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On Hamiltonian - Connected Graphs

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ON HAMILTONIAN - CONNECTED GRAPHS

by

James E. Williamson

A Dissertation
Submitted to the
Faculty of The Graduate College
in partial fulfillment
of the
Degree of Doctor of Philosophy

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James E. Williamson
For my parents,
who made certain that I did not fail fifth grade arithmetic,
and for Carol and Kimberly,
who gave me sufficient freedom and incentive to go this far.
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CHAPTER I

INTRODUCTION

Historically, most difficult mathematical problems have themselves been the catalyst for new areas of research. Fortunately many such new areas prove to have significance beyond the original problem. One such problem has been that of determining which graphs are hamiltonian, that is, the problem of determining which graphs $G$ possess a cycle containing all of the vertices of $G$. Many conditions sufficient to guarantee that a graph is hamiltonian are well known. Also some investigation has been made into which graphs are in some sense nearly hamiltonian. Since no useful characterization of hamiltonian graphs appears forthcoming, it is not surprising that certain subclasses of hamiltonian graphs have been singled out for study.

One such subclass of hamiltonian graphs is the family of hamiltonian-connected graphs introduced by Ore. A graph $G$ is said to be hamiltonian-connected if each pair $u, v$ of distinct vertices are joined by a $u,v$-path containing all the vertices of $G$. Since Ore introduced these graphs in 1963, many sufficient conditions have been found and certain other research has taken place. Chapter III of this thesis presents
a survey of what is known about sufficient conditions for hamiltonian-connected graphs. In addition, several possible interpretations are given to the meaning of the concept of nearly hamiltonian-connected graphs and a number of results along these lines are included. Bondy and others have recently undertaken some investigations in the realm of nearly hamiltonian graphs; however, no similar research has been done on nearly hamiltonian-connected graphs until this time.

A number of so-called graph valued functions are in common use in graph theory. Among them are the line graph, total graph, and the nth power of a graph. Any graph valued function \( f \) gives rise to a collection \( \{ f^n \} \) of graph valued functions (where it is usually the case that \( f^n = f(f^{n-1}) \), for \( n \geq 2 \)). If \( Q \) is a family of graphs and \( P \) is some property a graph might possess, then a possible area of research deals with determining for which value or values of \( n \) is it true that \( f^n(G) \) possesses property \( P \) for all graphs \( G \) in a given family \( Q \). Such research has occurred and the following examples are just a few of the results of this type.

It is known that if \( Q \) is the family of hamiltonian graphs then both the line graph and total graph functions produce hamiltonian graphs, so that \( f^n(G) \), for \( G \in Q \),
is hamiltonian for each nonnegative integer n. Behzad and Chartrand showed that if \( \mathcal{G} \) is the family of non-trivial connected graphs and \( f \) is the total graph function \( T \), then \( T^2(G) \) is hamiltonian. Beineke showed that if \( \mathcal{G} \) is the family of connected graphs, then \( G^3 \) is hamiltonian, for each \( G \in \mathcal{G} \). This result was improved by Karaganis and Sekanina who showed the same result with the property \( P \) strengthened to hamiltonian-connected. A further strengthening was made by Bhat and Kapoor.

The most celebrated result of this type is the Plummer, Nash-Williams conjecture recently settled by Fleischner; namely, if \( \mathcal{G} \) is the family of 2-connected graphs, then \( G^2 \) is hamiltonian for \( G \in \mathcal{G} \). Hobbs has also improved this result, strengthening \( P \) to hamiltonian-connected.

Chartrand undertook a similar investigation for the line graph function and the property of being hamiltonian. He showed that if \( \mathcal{G} \) is the family of connected graphs of order \( p \), then \( L^n(G) \) is hamiltonian, for each \( n \geq p - 3 \) and \( G \in \mathcal{G} \). Improved results were obtained by Chartrand and Wall by restricting the class of graphs \( \mathcal{G} \). If we let \( P \) be strengthened to hamiltonian-connected, then a new avenue of investigation is opened. This in fact is the subject of Chapter IV. It is shown there that if \( \mathcal{G} \) is the family of connected graphs which

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are not paths or cycles of order greater than four, then there exists an \( n \) such that \( L^k(G) \) is hamiltonian-connected for all \( k \geq n \) and \( G \in \mathcal{Q} \). This result is further improved by restricting the class \( \mathcal{Q} \), and the least integer \( n \) such that \( L^n(G) \) is hamiltonian-connected for each \( G \in \mathcal{Q} \) is determined exactly for a number of classes \( \mathcal{Q} \).

If \( G \) is a hamiltonian-connected graph, then each pair \( u, v \) of vertices of \( G \) are joined by a path of length \( p - 1 \) in \( G \). Also, since \( G \) is connected, each pair \( u, v \) of vertices of \( G \) are joined by a path of length \( k = d_G(u,v) \), where \( d_G(u,v) \) denotes the distance from \( u \) to \( v \). Thus one possible subclass of hamiltonian-connected graphs are those graphs \( G \) for which each pair \( u, v \) of vertices of \( G \) are joined by paths of each length \( l \), where \( d_G(u,v) \leq l \leq p - 1 \). Such graphs will be studied in Chapter V and will be called panconnected. Among the results contained there are several theorems pertaining to the panconnectedness of \( n \)th powers of certain families of graphs and a sufficient condition for panconnectedness, as well as an introductory discussion of panconnected line graphs.
CHAPTER II

BASIC NOTATION AND DEFINITIONS

In this chapter, we present some basic definitions and notation that will be used throughout this thesis. Additional more specialized definitions are placed appropriately throughout the remaining chapters. All other definitions will be consistent with [1], or will be presented as needed.

A graph $G$ is a finite nonempty set $V(G)$ of vertices and a set $E(G)$ of unordered pairs of distinct vertices called edges. We denote the edge joining vertex $u$ to the vertex $v$ by $uv$ or $vu$. The order of a graph $G$, denoted by $p$ or $|V(G)|$ is the number of elements in $V(G)$. The number of elements in $E(G)$ is called the size of $G$ and will be denoted by $q$ or $|E(G)|$. The graph of order one is called the trivial graph. A graph of size zero is called an empty graph.

Two vertices (edges) are said to be adjacent in a graph $G$ if they are joined by an edge (have exactly one common vertex) in $G$. If the vertices $u$ and $v$ are adjacent in $G$, then the vertex $u$ (also the vertex $v$) is said to be incident with the edge $uv$. The degree of a vertex $v$ in $G$, denoted $\deg_G v$, is the number of edges of $G$ incident with the vertex $v$. 

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A vertex $v$ is said to be an **even** (odd) vertex if the degree of $v$ in $G$ is even (odd). A vertex $v$ having degree one in $G$ is called an **end-vertex** of $G$. A graph $G$ is said to be regular of degree $r$ or $r$-regular if the degree of each vertex of $G$ is $r$. We denote by $\Delta(G)$ the **maximum degree** among the vertices of $G$, and we denote by $\delta(G)$ the **minimum degree** among the vertices of $G$.

A graph $G$ is **isomorphic** to a graph $H$ if there exists a one-to-one mapping $\phi$, called an isomorphism, from $V(G)$ onto $V(H)$ such that $\phi$ preserves adjacency and nonadjacency; in this case $G$ is said to be a **copy** of $H$. A graph $H$ is called a **subgraph** of a graph $G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Any graph isomorphic to a subgraph of $G$ is also called a subgraph of $G$.

If $H$ is a subgraph of a graph $G$ for which $V(H) = V(G)$, we say that $H$ is a **spanning subgraph** of $G$, or that the subgraph $H$ **spans** the graph $G$. A subgraph $H$ of a graph $G$ is called an **induced subgraph** of $G$ if each edge of $G$ joining vertices of $H$ is also an edge of $H$.

Two graphs $G$ and $H$ are said to be disjoint (edge-disjoint) if the vertex sets of $G$ and $H$ are disjoint (if the edge sets of $G$ and $H$ are disjoint). We denote by $G-u$ the subgraph of $G$ obtained by removing from $G$ the vertex $u$ and all the edges of $G$ which are incident with $u$. We denote by $G-e$ the subgraph
of $G$ obtained by removing the edge $e$ from $G$.

If $u$ and $v$ are (not necessarily distinct) vertices of a graph $G$, then a $u,v$-walk of $G$ is a finite, alternating sequence of vertices and edges of $G$, beginning with $u$ and ending with $v$, such that every edge is immediately preceded and succeeded by the two vertices of $G$ with which it is incident. A $u,v$-walk of a graph $G$ is called a closed walk (open walk) if $u = v$ ($u \neq v$). A $u,v$-trail is a $u,v$-walk in which no edge is repeated. A $u,v$-path is a $u,v$-walk in which no vertex is repeated. A path is said to have length $l$ if the path has $l$ edges. A vertex $w$ of a $u,v$-path is called an interior vertex of the path if $w \neq u$ and $w \neq v$. A non-trivial closed trail of $G$ is called a circuit of $G$. The circuit $v_1, v_2, \ldots, v_n, v_{n+1} = v_1$ is a cycle if $v_1, v_2, \ldots, v_n$ are distinct. An $n$-cycle is a cycle with $n$ vertices. A 3-cycle is called a triangle.

Two vertices $u$ and $v$ of a graph $G$ are said to be connected if there exists a $u,v$-path in $G$. A graph $G$ is said to be a connected graph if every two of its vertices are connected. A graph $G$ which is not connected is called disconnected. A component of a graph $G$ is a maximal connected subgraph of $G$. A vertex $v$ of a connected graph $G$ is called a cut-vertex of $G$ if $G - v$ is disconnected. An edge $e$ of a connected graph
G is called a bridge if G - e is disconnected. A non-trivial connected graph G is called a block if G has no cut-vertices. A block of a graph G is a subgraph which is itself a block and which is maximal with respect to that property. An end-block of a graph G is a block containing exactly one cut-vertex of G. For a connected graph G, we define the distance from a vertex u to a vertex v, denoted d_G(u,v), as the length of a shortest u,v-path in G.

The connectivity of a graph G, denoted κ(G), is the minimum number of vertices whose removal from G results in a disconnected graph or the trivial graph. The edge-connectivity, denoted κ₁(G), of a graph G is the minimum number of edges whose removal from G results in a disconnected or trivial graph. A set S of vertices (edges) of a connected graph G is called a cut-set if G - S is disconnected. A two element cut-set is called a cut-pair.

The following theorem due to Whitney [22] will be needed several times in the remaining chapters, as it furnishes an important link between the connectivity, edge-connectivity, and minimum degree of any graph.

**Theorem 2-A.** For any graph G, we have

\[ \kappa(G) \leq \kappa_1(G) \leq \delta(G). \]
The complement, denoted \( \overline{G} \), of a graph \( G \) is that graph whose vertex set is that of \( G \) and such that two vertices \( u \) and \( v \) are adjacent in \( \overline{G} \) if and only if \( u \) and \( v \) are not adjacent in \( G \). A tree is an acyclic connected graph. A graph of order \( p \) which is a cycle is denoted by \( C_p \). A complete graph is a graph such that every two distinct vertices are joined by an edge; the complete graph of order \( p \) is denoted by \( K_p \). A graph \( G \) is \( n \)-partite, for \( n \geq 2 \), if it is possible to partition \( V(G) \) into \( n \) subsets \( V_1, V_2, \ldots, V_n \), called partite sets, such that every edge of \( G \) joins a vertex of \( V_i \) to a vertex of \( V_j \), where \( i \neq j \). If \( n = 2 \), the graph \( G \) is said to be bipartite. A complete \( n \)-partite graph \( G \) is an \( n \)-partite graph with vertex set partition \( V_1, V_2, \ldots, V_n \) having the added property that if \( u \in V_i \) and \( v \in V_j \) for \( i \neq j \), then \( uv \in E(G) \). If \( |V_i| = p_i \) for each \( i \), where \( 1 \leq i \leq n \), then \( G \) is denoted by \( K(p_1, p_2, \ldots, p_n) \). If \( G \) and \( H \) are any graphs, then the join of \( G \) and \( H \), denoted \( G + H \), is the graph obtained by joining each vertex of a copy of \( G \) to each vertex of a disjoint copy of \( H \).

An eulerian trail of a connected graph \( G \) is an open trail of \( G \) containing all the edges of \( G \). An eulerian circuit of \( G \) is a circuit containing all the edges of \( G \). A graph possessing an eulerian circuit is
called an **eulerian graph**. The following characterization of eulerian graphs was known by Euler himself [11].

**Theorem 2-B.** Let $G$ be a connected non-trivial graph. Then $G$ is eulerian if and only if every vertex of $G$ is even.

The following theorem characterizes those graphs containing eulerian trails.

**Theorem 2-C.** Let $G$ be a non-trivial connected graph. Then $G$ contains an eulerian trail if and only if $G$ has exactly two odd vertices.

Theorem 2-B and Theorem 2-C have wide application in the results contained in Chapter IV. We state them here mainly for convenience. Also for convenience we state the following theorem, often called the "First Theorem of Graph Theory", as it is so basic that its application appears often in proofs of theorems.

**Theorem 2-D.** Let $G$ be a graph of order $p$ and size $q$ where $V(G) = \{v_1, v_2, \ldots, v_p\}$. Then

$$
\sum_{i=1}^{p} \deg v_i = 2q
$$

Proofs of the four theorems in this chapter can be

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found in [1].

As a matter of style, theorems appearing in the literature and used in this thesis will be labeled with letter designations. Also the symbol □ is used to denote the end of a proof.
A Hamiltonian path of a graph $G$ is a path containing every vertex of $G$. A graph $G$ is called Hamiltonian-connected if for every pair $u, v$ of distinct vertices of $G$, there exists a $u, v$-Hamiltonian path. This class of graphs was introduced by Ore [18] in 1963. If $G$ is a Hamiltonian-connected graph of order $p \geq 3$, then it is immediate that $G$ is Hamiltonian. As with Hamiltonian graphs, Hamiltonian-connected graphs have not been characterized.

In the following chapters, we shall study Hamiltonian-connected graphs and several subclasses of Hamiltonian-connected graphs. In this chapter, however, we shall attempt to gain some insight into Hamiltonian-connected graphs by considering several examples of graphs which are "not quite" Hamiltonian-connected. There is no natural definition of graphs which are "nearly" Hamiltonian-connected; thus we shall consider graphs which are nearly Hamiltonian-connected from several points of view. In the process, we shall review the several sufficient conditions which have been derived for Hamiltonian-connected graphs.
Section 3.1

Some Important Examples of Nonhamiltonian-connected Graphs

The purpose of this section is to set down three classes of nonhamiltonian-connected graphs which play a recurrent role in that which is to follow.

The first of these classes is the class of complete bipartite graphs $K(p_1, p_2)$, where $p_1 = p_2$. That is, the graphs $K(p/2, p/2)$, where $p$ is the order of the complete bipartite graph. The graph $K(p/2, p/2)$ is not hamiltonian-connected, for if we let the partite sets be $U$ and $V$, where $U = \{u_1, u_2, \ldots, u_{p/2}\}$ and $V = \{v_1, v_2, \ldots, v_{p/2}\}$, then no $v_1, v_2$-hamiltonian path exists. This follows from the fact that any $v_1, v_2$-hamiltonian path must have odd length, and no $v_1, v_2$-path of odd length exists in $K(p/2, p/2)$. In fact, if $u$ and $v$ belong to the same partite set, then no $u, v$-hamiltonian path exists.

The second class of graphs to be considered are the graphs of order $p \geq 4$ (which we shall denote by $K_r | K_s$, where $r \geq 3$, $s \geq 3$, and $r+s = p+2$), obtained by identifying an edge in a copy of $K_r$ with an edge in a copy of $K_s$. That is, $K_r | K_s$ consists of a copy of $K_r$ and a copy of $K_s$ in which a pair $u_1, v_1$ of vertices of $K_r$ are identified with, respectively, a pair $u_2, v_2$.
of vertices of $K_s$; therefore, edges $u_1v_1$ and $u_2v_2$ are replaced by a single edge in $K_r \mid K_s$. Figure 3.1 shows the smallest member of this class, namely, $K_3 \mid K_3$.

$$K_3 \mid K_3 : \quad \begin{array}{cc}
\bullet & \bullet \\
\mid & \\
\bullet & \bullet \\
\end{array}$$

$u_1 = v_1$

$u_2 = v_2$

Figure 3.1

For any $r$ and $s$, where $r \geq 3$ and $s \geq 3$, the graph $K_r \mid K_s$ has connectivity exactly two. That is, if $u$ and $v$ are the vertices of degree $p-1$, then $K_r \mid K_s - \{u,v\}$ consists of disjoint copies of $K_{r-2}$ and $K_{s-2}$. Thus it follows that $K_r \mid K_s$ has connectivity two since it does not have a cut-vertex. Furthermore, the graph $K_r \mid K_s$ for any $r, s \geq 3$ is not hamiltonian-connected since no $u,v$-hamiltonian path exists. This follows since any $u,v$-path can contain only vertices of a single component of $K_r \mid K_s - \{u,v\}$ as its interior vertices.

The last class of graphs to be listed here is the class obtained by taking the join of any graph $G$ of order $n \geq 2$ and the graph $\overline{K_n}$. These graphs are
denoted $G + \overline{K}_n$, which is consistent with previous notation. The graph $K_{p/2} + \overline{K}_{p/2}$ of order $p \geq 4$ and $p$ even is the graph belonging to this class for which we shall have the greatest need; however, we observe that $G + \overline{K}_n$ is not hamiltonian-connected for any graph $G$ of order $n$. To see this, we observe that if $u$ and $v$ are any two vertices of $G + \overline{K}_n$ belonging to the subgraph isomorphic to $G$, then any $u, v$-hamiltonian path would contain $2n$ vertices and include each of the $n$ vertices of $\overline{K}_n$ as interior vertices, which is impossible. Therefore, $G + \overline{K}_n$ is not hamiltonian-connected.

Section 3.2

Nearly Hamiltonian-connected Graphs and Sufficient Conditions for Hamiltonian-connectedness

Historically, the first sufficient condition for hamiltonian-connectedness appeared in Ore's introductory paper.

Theorem 3-A. (Ore) If $G$ is a graph of order $p \geq 3$ such that $\deg_G u + \deg_G v \geq p + 1$ for every pair $u, v$ of nonadjacent vertices, then $G$ is hamiltonian-connected.

Analogous to this result is an earlier theorem of Ore [16] which gives a sufficient condition for hamiltonian graphs.
Theorem 3-B. (Ore) If $G$ is a graph of order $p \geq 3$ such that $\deg_G u + \deg_G v \geq p$ for every pair $u, v$ of nonadjacent vertices, then $G$ is hamiltonian.

This is the first of several pairs of theorems, one giving a sufficient condition for a graph to be hamiltonian-connected and one giving an analogous condition for a graph to be hamiltonian. This pairing of theorems suggests one possible interpretation of nearly hamiltonian-connected graphs: A graph $G$ is said to be nearly hamiltonian-connected with respect to Theorem $X$ if $G$ is not hamiltonian-connected but is hamiltonian by Theorem $X$. For example, the graph $K(p/2, p/2)$, which has been shown to be nonhamiltonian-connected in the previous section, is nearly hamiltonian-connected with respect to Ore's theorem (Theorem 3-B), since each pair $u, v$ of nonadjacent vertices has $\deg u + \deg v = p$.

The class of graphs $K_r \mid K_s$, where $r, s \geq 3$, is also a class of graphs which are nearly hamiltonian-connected with respect to Theorem 3-B, for if $v_1$ and $v_2$ are nonadjacent vertices of $K_r \mid K_s$, then $\deg v_1 + \deg v_2 = (r-1) + (s-1) = p$.

In both cases above we note that each pair of nonadjacent vertices of the given graphs have degree sum equal to $p$. There do exist, however, graphs which are nearly hamiltonian-connected with respect to Theorem 3-B.
for which some of their pairs of nonadjacent vertices have degree sum strictly greater than $p$. For instance, if we let $G$ be any graph of order $p/2$ (thus $p$ is even) and size $k$, where $1 \leq k < p(p-2)/2$, then $G + K_{p/2}$ is nearly hamiltonian-connected with respect to Theorem 3-B and there exist pairs $u, v$ of nonadjacent vertices for which $\deg u + \deg v > p$.

Naturally, no graph which is nearly hamiltonian-connected with respect to Theorem 3-B can have all of its pairs of nonadjacent vertices $u, v$ such that $\deg u + \deg v > p$, for then Theorem 3-A implies that $G$ is hamiltonian-connected. Thus we might ask whether it is possible for a graph $G$ to possess a single pair $u, v$ of vertices such that $\deg u + \deg v = p$, with all other nonadjacent pairs (if any) having degree sum greater than $p$. The following theorem deals with this question.

**Theorem 3-1.** Let $G$ be a nonhamiltonian-connected graph for which there exists a unique pair $u, v$ of nonadjacent vertices such that $\deg_G u + \deg_G v = p$ and such that for all pairs $w_1, w_2$ of nonadjacent vertices different from the pair $u, v$, we have $\deg w_1 + \deg w_2 > p$. Then $G$ is the graph $K_3 \mid K_3$.

**Proof.** Let $G$ be a graph which satisfies the hypotheses. Then by Theorem 3-B, the graph $G$ is hamiltonian. Let
C be a hamiltonian cycle of G where

\[ C: v_1, v_2, \ldots, v_p, v_1 \]

and suppose \( v_i, v_j \) are not joined by a hamiltonian path. Then by an argument contained in the proof (See [18]) of Ore's Theorem 3-B, it follows that \( \deg_G v_{i+1} + \deg_G v_{j+1} \leq p \). Thus the pair \( v_{i+1}, v_{j+1} \) must be the pair \( u, v \) and \( \deg_G v_{i+1} + \deg_G v_{j+1} = p \).

Since this argument can be reversed relative to the cycle \( C \), we must also have \( \deg_G v_{i-1} + \deg_G v_{j-1} \leq p \), so that the pair \( v_{i-1}, v_{j-1} \) must also be the pair \( u, v \). From this it follows that \( G \) must have order four and that \( G = K_3 \mid K_3 \), the only graph of order four which is nearly hamiltonian-connected with respect to Theorem 3-B.

\[ \square \]

Ore's theorems have the following theorems as corollaries.

**Theorem 3-C.** If \( G \) is a graph of order \( p \geq 3 \) such that \( \delta(G) \geq (p+1)/2 \), then \( G \) is hamiltonian-connected.

**Theorem 3-D.** (Dirac) If \( G \) is a graph of order \( p \geq 3 \) such that \( \delta(G) \geq p/2 \), then \( G \) is hamiltonian.

As indicated, Theorem 3-D is initially due to Dirac.
[10] who gave an independent proof of the result eight years prior to Ore's Theorem 3-B, of which it is a corollary.

Observe that if $G$ is a nearly hamiltonian-connected graph with respect to Theorem 3-D, then $p$ is even, for otherwise $\delta(G) \geq p/2$ also implies that $\delta(G) \geq (p+1)/2$, which is impossible, since then $G$ is hamiltonian-connected by Theorem 3-C.

The graph $K(p/2, p/2)$ is nearly hamiltonian-connected with respect to Theorem 3-D and is in fact $p/2$-regular.

The graph $K_n \mid K_n$, where $n = (p+2)/2$, also has minimum degree $p/2$ so that it is nearly hamiltonian-connected with respect to Theorem 3-D. It does, however, have two vertices of degree greater than $p/2$.

Now if we let $G$ be the graph of order $n$ consisting of a path with $k$ vertices, where $2 \leq k \leq n$, and $n-k$ isolated vertices, then $G + \overline{K}_n$ has minimum degree at least $p/2$, is nearly hamiltonian-connected with respect to Theorem 3-D and has $k$ vertices whose degree exceeds $p/2$.

These examples show that graphs $G$ exist which are nearly hamiltonian-connected with respect to Theorem 3-D and either have no vertices of degree greater than $p/2$ or $k$ vertices of degree greater than $p/2$, where $2 \leq k \leq p/2$. It is not known whether graphs $G$ exist.
which are nearly hamiltonian-connected with respect to Theorem 3-D and have precisely one vertex of degree greater than \( p/2 \). However, if such a graph exists, it must have order \( p \geq 12 \).

On the other hand, no nearly hamiltonian-connected graph can exist having more than \( p/2 + 1 \) vertices of degree greater than \( p/2 \) by a theorem of Chartrand, Kapoor, and Kronk [7], which shall appear later in this chapter (cf. Theorem 3-E). The following theorem completes the analysis (except for the one aforementioned case) of graphs which are nearly hamiltonian-connected with respect to Theorem 3-D, regarding constraints on the number of vertices of degree greater than \( p/2 \).

**Theorem 3-2.** There exists no graph \( G \) which is nearly hamiltonian-connected with respect to Theorem 3-D, whose degree sequence satisfies \( d_1 = d_2 = \ldots = d_{p/2 - 1} = p/2 \), and \( p/2 + 1 \leq d_{p/2} \leq \ldots \leq d_p \).

**Proof.** Suppose \( G \) is a graph which is nearly hamiltonian-connected with respect to Theorem 3-D and whose degree sequence satisfies the hypotheses; hence \( G \) has precisely \( p/2 + 1 \) vertices of degree greater than \( p/2 \).

Let \( j \) be any integer such that \( 1 \leq j < p/2 \) and \( d_j \leq j + 1 \); then \( p/2 - 1 \leq j \) since \( d_j = p/2 \) for \( j < p/2 \), so that \( j = p/2 - 1 \) is the only integer less than \( p/2 \) for which \( d_j \leq j + 1 \) holds. Thus if there
exists a \( k \) for which \( d_k \leq k + 1 \), and \( k \neq j \), then \( k \geq p/2 \), from which it follows by hypothesis that \( d_k \geq p/2 + 1 \). Therefore, \( d_j + d_k \geq p/2 + (p/2+1) = p + 1 \); thus by a theorem of Lick (cf. Theorem 3-G), the graph \( G \) is hamiltonian-connected, a contradiction. Therefore, no such graph \( G \) exists.

\[ \square \]


**Theorem 3-E.** (Chartrand, Kapoor and Kronk; Lick) If \( G \) is a graph of order \( p \geq 3 \) such that for every \( j \) with \( 2 \leq j \leq p/2 \), the number of vertices of degree not exceeding \( j \) is less than \( j - 1 \), then \( G \) is hamiltonian-connected.

The analogous result for hamiltonian graphs is a theorem due to Posa [20].

**Theorem 3-F.** Let \( G \) be a graph of order \( p \geq 3 \) such that for every integer \( j \) with \( 1 \leq j < p/2 \), the number of vertices of degree not exceeding \( j \) is less than \( j \). Then \( G \) is hamiltonian.

Again, the graph \( K(p/2, p/2) \) serves as an example of a graph which is nearly hamiltonian-connected with
respect to Theorem 3-F, as it vacuously satisfies the hypothesis. Less trivially, we turn to the class of graphs $K_r \mid K_s$ and consider those members for which $r < s$. Since $r + s = p + 2$ and $r < s = p + 2 - r$, we have $r < p/2 + 1$. Moreover, the degree sequence satisfies $d_1 = d_2 = \ldots = d_{r-2} = r - 1$, $d_{r-1} = d_r = \ldots = d_{r+s-4} = s - 1$, and $d_{p-1} = d_p = p - 1$. Since $r < p/2 + 1$, we have $r - 1 < p/2$, so the hypothesis of Theorem 3-F is satisfied. That is, for any integer $j < r - 1 < p/2$, there are no vertices of degree at most $j$, for $j = r - 1$, there are exactly $j - 1 = r - 2$ vertices of degree not exceeding (equal to) $r - 1$, and since $d_{r-1} = \ldots = d_{[p/2]} = s - 1$, and $s - 1 \geq p/2 + 1$, the number of vertices of degree not exceeding $j$ for $r < j < p/2$ is exactly $r - 2$. Therefore $K_r \mid K_s$ for $r < s$ is nearly hamiltonian-connected with respect to Theorem 3-F.

Actually, the above discussion shows more than was stated and we present the result in a theorem. The proof is immediate from the above argument.

**Theorem 3-3.** There exist graphs $G$ which are nearly hamiltonian-connected with respect to Theorem 3-F having the property that for some $j$, where $2 \leq j < p/2$, the graph $G$ contains precisely $j - 1$ vertices whose degree does not exceed $j$. 

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We now present a class of nearly hamiltonian-connected graphs with respect to Theorem 3-F which shows that "for some j" in Theorem 3-3 can be replaced by "for each j". Let $p$ be chosen so that $p$ is even and at least four, and let $H$ be a copy of $K_{(p+2)/2}$, where $V(H) = \{v_1, v_2, \ldots, v_{(p+2)/2}\}$ and let $W = \{u_1, u_2, \ldots, u_{(p-2)/2}\}$ be a disjoint set of vertices. We obtain a graph $G$ of order $p$ from $H$ and $W$ by joining each vertex $u_i$, where $1 \leq i \leq (p-2)/2$, to the vertices $v_1, v_2, \ldots, v_{i+1}$. Then $\text{deg}_G u_i = i + 1$, for each $i$, where $1 \leq i \leq (p-2)/2$. Furthermore, $\text{deg}_G v_j \geq p/2$ for all $j = 1, 2, \ldots, (p+2)/2$. By construction, $G$ contains a vertex of degree two and is thus not hamiltonian-connected, but it is nearly hamiltonian-connected with respect to Theorem 3-F.

In 1970, Lick [13] presented the following conditions which are sufficient for hamiltonian-connectedness.

**Theorem 3-G.** (Lick) Let $G$ be any graph of order $p \geq 4$ whose vertices have degrees $d_1, d_2, \ldots, d_p$, where $d_i \leq d_{i+1}$, $i = 1, 2, \ldots, p-1$, and suppose whenever $d_j \leq j + 1$ and $d_k \leq k + 1$, $j \neq k$, that $d_j + d_k \geq p + 1$. Then $G$ is hamiltonian-connected.

Lick's result is the analog to the following theorem.
due to Bondy [3].

**Theorem 3-H.** Let $G$ be a graph on $p \geq 3$ vertices whose degree sequence satisfies $d_i \leq i$, $d_j \leq j$ ($i \neq j$) implies $d_i + d_j \geq p$. Then $G$ is hamiltonian.

For convenience, we shall postpone the discussion of graphs $G$ which are nearly hamiltonian-connected with respect to Theorem 3-H until after the next pair of results have been introduced.

The following theorem is another theorem presenting a sufficient condition for hamiltonian-connectedness.

**Theorem 3-4.** Let $G$ be a graph on $p \geq 4$ vertices whose vertices have degrees $d_1, d_2, \ldots, d_p$, where $d_i \leq d_{i+1}$, for $i = 1, 2, \ldots, p-1$, and suppose $d_k \leq k + 1 < (p+1)/2$ implies $d_{p-k} \geq p - k$. Then $G$ is hamiltonian-connected.

This result is an analog of the following theorem due to Chvátal [8].

**Theorem 3-I.** (Chvátal) Let $G$ be a graph on $p \geq 3$ vertices whose degree sequence satisfies: $d_k \leq k < p/2$ implies $d_{p-k} \geq p - k$. Then $G$ is hamiltonian.

**Proof of Theorem 3-4.** Let $G$ be any graph which satisfies the hypotheses of the theorem and let $u$ and $v$ be any two distinct vertices of $G$. We must show that
G contains a u,v-hamiltonian path. We proceed by cases.

**Case i. Suppose uv is an edge of G.**

Let H be the graph of order p + 1 obtained from G by subdividing the edge uv. That is, the edge uv of G is replaced by a vertex w together with the edges uw and wv. The graph H has order p + 1 and its degree sequence satisfies \( d'_1 = 2 \) and \( d'_{i+1} = d_i \) for all \( i \), where \( 1 \leq i \leq p \).

Let \( k \) be any integer such that \( 1 \leq k < (p-1)/2 \), and \( d'_{k+1} \leq k + 1 \). Since \( d'_k = d'_{k+1} \), it follows that \( d_k \leq k + 1 < (p+1)/2 \). Thus by hypothesis we have \( d'_{p-k-1} \geq p - k \). From this it follows that \( d'_{p-k} = d'_{p-k-1} \geq p - k \). Since \( d'_{p-k} = d'_{(p+1)-(k+1)} \geq p - k = (p + 1) - (k + 1) \), the hypothesis of Theorem 3-I is satisfied in this case, so that H is hamiltonian.

**Case ii. Suppose uv is not an edge of G.**

In this case we obtain a new graph \( G_1 \) from G by adding the edge uv (i.e., \( G_1 = G + uv \)). The hypotheses on G also hold for \( G_1 \); thus we may apply Case i to the graph \( G_1 \), obtaining the hamiltonian graph H.

In any case, the graph H so obtained is hamiltonian by Theorem 3-I. Since every hamiltonian cycle of H must
contain the vertex \( w \) and the edges \( uw \) and \( vw \), any hamiltonian cycle of \( H \) gives rise to a \( u,v \)-hamiltonian path in \( G \). Hence \( G \) is hamiltonian-connected.

\[ \square \]

We comment that Theorem 3-4 and Theorem 3-1 are the most general theorems known which respectively provide sufficient conditions for hamiltonian-connected graphs and hamiltonian graphs. In fact, the following sequence of implications concerning hamiltonian-connectedness can be established, where each implies the succeeding result: 3-4, 3-G, 3-E, 3-A, and 3-C. Likewise, the following sequence exists for hamiltonian graphs: 3-I, 3-H, 3-F, 3-B, and 3-D.

The value of Theorems 3-G and 3-4 can be found in their ability to detect hamiltonian-connected graphs which have a great many vertices of relatively small degree. Similarly, Theorems 3-H and 3-I are able to detect hamiltonian graphs having a great number of vertices with relatively small degree. The graphs \( K(p/2, p/2) \), \( G + \overline{K}_n \), where \( G \) is any graph of order \( n \), and \( K_{p/2} + \overline{K}_{p/2} \) are all graphs which are nearly hamiltonian-connected with respect to Theorem 3-H and with respect to Theorem 3-I since they satisfy the respective conditions vacuously. Also the family of graphs \( K_r | K_s \) for any \( r \) and \( s \) at least three are
nearly hamiltonian-connected with respect to both Theorems 3-H and 3-I. This is not surprising since all of the aforementioned families have been found to be nearly hamiltonian-connected with respect to at least one of Theorems 3-B, 3-D, and 3-F, and each of Theorems 3-H and 3-I imply all of these theorems.

Unfortunately, no "reasonable" class of graphs which are nearly hamiltonian-connected with respect to either Theorem 3-H or Theorem 3-I can easily be described. However, consider the following two graphical sequences:

\[ s_1: 2, 3, 3, 5, 5, 5, 5, 6 \quad \text{and} \]
\[ s_2: 2, 2, 3, 4, 5, 6, 6, 6 \]

The sequence \( s_1 \) satisfies the conditions of Bondy's Theorem (Theorem 3-H) and thus is a graphical sequence for a family of graphs which are nearly hamiltonian-connected with respect to Theorem 3-H, since no such graph containing a vertex of degree two can be hamiltonian-connected. Furthermore, the sequence \( s_1 \) does not describe a family of graphs which are nearly hamiltonian-connected with respect to any of the previous theorems (Theorems 3-B, 3-D, and 3-F). Likewise, the graphical sequence \( s_2 \) is the graphical sequence of a family of graphs which is nearly hamiltonian-connected with respect to Theorems 3-I but not with respect to any of the previous theorems including Theorem 3-H.
We conclude this portion of the discussion with the following graphical sequences:

\[ s_3: 3, 4, 4, 5, 6, 6, 6, 6 , \text{ and } s_4: 3, 3, 4, 5, 6, 6, 6, 7 \]

The graphical sequence \( s_3 \) describes a family of graphs which are hamiltonian-connected by Lick's theorem (Theorem 3-G) and which could not have been detected by any of the previous theorems (3-A, 3-C, or 3-E). Furthermore, the graphical sequence \( s_4 \) describes a family of graphs which are hamiltonian-connected by Theorem 3-4 that could not be detected by any of the previous theorems for hamiltonian-connected graphs, including Theorem 3-G.

Before presenting the next pair of results we need a definition. The maximum order among the induced regular subgraphs of degree zero (induced empty subgraphs) of a graph \( G \) is called the independence number and is denoted by \( \beta(G) \). Recall that \( \kappa(G) \) denotes the connectivity of a graph \( G \). In 1972, Chvátal and Erdős presented the following pair of theorems in \([9]\).

**Theorem 3-J.** (Chvátal and Erdős) If \( G \) is a graph for which \( \beta(G) < \kappa(G) \), then \( G \) is hamiltonian-connected.

**Theorem 3-K.** (Chvátal and Erdős) If \( G \) is a graph for which \( \beta(G) \leq \kappa(G) \), then \( G \) is hamiltonian.
Once again the graph $K(p/2, p/2)$ plays a role in the discussion. We have $\beta(G) = \kappa(G) = p/2$ for the graph $G = K(p/2, p/2)$, so that $K(p/2, p/2)$ is nearly hamiltonian-connected with respect to Theorem 3-K.

Recall that the family of graphs $K_r \mid K_s$ for any $r$ and $s$ at least three is a family of graphs having connectivity two. We observe that the independence number of any graph in this family is also two. Thus $\beta(G) = \kappa(G) = 2$, for $G = K_r \mid K_s$, where $r, s \geq 3$, so that these graphs are also nearly hamiltonian-connected with respect to Theorem 3-K.

We observe that no graph $G$ which is nearly hamiltonian-connected with respect to any theorem can exist for which $\kappa(G) \leq 1$ or $\kappa(G) > p/2$. In the first case $G$ would not be hamiltonian and in the second case $G$ would be hamiltonian-connected by Theorem 3-C, since $\kappa(G) > p/2$ implies $\delta(G) \geq (p+1)/2$ by Theorem 2-A. Thus we might ask whether nearly hamiltonian-connected graphs $G$ exist for which $\beta(G) = \kappa(G) = n$ and $2 < n < p/2$. Unfortunately, no example of such a graph is known.

We now present the last of the seven pairs of theorems which are known to imply a graph $G$ is hamiltonian-connected (respectively, hamiltonian).

**Theorem 3-L.** (Ore) Let $G$ be a graph with $p \geq 3$
vertices and \( q \) edges such that \( q \geq \binom{p-1}{2} + 3 \).
Then \( G \) is hamiltonian-connected.

**Theorem 3-M. (Ore)** Let \( G \) be a graph with \( p \geq 3 \) vertices and \( q \) edges such that \( q \geq \binom{p-1}{2} + 2 \).
Then \( G \) is hamiltonian.

Theorem 3-L appears in Ore's original paper on hamiltonian-connected graphs and Theorem 3-M appears in [17].

Note that if \( G \) is nearly hamiltonian-connected with respect to Theorem 3-M, then the size of \( G \) must be \( \binom{p-1}{2} + 2 \).

Consider the graph \( G = K_3 \mid K_{p-1} \) for any \( p \geq 4 \).
This graph \( G \) has size \( q = \binom{p-1}{2} + 2 \), so that \( G \) is nearly hamiltonian-connected with respect to Theorem 3-L.

Now consider the member of the family \( K_{p/2} + \overline{K}_{p/2} \) for which \( p = 6 \) (i.e. the graph \( K_3 + \overline{K}_3 \)). It can easily be verified that \( K_3 + \overline{K}_3 \) has size 12 and that \( \binom{5}{2} + 2 = 12 \) also. Thus \( K_3 + \overline{K}_3 \) is also nearly hamiltonian-connected with respect to Theorem 3-M.

The property of being nearly hamiltonian-connected with respect to Theorem 3-L is apparently the only property of this type which appears in the literature and it was investigated by Ore in his original paper. He showed there that with the exception of the aforementioned...
graphs, every graph of order \( p \) and size \( \left( \binom{p-1}{2} \right) + 2 \)
is in fact hamiltonian-connected.

Section 3.3

Maximally Nonhamiltonian-connected Graphs

Those graphs which are nearly hamiltonian-connected with respect to Theorem 3-M, namely \( K_3 + \overline{K}_3 \) and the family \( K_3 \mid K_{p-1} \), as was previously noted, have the property that the addition of any edge between nonadjacent vertices yields a hamiltonian-connected graph. This observation suggests yet another possible interpretation of "nearly hamiltonian-connected graphs". If \( G \) is any graph which is not hamiltonian-connected and has the property that \( G + uv \) is hamiltonian-connected for each pair \( u, v \) of nonadjacent vertices of \( G \), then we call \( G \) a maximally nonhamiltonian-connected graph.

The aforementioned graphs are not the only examples of such graphs; and in fact, it will be shown that they are members of larger families of maximally nonhamiltonian-connected graphs, namely \( K_{p/2} + \overline{K}_{p/2} \) and \( K_r \mid K_s \) for \( r, s \geq 3 \). We have previously observed that these latter graphs have connectivity two. We shall now say more about this.
**Theorem 3-5.** Let $G$ be a graph with $\kappa(G) = 2$. Then $G$ is maximally nonhamiltonian-connected if and only if $G = K_r \mid K_s$, for some $r$ and $s$ at least three.

**Proof.** Let $G$ be the graph $K_r \mid K_s$ for some $r$ and $s$ at least three. It has previously been established that $\kappa(G) = 2$ and the pair $u, v$ of vertices common to $K_r$ and $K_s$ are a cut-pair of vertices of $G$. Since we have already shown that $K_r \mid K_s$ is maximally nonhamiltonian-connected for the case $3 = r \leq s$, we henceforth assume $4 \leq r \leq s$. If $r \neq s$, let $G_1$ denote the component of $G - \{u,v\}$ which is isomorphic with $K_{r-2}$ and let $G_2$ denote the component of $G - \{u,v\}$ which is isomorphic with $K_{s-2}$. If $r = s$, then let $G_1$ be a component of $G - \{u,v\}$ and let $G_2$ be the other component. In any case, since $G_1$ and $G_2$ are complete graphs and since each of $u$ and $v$ is adjacent with each of the $r + s - 3$ vertices of $G$ distinct from itself, any edge $x$ added to $G$ must be such that $x = v_1v_2$, where $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$. It suffices to prove that $G + x$ is hamiltonian-connected, since $G$ is not hamiltonian-connected.

Let $w_1$ and $w_2$ be any two vertices of $G + x$, and suppose first that $w_1 = u$ and $w_2 = v$. Furthermore let $u_1$ and $u_2$ be vertices of $G_1$ and $G_2$, respectively, which are different from $v_1$ and $v_2$. 

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Since $G_1$ is complete, there exists in $G_1$ a $u_1, v_1$-hamiltonian path $P_1$. Likewise, in $G_2$ there exists a $u_2, v_2$-hamiltonian path $P_2$. Then in $G + x$, there exists a $u,v$-hamiltonian path consisting of $u$, the edge $uu_1$, the path $P_1$, the edge $x$, the path $P_2$, the edge $u_2v$, and $v$.

![Diagram](image)

*Figure 3.2*

Now if $w_1 = u$ and $w_2 = z$, where $z \in V(G_1)$, then in $G + x$ there exists a $w_1, w_2$-hamiltonian path consisting of a $u,v$-path of $G$ whose interior vertices are precisely the vertices of $G_2$, an edge $vz'$, where $z' \in V(G_1)$ and $z' \neq z$, and a $z', z$-hamiltonian path of $G_1$. Analogously, if $w_1 = u$ and $w_2 = z$, where $z \in V(G_2)$, or if $w_1 = v$ and $w_2 \in [V(G_1) \cup V(G_2)]$, then there exists a $w_1, w_2$-hamiltonian path in $G + x$.

Furthermore if $w_1$ and $w_2$ are both vertices of $G_1$, say, then there exists a $w_1, w_2$-path $P$ in $G$.
whose interior vertices are the vertices of $G_1$ different from $w_1$ and $w_2$, together with the vertices $u$ and $v$ such that $u$ and $v$ are consecutive on $P$. Therefore, since $G$ also contains a $u,v$-path whose interior vertices are precisely the vertices of $G_2$, the graph $G + x$ contains a $w_1, w_2$-hamiltonian path. Thus the graph $G + x$ is hamiltonian-connected, and since the edge $x = v_1v_2$ was chosen arbitrarily, $G$ is maximally nonhamiltonian-connected.

Conversely, assume that a graph $G$ is maximally nonhamiltonian-connected and $\kappa(G) = 2$. Necessarily, $G$ is not complete and has order at least 4. Let $u$ and $v$ be a cut-pair of vertices, let $G_1$ denote a component of $G - \{u,v\}$, and let $G_2$ denote the remaining components of $G - \{u,v\}$. Note that the pair $u,v$ of vertices are not joined by a hamiltonian path in $G$, since any $u,v$-hamiltonian path must contain at least one edge incident with a vertex of $G_1$ and a vertex of $G_2$, which is impossible. By hypothesis, $G$ is maximally nonhamiltonian-connected so that the addition of any edge $x$ to $G$ yields a hamiltonian-connected graph. Necessarily, any edge $x$ added to $G$ must be an edge joining a vertex of $G_1$ to a vertex of $G_2$, for otherwise $G + x$ contains no $u,v$-hamiltonian path by the previous argument. This of course implies that $G_1$ is complete, $G_2$ is complete and both $u$ and $v$ are
joined to every vertex of \( G \) different from themselves, for otherwise \( x \) can be added in such a manner that it does not join a vertex of \( G_1 \) with a vertex of \( G_2 \). It follows then that the vertices of \( G_1 \) together with \( \{ u, v \} \) induce a complete subgraph of \( G \), and the vertices of \( G_2 \) together with \( \{ u, v \} \) induce a complete subgraph of \( G \). But then \( G = K_r \upharpoonright K_s \), where \( r = |V(G_1)| + 2 \) and \( s = |V(G_2)| + 2 \), which completes the proof of the theorem.

\[ \Box \]

Just as Theorem 3-5 provides a characterization of maximally nonhamiltonian-connected graphs with \( \kappa(G) = 2 \) in terms of a now familiar family of graphs, Theorem 3-6 provides a characterization of maximally nonhamiltonian-connected graphs with \( \kappa(G) = p/2 \) in terms of a well known class of graphs.

**Theorem 3-6.** Let \( G \) be a graph of order \( p \) with \( \kappa(G) = p/2 \). Then \( G \) is maximally nonhamiltonian-connected if and only if \( G = K_{p/2} + \overline{K}_{p/2} \), for \( p \) even and at least four.

**Proof.** For convenience, we let \( n = p/2 \) throughout the proof. Suppose \( G \) is the graph \( K_n + \overline{K}_n \), and let the \( n \) mutually nonadjacent vertices of \( G \) be labeled
$u_1, u_2, \ldots, u_n$ and the $n$ remaining vertices be labeled $v_1, \bar{v}_1, v_2, \bar{v}_2, \ldots, v_n$. We note that $K_n + \overline{K}_n$ has connectivity $n$.

Consider any pair $u_i, u_j$ of vertices of $G$, where $1 \leq i < j \leq n$. Without loss of generality we may consider the pair $u_1$ and $u_n$. Then $G$ contains the path $u_1, v_1, v_2, u_2, \ldots, v_n, u_n$, which is a Hamiltonian path of $G$, so that in a like manner any pair of vertices $u_i, u_j$ are joined by a Hamiltonian path in $G$. Now consider any pair $u_i, v_j$ of vertices, where $1 \leq i \leq n$ and $1 \leq j \leq n$. We show there exists a Hamiltonian path in $G$. Again, without loss of generality we may consider the pair $u_1$ and $v_n$. Then $G$ contains the path $u_1, v_1, u_2, v_2, \ldots, u_n, v_n$, which is a Hamiltonian path of $G$, as desired.

On the other hand, for any pair $v_i, v_j$ of vertices, where $1 \leq i < j \leq n$, there exists no Hamiltonian path in $G$. Assume to the contrary that such a path exists. Then each of the vertices $u_1, u_2, \ldots, u_n$ is a vertex of any $v_i, v_j$ - Hamiltonian path. Moreover, each occurrence of a vertex from the set $\{u_1, u_2, \ldots, u_n\}$ must be succeeded and preceded only by vertices from the set $\{v_1, v_2, \ldots, v_n\}$, since the vertices $u_1, u_2, \ldots, u_n$ are mutually nonadjacent in $G$. However, since each occurrence of a vertex from the set...
\[ \{u_1, u_2, \ldots, u_n\} \] must be an interior vertex of any \(v_i, v_j\) - hamiltonian path, there must be at least \(n + 1\) occurrences of vertices from the set \(\{v_1, v_2, \ldots, v_n\}\), which is impossible. Therefore, \(G\) contains no \(v_i, v_j\) - hamiltonian paths, where \(1 \leq i < j \leq n\), and hence \(G\) is not hamiltonian-connected.

Now let \(w_1, w_2\) be any pair of nonadjacent vertices of \(G\). Necessarily, \(w_1\) and \(w_2\) belong to the set \(\{u_1, u_2, \ldots, u_n\}\). Without loss of generality we may take \(w_1 = u_1\) and \(w_2 = u_2\). Consider the graph \(G + u_1u_2\). We must show that \(G + w_1w_2\) is hamiltonian-connected in order to verify that \(G\) is maximally nonhamiltonian-connected. It suffices to show that any pair of vertices \(v_i, v_j\), where \(1 \leq i < j \leq n\) are now joined by a hamiltonian path, since every other pair of vertices of \(G\) (and hence \(G + u_1u_2\)) have been shown to be joined by a hamiltonian path. Again, without loss of generality, we may take the pair \(v_i, v_j\) to be the pair \(v_1, v_n\). Then \(G + u_1u_2\) contains the \(v_1, v_n\) - path \(v_1, u_1, u_2, v_2, \ldots, u_n, v_n\) which is a \(v_1, v_n\) - hamiltonian path. Therefore \(G + u_1u_n\) is hamiltonian-connected and \(G\) is maximally nonhamiltonian-connected.

Conversely, let \(G\) be a graph of order \(p\) and connectivity \(n = p/2\) which is maximally nonhamiltonian-connected; thus \(G\) has even order. By Theorem 3-J, we must have \(\beta(G) \geq \kappa(G) = n\) since \(G\) is not hamiltonian-
Let the vertex set of $G$ be partitioned into two sets $U$ and $V$, where $U = \{u_1, u_2, \ldots, u_m\}$ and $V = \{v_1, v_2, \ldots, v_{p-m}\}$, where $m = \beta(G) \geq n = p/2$, and $U$ induces an empty subgraph of $G$. Suppose $m > n = p/2$; then since each vertex of $U$ can only be joined to vertices of $V$, each vertex $u \in U$ satisfies $\deg_G u \leq p - m < n = p/2$. But $\delta(G) < p/2$ is impossible since then $\delta(G) < \kappa(G)$ which contradicts Theorem 2-A. Therefore, $m = n = p/2$. Moreover, by Theorem 2-A, the minimum degree of $G$ must satisfy $\delta(G) \geq n$, from which it follows that every vertex of $U$ is adjacent with every vertex of $V$.

The proof will then be complete if we can show that the set $V$ induces a complete subgraph of $G$. Assume to the contrary that $V$ does not induce a complete subgraph of $G$, and suppose $v_1$ and $v_2$ are nonadjacent in $G$. Since $G$ is maximally nonhamiltonian-connected, there must exist a $v_1, v_2$-hamiltonian path in $G$, for if not, then $G + v_1v_2$ is not hamiltonian-connected, contradicting the hypotheses. However, no $v_1, v_2$-hamiltonian path can exist in $G$ since any such path in $G$ must include all $p/2$ vertices of $U$ as nonconsecutive interior vertices. Thus the vertices $v_1$ and $v_2$ must be joined by an edge and $V$ must induce a complete subgraph in $G$. Hence $G$ is the graph $K_n + \overline{K_n}$ and the
proof is complete.

A natural question to ask at this point is whether any maximally nonhamiltonian-connected graph exists aside from those of the two families considered by the previous theorems. It can only be reported that no such graphs have been found.

Section 3.4

Nearly Nonexistent Hypohamiltonian-connected Graphs

In the previous section we considered graphs $G$ which were nearly hamiltonian-connected in the sense that adding an edge to $G$ made the resultant graph hamiltonian-connected. This notion parallels that of work done by Bondy [4] on graphs which are not hamiltonian, but become hamiltonian upon the addition of any edge. These parallel ideas are certainly not unnatural approaches, since it is "easier" for a graph to possess a fairly complicated structural property, such as that of being hamiltonian or hamiltonian-connected, if the graph has many edges. Similarly, it is easier for a subgraph of a graph to have such a property since it might then be possible to ignore certain "bad" features of the entire graph.
The concept of hypohamiltonian graphs has evolved in just such a manner. A graph $G$ is said to be hypohamiltonian if it is not hamiltonian but $G - v$ is hamiltonian for every vertex $v$ of $G$. Thus in some sense, a hypohamiltonian graph is a nearly hamiltonian graph. It is reasonable to consider the possibility of a parallel concept as a third possible interpretation for the meaning of the term nearly hamiltonian-connected. Unfortunately, the parallelism between the hamiltonian and hamiltonian-connected properties breaks down at this point.

**Theorem 3-7.** If $G$ is any graph of order $p \geq 3$ having the property that $G - v$ is hamiltonian-connected for each vertex $v$ of $G$, then $G$ is hamiltonian-connected.

**Proof.** The only graphs $G$ of order $p = 3$ or $p = 4$ satisfying the hypotheses are the graphs $K_3$ and $K_4$, respectively, from which the conclusion follows immediately. Henceforth, we assume $p \geq 5$.

Let $v$ and $w$ be arbitrary distinct vertices of $G$, and let $H = G - v$. By hypothesis, $H$ is hamiltonian-connected and by assumption $H$ has order at least four. Therefore, each vertex of $H$ must have degree at least three in $H$ and thus at least three in $G$. Since $v$ was chosen arbitrarily, we have $\delta(G) \geq 3$.

Let $v_1$ and $v_2$ be two distinct vertices of $H$ each of which is adjacent in $G$ to the vertex $v$. If

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\( w = v_1 \), then since \( H \) is hamiltonian-connected there exists in \( H \) a \( v_2, v_1 \)-hamiltonian path \( P \) so that \( P \) together with the edge \( vv_2 \) produces a \( v, v_1 \)-hamiltonian path in \( G \). If \( w \neq v_1 \), then there exists in \( H \) a \( v_1, w \)-hamiltonian path \( Q \) so that \( Q \) together with the edge \( vv_1 \) produces a \( v, w \)-hamiltonian path in \( G \). Therefore, the vertex \( v \) is joined by a hamiltonian path to every other vertex of \( G \), from which the result is immediate since \( v \) was chosen arbitrarily.

\[ \square \]

It follows then from the above result that the nearly hamiltonian concept of hypohamiltonian cannot be carried over as a nearly hamiltonian-connected concept. We conclude with the comment that the graph consisting of two isolated vertices has the property that \( G - v \) is hamiltonian-connected for each vertex \( v \) of \( G \), which shows that the concept of "hypohamiltonian-connected" graphs is indeed nearly pointless.
LINE GRAPHS AND HAMILTONIAN-CONNECTED GRAPHS

With every graph $G$ having at least one edge, there is associated another graph $L(G)$, called the line graph of $G$, whose vertex set can be put in one-to-one correspondence with the edge set of $G$, with two vertices of $L(G)$ adjacent if and only if the corresponding edges of $G$ are adjacent. By $L^2(G)$ we shall mean $L(L(G))$, and for $n \geq 3$, the graph $L^n(G) = L(L^{n-1}(G))$. The graph $L^n(G)$ is called the $n$th-iterated line graph of $G$. For convenience, the symbols $L^1(G)$ and $L^0(G)$ denote $L(G)$ and $G$, respectively.

In [5] Chartrand defined $h(G)$ to be the smallest non-negative integer $n$ such that $L^n(G)$ is hamiltonian. The symbol $h(G)$ is called the hamiltonian-index of a graph $G$. It was shown that for any connected graph $G$, which is not a path, the hamiltonian-index exists and is no greater than $p - 3$, where $p$ is the order of $G$. In addition, in [6] Chartrand and Wall improved the bound on $h(G)$ for certain graphs $G$ and determined $h(G)$ in the case that $G$ is a tree. The object of this chapter is to investigate the relationship between iterated line graphs and hamiltonian-connectedness.
Section 4.1

Preliminary Results

We remark that our discussion shall be restricted to nontrivial connected graphs $G$, since the iterated line graphs, $L^n(G)$, for each non-negative integer $n$, are connected if $G$ is connected.

Let $G$ be a graph of order $p$ with $q$ edges, and let $e$ and $f$ be edges of $G$. Then an ordering

$$s: e_1, e_2, \ldots, e_q$$

of the edges of $G$ is called a hamiltonian-path ordering (or an $e,f$-hamiltonian-path ordering) if $s$ has the following properties:

1. $e_1 = e$ and $e_q = f$, and
2. every two consecutive edges in the ordering $s$ are adjacent.

A graph is called hamiltonian-path ordered if there exists a hamiltonian-path ordering for each distinct pair of edges of $G$. The first theorem of this section is a consequence of these definitions.

Theorem 4-1. The line graph $L(G)$ of a graph $G$ is hamiltonian-connected if and only if $G$ is hamiltonian-path ordered.
Proof. Suppose $G$ is hamiltonian-path ordered, and let $v$ and $w$ be any two vertices of $L(G)$. Then in $G$ there exist edges $e$ and $f$ corresponding under the line graph function, respectively, to $v$ and $w$ in $L(G)$. Let $s$ be an $e,f$-hamiltonian-path ordering of the edges of $G$,

$$s: e = e_1, e_2, \ldots, e_q = f$$

and let $v_i$ be the vertex of $L(G)$ corresponding to the edge $e_i$ of $G$, for each $i = 1, 2, \ldots, q$.

By the definition of $s$, the edges $e_i$ and $e_{i+1}$ are adjacent in $G$ for each $i = 1, 2, \ldots, q-1$, so that $v_i v_{i+1}$ is an edge of $L(G)$, for each $i = 1, 2, \ldots, q-1$. Hence $s$ gives rise to a $v_1, v_q$-path $P(v_1, v_q)$ in $L(G)$. Therefore, since $v_1 = v$, $v_q = w$, and $P(v_1, v_q)$ spans $L(G)$, it follows that $P(v_1, v_q)$ is a $v,w$-hamiltonian path of $L(G)$. This implies that $L(G)$ is hamiltonian-connected.

Conversely, suppose $L(G)$ is hamiltonian-connected, and $e$ and $f$ are any two edges of $G$. Further, let $v$ and $w$ be vertices of $L(G)$ corresponding to $e$ and $f$, respectively. Since $L(G)$ is hamiltonian-connected, there exists a $v,w$-hamiltonian path

$$P(v,w): v = v_1, v_2, \ldots, v_q = w.$$
Relative to $P(v,w)$, the edges of $G$ can be labeled $e_1, e_2, \ldots, e_q$, such that $e_i$ corresponds to $v_i$, for each $i = 1, 2, \ldots, q$. However, the ordering $s: e_1, e_2, \ldots, e_q$ is an $e,f$-hamiltonian-path ordering of the edges of $G$, since $e = e_1$, the edge $f = e_q$, and $v_i v_{i+1}$ is an edge of $L(G)$, for each $i = 1, 2, \ldots, q-1$. This implies that $e_i$ and $e_{i+1}$ are adjacent in $G$, for each $i = 1, 2, \ldots, q-1$. Since $e$ and $f$ were chosen arbitrarily, $G$ is hamiltonian-path ordered.

\[ \square \]

Suppose $G$ is a graph such that $L^k(G)$ is hamiltonian-connected for some nonnegative integer $k$. A highly relevant question pertaining to the current investigations is whether, in fact, $L^n(G)$ is hamiltonian-connected for each integer $n \geq k$. Theorem 4-2 and its corollary take up this matter.

**Theorem 4-2.** If $G$ is a non-trivial hamiltonian-connected graph, then $L(G)$ is hamiltonian-connected.

**Proof.** If $G$ is the graph $K_2$ or the graph $K_3$, then $L(G)$ is the graph $K_1$ or $K_3$, respectively, and the result follows immediately. Henceforth, then, we assume that $G$ is a hamiltonian-connected graph of order $p \geq 4$. 

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Let $v_1$ and $v_2$ be any two vertices of $L(G)$, and let $e$ and $f$ be the edges of $G$ which correspond, respectively, to $v_1$ and $v_2$. To prove that $L(G)$ is hamiltonian-connected, it is sufficient to determine an $e,f$-hamiltonian-path ordering $s$ of the edges of $G$, since the result then follows from Theorem 4-1. We proceed by cases.

Case i. The edges $e$ and $f$ are adjacent in $G$.

Let $u$ be the vertex of $G$ which is incident with $e$ but is not incident with $f$, and let $v$ be the vertex of $G$ which is incident with $f$ but is not incident with $e$. By hypothesis, $G$ is hamiltonian-connected, so that there exists in $G$ a $u,v$-hamiltonian path

$$P(u,v): u = u_1, u_2, \ldots, u_p = v.$$ 

Observe that every edge of $G$ is incident with some vertex of $P(u,v)$, since $P(u,v)$ spans $G$, and that if $uv$ is an edge of $G$, then it is not an edge of $P(u,v)$. From this path, the desired $e,f$-hamiltonian-path ordering $s$ can be constructed.

If the edge $e$ is the first edge of the hamiltonian path $P(u,v)$, then $f$ is not an edge of $P(u,v)$, since $e$ and $f$ are adjacent at $u_2$, and $p \geq 4$. In this case, select $e$ as the first edge of the ordering
Next, select in any order all those edges of $G$ not previously selected which are incident with $u_2$, excluding $u_2u_3$ and $f$. Follow these edges in $s$ with $u_2u_3$. Continue this process at each successive vertex of $P(u,v)$; that is, at $u_i$, $i = 3, 4, \ldots, p-1$, select in any order all those edges of $G$ which are incident with $u_i$, excluding $u_iu_{i+1}$, and follow these in $s$ by the edge $u_iu_{i+1}$. The last such edge selected is the edge $u_{p-1}u_p$. The ordering $s$ may be completed by the edge $uv = u_1u_p$, if it is an edge of $G$, and finally the edge $f$, since it is incident with $u_p = v$ and is the only edge of $G$ not previously selected. The ordering $s$ given above satisfies the conditions for an $e,f$-hamiltonian-path ordering, so that in this case, as noted, Theorem 4-1 implies that $L(G)$ is hamiltonian-connected.

Suppose that $e$ is not the first edge of $P(u,v)$. In this case, the procedure used above can be modified to produce the desired ordering $s$. Select $e$ as the first edge of $s$. Then select in any order all those edges of $G$ which are incident with $u_1$, excluding $u_1u_2$. Necessarily, if $u_1u_p$ is an edge of $G$ it will then be selected. Follow these edges in $s$ with the edge $u_1u_2$. As before, this process may be continued at each successive vertex $u_i$, $i = 2, 3, \ldots, p-1$. That is, at $u_i$, select in any order all those edges of $G$ not
previously selected which are incident with $u_1$, excluding $u_1u_{i+1}$, and $f$, if $f$ is incident with $u_i$, followed by the edge $u_1u_{i+1}$. The last such edge selected is the edge $u_{p-1}u_p$. If the edge $f$ is the edge $u_{p-1}u_p$, then the ordering $s$ is complete. If not, then $f$ is the only edge of $G$ not previously selected and since it is incident with $u_p$, it can be chosen last in the ordering $s$. Once again, $s$ has the necessary properties of an $e,f$-hamiltonian-path ordering. This completes the proof of this case.

Case ii. The edges $e$ and $f$ are not adjacent in $G$.

Suppose that $e$ and $f$ are nonadjacent edges and there exists a vertex $u$ incident with $e$ and a vertex $v$ incident with $f$, so that $uv$ is not an edge of $G$. Again, let $P(u,v)$ be a $u,v$-hamiltonian path, where

$$P(u,v): u = u_1, u_2, \ldots, u_p = v.$$

Then an analogous procedure to that of Case i may be applied to obtain the ordering $s$. That is, first select the edge $e$ in the ordering $s$. If $e \neq u_1u_2$, then select $u_1u_2$ as the second edge of $s$. Next, select in any order all those edges of $G$ not previously selected which are incident with $u_2$, excluding $u_2u_3$, and $f$ if it is incident with $u_2$, followed by the edge $u_2u_3$. Then at each successive vertex $u_i$, $i = 3, 4, \ldots, p-1$, 

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select in any order all those edges of $G$ which are
incident with $u_i$, and not previously selected,
excluding $u_iu_{i+1}$ and $f$, if $f$ is incident with $u_i$,
followed by $u_iu_{i+1}$. The last such edge selected is
$u_pu_p$. If $f = u_pu_p$, then every edge of $G$ has
been selected and the ordering $s$ is complete. Otherwise,
the edge $f$ may be selected as the last edge of $s$.

Now if the vertices $u$ and $v$ cannot be selected
so that $u$ and $v$ are incident, respectively, with $e$
and $f$ and $uv$ is not an edge of $G$, then the vertices
incident with the edges $e$ and $f$ induce a subgraph of
$G$ which is isomorphic with $K_4$. In this case, let $u$
and $v$ be chosen arbitrarily as vertices incident,
respectively, with the edges $e$ and $f$, and let $P(u,v)$
be a $u,v$-hamiltonian path, where

$$P(u,v): u = u_1, u_2, \ldots, u_p = v.$$  

If the first edge of $P(u,v)$ is the edge $e$, then
the ordering $s$ can be constructed by selecting first
the edge $e$, followed by the edges $uv$ and $vu_2$, in
that order. These edges exist since $e = u_1u_2$ and $f$ lie
in a complete subgraph of $G$ of order at least four. Then
beginning with $u_2$, and continuing successively for each
$u_i$, $i = 3, 4, \ldots, p-1$, we can apply the process used
in the first part of Case ii to complete the ordering $s$.

Now if the edge $e$ is not the first edge of $P(u,v)$,
then the ordering $s$ can be constructed by selecting first the edge $e$, followed by the edges $uv$ and $u_1u_2$, in that order, and the procedure used in the first part of Case ii continued successively at each vertex $u_i$, $i = 2, 3, \ldots, p-1$.

Since each method of constructing the ordering $s$ of the edges of $G$ in Case ii gives an $e,f$-hamiltonian-path ordering, Case ii and the theorem are complete.

\[ \square \]

Before stating the corollary to Theorem 4-2, we make a few observations. Suppose the graph $G$ is the graph $P_{n+1}$, where $n$ is some nonnegative integer. Then $L(G) = P_n$ and $L^2(G) = P_{n-1}$; in fact, $L^n(G) = P_{(n+1)-n} = P_1$. That is, the graph $G = P_{n+1}$ has the property that $L^n(G)$ is trivial. Moreover, Whitney [22] proved that if $G_1$ and $G_2$ are two nontrivial connected graphs which are neither $K_3$ nor $K(1,3)$, then $L(G_1) = L(G_2)$ if and only if $G_1 = G_2$. Thus, if $L^k(G) = P_{n-k+1}$, where $0 \leq k \leq n$, then $G = P_{n+1}$ is the unique graph such that $L^n(G)$ is trivial.

**Corollary 4-2-1.** Let $G$ be any graph which is not a path such that $L^k(G)$ is hamiltonian-connected for some nonnegative integer $k$. Then $L^n(G)$ is hamiltonian-connected for every integer $n \geq k$.
This corollary is an immediate consequence of the above argument and Theorem 4-2.

Let \( G \) be any connected graph with \( \delta(G) \geq 3 \). Then each vertex \( w \) of \( G \) is incident with precisely \( \deg_G w \) edges of \( G \). These edges of \( G \) correspond to \( \deg_G w \) mutually adjacent vertices in \( L(G) \); that is, a complete subgraph \( K(w) \) of order \( \deg_G w \) in \( L(G) \).

Since complete subgraphs of order at least three are hamiltonian, it follows that \( K(w) \) contains a spanning cycle \( C(w) \). The continuing importance of these observations makes it convenient to define a certain class of subgraphs of \( L(G) \).

Let \( G \) be any connected graph with \( \delta(G) \geq 3 \). For each vertex \( w \) of \( G \), let \( C(w) \) be a spanning cycle of \( K(w) \) in \( L(G) \). Then the subgraph, denoted by \( F(G) \), of \( L(G) \), defined by

\[
V(F(G)) = \bigcup_{w \in V(G)} V(C(w)), \quad \text{and} \quad E(F(G)) = \bigcup_{w \in V(G)} E(C(w))
\]

is called a frame of \( L(G) \).

**Theorem 4-3.** Let \( G \) be a connected graph with \( \delta(G) \geq 3 \), and let \( F(G) \) be a frame of \( L(G) \). Then \( F(G) \) is a spanning, 4-regular, eulerian subgraph of \( L(G) \).
Proof. Every edge of \( G \) is incident with some vertex \( w \) of \( G \), so that the corresponding vertex of \( L(G) \) is on a cycle \( C(w) \) which spans the complete subgraph \( K(w) \) in \( L(G) \), and hence is a vertex of \( F(G) \). Therefore, \( F(G) \) is a spanning subgraph of \( L(G) \).

Furthermore, any two cycles \( C(w_1) \) and \( C(w_2) \) of \( F(G) \) meet in at most one vertex, namely, the vertex of \( L(G) \) corresponding to the edge \( w_1w_2 \) of \( G \), if it exists. Thus, no two cycles \( C(w_1) \) and \( C(w_2) \) of \( F(G) \) share an edge of \( F(G) \). Moreover, every vertex \( u \) of \( L(G) \) corresponds to an edge \( w_1w_2 \) of \( G \), so that the cycles \( C(w_1) \) and \( C(w_2) \), and only these cycles, meet in the vertex \( u \). Therefore, every vertex of \( F(G) \) has even degree. In fact, \( F(G) \) is 4-regular since each of the two cycles meeting at a vertex contribute two to the degree of the vertex in \( F(G) \).

Finally, if \( v_1 \) and \( v_2 \) are vertices of \( F(G) \), then for some vertices \( w_1 \) and \( w_2 \) of \( G \) (not necessarily distinct) \( v_1 \) is on the cycle \( C(w_1) \) and \( v_2 \) is on the cycle \( C(w_2) \) in \( F(G) \). Since \( G \) is connected, \( G \) contains a \( w_1, w_2 \)-path \( P(w_1, w_2) \). However, the subgraph \( H \) of \( F(G) \) defined by

\[
V(H) = \bigcup V(C(w)) \quad \text{and} \quad E(H) = \bigcup E(C(w)) ,
\]

where both unions are taken over all \( w \in V(P(w_1, w_2)) \), is a connected subgraph of \( F(G) \) containing a \( v_1, v_2 \)-path.
Therefore, \( F(G) \) contains a \( v_1, v_2 \) - path and is connected. This also implies \( F(G) \) is eulerian, since it was previously observed that each vertex of \( F(G) \) has even degree (cf. Theorem 2-B).

\[ \square \]

We remark that the definition of a frame permits flexibility in its structure, in the sense that for each vertex \( w \) of \( G \), the corresponding subgraph \( K(w) \) of \( L(G) \) contains more than one spanning cycle if \( \deg_G w \geq 4 \). Thus the choice of \( C(w) \) is not necessarily unique. In later arguments it will be convenient to specify certain cycles \( C(w) \) of the complete subgraph \( K(w) \) to be subgraphs of a given frame \( F(G) \). To this end, we state the following two lemmas, whose validity is an immediate consequence of the remark above.

**Lemma 4-1.** Let \( G \) be a connected graph with \( \delta(G) \geq 3 \), and let \( C'(w_1), \ldots, C'(w_k) \) be spanning cycles, respectively, of the complete subgraphs \( K(w_1), \ldots, K(w_k) \) of \( L(G) \). Then there exists a frame \( F(G) \) of \( L(G) \) with \( C'(w_1), \ldots, C'(w_k) \) as subgraphs of \( F(G) \).

**Lemma 4-2.** Let \( G \) be a connected graph with \( \delta(G) \geq 3 \), and let \( F(G) \) be a frame of \( L(G) \). Then for any edge \( x \) of \( F(G) \) there exists an eulerian circuit of \( F(G) \) whose first edge is \( x \).
Theorem 4-4, which follows, is somewhat technical in nature and is vital to the proof of Theorem 4-5.

**Theorem 4-4.** Let $G$ be a connected graph such that $\delta(G) \geq 4$. Then for any pair of vertices $u$ and $v$ of $L(G)$, there exists in $L(G)$ a spanning subgraph $H(u,v)$ with $\delta(H(u,v)) \geq 4$, containing a $u,v$-eulerian trail.

**Proof.** Let $u$ and $v$ be a pair of distinct vertices of $L(G)$, and let $P(u,v)$ be a shortest $u,v$-path in $L(G)$, say

$$P(u,v): u = v_1, v_2, \ldots, v_n = v.$$ 

*Claim:* The path $P(u,v)$ has no more than two vertices in common with any complete subgraph $K(w)$.

For $n = 2$, the result is clear, hence assume $n \geq 3$. Assume to the contrary that three integers $i$, $j$, and $k$ exist, where $1 \leq i < j < k \leq n$, such that $v_i$, $v_j$, and $v_k$ are vertices in the same complete subgraph $K(w)$. Then $v_i v_k$ is an edge of $K(w)$, and hence

$$P'(u,v): u = v_1, \ldots, v_i, v_k, \ldots, v_n = v$$

is a $u,v$-path in $L(G)$ whose length is less than that of $P(u,v)$, which is a contradiction. This verifies the claim.

Now since no more than two vertices of $P(u,v)$ lie in any complete subgraph $K(w)$ of $L(G)$, no more than
one edge of $P(u,v)$ lies in any complete subgraph $K(w)$ of $L(G)$, so that by Lemma 4-1, there exists a frame $F(G)$ of $G$ which is edge disjoint from $P(u,v)$. This follows since there exist spanning cycles of any complete graph which avoid a given edge, provided the order of the complete graph is at least four, as is the case here for each $K(w)$.

Then $F(G)$ and $P(u,v)$ together determine a subgraph $H(u,v)$ where

$$V(H(u,v)) = V(F(G)) \cup V(P(u,v)),$$
$$E(H(u,v)) = E(F(G)) \cup E(P(u,v)).$$

The subgraph $H(u,v)$ is connected and spans $L(G)$ since $F(G)$ is connected and spans $L(G)$. Any vertex $w'$ of $H(u,v)$ has degree four if $w'$ is not a vertex of $P(u,v)$. If $w' = u$ or $w' = v$, then $\deg_H w' = 5$; otherwise, $\deg_H w' = 6$. Thus, the subgraph $H(u,v)$ has precisely two vertices of odd degree, namely $u$ and $v$.

Therefore $H(u,v)$ is a non-trivial connected graph with exactly two vertices of odd degree, so that $H(u,v)$ contains an eulerian $u,v$-trail (cf. Theorem 2-C) and is the desired subgraph of $L(G)$. Note that since $F(G)$ is $4$-regular, $\delta(H(u,v)) \geq 4$.

□
Theorem 4-5 contains the heart of the relationship between hamiltonian-connectedness and iterated line graphs which is summarized in Theorem 4-6.

**Theorem 4-5.** Let $G$ be a connected graph such that $\delta(G) \geq 4$. Then $L^2(G)$ is hamiltonian-connected.

**Proof.** By Theorem 4-1, $L^2(G)$ is hamiltonian-connected if $L(G)$ is hamiltonian-path ordered. Thus it is sufficient to show that for any pair of edges $e$ and $f$ of $L(G)$, there exists an $e,f$-hamiltonian-path ordering $s$ of the edges of $L(G)$.

We claim that if $u$ and $v$ are distinct vertices of $L(G)$ which are incident respectively with $e$ and $f$, and there exists a spanning $u,v$-trail, denoted $T(u,v)$, in $L(G)$, such that the first edge of $T(u,v)$ is $e$, the last edge of $T(u,v)$ is $f$, and both $u$ and $v$ are encountered at least once in the interior of $T(u,v)$, then there exists an $e,f$-hamiltonian-path ordering $s$ of the edges of $L(G)$. To see that this implication is true, suppose such a trail $T(u,v)$ exists and has its vertices and edges labeled consecutively as $T(u,v)$ is traversed from $u$ to $v$. That is,

$$T(u,v): u_1, x_1, u_2, x_2, \ldots, u_k, x_k, u_{k+1},$$

where $u_1 = u$, the edge $x_1 = e$, the edge $x_k = f$, and $u_{k+1} = v$, with the vertices $u_i$ not necessarily
distinct.

The ordering \( s \) of the edges of \( L(G) \) is begun by first selecting the edge \( e \). Next, all those edges of \( L(G) \) which are incident with \( u_2 \) and are not edges of \( T(u,v) \) are selected and placed in the ordering \( s \), immediately after the edge \( e \) and in any order desired, followed by the edge \( x_2 \). The ordering \( s \) is extended by following this procedure at each consecutively labeled vertex \( u_i \) of \( T(u,v) \), where \( 3 \leq i \leq k \). That is, at each successive \( u_i \) the ordering \( s \) is continued by placing in it, in any order, all those edges of \( L(G) \) which are incident with \( u_i \) and are neither edges of \( T(u,v) \) nor edges previously selected, followed by the edge \( x_i \). Observe that since the vertices \( u \) and \( v \) must be encountered at least once in the interior of \( T(u,v) \), all those edges of \( L(G) \) incident with either \( u \) or \( v \) have been ordered. In addition, \( s \) contains all of the edges of \( L(G) \), since \( T(u,v) \) spans \( L(G) \). Furthermore, the last edge placed in \( s \) is \( x_k \), which by assumption is \( f \), so that \( s \) is an \( e,f \)-Hamiltonian-path ordering, since \( s \) has the property that each two consecutive edges are adjacent.

Therefore, the proof will be complete if it can be shown that for any pair \( e,f \) of edges of \( L(G) \), and distinct vertices \( u \) and \( v \), incident respectively with \( e \) and \( f \), that a spanning \( u,v \)-trail with the
aforementioned properties exists.

Hence, let e and f be any pair of edges in L(G), and let u and v be distinct vertices adjacent, respectively, with e and f. By hypothesis, G is connected and \( \delta(G) \geq 4 \); hence by Theorem 4-4, there exists in L(G) a spanning subgraph, denoted H(u,v), with a u,v-eulerian trail. Recall that H(u,v) was formed from a frame F(G) and a shortest u,v-path P(u,v) which were edge disjoint. Further recall that since \( \delta(H(u,v)) \geq 4 \), both u and v must be encountered on the interior of any u,v-eulerian trail T(u,v) of H(u,v). We now proceed by cases.

Case i. The edges e and f are adjacent in L(G).

In this case, let u be the common vertex of e and f, and let v be the remaining vertex incident with f. Note that P(u,v) has f as its only edge. Hence, if T(u,v) is a u,v-eulerian trail of H(u,v), it can be assumed to consist of a u,u-eulerian circuit of F(G) followed by the edge f and the vertex v, so that its last edge is f. Furthermore, the edge e can be assumed to be an edge of the frame F(G), by Lemma 4-1. This follows since if e and f are in different complete subgraphs K(w_1) and K(w_2), for \( w_1 \neq w_2 \), respectively, then K(w_1) contains no edge of P(u,v) and thus by Lemma 4-1, the frame F(G) can be chosen with a cycle
C(w₁) of K(w₁) containing the edge e. Furthermore, if e and f lie in a common K(w), then it is always possible to find a spanning cycle C(w), and hence a frame F(G), which does not contain the edge f, but does contain the edge e, since the order of K(w) is at least four. Moreover, by Lemma 4-2, there exists a u,u-eulerian circuit of F(G) whose first edge is the edge e. Therefore in the case that e and f are adjacent, there exists a T(u,v) with the desired properties.

Case ii. The edges e and f are not adjacent but lie in the same complete subgraph K(w) determined by the mutual adjacencies of edges of G at the same vertex w of G.

Now for any non-adjacent pair of edges of a complete graph of order four or more, there exists a hamiltonian cycle containing these edges. Hence let C₁(w) be a hamiltonian cycle of K(w) containing e and f, and let F(G) be a frame using the cycle C₁(w) in the complete subgraph K(w). Moreover, let u be a vertex of C₁(w) which is incident with e and let v be chosen so that the u,v-path of C₁(w) containing the edge e does not contain the edge f, i.e., in C₁(w) the vertex v is encountered prior to f. (See Figure 4.1)
In this case, $P(u,v)$ has uv as its only edge. Since $e$ and $f$ are not adjacent, the edge uv is not an edge of the cycle $C_1(w)$, and therefore is not an edge of $F(G)$.

$$C_1(w) :$$

```
 u -- e -- v
 |    |
 f -- v
```

Figure 4.1

Then as in the proof of Theorem 4-4, $F(G)$ together with the edge uv constitute the subgraph $H(u,v)$.

Consider the trail $T'(u,v)$ of $H(u,v)$ which consists of the $u,v$-path along $C_1(w)$ whose first edge is $e$, followed by the edge $vu$, and followed by the $u,v$-path along $C_1(w)$ whose last edge is $f$. That is,

$$T'(u,v) : v, e, ..., v, vu, u, ..., f, v.$$ 

If the edges of $T'(u,v)$ are removed from $H(u,v)$, the remaining portion of $H(u,v)$ consists of one or more components having the property that each is eulerian and contains an eulerian circuit beginning and ending at a vertex of $T'(u,v)$. Hence it is possible to expand $T'(u,v)$ to a $u,v$-eulerian trail $T(u,v)$ of $H(u,v)$.
merely by inserting the aforementioned eulerian circuits in place of an interior vertex of \( T'(u,v) \) which is common to \( T'(u,v) \) and one of these circuits. Therefore in this case also, a \( u,v \)-eulerian trail \( T(u,v) \) exists whose first edge is \( e \) and last edge is \( f \).

**Case iii.** The edges \( e \) and \( f \) are not adjacent and are not in the same complete subgraph \( K(w) \) determined by the mutual adjacencies of edges of \( G \) at the same vertex \( w \) of \( G \).

Suppose \( e \) is an edge of the complete subgraph \( K(w_1) \) and \( f \) is an edge of the complete subgraph \( K(w_2) \), where \( w_1 \neq w_2 \). Choose \( u \) and \( v \) to be vertices incident, respectively, with \( e \) and \( f \) such that the length of a shortest \( u,v \)-path is minimized, and consider this path \( P(u,v) \). If either \( e \) or \( f \) is an edge of \( P(u,v) \), then \( P(u,v) - u \) or \( P(u,v) - v \) is a path beginning at a vertex incident with \( e \) and ending at a vertex incident with \( f \) which is shorter than \( P(u,v) \). This is a contradiction of the choice of \( u \) and \( v \).

Moreover, as in the proof of Theorem 4-4, by Lemma 4-1, there exists a frame \( F(G) \) which contains \( e \) and \( f \) in its edge set and is edge disjoint from \( P(u,v) \). This \( F(G) \) together with \( P(u,v) \) forms the desired subgraph \( H(u,v) \) of Theorem 4-4.

Now consider the trail \( T'(u,v) \) of \( H(u,v) \) which
consists of $C(w_1)$ traversed from $u$ so that the first edge used is $e$, followed by $P(u,v)$, and followed by $C(w_2)$ traversed from $v$ so that $f$ is the last edge used. That is, we have

$$T'(u,v): u, e, ..., u, ..., v, ..., f, v.$$ 

\[ C(w_1) \quad P(u,v) \quad C(w_2) \]

In a manner completely analogous to Case ii, if the edges of $T'(u,v)$ are removed from $H(u,v)$, the remaining portion of $H(u,v)$ consists of one or more components having the property that each is eulerian and contains an eulerian circuit beginning and ending at a vertex of $T'(u,v)$. Hence $T'(u,v)$ can be expanded to a $u,v$-eulerian trail $T(u,v)$ of $H(u,v)$ whose first edge is $e$ and last edge is $f$. This completes Case iii and the proof of the theorem.

We now define the key concept of this chapter. The hamiltonian-connected index of a graph $G$ is defined as the smallest nonnegative integer $n$ such that $L^n(G)$ is hamiltonian-connected and is denoted by $hc(G)$. The next theorem summarizes the results of this section.

\textbf{Theorem 4-6.} Let $G$ be a connected graph such that $\delta(G) \geq 3$. Then $hc(G) \leq 3$. 

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Proof. Since $G$ is connected and $\delta(G) \geq 3$, the line graph $L(G)$ is connected and $\delta(L(G)) \geq 4$. Hence we may apply Theorem 4-5 to $L(G)$ to conclude that $L^2(L(G))$, or $L^3(G)$, is hamiltonian-connected. Thus $hc(G) \leq 3$.

The bound given for $hc(G)$ in Theorem 4-6 is in general "best possible" for connected graphs $G$ with $\delta(G) \geq 3$, for consider the graph $G$ and its line graph $L(G)$ shown in Figure 4.2.

![Figure 4.2](image-url)
The graph \( L(G) \) is not hamiltonian-path ordered since no \( e,f \)-hamiltonian-path ordering exists for the edges \( e \) and \( f \) as indicated. Suppose to the contrary that an \( e,f \)-hamiltonian-path ordering \( s \) exists. For convenience, let the components of \( L(G) - \{ e, f \} \) be denoted by \( G_1 \) and \( G_2 \), where \( G_1 \) is the component containing the edge \( x_1 \). Since \( s \) exists, its first edge is \( e \). The next edge of \( s \) must be an element of \( \{ x_1, x_2, x_3, x_4, x_5 \} \). If this edge is \( x_1, x_2 \), or \( x_3 \), then \( s \) continues with edges of \( G_1 \) until \( f \) is encountered, which precedes any edge of \( G_2 \) in \( s \); but this is a contradiction. A similar contradiction is reached if one of the edges \( \{ x_4, x_5 \} \) is the second edge of \( s \). However, if \( L(G) \) is not hamiltonian-path ordered, then by Theorem 4-1, the graph \( L^2(G) \) is not hamiltonian-connected. Moreover, since \( \delta(L(G)) = 4 \) and \( L(G) \) is connected, \( L^3(G) \) is hamiltonian-connected by Theorem 4-5. Therefore \( hc(G) = 3 \), and Theorem 4-6 cannot be improved.

The fact that Theorem 4-6 is, in general, best possible gives rise to the question of whether connected graphs \( G \) with \( \delta(G) \geq 3 \) can actually be classified in such a manner as to allow either the exact value of \( hc(G) \) to be determined or an improved bound on \( hc(G) \) to be determined. This question will motivate the discussion contained in Section 4.3.
In addition, it becomes evident that nothing has been said pertaining to $hc(G)$ for connected graphs $G$ such that $\delta(G) < 3$. This topic will be taken up in the next section.

Section 4.2

The Hamiltonian-connected Index of Graphs With Minimum Degree Less Than Three

In Section 4.1, a bound was given for $hc(G)$ for those connected graphs $G$ with minimum degree at least three. We wish now to consider those connected graphs whose minimum degree is less than three.

If the graph $G$ is the path $P_n$ then from the discussion preceding Corollary 4-2-1, $L^{n-2}(P_n) = P_{n-(n-2)} = P_2$ which is hamiltonian-connected. Furthermore, $L^{n-3}(P_n) = P_3$ which is not hamiltonian-connected. Thus we have $hc(P_n) = n - 2$.

On the other hand, if the graph $G$ is the cycle $C_p$ of order $p$, then $L^n(C_p) = C_p$, for every integer $n \geq 0$. Thus for those cycles of order $p \geq 4$, the hamiltonian-connected index does not exist. Of course, $hc(K_3) = 0$, since $K_3$ is hamiltonian-connected. Also, if $G = K(1,3)$, then $hc(G) = 1$, since $K(1,3)$ is not hamiltonian-connected, but $L(K(1,3)) = K_3$ which is hamiltonian-connected.

The graphs just discussed, namely paths, cycles, and
\(K(1,3)\), share the property that they are the only graphs for which no integer \(n\) exists such that \(L^n(G)\) has minimum degree at least three. Furthermore, for any graph \(G\) of order \(p \geq 4\) it is known that \(\delta(G) \geq 3\) is a necessary condition for hamiltonian-connectedness. Therefore, since all those connected graphs \(G\) having order at least four and \(\delta(G) < 3\) cannot be hamiltonian-connected, it is necessary to determine the smallest integer \(n\) such that \(\delta(L^n(G)) \geq 3\), which will clearly be a lower bound for \(hc(G)\). The establishment of this lower bound is the object of Theorem 4-7; however, a number of definitions must be given first.

For any connected graph \(G\) which is not a cycle, a path, or \(K(1,3)\), let \(\delta_3(G)\) be the smallest nonnegative integer \(n\) such that \(\delta(L^n(G)) \geq 3\). A poly-vertex of a graph \(G\) is a vertex having degree at least three. A \(k\)-end path of a graph \(G\) is a path joining a poly-vertex of degree \(k\) with a vertex of degree one such that all other vertices of the path have degree two in \(G\). A poly-path of a graph \(G\) is a path joining two poly-vertices, any other vertex of which has degree two in \(G\). An end-cycle of a graph \(G\) is an end-block which is a cycle.

For convenience, the following notations will be used:

(i) If \(P\) is a path in a graph \(G\), denote the length of \(P\) by \(l(P)\).
(ii) Let $P_k$ denote a $k$-end path of a graph $G$, let $P_k^*(G)$, or $P_k^*$, denote the set of all $k$-end paths of $G$, and let $P^*(G) = \bigcup_{k \geq 3} P_k^*(G)$ be the set of all $k$-end paths of $G$.

(iii) Let $Q_{k,\ell}$ denote a poly-path of $G$ whose poly-vertices have degree $k$ and degree $\ell$ in $G$, where $3 \leq k \leq \ell$, and let $Q_{k,\ell}^*(G)$ and $Q^*(G) = \bigcup_{3 \leq k \leq \ell} Q_{k,\ell}^*(G)$ be defined analogously to $P_k^*(G)$ and $P^*(G)$.

(iv) Let $R_k$ denote an end-cycle of $G$ with $k \geq 3$ vertices, and analogously define $R_k^*(G)$ and $R^*(G) = \bigcup_{k \geq 3} R_k^*(G)$.

(v) Let

\[
\begin{align*}
    n_1 &= \begin{cases} 
        0, & \text{if } P_3^*(G) = \emptyset; \\
        \max \{ \ell(P_3), \text{ where } P_3 \in P_3^*(G) \} & \text{otherwise},
    \end{cases} \\
    n_2 &= \begin{cases} 
        0, & \text{if } [P^*(G) \setminus P_3^*(G) = \emptyset; \\
        \max \{ \ell(P), \text{ where } P \in [P^*(G) \setminus P_3^*(G)] \} & \text{otherwise},
    \end{cases} \\
    n_3 &= \begin{cases} 
        0, & \text{if } Q^*(G) = \emptyset; \\
        \max \{ \ell(Q), \text{ where } Q \in Q^*(G) \} & \text{otherwise},
    \end{cases} \\
    n_4 &= \begin{cases} 
        0, & \text{if } R^*(G) = \emptyset; \\
        \max \{ |V(R)|, \text{ where } R \in R^*(G) \} & \text{otherwise}.
    \end{cases}
\end{align*}
\]
We are now prepared for our next result.

**Theorem 4-7.** Let $G$ be a connected graph which is not a cycle, a path, or $K(1,3)$. Then

$$\delta_3(G) = \begin{cases} 0, & \text{if } \delta(G) \geq 3; \\ \max \{ n_1 + 1, n_2, n_3 - 1, n_4 - 1 \}, & \text{otherwise}. \end{cases}$$

**Proof.** If $\delta(G) \geq 3$, then by definition, $\delta_3(G) = 0$. Hence, assume $\delta(G) < 3$. Then $G$ has vertices of degree one or two, so that $P^*(G) \cup Q^*(G) \cup R^*(G)$ is nonempty, and $\delta_3(G) \geq 1$.

If $P_k$ is a $k$-end path of $G$, where $k \geq 3$, of length $m > 1$, then there is associated with $P_k$ in $L(G)$, a $k$-end path of length $m - 1$; and if $P_k$ is a $k$-end path of $G$ of length one, then there is associated with $P_k$ in $L(G)$ a vertex of degree $k - 1$.

If $Q_{k,\ell}$ is a $k,\ell$-poly-path in $G$, where $3 \leq k \leq \ell$, of length $m > 1$, then there is associated with $Q_{k,\ell}$ in $L(G)$, a $k,\ell$-poly-path of length $m - 1$. If $R_k$ is an end-cycle of length $k \geq 3$ in $G$, then there is associated with $R_k$ in $L(G)$ a poly-path of length $k - 1$.

Let $M = \max \{ n_1 + 1, n_2, n_3 - 1, n_4 - 1 \}$. We now show $\delta_3(G) \geq M$.

Suppose $P_3^*(G) \neq \emptyset$. Then $G$ contains a 3-end path $P_3$. Assume $P_3$ is chosen so that $\ell(P_3) = n_1$; that is, $\ell(P_3)$ is maximum among all 3-end paths of $G$. 

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Then, \( L^{n_1}(G) \) has a vertex of degree two, so that \( \delta_3(G) > n_1 \). That is, \( \delta_3(G) \geq n_1 + 1 \).

Suppose \([P^*(G) \setminus P_3^*(G)] \neq \emptyset\). Then \( G \) contains a \( k \)-end path \( P \), for some \( k \geq 4 \). Assume \( P \) is chosen so that \( \ell(P) = n_2 \) among all \( k \)-end paths of \( G \), where \( k \geq 4 \), and independent of \( k \). Then \( L^{n_2-1}(G) \) has a \( k \)-end path of length one, so that \( \delta_3(G) > n_2 - 1 \). That is, \( \delta_3(G) \geq n_2 \).

Similarly, if \( Q^*(G) \neq \emptyset \), then \( G \) contains a \( k,l \)-poly-path, where \( 3 \leq k \leq l \). Let \( Q_{k,l} \) be a member of \( Q^*(G) \) such that \( \ell(Q_{k,l}) = n_3 \). If \( n_3 = 1 \), then every poly-path of \( G \) is an edge joining polyvertices of \( G \), so that \( \delta_3(G) \geq 0 = n_3 - 1 \). If \( n_3 > 1 \), then \( L^{n_3-2}(G) \) contains a \( k,l \)-poly-path of length two. Thus \( \delta_3(G) > n_3 - 2 \); that is, \( \delta_3(G) \geq n_3 - 1 \).

Finally, if \( R^*(G) \neq \emptyset \), then \( G \) contains an end-cycle of \( G \). Let \( R \) be an end-cycle of \( G \) such that \( |V(R)| = n_4 \). Then in \( L(G) \) there is associated with \( R \), a poly-path \( Q \) of length \( n_4 - 1 \), where of course \( n_4 - 1 \geq 2 \). Then by the preceding remark, \( \delta_3(L(G)) \geq (n_4 - 1) - 1 = n_4 - 2 \), so that \( \delta_3(G) \geq n_4 - 1 \).

Since \( \delta_3(G) \) is at least as great as \( n_1 + 1, n_2, n_3 - 1, \) and \( n_4 - 1 \), we have \( \delta_3(G) \geq M \).

Now suppose that \( \delta_3(G) > M \). Then \( L^M(G) \) contains
a vertex of degree one or two. However, this is clearly impossible since any end-path, poly-path, or end-cycle of $L^M(G)$ must be generated by an end-path, poly-path, or end-cycle of $G$. But this would contradict the choice of $M$, so the proof is complete.

Combining the results of Theorems 4-6 and 4-7, we obtain a corollary.

**Corollary 4-7-1.** If $G$ is any connected graph which is neither a cycle, a path, nor $K(1,3)$, then $hc(G)$ is finite and the following holds:

$$\delta_3(G) \leq hc(G) \leq \delta_3(G) + 3.$$

**Section 4.3**

The Determination Of The Hamiltonian-connected Index For Certain Graphs $G$

In the preceding two sections we have determined the hamiltonian-connected index for paths, $K_3$, and $K(1,3)$. In general, however, for connected graphs $G$ which are not paths, cycles, or $K(1,3)$ it was shown only that $hc(G)$ satisfied the inequality $\delta_3(G) \leq hc(G) \leq \delta_3(G) + 3$. The purpose of this section is to determine the exact value of $hc(G)$ for graphs $G$ belonging to several...
additional classes.

To this end, we recall a concept from Chapter II. The edge-connectivity of a graph $G$, denoted $\kappa_1(G)$, is the minimum number of edges whose removal from $G$ results in a disconnected or trivial graph.

**Theorem 4-8.** If $G$ is a graph with $\delta(G) \geq 3$ and $\kappa_1(G) = 2$, then $hc(G) = 2$.

**Proof.** Let $G$ be a graph with $\delta(G) \geq 3$ and $\kappa_1(G) = 2$, and let $\{e_1, e_2\}$ be a cut-pair of edges of $G$. Let the components of $G - \{e_1, e_2\}$ be denoted by $G_1$ and $G_2$. If $L(G)$ is hamiltonian-connected, there exists an $e_1, e_2$-hamiltonian-path ordering $s$ of the edges of $G$ by Theorem 4-1. Necessarily, the first edge of $s$ is $e_1$. Without loss of generality, we may assume the second edge of $s$ is an edge of $G_1$ adjacent with $e_1$. Now any edge of $G_1$ can be followed in $s$ only by other edges of $G_1$, or by the edge $e_2$. Furthermore, if $x$ is the last edge of $G_1$ appearing in $s$, then $e_2$ must appear next in $s$. Thus, $G_2$ must be trivial, or $e_2$ does not appear last in $s$. However, if $G_2$ is trivial, then $G$ contains a vertex of degree two, which is impossible. Hence $L(G)$ is not hamiltonian-connected, and therefore $hc(G) \geq 2$.

The proof will be complete if it can be shown that $hc(G) \leq 2$. It suffices to prove that $L(G)$ is
hamiltonian-path ordered. Now let e and f be arbitrary edges in L(G).

Case i. The edges e and f are adjacent in L(G).

Let F(G) be a frame of L(G) such that e and f are edges of F(G). This can be done by Lemma 4-1. Furthermore, let u be the vertex incident with e and not with f, and let v be the vertex incident with f and not e. Now consider the subgraph H = F(G) - {e,f}. If H is connected, then H is a spanning subgraph of L(G) with a u,v-eulerian trail, and it is possible to find an e,f-hamiltonian-path ordering s of the edges of L(G). This follows by a procedure similar to that of Theorem 4-5; namely, if T(u,v) is a u,v-eulerian trail of the spanning subgraph H, where the vertices and edges of T(u,v) have been labeled (the vertices may not be distinct) as T(u,v) is traversed from u to v,

T(u,v): u = u_1, x_1, u_2, x_2, ..., x_k, u_{k+1} = v,

then an ordering of the edges of L(G) can be accomplished. This can be done by first placing in s all those edges of L(G) which are incident with u = u_1, and are neither edges of T(u,v) nor the edge f, in such a manner that the edge e is taken first followed by the edge x_1. Then, successively, for each vertex u_i, where 2 ≤ i ≤ k, we place in s all those edges of L(G), not previously
chosen, which are neither edges of $T(u,v)$ nor the edge $f$, and are incident with $u_1$, followed by the edge $x_1$, and complete $s$ with the edge $f$. Hence it remains to be shown in this case that $H$ is connected.

Assume to the contrary that $H$ is disconnected. Then $z$, the vertex common to $e$ and $f$, must be a cut-vertex of $F(G)$. However, if $z$ is a cut-vertex of the frame $F(G)$, then we claim that $z$ is also a cut-vertex of $L(G)$. Suppose to the contrary that $z$ is a cut-vertex of the frame $F(G)$ which is not a cut-vertex of $L(G)$.

Note that each edge of $L(G)$ is either an edge of $F(G)$, or it is a chord of a cycle $C(w)$, for some vertex $w$ of $G$. Now let $H_1$ and $H_2$ denote the two components of $F(G) - z$, and let $v_1$ and $v_2$ be vertices of $H_1$ and $H_2$, respectively. Since $z$ is not a cut-vertex of $L(G)$, there exists a $v_1, v_2$-path in $L(G)$ which does not contain the vertex $z$. Now on this $v_1, v_2$-path there exists an edge $x = z_1z_2$ such that $z_1$ is a vertex of $H_1$ and $z_2$ is a vertex of $H_2$. Since $x$ is not an edge of $F(G) - z$, and hence not an edge of $F(G)$, the edge $x$ must be a chord of some cycle $C(w)$ of $F(G)$. But then there exists a $v_1, v_2$-path in $F(G)$ consisting of a $v_1, z_1$-path in $H_1$, a subpath of $C(w)$ joining $z_1$ and $z_2$ and not containing $z$, and a $z_2, v_2$-path of $H_2$. This, however,
contradicts the assumption that \( z \) was a cut-vertex of \( F(G) \). Therefore, if \( F(G) \) has a cut-vertex, then \( L(G) \) has a cut-vertex. However, the assumption that \( H \) is disconnected implies \( F(G) \) has a cut-vertex, so that then \( L(G) \) also has a cut-vertex. But this is impossible, since \( K_1(G) = 2 \), which implies that \( K(L(G)) = 2 \). Therefore \( H \) must be connected. This completes Case i.

Case ii. The edges \( e \) and \( f \) are not adjacent in \( L(G) \) but lie in the same complete subgraph \( K(w) \).

In this case, \( K(w) \) must have order at least four. Let \( x \) be an edge of \( K(w) \) which is adjacent with the edge \( e \) at a vertex \( u \) and adjacent with \( f \) at a vertex \( v \). Then there exists a spanning cycle \( C(w) \) of \( K(w) \) which contains the edge \( x \), but neither the edge \( e \) nor the edge \( f \); that is, since \( K(w) \) is a complete graph of order at least four, there exists a spanning cycle which contains any given edge \( x \), and fails to contain two nonadjacent edges, each of which is adjacent with \( x \). Then by Lemma 4-1, there exists a frame \( F(G) \) of \( L(G) \) which contains the cycle \( C(w) \).

Now consider the subgraph \( H = F(G) - x \). The subgraph \( H \) is a spanning subgraph of \( L(G) \), and if it is connected, then \( H \) has a \( u,v \)-eulerian trail and therefore there exists an \( e,f \)-hamiltonian-path ordering \( s \)
by the method described in Case i, since e and f are not edges of H. However, if H is disconnected, then x is a bridge of F(G). But this is impossible, since F(G) is eulerian. Therefore, H is connected and Case ii is complete.

Case iii. The edges e and f are not adjacent and do not lie in the same complete subgraph K(w).

Let e be an edge of the complete subgraph K(w₁), and let f be an edge of the complete subgraph K(w₂), where w₁ ≠ w₂.

Subcase iiiia. The complete subgraphs K(w₁) and K(w₂) have order at least four.

Let u and v be vertices incident, respectively, with the edges e and f, such that the length of a u,v-path is minimum, and let P(u,v) be such a path. By an argument contained in Theorem 4-5, the edges e and f do not belong to P(u,v). Moreover, recalling an argument of Theorem 4-4, we conclude that no more than one edge of P(u,v) may lie in a single complete subgraph K(w) for any vertex w of G. Then for each vertex w of G such that K(w) contains an edge x of P(u,v), there exists a spanning cycle C(w) containing the edge x. Furthermore, there exists a cycle C(w₁) which contains an edge of P(u,v) but fails to contain the edge e, if P(u,v) has an edge in common with K(w₁).

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Similarly, there exists a spanning cycle $C(w_2)$ of $K(w_2)$ which contains an edge of $P(u,v)$ but fails to contain the edge $f$ if such exists in $K(w_2)$. Then by Lemma 4-1, there exists a frame $F(G)$ of $L(G)$ using the above specified cycles. That is, there exists a frame $F(G)$ which does not contain the edges $e$ or $f$, but does contain every edge of the path $P(u,v)$.

Let $H = F(G) - E(P(u,v))$. Then $H$ is a spanning subgraph of $L(G)$ with precisely two vertices of odd degree, namely $u$ and $v$, so that if $H$ is connected, then $H$ contains a spanning $u,v$-eulerian trail of $L(G)$, from which it follows that there exists an $e,f$-hamiltonian-path ordering $s$, as in Case i, since $e$ and $f$ are not edges of $H$.

By Theorem 4-3, the frame $F(G)$ is connected, so that for any pair of vertices $v_1$ and $v_2$ of $F(G)$, there exists a $v_1,v_2$-path $P(v_1,v_2)$ in $F(G)$. Now if $P(v_1,v_2)$ is also a path of $H$, then $H$ is connected. Otherwise, let $x = u_1u_2$ be the first edge encountered on $P(v_1,v_2)$ which is also an edge of $P(u,v)$ and thus not an edge of $H$, and suppose this edge $x$ is an edge of $C(w)$, for some vertex $w$ of $G$, where $C(w)$ is the cycle used to span $K(w)$ in $F(G)$. Then $C(w) - x$ is a $u_1,u_2$-path using only edges of $H$, and hence $C(w) - x$, together with the $v_1,u_1$-subpath of $P(v_1,v_2)$, forms a $v_1,u_2$-path, all of whose
edges are edges of $H$. Thus, the path $P(v_1, v_2)$ can be replaced by a $v_1, v_2$-path $P'(v_1, v_2)$ which does not include the edge $x$. Now if $P'(v_1, v_2)$ is a path of $H$, then $H$ is connected. Otherwise, let $y$ be the first edge of $P'(v_1, v_2)$ which is not an edge of $H$. Necessarily, $y$ is the next edge of $P(v_1, v_2)$ encountered after the edge $x$ which is not an edge of $H$. In a similar manner, the path $P'(v_1, v_2)$ can be replaced by a $v_1, v_2$-path $P''(v_1, v_2)$ which does not contain the edge $y$ (or $x$). In this manner each successive edge of $P(v_1, v_2)$ which is not an edge of $H$ gives rise to a new $v_1, v_2$-path which does not contain this or any of the previous such edges. That is, we can determine a $v_1, v_2$-path using only edges of $H$ so that $H$ is connected. This completes the proof of Subcase iiiia.

Subcase iiiib. One or more of the complete subgraphs $K(w_1)$ and $K(w_2)$ has order three.

We may in fact suppose that both $K(w_1)$ and $K(w_2)$ have order three, since the proof would be the same if one of $K(w_1)$ or $K(w_2)$ had order greater than three. As in Subcase iiiia, let $u$ and $v$ be vertices incident, respectively, with the edges $e$ and $f$, such that the length of a $u, v$-path is minimum, and let $P(u, v)$ be such a path. Then also as in Subcase iiiia, a frame $F(G)$
can be determined so that the edges of $P(u,v)$ are edges of $F(G)$. However in this subcase, $F(G)$ can only be chosen with the edges $e$ and $f$ as edges of $F(G)$, since $K(w_1)$ and $K(w_2)$ are triangles.

Let $H = (F(G) - E(P(u,v))) - e$. As in the other cases, $H$ is a spanning subgraph of $L(G)$ with precisely two odd vertices, so that if $H$ is connected, then there exists a spanning eulerian trail of $L(G)$. Note that the removal from $F(G)$ of the edges of $P(u,v)$ cannot disconnect $H$, by the argument used in Subcase iiia. (That part of Subcase iiia was independent of the orders of $K(w_1)$ and $K(w_2)$.) Hence, if $H$ is disconnected, it follows that $F(G) - \{e, x_1\}$ is disconnected, where $x_1$ is the edge of $P(u,v)$ which is adjacent with $e$ at the vertex $u$. However, this implies that $u$ is a cut-vertex of $F(G)$. But this is impossible by the argument used in Case i. Hence the subgraph $H$ of $L(G)$ is a connected spanning subgraph containing an eulerian trail. We remark that $H$ contains an eulerian trail beginning at the vertex $u'$, which is the vertex incident with $e$ and different from $u$, and ending at the vertex $v$.

Suppose that the edge $f$ is not a bridge of the subgraph $H$. Then the subgraph $H' = H - f$ is a spanning connected subgraph of $L(G)$ containing a $u', v'$ - eulerian trail, where $v'$ is the vertex incident
with \( f \) and different from \( v \). Since \( e \) and \( f \) are not edges of this \( u',v' \)-trail, but are incident, respectively, with the vertices \( u' \) and \( v' \), the \( u',v' \)-trail of \( H' \) gives rise to an \( e,f \)-Hamiltonian-path ordering of the edges of \( L(G) \), as in Case i.

On the other hand, suppose \( f \) is a bridge of \( H \). Let \( H_1 \) denote the component of \( H - f \) which contains the vertex \( v \), and let \( H_2 \) be the remaining component of \( H - f \). Furthermore, denote by \( x_k \) the edge of \( P(u,v) \) which is adjacent at \( v \) with the edge \( f \), and let \( v_k \) be the vertex incident with \( x_k \) and different from \( v \) (i.e., \( x_k = v_kv \)). Also, using the previously established notation, the edge \( f = vv' \). Since \( f \) is a bridge, it follows that the edge \( x_k \) is an edge of the triangle \( K(w_2) \); otherwise, the two remaining edges of \( K(w_2) \) in \( H \) comprise a second \( v',v \)-path, which is impossible. Therefore, the edges of \( K(w_2) \) are \( x_k, f \), and \( v_kv' \). Suppose that \( v_k \) is a vertex of the component \( H_1 \) of \( H - f \) (cf. Figure 4.3). This is impossible, however. For then the edge \( v_kv' \) of \( K(w_2) \), which is an edge of \( H \) joins a vertex of \( H_1 \) with a vertex of \( H_2 \), so that \( f \) is not a bridge of \( H \).
Therefore, the edge $x_k$ must join the vertex $v$ in $H_1$ with the vertex $v_k$, necessarily lying in $H_2$ (cf. Figure 4.4). Since the edge $v_kv'$ exists in $H$ and the edge $x_k$ of $L(G)$ does not, we may define the subgraph $H' = (H + x_k) - \{v_kv', f\}$ of $L(G)$. It is immediate that the subgraph $H'$ is a connected subgraph of $L(G)$; otherwise, the vertex $v'$ is a cut-vertex.
of \( L(G) \) by the argument used in Case i. But this is impossible since \( L(G) \) has no cut-vertices. Moreover, \( H' \) spans \( L(G) \) and contains a \( u', v' \)-eulerian trail. Since the edges \( e \) and \( f \) are not edges of \( H' \), but are incident, respectively, with \( u' \) and \( v' \), this \( u', v' \)-eulerian trail of \( H' \) gives rise to an \( e, f \)-hamiltonian-path ordering of the edges of \( L(G) \) as in Case i. This completes the proof of Subcase iiib and the theorem.

\[ \square \]

From the preceding theorem we obtain the following corollary which considers the value of \( hc(G) \) for graphs \( G \) having \( \delta(G) \geq 3 \) and containing a bridge which is incident with a vertex of degree 3. Theorem 4-9 which follows the corollary handles the remaining graphs \( G \) such that \( \delta(G) \geq 3 \) and \( \kappa_1(G) = 1 \).

**Corollary 4-8-1.** If \( G \) is a connected graph with \( \delta(G) = 3 \), and \( G \) contains a bridge incident with a vertex of degree 3, then \( hc(G) = 3 \).

**Proof.** Let \( G \) be a graph which satisfies the hypotheses, and let \( w \) be a vertex of \( G \) of degree three having incident edges \( x_1, x_2, \) and \( x_3 \), where \( x_1 \) is a bridge of \( G \). Then in \( L(G) \), the vertex \( v_1 \), corresponding to the edge \( x_1 \), is a cut-vertex.
Furthermore, if \( v_2 \) and \( v_3 \) are the vertices of \( L(G) \) corresponding respectively to \( x_2 \) and \( x_3 \), then \( v_1v_2 \) and \( v_1v_3 \) are edges of \( L(G) \) which constitute a two-element cut-set of edges. That is, \( (L(G) - v_1v_2) - v_1v_3 \) is disconnected, so that \( \kappa_1(L(G)) \leq 2 \). However, \( \kappa_1(L(G)) > 1 \), for if \( \kappa_1(L(G)) = 1 \), then \( G \) must contain two bridges incident with a vertex of degree two in \( G \). Therefore \( \kappa_1(L(G)) = 2 \), so that Theorem 4-8 applied to \( L(G) \) gives \( hc(L(G)) = 2 \), from which it follows that \( hc(G) = 3 \).

\[ \square \]

**Theorem 4-9.** If \( G \) is a graph with \( \delta(G) \geq 3 \) and \( \kappa_1(G) = 1 \), and \( G \) contains no bridge incident with a vertex of degree three, then \( hc(G) = 2 \).

**Proof.** Let \( G \) be a graph satisfying the hypotheses. Since \( G \) has \( \kappa_1(G) = 1 \), \( G \) contains a bridge. Thus \( L(G) \) has a cut-vertex, and therefore \( L(G) \) is not hamiltonian-connected. Hence \( hc(G) \geq 2 \), for such a graph \( G \).

It remains then to show that \( hc(G) \leq 2 \). Once again, it suffices to prove that \( L(G) \) is hamiltonian-path ordered. We now let \( e \) and \( f \) be arbitrary edges of \( L(G) \) and proceed by cases.

**Case i.** Suppose \( e \) and \( f \) are adjacent edges in \( L(G) \).
Subcase ia. The edges e and f are edges of the complete subgraph K(w), and K(w) is a triangle.

Let F(G) be a frame of L(G). Then necessarily, e and f are edges of F(G), since the spanning cycle C(w) of K(w) must contain all the edges of K(w).

Let u be the vertex incident with e which is not incident with f, and let v be the vertex incident with f which is not incident with e. Furthermore, let H = (F(G) - e) - f. Then H contains a spanning u,v-eulerian trail from which the desired e,f-hamiltonian-path ordering s can be formed using the method of Case i in Theorem 4-8, if H is connected. But H is connected in this case, for otherwise the vertex u' incident with both the edges e and f is a cut-vertex, which is impossible since then the vertex w of G has degree three and is incident with a bridge of G.

Subcase ib. The edges e and f are edges of the complete subgraph K(w), and K(w) has order at least four.

Let F(G) be a frame of G such that e is an edge of F(G) and f is not an edge of F(G). This is possible since we can always choose a spanning cycle C(w) of K(w) such that one of two adjacent edges is on C(w) and the other is not, provided the order of K(w) is at least four; thus by Lemma 4-1, the frame F(G)
with the aforementioned properties exists. Let \( u \) and \( v \) be the vertices incident with the edge \( e \), and consider \( H = F(G) - e \). The graph \( H \) must be a connected subgraph of \( F(G) \) in this case, for otherwise \( e \) is a bridge of \( F(G) \). But this is impossible since \( C(w) - e \) is a \( u,v \)-path in \( F(G) \) which does not contain the edge \( e \). Hence, \( H \) is a connected spanning subgraph of \( L(G) \) with a \( u,v \)-eulerian trail which again produces the desired \( e,f \)-hamiltonian-path ordering \( s \) as in Case i of Theorem 4-8.

**Subcase ic.** The edges \( e \) and \( f \) are adjacent edges of \( L(G) \) and \( e \) is an edge of the complete subgraph \( K(w_1) \) and \( f \) is an edge of the complete subgraph \( K(w_2) \).

Let \( F(G) \) be a frame such that the cycle \( C(w_1) \) spanning \( K(w_1) \) is a subgraph of \( F(G) \) and contains the edge \( e \), and the cycle \( C(w_2) \) spanning \( K(w_2) \) is a subgraph of \( F(G) \) and contains the edge \( f \). This frame \( F(G) \) exists by Lemma 4-1. Furthermore, let \( u \) be the vertex incident with the edge \( e \) and not \( f \), and let \( v \) be the vertex incident with \( f \) and not \( e \). Applying the reasoning of Subcase ib to each of \( C(w_1) \) and \( C(w_2) \) we see that the subgraph \( H = F(G) - \{ e,f \} \) is connected, and hence produces an \( e,f \)-hamiltonian-path ordering \( s \) by the method of Theorem 4-8, Case i.

**Case ii.** Suppose \( e \) and \( f \) are nonadjacent edges of
the complete subgraph \(K(w)\).

The proof of the existence of an \(e,f\)-hamiltonian-path ordering \(s\) in this case is identical to the proof of Case ii of Theorem 4-8.

Case iii. Suppose the edges \(e\) and \(f\) are not adjacent and do not lie in the same complete subgraph \(K(w)\).

Subcase iiiia. The edge \(e\) is an edge of the complete subgraph \(K(w_1)\), the edge \(f\) is an edge of the complete subgraph \(K(w_2)\), where \(w_1 \neq w_2\), and both \(K(w_1)\) and \(K(w_2)\) have order at least four.

The proof is identical with the proof of Subcase iiiia of Theorem 4-8.

Subcase iiiib. The edge \(e\) is an edge of the complete subgraph \(K(w_1)\), the edge \(f\) is an edge of the complete subgraph \(K(w_2)\), where \(w_1 \neq w_2\), and at least one of \(K(w_1)\) and \(K(w_2)\) has order three.

The proof here is almost identical with the proof of Subcase iiiib of Theorem 4-8. Recall that it was assumed without loss of generality that both \(K(w_1)\) and \(K(w_2)\) were triangles. The only difference from the proof in Subcase iiiib of Theorem 4-8 is a slight variation of the proof that \((F(G) - e) - x_1\) is connected (cf. Subcase iiiib, Theorem 4-8). It was argued there that if \((F(G) - e) - x_1\) is disconnected then the edges \(e\) and
\( x_1 \) are both edges of the triangle \( K(w_1) \), and that their common vertex \( u \) is a cut-vertex of \( (F(G) - e) - x_1 \). The method used to prove that \( u \) could not be a cut-vertex is not valid here. Instead, we argue that if \( u \) is a cut-vertex of \( (F(G) - e) - x_1 \) then since \( u \) is a vertex of the triangle \( K(w_1) \) the vertex \( w_1 \) of \( G \) has degree three and is incident with a bridge of \( G \). But this is a contradiction of the hypotheses. Hence \( u \) cannot be a cut-vertex of \( (F(G) - e) - x_1 \) so that this graph is connected. With this small change, the proof of Subcase iiib of Theorem 4-8 supplies the proof necessary here. That is, Subcase iiib and the theorem are complete.

The preceding results of this section calculated exact values for the hamiltonian-connected index of those graphs \( G \) with \( \delta(G) \geq 3 \) and \( \kappa_1(G) \) equal to one or two; hence to complete the discussion relating these two concepts, it is in order to consider the hamiltonian-connected index of those graphs with \( 3 \leq \kappa_1(G) \leq \delta(G) \). Among such graphs are the complete graphs \( K_p \), where \( p \geq 4 \), and the complete bipartite graphs \( K(m,n) \), where \( m \geq n \geq 3 \). Since every complete graph is a hamiltonian-connected graph, there exist graphs \( G \) with arbitrarily large edge-connectivity with \( \text{hc}(G) = 0 \). On the other hand, since no complete bipartite graph

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\( K(m,n) \), where \( m \geq n \geq 2 \), is hamiltonian-connected there exist graphs \( G \) with arbitrarily large edge-connectivity having \( \text{hc}(G) > 0 \). That in fact \( \text{hc}(K(m,n)) = 1 \) for \( m \geq n \geq 2 \) follows from Theorem 4-1. Moreover, no graph \( G \) with \( \kappa_1(G) \geq 3 \) is known whose hamiltonian-connected index exceeds one. These facts lead one to conjecture that if \( G \) is any graph with \( \kappa_1(G) \geq 3 \), then \( \text{hc}(G) \leq 1 \). Though the validity of this statement is not known, the following result is true and is really no more than a corollary of Theorem 4-8.

**Corollary 4-8-2.** If \( G \) is any graph with \( \kappa_1(G) \geq 3 \), then \( \text{hc}(G) \leq 2 \).

**Proof.** In the proof of Theorem 4-8, the inequality \( \text{hc}(G) \leq 2 \), where \( G \) is a graph with \( \delta(G) \geq 3 \), followed from \( \kappa_1(G) \geq 2 \). From this observation the present corollary follows immediately.

We close the discussions of this section and of this chapter by returning to the graphs of Section 4.2; that is, we give some further results about the hamiltonian-connected indices of connected graphs \( G \) with \( \delta(G) < 3 \), based on the preceding results of this section. In order to state these next results it is useful to partition all connected graphs \( G \) with \( \delta(G) < 3 \) into five classes.
Class I: \( G \) contains a poly-path \( Q \) such that
(i) \( \ell(Q) = \delta_3(G) + 1 \),
(ii) every edge of \( Q \) is a bridge of \( G \), and
(iii) at least one of the poly-vertices of \( Q \) is a vertex of degree three in \( G \).

Class II: \( G \) contains a poly-path \( Q \) such that
(i) \( \ell(Q) = \delta_3(G) + 1 \),
(ii) every edge of \( Q \) is a bridge of \( G \),
(iii) both poly-vertices of \( Q \) have degree at least four in \( G \), and
(iv) \( G \) is not a member of Class I.

Class III: \( G \) contains poly-paths \( Q \) and \( Q' \) such that
(i) \( \ell(Q) = \ell(Q') = \delta_3(G) + 1 \),
(ii) every pair \( x, x' \) of edges where \( x \) is an edge of \( Q \) and \( x' \) is an edge of \( Q' \) is a cut-pair of edges, and
(iii) \( G \) is not a member of Classes I or II.

Class IV: \( G \) contains a poly-path \( Q \) such that
(i) \( \ell(Q) = \delta_3(G) \),
(ii) every edge of \( Q \) is a bridge of \( G \),
(iii) at least one of the poly-vertices of \( Q \) is a vertex of degree three in \( G \), and
(iv) \( G \) is not a member of Classes I, II, or III.
Class V: G is not a member of Classes I - IV.

Having made this partition into classes, we now state and prove several theorems concerning connected graphs G with $\delta(G) < 3$. For convenience of notation, we let $H$ denote the $k$th-iterated line graph of each graph $G$ (where $k = \delta_3(G)$) in the following proofs; i.e. $H = L^k(G)$.

Proposition 4-1. If $G$ is a connected graph with $\delta(G) < 3$, belonging to Class I, then $\text{hc}(G) = \delta_3(G) + 3$.

Proof. Since $G$ is a member of Class I, the graph $G$ contains a poly-path $Q$ of length $\delta_3(G) + 1$ such that every edge of $Q$ is a bridge. From this it follows that $H'$, the $(\delta_3(G) - 1)$st-iterated line graph of $G$ contains a poly-path $Q'$ of length $(\delta_3(G) + 1) - (\delta_3(G) - 1) = 2$, such that both of its edges are bridges. Furthermore then $H$, the $\delta_3(G)$th-iterated line graph of $G$, contains a bridge; that is, the poly-path $Q'$ of $H'$ gives rise to a poly-path of length one in $H$, whose single edge must be a bridge.

Moreover, since $G$ is a member of Class I, the poly-path $Q$ is incident with a poly-vertex of degree three in $G$. Therefore, the bridge contained in $H$ must also be incident with a vertex of degree three. From this it follows that the graph $H$ satisfies the hypotheses of
Corollary 4-8-1, so that \( hc(H) = 3 \). Therefore,
\[
hc(G) = \delta_3(G) + hc(H) = \delta_3(G) + 3,
\]
as desired. \( \Box \)

**Proposition 4-2.** If \( G \) is a connected graph with \( \delta(G) < 3 \), belonging to Class II, then \( hc(G) = \delta_3(G) + 2 \).

**Proof.** Let \( G \) be a graph which satisfies the hypotheses of the theorem. By an argument similar to that of Proposition 4-1, the graph \( H \) must contain a bridge which is incident only with vertices of degree at least four. Since \( G \) is not a member of Class I, the graph \( G \) does not contain a bridge incident with a vertex of degree three. Thus, by Theorem 4-9, the graph \( H \) has \( hc(H) = 2 \), from which it follows that \( hc(G) = \delta_3(G) + hc(H) = \delta_3(G) + 2 \). \( \Box \)

**Proposition 4-3.** If \( G \) is a connected graph with \( \delta(G) < 3 \), belonging to Class III, then \( hc(G) = \delta_3(G) + 2 \).

**Proof.** By hypothesis, the graph \( G \) contains poly-paths \( Q \) and \( Q' \) each of length \( \delta_3(G) + 1 \), such that each pair \( x, x' \) of edges, where \( x \) is an edge of \( Q \) and \( x' \) is an edge of \( Q' \), is a cut-pair of edges of \( G \). Then the graph \( H \) contains a pair of edges whose removal disconnects the graph. That is, \( K_1(H) \leq 2 \).
Suppose that $κ_1(H) < 2$. Since $H$ must be connected, we must have $κ_1(H) = 1$; that is, the graph $H$ contains a bridge. This implies that the graph $G$ contains a poly-path $Q$ of length $δ_3(G) + 1$, and thus belongs to Class I or Class II, contradicting the hypothesis that $G$ belongs to Class III. Therefore, in fact $κ_1(H) = 2$, so that by Theorem 4-8, $hc(H) = 2$, from which it follows that $hc(G) = δ_3(G) + 2$.

Proposition 4-4. If $G$ is a connected graph with $δ(G) < 3$, belonging to Class IV, then $hc(G) = δ_3(G) + 2$.

Proof. Let $G$ be a graph satisfying the hypotheses. Since $G$ does not belong to Class I or Class II, the graph $H$ has $κ_1(H) ≥ 2$. Furthermore, since $G$ contains a poly-path $Q$ such that $l(Q) = δ_3(G)$, every edge of which is a bridge, and such that at least one poly-vertex of $Q$ has degree three in $G$, then the $(δ_3(G) - 1)$st-iterated line graph of $G$ contains a bridge incident with two poly-vertices, at least one of which has degree three.

Therefore, the graph $H$ has a cut-vertex incident with two edges whose removal disconnects $H$. Hence $κ_1(H) = 2$, so that by Theorem 4-8, $hc(H) = 2$. Thus $hc(G) = δ_3(G) + 2$.
Proposition 4-5. If \( G \) is a connected graph with \( \delta(G) < 3 \), belonging to Class \( V \), then \( \delta_3(G) \leq hc(G) \leq \delta_3(G) + 2 \).

Proof. Since \( G \) is a graph belonging to Class \( V \), it suffices to prove that the graph \( H \) has edge-connectivity at least three, so that by Corollary 4-8-2, the result follows.

Suppose to the contrary that \( \kappa_1(H) < 3 \). Then either \( \kappa_1(H) \) is one or two, since \( G \) is a connected graph. If the edge-connectivity of \( H \) is one, by the argument contained in Proposition 4-3, the graph \( G \) must belong to Class I or Class II, which is impossible. Therefore, we assume \( \kappa_1(H) = 2 \), and that the pair \( x_1, x_2 \) of edges of \( H \) are a cut-pair of edges. If the edges \( x_1 \) and \( x_2 \) are nonadjacent edges of \( H \), then the \( (\delta_3(G) - 1) \)st-iterated line graph must contain a pair of poly-paths of length two, such that the removal of one edge from each disconnects the graph (cf. Figure 4.5).

![Figure 4.5](image-url)
It follows that $G$ then belongs to Class III, which is also impossible. On the other hand, if $H$ contains only pairs $x_1, x_2$ of edges which are adjacent, then the vertex common to any such pair is a cut-vertex of $H$. Thus the graph is as shown in Figure 4.6. But this

![Figure 4.6](image_url)

implies that the graph $G$ has a poly-path $Q$ of length $\delta_3(G)$, every edge of which is a bridge, and having one of its poly-vertices of degree three in $G$. Moreover, since $H$ contains no cut-pairs of nonadjacent edges, the graph $G$ cannot belong to Class III. Therefore, $G$ belongs to Class IV. Since this is also impossible, we must have $\kappa_1(H) \geq 3$, which completes the proof of the theorem.

$\Box$
We recall that a graph G is connected if for every pair u, v of vertices of G, there exists a u,v-path. If u and v are vertices of a connected graph G, then the distance $d_G(u,v) = d(u,v)$ between u and v is the length of a shortest u,v-path in G. Hence if G has order p, and $\ell$ is the length of a u,v-path in G, then $d(u,v) \leq \ell \leq p - 1$. We define a connected graph G to be panconnected if for each pair u, v of vertices in G, there exists a u,v-path of length $\ell$ for each $\ell$ such that $d(u,v) \leq \ell \leq p - 1$.

In [2] Bondy introduced the concept of edge-pancyclic graphs. A graph is said to be edge-pancyclic if each edge of G is contained in a cycle of every length $\ell$, where $3 \leq \ell \leq p$. If a graph is panconnected, then it is edge-pancyclic; however, the concepts are not equivalent, since the graph shown in Figure 5.1 is edge-pancyclic but not panconnected. In fact, the join of $\overline{K_2}$ with $2K_n$ for each integer $n \geq 3$ gives an infinite class of edge-pancyclic graphs which are not panconnected (Figure 5.1 is the case $n = 3$).

If a graph is panconnected, then, by definition, it is hamiltonian-connected; moreover, these concepts are not,
in general, equivalent since infinite classes of graphs are known which are hamiltonian-connected but not pan-connected. More will be said about such classes later in the chapter. Figure 5.2 shows a graph which is
hamiltonian-connected but not panconnected. Much of the material in this chapter will be devoted to showing that conditions sufficient to guarantee the hamiltonian-connectedness of a graph or a family of graphs also (possibly with slight strengthening) imply their pan-connectedness.

Section 5.1

Preliminary Results

Whenever a new property of graphs is introduced, it is customary and useful to determine which, if any, of the well-known families of graphs possess this property. We remark that $K_p$, where $p \geq 1$, is panconnected, and that $K(p_1, p_2)$ where $1 \leq p_1 \leq p_2$, is not hamiltonian-connected (unless $p_1 = p_2 = 1$) and hence is not pan-connected. Now, both the complete graphs and the complete bipartite graphs belong to the family of complete $n$-partite graphs. The first theorem of this chapter characterizes those complete $n$-partite graphs $(n \geq 3)$ which are panconnected and at the same time furnishes a characterization of the complete $n$-partite graphs which are hamiltonian-connected, since the concepts are shown to be equivalent for this family.

Theorem 5-1. Let $G$ be the complete $n$-partite graph $K(p_1, p_2, \ldots, p_n)$, where $n \geq 3$ and $p_1 \leq p_2 \leq \ldots \leq p_n$. 

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Then the following are equivalent:

1. \( G \) is panconnected.
2. \( G \) is hamiltonian-connected.
3. \( p \geq 2p_n + 1 \), where \( p \) is the order of \( G \).

**Proof.** Let \( G = K(p_1, p_2, \ldots, p_n) \), where \( n \geq 3 \) and \( p_1 \leq p_2 \leq \ldots \leq p_n \), and let \( V_1, V_2, \ldots, V_n \) denote the respective partite sets of \( G \), then \( |V_i| = p_i \) for each \( i \), where \( 1 \leq i \leq n \).

That statement 1 implies statement 2 is a consequence of the definitions of panconnectedness and hamiltonian-connectedness for any graph \( G \).

We now show that statement 2 implies statement 3. Let \( u \) be a vertex of the partite set \( V_1 \) and let \( v \) be a vertex of the partite set \( V_2 \). Since \( G \) is hamiltonian-connected, there exists a \( u,v \)-hamiltonian path \( P(u,v) \) in \( G \). Since \( P(u,v) \) spans \( G \), every vertex of the partite set \( V_n \) is a vertex of \( P(u,v) \). Furthermore, no two vertices of \( V_n \) can appear consecutively on \( P(u,v) \), since no two vertices of \( V_n \) are adjacent in \( G \). Hence the number of vertices in \( V_n \), namely \( p_n \), must be less than the number of vertices in the remainder of \( G \). That is, \( p_n < p - p_n \). Therefore, \( 2p_n < p \), or \( p \geq 2p_n + 1 \).

Finally, we show that statement 3 implies statement 1. Assume \( p \geq 2p_n + 1 \) but that \( G \) is not panconnected.
Hence there exists a pair \( u, v \) of vertices of \( G \) and an integer \( k \), where \( d_G(u,v) < k \leq p - 1 \) such that there exists a \( u,v \)-path \( P(u,v) \) with \( k \) vertices, but no \( u,v \)-path with \( k + 1 \) vertices. Let the \( u,v \)-path \( P(u,v) \) be labeled

\[
P(u,v): \ u = v_1, \ v_2, \ldots, \ v_k = v.
\]

By the choice of \( u, v \), and \( k \), there exist vertices of \( G \) which are not vertices of \( P(u,v) \). Let \( t = p - k \), and let \( W = \{ w_1, w_2, \ldots, w_t \} \) be the set of vertices of \( G \) which are not vertices of \( P(u,v) \). Consider \( w_1 \), and suppose \( w_1 \) is a vertex of the partite set \( V_i \), for some \( i \), where \( 1 \leq i \leq n \). Then it follows that the vertices of \( P(u,v) \) are alternately vertices of the partite set \( V_i \); for if two consecutive vertices \( v_j \) and \( v_{j+1} \) of \( P(u,v) \), where \( 1 \leq j < k \), are not vertices of \( V_i \), then

\[
P'(u,v): \ u = v_1, v_2, \ldots, v_j, w_1, v_{j+1}, \ldots, v_k = v
\]
is a \( u,v \)-path with \( k + 1 \) vertices, which contradicts the choice of \( k \).

If \( w_1 \) is the only vertex of \( G \) in the set \( W \), then \( k = p - 1 \), and the partite set \( V_i \) has at least \( \lceil (p-1)/2 \rceil + 1 = \lceil (p+1)/2 \rceil \) vertices, since \( w \) is a vertex of \( V_i \) and at least \( \lceil (p-1)/2 \rceil \) vertices of \( P(u,v) \) are vertices of \( V_i \). Then \( p_i \), the order of
$V_i$, must be at least $[(p+1)/2]$. That is, $p_i \geq [(p+1)/2] \geq p/2$, so that $2p_i \geq p$. From this it follows that $2p_n \geq 2p_i \geq p$, which gives a contradiction.

On the other hand, if $w_i$ is not the only vertex of $G$ in the set $W$, then we claim that in fact $W \subseteq V_i$. Suppose this is not the case. Then some vertex of $W$, say $w_2$, is a vertex of the partite set $V_j$, where $1 \leq j \leq n$, and $j \neq i$. However, by a previous argument, it follows that the vertices of $P(u,v)$ are alternately vertices of $V_j$. Hence the path $P(u,v)$ must alternate its vertices between $V_i$ and $V_j$; that is, if $w_m \in V_i$, then $v_{m+1} \in V_j$, or if $v_m \in V_j$, then $v_{m+1} \in V_i$, for each $m$, where $1 \leq m < k$. But this is impossible, for if $u'$ is any vertex of $G$ which is not a vertex of $V_i$ or $V_j$ (such vertices exist, since $n \geq 3$), then

$$P''(u,v): u = v_1, u', v_2, v_3, \ldots, v_k = v$$

is a $u,v$-path with $k + 1$ vertices, contradicting the choice of $k$. Thus every vertex of $W$ must be a vertex of the partite set $V_i$, as claimed.

From this, it follows that the partite set $V_i$ has at least $[k/2] + t$ vertices. That is, $p_i \geq [k/2] + t$. But then, $2p_n \geq 2p_i \geq 2([k/2] + t) \geq 2((k-1)/2) + 2t = k + 2t - 1 = (k+t) + (t-1) \geq k + t = p$.

Again we have a contradiction. Therefore $G$ is
panconnected, and statement 3 implies statement 1. This completes the proof of the theorem.

The \textit{nth power} $G^n$ of a connected graph $G$ is the graph whose vertex set is that of $G$ and with the property that two distinct vertices are adjacent in $G^n$ if and only if the distance between these vertices in $G$ is at most $n$. It was shown independently by Karaganis [12] and Sekanina [21] that if $G$ is a connected graph of order $p$, then $G^3$ is hamiltonian-connected, i.e., every two distinct vertices are connected by a path of length $p - 1$. The object of Theorem 5-2 is to present a strengthening of that result.

\textbf{Theorem 5-2.} If $G$ is a connected graph, then $G^3$ is panconnected.

\textbf{Proof.} We proceed by induction on the order $p$ of connected graphs. For small values of $p$, the result follows immediately. Assume for all connected graphs $H$ of order less than $p$ that $H^3$ is panconnected. Let $G$ be a connected graph of order $p$, and let $u$ and $v$ be any two distinct vertices of $G$. Let $T$ be a spanning tree of $G$ such that $d_T(u,v) = d_G(u,v)$. (The existence of such a tree is shown in [19, p. 103].) It is now sufficient to verify that there exists in $T^3$ a $u,v$-path.

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of length \( l \) for each \( l \) such that \( d_T(u,v) \leq l \leq p - 1 \).

We consider two cases depending on whether \( u \) and \( v \) are adjacent in \( T \).

**Case i.** Assume \( u \) and \( v \) are adjacent in \( T \).

Here we show that there exists in \( T^3 \) a \( u,v \)-path of every positive length not exceeding \( p - 1 \). Such a path of length one is obvious. Let \( T_1 \) and \( T_2 \) be the components containing \( u \) and \( v \), respectively, in the graph \( T - uv \), and let \( p_i, i = 1, 2 \), be the order of \( T_i \).

Suppose first, that either \( p_1 = 1 \) or \( p_2 = 1 \), say the former. Let \( v_1 \) be a vertex of \( T_2 \) adjacent with \( v \). Since \( p_2 = p - 1 \), it follows by the induction hypothesis that there exist \( v,v_1 \)-paths in \( T_2^3 \) of length \( l_2 \) for all \( l_2 \) such that \( 1 \leq l_2 \leq p - 2 \). However, \( u \) and \( v_1 \) are adjacent in \( T^3 \); hence in \( T^3 \), there are \( u,v \)-paths of length \( l \) for all \( l \) such that \( 2 \leq l \leq p - 1 \).

We now assume that \( p_1 \geq 2 \) and \( p_2 \geq 2 \). Let \( u_1 \) be adjacent with \( u \) in \( T_1 \), and let \( v_1 \) be adjacent with \( v \) in \( T_2 \). Thus \( d_T(u_1,v_1) = 3 \), and \( u_1 \) and \( v_1 \) are adjacent in \( T^3 \).

By the induction hypothesis, there exists in \( T_1^3 \) a \( u,u_1 \)-path of length \( l_1 \) for each \( l_1 \) such that \( 1 \leq l_1 \leq p_1 - 1 \), and there exists in \( T_2^3 \) a \( v,v_1 \)-path.
of length \( l_2 \) for each \( l_2 \) such that \( 1 \leq l_2 \leq p_2 - 1 \). Hence by combining such a \( u, u_1 \)-path, such a \( v, v_1 \)-path, and the edge \( u_1v_1 \) of \( T^3 \), we obtain \( u,v \)-paths in \( T^3 \) of length \( \ell \) for each \( \ell \) such that \( 3 \leq \ell \leq (p_1 - 1) + (p_2 - 1) + 1 = p_1 + p_2 - 1 = p - 1 \). Furthermore, \( u,v_1,v \) is a \( u,v \)-path of length two in \( T^3 \). Therefore, the result follows in this case.

Case ii. Assume \( u \) and \( v \) are not adjacent in \( T \).

On the unique \( u,v \)-path \( P \) in \( T \), select the vertex \( w \) on \( P \) adjacent with \( u \). As in Case i, denote by \( T_1 \) and \( T_2 \) the components of \( T - uw \) containing \( u \) and \( v \), respectively, and let \( p_i \), \( i = 1, 2 \), represent the order of \( T_i \).

Assume that \( p_1 = 1 \) so that \( p_2 = p - 1 \). Since, by the induction hypothesis, \( T_2 \) is panconnected, \( T_2 \) (and thus \( T^3 \)) contains \( w,v \)-paths of length \( l_2 \) for all \( l_2 \) such that \( d_{T^3}(w,v) \leq l_2 \leq p - 2 \). Combining such a \( w,v \)-path with edge \( uw \), we obtain \( u,v \)-paths in \( T^3 \) of length \( \ell \) for each \( \ell \) such that \( 1 + d_{T^3}(w,v) \leq \ell \leq p - 1 \). Observe that either \( d_{T^3}(u,v) = d_{T^3}(w,v) \) or \( d_{T^3}(u,v) = 1 + d_{T^3}(w,v) \). Hence there exist \( u,v \)-paths in \( T^3 \) of length \( \ell \) for each \( \ell \) such that \( 1 + d_{T^3}(u,v) \leq \ell \leq p - 1 \). Since \( T^3 \) must by definition contain \( u,v \)-paths of length \( d_{T^3}(u,v) \) we have the desired result.
Henceforth, we assume $p_1 > 1$, and let $u'$ be a vertex of $T_1$ adjacent with $u$. By the induction hypothesis, $T_1^3$ is panconnected and thus contains $u, u'$ - paths of length $\ell_1$ for all $\ell_1$ such that $1 \leq \ell_1 \leq p_1 - 1$. Also, $T_2^3$ is panconnected, from which it follows that there exist $w, v$ - paths of length $\ell_2$ for all $\ell_2$ such that $d_{T_3}(w, v) \leq \ell_2 \leq p_2 - 1$. Combining the above $u, u'$ - paths with the $w, v$ - paths and the edge $u'w$, we obtain $u, v$ - paths in $T^3$ of length $\ell$ for all $\ell$ such that $2 + d_{T_3}(w, v) \leq \ell \leq (p_1 - 1) + (p_2 - 1) + 1 = p - 1$. If $d_{T_3}(u, v) = 1 + d_{T_3}(w, v)$, then this inequality gives the existence of $u, v$ - paths of all lengths $\ell$ such that $1 + d_{T_3}(u, v) \leq \ell \leq p - 1$. Thus once again the desired result has been obtained, since $T_3^3$ must contain $u, v$ - paths of length $d_{T_3}(u, v)$. However, if $d_{T_3}(u, v) = d_{T_3}(w, v)$, then a $w, v$ - path of length $d_{T_3}(w, v)$ in $T_2^3$ together with the edge $uw$ produces a $u, v$ - path in $T^3$ of length $1 + d_{T_3}(u, v)$, so that this case and the proof are complete.

Since there exist graphs $G$ whose squares are not hamiltonian-connected, there exist graphs $G$ whose squares are not panconnected. However, it can be observed that if $G$ is a hamiltonian graph, then its square is hamiltonian-connected. This result is
Theorem 5-3. If $G$ is a hamiltonian graph, then $G^2$ is panconnected.

Proof. Let $G$ be a hamiltonian graph and let $u$ and $v$ be a pair of vertices of $G$. Furthermore, let $C$ be a hamiltonian cycle of $G$, and suppose that the vertices of $G$ are labeled $v_1, v_2, \ldots, v_p$, consecutively along the cycle $C$. Without loss of generality, we assume $u$ has been labeled $v_1$ and $v$ has been labeled $v_k$, for some $k$, $2 \leq k < p$.

Let $P$ be a $u,v$-path in $G$ such that the length of $P$ is $d_G(u,v)$. Then in $G^2$, the subgraph induced by the vertices of $P$ contains a $u,v$-path $P'$ of length $m$, where $m = \left\lceil \frac{d_G(u,v)}{2} \right\rceil$, which, with at most one exception, consists of edges of $G^2$ which are not edges of $G$. The path $P'$ is necessarily a shortest $u,v$-path in $G^2$, so that $d_{G^2}(u,v) = m$.

We show that there exists a $u,v$-path of length $k$, for each $k$ such that $d_{G^2}(u,v) \leq k \leq p-1$, by means of three cases, depending on the value of $k$.

Case i. Suppose $k$ is such that $d_{G^2}(u,v) \leq k \leq d_G(u,v)$.

Let $H$ be the subgraph of $G^2$ defined by $V(H) = V(P)$ and $E(H) = E(P) \cup E(P')$. Since both $P'$ and $P$ are also subgraphs of $H$, the subgraph $H$ contains $u,v$-paths...
of lengths $d^2_G(u,v)$ and $d_G(u,v)$. Now suppose $H$ contains a $u,v$-path $P''$ of length $\ell$, where $d^2_G(u,v) \leq \ell < d_G(u,v)$. Then $P''$ contains in its edge set an edge $e$ which is an edge of $P'$ but not an edge of $P$. However, the edge $e$ is adjacent with a pair of edges $f_1$ and $f_2$ which are mutually adjacent edges of $P$, such that the subgraph induced by $e$, $f_1$, and $f_2$ is a triangle in $H$. Then $P'' - e$, together with $f_1$, $f_2$, and their common vertex, is a $u,v$-path of length $\ell + 1$ in $H$. Therefore $H$, and hence $G^2$, contains $u,v$-paths of all lengths $\ell$, where $d^2_G(u,v) \leq \ell < d_G(u,v)$.

**Case ii.** Suppose $\ell$ is such that $d_G(u,v) \leq \ell \leq k - 1$.

Recall that $u = v_1$ and $v = v_k$. There exists a $u,v$-path of length $k - 1$ using only edges of $C$, so that indeed $d_G(u,v) \leq k - 1$. If $d_G(u,v) = k - 1$, the result follows immediately in this case. Hence we assume $d_G(u,v) < k - 1$.

Let $v_i$ be the first vertex encountered on $P$ such that $1 \leq i \leq k$, and $v_j$, the next vertex encountered on $P$, is not $v_{i+1}$. Furthermore, let $m$ be the smallest integer such that $m > i$ and $v_m$ lies on $P$. Necessarily, $m \leq k$. Note that $m \neq i + 1$, for otherwise $P$ is not a shortest path in $G$. Now $v_i v_{i+1}$ and $v_i v_j$ are edges of $G$, so that $v_{i+1} v_j$ is an edge of $G^2$. Therefore, $G^2$ contains $u,v$-paths of all lengths $\ell$, where $d^2_G(u,v) \leq \ell < d_G(u,v)$.
\[ G^2. \] Hence if \( P \) is written

\[ P: P_i, P_j \]

where \( P_i \) is the \( v_1, v_i \) - subpath of \( P \), and \( P_j \) is the \( v_j, v_k \) - subpath of \( P \), then the path \( P^* \) written

\[ P^*: P_i, v_{i+1}, P_j \]

is a \( u, v \) - path of length one greater than that of \( P \).

For each \( t, \ i < t < m - i \), \( v_{t-1}v_t \) and \( v_tv_{t+1} \) are edges of \( G \), and hence \( v_{t-1}v_{t+1} \) is an edge of \( G^2 \). Thus if \( m \geq i + 2 \), the edge \( v_i v_{i+1} \) of \( P^* \) may be replaced by \( v_i v_{i+2}, v_{i+2}, \) and \( v_{i+2}v_{i+1} \) to obtain a \( u, v \) - path of length one greater than \( P^* \). Similarly, this may be repeated for each \( t, \ i < t < m - 1 \), until a path \( Q \) is obtained which has length \( m - i - 1 \) greater than \( P \) and uses all the vertices of \( G \) which lie between \( v_i \) and \( v_m \) on \( C \). That is, \( Q \) may be written

\[ Q: P_i, v_{i+2}, v_{i+4}, \ldots, v_i+2n, v_i+2n+1, \]

\[ v_i+2n-1, \ldots, v_{i+1}, P_j, \]

where \( n = (m - i)/2 - 1 \), and \( m - i \) is even, or

\[ Q: P_i, v_{i+2}, v_{i+4}, \ldots, v_i+2n, v_i+2n-1, \]

\[ v_i+2n-3, \ldots, v_{i+1}, P_j, \]

where \( n = (m - i - 1)/2 \), and \( m - i \) is odd.
Analogously, at each successive vertex $v_s$ of $P$, where $1 \leq s \leq k$, and the next vertex encountered on $P$ is not $v_{s+1}$, the process produces a sequence of paths, each one greater in length than the previous path until all the vertices of $G$ which lie on $C$ between $v_i$ and $v_k$ have been used.

**Case iii.** Suppose $\ell$ is such that $k \leq \ell \leq p - 1$.

Let $R$ denote the $u,v$-path of $G$ such that $V(R) = \{v_1, v_2, \ldots, v_k\}$. Then $R$ is a $u,v$-path of length $k - 1$. Now $v_{k-1}v_k$ is an edge of $G$ and of $R$. Also $v_kv_{k+1}$ is an edge of $G$, so that $v_{k-1}v_{k+1}$ is an edge of $G^2$. Then the edge $v_{k-1}v_k$ of $R$ can be replaced by $v_{k-1}v_{k+1}$, $v_{k+1}$, and $v_{k+1}v_k$ to obtain a path $R'$ of length $k$ in $G^2$. Similarly, the edge $v_kv_{k+1}$ of $R'$ may be replaced by $v_kv_{k+2}$, $v_{k+2}$, and $v_{k+2}v_{k+1}$ to obtain a path $R''$ of length $k + 1$ in $G^2$. Furthermore, for each $s$, $k < s < p$, $v_{s-1}v_s$ is an edge of $G$, $v_sv_{s+1}$ is an edge of $G$, and $v_{s-1}v_{s+1}$ is an edge of $G^2$. Applying this fact recursively from $R'$, the path $R''$ being the first such application, a sequence of paths of all lengths $\ell$, where $k \leq \ell \leq p - 1$ is obtained. This completes the case and the proof, since $u$ and $v$ were chosen arbitrarily.

□
We remark that by Theorem 5-3, the square of any cycle is a panconnected graph. In fact, if $G = C_p$, where $p \geq 5$, then $G^2$ is panconnected and it is 4-regular, so that there exist many graphs with small minimum degree which are panconnected. Moreover, if $G$ is a panconnected graph of order $p \geq 4$, then $\delta(G) \geq 3$. This follows from the fact that if $G$ is panconnected, then $G$ is hamiltonian-connected, whence it is known that $\delta(G) \geq 3$.

The question which naturally arises from these observations is whether panconnected graphs exist having vertices of degree three, and if so, how many such vertices can there be. The answer to this question is that such graphs exist and in fact may have nearly all of their vertices of degree three.

Consider the family of wheels $W_n$, where $n \geq 4$. The wheel $W_n$ is defined to be the join of the graph $K_1$ with the cycle $C_{n-1}$, where $n \geq 4$. The wheel $W_4$ is the complete graph $K_4$, which is 3-regular and panconnected. The wheel $W_n$, where $n \geq 5$, is a graph with $n - 1$ vertices of degree three and a single vertex of degree $n - 1$ and is panconnected. Thus there exist panconnected graphs of order $p \geq 4$, which are regular of degree three and also panconnected graphs of order $p \geq 4$ having at most one vertex of degree greater than three.

Theorem 5-4 shows that the only 3-regular
panconnected graph is the wheel $W_4$, and that if $G$ is a panconnected graph of order $p \geq 4$, having at most one vertex of degree greater than three, then $G$ is the wheel $W_p$.

**Theorem 5-4.** If $G$ is a panconnected graph of order $p \geq 4$ having at most one vertex of degree exceeding 3, then $G$ is the wheel $W_p$.

**Proof.** Suppose $G$ is a panconnected cubic graph, and let $v$ be a vertex of $G$, and $v_1, v_2, v_3$ be the vertices of $G$ to which $v$ is adjacent. Since $v$ and $v_1$ are adjacent in $G$, there must exist a $v_1, v$-path of length two in $G$ because $G$ is panconnected. Necessarily, this $v_1, v$-path must be $v_1, v_2, v$ or $v_1, v_3, v$. Without loss of generality, we may assume the former, so that $v_1v_2$ is an edge of $G$. (cf. Figure 5.3)

![Figure 5.3](image)
In a like manner, there must exist a $v_3, v$-path of length two in $G$, which must necessarily be either $v, v_1, v_3$ or $v, v_2, v_3$. Again, without loss of generality, we may assume the former, so that $v_1v_3$ is an edge of $G$. (cf. Figure 5.4)

![Figure 5.4](image)

Since $G$ is cubic, $G$ contains an edge incident with $v_2$ and a vertex of $G$ different from $v_1$ and $v$. If $v_2v_3$ is an edge of $G$ then $G$ is the graph $W_4$, and we are done. Hence, suppose to the contrary that $v_2u$ is an edge of $G$ where $u$ is distinct from $v, v_1, v_2,$ and $v_3$. (cf. Figure 5.5)

![Figure 5.5](image)
Since the vertices $v_2$ and $u$ are adjacent, there
must exist a $v_2, u$-path in $G$ of length two. Once
again it follows that this $v_2, u$-path must be $v_2, v_1, u$
or $v_2, v, u$. However, in either case, then $G$ contains
a vertex of degree greater than three, which is impossible.
Thus no such vertex $u$ can exist, and $v_2$ must be
adjacent with $v_3$, which in turn implies that $G$ is
the graph $W_4$.

Now suppose that $G$ is a panconnected graph with a
single vertex of degree greater than three. Let $v$
denote this vertex and let $v_1, v_2, \ldots, v_k$
be the
vertices of $G$ adjacent with $v$, where it is assumed
that $\deg_G v = k > 3$.

Since each $v_i$, where $1 \leq i \leq k$, is adjacent with
$v$, the graph $G$ must contain $v_i, v$-paths of length
two, which necessarily are of the form $v_i, v_j, v$, where
$1 \leq j \leq k$, and $j \neq i$. Thus $v_i v_j$ is an edge of $G$.
That is, for each $v_i$, where $1 \leq i \leq k$, the vertex
$v_i$ is adjacent with at least one $v_j$, for $1 \leq j \leq k$,
and $j \neq i$.

Suppose every $v_i$ is joined only to vertices $v_j$,
where $1 \leq j \leq k$, and $j \neq i$, and of course to $v$.
Then $G - v$ is a 2-regular subgraph of $G$. If $G - v$
is a cycle, then $G$ is a wheel, which is the desired
result. Hence assume that $G - v$ is 2-regular and not
a cycle. Then $G - v$ is a disconnected graph, each
component of which is a cycle. This implies, however, that \( v \) is a cut-vertex of \( G \), which is impossible since \( G \) is panconnected. Therefore, \( G - v \) is a cycle and \( G \) is a wheel.

Let us now assume that at least one \( v_i, \ 1 \leq i \leq k \), say \( v_1 \), is adjacent with a vertex \( u \) different from \( v \) and \( v_j \) for each \( j \), where \( 1 < j \leq k \). From the previous arguments, the remaining edge incident with \( v_1 \) must be an edge \( v_1v_j \), where \( 2 \leq j \leq k \), say \( v_1v_2 \) (cf. Figure 5.6).

![Figure 5.6](image)

Now since \( u \) and \( v_1 \) are adjacent in \( G \), there must exist \( v_1, u \)-paths of length two in \( G \). Again, it may be seen that such a path must necessarily be \( v_1, v, u \) or \( v_1, v_2, u \). However, no edge \( vu \) exists in \( G \), since the only vertices adjacent with \( v \) in \( u \) are the vertices \( v_i \), where \( 1 \leq i \leq k \), and \( u \) was assumed distinct from these. Hence the \( v_1, u \)-path of length two in \( G \) is \( v_1, v_2, u \), so that the edge \( v_2u \) is an edge.
Now the degree of $u$ in $G$ is three, so that there exists a vertex $w$ adjacent with $u$, where $w$ is different from $v$, $v_1$, and $v_2$, but not necessarily different from $v_i$, for $3 \leq i \leq k$. In any case, since $u$ and $w$ are adjacent in $G$, there exist $u,w$-paths of length two in $G$. But this is impossible, since then either $v_1$ or $v_2$ must have degree at least four, which contradicts the assumption that $G$ has a single vertex of degree greater than three. This implies that the vertex $u$ cannot exist in $G$. That is, each $v_i$ is adjacent only to vertices $v_j$, where $1 \leq j \leq k$, and $j \neq i$. As previously observed, then $G$ is a wheel and the proof is complete.
Section 5.2

Sufficient Conditions for Panconnectedness

Recall that in Section 3.2, two theorems of Ore were presented, namely Theorem 3-A and Theorem 3-B, which respectively gave a sufficient condition for a graph to be hamiltonian-connected or hamiltonian. Theorem 3-C and Theorem 3-D, the latter due to Dirac, also respectively gave a sufficient condition for a graph to be hamiltonian-connected or hamiltonian. The purpose of this section will be to investigate "Ore-type" and "Dirac-type" sufficient conditions for a graph $G$ to be panconnected.

We begin by considering a family of graphs $G$ of order $p$, where $p$ is odd and at least seven. Let $G_1$ be the empty graph on the vertices $\{v_1, v_2, \ldots, v_{(p-1)/2}\}$, and let $G_2$ be the disconnected graph on the vertices $\{u, v, w_1, w_2, \ldots, w_{(p-3)/2}\}$ having two components, one of which is isomorphic to $P_2$ and is induced by the vertices $u, v$ and the other of which is isomorphic to $P_{(p-3)/2}$ and is induced by the vertices $w_1, w_2, \ldots, w_{(p-3)/2}$. Now let $G$ be the join of $G_1$ and $G_2$. Note that Figure 5.2 is such a graph $G$ for the case $p = 9$.

Now each of the vertices $u$, $v$, and $v_i$, where $1 \leq i \leq (p-1)/2$ has degree $(p+1)/2$, and each $w_j$, where $1 \leq j \leq (p-3)/2$, has degree at least $(p+1)/2$. Thus $\delta(G) = (p+1)/2$, and moreover, $G$ is not
panconnected since it contains no \( u,v \)-path of length three.

From this family of non-panconnected graphs it is clear that any "Dirac-type" sufficient condition for pan-connectedness must require the minimum degree of any graph \( G \) of order \( p \geq 3 \) to be at least \( k \), where \( k \geq (p+2)/2 \). In fact, such a result can be found in Theorem 5-5.

**Theorem 5-5.** If \( G \) is any graph of order \( p \geq 4 \) such that \( \delta(G) \geq (p+2)/2 \), then \( G \) is panconnected.

**Proof.** The graph \( G = K_4 \) is the only graph of order four for which \( \delta(G) \geq (4+2)/2 = 3 \), and \( K_4 \) is panconnected. Henceforth, we assume that \( p \geq 5 \). Let \( G \) be any graph of order \( p \geq 5 \) such that \( \delta(G) \geq (p+2)/2 \), and let \( u \) and \( v \) be any two distinct vertices of \( G \).

The graph \( H = G - \{u,v\} \) is a graph of order \( p' \geq 3 \) and such that \( \delta(H) \geq p'/2 \), where \( p' = p - 2 \), so that \( H \) is hamiltonian by Theorem 3-D. Let \( C: v_1, v_2, \ldots, v_{p'}, v_1 \) be a hamiltonian cycle of \( H \).

Now let \( k \) be any integer such that \( 3 \leq k \leq p \) and suppose that there exists no \( u,v \)-path with \( k \) vertices in \( G \). Now if \( i \) is any integer such that \( 1 \leq i \leq p' \) and \( uv_i \) is an edge of \( G \), then \( v_{i+k-3} \) (where the index is taken mod \( p' \)) is not adjacent with \( v \) in \( G \), for otherwise \( u, v_1, v_{i+1}, \ldots, v_{i+k-3}, v \) is a \( u,v \)-path with \( k \) vertices, which is impossible. Thus for each
vertex on \( C \) which is adjacent with \( u \), there is a vertex on \( C \) which is not adjacent with \( v \).

However, since \( u \) is adjacent with at least \( p/2 \) vertices on \( C \), the vertex \( v \) is adjacent with at most \( p' - p/2 \) vertices on \( C \), so that in \( G \) there are at most \( p' - p/2 + 1 \) edges incident with \( v \). Hence, \( \deg_v \leq p' - p/2 + 1 = (p - 2) - p/2 + 1 = p/2 - 1 \).

But this is a contradiction, since \( \deg_v \geq p/2 + 1 \), by hypothesis. Therefore, for each integer \( k \) such that \( 3 \leq k \leq p \), there exists a \( u,v \)-path with \( k \) vertices in \( G \), so that there exist \( u,v \)-paths in \( G \) of each length \( \ell \), where \( 2 \leq \ell \leq p - 1 \). It follows then that there exist \( u,v \)-paths of each length \( \ell \), where \( d_G(u,v) \leq \ell \leq p - 1 \), since \( d_G(u,v) \leq 2 \) and the edge \( uv \) is a path of length one if \( d_G(u,v) = 1 \). Since \( u \) and \( v \) were chosen arbitrarily, \( G \) is panconnected.

\( \square \)

We remark that Theorem 5-5 is in general best possible by virtue of the family of graphs presented just prior to the theorem. Since we have now stated and proved the best possible "Dirac-type" theorem for panconnectedness, we shall do likewise for an "Ore-type" result.

Before doing so, however, consider the following family of graphs \( G \) of order \( p \geq 4 \). Let \( V(G) = \{ u, v, v_1, v_2, ..., v_{p-2} \} \) and \( E(G) = \{ uv \} \cup \{ v_i v_j \mid 1 \leq i \leq p - 2 \} \).
By construction, no edge $uv_j$ exists for $[p/2] \leq j \leq p-2$; furthermore, $\deg_G u = \lceil (p-2)/2 \rceil + 1 = \lceil p/2 \rceil$, and $\deg_G v_j = (p-3) + 1 = p-2$, for each $j$, where $[p/2] \leq j \leq p-2$, so that $u$ and $v_j$ are nonadjacent vertices such that $\deg_G u + \deg_G v_j = \lceil p/2 \rceil + p-2 = \lceil (3p-4)/2 \rceil$. Moreover, $G$ is not panconnected since there exists no $u,v$-path of length two. Thus any "Ore-type" sufficient condition concerning nonadjacent vertices of a graph $G$ of order $p \geq 3$ must be such that $\deg_G u + \deg_G v \geq \ell$ for each pair $u,v$ of nonadjacent vertices of $G$, where $\ell > \lceil (3p-4)/2 \rceil$. That is, an appropriate choice for $\ell$ must be at least $\lceil (3p-2)/2 \rceil$. That $\lceil (3p-2)/2 \rceil$ is in fact sufficient is shown by Corollary 5-5-1 which follows Proposition 5-1.

**Proposition 5-1.** If $G$ is any graph of order $p \geq 4$ such that for any pair $u,v$ of nonadjacent vertices of $G$, we have $\deg_G u + \deg_G v \geq (3p-2)/2$, then $\delta(G) \geq (p+2)/2$.

**Proof.** Suppose the result is false. Then there exists a graph $G$ of order $p \geq 4$ satisfying the hypothesis which has a vertex of degree less than $(p+2)/2$. Let $w$ be a vertex such that $\deg_G w < (p+2)/2$.

Since $\deg_G w < (p+2)/2 \leq p-1$ for $p \geq 4$, there exists a vertex $w_1$ of $G$ which is not adjacent with $w$. 

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Then 

$$\deg_G w + \deg_G w_1 < \frac{(p+2)}{2} + (p-2)$$

$$= \frac{(3p-2)}{2},$$

since $$\deg_G w_1 \leq p-2$$ follows from the fact that $$w_1$$ is at least not adjacent with $$w$$.

But this is impossible, since $$\deg_G w + \deg_G w_1 < \frac{(3p-2)}{2}$$ contradicts the fact that $$\deg_G w + \deg_G w_1 \geq \frac{(3p-2)}{2}$$. Thus the degree of every vertex of $$G$$ must be at least $$\frac{(p+2)}{2}$$, that is, $$\delta(G) \geq \frac{(p+2)}{2}$$.

We now state and prove the "Ore-type" result, which is best possible by virtue of the family of graphs introduced just prior to Proposition 5-1.

**Corollary 5-5-1.** If $$G$$ is any graph of order $$p \geq 4$$ such that for any pair $$u, v$$ of nonadjacent vertices of $$G$$, we have $$\deg_G u + \deg_G v \geq \frac{(3p-2)}{2}$$, then $$G$$ is pancyclic.

**Proof.** Let $$G$$ be any graph satisfying the hypothesis. Then by Proposition 5-1, the graph $$G$$ has $$\delta(G) \geq \frac{(p+2)}{2}$$. Hence by Theorem 5-5, the graph $$G$$ is pancyclic.

In closing this section we note two oddities concerning the "Dirac-type" result obtained in Theorem 5-5 and the
"Ore-type" result obtained in Corollary 5-5-1. The first of these oddities is that the previous theorems dealing with hamiltonian and hamiltonian-connected graphs were such that the "Dirac-type" result was always a corollary of the "Ore-type" result. In particular, Theorem 3-C is a corollary of Theorem 3-A and Theorem 3-D is a corollary of Theorem 3-B. This of course has been reversed for panconnectedness since the "Dirac-type" result implies the "Ore-type" result.

The second observed oddity is that the critical parameters in Theorems 3-A and 3-C and also in Theorems 3-B and 3-D were related in both cases by a factor of 1/2, whereas the critical parameters in Theorems 5-5 and Corollary 5-5-1 are respectively \((p+2)/2\) and \((3p-2)/2\).

Section 5.3

Line Graphs And Panconnected Graphs

Since the concept of panconnectedness is related to hamiltonian-connectedness, it is logical to consider paralleling the discussion of the hamiltonian-connected index of a graph in Chapter IV with a similar discussion involving panconnectedness. Hence we define the panconnected index of a graph \(G\) to be the smallest nonnegative integer \(n\) such that \(L^n(G)\) is panconnected, and if such an integer \(n\) exists, we denote it by \(pc(G)\).
It follows that if $G$ is a graph for which $pc(G)$ exists, then $hc(G)$ exists and $hc(G) \leq pc(G)$. There exist graphs for which equality holds; for example, $hc(G) = pc(G) = 0$, if $G = K_3$, and $hc(G) = pc(G) = n - 2$, if $G = P_n$. However, there also exist graphs $G$ for which the inequality between $hc(G)$ and $pc(G)$ is strict. In [15] Moon presented a class of hamiltonian-connected graphs containing the graphs $G_1$ and $G_2$ shown in Figure 5.8.

Figure 5.8

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Since $G_1$ and $G_2$ are Hamiltonian-connected, their Hamiltonian-connected indices are zero. On the other hand, the panconnected indices of $G_1$ and $G_2$ are positive since the vertices $u$ and $v$ (cf. Figure 5.8) are adjacent in $G_1$ and no $u,v$-path of length two exists in $G_1$. Moreover, the line graph $L(G_1)$ shown in Figure 5.8 is panconnected so that $pc(G_1) = 1$, whereas $L(G_2)$ is not panconnected since the vertices $z_1$ and $z_2$ are not joined in $L(G_2)$ by a path of length three.

From this example it can be seen that in general, $pc(G)$ may exceed $hc(G)$ by as much as two. No attempt will be made here to further investigate the relationship between $hc(G)$ and $pc(G)$ for an arbitrary graph $G$. Also, no attempt shall be made to give a result analogous to that of Theorem 4-6, which showed that $hc(G) \leq 3$ for any connected graph $G$ with $\delta(G) \geq 3$. We shall, however, state and prove a result analogous to Theorem 4-2.

**Theorem 5-6.** If $G$ is a panconnected graph of order $p \geq 2$, then $L(G)$ is a panconnected graph.

**Proof.** The only panconnected graphs of order $p = 2$ or $p = 3$ are respectively $K_2$ and $K_3$, and the result follows immediately in each case. Henceforth, we assume that $G$ is a panconnected graph of order $p \geq 4$ with $q$ edges.

Since $G$ is panconnected, it is connected, from which
it follows that \( L(G) \) is connected. Thus if \( v_1 \) and \( v_2 \) are any two vertices of \( L(G) \), there exists in \( L(G) \) a \( v_1, v_2 \)-path of length \( k = d_{L(G)}(v_1, v_2) \). Furthermore, if \( e \) and \( f \) are edges of \( G \) which correspond respectively with \( v_1 \) and \( v_2 \), then there exist vertices \( u \) and \( v \) (not necessarily distinct) which are incident respectively with \( e \) and \( f \), such that \( d_G(u, v) = k - 1 \).

If \( k = 1 \), then the edges \( e \) and \( f \) are adjacent in \( G \), and since \( \delta(G) \geq 3 \), there exists an edge \( x \) in \( G \) mutually adjacent with \( e \) and \( f \) so that \( v_1, v_x, v_2 \) is a \( v_1, v_2 \)-path of length two in \( L(G) \), where \( v_x \) is the vertex of \( L(G) \) corresponding to the edge \( x \) of \( G \). Hence it will be sufficient (and necessary) to show that there exist \( v_1, v_2 \)-paths in \( L(G) \) of each length \( \ell \), where \( 2 \leq k \leq \ell \leq q - 1 \), for if \( v_1v_2 \) is an edge of \( L(G) \), then \( L(G) - v_1v_2 \) must have \( v_1, v_2 \)-paths of all lengths \( \ell \), where \( 2 \leq \ell \leq q - 1 \). We proceed by cases.

**Case i.** Suppose \( \ell \) is such that \( 2 \leq k \leq \ell \leq p \).

Since \( G \) is panconnected and there exist vertices \( u \) and \( v \) incident respectively with the edges \( e \) and \( f \) such that \( d_G(u, v) = k - 1 \), the graph \( G \) must contain \( u, v \)-paths of every length \( m \), where \( k - 1 \leq m \leq p - 1 \). Let \( P(u, v) \) denote a \( u, v \)-path of length \( m \), where

\[
P(u, v): u = u_1, u_2, \ldots, u_{m+1} = v.
\]
Then \( P(u,v) \) gives rise to an ordering

\[ s: u_1 u_2, u_2 u_3, \ldots, u_m u_{m+1} \]

of \( m \) edges of \( G \) having the property that \( u_i u_{i+1} \) is adjacent with \( u_{i+1} u_{i+2} \) for each \( i \), where \( 1 \leq i < m \).

If both \( u_1 u_2 = e \) and \( u_m u_{m+1} = f \) hold, then \( m \geq k + 1 \), for otherwise we have a contradiction of the choice of \( u \) and \( v \). Then since \( k \geq 2 \), we have \( m \geq 3 \). Thus both \( u_2 \) and \( u_3 \) are interior vertices of the path \( P(u,v) \), and since \( \delta(G) \geq 3 \), there exist edges \( x_2 \) and \( x_3 \) of \( G \) which are incident respectively with \( u_2 \) and \( u_3 \) and are not edges of \( P(u,v) \). Hence the ordering \( s \) together with the edges \( x_2 \) and \( x_3 \) gives rise to an ordering \( s_1 \), where

\[ s_1: e = u_1 u_2, x_2, u_2 u_3, x_3, u_3 u_4, \ldots, u_m u_{m+1} = f. \]

However, the ordering \( s_1 \) produces a \( v_1, v_2 \) - path in \( L(G) \) with \( m + 2 \) vertices, since \( s_1 \) contains \( m + 2 \) edges.

Similarly, if exactly one of \( u_1 u_2 = e \) and \( u_m u_{m+1} = f \) holds, say \( u_m u_{m+1} = f \), then \( m \geq k \), for otherwise we have a contradiction of the choice of \( u \) and \( v \). Then since \( k \geq 2 \), we have \( m \geq 2 \), so that \( u_2 \) is an interior vertex of \( P(u,v) \). Then since \( \delta(G) \geq 3 \), there exists an edge \( x_2 \) of \( G \) incident with \( u_2 \) which is not an edge of \( P(u,v) \cup \{ e \} \). Therefore, the ordering \( s \)
together with the edges $e$ and $x_2$ gives rise to an ordering $s_2$, where

$$s_2: e, u_1 u_2, x_2, u_2 u_3, \ldots, u_m u_{m+1} = f,$$

which produces a $v_1, v_2$ - path in $L(G)$ with $m + 2$ vertices.

On the other hand, if neither $u_1 u_2 = e$ nor $u_m u_{m+1} = f$, then the ordering $s$ together with the edges $e$ and $f$ gives rise to an ordering $s_3$, where

$$s_3: e, u_1 u_2, u_2 u_3, \ldots, u_m u_{m+1}, f,$$

which produces a $v_1, v_2$ - path in $L(G)$ with $m + 2$ vertices.

In any case, then we have established that for each $m$, where $k - 1 \leq m \leq p - 1$, there exists in $L(G)$ a $v_1, v_2$ - path with $m + 2$ vertices, and hence $m + 1$ edges. Therefore, for each $\ell$, where $2 \leq k \leq \ell \leq p$, there exists a $v_1, v_2$ - path in $L(G)$ of length $\ell$.

Case ii. Suppose $\ell$ is such that $p \leq \ell \leq q - 1$.

Since the graph $G$ is panconnected, it is hamiltonian-connected, so that if $u$ and $v$ are any two vertices of $G$, there exists a hamiltonian path joining $u$ and $v$; that is, a $u, v$ - path of length $p - 1$. Once again let $v_1$ and $v_2$ be any two vertices of $L(G)$ and let $e$ and $f$ be the edges of $G$ which correspond, respectively, with
Furthermore, let $u$ and $v$ be distinct vertices of $G$ which are incident respectively with $e$ and $f$, and let $P(u,v)$ denote a $u,v$-hamiltonian path of $G$, where

$$P(u,v): u = u_1, u_2, \ldots, u_p = v.$$  

Then $P(u,v)$ gives rise to an ordering

$$s: u_1u_2, u_2u_3, \ldots, u_{p-1}u_p$$

of $p - 1$ edges of $G$.

Subcase iia. Suppose $u_1u_2 = e$ and $u_{p-1}u_p = f$.

Furthermore, suppose $u_1u_p = uv$ is not an edge of $G$. If the vertices $u$ and $v$ were not chosen or cannot be chosen so that $u$ and $v$ are incident, respectively, with $e$ and $f$ and $uv$ is not an edge of $G$, then the vertices incident with $e$ and $f$ induce a subgraph of $G$ which is isomorphic with $K_4$ (cf. Figure 5.9). In this

![Figure 5.9](image-url)

Figure 5.9

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case, however, there exists a $u_1, u_{p-1}$-hamiltonian path of $G$ which does not contain either the edge $e$ or the edge $f$. This situation will be handled by Subcase iic.

We then observe that if $x$ is any edge of $G$ which is not an edge of $P(u,v)$, then $x$ is incident with a vertex $u_i$ of $P(u,v)$, where $2 \leq i \leq p - 1$. Now let $X = \{x_1, x_2, \ldots, x_t\}$, where $t = q - p + 1$, be the set of edges of $G$ which are not edges of $P(u,v)$. Then the edge $x_1$ is incident with $u_i$, for some $i$, where $2 \leq i \leq p - 1$, so that the ordering $s$ together with the edge $x_1$ gives rise to an ordering $s_1$, where

$$s_1: \ e = u_1 u_2, u_2 u_3, \ldots, u_{i-1} u_i, x_1,$$

$$u_i u_{i+1}, \ldots, u_{p-1} u_p = f,$$

which produces a $v_1, v_2$-path with $p$ vertices (length $p - 1$) in $L(G)$.

Moreover, the edge $x_2$ is incident with some vertex $u_j$ of $P(u,v)$, where $2 \leq j \leq p - 1$. If $j \neq i$ (say $j > i$), then the ordering $s$ together with the edges $x_1$ and $x_2$ produce an ordering $s_2$ of the form

$$s_2: \ e = u_1 u_2, u_2 u_3, \ldots, u_{i-1} u_i, x_1, u_i u_{i+1}, \ldots,$$

$$u_{j-1} u_j, x_2, u_j u_{j+1}, \ldots, u_{p-1} u_p = f.$$

If $j = i$, then $s_2$ has the form

$$s_2: \ e = u_1 u_2, u_2 u_3, \ldots, u_{i-1} u_i, x_1, x_2, u_i u_{i+1},$$

$$\ldots, u_{p-1} u_p = f.$$
In either case, $s_2$ produces a $v_1, v_2$-path of length $p$ in $L(G)$.

In fact, the ordering $s$, together with the edges $x_1, x_2, \ldots, x_m$, where $1 \leq m \leq t$, produce an ordering $s_m$ of $(p-1)+m$ edges of $G$ which gives rise to a $v_1, v_2$-path of length $(p-1)+m-1$ in $L(G)$. That is, there exist in $L(G)$ paths from $v_1$ to $v_2$ of each length $\ell$, where $p-1 \leq \ell \leq (p-1)+t-1$, and since $t = q-p+1$, the integer $(p-1)+t-1 = (p-1)+(q-p+1)-1 = q-1$. Hence we have the desired result in this case.

**Subcase iib.** Suppose exactly one of $u_1u_2 = e$ and $u_{p-1}u_p = f$ holds.

Without loss of generality, we may suppose that $u_{p-1}u_p = f$. In this case (as in Subcase iic to follow), every edge $x$ which is not an edge of $P(u,v)$ is either incident with a vertex $u_i$ of $P(u,v)$, where $2 \leq i \leq p-1$, or $x$ is mutually adjacent at $u_1$ with $e$ and $u_1u_2$, presenting no problem.

Here the edges of $P(u,v)$ together with the edge $e$, give rise to an ordering

$$s: e, u_1u_2, u_2u_3, \ldots, u_{p-1}u_p = f$$

of $p$ edges of $G$ which produces a $v_1, v_2$-path of length $p-1$ in $L(G)$. Thus, if $X = \{x_1, x_2, \ldots, x_t\}$
where \( t = q - p \), is the set of edges of \( G \) which are not edges of \( P(u,v) \cup \{ e \} \), then we can produce an ordering \( s_m \) of the edges of \( P(u,v) \) together with \( x_1, x_2, \ldots, x_m \), where \( 1 \leq m \leq q - p \) as in Subcase iia. Moreover, the ordering \( s_m \) is an ordering of \( p + m \) edges of \( G \) which gives rise to a \( v_1, v_2 \) - path of length \( p + m - 1 \) in \( L(G) \). Hence in this case there exist \( v_1, v_2 \) - paths of each length \( \ell \), where \( p \leq \ell \leq p + m - 1 = p + (q - p) - 1 = q - 1 \), as desired.

**Subcase iic.** Suppose \( u_1u_2 \neq e \) and \( u_{p-1}u_p \neq f \).

In this case, the \( p - 1 \) edges of \( P(u,v) \) together with the edges \( e \) and \( f \) give rise to an ordering

\[
s: e, u_1u_2, u_2u_3, \ldots, u_{p-1}u_p, f,
\]

which produces a \( v_1, v_2 \) - path with \( p + 1 \) vertices (length \( p \)). If we then let \( X = \{ x_1, x_2, \ldots, x_t \} \), where \( t = q - p - 1 \), be the set of edges of \( G \) which are not edges of \( P(u,v) \cup \{ e, f \} \), we again may obtain for each \( m \), where \( 1 \leq m \leq t \), an ordering \( s_m \) of \( p + 1 + m \) edges, which produces a \( v_1, v_2 \) - path in \( L(G) \) of length \( p + m \). Therefore, we again obtain \( v_1, v_2 \) - paths of length \( \ell \), for each \( \ell \), where \( p + 1 \leq \ell \leq p + t = q - 1 \). This completes Case ii and the proof of the theorem.
Corollary 5-6-1. If \( G \) is a panconnected graph of order \( p \geq 3 \), then \( L^n(G) \) is panconnected for each nonnegative integer \( n \).

From this corollary it follows that if \( G \) is any connected graph which is not a path, and if \( pc(G) = n \), then \( L^k(G) \) is panconnected for each integer \( k \geq n \).

We remark that if \( G \) is a hamiltonian-connected graph of order \( p \) and size \( q \), then \( L(G) \) certainly has paths joining each pair of vertices for each length \( \ell \), where \( p \leq \ell \leq q - 1 \). This is in fact what Case ii of Theorem 5-6 showed. Moreover, if \( G \) is any hamiltonian-connected graph of order \( p \geq 3 \) and \( u \) and \( v \) are vertices of \( L(G) \) for which \( k = d_L(G) (u,v) \), then it can be shown that \( L(G) \) actually contains paths of all lengths \( \ell \), where \( k \leq \ell \leq 2k \). Thus if \( G \) is a hamiltonian-connected graph for which \( pc(G) > hc(G) \) there must exist vertices \( u \) and \( v \) of \( L(G) \) which are not joined by a path of length \( \ell \), for some \( \ell \), such that \( 2d_L(G) (u,v) < \ell < p \), where \( p \) is the order of \( G \). Thus since each successive iterated line graph of a hamiltonian-connected graph of order \( p \geq 4 \) has an increased minimum degree and an increased likelihood of paths of varying lengths joining its vertices it may be conjectured that \( pc(G) - hc(G) \) might have a relatively small upper bound for arbitrary graphs \( G \).
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