Local Connectivity in Graphs

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LOCAL CONNECTIVITY IN GRAPHS

by

Donald W. VanderJagt

A Dissertation
Submitted to the
Faculty of The Graduate College
in partial fulfillment
of the
Degree of Doctor of Philosophy

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Donald W. VanderJagt
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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>CHAPTER</th>
<th>TABLE OF CONTENTS</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>PRELIMINARIES.</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Introduction</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Basic Definitions and Notation</td>
<td>4</td>
</tr>
<tr>
<td>II</td>
<td>LOCALLY CONNECTED GRAPHS</td>
<td>9</td>
</tr>
<tr>
<td></td>
<td>Introduction</td>
<td>9</td>
</tr>
<tr>
<td></td>
<td>Locally Connected Graphs with Small Degrees I</td>
<td>11</td>
</tr>
<tr>
<td></td>
<td>Locally Connected Graphs with Small Degrees II</td>
<td>44</td>
</tr>
<tr>
<td></td>
<td>Locally Connected Graphs with Large Degrees</td>
<td>54</td>
</tr>
<tr>
<td>III</td>
<td>GRAPHS WITH PRESCRIBED LOCAL CONNECTIVITIES</td>
<td>56</td>
</tr>
<tr>
<td></td>
<td>Introduction</td>
<td>56</td>
</tr>
<tr>
<td></td>
<td>Some Remarks on Hamiltonian Graphs</td>
<td>61</td>
</tr>
<tr>
<td></td>
<td>Connectivity and Local Connectivity</td>
<td>65</td>
</tr>
<tr>
<td></td>
<td>Local Connectivity and Local Edge Connectivity</td>
<td>69</td>
</tr>
<tr>
<td></td>
<td>Edge Connectivity and Local Edge Connectivity</td>
<td>72</td>
</tr>
<tr>
<td></td>
<td>Critically and Minimally Locally Connected Graphs</td>
<td>85</td>
</tr>
<tr>
<td>IV</td>
<td>SUFFICIENT CONDITIONS FOR LOCAL CONNECTIVITY</td>
<td>90</td>
</tr>
<tr>
<td></td>
<td>Introduction</td>
<td>90</td>
</tr>
<tr>
<td></td>
<td>Sufficient Conditions for Locally Connected Graphs</td>
<td>95</td>
</tr>
</tbody>
</table>

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CHAPTER I

PRELIMINARIES

Introduction

Many of the properties studied in mathematics can be categorized as being either local or global. For example, the integrability of a function can be considered to be a global property since it is defined in terms of the domain of the function or some region contained in its domain. On the other hand, the continuity of a function is a local property in the sense that continuity is defined at a point and a function is said to be (globally) continuous if it is continuous at each point in its domain.

In some cases, the name given to the local property describes the (global) property which appears "locally." In algebra, "local rings" [16] and groups which are "locally cyclic" [14] are encountered. A topological vector space may be "locally compact" or "locally convex" [23]. More generally, a topological space may possess either of these last two properties or be "locally connected" [11]. As each of these terms
implies, a mathematical structure $S$ is said to have local property $P$ if for each $x \in S$, the substructure of $S$ generated by those elements of $S$ "sufficiently close" to $x$ possesses property $P$.

We now turn our attention to the mathematical structure "graph" and the property of connectedness. A locally connected graph is defined as one in which the neighborhood of every vertex is connected. It is the object of this dissertation to investigate graphs which are locally connected. The first paper which considered such graphs in some detail was by Chartrand and Pippert [7]. In this thesis, we extend several of their results and consider related questions not previously studied.

After defining some pertinent terms in Chapter I, we formally introduce the property of a graph being locally connected in Chapter II. For graphs having only vertices of "small degree", we extend a result of Chartrand and Pippert [7] and characterize such graphs. Special consideration is given to hamiltonian graphs.

In Chapter III we define the local connectivity of a graph and determine a graph of minimum order which has prescribed connectivity and local connectivity. Similarly we determine a graph of minimum order
with prescribed local connectivity and local edge connectivity, and a graph of minimum order with prescribed edge connectivity and local edge connectivity. In the last section of the chapter, graphs are investigated which are critically locally connected and minimally locally connected.

Chapter IV is concerned with conditions on degrees of vertices which are sufficient for a graph to be locally connected. Chartrand and Pippert [7] gave a lower bound for the minimum degree of a graph which is sufficient for the graph to be locally connected. Moreover, they showed that this number could not be reduced. We obtain sufficient conditions which allow some vertices to have smaller degrees. Furthermore, we determine the smallest degree a vertex may assume, the number of vertices which may have this degree, and show that all these results are best possible.
Basic Definitions and Notation

For completeness of presentation, we include here a substantial number of basic definitions and notation. Definitions of those terms not given here are found in [2] or [15]. Additional definitions will be given later as required.

The vertex set of a graph $G$ is denoted by $V(G)$; its edge set is denoted by $E(G)$. The order of $G$, usually denoted by $p$, is the number of elements in the vertex set; the size of $G$ is the number of edges in $G$, and is usually denoted by $q$. If $v \in V(G)$, the number of edges incident with $v$ is called the degree of $v$ and is denoted by $\deg_G v$, or $\deg v$ is the graph $G$ is clear from context. If $\deg v = 0$, then $v$ is called an isolated vertex; if $\deg v = 1$, then $v$ is called an end-vertex. A graph $H$ is said to be a subgraph of a graph $G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$; in this case we also say $G$ is a supergraph of $H$.

A non-decreasing sequence $(d_1, d_2, \ldots, d_p)$ of non-negative integers is the degree sequence of a graph $G$ if the vertices of $G$ can be labeled $v_1, v_2, \ldots, v_p$ so that $\deg v_i = d_i$ for $i = 1, 2, \ldots, p$. In this case, $d_1$
is called the minimum degree of $G$ and is denoted by $\delta(G)$; the maximum degree, $\Delta(G)$, is usually denoted by $\Delta(G)$. If each vertex of $G$ has degree $d$, then $G$ is called $d$-regular, or simply a regular graph. A graph of order $p$ in which each vertex is adjacent to each other vertex of the graph is called a complete graph and is denoted by $K_p$.

If $u$ and $v$ are vertices of a graph $G$, a $u$-$v$ walk in $G$ is an alternating sequence of vertices and edges of $G$, beginning with $u$ and ending with $v$ such that each edge is immediately preceded and succeeded by the two vertices it is incident with. It is customary to represent a walk by its vertices only, since the edges are then evident. A $u$-$v$ path is a $u$-$v$ walk in which no vertex is repeated. Two different $u$-$v$ paths are disjoint if they have no vertex in common other than $u$ and $v$; they are called edge-disjoint if they have no edges in common. A path on $n$ vertices is denoted by $P_n$.

A walk $v_1, v_2, \ldots, v_n, v_1$ is called a cycle if all $n$ vertices are distinct and $n \geq 3$. A cycle on $n$ vertices is denoted by $C_n$. A cycle $C$ in a graph $G$ is called a hamiltonian cycle if $C$ contains all the vertices of $G$; if such a cycle exists, then $G$ is called a hamiltonian graph.
A graph $G$ is said to be connected if for every pair $u,v$ of vertices, $G$ contains a $u$-$v$ path; otherwise $G$ is said to be disconnected. A component of $G$ is a maximal connected subgraph of $G$. The number of components of $G$ is denoted by $c(G)$. A vertex of a graph $G$ whose removal from $G$ results in a graph having a larger number of components is called a cut-vertex of $G$. A connected subgraph of $G$ which contains no cut-vertices and which has no connected supergraph in $G$ without cut-vertices is called a block of $G$.

If $G$ is a graph and $V_1 \subseteq V(G)$, then the subgraph of $G$ with vertex set $V_1$ which contains exactly those edges of $G$ which are incident with two vertices in $V_1$ is called the subgraph induced by $V_1$ and is denoted by $\langle V_1 \rangle$. Similarly, a subgraph may be induced by a set of edges.

A graph $G$ is called $n$-partite, $n \geq 2$, if $V(G)$ can be partitioned into non-empty subsets $V_1,V_2,\ldots,V_n$ so that each edge of $G$ is incident with a vertex in $V_i$ and a vertex in $V_j$, $i \neq j$. If $n = 2$, then $G$ is called bipartite. A $n$-partite graph with the additional property that if $u \in V_i$ and $v \in V_j$, $i \neq j$, then $uv \in E(G)$ is called a complete $n$-partite graph; for $n = 2$, the graph is complete bipartite. If $|V_i| = p_i$, 

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the complete n-partite graph is denoted by $K(p_1,p_2,\ldots,p_n)$.

A set $S$ of vertices (or edges) of a graph $G$ is said to separate vertices $u$ and $v$ of $G$ if the removal of $S$ from $G$ results in a disconnected graph in which $u$ and $v$ are in different components. The graph $G$ is $n$-connected, $n \geq 1$, if the removal of fewer than $n$ vertices results in neither a disconnected nor a trivial graph.

The number of edges in a path is called its length. In a connected graph $G$, if $u,v \in V(G)$, the distance between $u$ and $v$, denoted by $d(u,v)$, is the least length of a $u$-$v$ path in $G$. To emphasize that the distance between $u$ and $v$ is determined by paths in $G$, we sometimes use the notation $d_G(u,v)$. The set $V(G)$ is a metric space under this distance function.

The $n$th power $G^n$ of $G$ is the graph for which $V(G^n) = V(G)$ and $uv \in E(G)$ if and only if $1 \leq d_G(u,v) \leq n$. The graph $G^2$ is also called the square of $G$.

Let $G_1$ and $G_2$ be two disjoint graphs; that is, two graphs whose vertex sets are disjoint. The union of $G_1$ and $G_2$, denoted $G_1 \cup G_2$, is the graph for which $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. The join of $G_1$ and $G_2$, denoted
\( G_1 + G_2 \), is the graph for which \( V(G_1 + G_2) = V(G_1) \cup V(G_2) \) and \( E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup X \), where \( X = \{uv | u \in V(G_1) \text{ and } v \in V(G_2)\} \). The union and join of a finite number of mutually disjoint graphs are defined analogously. The union of \( n \) graphs, each of which is isomorphic to \( G \) is, sometimes denoted by \( nG \); if \( G \) is connected, the graph \( nG \) has \( n \) components, each of which is isomorphic to \( G \). One particularly useful graph in the discussion which follows is a \textit{wheel}, defined by \( W_n = K_1 + C_{n-1} \), for \( n \geq 4 \).

A graph is called \textit{planar} if it can be embedded in the plane. A planar graph \( G \) is said to be \textit{maximal planar} if for every pair of non-adjacent vertices \( u \) and \( v \) of \( G \), the graph \( G + uv \) is non-planar. If a graph can be embedded in the plane so that all of its vertices lie on the boundary of a single region, then \( G \) is called \textit{outerplanar}.

Finally, there are two commonly-used arithmetical symbols we will need. For a real number \( x \), we denote the largest integer not exceeding \( x \) by \([x]\), and the least integer not less than \( x \) by \((x)\). We use the symbol \( \square \) to designate the end of a proof.
CHAPTER II

LOCALLY CONNECTED GRAPHS

Introduction

Let G be a graph without isolated vertices and let v be a vertex of G. The neighboring vertices of v, denoted N(v), are those vertices of G adjacent with v. The neighborhood of v, \( \langle N(v) \rangle \), is the subgraph of G induced by the neighboring vertices of v. We say that G has local property P if for each \( v \in V(G) \), the subgraph \( \langle N(v) \rangle \) has property P. Thus, we consider a graph to have a given local property if each "deleted neighborhood of radius one" has this property.

Some aspects of the properties of neighborhoods in graphs have already been investigated. Agakishieva [1] examined graphs with isomorphic neighborhoods; Vetukhnovskii [24, 25] studied the problem of covering a graph with a system of neighborhoods. Our interest, however, lies with graphs in which each neighborhood possesses a property which is usually considered to be a global property of a graph. One such property,
"locally hamiltonian", was studied by Skupiën [21, 22]. As expected, a graph G is \textit{locally hamiltonian} if the neighborhood of each vertex of G is hamiltonian.

One of the most fundamental properties that a graph may possess is that of being connected. It is the local analogue of this property we shall investigate. A graph G without isolated vertices is said to be \textit{locally connected} if the neighborhood of every vertex of G is connected. We shall restrict our attention to graphs without isolated vertices so each vertex will have a well-defined neighborhood.

It should be noted that the properties of being connected and of being \textit{locally connected} are independent. An n-cycle, \( n > 3 \), is connected but is not \textit{locally connected} since each neighborhood consists of two isolated vertices. Conversely, a disconnected graph may be \textit{locally connected}; the graph \( 2K_3 \), which consists of two components each of which is a triangle, is an example of such a graph.
Locally Connected Graphs with Small Degrees I

As noted above, there exist graphs which are disconnected and locally connected and there exist graphs which are connected but not locally connected. Although a graph may be neither connected nor locally connected (e.g., consider $2C_4$), we are interested in graphs which are both connected and locally connected. Chartrand and Pippert [7] obtained the following results.

**Theorem 2.A.** Let $G$ be a connected, locally connected graph with at least three vertices. If $\Delta(G) \geq 4$, then $G$ is hamiltonian unless $G = K(1,1,3)$.

**Theorem 2.B.** If a graph $G$ is locally connected, then every component of $G$, other than $K_2$, is 2-connected.

Although Theorem 2.A provides a necessary condition to be satisfied by a graph of small maximum degree, which is both connected and locally connected, it is immediate that such a condition is not sufficient. For example, a cycle of length four is hamiltonian and has maximum degree two, but is not locally connected. The following result characterizes graphs which are
connected, locally connected and have maximum degree no greater than four.

**Theorem 2.1.** Let $G$ be a graph which has $p \geq 3$ and $\Delta(G) \leq 4$. Then $G$ is connected and locally connected if and only if $G = K(1,1,3)$, $K_3$, $K_4$, $K_4-e$, or is one of the graphs given in Figs. 2.1-2.9, 2.12, 2.13, 2.18, 2.21.

That the graphs described in Theorem 2.1 are connected and locally connected is a direct result of the definitions and consideration of each of the graphs and the resulting neighborhoods. The converse is considerably longer and involves the investigation of several cases. For convenience, we separate the proof into four parts as indicated by the four lemmas below. The proof is then immediate from these four lemmas.

**Lemma 2.2.** Let $G$ be a graph which is connected, locally connected, and has $\Delta(G) \leq 3$. If $p \geq 3$, then $G = K_3$, $K_4$, or $K_4-e$.

**Proof.** By Theorem 2.1, either $G = K(1,1,3)$ or $G$ contains a hamiltonian cycle, call it $C: v_1, v_2, \ldots, v_p, v_1$. Suppose $C$ has two adjacent vertices of degree 2, say $v_1$ and $v_2$. Then $v_1$ is adjacent to $v_3$ since $\langle N(v_2) \rangle$ is
connected. Thus \( G = K_3 \).

If \( C \) contains adjacent vertices of degree 2 and 3, respectively, we may assume \( \deg v_1 = 3 \) and \( \deg v_2 = 2 \), so \( p \geq 4 \). Since \( \langle N(v_2) \rangle \) is connected, \( v_3 \) is adjacent to \( v_1 \). Also, \( v_p \) is adjacent to \( v_3 \) since \( \langle N(v_1) \rangle \) is connected. But, \( \deg v_3 \leq \Delta(G) \leq 3 \), so \( v_p = v_4 \). Thus, \( p = 4 \) and \( G = K_4 - e \).

The only other alternative for \( G \) is to be regular of degree 3 and have \( p \geq 4 \). If \( v_2 \) is adjacent to \( v_j \) for some \( 5 \leq j \leq p-1 \), then at least one of \( v_1, v_3, v_j \) must be adjacent to the other two vertices since \( \langle N(v_2) \rangle \) is connected. But \( \Delta(G) = 3 \), so this is impossible. Hence, \( v_2 \) is adjacent to \( v_4 \) or \( v_p \); we may assume without loss of generality that \( v_2 \) is adjacent to \( v_4 \). If \( p = 4 \), we must have \( G = K_4 \). Otherwise, since \( \langle N(v_4) \rangle \) is connected, \( v_5 \) must be adjacent to \( v_3 \). But this is impossible since \( \langle N(v_2) \rangle \) is not connected.

\( \square \)

Lemma 2.3. Let \( G \) be a graph which is connected, locally connected with \( \Delta(G) = 4, \delta(G) \leq 2 \), and \( p \geq 3 \). Then \( G = K(l, l, 3) \) or \( G \) is one of the graphs in Figs. 2.1-2.6.
Proof. For $\delta(G) = 1$, let $v_1$ be a vertex of $G$ of degree 1. Let $v_2$ be the vertex adjacent to $v_1$. Since $\langle N(v_2) \rangle$ is connected, $v_2$ also must have degree 1. Thus $G = K_2$. But $p \geq 3$, and therefore this is impossible.

Let $\delta(G) = 2$. By Theorem 2.4, either $G = K(1,1,3)$ or $G$ is hamiltonian. Assume $G \neq K(1,1,3)$. Then we may let $C: v_1, v_2, \ldots, v_p, v_1$ be a hamiltonian cycle in $G$. Suppose there are two consecutive vertices on $C$ of degree 2. Let $v_1$ and $v_2$ denote such vertices. Now $\Delta(G) = 4$, so $p \geq 5$. Since $\langle N(v_2) \rangle$ is connected, $v_1$ is adjacent to $v_3$. But $\deg v_1 = 2$, so this is an impossibility.

Suppose $C$ has two consecutive vertices of degree 2 and 3; say $\deg v_2 = 2$ and $\deg v_3 = 3$. Since $\langle N(v_2) \rangle$ is connected, we must have $v_1$ adjacent to $v_3$. Also, $v_1$ is adjacent to $v_4$ because $\langle N(v_3) \rangle$ is connected. Since $\langle N(v_1) \rangle$ is connected, $v_p$ is adjacent to $v_4$. This process may be continued, obtaining one of the graphs in Fig. 2.1 depending on the order of $G$. Thus, if $C$ contains consecutive vertices of degree 2 and 3, $G$ must be the graph in Fig. 2.1(a) if $p$ is odd or the graph in Fig. 2.1(b) if $p$ is even.

The only other alternative is for $G$ to contain a vertex, say $v_2$, of degree 2 while $\deg v_1 = \deg v_3 = 4$. 

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Figure 2.1

(a.) p odd

(b.) p even

Figure 2.2

Figure 2.3

Figure 2.4

Figure 2.5

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Now $\langle N(v_2) \rangle$ is connected, so $v_1$ is adjacent to $v_3$. Let $N(v_1) = \{v_p, v_2, v_3, u\}$. Since $\langle N(v_1) \rangle$ is connected and $\deg v_2 = 2$, $v_3$ must be adjacent to $v_p$ or $u$.

**Case 1. Assume $v_3$ is adjacent to $v_p$.**

If $p = 5$, then $G$ is the graph in Fig. 2.2. Suppose henceforth that $p \geq 6$. Assume $v_4 \neq u$. Then $v_4$ must be adjacent to $v_p$ since $\langle N(v_3) \rangle$ is connected. Now $\langle N(v_1) \rangle$ is connected, so $u$ is adjacent to $v_p$. But $\deg v_p \leq 4$, so $u = v_{p-1}$. If $p = 6$, then $G$ is the graph in Fig. 2.3. For $p \geq 7$, since $\langle N(v_{p-1}) \rangle$ is connected, at least one of $v_p, v_1$ must be adjacent to one of the other vertices in $N(v_{p-1})$. Since $v_p$ and $v_1$ already have maximum degree, $N(v_{p-1})$ must contain a vertex which is adjacent to $v_{p-2}$ and to either $v_p$ or $v_1$. Since $v_2$ and $v_3$ already have maximum degree, this vertex must be $v_4$. But $\deg v_4 \leq 4$, so we must have $v_{p-2} = v_5$. Thus $p = 7$ and $G$ must be the graph in Fig. 2.4.

It remains to consider the possibility in Case 1 where $p \geq 6$ and $v_4 = u$. Since $\langle N(v_4) \rangle$ is connected and $v_5$ cannot be adjacent to $v_1$ or $v_3$, there must be another vertex in $N(v_4)$ which is adjacent to $v_5$ and to either $v_1$ or $v_3$. Since $v_1$ and $v_3$ have maximum degree, this vertex must be $v_p$. But $\deg v_p \geq 4$, so $G$ must be the graph in Fig. 2.5.
Case 2. Assume $v_3$ is adjacent to $u$ but not to $v_p$.

Now $p \neq 5$, for otherwise $\deg v_3 = 3$, contradicting the hypothesis that $\deg v_3 = 4$. Hence $p \geq 6$. Also $\langle N(v_1) \rangle$ is connected, so $v_p$ must be adjacent to $u$.

Since $\deg u \leq 4$, either $u = v_{p-1}$ or $u = v_4$. But if $u = v_{p-1}$, then $v_4$ must be adjacent to $v_{p-1}$ since $\langle N(v_3) \rangle$ is connected; thus $G$ is the graph in Fig. 2.6.

Hence, we consider $u = v_4$. Since $\deg v_3 = 4$, $N(v_3)$ contains a vertex other than $v_1, v_2,$ and $v_4$ which must be adjacent to at least one of $v_1, v_2, v_4$. Since $v_p \notin N(v_3)$ and $\deg v_4 \leq 4$, we must have $v_5 \in N(v_3)$. If $p = 6$, then $G$ is the graph in Fig. 2.3.

Let $p \geq 7$. Since $\langle N(v_p) \rangle$ is connected, and $v_{p-1}$ is not adjacent to $v_1$ or $v_4$, it follows that $N(v_p)$ must contain a vertex which is adjacent to $v_{p-1}$ and to either $v_1$ or $v_4$. This vertex must be $v_5$. But $\deg v_5 \leq 4$, so $v_5 = v_{p-2}$ and $G$ must be the graph in Fig. 2.4.

Lemma 2.4. Let $G$ be a graph which is connected, locally connected with $\Delta(G) = 4$ and $\delta(G) = 3$. Then $G$ is one of the graphs in Figs. 2.7-2.9.

Proof. Again $G$ is hamiltonian, so we may take $C: v_1, v_2, \ldots, v_p, v_1$ to be a hamiltonian cycle in $G$. 

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Since $\Delta(G) = 4$, necessarily $p \geq 5$. First we consider the case in which there are two consecutive vertices on $C$ both of which have degree 3. Suppose $v_2$ and $v_3$ are two such vertices. Certainly $N(v_2) \cap N(v_3) \neq \emptyset$, for otherwise neither $\langle N(v_2) \rangle$ nor $\langle N(v_3) \rangle$ is connected. If $|N(v_2) \cap N(v_3)| = 2$, then we must have $N(v_2) \cap N(v_3) = \{v_1, v_4\}$. Since $p \geq 5$, it follows that $v_5$ cannot be adjacent to $v_2$ or $v_3$ because $\deg v_2 = \deg v_3 = 3$. But $\langle N(v_4) \rangle$ is connected, so $N(v_4)$ must contain another vertex adjacent to $v_5$ and to either $v_2$ or $v_3$. This vertex must be $v_1$ since it is the only vertex, other than $v_4$, adjacent to $v_2$ or $v_3$ which does not have maximum degree. But $\deg v_1 \leq 4$, so $v_5 = v_p$. But then $\deg v_5 = 2$, contradicting the hypothesis that every vertex of $G$ has degree 3 or 4.

It remains to consider the case in which $|N(v_2) \cap N(v_3)| = 1$, where $\deg v_2 = \deg v_3 = 3$. Let $N(v_2) \cap N(v_3) = v'$. If $v' \notin \{v_1, v_4\}$, we may assume, without loss of generality, that $v' = v_1$ and $v_2v_4 \notin E(G)$. But $\langle N(v_3) \rangle$ is connected, so $v_1v_4 \in E(G)$. Let $N(v_2) = \{v_1, v_3, w\}$. Since $\langle N(v_2) \rangle$ is connected, necessarily $v_1$ is adjacent to $w$. But $\deg v_1 \leq 4$, so $w = v_p$. If $p = 5$, then $G$ is the graph in Fig. 2.7. For $p \geq 6$, since $\langle N(v_p) \rangle$ is connected,
$v_1$ or $v_2$ must be adjacent to at least one of the other vertices in $N(v_p)$. But $v_1$ and $v_2$ already have maximum degree, so the vertex in $N(v_p)$, other than $v_1,v_2,v_{p-1}$, must be adjacent to $v_{p-1}$ and to either $v_1$ or $v_2$. This vertex must be $v_4$. But $\deg v_4 \leq 4$, so $v_{p-1}$ and $v_4$ must be consecutive on $C$. Hence $p = 6$. But then $\deg v_5 = 2$, which, by hypothesis, is not possible.

Thus we may assume that $v' \not\in \{v_1,v_4\}$.

Since $\langle N(v_2) \rangle$ and $\langle N(v_3) \rangle$ are connected, we must have $v'$ adjacent to both $v_1$ and $v_4$. But $\deg v' \leq 4$, so $v'$ must be consecutive with $v_1$ and $v_4$ on $C$; thus $p = 5$. Now each vertex of $G$ has degree 3 or 4, so $v_1$ is adjacent to $v_4$. Therefore $G$ must be the graph in Fig. 2.7.

We now may assume that $C$ does not contain consecutive vertices of degree 3. Since $C$ must contain consecutive vertices of degree 3 and 4, we may assume, without loss of generality, that $\deg v_2 = 4$ and $\deg v_3 = 3$. Let $N(v_3) = \{v_2,v_4,u\}$. We consider two cases determined by the location of $u$ on $C$.

**Case 1. Assume** $u \not\in \{v_1,v_5\}$.

Since $\langle N(v_3) \rangle$ is connected, $u$ is adjacent to $v_2$ or $v_4$. We may assume $\deg v_4 = 4$, since otherwise $C$ has two consecutive vertices of degree 3, namely $v_3$ and
Figure 6

Figure 7

Figure 8

Figure 9

Figure 10

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v_4$, a situation which was considered above. Thus \( \deg v_2 = \deg v_4 = 4 \), so we may assume without loss of
generality, that \( u \) is adjacent to \( v_2 \). Since \( \langle N(v_3) \rangle \)
is connected, \( v_4 \) is adjacent to \( u \) or \( v_2 \). If \( v_4 \) is
adjacent with \( u \), then \( u = v_5 \) since \( \deg u \leq 4 \). But we
are considering the case where \( u \not\in \{v_1, v_5\} \), so we must
have \( v_4 \) adjacent with \( v_2 \) but not with \( u \). Since \( \langle N(v_2) \rangle \)
is connected, \( v_1 \) is adjacent to either \( v_4 \) or \( u \). Suppose
\( v_1 v_4 \in E(G) \). Since \( u \neq v_5 \), it follows that \( v_5 \) must be
adjacent to \( v_1 \) because \( \langle N(v_4) \rangle \) is connected. If
\( u = v_p \), then we must have \( p = 6 \); otherwise \( \langle N(v_p) \rangle \)
is not connected. In this case, \( G \) is the graph in
Fig. 2.8. For \( u \neq v_p \), we must have \( v_p \) adjacent to
\( v_5 \) since \( \langle N(v_1) \rangle \) is connected. But this is impossible
since neither of \( v_2, v_3 \) is adjacent to the other two
vertices in \( N(u) \), so \( \langle N(u) \rangle \) is not connected.

Thus, we may assume that \( v_1 \) is adjacent to \( u \) but
not to \( v_4 \). But \( \deg u \leq 4 \), so we must have \( u = v_p \). As
was noted above, \( \deg v_4 = 4 \), so \( p \geq 7 \), since \( v_4 \) is
not adjacent to \( v_p \) or \( v_1 \). Now \( \langle N(v_4) \rangle \) is connected
so at least one of \( v_2 \) and \( v_3 \) is adjacent to one of the
other two vertices in \( N(v_4) \). But this is impossible
since the only vertex adjacent to \( v_2 \) or \( v_3 \) which
does not have maximum degree is \( v_1 \), and, by
assumption, \( v_1 \not\in N(v_4) \).
Case 2. Assume \( u \in \{v_1, v_5\} \).

Since \( \deg v_4 = \deg v_2 = 4 \), we may assume, without loss of generality, that \( u = v_1 \). Now \( \langle N(v_3) \rangle \) is connected, so \( v_4 \) is adjacent to \( v_1 \) or \( v_2 \).

Subcase 2(a). Suppose \( v_4 \) is adjacent to \( v_1 \).

If \( p = 5 \), then \( G \) is the graph in Fig. 2.9, since \( \deg v_2 = 4 \). Let \( p \geq 6 \). Since \( \langle N(v_1) \rangle \) is connected, \( v_p \) is adjacent to \( v_4 \) or \( v_2 \). If \( v_p \) is adjacent to \( v_4 \), then \( v_5 \) is adjacent to \( v_p \) because \( \langle N(v_4) \rangle \) is connected. By hypothesis, \( \deg v_2 = 4 \), and since \( \langle N(v_2) \rangle \) is connected, we must have \( v_1 \) or \( v_3 \) adjacent to at least one of the other vertices in \( N(v_2) \). If \( p \geq 7 \), this is impossible since the only vertices adjacent to \( v_1 \) or \( v_3 \) are \( v_p \) and \( v_4 \), both of which have maximum degree and so cannot be in \( N(v_2) \). If \( p = 6 \), then \( G \) is determined since \( v_2 \) must be adjacent to both \( v_5 \) and \( v_6 \). But this is the graph in Fig. 2.8.

Thus, we consider \( v_p \) adjacent to \( v_2 \) but \( v_p \) not adjacent to \( v_4 \). Now \( \langle N(v_4) \rangle \) is connected, so at least one of \( v_1 \) and \( v_3 \) must be adjacent to at least one of the other vertices in \( N(v_4) \). Since \( v_1 \) and \( v_3 \) already have maximum degree and \( v_p \not\in N(v_4) \), we must have \( v_2 \in N(v_4) \) and \( v_5 \) adjacent to \( v_2 \). But this is impossible because \( \deg v_2 = 4 \).
Subcase 2(b). Assume $v_4$ is adjacent to $v_2$ but not to $v_1$.

If $p = 5$, then $G$ is the graph in Fig. 2.7. Hence we consider $p \geq 6$. Now $\deg v_2 = 4$, by hypothesis, so $N(v_2)$ contains some vertex other than $v_1, v_3,$ and $v_4$; call such a vertex $w$. If $w = v_p$, since $\langle N(v_4) \rangle$ is connected, at least one of $v_2$ and $v_3$ must be adjacent to one of the other vertices in $N(v_4)$. But $v_2$ and $v_3$ already are of maximum degree, so one of the vertices adjacent to $v_2$ or $v_3$ must be in $N(v_4)$ and be adjacent to $v_5$. Since $v_1 \not\in N(v_4)$, we must have $v_p$ in $N(v_4)$ and be adjacent to $v_5$. But $\deg v_p \leq 4$, so necessarily $p = 6$. Also, $\deg v_5 \neq 2$, so $v_5$ is adjacent to $v_1$.

The graph $G$ is isomorphic to the graph of Fig. 2.8, which can be seen by permuting vertices $v_1$ and $v_6$ while fixing the remaining vertices.

If $w = v_5$, since $\langle N(v_1) \rangle$ is connected, at least one of $v_2, v_3$ is adjacent to at least one of the other vertices in $N(v_1)$. But $v_2$ and $v_3$ have maximum degree, so we must have one of $v_2, v_3$ adjacent to a vertex in $N(v_1)$ which must be adjacent to $v_p$. This vertex cannot be $v_4$, so it must be $v_5$; that is, $v_5$ is adjacent to both $v_1$ and $v_p$. But $\deg v_5 \leq 4$, so $p = 6$. Also $\deg v_4 = 4$, so $v_4v_6 \in E(G)$. This graph is the graph in Fig. 2.8, which can be seen by replacing $v_5$ by $v_6$, $v_1$ by $v_5$, $v_6$ by $v_1$ and fixing the remaining vertices.
It remains to consider the case where $N(v_2) = \{v_1, v_3, v_4, v_j\}$ for some $j$ such that $6 \leq j \leq p-1$. For this to occur, we must have $p \geq 7$. Now $\langle N(v_2) \rangle$ is connected, so $v_j$ must be adjacent to either $v_4$ or $v_1$. If $v_j$ is adjacent to $v_4$, then $v_5$ is adjacent to $v_j$, since $\langle N(v_4) \rangle$ is connected. But $\deg v_j \leq 4$, so $v_j = v_6$.

Now $\langle N(v_1) \rangle$ is connected and $v_p$ is not adjacent to $v_2$ or $v_3$, so $N(v_1)$ must contain a vertex which is adjacent to $v_p$ and to either $v_2$ or $v_3$. But $v_1v_4 \not\in E(G)$, $\deg v_3 = 3$, $\deg v_2 = 4$, and $\deg v_6 \leq 4$, so this is impossible. Thus we must have $v_j$ adjacent to $v_1$.

Since $\langle N(v_1) \rangle$ is connected, $v_p$ must be adjacent to $v_j$, so $v_j = v_{p-1}$. Now $\langle N(v_{p-1}) \rangle$ is connected, so $v_{p-2}$ is adjacent to $v_p$. But $\langle N(v_4) \rangle$ is connected, so $v_2$ or $v_3$ must be adjacent to $v_5$ or be adjacent to the other vertex in $N(v_4)$, which must be adjacent to $v_5$.

Since $v_1 \not\in N(v_4)$ and $v_{p-1} \not\in N(v_4)$, this is impossible.

**Lemma 2.5.** Let $G$ be a graph which is connected, locally connected, and is regular of degree 4. Then $G$ is one of the graphs in Figs. 2.12, 2.13, 2.17, 2.21.

**Proof.** By Theorem 2.A, the graph $G$ contains a hamiltonian cycle $C$. Then $H = G - E(C)$ is regular of degree 2, so consists of $k$ disjoint cycles, for some $k \geq 1$. Let $H = C_1 \cup C_2 \cup ... \cup C_k$, where $C_i$ is a
cycle \((1 \leq i \leq k)\). Since a cycle has at least three vertices, \(|E(C_i)| \geq 3\). To construct \(G\) from \(H\), it suffices to determine the hamiltonian cycle \(C\).

Suppose \(k \geq 2\). Let \(C_1: u_1, u_2, \ldots, u_p, u_1\) and \(C_2: v_1, v_2, \ldots, v_p, v_1\). Assume further that \(u_2v_2\) is on hamiltonian cycle \(C\). Since \(\operatorname{deg} v_2 = 4\), there is a vertex \(v\) (other than \(u_2, v_1, v_3\)) to which \(v_2\) is adjacent.

**Case 1. Assume \(v\) is on cycle \(C_2\), say \(v = v_j\).**

Now \(v_j\) is not adjacent to \(u_2\), for otherwise the hamiltonian cycle \(C\) contains only three vertices. But \(\langle N(v_2) \rangle\) is connected, so \(v_j\) is adjacent to at least one of \(v_1\) and \(v_3\). If \(v_j\) is adjacent to exactly one of \(v_1\) and \(v_3\), we may assume that \(v_j\) is adjacent to \(v_3\) but not to \(v_1\). Now \(v_3\) cannot be adjacent to \(u_2\), for otherwise \(C\) contains only four vertices. Since \(\langle N(v_2) \rangle\) is connected, we must have \(v_1\) adjacent to both \(u_2\) and \(v_3\). This is impossible since the hamiltonian cycle \(C\) does not contain \(u_1\).

Thus, we must have \(v_j\) adjacent to both \(v_1\) and \(v_3\). Since \(\operatorname{deg} v_j = 4\), the vertex \(v_j\) must be consecutive on \(C_2\) with at least one of \(v_1\) and \(v_3\); without loss of generality, we assume \(v_j = v_4\) and \(v_1v_4 \in E(C)\). Since \(\langle N(v_2) \rangle\) is connected and \(C\) is hamiltonian, \(u_2\) must be adjacent to \(v_3\). But \(\langle N(u_2) \rangle\) is connected, so \(v_3\) must
be adjacent to one of $u_1$ and $u_3$; without loss of
generality, say the latter. Then $u_1$ must be adjacent
to $u_3$. Since $\langle N(v_4) \rangle$ is connected, $v_5$ must be adjacent
to $v_1$; since $\langle N(u_3) \rangle$ is connected, $u_4$ must be adjacent
to $u_1$.

We continue this process in each of the cycles.
The process must terminate, resulting in the graph of
Fig. 2.10. Since $\deg v_{j+1} = 4$, there is a vertex $v'$
to which $v_{j+1}$ is adjacent ($v' \neq v_{j-1}, v_j, v_{j+2}$). Now
$v' \neq u_{i-1}$, since otherwise the hamiltonian cycle $C$
does not contain $u_i$, an impossibility. If $v' = u_i$,
then $u_i$ must be adjacent to $v_j$ because $\langle N(v_{j+1}) \rangle$ is
connected, and $u_{i-1}$ must be adjacent to $v_j$ since
$\langle N(u_i) \rangle$ is connected. Then $C$ contains exactly those
vertices on cycles $C_1$ and $C_2$. This is impossible
unless $k = 2$.

If $k \geq 3$, then $v'$ must be a vertex on $C_i$, for some
$i \geq 3$; without loss of generality, say $v' = w_2 \in V(C_3)$.
Then $w_2$ is adjacent to $v_j$ since $\langle N(v_{j+1}) \rangle$ is connected,
and one of $w_1$ and $w_3$, say $w_3$, is adjacent to $v_j$ since
$\langle N(w_2) \rangle$ is connected. Thus, $w_1$ is adjacent to $w_3$, and
we may continue this process on $C_3$ as above. Ulti-
mately the process terminates, and we obtain the graph
in Fig. 2.11. This completes Case 1.
Case 2. Assume \( v \) is on cycle \( C_i \), \( i \neq 1,2 \); without loss of generality, say \( v = v_2 \in V(C_3) \).

Since \( \langle N(v_2) \rangle \) is connected, one of \( v_1, v_3 \) is adjacent to one of \( u_2, w_2 \); without loss of generality, say \( v_1 \) is adjacent to \( w_2 \). Also, \( \langle N(w_2) \rangle \) is connected, so \( v_1 \) is adjacent to one of \( w_1 \) and \( w_3 \); without loss of generality, say \( v_1 \) is adjacent to \( w_3 \). Then \( \langle N(v_2) \rangle \) is not connected since neither \( u_2 \) nor \( v_3 \) is adjacent to \( v_1 \) or \( w_2 \) unless \( C_2 \) is a triangle, in which case \( v_3 \) is adjacent to \( v_1 \). If \( C_2 \) is a triangle, then \( u_2 \) is adjacent to \( v_3 \) since \( \langle N(v_2) \rangle \) is connected, and since \( \langle N(u_2) \rangle \) is connected, we may take \( u_3 \) to be adjacent to \( v_3 \). Continuing this process as in Case 1, we obtain the result that if a cycle \( C_i \) contains a vertex which is adjacent to two vertices on distinct cycles, then \( C_i \) must be a triangle. With this restriction, the construction of \( C \) follows the procedure in Case 1; in fact, Case 2 is Case 1 when cycle \( C_2 \) is taken to be a triangle.

Case 3. Assume \( v = u_j \in V(C_3) \), \( j \neq 1,3 \).

Since \( C \) is hamiltonian, \( u_j \) cannot be adjacent to \( u_2 \). But \( \langle N(v_2) \rangle \) is connected, so \( u_j \) is adjacent to either \( v_1 \) or \( v_3 \); without loss of generality, say \( u_j \) is adjacent to \( v_1 \). Again since \( \langle N(v_2) \rangle \) is connected, one of \( u_2 \) and \( v_3 \) is adjacent to \( v_1 \). If \( u_2 \) is
adjacent to $v_1$, then $C$ is not hamiltonian, so $v_3$ must be adjacent to $v_1$. But the connectedness of $\langle N(v_2) \rangle$ necessitates that $u_2$ be adjacent to $v_3$, which again contradicts the hamiltonian property of $C$. Thus, Case 3 cannot occur.

**Case 4.** Assume $v = u_1$ or $u_3$; without loss of generality, say $v = u_3$.

Since $\langle N(v_2) \rangle$ is connected, one of $v_1$ and $v_3$ is adjacent with one of $u_2$ and $u_3$; without loss of generality, let $v_3$ be adjacent to $u_3$. Unless $C_2$ is a triangle, we must have $v_1$ adjacent to $u_2$ or $v_1$ adjacent to $v_3$.

**Subcase 4(a).** Suppose $C_2$ is a triangle.

Since $\langle N(u_3) \rangle$ is connected, $u_4$ is adjacent to $u_2$ or $v_3$. If $u_4$ is adjacent to $u_2$, then $u_1$ must be adjacent to $u_4$ since $\langle N(u_2) \rangle$ is connected. Continuing in this manner, we see that this situation is the same as that considered above in Case 1 and illustrated in Fig. 2.10. Thus we consider $u_4$ adjacent to $v_3$. Now $\langle N(u_2) \rangle$ is connected so at least one of $u_3$ and $v_2$ must be adjacent to one of the other vertices in $N(u_2)$. Since $u_3$ and $v_2$ already have maximum degree, one of the vertices adjacent to $u_3$ or $v_2$ must be in $N(u_2)$ and adjacent to $u_1$. This vertex must be $v_1$.
that is, v is adjacent to $u_1$ and $u_2$. But $\langle N(u_1) \rangle$ is not connected since neither of $v_1$ and $u_2$ is adjacent to either of the other two vertices in $N(u_1)$. This is impossible, so no new graphs are obtained in this case.

Subcase 4(b). Suppose $v_1u_2 \in E(C)$.

Since $\langle N(u_2) \rangle$ is connected, either $C_1$ is a triangle or $u_1$ is adjacent to $v_1$. But by symmetry, the situation for $C_1$ being a triangle was considered in Subcase 4(a); hence we may assume $C_1$ is not a triangle. Thus, $u_1$ is adjacent to $v_1$. Since $\langle N(u_3) \rangle$ is connected, $u_4$ is adjacent to $v_3$. Then, since $C_2$ is not a triangle, $v_4$ is adjacent to $u_4$ because $\langle N(v_3) \rangle$ is connected. We continue in this manner. Without loss of generality, suppose the number of vertices on $C_1$ does not exceed the number of vertices on $C_2$. Let $C_1$ contain $k$ vertices, $k \geq 4$. Now $\langle N(u_k) \rangle$ is connected so $u_1v_k \in C$. This is possible if and only if $H$ consists of exactly two cycles of the same length. The graph $G$ is given in Fig. 2.12.

Subcase 4(c). Suppose $v_1v_3 \in E(C)$.

Then $v_4$ is adjacent to $v_1$ since $\langle N(v_3) \rangle$ is connected. Also, $u_4$ is adjacent to $u_2$ since $\langle N(u_3) \rangle$ is connected, and the connectedness of $\langle N(u_2) \rangle$ implies
is adjacent to $u^\parallel$. This situation is the same as in Case 1. The resulting graph $G$, given in Fig. 2.11, is the same as the graph given in Fig. 2.13.

This completes Case 4 and consequently the situation when $k \geq 2$. It remains to determine $G$ if $k = 1$, i.e., if $H$ is hamiltonian. In this case $G$ consists of two hamiltonian cycles, namely $H$ and $C$.

Let the hamiltonian cycle $H$ of length $p \geq 5$ be given. To determine another hamiltonian cycle $C$ on the same $p$ vertices and hence the graph $G$, we consider three cases determined by the location on $H$ of the two neighboring vertices of a given vertex which are not consecutive with this vertex on $H$. Let $v_3$ denote the given vertex, so $N(v_3)$ consists of $v_2, v_4$ and two other vertices of $H$, say $w$ and $w'$. The three cases are determined by the cardinality of $\{w, w'\} \cap \{v_1, v_5\}$, which can be 2, 1, or 0.

**Case 5.** Assume $|\{w, w'\} \cap \{v_1, v_5\}| = 2$, that is $v_1v_3, v_3v_5 \in E(C)$.

Now $v_1v_5 \not\in E(C)$, for otherwise $C$ is a cycle of length three so is not hamiltonian. Without loss of generality, either $v_2v_5 \in E(C)$ or $v_2v_4 \in E(C)$ since $\langle N(v_3) \rangle$ is connected.
Subcase 5(a). Assume $v_2 v_5 \notin E(C)$.

If $p = 5$, then necessarily $G = K_5$. For $p \neq 5$, $\langle N(v_5) \rangle$ is connected so $v_6$ is adjacent to $v_2$ or $v_4$.

If $v_6$ is adjacent to $v_2$, then since $\langle N(v_4) \rangle$ is connected, at least one of the vertices in $N(v_4)$, other than $v_3$ and $v_5$, must be adjacent to $v_3$ or $v_5$, so must be $v_1$ or $v_6$. If $p = 6$, then both $v_1$ and $v_6$ are in $N(v_4)$ and $G$ is the graph given in Fig. 2.14. If $p \neq 6$, then exactly one of $v_1$ and $v_6$ is in $N(v_4)$. If $v_1 \notin N(v_4)$, then $v_6 \notin N(v_4)$, since $\langle N(v_4) \rangle$ is connected.

Now $p \neq 7$ since $\deg v_6 = 3$. Similarly, $p \neq 8$. But then $\langle N(v_6) \rangle$ is not connected since neither $v_2$ nor $v_5$ is adjacent to either of the other two vertices in $N(v_6)$. Thus $v_1 \notin N(v_4)$, so $v_6 \notin N(v_4)$. Since $\langle N(v_6) \rangle$ is connected, $v_7$ is adjacent to $v_4$. But $G$ is regular of degree 4, so $p \geq 9$. Now $\langle N(v_7) \rangle$ is not connected since neither $v_4$ nor $v_6$ is adjacent to either of the other two vertices in $N(v_7)$. This is impossible.

Thus we must have $v_6$ adjacent to $v_4$ and $p \geq 6$.

Since $v_4 v_6 \in E(C)$, if $p = 6$, $G$ is the graph in Fig. 2.14. If $p = 7$, then $G$ is the graph given in Fig. 2.15. For $p \geq 8$, since $\langle N(v_4) \rangle$ is connected, either (i) there is an integer $j \geq 8$ such that $v_4 v_j, v_6 v_j \in E(C)$, (ii) $v_4$ is adjacent to $v_1$, (iii) $v_4$ is adjacent to $v_2$, or (iv) $v_4$ is adjacent to $v_7$. 

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Figure 2.13

Figure 2.14

Figure 2.15

Figure 2.16

Figure 2.17

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Now $v_4$ and $v_6$ cannot be adjacent to some $v_j$ ($7 \leq j \leq p$) for otherwise $v_4, v_j, v_6, v_4$ is a cycle in $G$ containing no edges of $H$ which is not hamiltonian; thus (i) cannot occur. Also $v_4$ is not adjacent to $v_1$ for otherwise the connectedness of $\langle N(v_1) \rangle$ implies that $v_p$ is adjacent to $v_2$ and hence $\langle N(v_6) \rangle$ is not connected. Furthermore, $v_4$ cannot be adjacent to $v_2$ since otherwise $\langle N(v_6) \rangle$ would not be connected. Thus $v_4$ is adjacent to $v_7$.

Now $v_2$ is not adjacent to $v_6$ or else $\langle N(v_1) \rangle$ is not connected. If there is a vertex $v_j$ ($j \geq 8$) to which $v_2$ and $v_1$ are adjacent, then $v_j, v_1, v_3, v_5, v_2, v_j$ is a non-hamiltonian cycle in $G$ containing no edges of $H$, which is impossible. In addition, $v_2$ is not adjacent to $v_7$ since $\langle N(v_2) \rangle$ is connected. Thus $v_2$ is adjacent to $v_p$. If $p = 8$, then $G$ is the graph given in Fig. 2.16. For $p \geq 9$, the vertex in $N(v_1)$ other than $v_2, v_3, v_p$, call it $v'$, must also be adjacent to $v_p$, since $\langle N(v_1) \rangle$ is connected. This vertex must be $v_{p-1}$, for otherwise $v', v_1, v_3, v_5, v_2, v_p, v'$ is a non-hamiltonian cycle in $G$ containing no edges of $H$.

Thus $v_{p-1}$ is adjacent to $v_1$. Similarly, the vertex in $N(v_6)$ other than $v_4, v_5, v_7$ must be adjacent to $v_7$ since $\langle N(v_6) \rangle$ is connected. This vertex must be $v_8$, for otherwise $G$ contains a non-hamiltonian cycle.
Continue this process until we obtain one of the graphs in Fig. 2.18. Since G is regular of degree 4, the remaining edges of G are determined in each case. Thus G is the graph given in Fig. 2.17. An isomorphic formulation of G is given in Fig. 2.19. This completes Subcase 5(a).

Subcase 5(b). Assume $v_2v_4 \in E(C)$ and $v_1v_5, v_2v_5 \notin E(C)$.

Since $\deg v_5 = 4$, it follows that $p \geq 7$. Let $N(v_5) = \{v_3, v_4, v_6, v_j\}$, where $j \geq 7$. Since $\langle N(v_5) \rangle$ is connected, one of $v_j$ and $v_6$ is adjacent to one of $v_3$ and $v_4$, so $v_4$ is adjacent to $v_6$ or $v_j$, where $j \geq 7$. In the latter case, $v_j$ is adjacent to both $v_5$ and $v_4$ and, since $\langle N(v_5) \rangle$ is connected, $v_6$ must be adjacent to $v_j$. Now $\deg v_j = 4$, so $v_jv_6 \in E(H)$, that is, $j = 7$. But then $v_3, v_5, v_7, v_4$ is the path which determined Subcase 5(a), so this situation has been considered above. Hence we consider $v_4$ adjacent to $v_6$. Again, since $\langle N(v_5) \rangle$ is connected, $v_j$ is adjacent to $v_6$. If $j \geq 9$, then $\langle N(v_6) \rangle$ is not connected. If $v_j = v_8$, then $v_4, v_6, v_8, v_5$ is the path which determined Subcase 5(a), and therefore has already been considered.

Thus, it remains to consider $v_j = v_7$.

Let $N(v_6) = \{v_4, v_5, v_7, v_i\}$. Since $\langle N(v_6) \rangle$ is connected, $v_i$ is adjacent to $v_7$, so $8 \leq i \leq p$. If $i \geq 10$,
Figure 2.18

Figure 2.19

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has been considered in Subcase 5(a). Thus $v_i = v_8$. We continue in this manner until one of the graphs in Fig. 2.20 is obtained. The graph in Fig. 2.20(a) is regular of degree 4, so $v_p$ must be adjacent to both $v_{p-2}$ and $v_2$. But then $v_2, v_4, v_6, \ldots, v_{2i}, \ldots$, $v_p, v, v_2$ is a non-hamiltonian cycle in $G$ which contains no edges of $H$. This is impossible, so this situation cannot occur. For the graph in Fig. 2.20(b), we have $\deg v_p = 4$, so $v_p$ is adjacent to $v_{p-2}$ and $v_2$; consequently $v_{p-1}$ is adjacent to $v_1$. The resulting graph $G$ can be realized as either of the graphs in Fig. 2.21. This completes Subcase 5(b) and hence Case 5.

Case 6. Assume $|\{w, w'\} \cap \{v_1v_5\}| = 1$, say $v_3v_5 \in E(C)$ and $v_jv_5 \in E(C)$ for some $j, 6 \leq j \leq p$.

Since $C$ is hamiltonian, $v_jv_5 \not\in E(C)$. Since $\langle N(v_3) \rangle$ is connected, $v_j$ is adjacent to either $v_4$ or $v_2$.

Subcase 6(a). Assume $v_4v_j \in E(C)$.

Since $\langle N(v_3) \rangle$ is connected, $v_2$ is adjacent to $v_4$ or $v_5$. Suppose $v_2v_4 \in E(C)$. Now we may assume $v_j \neq v_1$, for otherwise $v_5, v_3, v_1, v_4$ is the path which determined Subcase 1(a) and thus has been considered.

Also, if $v_j = v_6$, then path $v_2, v_4, v_6, v_3$ indicates that
Figure 2.20

(a) $p$ even

(b) $p$ odd

Figure 2.21 $p$ odd

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this situation has been considered above. But for
7 \leq j \leq p, neither \(v_{j-1}\) nor \(v_{j+1}\) is adjacent to \(v_3\)
or \(v_4\), so \(\langle N(v_j)\rangle\) is not connected, an impossibility.
Hence we cannot have \(v_2v_4 \in E(C)\), so necessarily
\(v_2v_5 \in E(C)\).

We now introduce a procedure which will be used frequently to show that the situation being analyzed
has been considered previously. Recall that in con­structing the graph \(G\), we are beginning with the ver­tices labeled relative to their position on the hamil­tonian cycle \(H\) and we construct the hamiltonian cycle
\(C\) by determining the \(p\) edges of \(C\). Sometimes it is
instructive to interchange the roles of \(H\) and \(C\) so the
vertices are ordered according to their location on \(C\)
instead of their location on \(H\). When this is done,
after a few edges of \(C\) have been determined, it may
happen that the portion of \(G\) constructed thus far is
exactly that considered in a previous case with the
roles of \(H\) and \(C\) interchanged. If this occurs, we
need not analyze such a situation further; we consider
the two situations to be "equivalent." In the case
under consideration, the vertices \(v_2, v_5, v_3, v_4\) are
consecutive on hamiltonian cycle \(C\) and the hamiltonian
cycle \(H\) contains the path \(v_2, v_3, v_4, v_5\). Thus, this
situation is equivalent to Subcase 5(a) considered above.

Subcase 6(b). Assume $v_{2}v_{j} \in E(C)$. Since $C$ is hamiltonian, $v_{2}v_{5} \not\in E(C)$. Since $\langle N(v_{3}) \rangle$ is connected, at least one of $v_{2}$ and $v_{j}$ is adjacent to one of $v_{3}$ and $v_{5}$; that is (i) $v_{j}$ is adjacent to $v_{4}$, (ii) $v_{2}$ is adjacent to $v_{4}$, or (iii) $v_{j}$ is adjacent to $v_{5}$.

If $v_{j}$ is adjacent to $v_{4}$, since $\deg v_{j} = 4$, we must have $v_{j} = v_{1}$. But then $C$ contains the path $v_{5}, v_{3}, v_{1}, v_{4}$, so this case has been considered above in Subcase 5(a). Likewise, if $v_{2}$ is adjacent to $v_{4}$, the path $v_{4}, v_{2}, v_{j}, v_{3}$ in $C$ is equivalent to the path $v_{5}, v_{3}, v_{j}, v_{4}$ which determined Subcase 6(a). Thus, we need only consider $v_{j}$ adjacent to $v_{5}$. But $\deg v_{j} = 4$, so we must have $v_{j} = v_{6}$. Since $\langle N(v_{6}) \rangle$ is connected, either $v_{7}$ is adjacent to $v_{5}$ or $v_{2}$. If $v_{7}v_{5} \in E(C)$, then the path $v_{7}, v_{5}, v_{3}, v_{6}$ in $C$ shows that this case has been considered above. Thus $v_{7}v_{2} \in E(C)$. As was done above, reorder the vertices of $G$ according to their location on $C$ instead of their location on $H$. It then follows that $G$ contains the path of length two and the edge which determined Subcase 5(b). Hence, this case is equivalent to Subcase 5(b) which has been
considered above. This completes Subcase 6(b) and hence Case 6.

**Case 7.** \(|\{w,w'\} \cap \{v_1,v_5\}| = 0, \text{ say } N(v_3) = \{v_2,v_4,v_i,v_j\}, \text{ where } 6 \leq i < j \leq p.\)

Since \(\langle N(v_3) \rangle\) is connected, at least one of \(v_i\) and \(v_j\) is adjacent to at least one of \(v_2\) and \(v_4\). Without loss of generality, we consider \(v_j\) to be adjacent to either \(v_2\) or \(v_4\).

**Subcase 7(a).** Assume \(v_j v_2 \in E(C)\).

Since \(C\) is hamiltonian, \(v_2 v_i \notin E(C)\). But \(\langle N(v_3) \rangle\) is connected, so \(v_i\) is adjacent to \(v_4\) or \(v_j\). If \(v_4 v_i \in E(C)\), then \(v_2 v_4 \notin E(C)\) since \(C\) is hamiltonian. But \(\langle N(v_3) \rangle\) is connected, so we must have \(i = j - 1\).

Reordering the vertices of \(G\) according to their location on \(C\), instead of on \(H\), we see that this situation has been considered in Subcase 5(b). Thus, we may consider \(v_1 v_j \in E(C)\) and \(v_4 v_i \notin E(C)\). But \(\deg v_j = 4\), so \(i = j - 1\). Since \(\langle N(v_3) \rangle\) is connected, necessarily \(v_4\) is adjacent to \(v_2\). Also \(\langle N(v_2) \rangle\) is connected, so \(v_1\) is adjacent to \(v_4\) or \(v_j\).

If \(v_1 v_4 \in E(C)\), then, reordering the vertices according to their location on \(C\), we recognize that this situation is equivalent to that which determined Subcase 5(a). Thus we may assume \(v_1 v_j \in E(C)\). Since
$\deg v_j = 4$, we must have $v_j = v_p$. But, then $C$ contains the path $v_4, v_2, v_p, v_3$ so again we have a situation equivalent to one considered above.

**Subcase 7 (b).** Assume $v_4 v_j \in E(C)$ and $v_2 v_j, v_4 v_i \notin E(C)$.

Since $\langle N(v_3) \rangle$ is connected, $v_2$ is adjacent to either $v_i$ or $v_4$. But if $v_2 v_i \in E(C)$, by reordering the vertices according to their location on $C$, we observe that this situation is equivalent to Case 5 considered above. Thus, we must have $v_2 v_4 \in E(C)$, and we may assume $v_2 v_i \notin E(C)$. But $\langle N(v_3) \rangle$ is connected, so both $v_4$ and $v_i$ must be adjacent to $v_j$. Since $\deg v_j = 4$, necessarily $i = j-1$. Since $\langle N(v_4) \rangle$ is connected, we must have $v_2 v_5 \in E(C)$. Again, by reordering the vertices of $G$ according to their location on $C$, we see that this case is equivalent to one considered above since it contains the path $v_5, v_4, v_3, v_2$. This completes Subcase 7(b) and hence Case 7.

We have now determined that if the graph $G$ is connected, locally connected, and is regular of degree 4, it must be one of the graphs given in Figs. 2.12, 2.13, 2.14, 2.15, 2.16, 2.17, 2.21 or $G = K_5$. But $K_5$ is a special case of the graph in Fig. 2.21; the graphs in Figs. 2.14, 2.15 and 2.16 are special cases of the graph in Fig. 2.17. Hence, $G$ must be one of the graphs...
given in Fig. 2.12, Fig. 2.13, (alternately Fig. 2.11), Fig. 2.17 (alternately Fig. 2.19), or Fig. 2.21. This completes the proof of Lemma 2.5, and hence of Theorem 2.1.
Locally Connected Graphs with Small Degrees II

In Theorem 2.1, it was determined that if G is connected, locally connected, and has \( \Delta(G) \leq 4 \), then with a single exception, G must be hamiltonian. We might expect that a few more "exceptions" arise when \( \Delta(G) \) is allowed to increase, but perhaps only finitely many. However, this is not so. In fact, a slight increase in \( \Delta(G) \) produces an infinite class of non-hamiltonian graphs, as evidenced in the following result. The class of graphs obtained in this theorem includes the non-hamiltonian graph obtained when \( \Delta(G) \) was taken to be at most 4, since \( G_5 = K(1,1,3) \).

**Theorem 2.6.** Let G be a graph which is connected, locally connected, and has \( p \geq 3 \). Suppose G has at most one vertex of degree 5, all other vertices being of degree at most 4. Then G is hamiltonian or \( G = G_k \) for some \( k \geq 5 \), where \( G_k \) is one of the non-hamiltonian graphs in Fig. 2.22.

**Proof.** Since G is connected and locally connected, by Theorem 2.B, G is 2-connected, so that G contains a cycle. Assume G is not hamiltonian, and let C be a
cycle of maximum length in \( G \). Since \( G \) is not hamiltonian, there is a vertex of \( G \) not on \( C \). Since \( G \) is connected, there are vertices \( v_3 \) and \( v_k \) of \( G \) such that \( v_3 \in V(C), v_k \not\in V(C), \) and \( v_3v_k \in E(G) \). Let \( v_2, v_4 \in V(G) \) be such that \( v_2, v_3, \) and \( v_4 \) are consecutive on \( C \). Since \( C \) is a cycle of maximum length, \( v_2v_k \not\in E(G) \). Similarly, \( v_4v_k \not\in E(G) \). Now \( \langle N(v_3) \rangle \) is connected, so \( \deg v_3 > 3 \).

![Diagram](image)

**Case 1.** Assume \( \deg v_3 = 4 \).

Since \( \langle N(v_3) \rangle \) is connected, there is a vertex \( v_1 \) of \( G \) such that \( v_3v_1 \in E(G), v_1v_k \in E(G), \) and \( v_1 \) is adjacent to at least one of \( v_2 \) and \( v_4 \). Without loss of generality, assume \( v_1v_2 \in E(G) \). Since \( v_1v_2, v_1v_3 \in E(G) \), and \( C \) is a cycle of maximum length, necessarily \( v_1 \in V(C) \). Since \( |N(v_3)| = 4 \) and \( \langle N(v_3) \rangle \) is connected, \( v_4 \) must be adjacent to at least one of \( v_1 \) and \( v_2 \).
Suppose $v_4$ is not adjacent to $v_1$. Then $v_2v_4 \notin E(G)$ since $\langle N(v_3) \rangle$ is connected. There must be at least one vertex on $C$ between $v_1$ and $v_2$ since otherwise $v_1,\ldots,v_4,v_2,v_3,v_1$ is a cycle of length greater than that of $C$. Similarly there must be at least one vertex on $C$ between $v_1$ and $v_4$ since otherwise $v_1,\ldots,v_2,v_4,v_3,v_k,v_1$ is a cycle whose length is greater than $C$. Thus $\deg v_1 = 5$. Let $w_1,w_2 \in V(G)$ be such that $C$ is given by $C_v: v_1,w_1,\ldots,v_2,v_3,v_4,\ldots,w_2,v_1$. Now $w_2v_2 \notin E(G)$ since otherwise $\deg v_2 = 5$, which is impossible since $\deg v_1 = 5$ and $G$ has only one vertex of degree 5. Also, $v_3$ is not adjacent to $w_1$ or $w_2$ since $\deg v_3 = 4$. Since $C$ is a cycle of maximum length, $w_1v_4 \notin E(G)$ since otherwise $w_1,\ldots,v_2,v_3,v_k,v_1,w_2,\ldots,v_4,w_1$ is a cycle of length greater than that of $C$; now $w_1w_2 \notin E(G)$ since otherwise $w_2,\ldots,v_4,v_3,v_k,v_1,v_2,\ldots,w_1,w_2$ is a cycle of length greater than that of $C$. Also, $w_1v_kv_2w_k \notin E(G)$ since otherwise $w_i,\ldots,v_1,v_k,w_i$ for $i = 1,2$, is a cycle of length greater than $C$. Since $\deg v_1 = 5 = \Delta(G)$, $N(w) = \{w_1,w_2,v_2,v_3,v_k\}$. Thus, in $\langle N(v_1) \rangle$ the vertex $w_2$ is isolated; this is impossible since $\langle N(v_1) \rangle$ is connected. Therefore, it cannot occur that $v_4$ is not adjacent to $v_1$, i.e., $v_1v_4 \in E(G)$.
Since $\text{deg} v_1 \leq 5$, at least one of $v_1$ and $v_4$ or $v_1$ and $v_2$ are consecutive on $C$. If both $v_1v_2, v_1v_4 \in E(C)$, then $|V(C)| = 4$. In this case, $v_2v_4 \notin E(G)$ since otherwise cycle $v_4', v_1', v_k', v_3', v_2', v_4$ contradicts the choice of $C$. Thus $G$ contains the induced subgraph $G_5 = K(1,1,3)$, where $\{v_2', v_4', v_k'\}$ is the partite set of order three.

We claim $G = G_5$, for suppose not. Then there is a vertex $t$ of $G$ which is distinct from the five vertices in $G_5$ and is adjacent to at least one vertex of $G_5$ (since $G$ is connected). Since a cycle of length four has maximum length in $G$, the vertex $t$ cannot be adjacent to more than two vertices of $G_5$, other than $v_1$ and $v_3$. But $G$ has at most one vertex of degree 5 so $t$ cannot be adjacent to both $v_1$ and $v_3$. Thus $t$ must be adjacent to exactly one vertex of $G_5$. If $t$ is adjacent to $v_1$ (or $v_3$), then $\langle N(v_1) \rangle$ (respectively, $\langle N(v_3) \rangle$) is not connected, contrary to the hypothesis. If $t$ is adjacent to $v_4$ (or $v_2$ or $v_k$), then $t \in N(v_4)$ (respectively $N(v_2)$ or $N(v_k)$), which induces a connected graph. Thus there is a $t-v_3$ path in $\langle N(v_4) \rangle$ (respectively $\langle N(v_2) \rangle$ or $\langle N(v_k) \rangle$) which has length at least two. But this gives rise to a path in $G$ of length greater than four, which is impossible.
by the definition of C. Therefore, such a vertex t cannot exist, so \( G = G_5 = K(1,1,3) \).

We now consider the case where exactly one of \( v_2 \) and \( v_4 \) is consecutive with \( v_1 \) on \( C \). Without loss of generality, say \( v_1 \) and \( v_2 \) are consecutive on \( C \), while \( v_1 \) and \( v_4 \) are not consecutive on \( C \). Let \( v_{k-1} \in V(G) \) be such that \( C: v_1, v_2, v_3, \ldots, v_{k-1}, v_1 \). Now \( v_{k-1}, v_k \not\in E(G) \) since otherwise \( v_1, v_2, v_3, \ldots, v_{k-1}, v_k, v_1 \) is a cycle of length greater than that of \( C \). Also, \( v_2, v_{k-1} \not\in E(G) \) since otherwise \( v_1, v_k, v_3, v_4, \ldots, v_{k-1}, v_2, v_1 \) is a cycle of length greater than that of \( C \).

Since \( \deg v_3 = 4 \), we must have \( v_3 v_{k-1} \not\in E(G) \). Now \( \langle N(v_1) \rangle \) is connected, so necessarily \( v_4, v_{k-1} \in E(G) \). Also \( v_2, v_4 \not\in E(G) \), for otherwise \( v_1, v_k, v_3, v_2, v_4, \ldots, v_{k-1}, v_1 \) is a cycle of length greater than the length of \( C \). Thus \( G \) contains the induced subgraph \( G_6 \) of Fig. 2.22.

If \( v_4 \) and \( v_{k-1} \) are not consecutive on \( C \), let \( v_5 \in V(G) \) be such that \( C: v_1, v_2, v_3, v_4, v_5, \ldots, v_{k-1}, v_1 \). Now \( v_5, v_k \not\in E(G) \) since otherwise the cycle \( v_1, v_2, v_3, v_k, v_5, \ldots, v_{k-1}, v_4, v_1 \) contradicts the definition of \( C \); similarly \( v_2, v_5 \not\in E(G) \). Since \( \deg v_1 = 5 \), we have \( v_1 v_5 \not\in E(G) \); since \( G \) has at most one vertex of degree 5, it follows that \( v_3, v_5 \not\in E(G) \). Since
deg \( v_4 = 4 \), it follows that \( N(v_4) = \{v_1, v_3, v_5, v_{k-1}\} \); but \( \langle N(v_4) \rangle \) is connected so \( v_5v_{k-1} \in E(G) \). Thus, \( G \) contains the induced subgraph \( G_7 \) of Fig. 2.22.

If \( v_5 \) and \( v_{k-1} \) are not consecutive on \( C \), let \( v_{k-2} \) be the vertex on cycle \( C \) so that \( v_{k-2}, v_{k-1}, v_1 \) are consecutive on \( C \). As in the previous paragraph, \( v_{k-2} \) is not adjacent to vertices \( v_2 \) and \( v_k \) since \( C \) is a cycle of maximum length and \( v_{k-2} \) is adjacent to none of the vertices \( v_1, v_3, v_4, v_{k-1} \) since each of these vertices already has maximum possible degree. However \( v_{k-2} \) is adjacent to \( v_5 \) since \( \langle N(v_{k-1}) \rangle \) is connected. Thus \( G \) contains the induced subgraph \( G_8 \) of Fig. 2.22.

If \( v_5 \) and \( v_{k-2} \) are not consecutive on \( C \), then a vertex \( v_6 \) can be obtained which precedes \( v_5 \) on the \( v_{k-2} - v_5 \) path determined by \( C \) which does not contain \( v_{k-1} \). Proceeding as above, \( G \) contains the induced subgraph \( G_9 \) of Fig. 2.22. This process must terminate. Thus \( G \) necessarily contains an induced subgraph \( G_k \) for some \( k \geq 1 \), but does not contain \( G_{k+1} \) as an induced subgraph. We will show that \( G = G_k \), that is, \( G = G_k \).

Suppose \( G_k \neq G \). Since \( G \) is connected, there is a vertex \( u \) of \( G \) such that \( u \notin V(G_k) \) and \( u \) is adjacent to at least one vertex of \( G_k \). Now \( u \) is not adjacent to \( v_1 \) since \( \deg v_1 = 5 \). Since \( v_1 \) is the only vertex
of $G$ which has degree 5, the vertex $u$ is not adjacent to $v_3, v_4, \ldots, v_{i-1}, v_{i+2}, \ldots, v_{k-1}$ (where $i = \left\lfloor \frac{k+3}{2} \right\rfloor$) because each of these vertices has degree 4. Thus $u$ is adjacent to at least one of $v_2, v_i, v_{i+1}$, and $v_k$.

We claim $u$ cannot be adjacent to $v_k$ since if $uv_k \in E(G)$, then $u, v_3 \in N(v_k)$. Since $\langle N(v_k) \rangle$ is connected, there is a $u-v_1$ path in $\langle N(v_k) \rangle$. The vertex of $P$ which precedes $v_1$ must also be adjacent to $v_k$ since $P$ is in $\langle N(v_k) \rangle$. This vertex must be in $G_k$ since the degree of $v_1$ in $G_k$ is 5 ($= \Delta(G)$); the only vertex of $G_k$ adjacent to both $v_1$ and $v_k$ is $v_3$, which cannot be adjacent to $u$ since $\deg_{G_k} v_3 = 4$. Thus such a vertex does not exist; that is, $u$ must be adjacent to $v_1$. This is also impossible since $\deg_{G_k} v_1 = 5 = \Delta(G)$. Hence $u$ cannot be adjacent to $v_k$. Similarly, $u$ cannot be adjacent to $v_2$.

Thus $u$ is adjacent to at most $v_1$ and $v_{i+1}$, where $i = \left\lfloor \frac{k+3}{2} \right\rfloor$. Now $u$ cannot be adjacent to only $v_i$. Otherwise $\langle N(v_i) \rangle$ is not connected. Similarly, $u$ cannot be adjacent to only $v_{i+1}$. Therefore, $u$ is adjacent to exactly two vertices of $G_k$, namely $v_1$ and $v_{i+1}$. But then $\langle V(G_k) \cup \{u\} \rangle = G_{k+1}$, which is impossible since, by assumption, among the graphs $G_i$, the graph $G_k$ is the induced subgraph of $G$ having

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greatest order. Thus such a vertex u does not exist, so \( G = G_k \).

**Case 2. Assume** \( \deg v_3 = 5 \).

Since \( \deg v_3 = 5 \), there are vertices \( v_1 \) and \( v_{k-1} \) of \( G \) such that \( N(v_3) = \{v_1, v_2, v_4, v_{k-1}, v_k\} \). If neither \( v_1 \) nor \( v_{k-1} \) is on \( C \), then neither of \( v_1 \) and \( v_{k-1} \) is adjacent to \( v_4 \) or \( v_2 \) since \( C \) is a cycle of maximum length. But then \( \langle N(v_3) \rangle \) is disconnected since \( \langle N(v_3) \rangle \) contains no \( v_1 \)-\( v_k \) path. Thus, at least one of \( v_1 \) and \( v_{k-1} \) is on \( C \).

Suppose exactly one of \( v_1 \) and \( v_{k-1} \) is on \( C \).

Without loss of generality, say \( v_1 \in V(C) \) but \( v_{k-1} \notin V(C) \). Then \( v_2, v_{k-1}, v_4, v_{k-1} \notin E(C) \) since \( C \) has maximum length. Since \( \langle N(v_3) \rangle \) is connected, at least one of \( v_{k-1} \) and \( v_k \) is adjacent to \( v_1 \); without loss of generality, say \( v_1 v_k \in E(G) \). Since \( \langle N(v_3) \rangle \) is connected, at least one of \( v_2 \) and \( v_4 \) is adjacent to \( v_1 \); without loss of generality, say \( v_1 v_2 \in E(G) \). But \( G \) has only one vertex of degree 5, namely \( v_3 \), so \( \deg v_1 \leq 4 \). Thus, \( v_1 \) and \( v_2 \) must be consecutive vertices on \( C \). Now \( v_2 v_4 \notin E(G) \), for otherwise \( v_1, v_k, v_3, v_2, v_4, \ldots, v_1 \) is a cycle of length greater than that of \( C \). Since \( N(v_3) \) contains \( v_4 \), which is not adjacent to \( v_2, v_{k-1} \), or \( v_k \), necessarily \( v_1 v_4 \in E(G) \).
But \( \deg v_1 \leq 4 \) so \( v_1 \) and \( v_4 \) must be consecutive vertices on \( C \). Thus \( C \) has length four. Now 

\[ v_{k-1}v_k \not\in E(G), \text{ for otherwise } v_1,v_4,v_3,v_{k-1},v_k,v_1 \text{ is a cycle of length five. Also } \deg v_1 \leq 4 \text{ so } v_1v_{k-1} \not\in E(G). \] 

Thus \( v_{k-1} \) is an isolated vertex in \( \langle N(v_3) \rangle \), which is impossible. Hence it is not possible for exactly one of \( v_1 \) and \( v_{k-1} \) to be on \( C \).

Since both \( v_1 \) and \( v_{k-1} \) are on \( C \), the cycle \( C \) determines two \( v_2-v_4 \) paths. Let \( P \) be the \( v_2-v_4 \) path not containing \( v_3 \). Without loss of generality, assume \( v_1 \) precedes \( v_{k-1} \) on \( P \). Since \( \langle N(v_3) \rangle \) is connected, at least one of \( v_1 \) and \( v_{k-1} \) is adjacent to \( v_k \). Without loss of generality, say \( v_1v_k \in E(G) \). Since \( \deg u \leq 4 \), we have \( v_1v_4 \not\in E(G) \).

Suppose \( v_1 \) and \( v_{k-1} \) are not consecutive on \( C \). On the \( v_1-v_{k-1} \) path in \( C \) which does not contain \( v_3 \), let \( w_1 \) be the first vertex following \( v_1 \). Then \( w_1v_3 \not\in E(G) \) since \( \deg v_3 = 5 \), and \( w_1v_k \not\in E(G) \) for otherwise 

\[ w_1,\ldots,v_{k-1},\ldots,v_4,v_3,v_2,\ldots,v_1,v_k,w_1 \] 

is a cycle of length greater than the length of \( C \). If \( v_1 \) and \( v_2 \) are not consecutive on \( C \), then let \( w_2 \) denote the first vertex following \( v_1 \) on the \( v_1-v_2 \) path in \( C \) which does not contain \( v_3 \). Now \( w_2 \) is not adjacent to \( v_3 \) since \( \deg v_3 = 5 \), and \( w_2 \) is not adjacent to \( v_k \) since other-
wise the existence of the cycle $v_1, \ldots, w_2, v_k, v_1$ contradicts the definition of $C$. But then $\langle N(v_1) \rangle$ contains no $w_2-v_k$ path, an impossibility. Thus $v_1$ and $v_2$ are consecutive vertices on $C$. Since $\langle N(v_1) \rangle$ is connected, necessarily $w_1v_2 \in E(G)$. But then $w_1, \ldots, v_{k-1}, v_3,v_k,v_1,v_2,w_1$ is a cycle whose length is greater than that of $C$, which is impossible. Thus $v_1$ and $v_{k-1}$ must be consecutive on $C$.

If $v_{k-1}v_k \notin E(G)$, then $v_1,v_2,\ldots,v_{k-1},v_k,v_1$ is a cycle whose length exceeds that of $C$, which is impossible. Also, $v_2v_{k-1} \notin E(G)$, for otherwise $v_1,v_k,v_3,\ldots,v_{k-1},v_2,v_1$ is a cycle of length greater than the length of $C$. Similarly, $v_2v_4 \notin E(G)$, for otherwise $v_2,\ldots,v_1,v_k,v_3,v_{k-1},\ldots,v_4,v_2$ is a cycle whose length is greater than that of $C$. But $\langle N(v_3) \rangle$ is connected, so $v_4v_{k-1}$ and $v_1v_2$ are edges of $G$.

Since $\deg v_1 \leq 4$, the vertices $v_1$ and $v_2$ are consecutive on $C$.

Therefore, $G$ contains the induced subgraph $G_6$ of Fig. 2.22. (This can be accomplished by interchanging $v_3$ and $v_4$ with $v_1$ and $v_{k-1}$, respectively.) Proceeding as in Case 1, we see that $G = G_p$. ■
Locally Connected Graphs with Large Degrees

In the previous two sections we have encountered only locally connected graphs whose vertices were of small degree. All locally connected graphs considered thus far have had at most one vertex of degree 5 with all other vertices of degree 4 or less. There are, however, large classes of locally connected graphs whose vertices have arbitrarily large degree. It can be shown, for example, that maximal planar graphs are locally connected \[21,22\]; this result will be discussed further in Chapter III. In \[9\], Chvátal noted the following result.

Proposition 2.C. The square of any graph is locally connected.

There are classes of graphs which are readily determined to be not locally connected. Since the neighborhood of a cut-vertex of a graph is necessarily disconnected, no locally connected graph can contain a cut-vertex; that is, each component of a locally connected graph is a block. This is not a sufficient condition for a graph to be locally connected, since
a cycle of length at least 4 is a block but is not locally connected.

Presently it suffices to note that the class of locally connected graphs is certainly not trivial.
CHAPTER III

GRAPHS WITH PRESCRIBED LOCAL CONNECTIVITIES

Introduction

There are many properties associated with a graph which are variations and generalizations of the property of being connected. In this chapter we will study the local analogues of several of these variations.

A graph is said to be \textit{locally n-connected} if the neighborhood of each vertex is n-connected; i.e., if the neighborhood of each vertex has the property that the removal of fewer than n vertices results in neither a disconnected nor a trivial graph. To show that a graph is locally n-connected using this definition is usually rather difficult since in each neighborhood we must consider the subgraph obtained when each k-element subset of vertices, $k < n$, is removed. Most techniques used to show a graph is n-connected are variations of the following classical result of Menger [17].

\textbf{Theorem 3.A.} If $u$ and $v$ are distinct non-adjacent vertices of a graph $G$, then the maximum number of
disjoint u-v paths in G equals the minimum number of vertices of G which separate u and v.

One particularly useful extension of Menger's Theorem is the following characterization of n-connected graphs due to Whitney [27].

**Theorem 3.B.** A graph G is n-connected if and only if for each pair u,v of distinct vertices there are at least n disjoint u-v paths in G.

We will use Theorem 3.B to prove the following result.

**Theorem 3.1.** Let n > 2 be a positive integer and let G be a graph for which δ(G) > n. Then G^n is locally (n-1)-connected.

**Proof.** Let v ∈ V(G) and let G_k = \{u ∈ V(G) | d_G(u,v) = k\}. Since δ(G) ≥ n, we must have |V(G_1)| ≥ n; let v_1,...,v_n be distinct vertices in G_1. To show G^n is locally (n-1)-connected, we use Theorem 3.B to verify that the neighborhood of v in G^n is (n-1)-connected; i.e., we show that between any two vertices adjacent to v in G^n, there are at least n-1 disjoint paths. Let \langle N(v) \rangle denote the neighborhood of v in G^n, so \langle N(v) \rangle = \{u ∈ V(G) | u ∈ V(G_j) for some j = 1,2,...,n\}. Now if u ∈ V(G_j), then d_G(u,v) = j.
so in $G^n$, $u$ is adjacent to all vertices in $G_1, \ldots, G_{n-j}$.
In particular, each vertex in $G_1$ is adjacent in $G^n$ to all vertices in $G_1, \ldots, G_{n-1}$.

Let $u$ and $w$ be two distinct vertices in $N(v)$. If neither of $u$ and $w$ is in $G_n$, then $u, v_j, w$ is a $u$-$w$ path in $G^n$ for each $v_j \neq u, w$ in $G_1$. If not both $u$ and $w$ are in $G_1$, then there are at least $n-1$ such $u$-$w$ paths. If both $u$ and $w$ are in $G_1$, then there are only $n-2$ such paths, but the edge $uv$ provides an additional path.

If exactly one of $u$ and $w$ is in $G_n$, we may assume, without loss of generality, that $u \in V(G_n)$ and $w \not\in V(G_n)$. Furthermore, we may let $v_1 = u_1 u_2, \ldots, u_{n-1}, u_n = u$ be a $v_1$-$u$ path in $G$, where $u_j \in V(G_j)$. Then in $G^n$ there are $(n-2)$ $u$-$w$ paths of the form $u, u_j, v_j, w$, where $u_j \in V(G_j)$ and $2 \leq j \leq n-1$. Another $u$-$w$ path is $u, v_1, w$, and all $n-1$ of these paths are disjoint unless $w = u_j$ for some $j$. In this case, $u, u_j, v_j, w$ is not a path, but replacing it by the edge $uw$ provides $n-1$ distinct paths.

It remains to consider the situation when both $u$ and $w$ are in $G_n$. As above, we may assume, without loss of generality, that $v_1 = u_1 u_2, \ldots, u_{n-1}, u_n = u$ is a $v_1$-$u$ path in $G$, and $v_n = w_1, w_2, \ldots, w_{n-1}, w_n = w$ is a $v_n$-$w$ path in $G$, where $u_j, w_j \in V(G_j)$. If $u_j \neq w_j$ for
each \( j = 1, 2, \ldots, n-1 \), then the \( n-2 \) paths in \( G^n \) of the form \( u, u_j, v_j, w_j, w \), for \( j = 2, 3, \ldots, n-1 \) are disjoint u-w paths. Also, the path \( u, u_1, w_1, w \) is a u-w path in \( G^n \) disjoint from the other \( n-2 \) paths. Thus, there are at least \( n-1 \) disjoint u-w paths in \( G^n \).

It remains to consider the situation where the \( v_1-u \) path and the \( v_n-w \) path intersect. Let \( u_k = w_k \) be the vertex of largest index in which the two paths intersect. Without loss of generality, we may assume the two paths are identical prior to \( u_k = w_k \); that is, \( u_j = w_j \) for \( j = 1, 2, \ldots, k \). Then \( u, u_i, w_i, w \) is a u-w path of length two, for \( i = 1, 2, \ldots, k \). Since 
\[
d(u_{k+j}, w_{n-j}) = n - k < n,
\]
we must have \( u_{k+j} \) adjacent to \( w_{n-j} \) in \( G^n \). Thus, \( G^n \) contains \( n - k - 1 \) paths of the form \( u, u_{k+j}, w_{n-j}, w \), where \( j = 1, 2, \ldots, n-k-1 \).

Therefore, \( G^n \) contains at least \( n-1 \) u-w paths; furthermore, all of these paths are disjoint.

By Theorem 3.1, a large class of locally \((n-1)\)-connected graphs can be obtained. In particular, if \( G \) is a graph without isolated vertices or end-vertices, then \( G^2 \) must be locally connected. As indicated in Proposition 2.C, even if \( G \) has end-vertices, \( G^2 \) is locally connected.

Thus far we have discussed how the removal of a collection of vertices affects the graph. We now
consider an analogous situation when a collection of edges is deleted. A graph $G$ is said to be \textit{n-edge connected} if the removal of fewer than $n$ edges of $G$ results in neither a disconnected nor a trivial graph. A graph $G$ is called \textit{locally n-edge connected} if the neighborhood of each of its vertices is $n$-edge connected. The following results [12,13] are edge analogues of Theorems 3.A and 3.B.

\textbf{Theorem 3.C.} If $u$ and $v$ are distinct vertices of a graph $G$, then the maximum number of edge-disjoint $u$-$v$ paths in $G$ equals the minimum number of edges of $G$ which separate $u$ and $v$.

\textbf{Theorem 3.D.} A graph $G$ is $n$-edge connected if and only if for every two distinct vertices $u$ and $v$ of $G$, there exist at least $n$ edge-disjoint $u$-$v$ paths in $G$.

Theorems 3.B and 3.D will be used repeatedly in the remainder of this chapter to show that a graph is $n$-connected, $n$-edge connected, or has either of the corresponding local properties.
Some Remarks on Hamiltonian Graphs

One of the most outstanding unsolved problems in graph theory is to obtain a reasonable characterization of hamiltonian graphs. In Chapter II we obtained a class of non-hamiltonian graphs in which each graph was both connected and locally connected, namely the graphs $G_k$ in Fig. 2.22. Each of these graphs has minimum degree 2. In fact, each contains two vertices which are mutually adjacent to two vertices of degree 2. This may suggest that if $G$ is connected, locally connected, and has minimum degree sufficiently large, then $G$ must be hamiltonian. However, this is not so, as the following example illustrates.

Let $n (\geq 5)$ be a positive integer and let $H$ be the graph in Fig. 3.1, where $\text{deg} \ v = \lfloor n/2 \rfloor$ and each vertex whose distance from $v$ is two, has $n-1$ end-vertices adjacent to it. Let $G = H^2$. Thus $G$ is connected and, by Proposition 2.C, the graph $G$ is locally connected. Also, $\delta(G) \geq n$; i.e., the minimum degree of $G$ is arbitrarily large. But $G$ is not hamiltonian, for suppose otherwise. Let $C$ be a hamiltonian cycle in $G$ and let $u$ be adjacent to $v$ in $H$ and $w \neq v$ be adjacent to $u$.
in \( H \). Let \( w_1, \ldots, w_{n-1} \) denote the end-vertices adjacent to \( w \) and let \( v_1 \) and \( v_2 \) denote the immediate predecessor and successor of \( w \) in \( C \), respectively. If \( v_1 \neq v \), then \( v_1 = u \) or \( v_1 = w_j \) for some \( j \) (\( 1 \leq j \leq n-1 \)). If \( v_1 = u \), then \( v_2 = v \) since \( C \) is hamiltonian. If \( v_1 = w_j \) for some \( j \), then the first vertex preceding \( w \) in \( C \) which is not adjacent to \( w \) in \( H \) must be \( u \). Thus \( v_2 = v \). Therefore, in any case, the vertex \( v \) must be consecutive on \( C \) with \( w \). This is so for each of the \( \lfloor n/2 \rfloor \) vertices whose distance in \( H \) from \( v \) is two. Since \( n \geq 5 \), we must have at least 3 such vertices. But \( v \) can appear only once on \( C \) so there can be only two vertices consecutive with \( v \) on \( C \). Thus, this situation is impossible so \( G \) is not hamiltonian.

The previous example shows that a connected, locally connected graph of arbitrarily large minimum degree need not be hamiltonian. Now the cube of every

![Diagram](image-url)
connected graph of order at least three must be hamiltonian [20]. By Theorem 3.1, the cube of a graph whose minimum degree is at least 3 is locally 2-connected. It can also be observed that every connected, locally 2-connected graph encountered thus far has been hamiltonian. Thus, we may conjecture that every connected, locally 2-connected graph is hamiltonian. However, we can also show this conjecture is false.

As mentioned previously, Skupień has studied graphs which are locally hamiltonian. In [21], he proved the following.

**Theorem 3.E.** If a graph G of order \( p \geq 3 \) is maximal planar, then G is locally hamiltonian.

Since a hamiltonian graph is a priori 2-connected, by Theorem 3.E every maximal planar graph of order at least 3 is locally 2-connected. Thus the question as to whether every connected, locally 2-connected graph is hamiltonian can be answered in the negative if we can exhibit a graph which is connected and maximal planar but not hamiltonian. Such a graph, due to C. N. Reynolds, may be found in [26]. Thus, we have an example of a connected, 2-connected graph which is not hamiltonian. However, we are left with the following.
Open Question: If there a positive integer $n \geq 3$ such that if $G$ is an arbitrary connected, locally $n$-connected graph, then $G$ must be hamiltonian?
Connectivity and Local Connectivity

One particularly useful variation of the property of being connected is the connectivity of a graph. The connectivity of a graph $G$, denoted by $\kappa(G)$, is the maximum $n$ for which $G$ is $n$-connected. Thus, the connectivity of $G$ is the minimum number of vertices whose removal from $G$ results in a disconnected or trivial graph. The local connectivity of a graph $G$, denoted by $\kappa_0(G)$, is the maximum $n$ for which $G$ is locally $n$-connected. Note that $K_2$ is the only locally connected graph with local connectivity 0. The following result is due to Chartrand and Pippert [7].

**Theorem 3.5.** If a graph $G$ is locally $n$-connected, $n \geq 1$, then every component of $G$ other than $K_2$ is $(n+1)$-connected.

Thus, if $\kappa_0(G) = n$, then by Theorem 3.5 we must have $\kappa(G_1) \geq n+1$, where $G_1$ is any component of $G$. In particular, if $G$ is connected, then $\kappa(G) > \kappa_0(G)$. We will prove the validity of a converse of this result, namely, if $k$ and $m$ are integers with $0 \leq k < m$, then there is a graph $G$ for which $\kappa_0(G) = k$ and $\kappa(G) = m$. 

65
Furthermore, the minimum order of such a graph will be shown to be \(2m-k\).

**Lemma 3.2.** If a graph \(H\) is connected and locally \(k\)-connected, then \(G = H + v\) is locally \((k+1)\)-connected.

**Proof.** Since \(v\) is adjacent to each vertex of \(H\), the connectivity of \(<N(v)>\) is equal to \(\kappa(H)\), which we know is greater than \(\ell \kappa(H) \geq k\). Thus \(<N(v)>\) is \((k+1)\)-connected. Let \(u \in V(H)\). Since \(v\) is adjacent to each vertex of \(<N(u)>\), to disconnect \(<N(u)>\) in \(G\), it is necessary to remove \(v\) and those vertices which disconnect \(<N(u)>\) in \(H\). Hence \(<N(u)>\) in \(G\) has connectivity at least \(k+1\), so \(G\) is locally \((k+1)\)-connected. ■

With the assistance of the two previous results, we can obtain a graph with prescribed connectivity and local connectivity.

**Theorem 3.3.** Let \(k\) and \(m\) be integers with \(0 < k < m\). Let \(G\) denote the complete \((k+2)\)-partite graph \(K(1,1,\ldots,1,m-k,m-k)\). Then \(\kappa(G) = m\) and \(\ell \kappa(G) = k\).

**Proof.** Now \(K(m-k,m-k)\) has connectivity \(m-k\) and local connectivity zero. By \(k\) applications of Lemma 3.2, \(G\) is locally \(k\)-connected. Also, \(G\) is \(m\)-connected since to disconnect \(G\), all \(k\) of the singleton vertices must be removed along with a set of \(m-k\) vertices whose
removal disconnects $K(m-k, m-k)$. Thus $\kappa(G) \geq m$ and $L\kappa(G) \geq k$. But the removal of the indicated vertices does disconnect the required graphs, so $\kappa(G) = m$ and $L\kappa(G) = k$.

**Theorem 3.4.** Let $k$ and $m$ be integers with $0 \leq k < m$. If $G$ is a graph for which $\kappa(G) = m$ and $L\kappa(G) = k$, then $p(G) \geq 2m-k$.

**Proof.** Assume there is a graph $G$ for which $L\kappa(G) = k$, $\kappa(G) = m$, but $p(G) < 2m-k$. Since $L\kappa(G) = k$, there is a vertex $v$ of $G$ for which $\kappa(\langle N(v) \rangle) = k$. Let $T$ denote a set of $k$ vertices in $N(v)$ whose removal from $\langle N(v) \rangle$ results in a disconnected or trivial graph. Now $k < m$ and $\kappa(G) = m$, so $|N(v)| \geq m$; for otherwise, the removal of $|N(v)|$ vertices would isolate vertex $v$. Let $S = V(G) - (N(v) \cup \{v\})$. Then $|S \cup \{v\}| < 2m-k-m = m-k$, so $|S \cup \{v\} \cup T| < m-k+k = m$. But the removal of $S \cup \{v\} \cup T$ from $G$ results in a disconnected or trivial subgraph. Thus $\kappa(G) < m$, which contradicts the hypothesis.

From Theorems 3.4 and 3.5, we see that the complete $(k+2)$-partite graph $G = K(1,1,\ldots,1,m-k,m-k)$ is a graph of minimum order for which $\kappa(G) = m$ and $L\kappa(G) = k$, where $k$ and $m$ are arbitrary integers.
satisfying $0 \leq k < m$. For the special case when $m = k+1$, we obtain the graph $K_{k+2}$, for which we know that $\kappa(K_{k+2}) = k+1$ and $\ell\kappa(K_{k+2}) = k$. 

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Local Connectivity and Local Edge Connectivity

We now turn our attention to the edge analogue of local connectivity. The edge-connectivity of a graph $G$, denoted $\kappa_1(G)$, is the minimum number of edges whose removal from $G$ results in a disconnected or trivial graph. Thus a graph for which $\kappa_1(G) \geq n$ is $n$-edge connected; that is, the removal of fewer than $n$ edges results in neither a disconnected nor a trivial graph. The local edge connectivity of a graph $G$, denoted by $\ell \kappa_1(G)$, is the maximum $n$ for which $G$ is locally $n$-edge connected; that is, there is at least one vertex of $G$ whose neighborhood has edge-connectivity $n$ and no neighborhood of $G$ has edge-connectivity less than $n$.

It is well known [27] that, for any graph $H$, $\kappa(H) \leq \kappa_1(H)$. We now observe that a similar result is valid for local connectivities.

**Proposition 3.5.** If $G$ is a graph, then $\ell \kappa(G) \leq \ell \kappa_1(G)$.

**Proof.** Let $v \in V(G)$ be such that $\kappa_1(\langle N(v) \rangle) = \ell \kappa_1(G)$. Since $\kappa(H) \leq \kappa_1(H)$, in particular $\kappa(\langle N(v) \rangle) \leq \kappa_1(\langle N(v) \rangle)$. But $\ell \kappa(G) \leq \kappa(\langle N(v) \rangle)$, so $\ell \kappa(G) \leq \ell \kappa_1(G)$. 

69
By Proposition 3.5, it is meaningful to ask whether or not there is a graph with prescribed local connectivities. In particular, if $0 \leq k \leq m$, is there a graph $G$ such that $\kappa_1(G) = k$ and $\kappa_1(G) = m$? Consideration of the case where $k = 0$ and $m > 0$ shows that the answer is not affirmative in general, since for $\kappa_1(G) = 0$, there must exist a vertex whose neighborhood is disconnected or trivial. But then $\kappa_1(G) = 0$. The following result shows that by restricting $k$ to be positive, a graph with prescribed local connectivities can be found.

Theorem 3.6. Let $k$ and $m$ be integers such that $1 \leq k \leq m$. For $k = m$, let $G = K_{k+2}$; when $k < m$, let $G = 2K_{m-k+1} + K_{k+1}$. Then $\kappa(G) = k$ and $\kappa_1(G) = m$.

Proof. For each $v \in V(K_{k+2})$, we have $\langle N(v) \rangle = K_{k+1}$, so $\kappa\left(\langle N(v) \rangle \right) = \kappa_1\left(\langle N(v) \rangle \right) = k$. Thus, for $k = m$, $\kappa(G) = k$ and $\kappa_1(G) = k = m$.

Let $k < m$ and consider $G = 2K_{m-k+1} + K_{k+1}$. Each neighborhood in $G$ is either $K_{m+1}$ or $2K_{m-k+1} + K_{k}$. Since $\kappa(K_{m+1}) = m$ and $\kappa(2K_{m-k+1} + K_{k}) = k$, we have $\kappa(G) = k$. Also $\kappa_1(K_{m+1}) = m$ and $\kappa_1(2K_{m-k+1} + K_{k}) = \min(m, k(m-k+1)) = m$, so $\kappa_1(G) = m$. 

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It remains to determine a graph \( G \) of smallest order for which \( \ell \kappa(G) = k \) and \( \ell \kappa_1(G) = m \), where \( k \) is any positive integer no greater than integer \( m \). We use the following result of Chartrand and Harary [4] to show that the graphs in Theorem 3.6 do have minimum order.

**Theorem 3.6.** For any integers \( b \) and \( c \), \( 0 < b \leq c \), there exists a graph \( G \) with \( \kappa(G) = b \) and \( \kappa_1(G) = c \). Furthermore the smallest such graph is

1. \( K_{b+1} \) when \( b = c \),
2. \( 2K_{c-b+1} + K_b \) when \( b < c \).

**Theorem 3.7.** Let \( k \) and \( m \) be integers such that \( 1 \leq k \leq m \) and let \( G \) be a connected graph for which \( \ell \kappa(G) = k \) and \( \ell \kappa_1(G) = m \). If \( k = m \), then \( p(G) \geq m+2 \); otherwise, \( p(G) \geq 2m-k+3 \).

**Proof.** For \( k = m \), \( \delta(G) \geq \kappa_1(G) > \ell \kappa_1(G) = m \), since \( G \) is connected. Thus \( \delta(G) \geq m+1 \), so \( p(G) \geq m+2 \).

For \( 1 \leq k < m \), let \( G \) be a graph such that \( \ell \kappa(G) = k \) and \( \ell \kappa_1(G) = m \). Let \( v \in V(G) \) be such that \( \kappa(\langle N(v) \rangle) = k \). But \( \kappa_1(\langle N(v) \rangle) \geq m \), so \( \kappa_1(\langle N(v) \rangle) > k \). By Theorem 3.6, we must have \( p(\langle N(v) \rangle) \geq 2m-k+2 \). Therefore, \( p(G) \geq 2m-k+3 \). ■

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Edge Connectivity and Local Edge Connectivity

In the two previous sections we have considered the relationship between \( \kappa(G) \) and \( \ell\kappa(G) \) and between \( \ell\kappa(G) \) and \( \ell\kappa_1(G) \). The edge connectivity has been defined in the previous section; we now consider its relationship to the local edge connectivity.

**Proposition 3.8.** Let \( n \) be a non-negative integer and let \( G \) be a connected graph which is locally \( n \)-edge connected. Then \( G \) is \((n+1)\)-edge connected.

**Proof.** If \( G = K_2 \), then \( n = 0 \) since each neighborhood in \( K_2 \) is the trivial graph. Since \( K_2 \) is 1-edge connected, the result holds for \( G = K_2 \).

Suppose \( G(\neq K_2) \) is connected and locally \( n \)-edge connected, but not \((n+1)\)-edge connected. Then there is a set \( S \) of \( k(\leq n) \) edges whose removal from \( G \) disconnects the graph. Furthermore, we may take \( k = \kappa_1(G) \). Since the removal of fewer than \( k \) edges does not disconnect \( G \), it follows that \( G-S \) consists of two components; denote them by \( G_1 \) and \( G_2 \). Let \( u \) and \( v \) be vertices in \( G_1 \) and \( G_2 \), respectively, such \( uv \in S \). Since \( G \neq K_2 \), at least one of \( G_1 \) and \( G_2 \) has order at least two. Without loss of generality, let
\[ |V(G_1)| \geq 2. \] Then \( N(u) \) contains \( v \in V(G_2) \) and at least one vertex of \( G_1 \). Since there are at most \( k-1 \) edges in \( \langle N(u) \rangle \) joining vertices of \( G_1 \) with vertices of \( G_2 \), \( \langle N(u) \rangle \) is not \( k \)-edge connected, where \( k \leq n \). This contradicts the hypothesis that \( G \) is locally \( n \)-edge connected. \[ \Box \]

An immediate result of the preceding proposition is the following.

**Corollary 3.9.** If \( G \) is connected, then \( \kappa_1(G) < \kappa_1(G) \).

We now determine the edge connectivity and the local edge connectivity of the complete \((k+2)\)-partite graph \( K(l, l, \ldots, l, m-k, m-k) \), which we considered earlier. The following two lemmas will be useful.

**Lemma 3.10.** If \( H \) is connected and locally \( k \)-edge connected, then \( G = H+v \) is locally \((k+1)\)-edge connected.

**Proof.** Since \( v \) is adjacent to each vertex of \( H \),
\[ \kappa_1(\langle N(v) \rangle) = \kappa_1(H) > \kappa_1(H) \geq k. \] Thus \( \langle N(v) \rangle \) is \((k+1)\)-edge connected. Let \( u \in V(H) \) and let \( S \) be a separating set of edges in the subgraph \( \langle N(u) \rangle \) of \( G \). The set \( S \) must contain at least those edges of \( H \) which disconnect \( \langle N(u) \rangle \) in \( H \), so \( |S| \geq k \). But \( v \) is adjacent
to each vertex of $H$, and $S$ must contain at least one edge incident with $v$. (In fact, $S$ must contain at least those edges incident with the vertices in the smallest component of $(N(u))-S$ in $H$.) Thus $|S| \geq k+1$. Hence, $(N(u))$ has edge-connectivity at least $k+1$, so $G$ is locally $(k+1)$-edge connected. ■

The next lemma is more number-theoretic in nature.

**Lemma 3.11.** If $x, y,$ and $A$ are non-negative integers, with $x^2 + y^2 \neq 0$ and $A > \max \{x, y\}$, then $x(A-y) + y(A-x) \geq A$.

**Proof.** Since $A > \max \{x, y\}$, it follows that $A \geq x+1$. Thus

$$A(x+y-1) \geq (x+1)(x+y-1)$$

$$= x^2 + xy + y - 1$$

$$\geq x^2 + xy,$$ if $y \geq 1$.

Similarly, $A(x+y-1) \geq y^2 + xy$ if $x \geq 1$. At least one of $x^2 + xy$ and $y^2 + xy$ is no less than $2xy$. Thus, $A(x+y-1) \geq 2xy$, so $Ax + Ay = xy - xy \geq A$; that is, $x(A-y) + y(A-x) \geq A$. If exactly one of $x$ and $y$ is zero, the result is trivial. ■

**Proposition 3.12.** Let $k$ and $m$ be integers, with $0 \leq k < m$. If $G$ is the complete $(k+2)$-partite graph $K(1,1,\ldots,1,m-k,m-k)$, then $\kappa_1(G) = m$ and $\ell K_1(G) = k$. 
Proof. The complete bipartite graph $K(m-k, m-k)$ has edge connectivity $m-k$ and local edge connectivity zero. By $k$ applications of Lemma 3.10, $G$ is locally $k$-edge connected.

Let $V_1$ and $V_2$ denote the two partite sets of order $m-k$ and let $\{v_1, \ldots, v_k\}$ denote the collection of vertices in the singleton sets. For $u \in V_1$, the subgraph $\langle N(u) \rangle$ can be disconnected by the removal of $k$ edges, since a vertex in $V_2$ has degree $k$ in $\langle N(u) \rangle$. Thus, $\kappa_1(G) = k$.

Since $\deg_G u = m$ and $\kappa_1(G) \leq \delta(G)$, we must have $\kappa_1(G) \leq m$. To show $\kappa_1(G) = m$, we assume $\kappa_1(G) < m$. Then there is a set $T$ of edges of $G$ whose removal from $G$ disconnects $G$, where $|T| < m$. Let $G_1$ denote one of the resulting components, and let $G_2 = G - V(G_1)$. Since $\delta(G) = m$ and $\kappa_1(G) < m$, we must have $|V(G_1)| \geq 2$. Now $k \neq 0$, for otherwise $\kappa_1(G) = m$.

Let $G_1$ contain $g_1$ vertices from $V_1$, $g_2$ vertices from $V_2$, and (relabel, if necessary) $v_1, \ldots, v_j$. Then $G_2$ contains $m-k-g_1$ vertices from $G_1$, $m-k-g_2$ vertices from $V_2$, and $v_{j+1}, \ldots, v_k$. If $0 < j < k$, then $v_1 \in V(G_1)$ and $v_k \in V(G_2)$, so $v_1$ is adjacent to $2m-k-j-g_1-g_2$ vertices in $G_2$ and $v_k$ is adjacent to $g_1 + g_2 + j$ vertices in $G_1$. Thus, there are
2m-k-l (≥ m) edges incident with v₁ or vₖ which are in T. This is impossible since |T| < m. Thus, all of v₁, ..., vₖ are in G₁ or in G₂; say \{v₁, ..., vₖ\} ⊆ V(G₂). Then

|T| = k(g₁ + g₂) + g₁(m-k-g₂) + g₂(m-k-g₁)
    = g₁(m-g₂) + g₂(m-g₁).

As was noted above, k ≠ 0; so necessarily m > max(g₁, g₂). Also, g₁² + g₂² ≠ 0, so by Lemma 3.11, g₁(m-g₂) + g₂(m-g₁) ≥ m. Thus, |T| ≥ m, which contradicts the assumption that κ₁(G) < m. Therefore, κ₁(G) = m.

The previous proposition exhibits the existence of a graph with edge connectivity m, local edge connectivity k, and order 2m-k. If the present situation were analogous to the result above where the connectivity and local connectivity were compared, it would follow that if G is a graph for which κ₁(G) = m and rank(G) = k, then p(G) ≥ 2m-k. However, such a result does not exist since the graph in the following proposition has the required connectivities but has order less than 2m-k.

**Proposition 3.13.** Let k and m be integers such that 0 ≤ k < m/2 - 1. There exists a graph G with κ₁(G) = m, rank₁(G) = k and p(G) ≤ 2m-k. If m is even,
then \( p(G) < 2m-k \).

**Proof.** Since \( k < m/2 - 1 \), we must have \( k \leq \lfloor m/2 \rfloor - 1 \).

Let \( G' = (G_1 \cup G_2) + (G_3 \cup \{v\}) \), where \( G_1 = K_{\lfloor m/2 \rfloor} \) and \( G_2 = G_3 = K_{\lfloor m/2 \rfloor} \). Select \( k \) vertices from each of \( G_1 \) and \( G_2 \), say \( x_1, \ldots, x_k \in V(G_1) \) and \( y_1, \ldots, y_k \in V(G_2) \).

Let the graph \( G \) be defined by \( V(G) = V(G') \) and \( E(G) = E(G') \cup T \), where \( T = \{x_iy_i \mid 1 \leq i \leq k\} \). Since \( \deg v = m \), we must have \( \kappa_1(G) \leq m \). Since \( \kappa_1(G) \geq \kappa_1(G-T) = \kappa_1(G') \), to show \( \kappa_1(G) = m \), it suffices to show \( \kappa_1(G') \geq m \).

To show that \( G' \) is \( m \)-edge connected, we will use Theorem 3.D. Thus we must show that for each pair of distinct vertices of \( G' \), there are at least \( m \) edge-disjoint paths joining the two vertices.

Let \( u_i, w_i \in V(G_i) \). There are \( \lfloor m/2 \rfloor - 1 \) edge-disjoint \( v-u_i \) paths of the form \( v, u_j, u_i \) (where \( u_j \in V(G_1) \) for \( j \geq 2 \)), the edge \( vu_1 \), and \( \lfloor m/2 \rfloor \) edge-disjoint \( v-u_i \) paths of length three, where the interior vertices are from \( G_2 \) and \( G_3 \), respectively.

Thus, there are \( \lfloor m/2 \rfloor - 1 + 1 + \lfloor m/2 \rfloor = m \) edge-disjoint \( v-u_i \) paths in \( G \). Similarly, there are \( m \) edge-disjoint \( v-u_2 \) paths in \( G \). There are also \( m \) edge-disjoint \( v-u_3 \) paths (of length two) in \( G \), namely \( \lfloor m/2 \rfloor \) through \( G_1 \) and \( \lfloor m/2 \rfloor \) through \( G_2 \). Within \( G_1 \), there
are \([m/2] - 1\) edge-disjoint \(u_1-w_1\) paths; there are \([m/2]\) edge-disjoint \(u_1-w_1\) paths of length two (with interior vertex in \(G_3\)), and the path \(u_1,v,w_1\). Thus, there are \(m\) edge-disjoint \(u_1-w_1\) paths. Similarly there are \([m/2] - 1 + [m/2] + 1 \geq m\) edge-disjoint \(u_2-w_2\) paths, and there are \([m/2] - 1 + [m/2] + [m/2] \geq m\) edge-disjoint \(u_3-w_3\) paths. There are \([m/2] - 1\) edge-disjoint \(u_1-u_3\) paths of length two (with the interior vertex in \(G_3\)); the edge \(u_1u_3\); the path \(u_1,v_1,u_2,u_3\); and \([m/2] - 1\) edge-disjoint \(u_1-u_3\) paths of length three with interior vertex in \(G_1\) (but not \(u_1\)). Thus there are \(m\) edge-disjoint \(u_1-u_3\) paths in \(G\). Similarly, there are \(m\) edge-disjoint \(u_2-u_3\) paths, and \(m\) edge-disjoint \(u_1-u_2\) paths. Thus, \(\kappa_1(G') \geq m\), so \(\kappa_1(G) = m\).

To determine \(\ell\kappa_1(G)\), we first note that \(\kappa_1(\langle N(v)\rangle) = k\), so \(\ell\kappa_1(G) \leq k\). To show that \(\ell\kappa_1(G) = k\), we again use Theorem 3.D. Thus in each neighborhood of \(G\), we want to find at least \(k\) edge-disjoint paths between any two vertices in that neighborhood. That this indeed happens is readily determined since each neighborhood in \(G\) (other than \(\langle N(v)\rangle\)) is isomorphic to one of the graphs in Fig. 3.2. Each vertex of \(G_3\) has a neighborhood isomorphic to \(N_1\);
each of $u_1, \ldots, u_k, v_1, \ldots, v_k$ has a neighborhood isomorphic to $N_2$; the remaining vertices in $G_1$ and $G_2$ have neighborhoods isomorphic to $N_3$. By Theorem 3.6, we can show that $\kappa_1(N_1) \geq \lfloor m/2 \rfloor - 1 \geq k$, $\kappa_1(N_2) = \lfloor m/2 \rfloor > k$, and $\kappa_1(N_3) = \lfloor m/2 \rfloor > k$. Thus $\kappa_1(G) = k$.

It remains to show that $p(G) \leq 2m-k$. But $p(G) = 1 + \lfloor m/2 \rfloor + \lceil m/2 \rceil + \lfloor m/2 \rfloor = 1 + m + \lfloor m/2 \rfloor = 1 + 2m - \lfloor m/2 \rfloor$ and $k \leq \lfloor m/2 \rfloor - 1$, so $p(G) \leq 2m-k$. Since $k < m/2 - 1$, if $m$ is even, then $k < \lfloor m/2 \rfloor - 1$. In this case $p(G) < 2m-k$.

---

Figure 3.2

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Since \( k < m \), the largest value which \( k \) may assume is \( m-1 \). The complete graph \( K_{m+1} \) has the property that \( \ell \kappa_1(K_{m+1}) = m-1 (= k) \) and \( \kappa_1(K_{m+1}) = m \). Since any graph with edge-connectivity \( m \) must contain at least \( m+1 \) vertices, certainly \( K_{m+1} \) is a graph of minimum order with local edge-connectivity \( m-1 (= k) \) and edge-connectivity \( m \). Also \( p(K_{m+1}) = m+1 = 2m-(m-1) = 2m-k \), so for \( k = m-1 \) we have a graph \( G \) such that \( \ell \kappa_1(G) = k \), \( \kappa_1(G) = m \), and \( p(G) = 2m-k \). For \( k = 0 \), the graph \( G' \) in Proposition 3.13 has \( \ell \kappa_1(G') = 0 \), \( \kappa_1(G') = m \); but \( p(G') = \left\lfloor \frac{3m+2}{2} \right\rfloor \), which is less than \( 2m-k \) when \( k = 0 \). Thus it appears that the minimum order of a graph \( G \) with \( \ell \kappa_1(G) = k \) and \( \kappa_1(G) = m \) depends on the magnitude of \( k \) relative to \( m \). That this indeed is the case will be shown below.

**Theorem 3.14.** Let \( G \) be a graph such that \( \ell \kappa_1(G) = k \) and \( \kappa_1(G) = m \), where \( k \) and \( m \) are integers for which \( 0 \leq k < m \). Then \( p(G) \geq \min \left( \frac{3m+2}{2}, 2m-k \right) \).

**Proof.** We first consider the relative sizes of \( \left\lfloor \frac{3m+2}{2} \right\rfloor \) and \( 2m-k \). Now if \( k = m/2 - 1 \), then \( \frac{3m+2}{2} \) is integral, so \( \left\lfloor \frac{3m+2}{2} \right\rfloor = \left\lfloor \frac{4m+2}{2} - k - 1 \right\rfloor = 2m-k \). Also \( k \leq m/2 - 1 \) if and only if \( \left\lfloor \frac{3m+2}{2} \right\rfloor \leq 2m-k \). Therefore, the conclusion of this theorem can be restated as:
(1) \( p(G) \geq \left\lfloor \frac{3m+2}{2} \right\rfloor \) if \( k < m/2 - 1 \), and

(2) \( p(G) \geq 2m-k \) if \( k \geq m/2 - 1 \).

It is this formulation of the conclusion that we will justify.

Suppose there is a graph \( G \) with \( \kappa_1(G) = k \) and \( \kappa_1(G) = m \), but of order less than the desired minimum. Let \( v \in V(G) \) be such that \( \kappa_1(\langle N(v) \rangle) = k \) and let \( T \) be a separating set of \( k \) edges in \( \langle N(v) \rangle \). Denote a smallest component of \( \langle N(v) \rangle - T \) by \( G_1 \); let \( G_2 = \langle N(v) \rangle - G_1 \), and \( G_3 = G - (N(v) \cup \{v\}) \). Let \( r = |V(G_1)| \). Since \( \kappa_1(G) = m \), we must have \( \delta(G) \geq m \). In particular, \( \deg v \geq m \); say \( \deg v = m+t \), for \( t \geq 0 \). Then \( r \leq \left\lfloor \frac{m+t}{2} \right\rfloor \) and \( |V(G_2)| = m+t-r \).

Case 1. Suppose \( k < m/2 - 1 \).

Our assumption here, then, is that
\[
p(G) < \left\lfloor \frac{3m+2}{2} \right\rfloor.
\]
Thus \( |V(G_3)| = p(G) - 1 - \deg v < \left\lfloor \frac{3m+2}{2} \right\rfloor - 1 - m - t = \left\lfloor \frac{m}{2} \right\rfloor - t \).

If \( k = 0 \) or if \( k \geq 1 \) but there is a vertex \( w \) in \( G_1 \) which is not incident with an edge of \( T \), then
\[
\deg w \leq |V(G_1)| + |V(G_3)| < \left\lfloor \frac{m+t}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor - t
\]
= \left\lfloor \frac{m+t}{2} \right\rfloor + \{m/2\}
\leq \lfloor m/2 \rfloor + \lfloor m/2 \rfloor
= m,

contradicting \( \delta(G) \geq m \). Thus, we may assume that \( k \geq 1 \) and each vertex in \( G_1 \) is incident with at least one edge of \( T \). Let \( u \in V(G_1) \) be incident with the fewest number of edges in \( T \). The number of edges of \( T \) incident with \( u \) is no greater than \([k/r]\). Since \( \text{deg} \, u \geq m \), we must have \( V(G_3) \geq m - r - [k/r] \), so
\[
p(G) \geq r + (m-r) + (m-r-[k/r]) + 1
= 2m - r - [k/r] + 1.
\]

Let \( k = nr + r_0 \), where \( 0 \leq r_0 < r \), so \([k/r] = n\). Since \( k < m/2 - 1 \), we must have \( m > 2k + 2 \), so \( m \geq 2k + 3 \). Then
\[
m \geq 2(k-r_0) + 3
= 2(rn + 1) + 1
\geq 2(r+n) + 1, \text{ since } n \geq 1 \text{ and } r \geq 1,
\]
so \( 4m - 2r - 2n + 2 \geq 3m + 3 \), or \( 2m - r - n + 1 \geq \frac{3m+3}{2} \).

But \( \frac{3m+3}{2} \geq \left\lfloor \frac{3m+2}{2} \right\rfloor \), so \( 2m-r-n+1 \geq \left\lfloor \frac{3m+2}{2} \right\rfloor \); that is,
\[
2m-r-[k/r]+1 \geq \left\lfloor \frac{3m+2}{2} \right\rfloor. \text{ Thus, } p(G) \geq \left\lfloor \frac{3m+2}{2} \right\rfloor,
\]
which contradicts the initial assumption on the order of \( G \).
Case 2. Suppose \( k \geq m/2 - 1 \).

In this case, our assumption is that \( p(G) < 2m-k \), where \( \ell \kappa_1(G) = k \) and \( \kappa_1(G) = m \). As in Case 1, let \( u \) be a vertex of \( G_1 \) incident with the fewest number of edges of \( T \). Then \( u \) is incident with at most \( \lfloor k/r \rfloor \) vertices in \( G_2 \). Since \( \deg u \geq m \), we must have
\[
|V(G_2)| \geq m-r-[k/r].
\]
Thus \( p(G) \geq 2m-r-[k/r]+1 \).

Let \( k = rn+r_0 \), where \( 0 \leq r_0 < r \), so \( [k/r] = n \). Then
\[
k+1 = rn+r_0+1 \geq rn+1 \geq r+n, \text{ since } r \geq 1 \text{ and } n \geq 1.
\]
Thus \( -r-[k/r]+1 \geq -k \), so \( p(G) \geq 2m-r-[k/r]+1 \geq 2m-k \). However, this contradicts our initial assumption on the order of \( G \).

By Theorem 3.14 and Propositions 3.12 and 3.13, for given integers \( k \) and \( m \), where \( 0 \leq k < m \), we can find a graph \( G \) of minimum order for which \( \ell \kappa_1(G) = k \) and \( \kappa_1(G) = m \). If \( k < m/2 - 1 \), the graph \( G \) constructed in the proof of Proposition 3.13 provides a minimal example. If \( k \geq m/2 - 1 \), the complete \((k+2)\)-partite graph \( K(1,1,\ldots,m-k,m-k) \) is such a minimal graph.

Of course, if \( k = m-1 \), this \((k+2)\)-partite graph is
\[
K_{k+2} = K_{m+1}.
\]
For \( k \leq m-2 \), another graph \( G \) of minimum order for which \( \ell \kappa_1(G) = k \) and \( \kappa_1(G) = m \) can be determined. Let \( G' = (G_1 \cup \{u\}) + (G_2 \cup \{z\}) \), where
\[
G_1 = K_{m-1} \text{ and } G_2 = K_{m-k-1}.
\]
Let \( G \) be the graph such

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that \( V(G) = V(G') \) and \( E(G) = E(G') \cup \{uv_i \mid v_i \in G_1, 1 \leq i \leq k \} \). Using Theorem 3.D, it is straightforward to show that \( \kappa_1(G) \geq m \). But \( \deg z = m \), so \( \kappa_1(G) = m \). By consideration of each of the five classes of neighborhoods in \( G \) and again using Theorem 3.D, we obtain \( \kappa_1(G) = k \). Since \( p(G) = 2m-k \), we have exhibited another minimal example if \( k \geq m/2 - 1 \).
Critically and Minimally Locally Connected Graphs

We have already seen that some locally connected graphs are "more locally connected" than others. That is, a locally connected graph may be locally n-connected for some $n > 1$. Hence, in this sense, the least locally connected graphs are the graphs having local connectivity 1. There is yet another method for measuring graphs which are just barely locally connected; we briefly discuss this in the present section.

We say that a graph $G$ is critically locally connected if $G$ is locally connected but for each vertex $v$ of $G$, the graph $G - v$ is not locally connected. The graph $G$ is minimally locally connected if $G$ is locally connected but for each edge $e$ of $G$, the graph $G - e$ is not locally connected.

In Chapter II, several classes of connected, locally connected graphs were obtained. Some of these graphs can be used to show that critically locally connected and minimally locally connected are independent concepts. For example, in Fig. 2.13 (page 33), if we take $k = 2$ and $C_{i_1} = C_{i_2} = C_5$, then the graph $G_1$ of Fig. 3.3 (a) is obtained. For each
v \in V(G_1), the graph $G_1-v$ is not locally connected, so $G_1$ is critically locally connected. Similarly, $G_1$ is minimally locally connected. The graph $G_2$ in Fig. 3.3(b) is also critically locally connected since it is locally connected and \( \langle N(v) \rangle \) is a disconnected neighborhood in $G_2 - v$; by the symmetry of $G_2$, each of its subgraphs obtained by the deletion of a single vertex fails to be locally connected. However, $G_2$ is not minimally locally connected since $G_2 - e$ is locally connected.

Several of the graphs in the previous chapter are minimally locally connected but not critically locally connected. The family of graphs $G_k$ in Fig. 2.22 as well as $C_p^2 (p \geq 7)$ are examples of such graphs. The family $P_{p-1} + \{v\}$ is still another such example. There are also many examples of graphs which are locally

![Figure 3.3](image)
connected but neither critically locally connected nor minimally locally connected; all complete graphs and wheels (on at least four vertices) are examples of such graphs.

The next proposition will be useful in giving us some information about critically locally connected graphs.

**Proposition 3.15.** If $G$ is a graph for which $\delta(G) \geq 2$ and $\kappa(G) = n$, then there is a vertex $v$ of $G$ for which $\kappa(G-v) = n-1$.

**Proof.** If $\kappa(G-v) = n$ for each $v \in V(G)$, then every neighborhood in $G-v$ has connectivity at least $n$. But then every neighborhood in $G$ has connectivity at least $n+1$, contradicting the hypothesis. ■

**Corollary 3.16.** A necessary condition that a locally connected graph $G$ be critically locally connected is that $\kappa(G) = 1$.

The necessary condition on $G$ given in the preceding corollary is certainly not a sufficient condition for a graph to be critically locally connected. In fact, each of the graphs $G_k, C_p, P_{p-1} + \{v\}$ presented above as examples of graphs which are minimally locally connected but not critically locally connected
connected also has local connectivity 1.

**Proposition 3.17.** If $G (\not= K_2)$ is a locally connected graph in which every neighborhood is a tree, then $G$ is minimally locally connected.

**Proof.** Let $e = uv$ be an edge of $G$. We must show that $G-e$ is not locally connected. To this end, we first show that $e$ is an edge of some triangle in $G$. Since $G \not= K_2$, either $\deg u > 1$ or $\deg v > 1$. Without loss of generality, say $\deg v > 1$; let $w \in N(v), w \not= u$. Since $\langle N(v) \rangle$ is connected, there is a $u-v$ path $P$ in $\langle N(v) \rangle$. Let $u_1$ be the vertex adjacent to $u$ on $P$. Then $u, u_1, v, u$ is a triangle in $G$ to which $e = uv$ belongs.

By hypothesis, $\langle N(u_1) \rangle$ is a tree. Since $e \in E(\langle N(u_1) \rangle)$, and since in a tree the deletion of any edge disconnects the tree, we must have that $\langle N(u_1) \rangle - e$ is disconnected. But then $G-e$ is not locally connected.

![Figure 3.4](image-url)
The sufficient condition given in the previous proposition is not also a necessary condition for a graph to be minimally locally connected. The graph $G$ given in Fig. 3.4 is minimally locally connected but not every neighborhood in $G$ is a tree.

The result in Proposition 3.17 is best possible in the sense that if $G$ is a locally connected graph in which all neighborhoods except one are trees, then $G$ need not be minimally locally connected; for example, let $G$ be a wheel.
CHAPTER IV

SUFFICIENT CONDITIONS FOR LOCAL CONNECTIVITY

Introduction

In this chapter we present several sufficient conditions (expressed in terms of degrees) for a graph to be locally connected. In order to place these results in their proper perspective, we review some of the history of sufficient conditions which have been developed for certain graph-theoretic properties. Hamiltonian graphs serve as a good example in this context.

In 1952 Dirac [10] discovered one of the most famous theorems dealing with hamiltonian graphs.

**Theorem 4.A.** If $G$ is a graph of order $p \geq 3$ such that $\delta(G) \geq p/2$, then $G$ is hamiltonian.

It is not difficult to show that the number $p/2$ cannot be improved, i.e., there exist graphs $G$ of order $p$ for which $\delta(G) = (p-1)/2$, and $G$ is not hamiltonian. However, Ore [18] improved Dirac's result by showing that a graph could have some vertices...
with degree less than \( p/2 \) and still necessarily be hamiltonian.

**Theorem 4.B.** If \( G \) is a graph of order \( p \geq 3 \) such that for every pair of nonadjacent vertices \( u \) and \( v \),

\[
\text{deg } u + \text{deg } v \geq p,
\]

then \( G \) is hamiltonian.

Pósa [19] then proceeded to provide a sufficient condition for hamiltonian graphs which allows more vertices of degree less than \( p/2 \), including some of quite small degree.

**Theorem 4.C.** Let \( G \) be a graph of order \( p \geq 3 \) such that for every integer \( j \) with \( 1 \leq j \leq p/2 \), the number of vertices of degree not exceeding \( j \) is less than \( j \). Then \( G \) is hamiltonian.

As is often the case, the stronger result is more difficult to apply. For example, although Theorem 4.A is weaker than Theorem 4.C, it is considerably easier to employ when checking whether a graph is hamiltonian.

The property of being hamiltonian has been generalized; a graph \( G \) of order \( p \geq 3 \) is called \( n \)-hamiltonian, \( 0 \leq n \leq p-3 \), if the removal of any \( k \) vertices, \( 0 \leq k \leq n \), results in a hamiltonian graph. Thus a 0-hamiltonian graph is simply a hamiltonian...
The three results above have been generalized [6] and appear in corresponding order below.

**Theorem 4.D.** Let $G$ be a graph of order $p \geq 3$ and let $0 \leq n \leq p-3$. If $\delta(G) \geq (p+n)/2$, then $G$ is $n$-hamiltonian.

**Theorem 4.E.** Let $G$ be a graph of order $p \geq 3$ and let $0 \leq n \leq p-3$. If for every pair of nonadjacent vertices $u$ and $v$, $\deg u + \deg v \geq p+n$, then $G$ is $n$-hamiltonian.

**Theorem 4.F.** Let $G$ be a graph of order $p \geq 3$ and let $0 \leq n \leq p-3$. If (1) for every $j$ such that $n+1 \leq j < (p+n-1)/2$ the number of vertices of degree not exceeding $j$ is less than $j-n$ and (2) the number of vertices of degree not exceeding $(p+n-1)/2$ does not exceed $(p-n-1)/2$, then $G$ is $n$-hamiltonian.

These results have been extended still further by Chvátal [8], who stated the sufficient condition in terms of the degree sequence of the graph.

**Theorem 4.G.** Let $G$ be a graph of order $p \geq 3$, let $0 \leq n \leq p-3$, and let $(d_1, d_2, \ldots, d_p)$ be the degree sequence of $G$. If for each $m$ ($0 \leq m \leq n$), $d_k \leq k+m < \frac{1}{2}(p+m)$ implies $d_{p-m-k} \geq p-k$, then $G$ is $n$-hamiltonian.
Since a hamiltonian graph is 2-connected, and more generally, a \((n-2)\)-hamiltonian graph is \(n\)-connected, each of the above results can be restated to give a sufficient condition for a graph to be \(n\)-connected. Instead of following this approach, we present a more direct sequence of results.

In [4], Chartrand and Harary presented a lower bound for \(\delta(G)\) which is a sufficient condition for the graph \(G\) to be \(n\)-connected.

**Theorem 4.H.** Let \(G\) be a graph of order \(p \geq 3\) and let \(1 \leq n \leq p-1\). If \(\delta(G) \geq (p+n-2)/2\), then \(G\) is \(n\)-connected.

Theorem 4.H cannot be improved in the sense that if the degree of each vertex of \(G\) is lowered by one, then \(G\) need not be \(n\)-connected. However, several results have been obtained to show that it is possible to lower the degrees of some vertices and still force \(G\) to be \(n\)-connected. Two such results are the following [3,5].

**Theorem 4.1.** Let \(G\) be a graph of order \(p \geq 2\) and let \(1 \leq n < p\). If (1) for every \(k\) such that \(n-1 \leq k < (p+n-3)/2\), the number of vertices of degree not exceeding \(k\) does not exceed \(k+1-n\), and (2) the
number of vertices of degree not exceeding \((p+n-3)/2\) does not exceed \(p-n\), then \(G\) is \(n\)-connected.

**Theorem 4.5.** Let \(G\) be a graph of order \(p \geq 2\), let \(1 \leq m < p\), and let \((d_1, d_2, \ldots, d_p)\) be the degree sequence for \(G\). If \(1 \leq k \leq p-d_{p-n+1} - 1\) implies \(d_k \geq k+n-1\), then \(G\) is \(n\)-connected.

With these ideas in mind, we now return to the concept in which we are chiefly interested: locally connected graphs.
Sufficient Conditions for Locally Connected Graphs

Chartrand and Pippert [7] have shown that if \( \delta(G) > \frac{2}{3}(p-1) \), then \( G \) is locally connected. Furthermore, they showed this number cannot be reduced. More generally, they obtained the following.

**Theorem 4.1K.** If \( G \) is a graph of order \( p \) such that for every pair \( u,v \) of vertices, \( \deg u + \deg v > \frac{4}{3}(p-1) \), then \( G \) is locally connected.

In this section we show that we may allow some vertices to have degree less than or equal to \( \frac{2}{3}(p-1) \), and still force \( G \) to be locally connected. In this connection, we will determine the smallest value which the degree of a single vertex may assume and still require \( G \) to be locally connected. Also we will investigate the number of vertices which may have this smallest degree to insure that \( G \) is locally connected.

To justify the results given below, it will be necessary to compare expressions, many of which involve the least integer function \( \left\lfloor \frac{1}{3}(p-1) \right\rfloor \). To assist in these comparisons, we present the following table, which may be verified by direct substitution.
<table>
<thead>
<tr>
<th>$f(p)$</th>
<th>$f(3n)$</th>
<th>$f(3n+1)$</th>
<th>$f(3n+2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{2}{3}(p-1)$</td>
<td>$2n-2/3$</td>
<td>$2n$</td>
<td>$2n+2/3$</td>
</tr>
<tr>
<td>$\left{\frac{1}{3}(p-1)\right}$</td>
<td>$n$</td>
<td>$n$</td>
<td>$n+1$</td>
</tr>
<tr>
<td>$2\left{\frac{1}{3}(p-1)\right}-1$</td>
<td>$2n-1$</td>
<td>$2n-1$</td>
<td>$2n+1$</td>
</tr>
<tr>
<td>$p-\left{\frac{1}{3}(p-1)\right}$</td>
<td>$2n$</td>
<td>$2n+1$</td>
<td>$2n+1$</td>
</tr>
<tr>
<td>$\left{\frac{2}{3}(p-1)\right}-\left{\frac{1}{3}(p-1)\right}$</td>
<td>$n$</td>
<td>$n$</td>
<td>$n$</td>
</tr>
<tr>
<td>$2\left{\frac{2}{3}(p-1)\right}-p$</td>
<td>$n$</td>
<td>$n-1$</td>
<td>$n$</td>
</tr>
<tr>
<td>$2p-2\left{\frac{2}{3}(p-1)\right}$</td>
<td>$2n$</td>
<td>$2n+2$</td>
<td>$2n+2$</td>
</tr>
<tr>
<td>$2\left{\frac{2}{3}(p-1)\right}-\frac{8}{3}(p-1)-2$</td>
<td>$2n-4/3$</td>
<td>$2n-2$</td>
<td>$2n-2/3$</td>
</tr>
</tbody>
</table>

**TABLE I**

**Proposition 4.1.** Let $G$ be a graph of order $p \geq 5$. If $G$ has one vertex of degree $2\left\{\frac{2}{3}(p-1)\right\}-2$ and all others of degree greater than $\frac{3}{2}(p-1)$, then $G$ need not be locally connected.

**Proof.** Let $k = \left\{\frac{3}{2}(p-1)\right\}$ and consider the graph $G = 2K_{k-1} + ([v] \cup K_{p+1-2k})$. Then $\deg v = 2(k-1) = 2\left\{\frac{3}{2}(p-1)\right\}-2$, and if $u(\neq v)$ is a vertex of $G$, $\deg u \geq (k-2) + 1 + (p+1-2k) = p-k = p-\left\{\frac{3}{2}(p-1)\right\} > \frac{3}{2}(p-1)$, by Table 1. Thus $G$ satisfies the degree requirements.
of the proposition. Since $\langle N(v) \rangle = 2^k k-1$, which is disconnected, \( G \) is not locally connected.

Thus, by Proposition 4.1, we may not allow even a single vertex to have degree as small as $2 \left\{ \frac{1}{3}(p-1) \right\} -2$
(with all other degrees greater than $\frac{5}{3}(p-1)$) and still be assured that the graph is locally connected. Hence we cannot expect a result for locally connected graphs which is similar to Pósa's result for Hamiltonian graphs. However, we may question whether a graph may have a vertex of degree $2 \left\{ \frac{1}{3}(p-1) \right\} -1$ and still be forced to be locally connected. The following result shows that in fact we may have more than a single vertex of degree $2 \left\{ \frac{1}{3}(p-1) \right\} -1$; we may have as many as $2 \left\{ \frac{5}{3}(p-1) \right\} -p-1$ such vertices providing the remaining vertices have sufficiently large degree.

**Theorem 4.2.** Let \( G \) be a graph of order \( p \) which has up to $2 \left\{ \frac{5}{3}(p-1) \right\} -p-1$ vertices of degree $2 \left\{ \frac{5}{3}(p-1) \right\} -1$ and all others of degree greater than $\frac{5}{3}(p-1)$. Then \( G \) is locally connected.

**Proof.** If \( p \equiv 2 \pmod{3} \), then by Table 1, we have $2 \left\{ \frac{5}{3}(p-1) \right\} -1 > \frac{5}{3}(p-1)$. Hence in this case $\delta(G) > \frac{5}{3}(p-1)$, so by Theorem 4.1, \( G \) is locally connected.
For \( p \equiv 0 \pmod{3} \) or \( p \equiv 1 \pmod{3} \), suppose \( G \) is not locally connected. Then there is a vertex \( v \) of \( G \) such that \( \langle N(v) \rangle \) is not connected.

**Case 1.** Suppose \( \deg v = 2\left\lfloor \frac{3}{3}(p-1) \right\rfloor - 1 \).

Let \( G_1 \) be a component of \( \langle N(v) \rangle \) of minimum order, say \( |V(G_1)| = r \). Then \( r \leq \frac{3}{2}(2\left\lfloor \frac{3}{3}(p-1) \right\rfloor - 1) \), so
\[
r \leq \left\lfloor \frac{3}{3}(p-1) \right\rfloor - 1,
\]
since \( r \) is integral. If \( u \in V(G_1) \), then \( \deg u \leq r + p - 2\left\lfloor \frac{3}{3}(p-1) \right\rfloor \leq p - 1 - \left\lfloor \frac{3}{3}(p-1) \right\rfloor \leq \frac{3}{3}(p-1) \), by Table 1. Thus each vertex of \( G_1 \) has degree at most \( \frac{3}{3}(p-1) \), so by hypothesis, the degree of each vertex of \( G_1 \) must be \( 2\left\lfloor \frac{3}{3}(p-1) \right\rfloor - 1 \). Therefore, \( r \leq 2\left\lfloor \frac{3}{3}(p-1) \right\rfloor - p - 2 \) since there are at most \( 2\left\lfloor \frac{3}{3}(p-1) \right\rfloor - p - 1 \) vertices of degree \( 2\left\lfloor \frac{3}{3}(p-1) \right\rfloor - 1 \), one of which is \( v \).

Hence \( \deg u \leq r + p - 2\left\lfloor \frac{3}{3}(p-1) \right\rfloor \leq 2\left(\left\lfloor \frac{3}{3}(p-1) \right\rfloor - \left\lfloor \frac{3}{3}(p-1) \right\rfloor \right) - 2 = 2\left(\left\lfloor \frac{3}{3}(p-1) \right\rfloor - 2 \right) \), by Table 1, since \( p \not\equiv 2 \pmod{3} \). But by hypothesis, no vertex of \( G \) has degree \( 2\left\lfloor \frac{3}{3}(p-1) \right\rfloor - 2 \), so Case 1 cannot happen.

**Case 2.** Suppose \( \deg v = k > \frac{3}{3}(p-1) \).

Select \( G_1 \) as in Case 1 so that \( r \leq k/2 \).

If \( u \in V(G_1) \), then \( \deg u \leq r + p - 1 - k \leq p - 1 - k/2 < p - 1 - \frac{3}{3}(p-1) = \frac{3}{3}(p-1) \). Thus \( r \leq 2\left\lfloor \frac{3}{3}(p-1) \right\rfloor - 1 \), so
\[
\deg u \leq r + p - 1 - k \leq 2\left\lfloor \frac{3}{3}(p-1) \right\rfloor - 2 - k < 2\left\lfloor \frac{3}{3}(p-1) \right\rfloor - \frac{3}{3}(p-1) - 2 < 2\left(\left\lfloor \frac{3}{3}(p-1) \right\rfloor - 1 \right),
\]
by Table 1. But, by hypothesis, this is impossible, so Case 2 cannot happen.
As we noted at the beginning of the proof of

Theorem 4.2, when \( p \equiv 2 (\text{mod } 3) \), then \( \delta(G) > \frac{2}{3}(p-1) \). But if \( p \equiv 2 (\text{mod } 3) \), then \( 2\left\lfloor \frac{2}{3}(p-1) \right\rfloor - 2 < \frac{2}{3}(p-1) < 2\left\lfloor \frac{2}{3}(p-1) \right\rfloor - 1 \) by Table 1. Thus by Proposition 4.1, when \( p \equiv 2 (\text{mod } 3) \), if \( G \) has as few as one vertex of degree less than or equal to \( \frac{2}{3}(p-1) \), then \( G \) need not be locally connected.

Proposition 4.3. If \( G \) has \( 2\left\lfloor \frac{2}{3}(p-1) \right\rfloor - p \) vertices of degree \( \left\lfloor \frac{2}{3}(p-1) \right\rfloor - 1 \) and all others of degree greater than \( \frac{2}{3}(p-1) \), then \( G \) need not be locally connected.

Proof. Let \( k = \left\lfloor \frac{2}{3}(p-1) \right\rfloor \) and consider

\[ G = (K_{2k-p-1} \cup K_{p-k}) + ([v] \cup K_{p-k}). \]

Then \( \deg v = \left\lfloor \frac{2}{3}(p-1) \right\rfloor - 1 \) and each of the \( 2\left\lfloor \frac{2}{3}(p-1) \right\rfloor - p \) vertices in \( K_{2k-p-1} \) has degree \( 2k-p-1+l+p-k = \left\lfloor \frac{2}{3}(p-1) \right\rfloor - 1 \), so \( G \) has \( 2\left\lfloor \frac{2}{3}(p-1) \right\rfloor - p \) vertices of degree \( \left\lfloor \frac{2}{3}(p-1) \right\rfloor - 1 \). If \( u \in K_{p-k} \), then \( \deg u \geq p-k+l+1+p-k = 2p-2\left\lfloor \frac{2}{3}(p-1) \right\rfloor > \frac{2}{3}(p-1) \) by Table 1. Thus \( G \) satisfies the degree requirements. But \( \langle N(v) \rangle = K_{2k-p-1} \cup K_{p-k} \), which is not connected. Thus, \( G \) is not locally connected.

If \( p \equiv 0 (\text{mod } 3) \), then by Theorem 4.2, \( G \) may have as many as \( 2\left\lfloor \frac{2}{3}(p-1) \right\rfloor - p-1 \) vertices of degree\n
\( 2\left\lfloor \frac{2}{3}(p-1) \right\rfloor - 1 \) and all others of degree greater than \( \frac{2}{3}(p-1) \), and necessarily \( G \) is locally connected. Now when \( p \equiv 0 (\text{mod } 3) \), \( 2\left\lfloor \frac{2}{3}(p-1) \right\rfloor - 1 = \left\lfloor \frac{2}{3}(p-1) \right\rfloor - 1 \), by
Table 1, so Theorem 4.2 is best possible. In summary, for \( p = 0 \pmod{3} \), by Proposition 4.1 if \( G \) has at least one vertex of degree \( \left\lceil \frac{2}{3}(p-1) \right\rceil - 2 \) and all others of degree greater than \( \frac{2}{3}(p-1) \), then \( G \) need not be locally connected. However, by Theorem 4.2, \( G \) may have up to \( 2\left\lceil \frac{2}{3}(p-1) \right\rceil - p - 1 \) vertices of degree \( 2\left\lceil \frac{2}{3}(p-1) \right\rceil - 1 \), with all other vertices of degree greater than \( \frac{2}{3}(p-1) \), and necessarily \( G \) is locally connected; furthermore, the number \( 2\left\lceil \frac{2}{3}(p-1) \right\rceil - p - 1 \) cannot be improved, i.e., it cannot be increased.

It remains to complete the analysis for \( p = 1 \pmod{3} \). In this case, Theorem 4.2 states that if \( G \) has a certain number of vertices of degree

\[
2\left\lceil \frac{2}{3}(p-1) \right\rceil - 1 = \frac{2}{3}(p-1) - 1
\]

and all others of degree greater than \( \frac{2}{3}(p-1) \), then \( G \) must be locally connected. Thus we want to determine what combination of vertices of degrees \( \frac{2}{3}(p-1) - 1 \) and \( \frac{2}{3}(p-1) \) with all other vertices having degree greater than \( \frac{2}{3}(p-1) \) will insure that \( G \) is locally connected. It is also desirable to obtain a result of this type which is best possible.

**Theorem 4.4.** Let \( p = 1 \pmod{3} \) and let \( k \) be such that \( 0 < k < \frac{2}{3}(p-1) - 1 \). If a graph \( G \) has \( k \) vertices of degree \( \frac{2}{3}(p-1) \) and \( \frac{2}{3}(p-1) - 1 - k \) vertices of degree \( \frac{2}{3}(p-1) - 1 \) with all other vertices of degree greater than
\( \frac{2}{3} (p-1) \), then \( G \) is locally connected.

**Proof.** Assume \( G \) is not locally connected and let \( v \) be a vertex of \( G \) for which \( \langle N(v) \rangle \) is not connected. We consider three cases determined by the degree of \( v \).

**Case 1.** Suppose \( \deg v = \frac{2}{3} (p-1) - 1 \).

Let \( G_1 \) be a component of \( \langle N(v) \rangle \) of smallest order, say \( |V(G_1)| = r \). Thus, \( r \leq \frac{1}{3} (p-1) - 1 \), since \( r \) is an integer and \( p \equiv 1 \pmod{3} \). Let \( u \in V(G_1) \). Then \( \deg u \leq r + p - \frac{2}{3} (p-1) \leq \frac{2}{3} (p-1) \). Thus each vertex in \( G_1 \) has degree \( \frac{2}{3} (p-1) - 1 \) or \( \frac{2}{3} (p-1) \), and since there are \( \frac{2}{3} (p-1) - 1 \) such vertices, one of which is \( v \), necessarily \( r \leq \frac{2}{3} (p-1) - 2 \). Thus \( \deg u \leq r + p - \frac{3}{3} (p-1) \leq \frac{2}{3} (p-1) - 1 \). But \( G \) contains \( \frac{2}{3} (p-1) - 1 - k \) vertices of degree \( \frac{2}{3} (p-1) - 1 \), one of which is \( v \), so \( r \leq \frac{2}{3} (p-1) - 2 - k \). Therefore, \( \deg u \leq r + p - \frac{3}{3} (p-1) \leq \frac{2}{3} (p-1) - 1 - k \). But \( k > 0 \), so \( \deg u \leq \frac{2}{3} (p-1) - 2 \), which by hypothesis is impossible. Thus Case 1 cannot happen.

**Case 2.** Suppose \( \deg v = \frac{2}{3} (p-1) \).

Let \( G_1 \) and \( r \) be as in Case 1. Then \( r \leq \frac{2}{3} (p-1) \). If \( u \in V(G_1) \), then \( \deg u \leq r + \frac{2}{3} (p-1) \leq \frac{2}{3} (p-1) \). But \( G \) has \( \frac{2}{3} (p-1) \) such vertices, one of which is \( v \), so \( r \leq \frac{2}{3} (p-1) - 2 \). Hence \( \deg u \leq r + \frac{2}{3} (p-1) \leq \frac{2}{3} (p-1) - 2 \), which by hypothesis is impossible. Thus Case 2 cannot occur.
Case 3. Suppose $\deg v = t > \frac{2}{3}(p-1)$.

Let $G_1$ and $r$ be as in Case 1, so $r \leq t/2$. For $u \in V(G_1)$, $\deg u \leq r+p-1-t \leq p-1-t/2 < \frac{2}{3}(p-1)$. Since $G$ has $\frac{2}{3}(p-1)-1-k$ such vertices, we must have $r \leq \frac{2}{3}(p-1)-1-k$. But then $\deg u \leq r+p-1-t \leq \frac{2}{3}(p-1)-1-k+(p-1)-t < \frac{2}{3}(p-1)-1-k$. Since $k > 0$, necessarily $\deg u < \frac{2}{3}(p-1)-2$, which is impossible. Thus Case 3 is also impossible, so the assumed graph $G$ cannot exist; that is, the theorem is valid.

We now show that the number of vertices of degree $\frac{2}{3}(p-1)-1$ cannot be increased; i.e., if the hypothesis of Theorem 4.4 is relaxed to allow just one additional vertex of degree $\frac{2}{3}(p-1)-1$, while retaining the other degree requirements, then $G$ need not be locally connected.

Proposition 4.5. Let $p \equiv 1 (\mod 3)$ and let $k$ be such that $0 < k < \frac{1}{3}(p-1)-1$. If a graph $G$ has $k$ vertices of degree $\frac{2}{3}(p-1)$ and $\frac{2}{3}(p-1)-k$ vertices of degree $\frac{2}{3}(p-1)-1$ and all other vertices of degree greater than $\frac{2}{3}(p-1)$, then $G$ need not be locally connected.

Proof. To justify this result we will construct such a graph $G$. To that end, let the graph $G'$ be defined by $G' = (G'_1 \cup G'_2) + (\{v\} \cup G'_3)$, where $G'_1, G'_2,$ and
$G'_3$ are complete graphs of order $\frac{1}{3}(p-1)-1$, $\frac{1}{3}(p-1)$, and $\frac{1}{3}(p-1)+1$, respectively. Thus $\deg v = \frac{2}{3}(p-1)-1$. We construct $G$ from $G'$ by deleting certain edges of $G'$. Let $V(G) = V(G')$ and select $\frac{1}{3}(p-1)-1-k$ vertices from $G'_1$. For each of these vertices so selected, we decrease its degree by one by deleting an incident edge which is also incident with a vertex in $G'_3$. We perform these deletions so that no vertex in $G'_3$ has degree decreased by more than one. This is possible since $|V(G'_3)| > |V(G'_1)|$. Let $G_1, G_2,$ and $G_3$ denote the subgraphs of $G$ corresponding to $G'_1$, $G'_2$, and $G'_3$ of $G'$. Then $G_1$ has $k$ vertices of degree $\frac{2}{3}(p-1)$ and $\frac{1}{3}(p-1)-1-k$ vertices of degree $\frac{2}{3}(p-1)-1$. Now all vertices of $G_2$ and $G_3$ have degrees at least $\frac{2}{3}(p-1)+1$.

Since $\deg v = \frac{2}{3}(p-1)-1$, the graph $G$ satisfies the degree requirements. Also $\langle N(v) \rangle = G_1 \cup G_2$, which is disconnected, so $G$ is not locally connected. ■

In Theorem 4.4 we considered those situations where, if the degree of a vertex did not exceed $\frac{2}{3}(p-1)$, then that vertex must have degree $\frac{2}{3}(p-1)$ or $\frac{2}{3}(p-1)-1$. In fact, we considered only those graphs which contained at least one vertex of degree $\frac{2}{3}(p-1)$ and at least one vertex of degree $\frac{2}{3}(p-1)-1$. Suppose now that all of the vertices of a graph $G$ which have degree at most
\( \frac{2}{3}(p-1) \) necessarily have degree \( \frac{2}{3}(p-1)-1 \), i.e., assume
G has \( \frac{2}{3}(p-1)-1 \) vertices of degree \( \frac{2}{3}(p-1)-1 \) and all
other vertices have degree greater than \( \frac{2}{3}(p-1) \). Since
an overall assumption in this discussion is that
\( p \equiv 1 \pmod{3} \), by Table 1 we have \( \frac{2}{3}(p-1)-1 = 2\left\lfloor \frac{2}{3}(p-1) \right\rfloor \). Hence, by Proposition 4.3, G need not be locally con­
nected. However, if G has one less vertex of degree
\( \frac{2}{3}(p-1) \), so that G has \( \frac{2}{3}(p-1)-2 = 2\left\lfloor \frac{2}{3}(p-1) \right\rfloor \) vertices
of degree \( \frac{2}{3}(p-1)-1 = 2\left\lfloor \frac{2}{3}(p-1) \right\rfloor -1 \), then by Theorem 4.2,
G must be locally connected. Hence, to insure that G
is locally connected, if all vertices have degree
exceeding \( \frac{2}{3}(p-1) \) except some which have degree
\( \frac{2}{3}(p-1)-1 \), then there can be at most \( \frac{2}{3}(p-1)-2 \) vertices
of degree \( \frac{2}{3}(p-1)-1 \).

We now consider the situation where the only
vertices whose degrees do not exceed \( \frac{2}{3}(p-1) \) must have
degree \( \frac{2}{3}(p-1) \).

**Theorem 4.6.** Let \( p \equiv 1 \pmod{3} \). If a graph G has no
more than \( \frac{2}{3}(p-1) \) vertices of degree \( \frac{2}{3}(p-1) \) and all
other vertices have degree greater than \( \frac{2}{3}(p-1) \), then G
is locally connected.

**Proof.** Suppose there is a graph G satisfying the
hypothesis which is not locally connected. Then there
is a vertex v of G such that \( \langle N(v) \rangle \) is not connected.
**Case 1.** Suppose \( \deg v = \frac{2}{3}(p-1) \).

Let \( \langle N(v) \rangle = G_1 \cup G_2 \) where \( G_1 \) is a component of \( \langle N(v) \rangle \) of minimum order, say \( |V(G_1)| = r \). Then \( r \leq \frac{2}{3}(p-1) \). If \( u \in V(G_1) \), then \( \deg u \leq r + \frac{1}{3}(p-1) \leq \frac{2}{3}(p-1) \). Thus each vertex of \( G_1 \) has degree \( \frac{2}{3}(p-1) \) since no vertex of \( G \) has smaller degree. But then \( r = \frac{2}{3}(p-1) \) and consequently \( |V(G_2)| = \frac{1}{3}(p-1) \). Thus if \( y \in V(G_2) \), then \( \deg y \leq \frac{2}{3}(p-1) \), so \( \deg y = \frac{2}{3}(p-1) \). Therefore, all vertices of \( G_2 \) have degree \( \frac{2}{3}(p-1) \). Also, \( \deg v = \frac{2}{3}(p-1) \), so \( G \) contains at least \( \frac{2}{3}(p-1) + 1 \) vertices of degree \( \frac{2}{3}(p-1) \), which by hypothesis is impossible.

**Case 2.** Suppose \( \deg v = t > \frac{2}{3}(p-1) \).

Let \( G_1 \) and \( r \) be as in Case 1, so \( r \leq t/2 \). If \( u \in V(G_1) \), then \( \deg u \leq r + p-1-t \leq p-1-t/2 \leq \frac{2}{3}(p-1) \). But no vertex of \( G \) has degree less than \( \frac{2}{3}(p-1) \), so Case 2 cannot happen.

We now show that Theorem 4.6 cannot be improved by allowing an additional vertex of degree \( \frac{2}{3}(p-1) \).

**Proposition 4.7.** Let \( p \equiv 1 \mod 3 \). If a graph \( G \) has \( \frac{2}{3}(p-1) + 1 \) vertices of degree \( \frac{2}{3}(p-1) \) and all other vertices of degree greater than \( \frac{2}{3}(p-1) \), then \( G \) need not be locally connected.
Proof. Let $r = (p-1)/3$ and consider $G = 2K_r + ([v] \cup K_r)$. Then $G$ has $2r+1 = \frac{3}{3}(p-1)+1$ vertices of degree $\frac{3}{3}(p-1)$ and all other vertices of degree $p-2$. Since $\langle N(v) \rangle = 2K_r$ is disconnected, $G$ is not locally connected.

This completes the analysis of the case $p \equiv 1 \pmod{3}$. Hence we have determined the number of vertices of degrees $\left\{\frac{3}{3}(p-1)\right\}-1$ and $\left\{\frac{3}{3}(p-1)\right\}$ which are allowed and still force $G$ to be locally connected.
BIBLIOGRAPHY


