, w_1u_2$, $w_2v_1, w_2v_2$ to $K_{n-1} + (K_2 \cup K_{p-n-1})$. Then the degree sequence of G satisfies

$$d_k = \begin{cases} 
  n + 2 & , \ k = 1, 2 \\
  p - 2 & , \ p - n - 2 \leq \ k \leq p - n + 1 \\
  p - 1 & , \ k \geq p - n + 2 \\
  p - 3 & , \ \text{otherwise.} 
\end{cases}$$

We first observe that if $k \geq 3$ and $k + n < (p + n)/2$, then $p - n - k > k \geq 3$. Therefore $d_{p-n-k} \geq d_k \geq d_3 \geq p - 3 \geq p - k$. Also, for $k = 1$ we have $d_k = 2 + n > k + n$. Now, for $k = 2$ we have $d_k = 2 + n = k + n$. However, $d_{p-n-k} = d_{p-n-2} = p - 2 = p - k$. Thus G is n-edge hamiltonian by Theorem 3-3. Clearly G is not (n+1)-edge hamiltonian.

We note that one can employ a proof technique similar to that used in Theorem 3-3 to prove Theorem 3-2 as well as a generalization of the aforementioned result of Pósa; however, this generalization (which is stated below) is itself a direct consequence of Theorem 3-3.
**Corollary 3-3-1.** Let $G$ be a graph of order $p \geq 3$ and let $0 \leq n \leq p - 3$. If for every integer $j$ with $n + 1 \leq j < (p + n)/2$, the number of vertices of degree not exceeding $j$ is less than $j - n$, then $G$ is $n$-edge hamiltonian.

It can be shown (see [2, p. 134]) that, for the special case $n = 0$, if a graph $G$ satisfies the hypothesis of Theorem 3-2, then $G$ satisfies the hypothesis of Corollary 3-3-1. Thus, for $n = 0$, Theorem 3-2 is a corollary of Corollary 3-3-1. However, this is not true in general for values of $n \geq 1$. If $n$ and $p$ are integers such that $p$ is even and $1 \leq n \leq p/2 - 2$, consider the graph $G$ of order $p$ constructed as follows. Begin with a copy of $K_{p/2-n}$, say $H_1$, and a copy of $K_{p/2+n}$ disjoint from $H_1$, say $H_2$. For each vertex $w$ of $H_1$, add edges joining $w$ to $n + 1$ vertices of $H_2$, subject only to the restriction that in the resulting graph $G$, every vertex of $H_2$ has degree at least $p/2 + n$. (This is possible since $n \leq p/2 - 2$.) Then $\deg_G u + \deg_G v \geq p + n$ for all pairs $u, v$ of nonadjacent vertices of $G$. Therefore $G$ satisfies the hypothesis of Theorem 3-2. However, $n + 1 \leq p/2 < (p + n)/2$ and the number of vertices of $G$ having degree not exceeding $p/2$ equals $p/2 - n$. Thus $G$ does not satisfy the hypothesis of Corollary 3-3-1. At the present time, it is
not known if Theorem 3-2 is implied by Theorem 3-3.

In [9], Chvátal and Erdős showed that a graph $G$ of order at least three is hamiltonian if $\kappa(G) \geq \beta(G)$. The final theorem in this section (whose proof is suggested by that given in [9]) gives an analogous sufficient condition for a graph to be 1-edge hamiltonian. The proof of Theorem 3-4 uses the result due to Dirac [11] that if a graph $G$ is $n$-connected ($n \geq 2$), then given any set of one edge and $n-1$ vertices there is a cycle in $G$ containing all $n$ members of this set.

**Theorem 3-4.** Let $G$ be a graph with at least four vertices. If $\kappa(G) \geq \beta(G) + 1$, then $G$ is 1-edge hamiltonian.

**Proof.** If $G$ is complete, then $G$ is 1-edge hamiltonian. Thus we may assume that $G$ is not complete so that $\beta(G) \geq 2$. Let $e = uv$ be an arbitrary edge of $G$. We wish to show that $G - e$ is hamiltonian. Assume, to the contrary, that $G - e$ is not hamiltonian.

Let $C$ be a longest cycle in $G - e$ which contains the vertices $u$ and $v$. By the aforementioned result of Dirac, since $\kappa(G - e) \geq \kappa(G) - 1 \geq \beta(G) (\geq 2)$, the cycle $C$ contains at least $\beta(G) + 1$ vertices. By assumption, $G - e$ is not hamiltonian. Thus there is a vertex $x$ of $G - e$ which does not lie on $C$.

Since $G$ is $(\beta(G) + 1)$-connected, there are
$\beta(G) + 1$ paths starting at $x$ and terminating in $C$ which are pairwise disjoint apart from $x$ and share with $C$ just their terminal vertices $x_1, x_2, \ldots, x_{\beta(G) + 1}$ (see [10]). We note that $e$ is an edge of none of these paths, implying that each of these paths is a path of $G - e$.

For each $i = 1, 2, \ldots, \beta(G) + 1$, let $y_i$ be the successor of $x_i$ in a fixed cyclic ordering of the vertices of $C$. No $y_i$ is adjacent to $x$ in $G - e$ (and hence not in $G$); for otherwise, we can replace the edge $x_i y_i$ in $C$ by the path from $x_i$ to $y_i$ outside of $C$ (through $x$) and obtain a cycle of $G - e$ longer than $C$ which contains $u$ and $v$. Consider the set \{\(x, y_1, y_2, \ldots, y_{\beta(G) + 1}\)\}. Since no $y_i$ is adjacent to $x$ in $G$, there are two edges $y_i y_j$ and $y_k y_l$ in $G$, where $3 \leq |\{y_i, y_j, y_k, y_l\}| \leq 4$. At least one of these edges, say $y_i y_j$, is an edge of $G - e$. By deleting the edges $x_i y_i$ and $x_j y_j$ from $C$ and adding the edge $y_i y_j$ together with the path from $x_i$ to $x_j$ outside of $C$ (through $x$), we obtain a cycle of $G - e$ longer than $C$ that contains the vertices $u$ and $v$. This presents a contradiction so that $G - e$ is hamiltonian. \(\diamond\)

We note that $K(G) \geq \beta(G)$ does not imply that $G$ is $1$-edge hamiltonian. Consider the graph $G$ obtained from $K(n, n+1)$, $n \geq 2$, by adding a single edge $e$ joining two vertices of $K(n, n+1)$ having degree $n$. Then
\( \kappa(G) = n = \beta(G) \) but \( G - e \) is not hamiltonian.

We remark in closing that there is no known result analogous to Theorem 3-4 concerning n-edge hamiltonian graphs for values of \( n \geq 2 \).

Section 3.2

n-Edge Hamiltonian-Connected Graphs

The concept of a hamiltonian-connected graph was introduced by Ore [18] in 1963. A graph \( G \) is called hamiltonian-connected if for every pair \( u,v \) of distinct vertices of \( G \), there exists a hamiltonian \( u-v \) path. If \( G \) is a hamiltonian-connected graph of order \( p \geq 3 \), then it is immediate that \( G \) is hamiltonian so that, obviously, every vertex of \( G \) lies on a hamiltonian cycle of \( G \). However, it is also true that every edge of \( G \) lies on some hamiltonian cycle of \( G \). A hamiltonian graph with the property that each of its edges belongs to a hamiltonian cycle of the graph is referred to as a strongly hamiltonian graph. Thus every hamiltonian-connected graph of order at least three is strongly hamiltonian.

A graph \( G \) with \( p \geq 4 \) vertices, \( 0 \leq n \leq p - 4 \), will be called n-edge hamiltonian-connected (strongly n-edge hamiltonian) if the removal of any \( n \) edges from \( G \) results in a hamiltonian-connected (strongly hamiltonian)
graph. Clearly, every n-edge hamiltonian-connected graph is also strongly n-edge hamiltonian. Moreover, if G is n-edge hamiltonian-connected (strongly n-edge hamiltonian), then G is m-edge hamiltonian-connected (strongly m-edge hamiltonian) for \( 0 \leq m \leq n \). It was shown by Ore [18] that if a graph G of order \( p \geq 4 \) has minimum degree at least \( (p + 1)/2 \), then G is hamiltonian-connected. As Theorem 3-5 indicates, such a graph is, in fact, \([(p - 4)/4]\)-edge hamiltonian-connected.

**Theorem 3-5.** Let \( p \) and \( n \) be nonnegative integers satisfying \( p \geq 4n + 4 \). If G is a graph of order \( p \) such that \( \delta(G) \geq (p + 1)/2 \), then G is n-edge hamiltonian-connected.

**Proof.** The case \( n = 0 \) follows by Ore's result. Thus we may assume that \( n \geq 1 \). Let \( E \) be an arbitrary set of \( n \) edges of G. We wish to show that \( G - E \) is hamiltonian-connected. It suffices to show that if \( j \) is an arbitrary integer with \( 2 \leq j \leq p/2 \), then the number of vertices of \( G - E \) of degree not exceeding \( j \) is less than \( j - 1 \) (see [17]). Let \( k \) denote the number of vertices of \( G - E \) of degree not exceeding \( j \). If \( k = 0 \), then \( k < j - 1 \). Thus we may assume that \( k \geq 1 \) and we label the vertices of G as \( v_1, v_2, \ldots, v_p \) so that for \( 1 \leq i \leq k \) we have \( (p + 1)/2 - n \leq \deg_{G-E} v_i \leq j \). Then
Consider a function $f$ defined for $x \geq 2$ by
\[ f(x) = \frac{4n}{x - 1} + 2x - C, \]
where $C$ is an arbitrary constant. We observe that since $f'(x) = \frac{-4n}{(x - 1)^2} + 2$, the function $f$ is monotonically increasing for $x \geq \sqrt{2n} + 1$. By hypothesis, $n < \frac{(p - 3)}{4}$ so that
\[ p - 2n + 1 > p - \frac{(p - 3)}{2} + 1 = \frac{(p + 3)}{2} + 1. \]
Therefore, $(p - 2n + 1)/2 > \frac{(p + 3)}{4} + 1/2 = \frac{(p - 3)}{4} + 2 > (n + 1) + 1$. Since $n + 1 > \sqrt{2n}$, we conclude that the function $f$ is monotonically increasing for $x \geq (p - 2n + 1)/2$.

We now proceed in cases.

**Case 1. Suppose $p$ is odd.** Then necessarily
\[ (p - 2n + 1)/2 \leq j \leq (p + 1)/2. \]
Since $\deg_{G} v_i \geq (p + 1)/2$ and $\deg_{G-E} v_i \leq j$, for $1 \leq i \leq k$, we have
\[ \deg_{G} v_i - \deg_{G-E} v_i \geq (p + 1)/2 - j \quad (1 \leq i \leq k). \]
Thus,
\[ k((p + 1)/2 - j) \leq \sum_{i=1}^{k} (\deg_{G} v_i - \deg_{G-E} v_i) \leq 2n \]
so that

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\[ k \leq \frac{2n}{(p + 1)/2 - j} = \frac{4n}{p + 1 - 2j}. \]

If \( p > \frac{4n}{(j - 1)} + 2j - 1 \), then \( p + 1 - 2j > \frac{4n}{(j - 1)} \) which implies that \( \frac{4n}{p + 1 - 2j} < j - 1 \).

Therefore Case 1 will be complete once we have shown that \( p > \frac{4n}{(j - 1)} + 2j - 1 \).

Since \( \frac{p - 2n + 1}{2} \leq j \leq \frac{p - 1}{2} \), by an earlier observation involving the monotonicity of \( f \), it remains only to show that

\[ p > \frac{4n}{(p - 3)/2} + (p - 1) - 1. \]

However, since \( p > 4n + 3 \), we have \( (p - 3)/8n > 1/2 \), so that

\[ 8n/(p - 3) + p < 2 + p, \]

which implies that

\[ p > \frac{4n}{(p - 3)/2} + (p - 1) - 1. \]

**Case 2.** Suppose \( p \) is even. Then necessarily \( \delta(G) \geq (p + 2)/2 \) and \( (p - 2n + 2)/2 \leq j \leq p/2 \). Since \( \text{deg}_{G}v_{i} \geq (p + 2)/2 \) and \( \text{deg}_{G-E}v_{i} \leq j \), for \( 1 \leq i \leq k \), we have

\[ \text{deg}_{G}v_{i} - \text{deg}_{G-E}v_{i} \geq (p + 2)/2 - j. \]

Thus,

\[ k((p + 2)/2 - j) \leq \sum_{i=1}^{k} (\text{deg}_{G}v_{i} - \text{deg}_{G-E}v_{i}) \leq 2n \]

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so that

\[ k \leq \frac{2n}{(p + 2)/2 - j} = \frac{4n}{p + 2 - 2j} . \]

If \( p > \frac{4n}{(j - 1) + 2j - 2} \), then \( p + 2 - 2j > \frac{4n}{(j - 1)} \) which implies that \( \frac{4n}{p + 2 - 2j} < j - 1 \).

Therefore Case 2 will be complete once we have shown that \( p > \frac{4n}{(j - 1) + 2j - 2} \).

Since \( (p - 2n + 2)/2 \leq j \leq p/2 \), by an earlier observation involving the monotonicity of \( f \), it remains only to show that

\[ p > \frac{4n}{(p - 2)/2} + p - 2 . \]

However, since \( p > 4n + 2 \), we have \( (p - 2)/8n > 1/2 \), so that

\[ \frac{8n}{(p - 2) + p} < 2 + p , \]

which implies that

\[ p > \frac{4n}{(p - 2)/2} + p - 2 . \]

**Corollary 3-5-1.** Let \( p \) and \( n \) be nonnegative integers satisfying \( p \geq 4n + 4 \). If \( G \) is a graph of order \( p \) such that \( \delta(G) \geq (p + 1)/2 \), then \( G \) is strongly \( n \)-edge hamiltonian.

According to Theorem 3-5, if \( n \) is a nonnegative integer and \( G \) is a graph of "sufficiently large" order \( p \) (in particular, \( p \geq 4n + 4 \)) such that
\( \delta(G) \geq (p + 1)/2 \), then \( G \) is \( n \)-edge hamiltonian-connected. For any positive integer \( n \), consider the graph \( G \) isomorphic to \( \overline{K}_{2n+1} + ((n + 1)K_2) \). Then \( G \) has order \( p = 4n + 3 \) and \( \delta(G) = 2n + 2 = (p + 1)/2 \). However, \( G \) is not \( n \)-edge hamiltonian-connected. Therefore we cannot improve Theorem 3-5 by replacing the condition \( p \geq 4n + 4 \) with the condition \( p \geq 4n + 3 \).

In Section 3.1, Theorem 3-2 gives a sufficient condition for a graph of order \( p \) to be \( n \)-edge hamiltonian, \( 0 \leq n \leq p - 3 \). The following theorem gives an analogous condition for a graph of order \( p \) to be \( n \)-edge hamiltonian-connected, \( 0 \leq n \leq p - 4 \). The proof of Theorem 3-6 uses the result of Ore [18] that a graph \( G \) of order \( p \geq 4 \) is hamiltonian-connected if the sum of the degrees of each pair of nonadjacent vertices of \( G \) is at least \( p + 1 \) and also uses the following observation. Suppose \( G \) is a graph of order \( p \geq 5 \) and \( F = \{e_1, e_2, \ldots, e_n\} \) is a set of \( n \) edges of \( G \), where \( 1 \leq n \leq p - 4 \). Let \( H \) be the graph defined by \( V(H) = V(G) \) and \( E(H) = F \). Then \( |E(H)| = \binom{p}{2} - n \geq \binom{p}{2} - (p - 4) = (p^2 - 3p + 8)/2 \). It is easily verified that this bound on \( |E(H)| \) implies that the sum of the degrees of each pair of nonadjacent vertices of \( H \) is at least \( p + 1 \). Therefore \( H \) is hamiltonian-connected, so that if \( u \) and \( v \) are vertices of \( H \), there is a hamiltonian \( u - v \) path in \( H \), say \( P: u = w_1, w_2, \ldots, w_p = v \). Then the ordering
S: \( u = v \)

of the vertices of \( G - F \) has the property that for \( 1 \leq j \leq n \), the two vertices incident with \( e_j \) in \( G \) are not consecutive vertices of \( S \).

**Theorem 3-6.** Let \( G \) be a graph of order \( p \geq 4 \) and let \( 0 \leq n \leq p - 4 \). If \( \text{deg } u + \text{deg } v \geq p + n + 1 \) for all pairs \( u, v \) of nonadjacent vertices, then \( G \) is \( n \)-edge hamiltonian-connected.

**Proof.** For \( n = 0 \) the theorem is true by Ore's result. Thus we may assume that \( n \geq 1 \) so that \( p \geq 5 \). Let \( F = \{ e_1, e_2, \ldots, e_n \} \) be an arbitrary set of \( n \) edges of \( G \). We wish to show that \( G - F \) is hamiltonian-connected. Assume, to the contrary, that \( G - F \) is not hamiltonian-connected. Thus there are vertices \( u \) and \( v \) such that \( G - F \) contains no hamiltonian \( u - v \) path.

As noted above, there exists an ordering \( S: w_1, w_2, \ldots, w_p \) of the vertices of \( G - F \) which satisfies the following property \( P \).

\((P)\). For \( 1 \leq j \leq n \), the two vertices incident with \( e_j \) in \( G \) are not consecutive vertices of the ordering. Moreover, \( u = w_1 \) and \( v = w_p \).

As in the proof of Theorem 3-2, we will say there is a gap in \( S \) between \( w_t \) and \( w_{t+1} \) (\( 1 \leq t \leq p - 1 \)) if \( w_t, w_{t+1} \not\in E(G - F) \).

Among all orderings of the vertices of \( G - F \) which
satisfy property $P$, let

$$S': v_1, v_2, \ldots, v_p$$

be an ordering with a minimum number of gaps. Since $G - F$ contains no Hamiltonian $u - v$ path, there is an integer $i$ satisfying $1 \leq i \leq p - 1$ such that $v_i v_{i+1} \notin E(G - F)$. This implies that $v_i v_{i+1} \notin E(G)$; for otherwise, $v_i v_{i+1} \in F$, which contradicts the manner in which $S'$ was chosen.

If $i \geq 2$ and $v_i v_k \in E(G - F)$, $1 \leq k \leq i - 1$, then $v_{i+1} v_{k+1} \notin E(G - F)$; for otherwise,

$$T: v_1, v_2, \ldots, v_{k-1}, v_i, v_{i-1}, \ldots, v_{k+1}, v_{i+1}, v_{i+2}, \ldots, v_p$$

is an ordering of the vertices of $G - F$ which satisfies property $P$ and which has at least one less gap than $S'$.

If $i \leq p - 2$ and $v_i v_k \in E(G - F)$, $i + 1 \leq k \leq p - 1$, then $v_{i+1} v_{k+1} \notin E(G - F)$; for otherwise,

$$T: v_1, v_2, \ldots, v_k, v_{k-1}, \ldots, v_{i+1}, v_{i+1}, v_{k+1}, v_{k+2}, \ldots, v_p$$

is an ordering of the vertices of $G - F$ which satisfies property $P$ and which has at least one less gap than $S'$.

Thus for each vertex $v_k$ in \{\(v_1, v_2, \ldots, v_{p-1}\)\} - \{\(v_i\)\} which is adjacent to $v_i$ in $G - F$, the vertex $v_{k+1}$ in \{\(v_2, v_3, \ldots, v_p\)\} - \{\(v_{i+1}\)\} is not adjacent to $v_{i+1}$ in $G - F$. Therefore there are at least deg$_{G - F} v_i - 1$ vertices in \{\(v_2, v_3, \ldots, v_p\)\} - \{\(v_{i+1}\)\} which are not adjacent.
to \( v_{i+1} \) in \( G - F \) so that

\[
\deg_{G-F} v_{i+1} \leq 1 + ((p - 2) - (\deg_{G-F} v_i - 1))
\]

or, equivalently,

\[
\deg_{G-F} v_i + \deg_{G-F} v_{i+1} \leq p.
\]

Since \( v_i v_{i+1} \notin E(G) \),

\[
\deg v_i + \deg v_{i+1} \leq (\deg_{G-F} v_i + \deg_{G-F} v_{i+1}) + n \leq p + n.
\]

This contradicts the hypothesis of the theorem so that \( G - F \) is hamiltonian-connected. \( \square \)

**Corollary 3-6-1.** Let \( G \) be a graph of order \( p \geq 4 \) and let \( 0 \leq n \leq p - 4 \). If \( \deg u + \deg v \geq p + n + 1 \) for all pairs \( u, v \) of nonadjacent vertices, then \( G \) is strongly \( n \)-edge hamiltonian.

**Corollary 3-6-2.** Let \( G \) be a graph of order \( p \geq 4 \) and let \( 0 \leq n \leq p - 4 \). If \( \delta(G) \geq (p + n + 1)/2 \), then \( G \) is \( n \)-edge hamiltonian-connected and therefore strongly \( n \)-edge hamiltonian.

We remark that the bound given in Theorem 3-6 is best possible. If \( n \) and \( p \) are integers satisfying \( 0 \leq n \leq p - 4 \) and \( G \) is the graph obtained from the complete graph \( K_{p-1} \) by adding a new vertex that is adjacent to exactly \( n + 2 \) vertices of \( K_{p-1} \), then
deg \( u + \deg v \geq p + n \) for all pairs \( u, v \) of nonadjacent vertices of \( G \). However, \( G \) is not \( n \)-edge hamiltonian-connected.

We now present a sufficient condition for a graph of order \( p \) to be \( n \)-edge hamiltonian-connected which is analogous to the result of Theorem 3-3 concerning \( n \)-edge hamiltonian graphs. The special case \( n = 0 \) in Theorem 3-7 is a result due to Berge [3, p. 218] and to Williamson [24, p. 24].

**Theorem 3-7.** If the degree sequence \( d_1, d_2, \ldots, d_p \) of a graph \( G \) satisfies \( 0 \leq n \leq p - 4 \) and

\[
d_k \leq k + n + 1 < \frac{(p + n + 1)}{2} = d_{p-(k+n+1)} \geq p - k,
\]

then \( G \) is \( n \)-edge hamiltonian-connected.

**Proof.** Assume the theorem is false. Hence there exists a graph \( G \) which satisfies the hypothesis of the theorem but contains a set \( F \) of \( n \) edges whose removal from \( G \) results in a graph which is not hamiltonian-connected.

Since for \( p \geq 4 \), the graph \( K_p \) is \( (p - 4) \)-edge hamiltonian-connected (by Corollary 3-6-2), we may assume, without loss of generality, that the addition of any edge \( f \) to \( G \) creates a graph \( G + f \) with the property that \( (G + f) - F \) is hamiltonian-connected. Since \( G - F \) is not hamiltonian-connected, \( G - F \) contains a pair \( u, v \) of vertices which are not joined by a hamiltonian path in
Among all pairs of nonadjacent vertices of $G$, let $u_1$ and $u_p$ be two nonadjacent vertices such that $\deg_{G-p} u_1 + \deg_{G-p} u_p$ is maximum. Suppose $\deg_{G-p} u_1 \leq \deg_{G-p} u_p$.

As we have already noted, the graph $(G + u_1 u_p) - F$ is hamiltonian-connected and so contains a hamiltonian $u - v$ path which necessarily contains the edge $u_1 u_p$. Thus $G - F$ contains two disjoint paths $P_1$ and $P_2$ with $V(G - F) = V(P_1) \cup V(P_2)$ where the vertices of $G - F$ can be labeled so that

$$P_1: u_1, u_2, \ldots, u_{l-1} \quad \text{and} \quad P_2: u_t, u_{t+1}, \ldots, u_p,$$

with $\{u_{l-1}, u_t\} = \{u, v\}$. Without loss of generality, we may assume that $u = u_{l-1}$ and $v = u_t$.

Now, if $u_1 u_i \in E(G - F)$ and $2 \leq i \leq t - 1$, then $u_{i-1} u_p \notin E(G - F)$; for otherwise,

$$u = u_{l-1}, u_{l-2}, \ldots, u_i, u_1, u_2, \ldots, u_{i-1}, u_p, u_{p-1}, \ldots, u_t = v$$

is a hamiltonian $u - v$ path of $G - F$.

Similarly, if $u_1 u_i \in E(G - F)$ and $t + 1 \leq i \leq p$, then $u_{i-1} u_p \notin E(G - F)$; for otherwise,

$$u = u_{l-1}, u_{l-2}, \ldots, u_i, u_1, u_i+1, \ldots, u_p, u_{i-1}, u_{i-2}, \ldots, u_t = v$$

is a hamiltonian $u - v$ path of $G - F$. Therefore there are at least $\deg_{G-F} u_1 - 1$ vertices of $G - F$ which are not adjacent to $u_p$ in $G - F$. Thus
\[ \deg_{G-F}u_p \leq (p - 1) - (\deg_{G-F}u_1 - 1) \] or, equivalently,

\[ \deg_{G-F}u_1 + \deg_{G-F}u_p < p + 1. \]

Since \( u_1u_p \not\in E(G) \), we have

\[ \deg_{G}u_1 + \deg_{G}u_p < p + n + 1. \]

Therefore \( \deg_{G}u_1 < (p + n + 1)/2 \). Let \( \deg_{G}u_1 = k + n + 1 \). Since \( G \) is \((n + 1)\)-hamiltonian (cf. Lemma 2-2), \( \delta(G) \geq 3 + n \). Thus \( k \geq 2 \).

Suppose \( u_1 \) is incident with \( m \) edges of \( F \) for some \( m \) satisfying \( 0 \leq m \leq n \). Then \( u_p \) is incident with at most \( n - m \) edges of \( F \). Since there are at least \( \deg_{G-F}u_1 - 1 \) vertices which are not adjacent to \( u_p \) in \( G - F \), there are at least \( (\deg_{G-F}u_1 - 1) - (n - m) \) vertices which are not adjacent to \( u_p \) in \( G \). But \( \deg_{G-F}u_1 = \deg_{G}u_1 - m \). Thus there are at least

\[ (\deg_{G}u_1 - m - 1) - (n - m) = \deg_{G}u_1 - n - 1 = k \]

vertices which are not adjacent to \( u_p \) in \( G \). From the manner in which \( u_1 \) and \( u_p \) were chosen, each of these \( k \) vertices has degree in \( G \) not exceeding \( \deg_{G}u_1 \). Since \( \deg_{G}u_1 = k + n + 1 \), we have

\[ d_k \leq k + n + 1 < (p + n + 1)/2. \]

Thus by hypothesis, \( d_{p-(k+n+1)} \geq p - k \). This implies that there are at least \( k + n + 2 \) vertices whose degrees
in \( G \) are at least \( p - k \). Let \( W \) be the set of all vertices of \( G \) whose degrees in \( G \) are at least \( p - k \). Since \( k + n + 1 < (p + n + 1)/2 \), we have that \( 2k + 2n + 2 < p + n + 1 \), so that \( k + n + 1 < p - k \). Thus \( \deg_G u_1 < p - k \) and \( u_1 \not\in W \). Since \( \deg_G u_1 = k + n + 1 \) and \( |W| \geq k + n + 2 \), there is a vertex \( w \in W \) which is not adjacent to \( u_1 \) in \( G \). But then \( \deg_G u_1 + \deg_G w \geq (k + n + 1) + (p - k) = p + n + 1 > \deg_G u_1 + \deg_G u_p \). This presents a contradiction, so that \( G \) is \( n \)-edge hamiltonian-connected.

**Corollary 3-7-1.** If the degree sequence \( d_1, d_2, \ldots, d_p \) of a graph \( G \) satisfies \( 0 \leq n \leq p - 4 \) and

\[
d_k \leq k + n + 1 < (p + n + 1)/2 = d_{p-(k+n+1)} \geq p - k ,
\]

then \( G \) is strongly \( n \)-edge hamiltonian.

We now show that if \( n \) and \( p \) are integers satisfying \( 1 \leq n \leq p - 6 \), then there is a graph \( G \) such that \( G \) is \( n \)-edge hamiltonian-connected by Theorem 3-7 but \( G \) is not \((n + 1)\)-edge hamiltonian-connected. We construct \( G \) by beginning with a copy of \( K_n + (K_2 \cup K_{p-n-2}) \). Let \( u_1, u_2, v_1, v_2 \) be four distinct vertices of degree \( p - 3 \) in this graph and let \( w_1 \) and \( w_2 \) be the two vertices of degree \( n + 1 \). We obtain \( G \) by adding the edges \( w_1 u_1, w_1 u_2, w_2 v_1, w_2 v_2 \) to \( K_n + (K_2 \cup K_{p-n-2}) \). Then the degree sequence of \( G \) satisfies
\[
\begin{align*}
\mathbf{d}_k &= \begin{cases} 
3 + n, & k = 1, 2 \\
p - 2, & p - n - 3 \leq k \leq p - n \\
p - 1, & k \geq p - n + 1 \\
p - 3, & \text{otherwise.}
\end{cases}
\end{align*}
\]

We first observe that if \( k \geq 3 \) and \( k + n + 1 < (p + n + 1)/2 \), then \( p - (k + n + 1) \geq k + 3 \). Therefore \( d_{p-(k+n+1)} \geq d_k \geq d_{3} \geq p - 3 \geq p - k \). Also, for \( k = 1 \) we have \( d_k = 3 + n > k + n + 1 \). Now, for \( k = 2 \) we have \( d_k = 3 + n = k + n + 1 \). However, \( d_{p-(k+n+1)} = d_{p-n-3} = p - 2 = p - k \). Thus \( G \) is n-edge hamiltonian-connected by Theorem 3-7. Clearly \( G \) is not \((n + 1)\)-edge hamiltonian-connected.

As a consequence of Theorem 3-7, we obtain a sufficient condition for a graph to be n-edge hamiltonian-connected (and therefore strongly n-edge hamiltonian) which parallels the condition given in Corollary 3-3-1 for a graph to be n-edge hamiltonian.

**Corollary 3-7-2.** Let \( G \) be a graph of order \( p \geq 4 \) and let \( 0 \leq n \leq p - 4 \). If for every integer \( j \) with \( n + 2 \leq j \leq (p + n + 1)/2 \), the number of vertices of degree not exceeding \( j \) is less than \( j - n - 1 \), then \( G \) is n-edge hamiltonian-connected and therefore strongly n-edge hamiltonian.

Examples completely analogous to those following.
Corollary 3-3-1 show that for every nonnegative integer \( n \), there are graphs which satisfy the hypothesis of Theorem 3-6 but not the hypothesis of Corollary 3-7-2. Thus Theorem 3-6 is not implied by Corollary 3-7-2.

It was shown by Chvátal and Erdös [9] that if a graph \( G \) has the property that \( \kappa(G) \geq \beta(G) + 1 \), then \( G \) is hamiltonian-connected. In order to prove an analogous sufficient condition for a graph to be 1-edge hamiltonian-connected, we will use a result of Plummer [21] that if a graph \( G \) is \( n \)-connected \( (n \geq 2) \), then given any sequence of \( n + 1 \) vertices \( a, b, v_1, v_2, \ldots, v_{n-1} \), there is an \( a - b \) path in \( G \) containing all vertices in \( \{v_1, v_2, \ldots, v_{n-1}\} \). We remark at this point that there is no known result analogous to Theorem 3-8 concerning \( n \)-edge hamiltonian-connected graphs for values of \( n \geq 2 \).

**Theorem 3-8.** Let \( G \) be a graph with at least five vertices. If \( \kappa(G) \geq \beta(G) + 2 \), then \( G \) is 1-edge hamiltonian-connected.

**Proof.** If \( G \) is complete, then \( G \) is 1-edge hamiltonian-connected. Thus we may assume that \( G \) is not complete so that \( \beta(G) \geq 2 \). Let \( e = uv \) be an arbitrary edge of \( G \). We wish to show that \( G - e \) is hamiltonian-connected. Assume, to the contrary, that \( G - e \) is not hamiltonian-connected. Thus there exist vertices \( y \) and \( z \) such that \( G - e \) contains no hamiltonian \( y - z \) path.
By the aforementioned result of Plummer, since $K(G - e) \geq K(G) - 1 \geq \beta(G) + 1 \geq 3$, there is a $y - z$ path in $G - e$ which contains the vertices $u$ and $v$. Let $P$ be a longest such path. Since $K(G - e) \geq \beta(G) + 1$, we may again apply Plummer's result to conclude that $P$ contains at least $\beta(G) + 2$ vertices. By assumption, $G - e$ contains no hamiltonian $y - z$ path. Thus there is a vertex $x$ of $G - e$ which does not lie on $P$.

Since $G$ is $(\beta(G) + 2)$-connected, there are $\beta(G) + 2$ paths starting at $x$ and terminating in $P$ which are pairwise disjoint apart from $x$ and share with $P$ just their terminal vertices $x_1, x_2, \ldots, x_{\beta(G)+2}$ (see [10]).

We note that $e$ is an edge of none of these paths, implying that each of these paths is a path of $G - e$. We may assume that $x_i \neq z$ for $i \leq \beta(G) + 1$. For each $i = 1, 2, \ldots, \beta(G) + 1$, let $y_i$ be the successor (in the direction from $y$ to $z$) of $x_i$ on $P$. No $y_i$ is adjacent to $x$ in $G - e$ (and hence not in $G$); for otherwise, we can replace the edge $x_iy_i$ in $P$ by the path from $x_i$ to $y_i$ outside of $P$ (through $x$) and obtain a $y - z$ path of $G - e$ longer than $P$ which contains $u$ and $v$. Consider the set $\{x, y_1, y_2, \ldots, y_{\beta(G)+1}\}$.

Since no $y_i$ is adjacent to $x$ in $G$, there are two edges $y_iy_j$ and $y_ky_\ell$ in $G$, where $3 \leq |\{y_i, y_j, y_k, y_\ell\}| \leq 4$. At least one of these edges, say $y_iy_j$, is an edge of $G - e$. Be deleting the edges $x_iy_i$ and
\( x_jy_j \) from \( P \) and adding the edge \( y_iy_j \) together with the path from \( x_i \) to \( x_j \) outside of \( P \) (through \( x \) ), we obtain a \( y - z \) path of \( G - e \) longer than \( P \) that contains the vertices \( u \) and \( v \). This presents a contradiction so that \( G - e \) is hamiltonian-connected.  

We note that \( K(G) \geq \beta(G) + 1 \) does not imply that \( G \) is 1-edge hamiltonian-connected. Consider the graph \( G \) obtained from \( K(n,n) \), \( n \geq 3 \), by adding an edge \( e \) joining two vertices of \( V_1 \) and an edge \( f \) joining two vertices of \( V_2 \), where \( V_1 \) and \( V_2 \) are the partite sets of \( K(n,n) \). Then \( K(G) = n = \beta(G) + 1 \) but neither \( G - e \) nor \( G - f \) is hamiltonian-connected.

As we observed earlier, if a graph is \( n \)-edge hamiltonian-connected, then it is also strongly \( n \)-edge hamiltonian. The converse is not true in general. For example, if \( n \) is any nonnegative integer, then for sufficiently large even values of \( p \), the graph \( K(p/2,p/2) \) is strongly \( n \)-edge hamiltonian but is not \( n \)-edge hamiltonian-connected. We now note that if a graph which is not complete is \( n \)-edge hamiltonian-connected, then the addition of any edge to the graph results in an \( n \)-edge hamiltonian-connected graph. As the graph \( K(p/2,p/2) \) illustrates, it is possible for a strongly \( n \)-edge hamiltonian graph to lose this property upon the addition of some edge to the graph. With this example in mind, we present Theorem 3-9.
**Theorem 3-9.** Let $G$ be a graph of order $p \geq 4$ and let $0 \leq n \leq p - 4$. Then $G$ is $n$-edge hamiltonian-connected if and only if $G$ is strongly $n$-edge hamiltonian and $G + uv$ is strongly $n$-edge hamiltonian for each pair $u, v$ of nonadjacent vertices of $G$.

**Proof.** Suppose $G$ is $n$-edge hamiltonian-connected. Then $G$ is strongly $n$-edge hamiltonian. If $u$ and $v$ are nonadjacent vertices of $G$, then $G + uv$ is $n$-edge hamiltonian-connected, which implies that $G + uv$ is strongly $n$-edge hamiltonian.

Conversely, suppose $G$ is strongly $n$-edge hamiltonian and $G + uv$ is strongly $n$-edge hamiltonian for each pair $u, v$ of nonadjacent vertices of $G$. Let $F$ be an arbitrary set of $n$ edges of $G$ and let $x$ and $y$ be vertices of $G$. We wish to show that $G - F$ contains a hamiltonian $x - y$ path. If $x$ and $y$ are adjacent in $G - F$, then there exists a hamiltonian cycle of $G - F$ containing the edge $xy$ since $G - F$ is strongly hamiltonian. Thus $G - F$ contains a hamiltonian $x - y$ path. So we may assume that $xy \notin E(G - F)$. Therefore either $x$ and $y$ are nonadjacent vertices of $G$ or $xy \in F$. If $x$ and $y$ are nonadjacent vertices of $G$, then there exists a hamiltonian cycle of $(G + xy) - F$ containing the edge $xy$ since $xy \notin F$ and $G + xy$ is strongly $n$-edge hamiltonian. Since
(G + xy) - F = (G - F) + xy, the graph G - F contains a hamiltonian x - y path. If, on the other hand, xy ∈ F, we consider the set F* = F - {xy}. Since G is strongly n-edge hamiltonian, the graph G - F* is strongly hamiltonian and therefore there exists a hamiltonian cycle of G - F* which contains the edge xy. Thus G - F contains a hamiltonian x - y path. ⊗

Section 3.3

Strongly Hamiltonian Graphs

With the aid of Theorem 3-9, we see that the collection of strongly hamiltonian graphs G of order p ≥ 3 which have the property that G + uv is also strongly hamiltonian for each pair u,v of nonadjacent vertices of G is precisely the collection of hamiltonian-connected graphs of order at least three. In this section we consider two other topics related to strongly hamiltonian graphs. In particular, we answer the following questions.

(1). What can be said about a graph that is not strongly hamiltonian but has the property that the addition of any edge between nonadjacent vertices of the graph results in a strongly hamiltonian graph?

(2). What can be said about a strongly hamiltonian
graph $G$ with the property that no strongly hamiltonian graph can be obtained from $G$ by adding a single edge to $G$?

In [24, p. 31], Williamson defined a class of graphs referred to as maximally non-hamiltonian-connected graphs. A graph $G$ is \textit{maximally non-hamiltonian-connected} if $G$ is not hamiltonian-connected and has the property that $G + uv$ is hamiltonian-connected for each pair $u,v$ of nonadjacent vertices of $G$. Williamson obtained the following results concerning maximally non-hamiltonian-connected graphs.

\textbf{THEOREM A.} Let $G$ be a graph with $\kappa(G) = 2$. Then $G$ is maximally non-hamiltonian-connected if and only if $G \cong K_2 + (K_r \cup K_s)$, for $r \geq 1$ and $s \geq 1$.

\textbf{THEOREM B.} Let $G$ be a graph of order $p$ with $\kappa(G) = p/2$. Then $G$ is maximally non-hamiltonian-connected if and only if $G \cong K_{p/2} + \overline{K}_{p/2}$, for $p$ even and at least four.

We will say that a graph $G$ is \textit{maximally non-strongly hamiltonian} if $G$ is not strongly hamiltonian and has the property that $G + uv$ is strongly hamiltonian for each pair $u,v$ of nonadjacent vertices of $G$. It is the somewhat surprising result of Theorem 3-10 that the concepts of a maximally non-hamiltonian-connected graph
and a maximally non-strongly hamiltonian graph are identical for graphs of order at least three.

**Theorem 3-10.** A graph G of order \( p \geq 3 \) is maximally non-hamiltonian-connected if and only if it is maximally non-strongly hamiltonian.

**Proof.** The result holds for \( p = 3 \). Thus we may assume that \( p \geq 4 \). Let G be a maximally non-hamiltonian-connected graph. Thus \( G + uv \) is hamiltonian-connected for each pair \( u,v \) of nonadjacent vertices of G.

Since \( p \geq 4 \), the graph \( G + uv \) is therefore strongly hamiltonian for each pair \( u,v \) of nonadjacent vertices of G. Moreover, G is not strongly hamiltonian; for otherwise, by Theorem 3-9 we have that G is hamiltonian-connected, contradicting our original hypothesis.

Conversely, let G be a graph which is maximally non-strongly hamiltonian. Since G is not strongly hamiltonian and \( p \geq 4 \), the graph G is not hamiltonian-connected. We wish to show that if u and v are nonadjacent vertices of G, then the graph \( G + uv \) is hamiltonian-connected. Let \( x,y \in V(G + uv) \). If x and y are adjacent in \( G + uv \), then there exists a hamiltonian cycle of \( G + uv \) containing the edge xy since \( G + uv \) is strongly hamiltonian. Thus \( G + uv \) contains a hamiltonian \( x - y \) path. If \( xy \not\in E(G + uv) \), then x and y are nonadjacent vertices of G. Thus \( G + xy \)
is strongly hamiltonian and there exists a hamiltonian cycle of $G + xy$ containing the edge $xy$. Therefore, $G$ contains a hamiltonian $x - y$ path so that $G + uv$ contains a hamiltonian $x - y$ path. $\Phi$

As a result of Theorems A and B and Theorem 3-10, we have two corollaries.

**Corollary 3-10-1.** Let $G$ be a graph with $\kappa(G) = 2$. Then $G$ is maximally non-strongly hamiltonian if and only if $G \cong K_2 + (K_r \cup K_s)$, for $r \geq 1$ and $s \geq 1$.

**Corollary 3-10-2.** Let $G$ be a graph of order $p$ with $\kappa(G) = p/2$. Then $G$ is maximally non-strongly hamiltonian if and only if $G \cong K_{p/2} + \overline{K_{p/2}}$, for $p$ even and at least four.

We note briefly that the maximally non-strongly hamiltonian graphs designated in the two corollaries above are all hamiltonian. The Petersen graph is an example of a maximally non-strongly hamiltonian graph which is not hamiltonian.

We now consider the second question mentioned at the beginning of this section. A graph $G$ will be called maximally strongly hamiltonian if $G$ is strongly hamiltonian but no strongly hamiltonian graph can be obtained from $G$ by adding a single edge to $G$. It is apparent that for $p \geq 3$, each of the graphs $K_p$, $C_p$, and
K(p/2, p/2) is maximally strongly hamiltonian, the last in the case p is even. As we shall see, these are the only maximally strongly hamiltonian graphs. In order to present this result, a few preliminary definitions will be useful.

Let G be a hamiltonian graph and C: v_1, v_2, ..., v_p, v_1 a hamiltonian cycle of G. With respect to this cycle, every edge of G either lies on C (and will be called simply a cycle edge in this context) or joins two nonconsecutive vertices of C and is referred to as a diagonal. Any cycle of G containing precisely one diagonal is an outer cycle of G (with respect to the fixed hamiltonian cycle C). Thus an outer n-cycle has n - 1 cycle edges and one diagonal.

**Theorem 3-11.** A graph G of order p ≥ 3 is maximally strongly hamiltonian if and only if G is isomorphic to C_p, K_p, or K(p/2, p/2), the last being possible only if p is even.

**Proof.** As noted above, each of the graphs C_p, K_p, and K(p/2, p/2) is maximally strongly hamiltonian.

Conversely, let G be a maximally strongly hamiltonian graph of order p ≥ 3. Since G is hamiltonian, G contains a hamiltonian cycle C: v_1, v_2, ..., v_p, v_1. If G consists only of the cycle C, then G ≅ C_p. Suppose then that G contains diagonals and therefore outer
cycles.

Suppose $G$ contains the diagonal $v_jv_k$. We now show that $G$ also contains the diagonal $v_{j+1}v_{k+1}$, where from this point on, all subscripts are to be taken modulo $p$. Since $v_jv_k \in E(G)$, the graph $G$ contains the hamiltonian path $v_{j+1}v_{j+2}v_{j+3} \ldots v_kv_{j-1}v_{j-2} \ldots v_{k+1}$. Thus $v_{j+1}v_{k+1} \in E(G)$; for otherwise, $v_{j+1}$ and $v_{k+1}$ are nonadjacent vertices of $G$ and $G + v_{j+1}v_{k+1}$ is strongly hamiltonian (since $G$ itself is strongly hamiltonian and $v_{j+1}v_{k+1}$ lies on a hamiltonian cycle of $G + v_{j+1}v_{k+1}$). By repeating this procedure we conclude that if $v_jv_k$ is a diagonal of $G$, then $v_{j+i}v_{k+i}$ is a diagonal of $G$, for $i = 1, 2, \ldots, p - 1$.

Let $n$ be the smallest value of $m$ such that $G$ contains an outer $m$-cycle. We next show that $n = 3$ or $n = 4$, for suppose that $n \geq 5$. From what we have seen, $v_1v_n$ and $v_2v_{n+1}$ are diagonals of $G$. Thus $G$ contains the hamiltonian path $v_3v_4v_5 \ldots v_nv_1v_2v_{n+1}v_{n+2}v_{n+3} \ldots v_p$. Therefore $v_pv_3 \in E(G)$; for otherwise, $v_p$ and $v_3$ are nonadjacent vertices of $G$ and $G + v_pv_3$ is strongly hamiltonian. But then the vertices $v_pv_1v_2v_3$ form an outer 4-cycle. This produces a contradiction; hence the smallest outer cycle of $G$ is an outer triangle or an outer 4-cycle.

Suppose that $G$ contains an outer triangle. We
prove here that $G \equiv K_p$. Since $G$ has an outer triangle, $G$ contains all the diagonals $v_i v_{i+2}$, for $i = 1, 2, \ldots, p$. Let $v_j$ and $v_k$ be any two nonconsecutive vertices of $C$. Without loss of generality we assume $k > j$. If $p - (k - j + 1)$ is odd, then $G$ contains the hamiltonian path $v_k, v_{k-1}, v_{k-2}, \ldots, v_{j+2}, v_{j+1}, v_{j-1}, v_{j-3}, \ldots, v_{k+3}, v_{k+2}, v_k, v_{k+4}, v_{k+6}, \ldots, v_{j-2}, v_j$. If $p - (k - j + 1)$ is even, then $G$ contains the hamiltonian path $v_k, v_{k-1}, v_{k-2}, \ldots, v_{j+2}, v_{j+1}, v_{j-1}, v_{j-3}, \ldots, v_{k+4}, v_{k+2}, v_{k+1}, v_{k+3}, v_{k+5}, \ldots, v_{j-2}, v_j$. Therefore $v_j v_k \in E(G)$; for otherwise, $v_j$ and $v_k$ are nonadjacent vertices of $G$ and $G + v_j v_k$ is strongly hamiltonian. Thus $G$ is complete since $v_j$ and $v_k$ were arbitrary nonconsecutive vertices of $C$. This completes the case where the smallest outer cycle is an outer triangle.

Before proceeding to the next case, it is convenient to make an observation. If $v_1 v_{\ell} \in E(G)$ for some $\ell$ satisfying $4 \leq \ell \leq p - 3$, then $v_1 v_{\ell+2} \in E(G)$. Indeed, if $v_1 v_{\ell} \in E(G)$, then $G$ contains all the diagonals $v_i v_{i+(\ell-1)}$, for $i = 1, 2, \ldots, p$. Therefore $G$ contains the hamiltonian path $v_1, v_{\ell}, v_{\ell+1}, v_2, v_3, v_4, \ldots, v_{\ell-1}, v_p, v_{p-1}, v_{p-2}, \ldots, v_{\ell+2}$. Thus $v_1 v_{\ell+2} \in E(G)$; for otherwise, $v_1$ and $v_{\ell+2}$ are nonadjacent vertices of $G$ and $G + v_1 v_{\ell+2}$ is strongly hamiltonian.

Assume now that $G$ contains no outer triangle and therefore $G$ contains a 4-cycle as its smallest outer
cycle. We wish to show that $p$ is even. Suppose $p$ is odd. Since $G$ has an outer 4-cycle, $G$ contains the diagonal $v_1v_4$. Since $p$ is odd, repeated application of the above observation results in the conclusion that $v_1v_{p-1} \in E(G)$. But then the vertices $v_{p-1}, v_p$, and $v_1$ form an outer triangle, which is a contradiction. Therefore, $G$ contains an even number of vertices.

We next verify that $G$ contains all edges of the type $v_jv_k$, where $j$ and $k$ are of opposite parity. From what we have already shown, it is sufficient to prove that $G$ contains all the diagonals $v_1v_r$, where $r$ is even. This follows readily from the above observation and the fact that $v_1v_4 \in E(G)$.

Finally, we show that $G$ contains none of the diagonals $v_jv_k$, where $j$ and $k$ are of the same parity. Again, it is sufficient to show that $G$ contains none of the diagonals $v_1v_s$, where $s$ is odd. Suppose, to the contrary, that $G$ has such an edge. If $s = 3$ or $s = p - 1$, then $G$ contains an outer triangle, which is a contradiction. So we may assume that $5 \leq s \leq p - 3$. Since $s$ is odd and $p$ is even, repeated application of the above observation results in the conclusion that $v_1v_{p-1} \in E(G)$, which again presents a contradiction.

Thus if the smallest outer cycle of $G$ is a 4-cycle, then $p$ is even and $v_jv_k$ is an edge of $G$ if and only if $j$ and $k$ are of opposite parity, i.e. two vertices

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of $G$ are adjacent if and only if one belongs to $V_1$ and the other to $V_2$, where

$$V_1 = \{v_{2m} | m = 1, 2, \ldots, p/2\}$$

and

$$V_2 = \{v_{2m-1} | m = 1, 2, \ldots, p/2\}.$$ 

Thus $G$ is isomorphic to the complete bipartite graph $K(p/2, p/2)$. ⊗

A hamiltonian graph $G$ is randomly hamiltonian from a vertex $v$ of $G$ if every path of $G$ beginning at $v$ can be extended to a hamiltonian $v - v$ cycle of $G$. A graph is called simply randomly hamiltonian if it is randomly hamiltonian from each of its vertices. Randomly hamiltonian graphs have been characterized (see [6]) and Corollary 3-11-1 combines this result with the result of Theorem 3-11.

**Corollary 3-11-1.** A graph $G$ is maximally strongly hamiltonian if and only if $G$ is randomly hamiltonian.
CHAPTER IV

LINE GRAPHS WITH HAMILTONIAN PROPERTIES

With each nonempty graph $G$ one can associate a graph $L(G)$, called the line graph of $G$, with the property that there exists a one-to-one correspondence between $E(G)$ and $V(L(G))$ such that two vertices of $L(G)$ are adjacent if and only if the corresponding edges of $G$ are adjacent. For convenience, $L^1(G)$ and $L^0(G)$ denote $L(G)$ and $G$, respectively. For integers $n \geq 2$, the iterated line graph $L^n(G)$ is defined to be $L(L^{n-1}(G))$. We investigate here the relationship between line graphs and a variety of hamiltonian properties. In this chapter, we assume that all graphs under consideration have no isolated vertices.

Section 4.1

Edge-Related Hamiltonian Properties Of Line Graphs

Among the first results concerning hamiltonian line graphs were sufficient conditions for a graph $G$ to have a hamiltonian line graph. For example, it was shown that if $G$ is hamiltonian or eulerian, then $L(G)$ is hamiltonian. In [13], Harary and Nash-Williams presented a characterization of graphs with hamiltonian line graphs.
which gave all previously known sufficient conditions as corollaries. It was shown that \( L(G) \) is hamiltonian if and only if \( G \) is isomorphic to \( K(1,n) \), for some \( n \geq 3 \), or there is a circuit in \( G \) which includes at least one endvertex of each edge of \( G \). We will say that a circuit \( C \) of a graph \( G \) is a dominating circuit of \( G \) if every edge of \( G \) is incident with at least one vertex of \( C \). The result of Harary and Nash-Williams was the first indication of the relationship between the existence of dominating circuits in a graph \( G \) and hamiltonian properties of \( L(G) \). In this section we will look further into this relationship. In order to do so, we first exhibit how a dominating circuit may be determined in a graph \( G \) (when \( G \neq K(1,n) \)) from a given hamiltonian cycle of \( L(G) \). The particular dominating circuit which we will construct for \( G \) will be used in later material.

Suppose we begin with a hamiltonian cycle \( x_1, x_2, \ldots, x_q, x_1 \) of the line graph of \( G \). Corresponding to this cycle is an ordering \( S: e_1, e_2, \ldots, e_q \) of the \( q \) edges of \( G \) where, for \( 1 \leq i \leq q \), the edge \( e_i \) is adjacent to \( e_{i+1} \) in \( G \) (subscripts expressed modulo \( q \)). For \( 1 \leq i \leq q - 1 \), let \( v_i \) be the vertex of \( G \) incident with both \( e_i \) and \( e_{i+1} \). We note that \( 1 \leq k \neq l \leq q - 1 \) does not necessarily imply that \( v_k \neq v_l \). Since \( G \) is not a star graph, there is a smallest integer \( j \).
exceeding \( l \) such that \( v_{j_1} \neq v_1 \). The vertex \( v_{j_1-l} \) is incident with \( e_{j_1} \), the vertex \( v_{j_1} \) is incident with \( e_{j_1} \), and \( v_{j_1-l} = v_1 \). Thus \( e_{j_1} = v_1v_{j_1} \). Now, let \( j_2 \) be the smallest integer exceeding \( j_1 \) (if it exists) such that \( v_{j_2} \neq v_{j_1} \). The vertex \( v_{j_2-l} \) is incident with \( e_{j_2} \), the vertex \( v_{j_2} \) is incident with \( e_{j_2} \), and \( v_{j_2-l} = v_{j_1} \). Thus \( e_{j_2} = v_{j_1}v_{j_2} \). We continue in this fashion to finally obtain the vertex \( v_{j_t} \) such that \( e_{j_t} = v_{j_{t-(t-1)}}v_{j_t} \), where \( v_{j_t} = v_{q-1} \).

Since every edge of \( G \) appears exactly once in \( S \) and since \( 1 < j_1 < j_2 < \ldots < j_t < q - 1 \), this construction yields a trail \( T: v_1e_{j_1}v_{j_1}e_{j_2}v_{j_2} \ldots v_{j_{t-(t-1)}}e_{j_t}v_{j_t} = v_{q-1} \) in \( G \) with the properties that

(i) every edge of \( G \) is incident with a vertex of \( T \), and

(ii) neither \( e_1 \) nor \( e_q \) is an edge of \( T \).

We emphasize that, since \( T \) is a trail, no edge is repeated in \( T \). However, this may not be the case for the vertices of \( T \). Let \( w \) be the vertex of \( G \) incident with both \( e_1 \) and \( e_q \). We now consider four possible (and exhaustive) cases.

**Case 1.** Suppose \( w = v_1 = v_{q-1} \). Then \( T \) itself is a dominating circuit of \( G \).
Case 2. Suppose \( w = v_1 \) and \( w \neq v_{q-1} \). Since \( e_q \) is incident with both \( w \) and \( v_{q-1} \), we have that \( e_q = v_{q-1}w = v_{q-1}v_1 \). Thus \( C: T, e_q, v_1 \) is a dominating circuit of \( G \).

Case 3. Suppose \( w = v_{q-1} \) and \( w \neq v_1 \). Since \( e_1 \) is incident with both \( w \) and \( v_1 \), we have that \( e_1 = wv_1 = v_{q-1}v_1 \). Thus \( C: T, e_1, v_1 \) is a dominating circuit of \( G \).

Case 4. Suppose \( w \neq v_{q-1} \) and \( w \neq v_1 \). Since \( e_q \) is incident with both \( w \) and \( v_{q-1} \), we have that \( e_q = v_{q-1}w \). Since \( e_1 \) is incident with both \( w \) and \( v_1 \), we have that \( e_1 = wv_1 \). Thus \( v_1 \neq v_{q-1} \). Hence \( C: T, e_q, w, e_1, v_1 \) is a dominating circuit of \( G \).

The first hamiltonian property that we will consider in line graphs is the property of being strongly hamiltonian. From the definition of line graph it is obvious that a graph \( G \) has a strongly hamiltonian line graph if and only if for each pair \( e, f \) of adjacent edges of \( G \), there exists an ordering \( S: e_1, e_2, \ldots, e_q \) of the \( q \) edges of \( G \) where, for \( 1 \leq i \leq q \), the edge \( e_i \) is adjacent to \( e_{i+1} \) in \( G \) (subscripts expressed modulo \( q \)) and in which \( e \) and \( f \) are consecutive edges. As we shall see, this statement involving orderings of the edges of \( G \) can be translated to one involving dominating
circuits of $G$.

Suppose $C: w_1, w_2, \ldots, w_s, w_1$ is a dominating circuit of a graph $G$ and $e$ and $f$ are adjacent edges of $G$ corresponding to the (adjacent) vertices $x$ and $y$ of $L(G)$, respectively. Since $C$ is a circuit, if $i \neq j$ then $w_i w_{i+1} \neq w_j w_{j+1}$ (subscripts expressed modulo $s$). However, $w_i$ and $w_j$ are not necessarily distinct if $i \neq j$. In what follows, we will let $v$ and $v'$ denote vertices of $G$ which may or may not be on $C$.

(A1). If $e = w_1 v$, $f = w_1 v'$, and $e, f \notin E(C)$, consider the following ordering $S$ of the edges of $G$. Begin by selecting $e, f$ and then, in any order, all edges of $G$ incident with $w_1$ that are neither edges of $C$ nor previously selected edges, followed by the edge $w_1 w_2$. At each successive $w_i$, $2 \leq i \leq s$, select, in any order, all edges of $G$ incident with $w_i$ that are neither edges of $C$ nor previously selected edges, followed by the edge $w_i w_{i+1}$ (subscripts expressed modulo $s$). This process terminates with the edge $w_s w_1$.

(A2). If $e = w_1 w_2$, $f = w_2 v$, and $f \notin E(C)$---or---$f = w_1 w_2$, $e = w_2 v$, and $e \notin E(C)$, consider the following ordering $S$ of the edges of $G$. Begin by selecting, in any order, all edges of $G$ incident with $w_1$ that are not edges of $C$. 

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followed by the edges $w_1w_2$ and $w_2v$ in that order. Next select, in any order, all edges of $G$ incident with $w_2$ that are neither edges of $C$ nor previously selected edges, followed by the edge $w_2w_3$. At each successive $w_i$, $3 \leq i \leq s$, select, in any order, all edges of $G$ incident with $w_i$ that are neither edges of $C$ nor previously selected edges, followed by the edge $w_iw_{i+1}$ (subscripts expressed modulo $s$). This process terminates with the edge $w_sw_1$.

(A3). If $e = w_1w_2$, $f = w_2w_3$, and every edge of $G$ incident with $w_2$ is also incident with $w_k$ for some $k \neq 2$ (where $w_2$ and $w_k$ are not necessarily distinct), consider the following ordering $S$ of the edges of $G$. Begin by selecting, in any order, all edges of $G$ incident with $w_1$ that are not edges of $C$, followed by the edges $e$ and $f$, in that order. At each successive $w_i$, $3 \leq i \leq s$, select, in any order, all edges of $G$ incident with $w_i$ that are neither edges of $C$ nor previously selected edges, followed by the edge $w_iw_{i+1}$ (subscripts expressed modulo $s$). This process terminates with the edge $w_sw_1$.

We now observe that in each case the ordering $S$ has the properties that every edge of $G$ appears exactly once
in $S$, consecutive edges of $S$ are adjacent in $G$, the first and last edges of $S$ are adjacent in $G$, and $e,f$ are consecutive edges of $S$. This implies that there is a hamiltonian cycle of $L(G)$ which contains the edge $xy$. Thus a sufficient condition for the existence of a hamiltonian cycle of $L(G)$ which contains the edge $xy$ can be given in terms of dominating circuits of $G$.

Before stating this result in Theorem 4-1, we clarify a matter of terminology. Suppose a graph $G$ contains a circuit $C$, i.e. an eulerian subgraph. By labeling the vertices of $C$ we shall mean assigning labels, say $w_1,w_2,\ldots,w_s$, to the vertices of $C$, where a vertex of $C$ may receive more than one label, so that

$$w_1,w_2,\ldots,w_s,w_1$$

is a circuit of $G$ and $h \in E(C)$ if and only if

$h = w_iw_{i+1}$, for some $i$ satisfying $1 \leq i \leq s$ (subscripts expressed modulo $s$).

Theorem 4-1. Let $e$ and $f$ be adjacent edges of a graph $G$ corresponding to the (adjacent) vertices $x$ and $y$ of $L(G)$, respectively. If $G \cong K(l,n)$, for some $n \geq 3$, or if there is a dominating circuit $C$ of $G$ whose vertices can be labeled so that (A1), (A2), or (A3) holds, then there is a hamiltonian cycle of $L(G)$ which contains the edge $xy$.
We now show that the sufficient condition given in Theorem 4-1 is, in fact, a necessary and sufficient condition.

Theorem 4-2. Let \( xy \in E(L(G)) \) where \( x \) and \( y \) correspond to the (adjacent) edges \( e \) and \( f \) of a graph \( G \), respectively. If there exists a Hamiltonian cycle of \( L(G) \) which contains the edge \( xy \), then \( G \cong K(1,n) \), for some \( n \geq 3 \), or there is a dominating circuit \( C \) of \( G \) whose vertices can be labeled so that (A1), (A2), or (A3) holds.

Proof. The theorem obviously holds if \( |E(G)| = 3 \). Therefore we will assume \( |E(G)| \geq 4 \) and \( G \neq K(1,n) \).

Let \( x_1,x_2=x,x_3=y,...,x_q,x_1 \) be a Hamiltonian cycle of \( L(G) \) which contains the edge \( xy \). Corresponding to this cycle is an ordering \( S: e_1,e_2=e,e_3=f,...,e_q \) of the \( q \) edges of \( G \) where, for \( 1 \leq i \leq q \), the edge \( e_i \) is adjacent to \( e_{i+1} \) in \( G \) (subscripts expressed modulo \( q \)). For \( 1 \leq i \leq q - 1 \), let \( v_i \) be the vertex of \( G \) incident with both \( e_i \) and \( e_{i+1} \). Then, using the construction and notation introduced in the first part of this section, we know that one of the following sequences of vertices and edges of \( G \)
is a dominating circuit \( C \) of \( G \) which satisfies the following conditions:

(i) \( 1 < j_1 < j_2 < ... < j_t \leq q - 1 \),

(ii) for \( 1 \leq i \leq j \), the edge \( e_i \) is incident with \( v_i \),

(iii) for \( 1 \leq k \leq t - 1 \) and \( j_k \leq i \leq j_{(k+1)} \),
    the edge \( e_i \) is incident with \( v_{j_k} \),

(iv) for \( j_t \leq i \leq q \), the edge \( e_i \) is incident with \( v_{j_t} \).

We now show that the vertices of \( C \) can be labeled so that (A1), (A2), or (A3) holds. We consider four possible (and exhaustive) cases.

Case 1. Suppose \( v_1 = v_2 = v_3 \). In this case, \( j_1 \geq 4 \).

Thus \( e_2 (= e) \) and \( e_3 (= f) \) are incident with \( v_1 \) and \( E(C) \subseteq E(G) - \{e_2, e_3\} = E(G) - \{e, f\} \). This implies that \( V(C) \) can be labeled so that (A1) holds.
Case 2. Suppose \( v_1 = v_2 \) and \( v_2 \neq v_3 \). In this case, 
\( j_1 = 3 \). Thus \( e_2 (= e) \) is incident with \( v_1 \), \( e_{j_1} = e_3 (= f) \), and \( E(C) \subseteq E(G) - \{e_2\} = E(G) - \{e\} \). This implies that \( V(C) \) can be labeled so that (A2) holds.

Case 3. Suppose \( v_1 \neq v_2 \) and \( v_2 = v_3 \). In this case, 
\( j_1 = 2 \) and \( j_2 \geq 4 \). Thus \( e_3 (= f) \) is incident with \( v_2 = v_{j_1} \), \( e_{j_1} = e_2 (= e) \), and \( E(C) \subseteq E(G) - \{e_3\} = E(G) - \{f\} \). This implies that \( V(C) \) can be labeled so that (A2) holds.

Case 4. Suppose \( v_1 \neq v_2 \) and \( v_2 \neq v_3 \). In this case, 
\( j_1 = 2 \) and \( j_2 = 3 \). Thus \( e_{j_1} = e_2 (= e) \) and \( e_{j_2} = e_3 (= f) \). Therefore, since \( C \) satisfies conditions (i)-(iv), we can label \( V(C) \) so that (A3) holds.

By combining the results of Theorems 4-1 and 4-2, we obtain a necessary and sufficient condition for a graph to have a strongly hamiltonian line graph. We state this as Theorem 4-3.

**Theorem 4-3.** The line graph \( L(G) \) of a graph \( G \) is strongly hamiltonian if and only if \( G \cong K(1,n) \), for some \( n \geq 3 \), or for each pair \( e,f \) of adjacent edges of \( G \), there is a dominating circuit \( C \) of \( G \) whose vertices can be labeled so that (A1), (A2), or (A3) holds.
For a graph $G$ with a relatively small number of dominating circuits, Theorem 4-3 may be applied fairly reasonably to show that $L(G)$ is not strongly hamiltonian. The graph $G$ of Figure 4.1 is a block with a spanning circuit. Yet $G$ fails to contain a dominating circuit with the necessary properties for the designated edges $e$ and $f$. Thus $L(G)$ is not strongly hamiltonian.

![Figure 4.1](image)

Some implications of the sufficiency condition of Theorem 4-3 will be given by Corollaries 4-3-1 through 4-3-3. In order to present Corollary 4-3-1, we first establish a lemma.

**Lemma 4-1.** If $G$ is a 2-connected eulerian graph and $e_1$ and $e_2$ are adjacent edges of $G$, then there exists an eulerian circuit $C: v_1, v_2, \ldots, v_s, v_1$ of $G$ such that $e_1 = v_1v_2$ and $e_2 = v_2v_3$.

**Proof.** Let $v_1$ be the vertex incident with $e_1$ but not with $e_2$, let $v_3$ be the vertex incident with $e_2$ but
not with $e_1$, and let $v_2$ be the vertex incident with both $e_1$ and $e_2$. Since $G$ is 2-connected, $G - v_2$ is connected. If $\deg_G v_2 > 2$, then the graph $G - \{e_1, e_2\}$ is a connected graph with exactly two odd vertices, $v_1$ and $v_3$. Thus $G - \{e_1, e_2\}$ contains an eulerian $v_3 - v_1$ trail $T$. Since $T$ is a spanning trail of $G$ which contains every edge of $G$ other than $e_1$ and $e_2$, the path $v_1, v_2, v_3$ of $G$ followed by the trail $T$ is the desired eulerian circuit of $G$. Therefore, we may assume $\deg_G v_2 = 2$. Then $G - v_2$ is a connected graph with exactly two odd vertices, $v_1$ and $v_3$, and so contains an eulerian $v_3 - v_1$ trail $T$. Since $T$ is a trail of $G$ which contains every edge of $G$ other than $e_1$ and $e_2$ and every vertex of $G$ other than $v_2$, the path $v_1, v_2, v_3$ of $G$ followed by $T$ is the desired eulerian circuit. 

**Corollary 4-3-1.** If a graph $G$ contains a 2-connected spanning circuit, then $L(G)$ is strongly hamiltonian.

**Proof.** Let $e$ and $f$ be an arbitrary pair of adjacent edges of $G$. By hypothesis, $G$ contains a 2-connected spanning circuit, say $C$, which is necessarily a dominating circuit of $G$. Suppose at least one of $e$ and $f$ is not an edge of $C$. Since $C$ is a spanning circuit of $G$, $V(C)$ can be labeled so that (A1) or (A2) holds. Suppose, on the other hand, that both $e$ and $f$
are edges of $C$. Since $C$ is a 2-connected spanning circuit of $G$, we may apply Lemma 4-1 to conclude that $V(C)$ can be labeled so that (A3) holds. An application of Theorem 4-3 now yields the desired result. \(\Box\)

**Corollary 4-3-2.** If a graph $G$ is hamiltonian, then $L(G)$ is strongly hamiltonian.

**Proof.** A hamiltonian cycle of $G$ is a 2-connected spanning circuit. \(\Box\)

**Corollary 4-3-3.** If a graph $G$ is strongly hamiltonian, then $L(G)$ is strongly hamiltonian.

We next consider 1-edge hamiltonian line graphs. Clearly, a graph $G$ has a 1-edge hamiltonian line graph if and only if for each pair $e,f$ of adjacent edges of $G$, there exists an ordering $S: e_1,e_2,\ldots,e_q$ of the $q$ edges of $G$ where, for $1 \leq i \leq q$, the edge $e_i$ is adjacent to $e_{i+1}$ in $G$ and $\{e_i, e_{i+1}\} \neq \{e, f\}$, where all subscripts are expressed modulo $q$. As in the case of strongly hamiltonian line graphs, we shall see that this statement involving orderings of the edges of $G$ can be translated to a statement involving dominating circuits of $G$.

Suppose $C: w_1,w_2,\ldots,w_s,w_1$ is a dominating circuit of a graph $G$ and $e$ and $f$ are adjacent edges of $G$ corresponding to the (adjacent) vertices $x$ and $y$ of
L(G), respectively. Thus if $i \neq j$, then $v_i w_{i+1} \neq w_j w_{j+1}$ (subscripts expressed modulo $s$) although $v_i$ and $w_j$ are not necessarily distinct. As before, in what follows we will let $v$ and $v'$ denote vertices of $G$ which may or may not be on $C$.

(B1). If $e = w_1 v$, $f = w_k v'$ ($k \neq 1$), and $e, f \notin E(C)$, consider the following ordering $S$ of the edges of $G$. Begin by selecting, in any order, all edges of $E(G) - \{f\}$ incident with $w_1$ that are not edges of $C$, followed by the edge $w_1 w_2$. At each successive $w_i$, $2 \leq i \leq s$, select, in any order, all edges of $G$ incident with $w_i$ that are neither edges of $C$ nor previously selected edges, followed by the edge $w_i w_{i+1}$ (subscripts expressed modulo $s$). This process terminates with the edge $w_s w_1$.

(B2). If $e = w_1 v$, $f = w_1 v'$, neither $e$ nor $f$ is an edge of $C$, and there exists $h \in E(G) - (E(C) \cup \{e, f\})$ such that $h$ is incident with $w_1$, then consider the following ordering $S$ of the edges of $G$. Begin by selecting $e, h, f$ in that order and then, in any order, all remaining edges of $G$ incident with $w_1$ that are not edges of $C$, followed by the edge $w_1 w_2$. At each successive $w_i$, $2 \leq i \leq s$, select, in any order,
all edges of $G$ incident with $w_i$ that are neither edges of $C$ nor previously selected edges, followed by the edge $w_{i+1}$. This process terminates with the edge $w_s w_1$.

(B3). If $e = w_i w_2$, $f = w_k v$ $(3 \leq k \leq s)$, and $f \not\in E(C)$---or---$f = w_i w_2$, $e = w_k v$ $(3 \leq k \leq s)$, and $e \not\in E(C)$, consider the following ordering $S$ of the edges of $G$. Begin by selecting, in any order, all edges of $E(G) - \{w_k v\}$ incident with $w_1$ that are not edges of $C$, followed by the edge $w_1 w_2$. Next select, in any order, all edges of $E(G) - \{w_k v\}$ incident with $w_2$ that are not edges of $C$, followed by the edge $w_2 w_3$. At each successive $w_i$, $3 \leq i \leq s$, select, in any order, all edges of $G$ incident with $w_i$ that are neither edges of $C$ nor previously selected edges, followed by the edge $w_i w_{i+1}$ (subscripts expressed modulo $s$). This process terminates with the edge $w_s w_1$.

(B4). If $e = w_i w_2$, $f = w_2 v$, $f \not\in E(C)$, and there exists $h \in E(G) - (E(C) \cup \{f\})$ such that $h$ is incident with $w_2$---or---$f = w_i w_2$, $e = w_2 v$, $e \not\in E(C)$, and there exists $h \in E(G) - (E(C) \cup \{e\})$ such that $h$ is incident with $w_2$, then
consider the following ordering \( S \) of the edges of \( G \). Begin by selecting, in any order, all edges of \( G \) incident with \( w_1 \) that are not edges of \( C \), followed by the edge \( w_1w_2 \). Next select \( h \) and \( w_2v \) in that order and then, in any order, all remaining edges of \( G \) incident with \( w_2 \) that are not edges of \( C \), followed by the edge \( w_2w_3 \). At each successive \( w_i \), \( 3 \leq i \leq s \), select, in any order, all edges of \( G \) incident with \( w_i \) that are neither edges of \( C \) nor previously selected edges, followed by the edge \( w_iw_{i+1} \) (subscripts expressed modulo \( s \)). This process terminates with the edge \( w_sw_1 \).

\( (B5) \). If \( e = w_1w_2 \) and \( f = w_kw_{k+1} \) \((3 \leq k \leq s - 1)\), consider the following ordering \( S \) of the edges of \( G \). Begin by selecting, in any order, all edges of \( G \) incident with \( w_1 \) that are not edges of \( C \), followed by the edge \( w_1w_2 \). At each successive \( w_i \), \( 2 \leq i \leq s \), select, in any order, all edges of \( G \) incident with \( w_i \) that are neither edges of \( C \) nor previously selected edges, followed by the edge \( w_iw_{i+1} \) (subscripts expressed modulo \( s \)). This process terminates with the edge \( w_sw_1 \).
If \( e = w_1w_2 \), \( f = w_2w_3 \), and there exists \( h \in E(G) - E(C) \) such that \( h \) is incident with \( w_2 \), consider the following ordering \( S \) of the edges of \( G \). Begin by selecting, in any order, all edges of \( G \) incident with \( w_1 \) that are not edges of \( C \), followed by the edge \( w_1w_2 \). Next select \( h \) and then, in any order, all remaining edges of \( G \) incident with \( w_2 \) that are not edges of \( C \), followed by the edge \( w_2w_3 \). At each successive \( w_i \), \( 3 \leq i \leq s \), select, in any order, all edges of \( G \) incident with \( w_i \) that are neither edges of \( C \) nor previously selected edges, followed by the edge \( w_iw_{i+1} \) (subscripts expressed modulo \( s \)). This process terminates with the edge \( w_sw_1 \).

We observe that in each case the ordering \( S \) has the properties that every edge of \( G \) appears exactly once in \( S \), consecutive edges of \( S \) as well as the first and last edges of \( S \) are adjacent in \( G \), and \( e \) and \( f \) are not consecutive edges of \( S \) nor does the pair \( e,f \) constitute the first and last edges of \( S \). This implies that \( L(G) - xy \) contains a hamiltonian cycle. Thus a sufficient condition for \( L(G) - xy \) to be hamiltonian can be given in terms of dominating circuits of \( G \).

**Theorem 4-4.** Let \( e \) and \( f \) be adjacent edges of a graph \( G \) corresponding to the (adjacent) vertices \( x \) and \( y \) of...
L(G), respectively. If \( G \cong K(1,n) \), for some \( n \geq 4 \), or if there is a dominating circuit \( C \) of \( G \) whose vertices can be labeled so that at least one of (B1) - (B6) holds, then \( L(G) - xy \) is hamiltonian.

As the next theorem will indicate, the sufficient condition given in Theorem 4-4 for \( L(G) - xy \) to be hamiltonian is also a necessary condition.

**Theorem 4-5.** Let \( xy \in E(L(G)) \) where \( x \) and \( y \) correspond to the (adjacent) edges \( e \) and \( f \) of a graph \( G \), respectively. If \( L(G) - xy \) is hamiltonian, then \( G \cong K(1,n) \), for some \( n \geq 4 \), or there is a dominating circuit \( C \) of \( G \) whose vertices can be labeled so that at least one of (B1) - (B6) holds.

**Proof.** Clearly, \( |E(G)| \geq 4 \) and it is easily verified that the theorem is true if \( |E(G)| = 4 \). Therefore we will assume that \( |E(G)| \geq 5 \) and that \( G \neq K(1,n) \).

Let \( x_1, x_2, x_3, \ldots, x_q = y, \ldots, x_q, x_1 \) be a hamiltonian cycle of \( L(G) - xy \). Since \( q \geq 5 \), we may assume that \( 4 \leq q - 1 \). Corresponding to this cycle is an ordering \( S: e_1, e_2 = e, e_3, \ldots, e_{q-1} = f, \ldots, e_q \) of the \( q \) edges of \( G \) where, for \( 1 \leq i \leq q \), the edge \( e_i \) is adjacent to \( e_{i+1} \) in \( G \) (subscripts expressed modulo \( q \)). For \( 1 \leq i \leq q - 1 \), let \( v_i \) be the vertex of \( G \) incident with both \( e_i \) and \( e_{i+1} \). Then, using the construction
and notation introduced in the first part of this section, we know that one of the following sequences of vertices and edges of $G$

$$v_1, e_{j_1}, v_{j_1}, e_{j_2}, v_{j_2}, \ldots, v_{j(t-1)}, e_{j_t}, v_{j_t} = v_{q-1}$$

$$v_1, e_{j_1}, v_{j_1}, e_{j_2}, v_{j_2}, \ldots, v_{j(t-1)}, e_{j_t}, v_{j_t} = v_{q-1}, e_{q}, v_1$$

$$v_1, e_{j_1}, v_{j_1}, e_{j_2}, v_{j_2}, \ldots, v_{j(t-1)}, e_{j_t}, v_{j_t} = v_{q-1}, e_1, v_1$$

$$v_1, e_{j_1}, v_{j_1}, e_{j_2}, v_{j_2}, \ldots, v_{j(t-1)}, e_{j_t}, v_{j_t} = v_{q-1}, e_q, e_1, v_1$$

is a dominating circuit $C$ of $G$ which satisfies the following conditions:

(i) $1 < j_1 < j_2 < \ldots < j_t \leq q - 1$,

(ii) for $1 \leq i \leq j_1$, the edge $e_i$ is incident with $v_1$,

(iii) for $1 \leq k \leq t - 1$ and $j_k \leq i \leq j(k+1)$, the edge $e_i$ is incident with $v_{j_k}$,

(iv) for $j_t \leq i \leq q$, the edge $e_i$ is incident with $v_{j_t}$,

(v) for $2 \leq i \leq q - 1$, the edge $e_i$ is an edge of $C$ if and only if $v_i \neq v_{i-1}$.

We will show that the vertices of $C$ can be labeled so that at least one of (B1) - (B6) holds. We consider four possible (and exhaustive) cases.
Case 1. Suppose \( v_1 = v_2 \) and \( v_{t-1} = v_t \). In this case, \( e, f \not\in E(C) \). If \( v_2 = v_3 = \ldots = v_{t-1} \), then \( e_2 (= e), e_3, \ldots, e_{t-1} (= f) \) are all edges of \( E(G) - E(C) \) that are incident with \( v_1 \), implying that \( V(C) \) can be labeled so that (B2) holds. Otherwise, one of three situations must occur. Either

(a) the edge \( e_2 (= e) \) is incident with \( v_1 \) and there exists an integer \( n \), with \( 1 \leq n \leq t - 1 \), such that \( e_n (= f) \) is incident with \( v_{j_n} \), in which case \( V(C) \) can be labeled so that (B1) holds, or

(b) the edge \( e_2 (= e) \) is incident with \( v_1, v_{j_t} \neq v_1 \), and \( e_t (= f) \) is incident with \( v_{j_t} \), in which case \( V(C) \) can be labeled so that (B1) holds, or

(c) the edge \( e_2 (= e) \) is incident with \( v_1 \), \( e_t (= f) \) is incident with \( v_1 \), and \( e_1, e_2 \) are edges of \( E(G) - E(C) \) that are incident with \( v_1 \), in which case \( V(C) \) can be labeled so that (B2) holds.

Case 2. Suppose \( v_1 = v_2 \) and \( v_{t-1} \neq v_t \). In this case, \( e \not\in E(C) \) and \( f \not\in E(C) \). If \( v_2 = v_3 = \ldots = v_{t-1} \), then \( e_2 (= e), e_3, \ldots, e_{t-1} \) are all edges of \( E(G) - E(C) \) that are incident with \( v_1 \). Moreover \( j_1 = t \) so that \( e_t (= f) = e_{j_1} = v_1v_{j_1} \). Thus \( V(C) \) can be labeled so that (B4) holds. Otherwise, one of three situations must occur. Either
(a) the edge $e_2 (= e)$ is incident with $v_1$ and there exists an integer $n$, with $2 \leq n \leq t - 1$, such that $e_t (= f) = e_j = v_{j_{(n-1)}}v_{j_n}$, in which case $V(C)$ can be labeled so that (B3) holds, or

(b) the edge $e_2 (= e)$ is incident with $v_1$, $v_{j_t} \neq v_1$, and $e_t (= f) = e_j = v_{j_{(t-1)}}v_{j_t}$, in which case $V(C)$ can be labeled so that (B3) holds, or

(c) the edge $e_2 (= e)$ is incident with $v_1$, $e_t (= f) = e_j = v_{j_{(t-1)}}v_1$, and $e_1, e_q$ are edges of $E(G) - E(C)$ that are incident with $v_1$, in which case $V(C)$ can be labeled so that (B4) holds.

Case 3. Suppose $v_1 \neq v_2$ and $v_{t-1} = v_t$. In this case, $e \in E(C)$ and $f \not\in E(C)$. If $v_2 = v_3 = \ldots = v_{t-1}$, then $e_2, e_4, \ldots, e_t (= f)$ are all edges of $E(G) - E(C)$ that are incident with $v_2$. Moreover, $j_1 = 2$ so that $e_2 (= e) = e_{j_1} = v_1v_{j_1} = v_1v_2$. Thus $V(C)$ can be labeled so that (B4) holds. Otherwise, one of three situations must occur. Either

(a) the edge $e_2 (= e) = e_{j_1} = v_1v_{j_1}$ and there exists an integer $n$, with $2 \leq n \leq t - 1$, such that $e_t (= f)$ is incident with $v_{j_n}$, in which case $V(C)$ can be labeled so that (B3) holds, or

(b) the edge $e_2 (= e) = e_{j_1} = v_1v_{j_1}$, $v_{j_t} \neq v_1$. 

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and \( e_t (= f) \) is incident with \( v_{j_t} \), in which case \( V(C) \) can be labeled so that (B3) holds, or

(c) the edge \( e_2 (= e) = e_{j_1} = v_1 v_{j_1} \), \( e_t (= f) \) is incident with \( v_1 \), and \( e_1, e_q \) are edges of \( E(G) - E(C) \) that are incident with \( v_1 \), in which case \( V(C) \) can be labeled so that (B4) holds.

Case 4. Suppose \( v_1 \neq v_2 \) and \( v_{t-1} \neq v_t \). In this case, \( e, f \in E(C) \). If \( v_2 = v_3 = \ldots = v_{t-1} \), then \( e_3, e_4, \ldots, e_{t-1} \) are all edges of \( E(G) - E(C) \) that are incident with \( v_2 \). Moreover, \( j_1 = 2 \) and \( j_2 = t \) so that

\[
e_2 (= e) = e_{j_1} = v_1 v_{j_1} = v_1 v_2 \quad \text{and} \quad e_t (= f) = e_{j_2} = v_{j_1} v_{j_2} = v_2 v_t .
\]

Thus \( V(C) \) can be labeled so that (B6) holds. Otherwise, one of three situations must occur. Either

(a) the edge \( e_2 (= e) = e_{j_1} = v_1 v_{j_1} \) and there exists an integer \( n \), with \( 3 \leq n \leq t - 1 \), such that \( e_t (= f) = e_{j_n} = v_j v_{j_n} \), in which case \( V(C) \) can be labeled so that (B5) holds, or

(b) the edge \( e_2 (= e) = e_{j_1} = v_1 v_{j_1} \), \( v_j \neq v_1 \), and \( e_t (= f) = e_{j_t} = v_j v_{j_t} \), in which case \( V(C) \) can be labeled so that (B5) holds, or

(c) the edge \( e_2 (= e) = e_{j_1} = v_1 v_{j_1} \), \( e_t (= f) = e_{j_t} = v_j v_{j_t} \), \( v_j \neq v_1 \), and \( e_1, e_q \) are edges of \( E(G) - E(C) \)
that are incident with $v_1$, in which case $V(C)$ can be labeled so that (B6) holds. ⊗

Theorem 4-6 combines the results of the two previous theorems to give a necessary and sufficient condition for a graph to have a 1-edge hamiltonian line graph.

**Theorem 4-6.** The line graph $L(G)$ of a graph $G$ is 1-edge hamiltonian if and only if $G \cong K(l,n)$, for some $n \geq 4$, or for each pair $e,f$ of adjacent edges of $G$, there is a dominating circuit $C$ of $G$ whose vertices can be labeled so that at least one of (B1) - (B6) holds.

A simple example of the necessity condition of Theorem 4-6 is given by the graph $G$ consisting of a cycle $C: v_1, v_2, \ldots, v_p, v_1$ together with a collection of vertices $w_1, w'_1, w_2, w'_2, \ldots, w_p, w'_p$, where $E(G) = E(C) \cup \{v_iw_i \mid 1 \leq i \leq p\} \cup \{v_iw'_i \mid 1 \leq i \leq p\}$. Then $\delta(L(G)) = 3$ and, since $C$ is a dominating circuit of $G$, $L(G)$ is hamiltonian. It is easily verified that $L(G)$ is not 1-edge hamiltonian. We observe that $G$ contains no dominating circuit with the necessary properties for the adjacent edges $v_iw_i$ and $v_iw'_i$ ($1 \leq i \leq p$).

**Corollary 4-6-1.** If a graph $G$ is 1-edge hamiltonian, then $L(G)$ is 1-edge hamiltonian and $1$-hamiltonian.

**Proof.** Let $e$ and $f$ be an arbitrary pair of adjacent
edges of G. Since G is 1-edge hamiltonian, there exists a hamiltonian cycle C of G - e. Then C is a dominating circuit of G whose vertices can be labeled so that at least one of (Bl) - (B4) holds. Thus L(G) is 1-edge hamiltonian.

Since G is 1-edge hamiltonian, G - e is hamiltonian for each edge e of G. Therefore L(G - e) is hamiltonian for each edge e of G or, equivalently, L(G) - x is hamiltonian for each vertex x of L(G). Since L(G) is hamiltonian, we conclude that L(G) is 1-hamiltonian. ®

It will subsequently be shown that if a graph G is 1-hamiltonian, then L(G) is also 1-hamiltonian. With this in mind, we present a second corollary of Theorem 4-6.

**Corollary 4-6-2.** If a graph G is 1-hamiltonian, then L(G) is 1-edge hamiltonian (and 1-hamiltonian).

**Proof.** Let e and f be an arbitrary pair of adjacent edges of G, where v is the vertex incident with both e and f. Since G is 1-hamiltonian, there exists a cycle C of G which contains every vertex of V(G) - {v}. Then C is a dominating circuit of G whose vertices can be labeled so that (Bl) holds. Thus G is 1-edge hamiltonian. ®

Chartrand [4] showed that the line graph function

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preserves the property of being 0-edge hamiltonian and we have seen that the property of being 1-edge hamiltonian is also preserved under the line graph function. We now observe that Corollary 4-3-2 improves Chartrand's result in the sense that if a graph G is 0-edge hamiltonian, then L(G) is strongly 0-edge hamiltonian. Therefore, it is perhaps natural to conjecture that if a graph G is 1-edge hamiltonian, then L(G) is strongly 1-edge hamiltonian. The final theorem of this section verifies this conjecture.

Theorem 4-7. If a graph G is 1-edge hamiltonian, then L(G) is strongly 1-edge hamiltonian.

Proof. Let wx and yz be distinct edges of L(G) and let e_1, e_2, e_3, and e_4 (not necessarily all distinct) be the edges of G corresponding to w, x, y, and z, respectively. We wish to show that there is a hamiltonian cycle of L(G) - yz that contains the edge wx. It suffices to exhibit an ordering S of the edges of G with the following properties:

(i) every edge of G appears exactly once in S,
(ii) consecutive edges of S are adjacent in G,
(iii) the first and last edges of S are adjacent in G,
(iv) e_1 and e_2 are either consecutive edges of S or one of e_1 and e_2 is the first edge.

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of S and the other is the last edge of S,

(v) \( e_3 \) and \( e_4 \) are not consecutive edges of \( S \) and \( e_3 \) and \( e_4 \) are not the first and last edges of \( S \).

We proceed by cases.

Case 1. Suppose \( |\{e_1,e_2,e_3,e_4\}| = 4 \) and \( e_1,e_2,e_3,e_4 \) are mutually adjacent. Let \( C: w_1,w_2,\ldots,w_p,w_1 \) be a hamiltonian cycle of \( G \) which does not contain the edge \( e_3 \). Without loss of generality, we may assume that \( w_1 \) is incident with \( e_i \), \( 1 \leq i \leq 4 \). An ordering \( S \) of the edges of \( G \) is constructed as follows. Begin by selecting \( e_1 \) (if it is not on \( C \)) and then \( e_2 \) (if it is not on \( C \)). Follow this by the edge \( w_1w_2 \). Next select, in any order, all edges of \( G \) incident with \( w_2 \) that are not edges of \( C \), followed by the edge \( w_2w_3 \). At each successive \( w_i \), \( 3 \leq i \leq p \), select, in any order, all edges of \( G \) incident with \( w_i \) that are neither edges of \( C \) nor previously selected edges, followed by the edge \( w_iw_{i+1} \) (subscripts expressed modulo \( p \)). This process terminates with the edge \( w_pw_1 \).

Case 2. Suppose \( |\{e_1,e_2,e_3,e_4\}| = 4 \) and exactly three of \( e_1,e_2,e_3,e_4 \) are mutually adjacent.

Subcase (i). Suppose \( e_1,e_2,e_3 \) are mutually adjacent. (The case in which \( e_1,e_2,e_4 \) are mutually adjacent is handled in a similar fashion.) Let \( C: w_1,w_2,\ldots,w_p,w_1 \)
be a hamiltonian cycle of $G$ which does not contain the edge $e_4$. Without loss of generality, we may assume that $w_1$ is incident with $e_i$, $1 \leq i \leq 3$, and that $e_3 = w_1w_k$ for some $k$ with $3 \leq k \leq p$. An ordering $S$ of the edges of $G$ is constructed as follows. Begin by selecting $e_1$ (if it is not on $C$) and then $e_2$ (if it is not on $C$), followed by the edge $w_1w_2$. Next select, in any order, all edges of $G$ incident with $w_2$ that are not edges of $C$, followed by the edge $w_2w_3$. At each successive $w_i$, $3 \leq i \neq k \leq p$, select, in any order, all edges of $G$ incident with $w_i$ that are neither edges of $C$ nor previously selected edges, followed by the edge $w_iw_{i+1}$ (subscripts expressed modulo $p$). At $w_k$ select, in any order, all edges of $G$ other than $e_4$ incident with $w_k$ that are neither edges of $C$ nor previously selected edges, followed by the edge $w_kw_{k+1}$ (subscripts expressed modulo $p$). This process terminates with the edge $w_pw_1$.

Subcase (ii). Suppose $e_2, e_3, e_4$ are mutually adjacent. (The case in which $e_1, e_3, e_4$ are mutually adjacent is handled in a similar fashion.) Let $C: w_1, w_2, \ldots, w_p, w_1$ be a hamiltonian cycle of $G$ which does not contain the edge $e_4$. Without loss of generality, we may assume that $w_1$ is incident with $e_i$, $2 \leq i \leq 4$, and that $e_2 = w_1w_k$ for some $k$ with $3 \leq k \leq p$. An ordering $S$ of the edges of $G$ is constructed as follows. Begin by
selecting, in any order, all edges of $G$ other than $e_4$ or $e_2$ incident with $w_1$ that are not edges of $C$, followed by the edge $w_1w_2$. Next select, in any order, all edges of $G$ other than $e_1$ incident with $w_2$ that are not edges of $C$, followed by the edge $w_2w_3$. At each successive $w_i$, $3 \leq i (\neq k) \leq p$, select, in any order, all edges of $G$ other than $e_1$ incident with $w_i$ that are neither edges of $C$ nor previously selected edges, followed by the edge $w_iw_{i+1}$ (subscripts expressed modulo $p$). At $w_k$, select $e_1$ (if it is not on $C$) and then $e_2$ (if it is not on $C$), followed by the edge $w_kw_{k+1}$ (subscripts expressed modulo $p$). This process terminates with the edge $w_pw_1$.

Case 3. Suppose $|\{e_1, e_2, e_3, e_4\}| = 4$ and no three of $e_1, e_2, e_3, e_4$ are mutually adjacent. Let $C: w_1, w_2, \ldots, w_p, w_1$ be a Hamiltonian cycle of $G$ which does not contain the edge $e_3$. Without loss of generality, we may assume that $w_1$ is incident with $e_3$ and $e_4$ and that $w_k$ is incident with $e_1$ and $e_2$ for some $k$ with $3 \leq k \leq p$. An ordering $S$ of the edges of $G$ is constructed as follows. Begin by selecting, in any order, all edges of $G$ other than $e_3$ incident with $w_1$ that are not edges of $C$, followed by the edge $w_1w_2$. Next select, in any order, all edges of $G$ other than $e_1$ or $e_2$ incident with $w_2$ that are not edges of $C$, followed
by the edge $w_2w_3$. At each successive $w_i$, $3 \leq i (\neq k) \leq p$, select, in any order, all edges of $G$ other than $e_1$ or $e_2$ incident with $w_i$ that are neither edges of $C$ nor previously selected edges, followed by the edge $w_iw_{i+1}$ (subscripts expressed modulo $p$). At $w_k$, select $e_1$ (if it is not on $C$) and then $e_2$ (if it is not on $C$), followed by the edge $w_kw_{k+1}$ (subscripts expressed modulo $p$). This process terminates with the edge $w_pw_1$.

**Case 4.** Suppose $|\{e_1, e_2, e_3, e_4\}| = 3$, where $e_2 = e_3$ and $e_1, e_2, e_4$ are mutually adjacent. (The cases in which $e_2 = e_4$, $e_1 = e_3$, or $e_1 = e_4$ and the three distinct edges are mutually adjacent are handled similarly.) Let $C: w_1, w_2, \ldots, w_p, w_1$ be a hamiltonian cycle of $G$ which does not contain the edge $e_1$. Without loss of generality, we may assume that $w_1$ is incident with $e_i$, $1 \leq i \leq 4$, that $e_4 = w_1w_k$ for some $k$ with $2 \leq k \leq p-1$, and that $e_2 = w_1w_\ell$ for some $\ell$ with $3 \leq \ell \leq p$. An ordering $S$ of the edges of $G$ is constructed as follows. Begin by selecting $e_2$ (if it is not on $C$) and then $e_1$, followed by the edge $w_1w_2$. Next select, in any order, all edges of $G$ incident with $w_2$ that are not on $C$, followed by the edge $w_2w_3$. At each successive $w_i$, $3 \leq i \leq p$, select, in any order, all edges of $G$ incident with $w_i$ that are neither edges
of C nor previously selected edges, followed by the edge \( w_iw_{i+1} \) (subscripts expressed modulo \( p \)). This process terminates with the edge \( wpw_1 \).

**Case 5.** Suppose \( |\{e_1, e_2, e_3, e_4\}| = 3 \), where \( e_2 = e_3 \) and \( e_1, e_2, e_4 \) are not mutually adjacent. (The cases in which \( e_2 = e_4 \), \( e_1 = e_3 \), or \( e_1 = e_4 \) and the three distinct edges are not mutually adjacent are handled similarly.) Let \( C: w_1w_2'...wpw_1 \) be a hamiltonian cycle of \( G \) which does not contain \( e_2 = e_3 \). Without loss of generality, we may assume that \( w_1 \) is incident with \( e_1 \) and \( e_2 \). An ordering \( S \) of the edges of \( G \) is constructed as follows. Begin by selecting \( e_1 \) (if it is not on \( C \)) and then \( e_2 \), followed by the edge \( w_1w_2 \). Next select, in any order, all edges of \( G \) incident with \( w_2 \) that are not edges of \( C \), followed by the edge \( w_2w_3 \). At each successive \( w_i \), \( 3 \leq i \leq p \), select, in any order, all edges of \( G \) incident with \( w_i \) that are neither edges of \( C \) nor previously selected edges, followed by the edge \( w_iw_{i+1} \) (subscripts expressed modulo \( p \)). This process terminates with the edge \( wpw_1 \). \( \otimes \)

Section 4.2

\textbf{n-Hamiltonian Line Graphs I}

As was seen in the previous section, it is often the case that fairly restrictive conditions must be satisfied.

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by a graph in order to insure that its line graph will possess some hamiltonian property. However, the situation changes somewhat when one considers the second iterated line graph of a graph $G$. In this case, we have an intermediate graph, namely the line graph $L(G)$, which often has very useful characteristics not depending upon extremely restrictive conditions being satisfied by the graph $G$. In this section we will concentrate on the relationship between second iterated line graphs and the property of being $n$-hamiltonian, $n \geq 0$. By using the structure of $L(G)$, we will be able to determine values of $n$ for which $L^2(G)$ is $n$-hamiltonian depending only on the minimum degree of the graph $G$. This result will then be improved for graphs $G$ which are 2-connected. Some preliminaries will be useful.

Let $G$ be any connected graph with $\delta(G) \geq 3$. Then each vertex $w$ of $G$ is incident with precisely $\deg_G w$ edges of $G$. These edges correspond to $\deg_G w$ mutually adjacent vertices of $L(G)$, that is, a complete subgraph denoted $K(w)$ having order $\deg_G w$ in $L(G)$. For each vertex $w$ of $G$, let $S(w)$ be any connected spanning subgraph of $K(w)$ in $L(G)$. Then the subgraph, denoted by $B(G)$, of $L(G)$, defined by

$$V(B(G)) = \bigcup_{w \in V(G)} V(S(w)),$$

$$E(B(G)) = \bigcup_{w \in V(G)} E(S(w))$$
is called a bonding subgraph of \( L(G) \).

Suppose \( G \) is a connected graph with \( \delta(G) \geq 3 \) and \( B(G) \) is a bonding subgraph of \( L(G) \). Every vertex \( u \) of \( L(G) \) corresponds to an edge \( w_1w_2 \) of \( G \), so that \( u \in V(K(w)) \) if and only if \( w = w_1 \) or \( w = w_2 \) (\( w \in V(G) \)). Thus \( u \in V(S(w)) \) if and only if \( w = w_1 \) or \( w = w_2 \). Since each edge of \( L(G) \) corresponds to a pair of adjacent edges of \( G \), each edge of \( L(G) \) lies in exactly one complete subgraph \( K(w) \) of \( L(G) \). Thus each edge of \( B(G) \) lies in exactly one subgraph \( S(w) \) of \( B(G) \). From these observations, we conclude that \( B(G) \) is a spanning subgraph of \( L(G) \) and if \( u \) is a vertex of \( L(G) \) corresponding to the edge \( w_1w_2 \) of \( G \), then

\[
\deg_{B(G)} u = \deg_{S(w_1)} u + \deg_{S(w_2)} u.
\]

Let \( v_1 \) and \( v_2 \) be vertices of \( B(G) \). Then for some vertices \( w_1 \) and \( w_2 \) of \( G \) (not necessarily distinct), \( v_1 \) is a vertex of \( S(w_1) \) and \( v_2 \) is a vertex of \( S(w_2) \) in \( B(G) \). If \( P \) is a \( w_1 - w_2 \) path in \( G \), then the subgraph \( H \) of \( B(G) \) defined by

\[
V(H) = \bigcup V(S(w)) \quad \text{and} \quad E(H) = \bigcup E(S(w)),
\]

where both unions are taken over all \( w \in V(P) \), is a connected subgraph of \( B(G) \) containing a \( v_1 - v_2 \) path.

The first theorem of this section is a consequence of the observations made in the previous paragraph.
Theorem 4-8. Let $G$ be a connected graph with $\delta(G) \geq 3$ and let $B(G)$ be a bonding subgraph of $L(G)$. Then:

(i) $B(G)$ is a connected spanning subgraph of $L(G)$,

(ii) if $G - w$ is connected, $w \in V(G)$, then $B(G) - E(S(w))$ is a connected spanning subgraph of $L(G)$,

(iii) if $S(w)$ is eulerian for each $w \in V(G)$, then $B(G)$ is a spanning eulerian subgraph of $L(G)$.

Proof. (i). From the observations above, $B(G)$ is a spanning subgraph of $L(G)$. If $v_1$ and $v_2$ are vertices of $B(G)$, then for some vertices $w_1$ and $w_2$ of $G$ (not necessarily distinct), $v_1$ is a vertex of $S(w_1)$ and $v_2$ is a vertex of $S(w_2)$. Since $G$ is connected, $G$ contains a $w_1 - w_2$ path. As we have seen, this implies that $B(G)$ contains a $v_1 - v_2$ path. Therefore $B(G)$ is connected.

(ii). Since $B(G) - E(S(w))$ is a spanning subgraph of $B(G)$ and, by part (i), $B(G)$ spans $L(G)$, it suffices to show that $B(G) - E(S(w))$ is connected. Let $v_1$ and $v_2$ be arbitrary vertices of $B(G) - E(S(w))$. Then there exist vertices $w_1$ and $w_2$ of $G$ (not necessarily distinct), with $w_1 \neq w$ and $w_2 \neq w$, such that $v_1$ is a vertex of $S(w_1)$ and $v_2$ is a vertex of $S(w_2)$. Since $G - w$ is connected, there is a $w_1 - w_2$ path in
which does not contain \( w \). From the observations made above, we conclude that \( B(G) - E(S(w)) \) contains a \( v_1 - v_2 \) path and is therefore connected.

(iii). By part (i), \( B(G) \) is a connected spanning subgraph of \( L(G) \). Therefore we need only show that every vertex of \( B(G) \) has even degree in \( B(G) \). However, we have seen that if \( u \) is a vertex of \( B(G) \) corresponding to the edge \( w_1w_2 \) of \( G \), then \( \deg_{B(G)}u = \deg_{S(w_1)}u + \deg_{S(w_2)}u \). Since \( u \) has even degree in each of the graphs \( S(w_1) \) and \( S(w_2) \), the result follows. \( \square \)

Before proceeding to Theorems 4-9 and 4-10, we note the following. If \( W \) is an arbitrary set of vertices of the line graph of a graph \( G \) (\( W \neq V(L(G)) \)) and \( E \) is the set of edges of \( G \) corresponding to the vertices in \( W \), then \( L(G - E) \equiv L(G) - W \). Thus, by the result of Harary and Nash-Williams given in Section 4.1, if \( G - E \) contains a spanning eulerian subgraph, then \( L(G) - W \) is hamiltonian.

**Theorem 4-9.** Let \( G \) be a connected graph with \( \delta(G) \geq 3 \). Then \( L^2(G) \) is \((\delta(G) - 3)\)-hamiltonian.

**Proof.** We wish to show that the removal of \( \delta(G) - 3 \) or fewer vertices from \( L^2(G) \) results in a hamiltonian graph. By the note above, it suffices to show that if \( E \) is an arbitrary set of edges of \( L(G) \) with

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0 \leq |E| \leq \delta(G) - 3$, then $L(G) - E$ contains a spanning eulerian subgraph.

Let $w$ be a vertex of $G$ and let $K(w)$ denote the corresponding complete subgraph of order $\deg_G w$ in $L(G)$.

We note that $|V(K(w))| \geq \delta(G) \geq 3$. Define $K_w$ to be the graph $K(w) - (E \cap E(K(w)))$, where $|E \cap E(K(w))| \leq \delta(G) - 3$. Then $V(K_w) = V(K(w))$. Since $E(K_w) = E(K(w)) - (E \cap E(K(w)))$, we have

$$|E(K_w)| = |E(K(w))| - |E \cap E(K(w))|$$

$$\geq |E(K(w))| - (\delta(G) - 3).$$

However,

$$|E(K(w))| = \left(\frac{|V(K(w))|}{2}\right) = \left(\frac{|V(K_w)|}{2}\right)$$

and

$$\delta(G) \leq |V(K(w))| = |V(K_w)|.$$

Thus,

$$|E(K_w)| \geq \left(\frac{|V(K_w)|}{2}\right) - |V(K_w)| + 3 =$$

$$\left(|V(K_w)|^2 - 3|V(K_w)| + 6\right)/2.$$

Therefore $K_w$ is hamiltonian and so contains a hamiltonian cycle $C$. We observe that $C$ is a spanning eulerian subgraph of $K(w)$ which contains no edges of $E$.

For each vertex $w$ of $G$, let $S(w)$ be a spanning eulerian subgraph of the complete subgraph $K(w)$ in $L(G)$.
which contains no edges of \( E \). By Theorem 4-8, the corresponding bonding subgraph \( B(G) \) of \( L(G) \) is a spanning eulerian subgraph of \( L(G) \). Since \( B(G) \) contains no edges of \( E \), we have, in fact, a spanning eulerian subgraph of \( L(G) - E \). 

The result given in Theorem 4-9 is best possible in the sense that for any integer \( d \geq 3 \), there is a connected graph \( G \) with \( \delta(G) = d \) such that \( L^2(G) \) is not \((d - 2)\)-hamiltonian. Consider the graph \( G \) constructed as follows. Begin with two disjoint copies of \( K_{d+1} \), say \( H_1 \) and \( H_2 \). Add a vertex \( v_1 \) adjacent to exactly \( d - 1 \) vertices of \( H_1 \) and a vertex \( v_2 \) adjacent to exactly \( d - 1 \) vertices of \( H_2 \), together with the edge \( v_1v_2 \). If \( u \) is the vertex of \( L(G) \) which corresponds to the edge \( v_1v_2 \) of \( G \), then there is a set \( E \) of \( d - 2 \) edges incident with \( u \) in \( L(G) \) whose removal from \( L(G) \) results in a graph with a nonterminal bridge. Thus \( L(L(G) - E) \) contains a cutvertex. Equivalently, if \( W \) is the set of \( d - 2 \) vertices of \( L^2(G) \) which correspond to the edges in \( E \), then \( L^2(G) - W \) contains a cutvertex and hence is not hamiltonian.

Let \( G \) be any connected graph with \( \delta(G) \geq 3 \), let \( E \) be an arbitrary set of edges of \( L(G) \), and let \( w \in V(G) \). We saw in the proof of Theorem 4-9 that if \( |E| \leq \delta(G) - 3 \), then there is a spanning eulerian
subgraph $S(w)$ of $K(w)$ which contains no edges of $E$
(where, as usual, $K(w)$ denotes the complete subgraph of
order $\deg_G w$ corresponding to $w$ in $L(G)$). We now ob-
serve that the existence of $S(w)$ does not depend on the
fact that $|E| \leq \delta(G) - 3$ but rather on the fact that
$|E \cap E(K(w))| \leq \delta(G) - 3$. This observation will be used
in the proof of Theorem 4-10.

It was shown in [16] that if $G$ is a graph of order
$p \geq 6$ with $\delta(G) \geq 2$ such that $\deg u + \deg v \geq p - 1$
for every pair $u, v$ of nonadjacent vertices, then $G$
contains a spanning eulerian subgraph. As a corollary,
we have that if $G$ is a graph of order $p \geq 6$ with
$\delta(G) \geq 2$ such that $|E(G)| \geq (p^2 - 3p + 4)/2$, then $G$
contains a spanning eulerian subgraph. It is easily veri-
fied that the latter result holds when the condition
$p \geq 6$ is replaced by $p \geq 4$ and this is used in the
proof of the following lemma.

Lemma 4-2. Let $G$ be a connected graph with $\delta(G) \geq 4$, 
let $E$ be an arbitrary set of edges of $L(G)$, and let $w \in V(G)$. If $|E_w| \leq \delta(G) - 2$ and $\delta(K(w) - E_w) \geq 2$, 
where $K(w)$ denotes the complete subgraph of order $\deg_G w$
corresponding to $w$ in $L(G)$ and $E_w = E \cap E(K(w))$, 
then there is a spanning eulerian subgraph $S(w)$ of $K(w)$
which contains no edges of $E$.

Proof. Define $K_w$ to be the graph $K(w) - E_w$. Then
\( V(K_w) = V(K(w)) \) and \( |V(K_w)| \geq \delta(G) \geq 4 \). Moreover, \( \delta(K_w) \geq 2 \). Since \( E(K_w) = E(K(w)) - E_w \), we have

\[
|E(K_w)| = |E(K(w))| - |E_w| \\
\geq |E(K(w))| - (\delta(G) - 2).
\]

However,

\[
|E(K(w))| = \left(\frac{|V(K(w))|}{2}\right) = \left(\frac{|V(K_w)|}{2}\right) \quad \text{and} \quad \delta(G) \leq |V(K_w)|.
\]

Thus

\[
|E(K_w)| \geq \left(\frac{|V(K_w)|}{2}\right) - |V(K_w)| + 2 = \\
\left(\frac{|V(K_w)|^2}{2} - 3|V(K_w)| + 4\right)/2.
\]

Therefore \( K_w \) contains a spanning eulerian subgraph \( S(w) \).

It remains to observe that \( S(w) \) is a spanning eulerian subgraph of \( K(w) \) which contains no edges of \( E \). \( \Box \)

We now present the final theorem of this section, which improves the result of Theorem 4-9 for 2-connected graphs with minimum degree at least four.

**Theorem 4-10.** Let \( G \) be a 2-connected graph with \( \delta(G) \geq 4 \). Then \( L^2(G) \) is \((2\delta(G) - 4)\)-hamiltonian.

**Proof.** For notational convenience, let \( d = \delta(G) \). We wish to show that the removal of \( 2d - 4 \) or fewer vertices
from $L^2(G)$ results in a hamiltonian graph. It suffices to show that if $E$ is an arbitrary set of edges of $L(G)$ with $0 \leq |E| \leq 2d - 4$, then $L(G) - E$ contains a spanning eulerian subgraph.

For each vertex $w$ of $G$, let $K(w)$ denote the complete subgraph of order $\deg_G w$ corresponding to $w$ in $L(G)$, let $E_w = E \cap E(K(w))$, and let $K_w = K(w) - E_w$. Since no two complete subgraphs $K(w_1)$ and $K(w_2)$ share an edge of $L(G)$, one of the following situations must occur:

(i) $|E_w| \leq d - 3$ for each vertex $w$ of $G$, 
(ii) there is a vertex $w_1$ of $G$ such that $|E_{w_1}| \geq d - 2$ and $|E_w| \leq d - 3$ for $w \in V(G) - \{w_1\}$,
(iii) there are distinct vertices $w_1$ and $w_2$ of $G$ such that $|E_{w_1}| = d - 2 = |E_{w_2}|$, $\delta(K_{w_2}) \geq 2$, and $|E_w| = 0$ for $w \in V(G) - \{w_1, w_2\}$,
(iv) there are distinct vertices $w_1$ and $w_2$ of $G$ such that $|E_{w_1}| = d - 2 = |E_{w_2}|$, $\delta(K_{w_1}) = 1 = \delta(K_{w_2})$, and $|E_w| = 0$ for $w \in V(G) - \{w_1, w_2\}$.

We proceed by considering these four possible cases.

Case (i). For each vertex $w$ of $G$, let $S(w)$ be a
spanning eulerian subgraph of \( K(w) \) which contains no edges of \( E \) (such subgraphs exist according to the observation made following Theorem 4-9). By Theorem 4-8, the bonding subgraph \( B(G) \) of \( L(G) \) defined by

\[
V(B(G)) = \bigcup_{w \in V(G)} V(S(w)) \quad \text{and} \quad E(B(G)) = \bigcup_{w \in V(G)} E(S(w))
\]

is a spanning eulerian subgraph of \( L(G) \). Since \( B(G) \) contains no edges of \( E \), we have, in fact, a spanning eulerian subgraph of \( L(G) - E \).

Case (ii) or Case (iii). For each vertex \( w \neq w_1 \) of \( G \), let \( S(w) \) be a spanning eulerian subgraph of \( K(w) \) which contains no edges of \( E \) (such subgraphs exist according to Lemma 4-2 and the observation made following Theorem 4-9). Let \( S(w_1) \) be any spanning eulerian subgraph of \( K(w_1) \). By Theorem 4-8, the bonding subgraph \( B(G) \) of \( L(G) \) defined by

\[
V(B(G)) = \bigcup_{w \in V(G)} V(S(w)) \quad \text{and} \quad E(B(G)) = \bigcup_{w \in V(G)} E(S(w))
\]

is a spanning eulerian subgraph of \( L(G) \). In particular, each vertex of \( B(G) \) has even degree in \( B(G) \). Since \( G - w_1 \) is connected, another application of Theorem 4-8 yields that \( B(G) - E(S(w_1)) \) is a connected spanning subgraph of \( L(G) \). Since every vertex of \( S(w_1) \) has even degree in \( S(w_1) \), every vertex of \( B(G) - E(S(w_1)) \) has even degree in \( B(G) - E(S(w_1)) \). Thus \( B(G) - E(S(w_1)) \)
is a spanning eulerian subgraph of $L(G)$. Since $B(G) - E(S(w_1))$ contains no edges of $E$, we have, in fact, a spanning eulerian subgraph of $L(G) - E$.

**Case (iv).** We observe that since $|E_{w_1}| = d - 2$ and $\delta(K_{w_1}) = 1$, the graph $K_{w_1}$ is composed of a copy of the complete graph of order $d - 1$, which we will denote by $H_1$, together with an endvertex adjacent to exactly one vertex of $H_1$. Let $e_1$ be the edge of $K_{w_1}$ incident with this endvertex. Similarly, the graph $K_{w_2}$ is composed of a copy of the complete graph of order $d - 1$, which we will denote by $H_2$, together with an endvertex adjacent to exactly one vertex of $H_2$. Let $e_2$ be the edge of $K_{w_2}$ incident with this endvertex. We note that $|V(H_i)| \geq 3$, for $i = 1, 2$.

For each vertex $w \in V(G) - \{w_1, w_2\}$, let $S(w)$ be any spanning cycle of $K(w)$. Since $|E_{w}| = 0$, $S(w)$ contains no edges of $E$. For $i = 1, 2$, let $C_i$ be any spanning cycle of $H_i$ and let $S(w_i)$ be the connected spanning subgraph of $K(w_i)$ induced by $E(C_i) \cup \{e_i\}$. Then $S(w_i)$ contains no edges of $E$ ($i = 1, 2$). By Theorem 4-8, the subgraph $B(G)$ of $L(G)$ defined by

$$V(B(G)) = \bigcup_{w \in V(G)} V(S(w)) \text{ and } E(B(G)) = \bigcup_{w \in V(G)} E(S(w))$$

is a connected spanning subgraph of $L(G)$. We note that
$B(G)$ contains no edges of $E$, every vertex of $B(G)$ incident with neither $e_1$ nor $e_2$ has even degree in $B(G)$, and every vertex of $B(G) - \{e_1, e_2\}$ has even degree in $B(G) - \{e_1, e_2\}$. Thus if $B(G) - \{e_1, e_2\}$ is connected, then it is a spanning eulerian subgraph of $L(G) - E$. Therefore we may assume that $B(G) - \{e_1, e_2\}$ is disconnected.

Since $G$ is 2-connected, we know by Theorem 4-8 that $B(G) - E(S(w^1))$ is connected so that $B(G) - e_i$ is connected, $i = 1, 2$. Thus $B(G) - \{e_1, e_2\}$ has two components, say $B_1$ and $B_2$, and in $B(G)$, each of $e_1$ and $e_2$ is incident with one vertex of $B_1$ and one vertex of $B_2$. Let $u_1$ and $u_2$ be the vertices of $B_1$ incident with $e_1$ and $e_2$, respectively, and let $v_1$ and $v_2$ be the vertices of $B_2$ incident with $e_1$ and $e_2$, respectively. We observe that $3 \leq |\{u_1, u_2, v_1, v_2\}| \leq 4$ and that

1. $S(w)$ is a subgraph of $B_1$ or $B_2$ ($w \neq w_1, w_2$)

and

2. $C_i$ is a subgraph of $B_1$ or $B_2$ ($i = 1, 2$).

Since $E = (E(K(w_1)) - E(K_{w_1})) \cup (E(K(w_2)) - E(K_{w_2}))$, the graph $L(G) - (E \cup \{e_1, e_2\})$ has two components, say $L_1$ and $L_2$, where $B_1$ and $B_2$ are spanning subgraphs of $L_1$ and $L_2$, respectively. Moreover,

1. $K(w)$ is a subgraph of $L_1$ or $L_2$ ($w \neq w_1, w_2$)

and
(2) \( H_1 \) is a subgraph of \( L_1 \) or \( L_2 \) \((i = 1, 2)\).

Let \( P(u_1, u_2) \) be a shortest \( u_1 - u_2 \) path in \( L_1 \) and let \( P(v_1, v_2) \) be a shortest \( v_1 - v_2 \) path in \( L_2 \). We note that \( P(u_1, u_2) \) and \( P(v_1, v_2) \) are disjoint and neither \( e_1 \) nor \( e_2 \) is an edge of \( P(u_1, u_2) \) or \( P(v_1, v_2) \). Furthermore, since \( P(u_1, u_2) \) is a shortest \( u_1 - u_2 \) path in \( L_1 \) and \( P(v_1, v_2) \) is a shortest \( v_1 - v_2 \) path in \( L_2 \), the set \( E(P(u_1, u_2)) \cup E(P(v_1, v_2)) \) contains at most one edge of \( K(w) \), for \( w \neq w_1, w_2 \), and at most one edge of \( H_i \), for \( i = 1, 2 \).

Suppose \( e \in E(P(u_1, u_2)) \cup E(P(v_1, v_2)) \). Then \( e \in E(K(w)) \), for some \( w \neq w_1, w_2 \), or \( e \in E(H_1) \), for \( i = 1 \) or \( 2 \). If \( e \in E(K(w)) \), for some \( w \neq w_1, w_2 \), but \( e \not\in E(S(w)) \), replace the subgraph \( S(w) \) of \( B(G) \) with a spanning cycle \( S'(w) \) of \( K(w) \) which contains \( e \). If \( e \in E(H_1) \), for \( i = 1 \) or \( 2 \), but \( e \not\in E(C_i) \), replace the subgraph \( C_i \) of \( B(G) \) with a spanning cycle \( C_i' \) of \( H_i \) which contains \( e \). Thus we may assume that \( P(u_1, u_2) \) is a path of \( B_1 \) and that \( P(v_1, v_2) \) is a path of \( B_2 \).

Recall that \( B(G) \) is a connected spanning subgraph of \( L(G) - E \), every vertex of \( B(G) \) incident with neither \( e_1 \) nor \( e_2 \) has even degree in \( B(G) \), and every vertex of \( B(G) - \{e_1, e_2\} \) has even degree in \( B(G) - \{e_1, e_2\} \). Thus \( B(G) - (E(P(u_1, u_2)) \cup E(P(v_1, v_2)) \) is a spanning subgraph of \( L(G) - E \) in which every vertex...
has even degree. Moreover, since \( d \geq 4 \), for each vertex \( w \) of \( G \) the subgraph of \( B(G) - \left( E(P(u_1, u_2)) \cup E(P(v_1, v_2)) \right) \) induced by \( V(K(w)) \) is a connected subgraph of \( K(w) \), so that by Theorem 4-8, \( B(G) - \left( E(P(u_1, u_2)) \cup E(P(v_1, v_2)) \right) \) is connected. Thus \( B(G) - \left( E(P(u_1, u_2)) \cup E(P(v_1, v_2)) \right) \) is a spanning eulerian subgraph of \( L(G) - E \). 

The bound given in Theorem 4-10 is sharp. For any integer \( d \geq 4 \), the connected graph \( G \) illustrated in Figure 4.2 has \( \delta(G) = d \). However, if \( u_i \) is the vertex of \( L(G) \) which corresponds to the edge \( e_i \) of \( G \) \((i = 1, 2)\), then there are disjoint sets \( E_1 \) and \( E_2 \), where \( E_1 \) is a set of \( d - 1 \) edges of \( L(G) \) incident with \( u_1 \) and \( E_2 \) is a set of \( d - 2 \) edges of \( L(G) \) incident with \( u_2 \), such that the removal of \( E_1 \cup E_2 \) from \( L(G) \) results in a graph with a nonterminal bridge. Thus \( L(L(G) - (E_1 \cup E_2)) \) contains a cutvertex. Equivalently, if \( W \) is the set of \( 2d - 3 \) vertices of \( L^2(G) \) which correspond to the edges in \( E_1 \cup E_2 \), then \( L^2(G) - W \) contains a cutvertex and hence is not hamiltonian.
Figure 4.2

Section 4.3

n-Hamiltonian Line Graphs II

In section 4.2 we saw that if \( n \) is a nonnegative integer and \( G \) is any connected graph with \( \delta(G) \geq n + 3 \), then \( L^2(G) \) is \( n \)-Hamiltonian. We now consider a question related to this result. If \( n \) is a nonnegative integer and \( G \) is a connected graph which is not a path, does there exist an integer \( k \geq 0 \) such that \( L^m(G) \) is \( n \)-Hamiltonian for every integer \( m \geq k \)? (It is known that if \( G \) is a connected graph which is not a path, then
$L^m(G)$ is defined for every integer $m \geq 0$. If $G$ is the path $P_\ell$ of order $\ell$, then $L^m(G) \cong P_{\ell-m}$ for $0 \leq m \leq \ell - 1$ and $L^m(G)$ is undefined for integers $m > \ell$. For $n = 0$, this question was first investigated in [4] where it was shown that for any connected graph $G$ which is not a path, such an integer $k$ exists and the smallest such integer, denoted $h(G)$, does not exceed $p - 3$, where $p$ is the order of $G$. Later, in [7], this bound on $h(G)$ was improved for certain graphs $G$ and $h(G)$ was determined in the case that $G$ is a tree. We will therefore restrict our attention to values of $n \geq 1$.

We begin with a few preliminary observations. If $G$ is the cycle $C_\ell$ of order $\ell$, then $L^m(G) \cong C_\ell$ for every integer $m \geq 0$. If $G$ is the graph $K(1,3)$, then $L^m(G) \cong C_3$ for every integer $m \geq 1$. Thus if $n$ is a positive integer and $G$ is a cycle or the graph $K(1,3)$, then there is no integer $k \geq 0$ such that $L^m(G)$ is $n$-hamiltonian for every integer $m \geq k$. We now note that paths, cycles, and the graph $K(1,3)$ are the only connected graphs $G$ for which no integer $d$ exists such that $L^d(G)$ has minimum degree at least three. The first theorem of this section answers the question posed in the preceding paragraph for all other connected graphs and all positive integers $n$. 

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Theorem 4-11. Let \( n \) be a positive integer and let \( G \) be a connected graph which is neither a path, cycle, nor \( K(1,3) \). Then there exists an integer \( k \geq 0 \) such that \( L^m(G) \) is \( n \)-hamiltonian for every integer \( m \geq k \).

Proof. We first observe that if \( e = uv \) is an edge of a graph \( H \) with \( \delta(H) \geq 3 \) and \( w \) is the vertex of \( L(H) \) which corresponds to \( e \), then \( \deg_{L(H)}w = \deg_Hu + \deg_Hv - 2 \geq \delta(H) + (\delta(H) - 2) \geq \delta(H) + 1 \). Thus \( \delta(L(H)) \geq 2\delta(H) - 2 \geq \delta(H) + 1 \). (Although \( \delta(L(H)) \geq 2\delta(H) - 2 \), for our purposes it is sufficient to employ the somewhat weaker inequality \( \delta(L(H)) \geq \delta(H) + 1 \).

Let \( G \) be a connected graph which is neither a path, cycle, nor \( K(1,3) \). Then there exists an integer \( d \) such that \( \delta(L^d(G)) \geq 3 \). Since \( L^{d+1}(G) \equiv L(L^d(G)) \) and \( \delta(L^d(G)) \geq 3 \), we have that \( \delta(L^{d+1}(G)) \geq \delta(L^d(G)) + 1 \geq 3 + 1 \). Similarly, \( \delta(L^{d+2}(G)) \geq 3 + 2 \) and, in general, \( \delta(L^{d+t}(G)) \geq 3 + t \) for every integer \( t \geq 0 \). Therefore, \( \delta(L^{d+n}(G)) \geq 3 + n \). Since \( L^{d+n}(G) \) is connected, we may apply Theorem 4-9 to conclude that \( L^{d+n+2}(G) \equiv L^2(L^{d+n}(G)) \) is \( n \)-hamiltonian. Moreover, if \( m \geq d + n + 2 \), a similar argument yields that \( L^m(G) \) is \( n \)-hamiltonian. If we set \( k = d + n + 2 \), the proof is complete. \( \Box \)

Theorem 4-11 deals with one aspect of the relationship between iterated line graphs and the property of...
being n-hamiltonian, for arbitrary values of $n \geq 1$. The object of the remainder of this section is to investigate this relationship in greater detail for the case $n = 1$.

Let $G$ be a connected graph which is not a path. We define the 1-hamiltonian index of $G$, denoted $h_1(G)$, to be the smallest nonnegative integer $k$ such that $L^m(G)$ is 1-hamiltonian for every integer $m \geq k$. As observed earlier, if $G$ is a cycle or the graph $K(1,3)$, then $h_1(G)$ does not exist. Otherwise, $h_1(G)$ exists by Theorem 4-11 and our purpose will be to determine bounds on (and, in some cases, the exact value of) the 1-hamiltonian index for graphs $G$ belonging to several classes of graphs. Pertinent to this work is the next lemma.

**Lemma 4-3.** If a graph $G$ is 1-hamiltonian, then $L(G)$ is 1-hamiltonian.

**Proof.** Since $G$ is hamiltonian, $L(G)$ is hamiltonian. Therefore we need only show that the removal of any vertex from $L(G)$ results in a hamiltonian graph. It suffices to show that if $uv$ is an arbitrary edge of $G$, then $G - uv$ contains a dominating circuit. Let $v, v_1, \ldots, v_s$ ($s \geq 2$) be the vertices of $G$ that are adjacent to $u$ in $G$. Since $G$ is 1-hamiltonian, the graph $G - u$ contains a hamiltonian cycle $C$, which is also a cycle of $G - uv$. The edge set of $G - uv$ consists of the edges of $G - u$ together with the edges $uv_1, \ldots, uv_s$.
Clearly, every edge of $G - u$ is incident with a vertex of $C$. Moreover, since $\{v_1, \ldots, v_s\} \subseteq V(C)$, each of the edges $uv_i$ ($1 \leq i \leq s$) is incident with a vertex of $C$. Thus $C$ is a dominating circuit of $G - uv$.

As a result of Lemma 4-3, we see that the 1-hamiltonian index of a graph $G$ is, in fact, simply the smallest nonnegative integer $k$ such that $L^k(G)$ is 1-hamiltonian. Combining this fact with the result of Theorem 4-9, we obtain our first bound on the 1-hamiltonian index of connected graphs with minimum degree at least three.

**Theorem 4-12.** Let $G$ be a connected graph with $\delta(G) \geq 3$. Then $h_1(G) \leq 3$.

**Proof.** Since $G$ is connected and $\delta(G) \geq 3$, the line graph $L(G)$ is connected and $\delta(L(G)) \geq 4$. Hence we may apply Theorem 4-9 to $L(G)$ to conclude that $L^3(G) \equiv L^2(L(G))$ is 1-hamiltonian. Thus $h_1(G) \leq 3$.

At this point we briefly consider the 1-hamiltonian index of connected graphs with minimum degree less than three which are neither paths, cycles, nor $K(1,3)$. Let $G$ be such a graph and let $d(G) = d$ denote the smallest integer such that $L^d(G)$ has minimum degree at least three. Applying Theorem 4-12 to $L^d(G)$, we conclude that the 1-hamiltonian index of $L^d(G)$ does not exceed three or, equivalently, that $h_1(G) \leq d(G) + 3$. Since a
necessary condition for a graph to be 1-hamiltonian is that the graph have minimum degree at least three, \( h_1(G) \geq d(G) \). Thus\[
d(G) \leq h_1(G) \leq d(G) + 3.\]

Our next result improves the bound given in Theorem 4-12 for one class of connected graphs with minimum degree at least three and, in addition, provides a lower bound on the 1-hamiltonian index of the remaining connected graphs with minimum degree at least three.

**Theorem 4-13.** Let \( G \) be a connected graph with \( \delta(G) \geq 3 \). Then \( h_1(G) \leq 2 \) if and only if \( G \) contains no cutvertex of degree three.

**Proof.** Assume that \( G \) is a connected graph with minimum degree at least three and no cutvertex of degree three. Since \( G \) is connected and \( \delta(G) \geq 3 \), we may apply Theorem 4-9 to conclude that \( L^2(G) \) is hamiltonian. Therefore, we need only show that the removal of any vertex from \( L^2(G) \) results in a hamiltonian graph. This will be the case if we can exhibit a spanning eulerian subgraph of \( L(G) - e \) for each edge \( e \) of \( L(G) \).

For each vertex \( w \) of \( G \), let \( K(w) \) denote the complete subgraph of order \( \delta_G(w) \) corresponding to \( w \) in \( L(G) \). Since \( e \) is an edge of \( L(G) \), there is a (unique) vertex \( u \) of \( G \) such that \( e \in E(K(u)) \). For
w ∈ V(G) − {u}, let S(w) be any spanning cycle of K(w). Now, if deg_G u ≥ 4, then there exists a spanning cycle S(u) of K(u) that does not contain the edge e. Thus the bonding subgraph B(G) of L(G) defined by

\[ V(B(G)) = \bigcup_{w \in V(G)} V(S(w)) \quad \text{and} \quad E(B(G)) = \bigcup_{w \in V(G)} E(S(w)) \]

is a spanning eulerian subgraph of L(G) − e. On the other hand, if deg_G u = 3, let S(u) = K(u). Then the corresponding bonding subgraph B(G) of L(G) is a spanning eulerian subgraph of L(G). Since u is not a cut-vertex of G, the graph B(G) − E(S(u)) is a connected spanning subgraph of L(G) which does not contain the edge e. It remains to observe that each vertex of B(G) − E(S(u)) has even degree in B(G) − E(S(u)).

In order to verify the converse, we let G be a connected graph with δ(G) ≥ 3 and assume that G has a cutvertex v of degree three. We wish to show that L^2(G) is not 1-hamiltonian. Since v is a cutvertex of degree three and δ(G) ≥ 3, one of the edges of G incident with v, say e, is a nonterminal bridge of G. Let f_1 and f_2 be the edges of G different from e that are incident with v and let w, w_1, and w_2 be the vertices of L(G) that correspond to e, f_1, and f_2, respectively.

Now, G − e is composed of two nontrivial components,
one of which, say $G_1$, contains the edges $f_1$ and $f_2$. Moreover, $f_1$ and $f_2$ are the only edges of $G_1$ which are adjacent to $e$ in $G$. Thus $L(G) - \{ww_1, ww_2\}$ is disconnected. However, we observe that $L(G) - ww_1$ is a bonding subgraph of $L(G)$ and thus is connected. Therefore $ww_2$ is a bridge of $L(G) - ww_1$. Moreover, $\delta(G) \geq 3$ so that $\delta(L(G)) \geq 4$. Therefore $L(G) - ww_1$ has minimum degree at least three and $ww_2$ is a nonterminal bridge of $L(G) - ww_1$. But then $L(L(G) - ww_1)$ contains a cutvertex and so is not hamiltonian. But this implies that $L^2(G)$ is not 1-hamiltonian, which completes the proof. 

A graph $G$ has a cutvertex of degree three if and only if $G$ contains a bridge incident with a vertex of degree three. Using this observation, we have two corollaries of Theorem 4-13 which together give the exact values of $h_1(G)$ for all graphs $G$ with $\delta(G) \geq 3$ and $\kappa_1(G) = 1$.

**Corollary 4-13-1.** If $G$ is a graph with $\delta(G) \geq 3$ and $\kappa_1(G) = 1$, and $G$ contains no bridge incident with a vertex of degree three, then $h_1(G) = 2$.

**Proof.** Let $G$ be a graph satisfying the hypothesis. According to Theorem 4-12, the 1-hamiltonian index of $G$ does not exceed two. Since $\kappa_1(G) = 1$ and $\delta(G) \geq 3$,
G contains a nonterminal bridge. Thus \( L(G) \) has a cut-vertex, and therefore \( L(G) \) is not 1-hamiltonian. Hence \( h^1(G) \geq 2 \) .

**Corollary 4-13-2.** If \( G \) is a connected graph with \( \delta(G) = 3 \), and \( G \) contains a bridge incident with a vertex of degree three, then \( h^1(G) = 3 \).

**Proof.** According to Theorem 4-12, the 1-hamiltonian index of \( G \) does not exceed three. An application of Theorem 4-13 to \( G \) yields that \( h^1(G) \geq 3 \).

We next determine the 1-hamiltonian index of graphs \( G \) with \( \delta(G) \geq 3 \) and \( k^1_1(G) = 2 \).

**Theorem 4-14.** If \( G \) is a graph with \( \delta(G) \geq 3 \) and \( k^1_1(G) = 2 \), then \( h^1_1(G) = 2 \).

**Proof.** Let \( G \) be a graph with \( \delta(G) \geq 3 \) and \( k^1_1(G) = 2 \). Then \( G \) contains no cutvertex of degree three so that by Theorem 4-13, we conclude that \( h^1_1(G) \leq 2 \).

Let \( e_1 \) and \( e_2 \) be a pair of edges of \( G \) whose removal from \( G \) results in a disconnected graph. Since \( k^1_1(G) = 2 \), the graph \( G - e_1 \) is connected and thus \( e_2 \) is a bridge of \( G - e_1 \). Moreover, since \( \delta(G) \geq 3 \), the graph \( G - e_1 \) has minimum degree at least two so that \( e_2 \) is a nonterminal bridge of \( G - e_1 \). Therefore \( L(G - e_1) \) contains a cutvertex and so is not hamiltonian. This
implies that $L(G)$ is not 1-hamiltonian and hence $h_1(G) \geq 2$.

We conclude this section with a discussion of those graphs $G$ satisfying $3 \leq k_1(G) \leq \delta(G)$. Clearly, if $G$ is such a graph, then $G$ contains no cutvertex of degree three and we know that $h_1(G) \leq 2$. Since every complete graph $K_p$ of order $p \geq 4$ is 1-hamiltonian, there exist graphs $G$ with arbitrarily large edge-connectivity for which $h_1(G) = 0$. On the other hand, no complete bipartite graph $K(m,n)$, where $3 \leq m \leq n$, is 1-hamiltonian. Therefore there exist graphs $G$ with arbitrarily large edge-connectivity for which $h_1(G) > 0$. However, it is easily verified that if $G$ is the graph $K(m,n)$, where $3 \leq m \leq n$, then $G$ contains a dominating circuit as does $G - e$ for each edge $e$ of $G$. Therefore $h_1(G) = 1$. It is presently unknown whether there exists a graph with edge-connectivity at least three for which the 1-hamiltonian index is two.
CHAPTER V

n-DISTANT HAMILTONIAN GRAPHS

The concept of a strongly hamiltonian graph is generalized in this chapter.

Section 5.1

An Introduction to n-Distant Hamiltonian Graphs

Recall that a hamiltonian graph $G$ is strongly hamiltonian if every edge of $G$ lies on some hamiltonian cycle of $G$. Equivalently, a graph $G$ is strongly hamiltonian if and only if whenever $u$ and $v$ are vertices of $G$ for which $d_G(u,v) \leq 1$, there is a hamiltonian cycle $C$ of $G$ such that $d_C(u,v) = d_G(u,v)$. We define a connected graph $G$ to be $n$-distant hamiltonian, $0 \leq n \leq \text{diam } G$, if whenever $u$ and $v$ are vertices of $G$ for which $d_G(u,v) \leq n$, there is a hamiltonian cycle $C$ of $G$ such that $d_C(u,v) = d_G(u,v)$. Then the 0-distant hamiltonian graphs are simply the hamiltonian graphs and, as observed above, the 1-distant hamiltonian graphs are the strongly hamiltonian graphs. Moreover, if a graph $G$ is $n$-distant hamiltonian, $0 \leq n \leq \text{diam } G$, then $G$ is $m$-distant hamiltonian for every integer $m$ satisfying $0 \leq m \leq n$. Several familiar classes of graphs illustrate
the concept of n-distant hamiltonian graphs. We observe that if G is a cycle or a complete graph of order at least three, then G is diam G-distant hamiltonian and if G has even order p ≥ 4 and is isomorphic to the complete bipartite graph K(p/2, p/2), then G is diam G-distant hamiltonian. However, if G is isomorphic to K₁ + Cₙ, n ≥ 6, then G is (diam G - 1)-distant hamiltonian but not diam G-distant hamiltonian.

Dirac showed that if G is a graph of order p ≥ 3 such that δ(G) ≥ p/2, then G is hamiltonian, i.e. G is 0-distant hamiltonian. Similarly, Ore showed that if G is a graph of order p ≥ 3 such that δ(G) ≥ (p + 1)/2, then G is hamiltonian-connected and thus G is 1-distant hamiltonian. The main purposes of the initial part of this section are to show that Ore's sufficient condition for a graph G to be 1-distant hamiltonian is, in fact, a sufficient condition for G to be diam G-distant hamiltonian and to specify exactly which graphs of order p ≥ 3 with minimum degree at least p/2 are not 1-distant hamiltonian. Theorem 5-1 provides an important step towards establishing these two results.

Before presenting Theorem 5-1, we introduce an additional notation. If w is a vertex of a graph G, let N[w] denote the set consisting of w together with all vertices of G which are adjacent to w in G.
Theorem 5-1. Let \( G \) be a graph of order \( p \geq 3 \) such that \( \delta(G) \geq p/2 \). If there is a pair \( u,v \) of distinct vertices of \( G \) which are joined by no path of length at least \( p-2 \), then \( p \) is even and at least six, and \( G \) is isomorphic to \( K_2 + (2K_{(p-2)/2}) \) or to \( \overline{K}_2 + (2K_{(p-2)/2}) \).

Proof. We first observe that \( p \) must be even; for if \( p \) were odd, then \( \delta(G) \geq (p+1)/2 \) which implies that \( G \) is hamiltonian-connected. But this contradicts the fact that \( G \) contains no hamiltonian \( u-v \) path. We wish to show that \( p \geq 6 \) and that \( G \) is isomorphic to \( K_2 + (2K_{(p-2)/2}) \) or to \( \overline{K}_2 + (2K_{(p-2)/2}) \).

Since \( \delta(G) \geq p/2 \), the graph \( G \) contains a hamiltonian cycle \( C \), which determines two \( u-v \) paths in \( G \). If \( p = 4 \), then one of these paths has length at least \( p-2 \). Thus \( p \geq 6 \). It is convenient to label the vertices of \( G \) so that

\[
C: u_1, u_2, \ldots, u_{l-2}, u_{l-1} = u, u_p, u_{p-1}, \ldots, u_{l+1}, u_l = v, u_1
\]

where \( l-1 \leq p/2 \). Since \( G \) contains no \( u-v \) path of length at least \( p-2 \), we have that \( l-1 \geq 3 \).

Then \( G \) contains the disjoint paths \( P_1 \) and \( P_2 \), where

\[
P_1: u_1, u_2, \ldots, u_{l-1} = u \quad \text{and} \quad P_2: v = u_l, u_{l+1}, \ldots, u_p,
\]

\(
V(G) = V(P_1) \cup V(P_2) \quad \text{and} \quad u_1, u_{l}, u_p, u_{l-1} \in E(G).
\)
(i) Now, if $u_i u_i \in E(G)$ and $2 \leq i \leq \ell - 1$, then 
$u_{i-1} u_p \not\in E(G)$; for otherwise,

$u = u_{\ell-1}, u_{\ell-2}, \ldots, u_i, u_i, u_{i+1}, \ldots, u_p, u_{i-1}, u_{i-2}, \ldots, u_\ell = v$

is a $u - v$ path of length $p - 1$.

(ii) Similarly, if $u_i u_i \in E(G)$ and $\ell + 1 \leq i \leq p$, then 
$u_{i-1} u_p \not\in E(G)$; for otherwise,

$u = u_{\ell-1}, u_{\ell-2}, \ldots, u_1, u_i, u_{i+1}, \ldots, u_p, u_{i-1}, u_{i-2}, \ldots, u_\ell = v$

is a $u - v$ path of length $p - 1$.

(iii) Moreover, if $u_i u_i \in E(G)$ and $3 \leq i \leq \ell - 1$, then 
$u_{i-2} u_p \not\in E(G)$; for otherwise,

$u = u_{\ell-1}, u_{\ell-2}, \ldots, u_i, u_i, u_{i+1}, \ldots, u_{i-2}, u_p, u_{i-1}, u_{i-2}, \ldots, u_\ell = v$

is a $u - v$ path of length $p - 2$, and,

(iv) if $u_i u_i \in E(G)$ and $\ell + 2 \leq i \leq p$, then 
$u_{i-2} u_p \not\in E(G)$; for otherwise,

$u = u_{\ell-1}, u_{\ell-2}, \ldots, u_i, u_i, u_{i+1}, \ldots, u_p, u_{i-2}, u_{i-3}, \ldots, u_\ell = v$

is a $u - v$ path of length $p - 2$.

Using observations (i) and (ii), we see that there are at least $\deg u_1 - 1$ vertices in the set \{u_1, u_2, \ldots, u_{p-1}\} that are not adjacent to $u_p$. Thus, $\deg u_p \leq (p - 1) - (\deg u_1 - 1)$, so that $\deg u_1 + \deg u_p \leq p$. Since $\delta(G) \geq p/2$, we have $\deg u_1 = p/2 = \deg u_p$.
Furthermore, 

(v) if \( u_i \in E(G) \), where \( 3 \leq i \leq l - 1 \) or \( l + 2 \leq i \leq p \), then \( u_{i-1} \in E(G) \); for otherwise, let \( k = \deg u_i - 1 \). Then there are \( k \) vertices \( u_j, u_{j_2}, \ldots, u_{j_k} \) that are adjacent to \( u_1 \) for which the following is true:

(a) \( 2 \leq j_1 < j_2 < \cdots < j_k \leq p \),
(b) \( j_n \neq l \) for \( 1 \leq n \leq k \), and
(c) there is an integer \( m \geq 2 \) such that 
\[
j_m > j_{m-1} + 1 \text{ and either } 4 \leq j_m \leq l - 1 \text{ or } l + 2 \leq j_m \leq p.
\]

According to observations (i) - (iv), \( u_p \) is adjacent to none of the \( k + 1 \) vertices \( u_2, u_3, \ldots, u_{j_k} \).

Therefore, \( \deg u_p \leq (p - 1) - (k + 1) = (p - l) - p/2 = (p - 2)/2 \), which is a contradiction.

We now show that \( u_1 u_{l-1} \in E(G) \). Assume, to the contrary, that \( u_1 u_{l-1} \notin E(G) \). Let \( j \) be the largest integer, \( 2 \leq j \leq l - 2 \), such that \( u_1 u_j \in E(G) \). Then, by observation (v), \( u_1 u_i \in E(G) \) for \( i = 2, 3, \ldots, j \).

Since \( u_1 u_l \in E(G) \), \( u_1 \) is adjacent to exactly \( j \) vertices in the set \( \{u_2, u_3, \ldots, u_l\} \). However, \( \deg u_1 = p/2 \) and \( j \leq l - 2 < p/2 \). Thus \( u_1 \) is adjacent to exactly \( p/2 - j \) vertices in the set \( \{u_{l+1}, u_{l+2}, \ldots, u_p\} \). Using observation (v), we conclude that
$u_1u_i \in E(G)$ if and only if

either $2 \leq i \leq j$ or $t \leq i \leq t + p/2 - j$.

Now, by observations (i) and (ii), $u_pu_i \notin E(G)$ if $1 \leq i \leq j - 1$ or $t \leq i \leq t + p/2 - j - 1$, where $t + p/2 - j - 1 \leq p - 2$. Since $|\{u_1, u_2, \ldots, u_{j-1}\} \cup \{u_t, u_{t+1}, \ldots, u_{t+p/2-j-1}\}| = p/2 - 1$ and $\deg u_p = p/2$, we conclude that

$u_pu_i \in E(G)$ if and only if

either $j \leq i \leq t - 1$ or $t + p/2 - j \leq i \leq p - 1$.

We now observe that $j \geq 3$; for otherwise, $j = 2$ and

$u = u_{t-1}, u_{t-2}, \ldots, u_2, u_p, u_{p-1}, \ldots, u_t = v$

is a $u - v$ path of length $p - 2$. Consider the vertex $u_{j-1}$. If $t + p/2 - j + 1 \leq i \leq p$, then $u_{j-1}u_i \notin E(G)$; for otherwise,

$v = u_t, u_{t+1}, \ldots, u_{i-1}, u_p, u_{p-1}, \ldots, u_i$

$u_{j-1}, u_{j-2}, \ldots, u_1, u_j, u_{j+1}, \ldots, u_{t-1} = u$

is a $v - u$ path of length $p - 1$. Similarly, if $j + 1 \leq i \leq t - 1$, then $u_{j-1}u_i \notin E(G)$; for otherwise,
is a \( v-u \) path of length \( p-1 \). Since
\[
[u_{j+1}, u_{j+2}, \ldots, u_{l-1}] \cup [u_{l-p/2-j+1}, u_{l+p/2-j+2}, \ldots, u_p] = p/2 - 1 \quad \text{and} \quad \deg u_{j-1} = p/2,
\]
we conclude that \( N[u_{j-1}] = N[u_1] \). In particular, \( u_{j-1}u_t \in E(G) \). But then
\[
u = u_{l-1}, u_{l-2}, \ldots, u_j, u_p, u_{p-1}, \ldots, u_{l+1}, u_1, u_2, \ldots, u_{j-1}, u = v
\]
is a \( u-v \) path of length \( p-1 \). This presents a contradiction so that \( u_1u_{l-1} \in E(G) \). Using observation (v) together with the facts that \( \deg u_1 = p/2 \) and \( u_1u_t \in E(G) \), we conclude that
\[
u_1u_i \in E(G) \, \text{if and only if} \, 2 \leq i \leq p/2 + 1.
\]
By observations (i) and (ii), \( u_pu_i \notin E(G) \) if \( 1 \leq i (\neq l-1) \leq p/2 \). Since \( \deg u_p = p/2 \), we conclude that
\[
u_pu_i \in E(G) \, \text{if and only if} \, i = l - 1 \, \text{or} \, p/2 + 1 \leq i \leq p - 1.
\]
Recall that by assumption, \( t \leq p/2 + 1 \). We next show that \( t = p/2 + 1 \). Assume, to the contrary, that \( t \leq p/2 \). Consider the vertex \( u_{l-2} \). If \( p/2 + 2 \leq i \leq p \), then \( u_{l-2}u_i \notin E(G) \); for otherwise,
is a \( u - v \) path of length \( p - 1 \). Since \( \deg u_{t-2} \geq p/2 \), we conclude that \( N[u_{t-2}] = N[u_1] \). In particular, \( u_{t-2}u_t \in E(G) \). But then

\[
\begin{align*}
\dot{u} &= u_{t-1}', u_1', u_2, \ldots, u_{t-2}', u_{i+1}', \ldots, u_p', u_1', u_2, \ldots, u_{t-2}', u_t = v
\end{align*}
\]

is a \( u - v \) path of length \( p - 1 \). This presents a contradiction, so that \( t = p/2 + 1 \). Thus \( u = u_{p/2} \) and \( v = u_{p/2+1} \). Moreover,

\[
\begin{align*}
u_i' &\in E(G) \text{ if and only if } 2 \leq i \leq p/2 + 1 \text{ and} \\
u_p' &\in E(G) \text{ if and only if } p/2 \leq i \leq p - 1.
\end{align*}
\]

We now observe that if \( 2 \leq i \leq p/2 - 1 \) and \( p/2 + 2 \leq j \leq p - 1 \), then \( u_i' u_j \not\in E(G) \); for otherwise,

\[
\begin{align*}
\dot{u} &= u_{p/2}', u_{p/2-1}', \ldots, u_{i+1}', u_1', u_2, \ldots, u_i' \\
u_j', u_j + 1', \ldots, u_{p/2}', u_{j-1}', u_{j-2}, \ldots, u_{p/2+1} = v
\end{align*}
\]

is a \( u - v \) path of length \( p - 1 \). Since \( \delta(G) = p/2 \), we conclude that if \( 2 \leq i \leq p/2 - 1 \) and \( p/2 + 2 \leq j \leq p - 1 \), then

\[
N[u_i] = N[u_1] \text{ and } N[u_j] = N[u_p].
\]
Thus, if $uv \in E(G)$, then $G$ is isomorphic to $K_2 + (2K_{(p-2)/2})$ and if $uv \notin E(G)$, then $G$ is isomorphic to $\overline{K}_2 + (2K_{(p-2)/2})$. \* 

We now present the first theorem directly related to the main topic of this chapter. The corollary following this result gives a sufficient condition for a graph $G$ to be diam $G$-distant hamiltonian.

Theorem 5-2. Let $G$ be a graph of order $p \geq 4$ such that $\delta(G) \geq (p + 1)/2$. Then for each pair $u, v$ of distinct vertices of $G$, there exists a hamiltonian cycle $C$ of $G$ such that $d_C(u, v) = 2$.

Proof. Let $u$ and $v$ be a pair of distinct vertices of $G$. Since $\delta(G) \geq (p + 1)/2$, there is a vertex $w$ such that $uw, vw \in E(G)$. Let $G^*$ be the graph $G - w$ of order $p^* = p - 1 \geq 3$. Then $\delta(G^*) \geq (p - 1)/2 = p^*/2$. If $G^*$ contains no $u - v$ path of length at least $p^* - 2$, then by Theorem 5-1, $p^*$ is even and at least six, and $G^*$ is isomorphic to $K_2 + (2K_{(p^*-2)/2})$ or to $\overline{K}_2 + (2K_{(p^*-2)/2})$. In either case, every vertex of $G^*$ other than $u$ or $v$ has degree $p^*/2 = (p - 1)/2$ in $G^*$. Therefore, since $\delta(G) \geq (p + 1)/2$, if $x \in V(G) - \{u, v, w\}$, then $wx \in E(G)$. Clearly, then, $G$ contains a hamiltonian cycle $C$ such that $d_C(u, v) = 2$. Thus we may assume that $G^*$ contains a $u - v$ path of length at
least \( p^* - 2 \), which implies that \( G \) contains a cycle \( C' \) of length at least \( p^* = p - 1 \) such that \( d_C(u, v) = 2 \). If \( C' \) is a hamiltonian cycle of \( G \), then the proof is complete. So we may assume that \( C' \) contains exactly \( p - 1 \) vertices of \( G \).

Label the vertices of \( C' \) so that \( C': w_1, w_2 = u, w_3, \ldots, w_{p-1} = v, w_1 \) where, then, \( w_1 = w \), and let \( x \) be the single vertex of \( G \) that is not on \( C' \). We now show that \( G \) contains a hamiltonian cycle \( C \) such that \( d_C(u, v) = 2 \). Assume, to the contrary, that no such cycle \( C \) exists. We first observe that if \( xw_i \in E(G) \), \( 2 \leq i \leq p - 2 \), then \( xw_{i+1} \notin E(G) \); for otherwise,

\[
C: w_1, w_2 = u, w_3, \ldots, w_i, x, w_{i+1}, w_{i+2}, \ldots, w_{p-1} = v, w_1
\]

is a hamiltonian cycle of \( G \) with \( d_C(u, v) = 2 \). Thus, \( p \) is odd and \( \text{deg}_G x = (p + 1)/2 \). Moreover, \( xw_i \in E(G) \) if and only if \( i = 1, 2, 4, 6, \ldots, p - 1 \). By a similar argument, \( \text{deg}_G w_1 = (p + 1)/2 \) and \( w_1w_i \in E(G) \) if and only if \( i = 2, 4, 6, \ldots, p - 1 \). We now observe that if \( p \geq 7 \) and \( i = 5, 7, \ldots, p - 2 \), then \( w_3w_i \notin E(G) \); for otherwise,

\[
C: w_1, w_2 = u, x, w_{i-1}, w_{i-2}, \ldots, w_3, w_i, w_{i+1}, \ldots, w_{p-1} = v, w_1
\]

is a hamiltonian cycle of \( G \) with \( d_C(u, v) = 2 \). But then \( w_3 \) is adjacent only to vertices in the set

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\{w_2, w_4, \ldots , w_{p-1}\} of cardinality \((p - 1)/2\), which contradicts the fact that \(\delta(G) \geq (p + 1)/2\).

**Corollary 5-2-1.** If \(G\) is a graph of order \(p \geq 3\) such that \(\delta(G) \geq (p + 1)/2\), then \(G\) is diam \(G\)-distant hamiltonian.

**Proof.** Since \(p \geq 3\) and \(\delta(G) \geq (p + 1)/2\), the graph \(G\) is 1-distant hamiltonian. Thus if \(G\) is complete, \(G\) is diam \(G\)-distant hamiltonian. If \(G\) is not complete, then \(p \geq 4\) and diam \(G = 2\) since \(\delta(G) \geq (p + 1)/2\). Thus it suffices to show that if \(u\) and \(v\) are distinct nonadjacent vertices of \(G\), then there exists a hamiltonian cycle \(C\) of \(G\) such that \(d_C(u,v) = 2\). This follows from Theorem 5-2.

We next turn our attention to the problem of determining which graphs of order \(p \geq 3\) with minimum degree at least \(p/2\) are not 1-distant hamiltonian. As mentioned earlier in this section, if a graph \(G\) of order \(p \geq 3\) has no vertices of degree less than \((p + 1)/2\), then \(G\) is hamiltonian-connected and thus is 1-distant hamiltonian. We find it convenient to first consider those graphs with minimum degree at least \(p/2\) which fail to be hamiltonian-connected.

**Theorem 5-3.** Let \(G\) be a graph of order \(p \geq 3\) with \(\delta(G) \geq p/2\). If \(G\) is not hamiltonian-connected, then
p is even and G is isomorphic to \( K_2 + (2K_{(p-2)/2}) \),
to \( \overline{K}_2 + (2K_{(p-2)/2}) \), or to \( H + \overline{K}_{p/2} \), where H is any graph of order \( p/2 \).

Proof. If \( \delta(G) \geq p/2 \) and \( p \) is odd, then \( \delta(G) \geq (p + 1)/2 \) and G is hamiltonian-connected. Thus if \( G \) is not hamiltonian-connected, \( p \) is even. Let \( u \) and \( v \) be vertices of \( G \) such that \( G \) contains no hamiltonian \( u - v \) path. If \( G \) contains no \( u - v \) path of length \( p - 2 \), then we may apply Theorem 5-1 to conclude that \( G \) is isomorphic to one of the graphs \( K_2 + (2K_{(p-2)/2}) \) or \( \overline{K}_2 + (2K_{(p-2)/2}) \). Thus we may assume that \( G \) contain a \( u - v \) path of length \( p - 2 \), say \( P: u = u_1, u_2, \ldots, u_{p-1} = v \). Let \( u_p \) be the single vertex of \( G \) which is not on \( P \). It suffices to show that \( \{u_2, u_4, \ldots, u_p\} \) is an independent set of vertices of \( G \).

We first observe that if \( u_p u_i \in E(G) \), \( 1 \leq i \leq p - 2 \), then \( u_p u_{i+1} \notin E(G) \); for otherwise,

\[ u = u_1, u_2, \ldots, u_i, u_p, u_{i+1}, u_{i+2}, \ldots, u_{p-1} = v \]

is a hamiltonian \( u - v \) path. Thus \( \deg u_p = p/2 \) and \( u_p u_i \in E(G) \) if and only if \( i = 1, 3, \ldots, p - 1 \). If \( p = 4 \), the proof is complete. Therefore, we may assume \( p \geq 6 \). We now observe that if \( i \) and \( j \) are even, \( 2 \leq i < j \leq p - 2 \), then \( u_i u_j \notin E(G) \); for otherwise,
is a hamiltonian $u-v$ path. Thus $\{u_2, u_4, \ldots, u_p\}$ is an independent set of vertices of $G$. $\diamond$

A corollary now follows.

**Corollary 5-3-1.** Let $G$ be a graph of order $p \geq 3$ with $\delta(G) \geq p/2$. Then $G$ is 1-distant hamiltonian if and only if either $p$ is odd, or $p$ is even and $G$ is isomorphic to neither $K_2 + (2K_{(p-2)/2})$ nor $H + \overline{K}_{p/2}$, where $H$ is any nonempty graph of order $p/2$.

The maximum possible diameter of a hamiltonian graph of order $p$ is $p/2$ which, of course, can only occur if $p$ is even. The diameter of the cycle $C_p$ of (even) order $p$ is $p/2$ and $C_p$ is $(p/2)$-distant hamiltonian.

We now note that $C_p$ is not the only hamiltonian graph of order $p$ with such a diameter. The following theorem gives the possible values of $n$ for which there exist $n$-distant hamiltonian graphs of order $p$ having diameter $p/2$.

**Theorem 5-4.** If $G$ is a hamiltonian graph of (even) order $p \neq 6$ such that $\text{diam } G = p/2$, then either

1. $G$ is isomorphic to $C_p$ (and thus $G$ is...
(p/2)-distant hamiltonian),

(2) G is 0-distant hamiltonian but not 1-distant hamiltonian, or

(3) G is 1-distant hamiltonian but not 2-distant hamiltonian.

Proof. Let G be a hamiltonian graph of (even) order p \neq 6 with diam G = p/2 and assume that neither (1) nor (2) holds. Then p \geq 8 and G is 1-distant hamiltonian. We wish to show that G is not 2-distant hamiltonian.

Let C* be a hamiltonian cycle of G and let u and v be vertices of G such that d_G(u,v) = p/2. Clearly, d_{C^*}(u,v) = p/2. Beginning with the vertex u, label the vertices of C* cyclically so that

C*: u, z_1, z_2, ..., z_{(p-2)/2}, v, w_{(p-2)/2}, w_{(p-4)/2}, ..., w_1, u.

Since d_G(u,v) = p/2, we have deg u = 2 = deg v. Now, by assumption, G does not consist only of the cycle C*. Therefore G contains a vertex with degree at least three.

Let \( k = \min \{i | \deg z_i \geq 3 \text{ or } \deg w_i \geq 3 \} \).

Without loss of generality, we assume that \( \deg z_k \geq 3 \).

We first observe that every hamiltonian cycle of G contains the path

R: z_k, z_{k-1}, ..., z_1, u, w_1, w_2, ..., w_k

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since \( \deg u = 2 \), and, because of the way in which \( k \) was chosen, if \( k \geq 2 \) and \( 1 \leq i \leq k - 1 \), then 
\[ \deg z_i = 2 = \deg w_i. \]
Next, since \( d_G(u, v) = p/2 \), we have \( 3 \leq \deg z_k \leq 4 \). However, \( z_kw_k \not\in E(G) \); otherwise, \( z_kw_k \) lies on no hamiltonian cycle of \( G \), which contradicts the fact that \( G \) is 1-distant hamiltonian. Therefore, it must be the case that \( \deg z_k = 3 \), where \( 1 \leq k \leq (p - 4)/2 \) and \( z_kw_{k+1} \in E(G) \). Since \( G \) is 1-distant hamiltonian, the edge \( z_kw_{k+1} \) lies on some hamiltonian cycle of \( G \) which also contains the path \( R \).
Thus the vertex \( w_k \) has degree at least three. Using the fact that \( d_G(u, v) = p/2 \), we conclude that \( \deg w_k = 3 \) and \( w_kz_{k+1} \in E(G) \). We now consider two cases.

Case 1. Suppose \( k \geq 2 \). We note that \( d_G(z_k, w_k) = 2 \).
Moreover, the paths
\[ P: z_k, z_{k+1}, w_k \] and \[ Q: z_k, w_{k+1}, w_k \]
are the only \( z_k - w_k \) distance paths in \( G \). Since the path \( R \) is contained in every hamiltonian cycle of \( G \), there is no hamiltonian cycle \( C \) of \( G \) for which \( d_C(z_k, w_k) = d_G(z_k, w_k) \). Thus \( G \) is not 2-distant hamiltonian.

Case 2. Suppose \( k = 1 \). We note that \( d_G(z_2, w_2) \leq 2 \).
Now, every hamiltonian cycle of \( G \) contains the path
Furthermore, $N[z_1] - \{z_1\} = \{z_2, u, w_2\} = N[w_1] - \{w_1\}$. Therefore, every Hamiltonian cycle of $G$ consists of one of the paths

$$P: z_2, z_1, u, w_1, w_2$$

or

$$Q: z_2, w_1, u, z_1, w_2$$

together with a $z_2 - w_2$ path of length $p - 4$. Since $p \geq 8$, we conclude that there is no Hamiltonian cycle $C$ of $G$ for which $d_C(z_2, w_2) = d_G(z_2, w_2)$. Thus $G$ is not 2-distant Hamiltonian.

For any even integer $p \geq 8$, let $C: u_1, u_2, \ldots, u_p, u_1$ be a cycle of order $p$. The graph $G$ obtained from $C$ by adding the edge $u_1 u_4$ has diameter $p/2$, is 0-distant Hamiltonian, and is not 1-distant Hamiltonian. If we now add the edge $u_3 u_p$, the diameter of the resulting graph $G^*$ is still $p/2$. Furthermore, $G^*$ is 1-distant Hamiltonian but not 2-distant Hamiltonian.

As an immediate corollary of the preceding theorem, $(p/2)$-distant Hamiltonian graphs can be characterized.

**Corollary 5-4-1.** Let $G$ be a graph of (even) order $p \neq 6$. Then $G$ is $(p/2)$-distant Hamiltonian if and only if $G$ is isomorphic to $C_p$.

We close this discussion by giving an example of a 3-distant Hamiltonian graph of order six other than $C_6$. Label the vertices of a 6-cycle $C$ so that
If the edges $u_1u_4$ and $u_3u_6$ are added to $C$, the resulting graph is 3-distant hamiltonian.

Every hamiltonian graph $G$ of order $p$ has diameter at least one and not exceeding $\lceil p/2 \rceil$. If $G$ is the complete graph $K_p$, then $\text{diam } G = 1$ and $G$ is $\text{diam } G$-distant hamiltonian. If $G$ is the cycle $C_p$, then $\text{diam } G = \lceil p/2 \rceil$ and $G$ is $\text{diam } G$-distant hamiltonian. The final theorem of this section will provide us with examples of $n$-distant hamiltonian graphs of order $p$ having diameter $n$ for integers $n$ and $p$ satisfying $p \geq 3$ and $1 \leq n \leq \lceil (p - 1)/3 \rceil$. A definition is necessary in order to present this theorem. For a positive integer $n$, the $n$-th power $G^n$ of a connected graph $G$ is that graph having the same vertex set as $G$ and in which two distinct vertices are adjacent if and only if their distance in $G$ is at most $n$. The graph $G^3$ is referred to as the cube of $G$. In the proof of Theorem 5-5 we use the result of Karaganis [15] and Sekanina [23] that the cube of a connected graph is hamiltonian-connected.

**Theorem 5-5.** If $G$ is a connected graph of order $p \geq 3$, then $G^3$ is $\text{diam } G^3$-distant hamiltonian.

**Proof.** Since $G^3$ is hamiltonian-connected and $p \geq 3$, the graph $G^3$ is 1-distant hamiltonian. Therefore it suffices to show that if $\text{diam } G^3 \geq 2$ and $u$ and $v$ are...
vertices of $G^3$ such that $2 \leq d_{G^3}(u,v) \leq \text{diam } G^3$, then there is a hamiltonian cycle $C$ of $G^3$ such that

$$d_C(u,v) = d_{G^3}(u,v).$$

Let $T$ be a spanning tree of $G$ which is distance-preserving from $u$ and let $P: u = z_0, z_1, \ldots, z_t = v$ be the unique $u - v$ path in $T$. If we set $d = d_{G^3}(u,v)$, then clearly $3d - 2 \leq t \leq 3d$. Let $T_u$ be the component of $T - z_0 z_1$ containing $u$ and let $T_v$ be the component of $T - z_{t-1} z_t$ containing $v$. We note that $V(T_u) \cap V(T_v) = \emptyset$, that $V(T_u) \cap V(P) = \{u\}$, and that $V(T_v) \cap V(P) = \{v\}$. Moreover, $V(T) = V(P) \cup V(T_u) \cup V(T_v)$.

Let $u'$ be any vertex of $T_u$ adjacent to $u$ in $T$ and let $v'$ be any vertex of $T_v$ adjacent to $v$ in $T$. If $T_u$ or $T_v$ is trivial, we define $u' = u$ or $v' = v$, respectively. Let $P_u$ be a hamiltonian $u' - u$ path of $T_u^3$ and let $P_v$ be a hamiltonian $v - v'$ path of $T_v^3$.

We now show that $P^3$ contains a $u - v$ path $R$ and a $z_{t-1} - z_1$ path $R'$, where $R$ and $R'$ are disjoint, $V(R) \cup V(R') = V(P)$, and the length of $R$ is $d$.

If $t = 3d$, choose $R$ and $R'$ as

$$R: z_0, z_3, \ldots, z_{3d}$$

and

$$R': z_{3d-1}, z_{3d-2}, z_{3d-4}, z_{3d-5}, \ldots, z_2, z_1.$$
If \( t = 3d - 1 \), choose \( R \) and \( R' \) as

\[
R: z_0, z_3, \ldots, z_{3d-3}, z_{3d-1}
\]

and

\[
R': z_{3d-2}, z_{3d-4}, z_{3d-5}, z_{3d-7}, z_{3d-8}, \ldots, z_2, z_1.
\]

If \( t = 3d - 2 \), choose \( R \) and \( R' \) as

\[
R: z_0, z_2, z_5, z_8, \ldots, z_{3d-4}, z_{3d-2}
\]

and

\[
R': z_{3d-3}, z_{3d-5}, z_{3d-6}, z_{3d-8}, z_{3d-9}, \ldots, z_4, z_3, z_1.
\]

Since \( d_T(v', z_{t-1}) \leq 2 \), the vertices \( v' \) and \( z_{t-1} \) are adjacent in \( T^3 \). Similarly, \( u' \) and \( z_1 \) are adjacent in \( T^3 \). Therefore \( T^3 \) contains the Hamiltonian cycle \( C \) formed by starting with the path \( R \) and then following with the path \( P_v \), the edge \( v'z_{t-1} \), the path \( R' \), the edge \( z_1u' \), and the path \( P_u \), in that order. Since \( T^3 \) is a spanning subgraph of \( G^3 \), the graph \( G^3 \) contains the Hamiltonian cycle \( C \). It remains to observe that \( d_C(u, v) = d = d_{G^3}(u, v) \). 

Let \( n \) and \( p \) be integers satisfying \( p \geq 3 \) and \( 1 \leq n \leq \lfloor (p - 1)/3 \rfloor \), and let \( P: v_0, v_1, \ldots, v_{3n-2} \) be a path of length \( 3n - 2 \). If \( T \) is the tree of order \( p \) obtained from \( P \) by adding \( p - (3n - 1) \) new vertices, each adjacent only to \( v_1 \), then \( T^3 \) is an \( n \)-distant Hamiltonian graph of order \( p \) having diameter \( n \).
Section 5.2

n-Distant Hamiltonian Line Graphs

In this section we return to the topic of line graphs to investigate n-distant hamiltonian line graphs. Our first concern will be to study the relationship between distance in a connected graph $G$ and distance in the line graph $L(G)$. In order to avoid unnecessary and cumbersome notation, we will adopt the convention that if $e$ is an edge of a graph $G$, then $e$ denotes the vertex of $L(G)$ which corresponds to $e$.

**Lemma 5-1.** If $e = u_1u_2$ and $f = v_1v_2$ are distinct edges of a connected graph $G$, then $d_{L(G)}(e, f) = \min_{1 \leq i, j \leq 2} \{d_G(u_i, v_j)\} + 1$.

**Proof.** If $e$ and $f$ are adjacent edges of $G$, the desired result obviously holds. So we may assume that $e$ and $f$ are nonadjacent edges of $G$. Choose $u \in \{u_1, u_2\}$ and $v \in \{v_1, v_2\}$ such that $d_G(u, v) = \min_{1 \leq i, j \leq 2} \{d_G(u_i, v_j)\}$. Set $d = d_G(u, v)$ where, then, $d \geq 1$, and let $P: u = z_0, z_1, \ldots, z_d = v$ be a $u - v$ distance path in $G$. By the way in which $u$ and $v$ were chosen, neither $e$ nor $f$ is an edge of $P$. Therefore,

$\overline{e, z_0 z_1 z_2 \ldots z_{d-1} z_d f}$
is an $\overline{e} - \overline{f}$ path in $L(G)$ of length $d + 1$. Thus
\[ d_{L(G)}(\overline{e}, \overline{f}) = d + 1 = \min_{1 \leq i, j \leq 2} \{ d_G(u_i, v_j) \} + 1. \]

Now, set $m = d_{L(G)}(\overline{e}, \overline{f})$ where, then, $m \geq 2$, and let $Q: \overline{e} = \overline{f}_0, \overline{f}_1, \ldots, \overline{f}_m = \overline{f}$ be an $\overline{e} - \overline{f}$ distance path in $L(G)$. For $0 \leq i \leq m - 1$, let $w_i$ be the vertex of $G$ incident with both $f_i$ and $f_{i+1}$. Then if $0 \leq k \leq l \leq m - 1$, the vertices $w_k$ and $w_l$ are distinct; for otherwise,
\[ \overline{e} = \overline{f}_0, \overline{f}_1, \ldots, \overline{f}_k, \overline{f}_{k+1}, \ldots, \overline{f}_m = \overline{f} \]
is an $\overline{e} - \overline{f}$ path in $L(G)$ of length less than $m$.

Therefore $f_i = w_{i-1} w_i$, for $1 \leq i \leq m - 1$, and $w_0, w_1, \ldots, w_{m-1}$ is a path of length $m - 1$ in $G$. Since $w_0 \in \{ u_1, u_2 \}$ and $w_{m-1} \in \{ v_1, v_2 \}$, we have
\[ \min_{1 \leq i, j \leq 2} \{ d_G(u_i, v_j) \} \leq d_G(w_0, w_{m-1}) \leq m - 1 \text{ or, equivalently,} \]
\[ \min_{1 \leq i, j \leq 2} \{ d_G(u_i, v_j) \} + 1 \leq d_{L(G)}(\overline{e}, \overline{f}) \quad \& \]

With the aid of Lemma 5-1, we obtain inequalities relating $\text{diam } G$ and $\text{diam } L(G)$ for an arbitrary nonempty connected graph $G$.

**Theorem 5-6.** For every nonempty connected graph $G$,
\[ \text{diam } G - 1 \leq \text{diam } L(G) \leq \text{diam } G + 1. \]
Proof. Let \( G \) be a nonempty connected graph. If \( L(G) \) is trivial, then clearly \( \text{diam} \ L(G) \leq \text{diam} \ G + 1 \). Otherwise, the inequality \( \text{diam} \ L(G) \leq \text{diam} \ G + 1 \) follows directly from the previous lemma. So it remains to show that \( \text{diam} \ L(G) \geq \text{diam} \ G - 1 \). Since this is true if \( \text{diam} \ G = 1 \), we may assume that \( \text{diam} \ G \geq 2 \). Let \( u \) and \( v \) be vertices of \( G \) such that \( d_G(u,v) = \text{diam} \ G \) and let \( P: u = z_0, z_1, \ldots, z_d = v \) be a \( u - v \) distance path in \( G \), where \( d = \text{diam} \ G \). By Lemma 5-1,

\[
\text{d}_L(G)(z_0 \overline{z_1}, \overline{z_{d-1}z_d}) = \min_{i=0,1}^{j=d-1,d} \{d_G(z_i,z_j)\} + 1
\]

\[
= (d - 2) + 1 = \text{diam} \ G - 1 .
\]

Therefore, \( \text{diam} \ L(G) \geq \text{diam} \ G - 1 \). \( \diamondsuit \)

These bounds on \( \text{diam} \ L(G) \) in terms of \( \text{diam} \ G \) cannot be improved. For example, if \( G \) is a complete graph of order at least four, then \( \text{diam} \ L(G) = \text{diam} \ G + 1 \). If \( G \) is any cycle, then \( \text{diam} \ L(G) = \text{diam} \ G \) since, in fact, \( G \cong L(G) \). Finally, if \( G \) is a nontrivial path, then \( \text{diam} \ L(G) = \text{diam} \ G - 1 \).

In Chapter IV we characterized those graphs with strongly hamiltonian line graphs. As a corollary of this result, we saw that if \( G \) is a hamiltonian graph, then \( L(G) \) is strongly hamiltonian. In our present terminology, if \( G \) is a 0-distant hamiltonian graph, then \( L(G) \) is
1-distant hamiltonian. It is with this in mind that we present the next theorem.

**Theorem 5-7.** Suppose $e$ and $f$ are distinct edges of a connected graph $G$ and $u$ and $v$ are vertices of $G$ incident with $e$ and $f$, respectively, such that $d_{L(G)}(e,f) = d_G(u,v) + 1$. If there exists a hamiltonian cycle $C$ of $G$ such that $d_C(u,v) = d_G(u,v)$, then there exists a hamiltonian cycle $C^*$ of $L(G)$ such that $d_{C^*}(e,f) = d_{L(G)}(e,f)$.

**Proof.** For notational purposes, let $d = d_G(u,v)$ and $p = |V(G)|$. If $d = 0$, the desired result follows from the fact that $L(G)$ is 1-distant hamiltonian. Therefore we may assume that $d \geq 1$. Let $C: u = z_0, z_1, \ldots, z_d = v, z_{d+1}, \ldots, z_{p-1}, z_0 = u$ be a hamiltonian cycle of $G$ such that $d_C(u,v) = d_G(u,v)$. If $d = 1$, then clearly every edge of $E(G) - E(C)$ that is incident with $z_i$, for some $i$ satisfying $0 \leq i \leq d$, is also incident with $z_j$, for some $j$ satisfying $d + 1 \leq j \leq p - 1$.

Suppose $d \geq 2$ and $h = z_i z_j$ is an edge of $E(G) - E(C)$, where $i + 1 < j$ and $0 \leq i \leq d$. Then $d + 1 \leq j \leq p - 1$; for otherwise, $u = z_0, z_1, \ldots, z_i', z_j, z_{j+1}', \ldots, z_d = v$ is a $u - v$ path in $G$ of length less than $d$. Thus every edge of $E(G) - E(C)$ that is incident with $z_i$, for some $i$ satisfying $0 \leq i \leq d$, is also incident with $z_j$, for some $j$ satisfying $d + 1 \leq j \leq p - 1$. 

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We wish to construct an ordering $S$ of the edges of $G$. In order to select the initial terms of $S$, we consider four possible cases.

**Case 1.** Suppose $e$ and $f$ are edges of $C$. Applying Lemma 5-1, we see that $e = z_{p-1}^0 z_{d+1}^\ast$, and $d + 1 < p - 1$. Begin the ordering $S$ by selecting, in any order, all edges of $G$ incident with $z_{p-1}$ that are not edges of $C$, followed by the edges $e, z_{0} z_{1}^\ast, \ldots, z_{d-1} z_{d}^\ast, f$ in that order.

**Case 2.** Suppose $e \in E(C)$ and $f \not\in E(C)$. Applying Lemma 5-1, we see that $e = z_{p-1}^0$ and that $f$ is not incident with $z_{p-1}$. Therefore, since $f \not\in E(C)$ and $f$ is incident with $z_{d}$, we note that by an earlier observation $d + 2 < p - 1$. Begin the ordering $S$ by selecting, in any order, all edges of $G$ incident with $z_{p-1}$ that are not edges of $C$, followed by the edges $e, z_{0} z_{1}^\ast, \ldots, z_{d-1} z_{d}^\ast, f, z_{d}^\ast z_{d+1}^\ast$ in that order.

**Case 3.** Suppose $e \not\in E(C)$ and $f \in E(C)$. Applying Lemma 5-1, we see that $f = z_{d} z_{d+1}^\ast$. Since $e \not\in E(C)$ and $e$ is incident with $z_{0}^\ast$, the edge $e$ is not incident with $z_{p-1}$. Therefore, we note that by an earlier observation $d + 1 \leq p - 2$. Begin the ordering $S$ by selecting, in any order, all edges of $G$ incident with $z_{p-1}$ that are not edges of $C$, followed by the edges.
Case 4. Suppose \( e \not\in E(C) \) and \( f \not\in E(C) \). Since \( e \not\in E(C) \) and \( e \) is incident with \( z_0 \), the edge \( e \) is not incident with \( z_{p-1} \). Since \( f \not\in E(C) \) and \( f \) is incident with \( z_d \), we note that by an earlier observation \( d + 2 \leq p - 1 \). Begin the ordering \( S \) by selecting, in any order, all edges of \( E(G) - \{f\} \) incident with \( z_{p-1} \) that are not edges of \( C \), followed by the edges \( z_{p-1}z_0, z_0^{-1}z_1, \ldots, z_{d-1}z_d, f, z_dz_{d+1} \) in that order.

In each of Cases 1 - 4, we now extend \( S \) as follows. At each successive \( z_i \), \( d + 1 \leq i \leq p - 2 \), select, in any order, all edges of \( G \) incident with \( z_i \) that are neither edges of \( C \) nor previously selected edges, followed by the edge \( z_iz_{i+1} \). This process terminates with the edge \( z_{p-2}z_{p-1} \). Then the ordering \( S \) has the properties that every edge of \( G \) appears exactly once in \( S \), consecutive edges of \( S \) are adjacent in \( G \), and the first and last edges of \( S \) are adjacent in \( G \). Therefore, there is a hamiltonian cycle of \( L(G) \), say \( C^* \), that corresponds naturally to \( S \). Furthermore, the cycle \( C^* \) contains the \( \overline{e} - \overline{f} \) path \( \overline{e}, z_0z_1, \ldots, z_{d-1}z_d, \overline{f} \) of length \( d + 1 \). Thus \( d_{C^*}(\overline{e}, \overline{f}) \leq d + 1 = d_{L(G)}(\overline{e}, \overline{f}) \). Since \( d_{L(G)}(\overline{e}, \overline{f}) \leq d_{C^*}(\overline{e}, \overline{f}) \), the proof is complete. \( \varnothing \)

We can now extend the result which states that a
sufficient condition for $L(G)$ to be 1-distant hamiltonian is that $G$ is 0-distant hamiltonian.

**Corollary 5-7-1.** If a graph $G$ is $n$-distant hamiltonian, $n \geq 0$, and $n + 1 \leq \text{diam } L(G)$, then $L(G)$ is $(n + 1)$-distant hamiltonian.

**Proof.** Since $G$ is hamiltonian, $L(G)$ is 1-distant hamiltonian. Therefore it suffices to show that if $n \geq 1$ and $\overline{e}, \overline{f}$ are vertices of $L(G)$ such that $2 \leq d_{L(G)}(\overline{e}, \overline{f}) \leq n + 1$, then there is a hamiltonian cycle $C^*$ of $L(G)$ such that $d_{C^*}(\overline{e}, \overline{f}) = d_{L(G)}(\overline{e}, \overline{f})$. By Lemma 5-1, there are vertices $u$ and $v$ of $G$ incident with $e$ and $f$, respectively, such that $d_{L(G)}(\overline{e}, \overline{f}) = d_G(u, v) + 1$. Since $d_{L(G)}(\overline{e}, \overline{f}) \leq n + 1$, we have that $d_G(u, v) \leq n$. Therefore there exists a hamiltonian cycle $C$ of $G$ such that $d_C(u, v) = d_G(u, v)$. We may now apply Theorem 5-7 to complete the proof. $\square$

We remark that, in terms of the statement of Corollary 5-7-1, the graph $L(G)$ need not be $(n + 2)$-distant hamiltonian in the case $n + 2 \leq \text{diam } L(G)$. For example, the graph $G$ of Figure 5.1 is 0-distant hamiltonian and the diameter of its line graph $L(G)$ is four. By Corollary 5-7-1, the graph $L(G)$ is 1-distant hamiltonian. However, $L(G)$ is not 2-distant hamiltonian since there is no hamiltonian cycle $C$ of $L(G)$ such that
\[ d_C(u, v) = d_{L(G)}(u, v) \]. We also note that \( \text{diam } L^2(G) = 4 \) so that by Corollary 5-7-1, \( L^2(G) \) is 2-distant hamiltonian. Since there is obviously no hamiltonian cycle \( C' \) of \( L^2(G) \) such that \( d_C(\overline{e, \overline{f}}) = d_{L^2(G)}(\overline{e, \overline{f}}) \), the graph \( L^2(G) \) is not 3-distant hamiltonian.

If \( G \) is a hamiltonian graph, then there is a largest integer \( n \), where \( 0 \leq n \leq \text{diam } G \), such that \( G \) is \( n \)-distant hamiltonian. In this context, we will define the deficiency of \( G \), denoted \( \text{def } G \), by

\[ \text{def } G = \text{diam } G - n. \]
For a connected graph $G$ that is not hamiltonian, we will set $\text{def } G = +\infty$. (We define $+\infty = +\infty$ and $n < +\infty$ for every integer $n$.) Then for any connected graph $G$ we have that $\text{def } G \geq 0$, where $\text{def } G = 0$ if and only if $G$ is diam $G$-distant hamiltonian. We can now combine the results of Theorem 5-6 and Corollary 5-7-1 to obtain a relationship between $\text{def } G$ and $\text{def } L(G)$ for any non-empty connected graph $G$. We do so in Theorem 5-8.

**Theorem 5-8.** If $G$ is a nonempty connected graph, then $\text{def } L(G) \leq \text{def } G$.

**Proof.** If $G$ is non-hamiltonian, the result follows. So we may assume that $G$ is hamiltonian and that $n$ is the largest integer, where $0 \leq n \leq \text{diam } G$, such that $G$ is $n$-distant hamiltonian. Then $\text{def } G = \text{diam } G - n$. Now, by Theorem 5-6, we have $\text{diam } G - 1 \leq \text{diam } L(G) \leq \text{diam } G + 1$.

**Case 1.** Suppose $\text{diam } L(G) = \text{diam } G + 1$. Then $G$ is $n$-distant hamiltonian and $n + 1 \leq \text{diam } G + 1 = \text{diam } L(G)$.

By Corollary 5-7-1, the graph $L(G)$ is $(n + 1)$-distant hamiltonian. Thus

$$\text{def } L(G) \leq \text{diam } L(G) - (n + 1) = \text{def } G.$$

**Case 2.** Suppose $\text{diam } L(G) = \text{diam } G$. If $n \geq 1$, then $G$ is $(n - 1)$-distant hamiltonian and
(n - 1) + 1 ≤ diam G = diam L(G). By Corollary 5-7-1, the graph L(G) is n-distant hamiltonian. Thus

\[ \text{def } L(G) ≤ \text{diam } L(G) - n = \text{def } G. \]

If \( n = 0 \), then since \( L(G) \) is 1-distant hamiltonian, we have

\[ \text{def } L(G) ≤ \text{diam } L(G) - 1 < \text{def } G. \]

Case 3. Suppose \( \text{diam } L(G) = \text{diam } G - 1 \). If \( n ≥ 2 \), then \( G \) is \((n - 2)\)-distant hamiltonian and \((n - 2) + 1 ≤ \text{diam } G - 1 = \text{diam } L(G)\). By Corollary 5-7-1, the graph \( L(G) \) is \((n - 1)\)-distant hamiltonian. Thus

\[ \text{def } L(G) ≤ \text{diam } L(G) - (n - 1) = \text{def } G. \]

If \( n ≤ 1 \), then since \( L(G) \) is 1-distant hamiltonian, we have

\[ \text{def } L(G) ≤ \text{diam } L(G) - 1 < \text{def } G. \]

It is worth pointing out that by Theorem 5-8, if a graph \( G \) is diam \( G \)-distant hamiltonian, then \( L(G) \) is diam \( L(G) \)-distant hamiltonian. In such a situation, \( \text{def } G = 0 = \text{def } L(G) \). As the graph \( G \) of Figure 5.2 illustrates, it is possible for a hamiltonian graph \( G \) and its line graph \( L(G) \) to satisfy \( \text{def } G = \text{def } L(G) \), even when \( \text{def } G > 0 \). Since diam \( G = 4 \) and \( G \) is 0-distant hamiltonian but not 1-distant hamiltonian (there is no
hamiltonian cycle $C$ of $G$ such that $d_C(u,v) = d_G(u,v)$), we have $\text{def } G = 4$. By Theorem 5-6, the diameter of $L(G)$ does not exceed five and, since $d_{L(G)}(x,y) = 5$, we have $\text{diam } L(G) = 5$. By Corollary 5-7-1, the graph $L(G)$ is 1-distant hamiltonian. However, $L(G)$ is not 2-distant hamiltonian since there is no hamiltonian cycle $C$ of $L(G)$ such that $d_C(w,z) = d_{L(G)}(w,z)$. Thus $\text{def } L(G) = 5 - 1 = 4$.

Figure 5.2
Recall that if $G$ is any connected graph other than a path, then $L^m(G)$ is defined for every integer $m \geq 0$. Thus with such a graph we can associate an infinite sequence $\{\text{def } L^m(G)\}_{m=0}^\infty$. According to Theorem 5-8, this sequence is a nonincreasing sequence in which the occurrence of a single term equal to zero implies that all following terms are also zero. Considering the discussion of the previous paragraph, it is natural to ask, for an arbitrary connected graph $G$ which is not a path, whether the sequence $\{\text{def } L^m(G)\}_{m=0}^\infty$ always converges to zero. The purpose of the remainder of this section is to answer this question.

In Theorem 5-9, we will restrict our attention to graphs $G$ with minimum degree at least four. In particular, this will mean that if $w$ is a vertex of $G$ and $K(w)$ is the corresponding complete subgraph of order $\deg_w$ in $L(G)$, then $K(w)$ has order at least four. The proof of Theorem 5-9 will depend heavily on the following two facts. First, if $H$ is a complete graph of order at least four and $e$ and $f$ are nonadjacent edges of $H$, then there is a hamiltonian cycle of $H$ which contains neither $e$ nor $f$ but does contain any other prescribed edge of $H$. Therefore it is also true that if $e$ and $f$ are adjacent edges of $H$, then there is a hamiltonian cycle of $H$ which contains $e$ but not $f$. 

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Theorem 5-9. Let $G$ be a connected graph such that $\delta(G) \geq 4$. Then $L^2(G)$ is diam $L^2(G)$-distant hamiltonian, i.e. $\text{def} L^2(G) = 0$.

Proof. Since $G$ is connected and $\delta(G) \geq 4$, the graph $L^2(G)$ is hamiltonian-connected (see [24, p. 56]) and thus is 1-distant hamiltonian. Therefore it suffices to show that if diam $L^2(G) \geq 2$ and $\overline{e,f}$ are vertices of $L^2(G)$ such that $2 \leq d_{L^2(G)}(\overline{e,f}) \leq \text{diam } L^2(G)$, then there is a hamiltonian cycle $C$ of $L^2(G)$ where $d_{L^2(G)}(\overline{e,f}) = d^2_{L^2(G)}(\overline{e,f})$.

According to Lemma 5-1, there are vertices $u$ and $v$ of $L(G)$ incident with $e$ and $f$, respectively, such that $d_{L^2(G)}(\overline{e,f}) = d_{L^2(G)}(u,v) + 1$. Let $d = d_{L^2(G)}(u,v)$ where, then, $d \geq 1$, and let $P$ be a $u-v$ distance path in $L(G)$. By Lemma 5-1, neither $e$ nor $f$ is an edge of $P$. Label the edges of $P$ as $f_1, f_2, \ldots, f_d$ so that the edge $f_1$ is incident with $u$, the edge $f_d$ is incident with $v$ and, in the case $d \geq 3$, the edges $f_i$ and $f_{i+1}$ are adjacent ($1 \leq i \leq d-1$).

For each vertex $w$ of $G$, let $K(w)$ denote the complete subgraph of order $\text{deg}_G w$ corresponding to $w$ in $L(G)$. Then each edge of $P$ lies in exactly one complete subgraph $K(w)$ of $L(G)$. Furthermore, since $P$ is a shortest $u-v$ path in $L(G)$, no more than one edge

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of \( P \) lies in any complete subgraph \( K(w) \) of \( L(G) \).

Let \( w_1, w_2, \ldots, w_d \) be the \( d \) distinct vertices of \( G \) such that \( f_1, f_2, \ldots, f_d \) lie in the complete subgraphs \( K(w_1), K(w_2), \ldots, K(w_d) \) of \( L(G) \), respectively. We wish to construct a spanning eulerian subgraph of \( L(G) \). In order to do so, we consider two cases.

**Case 1.** Suppose \( d \geq 2 \). Then \( e \) and \( f \) do not lie in the same complete subgraph \( K(w) \) of \( L(G) \). Moreover, \( e \) does not lie in \( K(w_i) \), for \( 2 \leq i \leq d \), and \( f \) does not lie in \( K(w_i) \), for \( 1 \leq i \leq d - 1 \). Let \( w' \) and \( w'' \) be the (distinct) vertices of \( G \) such that \( e \) and \( f \) lie in the complete subgraphs \( K(w') \) and \( K(w'') \) of \( L(G) \), respectively. Now, for each vertex \( w \in V(G) - (\{w_1, w_2, \ldots, w_d\} \cup \{w', w''\}) \), let \( S(w) \) be any spanning cycle of \( K(w) \). If \( d \geq 3 \) and \( 2 \leq i \leq d - 1 \), let \( S(w_i) \) be a spanning cycle of \( K(w_i) \) which contains the edge \( f_i \). If \( w_1 = w' \), then since \( e \) and \( f_1 \) are adjacent edges of \( K(w_1) \), we can choose a spanning cycle \( S(w_1) \) of \( K(w_1) \) which contains the edge \( f_1 \) but not the edge \( e \). Otherwise, let \( S(w_1) \) be a spanning cycle of \( K(w_1) \) which contains the edge \( f_1 \) and let \( S(w') \) be a spanning cycle of \( K(w') \) which does not contain the edge \( e \). Similarly, if \( w_d = w'' \), then since \( f \) and \( f_d \) are adjacent edges of \( K(w_d) \), we can choose a spanning cycle \( S(w_d) \) of \( K(w_d) \) which contains the edge \( f_d \) but not the
edge $f$. Otherwise, let $S(w_d)$ be a spanning cycle of $K(w_d)$ which contains the edge $f_d$ and let $S(w'')$ be a spanning cycle of $K(w'')$ which does not contain the edge $f$.

Case 2. Suppose $d = 1$. Let $w'$ and $w''$ (not necessarily distinct) be the vertices of $G$ such that $e$ and $f$ lie in the complete subgraphs $K(w')$ and $K(w'')$ of $L(G)$, respectively. For each vertex $w \in V(G) - ([w_1] \cup \{w', w''\})$, let $S(w)$ be any spanning cycle of $K(w)$.

Subcase (i). Assume $w_1 \neq w'$ and $w_1 \neq w''$. Then necessarily $w' \neq w''$. Let $S(w_1)$ be a spanning cycle of $K(w_1)$ which contains the edge $f_1$, let $S(w')$ be a spanning cycle of $K(w')$ which does not contain the edge $e$, and let $S(w'')$ be a spanning cycle of $K(w'')$ which does not contain the edge $f$.

Subcase (ii). Assume $w_1 = w'$ and $w_1 \neq w''$. Then let $S(w'')$ be a spanning cycle of $K(w'')$ which does not contain the edge $f$. Since $e$ and $f_1$ are adjacent edges of $K(w_1)$, we can choose a spanning cycle $S(w_1)$ of $K(w_1)$ which contains the edge $f_1$ but not the edge $e$.

Subcase (iii). Assume $w_1 \neq w'$ and $w_1 = w''$. Then let $S(w')$ be a spanning cycle of $K(w')$ which does not contain the edge $e$. Since $f$ and $f_1$ are adjacent
edges of $K(w_1)$, we can choose a spanning cycle $S(w_1)$ of $K(w_1)$ which contains the edge $f_1$ but not the edge $f$.

**Subcase (iv).** Assume $w' = w''$. Then necessarily $w' = w_1 = w''$, so that $e$, $f_1$, and $f$ are edges of $K(w_1)$. Since $e$ and $f$ are nonadjacent edges of $K(w_1)$, we can choose a spanning cycle $S(w_1)$ of $K(w_1)$ which contains neither $e$ nor $f$ but does contain $f_1$.

We now let $B(G)$ be the bonding subgraph of $L(G)$ defined by

$$V(B(G)) = \bigcup_{w \in V(G)} V(S(w)) \quad \text{and} \quad E(B(G)) = \bigcup_{w \in V(G)} E(S(w)).$$

By Theorem 4-8, the graph $B(G)$ is a spanning eulerian subgraph of $L(G)$. In particular, every vertex of $B(G)$ has even degree in $B(G)$. Furthermore, $E(P) \subseteq E(B(G))$.

Thus $B(G) - E(P)$ is a spanning subgraph of $L(G)$ with exactly two odd vertices, $u$ and $v$. Moreover, since $\delta(G) \geq 4$, for each vertex $w$ of $G$, the subgraph of $B(G) - E(P)$ induced by $V(K(w))$ is a connected subgraph of $K(w)$, so that by Theorem 4-8 (i), $B(G) - E(P)$ is connected. Therefore $B(G) - E(P)$ contains an eulerian $v - u$ trail $T$ where, then, $T$ is a spanning $v - u$ trail of $L(G)$. We note that $E(T) \cap (E(P) \cup \{e,f\}) = \emptyset$.

Label the vertices of $T$ so that $T: v = z_1, z_2, \ldots, z_t = u$,
where the vertices $z_i$ are not necessarily distinct.

In order to complete the proof, we now construct an ordering $S$ of the edges of $L(G)$. Begin by selecting the edges $e, f_1, \ldots, f_d, f$ in that order. Next select, in any order, all edges of $L(G)$ incident with $z_1$ that are neither edges of $T$ nor previously selected edges, followed by the edge $z_1z_2$. At each successive $z_i$, $2 \leq i \leq t-1$, select, in any order, all edges of $L(G)$ incident with $z_i$ that are neither edges of $T$ nor previously selected edges, followed by the edge $z_iz_{i+1}$. This process terminates with the edge $z_{t-1}z_t$. The ordering $S$ has the properties that every edge of $L(G)$ appears exactly once in $S$, consecutive edges of $S$ are adjacent in $L(G)$, and the first and last edges of $S$ are adjacent in $L(G)$. Therefore, there is a hamiltonian cycle of $L^2(G)$, say $C$, that corresponds naturally to $S$. Furthermore, the cycle $C$ contains the $\overline{e}-\overline{f}$ path $\overline{e}, \overline{f}_1, \ldots, \overline{f}_d, \overline{f}$ of length $d+1$. Thus $d_{L^2(G)}(\overline{e}, \overline{f}) = d + 1 = d_{L^2(G)}(\overline{e}, \overline{f})$. Since $d_{L^2(G)}(\overline{e}, \overline{f}) \leq d_{L^2(G)}(\overline{e}, \overline{f})$, the proof is complete. 

The graph of Figure 5.3 illustrates that if $G$ is a connected graph with $\delta(G) = 3$, then $L^2(G)$ need not be $\text{diam } L^2(G)$-distant hamiltonian. It is obvious that there is no hamiltonian cycle $C$ of $L^2(G)$ such that $d_{L^2(G)}(\overline{e}, \overline{f}) = d_{L^2(G)}(\overline{e}, \overline{f})$.
Before stating Corollary 5-9-1, we again mention the fact pointed out in Chapter IV that if $G$ is any connected graph other than a path, a cycle, or $K(1,3)$, then there is an integer $t$ such that the graph $L^t(G)$ has minimum degree at least three. Therefore, $\delta(L^{t+1}(G)) \geq 4$ and we may apply Theorem 5-9 to $L^{t+1}(G)$ to conclude that $\text{def } L^{t+3}(G) = 0$. On the other hand, if $G$ is a cycle, then $\text{def } L^0(G) = 0$ and if $G$ is the graph $K(1,3)$, then $\text{def } L^1(G) = 0$. We combine these observations with others made throughout this section in the statement of
Corollary 5-9-1.

Corollary 5-9-1. If $G$ is a connected graph which is not a path, then the sequence $\{\text{def } L^m(G)\}_{m=0}^{\infty}$ converges to zero.


