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Completions of Lattices with Semicomplementation

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Western Michigan University

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COMPLETIONS OF LATTICES
WITH
SEMICOMPLEMENTATION

by

Alan A. Bishop

A Dissertation
Submitted to the
Faculty of The Graduate College
in partial fulfillment
of the
Degree of Doctor of Philosophy

Western Michigan University
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August 1972
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Alan A. Bishop
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I. INTRODUCTION

Generalized Section Semicomplemented Lattices

In lattice theory, two classical problems are the investigation of congruences (and hence homomorphisms) and completion processes. We shall undertake, in this dissertation, the study of both of these problems for generalized section semicomplemented lattices. The major emphasis shall be on completion processes.

The two primary methods of completing a lattice $L$ are the completion by cuts and the completion by ideals. The former has the disadvantage of not preserving certain lattice properties such as modularity, while the latter does not preserve all existing lattice operations and is not idempotent (that is, applying the process several times one obtains successively larger lattices). In this paper, we shall be working primarily with a third process, the completion by complete ideals. This completion is idempotent and under certain conditions preserves important properties of the original lattice $L$. An explicit formula for the join in the lattice of complete ideals is not always available. One of the major goals of this paper is to find conditions under which this join can be described. We are then able to investigate properties of the completion. In these cases, we
frequently are able to obtain information about the completion by cuts of the lattice and also about the original lattice itself.

In lattice theory, there is a hierarchy of ideals. Unlike ring theory, there is not always a relationship between ideals and homomorphisms (and their corresponding congruence relations). One problem in lattice theory is to identify ideals which are homomorphism kernels. This was done by Grätzer and Schmidt for section complemented lattices. M. F. Janowitz investigated the problem in section semicomplemented lattices. A second goal of this paper is to investigate this problem in generalized section semicomplemented lattices. As the class of generalized section semicomplemented lattices includes both section complemented and section semicomplemented lattices, the results obtained are extensions of results of Grätzer and Schmidt and Janowitz.

In [5], Grätzer and Schmidt show that in section complemented lattices an ideal is a homomorphism kernel if and only if it is standard, and in any lattice, every standard ideal is a homomorphism kernel. In [6], Janowitz shows that every homomorphism kernel is a projective ideal, and in section complemented lattices, every projective ideal is a homomorphism kernel. It is then shown that this result holds for principal ideals in
atomistic lattices, and for complete ideals in conditionally complete, conditionally upper continuous atomistic lattices. It appears that the proofs depend upon atomistic lattices having the following property: If $0 \leq b < a$, then there exists $x$ such that $0 < x \leq a$ and $b \wedge x = 0$. The element $x$ is called a semicomplement of $b$ in $a$, and a lattice with this property is called a section semicomplemented (or SSC) lattice. In [8], Janowitz investigates several properties of SSC lattices, their congruence relations, and their completions by cuts. It is shown there that the completion by cuts of a distributive lattice $L$ is a Boolean algebra if and only if $L$ is both SSC and DSSC (DSSC is the dual concept). The proof involves showing that a distributive DSSC lattice is conditionally upper continuous, and also involves embedding the distributive SSC and DSSC lattice $L$ into its completion by complete ideals $K(L)$. It is then shown that for this lattice, $K(L)$ is the completion by cuts $\overline{L}$, and as it is known that $K(L)$ is distributive, the distributivity of $\overline{L}$ follows.

The completion by complete ideals $K(L)$ of a lattice $L$ apparently was investigated first by Peter Crawley [2]. He shows that if $L$ is conditionally upper continuous, then the join in $K(L)$ has an explicit formulation, $K(L)$ is upper continuous, and $K(L)$ is modular or distributive whenever $L$ has the same property. Moreover, a lattice

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is always embedded into its completion by complete ideals so as to preserve all existing meets and joins (the completion by cuts also has this property).

Because the existence of a least element is not assumed for generalized section semicomplemented lattices, we shall find it useful in our study of their completions to investigate the nature of the conditional completion by cuts \( \mathcal{C} \) and the completion by complete ideals with upper bounds \( \mathcal{B}(L) \). Both lattices are conditionally complete and are sublattices of \( \mathcal{L} \) and \( K(L) \), respectively.

In Chapter II, we investigate the properties of the completion \( \mathcal{L} \) for generalized section semicomplemented lattices. Then we develop the completions \( K(L) \) and \( \mathcal{B}(L) \), and determine conditions for the join in these lattices to have an explicit formulation. Chapter III is devoted to the investigation of these completions of conditionally upper continuous generalized section semicomplemented lattices. There we produce conditions so that \( \mathcal{B}(L) \) is a relatively complemented modular lattice. As a corollary of this, we obtain an alternate proof of the theorem that the completion by cuts of a distributive SSC-DSSC lattice is a Boolean algebra.

In Chapter IV, we investigate the completions of AC-lattices (i.e., atomistic lattices with the covering property). We show that \( K(L) \) preserves the property
of $L$ being a finite modular AC-lattice (or a strongly planar AC-lattice). We also show that in case $L$ is a modular atomistic lattice, the join of two elements $X$ and $Y$ in $K(L)$ has a particular formula if and only if the pair $(X,Y)$ is a dual modular pair.

Finally, in Chapter V, we present several results concerning standard ideals, homomorphism kernels and projective ideals, extending to generalized section semi-complemented lattices, results which appear in [6] and [8].

**Preliminary Notions**

In the remainder of this chapter we formally define gSSC lattices and list selected results from general lattice theory which will be needed later. We also present a list of examples of gSSC lattices.

**Definition 1.1.** In a lattice with a least element 0 and a greatest element 1, two elements $a$ and $b$ are complements if $a \lor b = 1$ and $a \land b = 0$. The lattice $L$ is complemented if $L$ has 0 and 1, and every element has a complement. It is relatively complemented if every interval $[a,b] = \{x \in L| a \leq x \leq b\}$ is complemented. A lattice with 0 is section complemented if every interval of the form $[0,x]$ is complemented. In general, a lattice $L$ is section complemented if $a \in L$ implies the existence
of $e \in L$ such that $e \leq a$ and $\{e\} = \{x \in L | x \geq e\}$ is a section complemented lattice with zero.

In a lattice $L$ with 0 and 1, if $a \land b = 0$ and $0 < a$, then $a$ is called a semicomplement of $b$. The lattice $L$ is semicomplemented if $b < 1$ implies the existence of a semicomplement $a$ of $b$. If every interval $[x,y]$ is semicomplemented in a lattice $L$, then $L$ is relatively semicomplemented, and $L$ is a section semicomplemented (SSC) lattice if $0 \in L$ and every interval of the form $[0,x]$ is semicomplemented.

We now introduce the following notion which extends the concept of an SSC lattice to a lattice $L$ which has no least element.

The lattice $L$ is a generalized section semicomplemented (gSSC) lattice if for every $x \in L$, there exists $e \in L$ such that $e \leq x$ and $\{e\} = \{y \in L | y \geq e\}$ is an SSC lattice.

We observe that $L$ is gSSC if and only if for every $x \in L$, there is an $e \in L$ such that $e \leq x$ and $e \leq b < a$ implies the existence of $c \in L$ such that $e < c \leq a$ and $b \land c = e$.

We say that $L$ is dual semicomplemented (resp. dual section semicomplemented (DSSC), generalized dual section semicomplemented (gDSSC)) whenever the dual of $L$ is semicomplemented (resp. SSC, gSSC).
Certainly any SSC lattice as well as any section complemented lattice is a gSSC lattice. An important class of SSC, and hence gSSC lattices, are the atomistic lattices.

Definition 1.2. In a lattice $L$, we say that $a$ covers $b$ if $a > b$ and the inequality $a > x > b$ fails for all $x \in L$. The element $p$ of $L$ is an atom if $p$ covers $0$, and an element $a$ is said to be a finite element if it is the join of finitely many atoms. A lattice with $0$ is atomic if every nonzero element $a$ contains an atom. An atomic lattice is atomistic (or relatively atomic) providing every nonzero element $a$ is the join of the atoms which it contains.

Lemma 1.3. (i) A lattice $L$ with $0$ is atomistic if and only if $L$ satisfies the following condition:

If $a < b$, then there exists an atom $p$ such that $p \not< a$ and $p \leq b$.

(ii) An atomic lattice $L$ is atomistic if and only if $L$ is an SSC lattice.

The proofs may be found in [10], Lemma (7.2) and Lemma (7.3), p. 30.

Definition 1.4. Let $L$ be a lattice and $a, b \in L$. We say that $(a, b)$ is a modular pair and write $(a, b)M$. 

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if \( c \leq b \) implies \( c \lor (a \land b) = (c \lor a) \land b \). If \((a,b)M\) holds for all \( a,b \in L \), then \( L \) is said to be modular. The pair \((a,b)\) is a dual modular pair, denoted \((a,b)M^*\), if \( c \geq b \) implies \( c \land (a \lor b) = (c \land a) \lor b \).

By [10], Lemma (1.2), p. 1, \( L \) is modular if and only if \((a,b)M^*\) holds for all \( a,b \in L \).

An element \( d \) of a lattice \( L \) is distributive if \( d \lor (x \land y) = (d \lor x) \land (d \lor y) \) for all \( x,y \in L \). The lattice \( L \) is distributive if every element is distributive.

We now describe two well known methods of lattice completion.

**Definition 1.5.** A lattice \( L \) is complete if every subset of \( L \) has a supremum and infimum in \( L \), and is conditionally complete if every bounded subset has a supremum and infimum. A completion of a lattice \( L \) is an embedding of \( L \) into a complete or conditionally complete lattice. If the embedding preserves all existing meets and joins, it is a regular embedding.

**Definition 1.6.** An ideal \( I \) of a lattice \( L \) is a non-empty subset of \( L \) such that
(i) a, b ∈ I implies a ∨ b ∈ I, and

(ii) a ∈ I and b ≤ a imply b ∈ I.

An ideal I is **principal** if I = \{x ∈ L | x ≤ a\} for some a ∈ L.

**Filters** (or dual ideals) and **principal filters** are defined dually.

We shall denote the principal ideal \{x ∈ L | x ≤ a\} by [a] except when [a] is considered as an element of a lattice. In that case, we shall use the notation J_a instead. Similarly, [a] denotes the principal filter generated by a. The set of all ideals of L shall be denoted by I(L).

It is well known that I(L), partially ordered by set inclusion, is a complete lattice (provided that ∅ is adjoined to I(L) when 0 ∉ L), and that L can be embedded into I(L) by the mapping x ↦ J_x. Moreover, I(L) is modular (resp. distributive) whenever L is modular (resp. distributive). In summary,

**Theorem 1.7.** Let L be a lattice. Then

(i) The lattice I(L) is a complete lattice.

(ii) The mapping x ↦ J_x embeds L into I(L).

(iii) The join of two elements I and J of I(L) is given by I ∨ J = \{x ∈ L | x ≤ a ∨ b, a ∈ I, b ∈ J\}.

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(iv) The meet in $I(L)$ is set intersection.

(v) If $L$ is modular (resp. distributive), then $I(L)$ is modular (resp. distributive).

Proofs can be found in [1], pp. 113-114.

We now consider the completion by cuts of a lattice.

Definition 1.8. Let $L$ be a lattice. If $X$ is a subset of $L$, $X^\cup$ denotes the set of all upper bounds of $X$, and $X^\downarrow$ denotes the set of all lower bounds of $X$. We shall let $\overline{L} = \{X \subseteq L | X = X^\cup \cap X^\downarrow\}$.

The elements of $\overline{L}$ are ideals (with the possible exception of $\emptyset$ when $L$ has no least element). The set $\overline{L}$ partially ordered by inclusion is a complete lattice called the completion by cuts of $L$.

We summarize the properties of $\overline{L}$. Details may be found in [1], pp. 126-127, and in [10], pp. 52-53.

Theorem 1.9. Let $L$ be a lattice and $\overline{L}$ its completion by cuts.

(i) Every principal ideal $J_a$ is an element of $\overline{L}$.

(ii) The mapping $a \mapsto J_a$ is a regular embedding of $L$ into $\overline{L}$.
(iii) If $L$ is complete, then $L$ is isomorphic to $\bar{L}$.

(iv) If $X, Y \in \bar{L}$, then $X \wedge Y = X \cap Y$ and $X \vee Y = (X \cup Y)^{ul}$.

It should be noted that, in general, $\bar{L}$ does not inherit modularity or distributivity from $L$ (see [2]).

We now describe some of the relationships between congruence relations and ideals of lattices. Complete details can be found in [1], [4], [5], [6], and [8].

**Definition 1.10.** An equivalence relation $\Theta$ of a lattice $L$ is called a congruence relation if $a \equiv b(\Theta)$ and $c \equiv d(\Theta)$ imply $a \vee c \equiv b \vee d(\Theta)$ and $a \wedge c \equiv b \wedge d(\Theta)$. The congruence class containing the element $a$ is denoted by $a/\Theta$, and the set of congruence classes by $L/\Theta$.

If $a, b \in L$, there exists a smallest congruence relation which identifies $a$ and $b$. This congruence is denoted by $\Theta_{a,b}$. Similarly, $\Theta_I$ denotes the smallest congruence relation identifying the elements of the ideal $I$. It is clear that if $a \equiv b(\Theta)$ then $x \equiv y(\Theta)$ for all $x, y \in [a \wedge b, a \vee b]$. Hence $\Theta_{a,b} = \Theta_{a \vee b, a \wedge b}$.

Following Grätzer and Schmidt ([4], Definition 1, p. 151) we define the notion of weak projectivity which is useful in describing $\Theta_{a,b}$. The notion is originally
due to R. P. Dilworth [3], who used it in the reverse sense.

**Definition 1.11.** Let $a, b, c, d$ be elements of a lattice $L$. The pair $a, b$ is said to be weakly projective into $c, d$ if there exists $x_1, x_2, \ldots, x_n \in L$ such that

$$
...([(a \lor b) \lor x_1] \land x_2) \lor x_3) \land \ldots) \lor x_n = c \lor d \quad \text{and}

...([(a \land b) \lor x_1] \land x_2) \lor x_3) \land \ldots) \lor x_n = c \land d.
$$

The notation $\overline{a, b} \rightarrow \overline{c, d}$ denotes that $a, b$ is weakly projective to $c, d$.

We observe that $a \equiv b(\emptyset)$ and $\overline{a, b} \rightarrow \overline{c, d}$ imply $c \equiv d(\emptyset)$.

**Theorem 1.12.** (Dilworth [3], Lemma 2.1). Let $a, b, c, d$ be elements of the lattice $L$. Then $c \equiv d(\emptyset a, b)$ if and only if there exists $y_i \in L$ with $c \lor d = y_0 \geq y_1 \geq \ldots \geq y_k = c \land d$ and $\overline{a, b} \rightarrow \overline{y_{i-1}, y_i}$ ($i = 1, 2, \ldots, k$).

**Theorem 1.13.** Let $I$ be an ideal of a lattice $L$. Then $x \equiv y(\emptyset I)$ if and only if $x \equiv y(\emptyset a, b)$ for some $a, b \in I$.

We now investigate special elements and ideals associated with congruence relations.
Definition 1.14. A triple \((a, b, c)\) of elements of a lattice \(L\) is a distributive triple, written \(D(a, b, c)\) if \((a \lor b) \land c = (a \land c) \lor (b \land c)\). It is called a dual distributive triple \((D^*(a, b, c))\) when \((a \land b) \lor c = (a \lor c) \land (b \lor c)\). With this notation, an element \(d\) is distributive (see Definition 1.6) if and only if \(D^*(a, b, d)\) holds for all \(a, b \in L\).

An element \(s\) of \(L\) is standard if \(D(a, s, b)\) for all \(a, b \in L\). If \(n \in L\), and \(D(a, b, n)\) and \(D^*(a, b, n)\) hold for all \(a, b \in L\) and all permutations of \(a, b\) and \(n\), then \(n\) is called a neutral element.

An ideal \(J\) of \(L\) is said to be distributive, standard or neutral according to whether it is a distributive, standard or neutral element of \(I(L)\). The ideal \(J\) is a homomorphism kernel if there is a congruence relation \(\Theta\) of \(L\) such that \(L/\Theta\) has a zero element \(0/\Theta\) and \(J = \{a \in L | a/\Theta = 0/\Theta\}\).

Following Janowitz ([6], Definition, p. 291), we say that an element \(b\) of \(L\) is subperspective to \(a\) if there exists \(x \in L\) such that \(b \lor x \leq a \lor x\) and \(b \land x \leq a \land x\). An ideal \(J\) of \(L\) is a projective ideal if \(a \in J\) and \(b\) subperspective to \(a\) imply \(b \in J\). Note that this definition of subperspectivity differs from that in [8] and [10].
Theorem 1.15. (Grätzer and Schmidt [5], Theorem 1, p. 28). The following conditions are equivalent upon an element $s$ of the lattice $L$:

(i) $s$ is a standard element.

(ii) The relation $\Theta_s$, defined by $x \equiv y(\Theta_s)$ if and only if $(x \land y) \lor s_1 = x \lor y$ for some $s_1 \leq s$, is a congruence relation.

(iii) For all $x, y \in L$

(a) $s \lor (x \land y) = (s \lor x) \land (s \lor y)$, and

(b) $s \land x = s \land y$ and $s \lor x = s \lor y$

imply $x = y$.

Theorem 1.16. (Grätzer and Schmidt [5], Theorem 2, p. 30). The following conditions are equivalent for an ideal $S$ of a lattice $L$:

(i) $S$ is a standard ideal.

(ii) For any ideal $I$, the elements of $S \lor I$ are of the form $s \lor x$ ($s \in S$, $x \in I$).

(iii) The relation $\Theta_S$, defined by $x \equiv y(\Theta_S)$ if and only if $(x \land y) \lor s = x \lor y$ for some $s \in S$, is a congruence relation.

Remark 1.17. (i) $\Theta_S$ defined in (iii) above is the smallest congruence relation identifying the elements of $S$, and $S = \{ s \in L \mid s/\Theta_S = 0/\Theta_S \} = \ker \Theta_S$. Thus
S is a homomorphism kernel when S is standard.

(ii) The ideal \([s]\) is standard if and only if \(s\) is a standard element of \(L\).

(iii) By Theorem 1.14 (ii) every standard element (resp. ideal) is a distributive element (resp. ideal).

**Theorem 1.18.** (Grätzer and Schmidt [5], Lemma 12, p. 41). An element \(n\) of the lattice \(L\) is neutral if and only if it is both standard and dual standard.

**Theorem 1.19.** (i) An element \(d\) of a lattice \(L\) is distributive if and only if \(x \equiv y(\Theta(d))\) is equivalent to \([ (x \land y) \lor d ] \land (x \lor y) = x \lor y\). In this case \((d)\) is the kernel of \(\Theta(d)\).

(ii) An ideal \(J\) of a lattice \(L\) is distributive if and only if the relation \(a \equiv b(\Theta)\) if \(a \lor t = b \lor t\) for some \(t \in J\) is a congruence relation. In this case, \(J\) is the kernel of \(\Theta\).

**Remark 1.20.** It should be noted that a principal ideal \((z)\) is a neutral, standard or distributive ideal according to whether \(z\) is a neutral, standard or distributive element.

**Theorem 1.21.** Let \(J\) be a homomorphism kernel of a lattice \(L\). Then \(J\) is a projective ideal.
We observe that the ideals listed in order of implication are: neutral, standard, distributive, homomorphism kernel, and projective.

Grätzer and Schmidt [5] show that in section complemented lattices, the notions of standard ideals and homomorphism kernels are equivalent, while in [6], Janowitz shows that under the same conditions, projective ideals and homomorphism kernels are the same. Here we prove the two results simultaneously by using a new approach.

**Theorem 1.22.** In a section complemented lattice $L$, the following conditions on an ideal $J$ are equivalent:

(i) $J$ is a standard ideal.

(ii) $J$ is a distributive ideal.

(iii) $J$ is a homomorphism kernel.

(iv) $J$ is a projective ideal.

**Proof:** By the previous discussion we need only show that (iv) implies (i). Let $J$ be a projective ideal and let $I$ be any ideal of $L$. If $w \in J \lor I$, then $w \leq a \lor x$ for some $a \in J$ and $x \in I$. Let $e$ be an element of $L$ such that $e \leq w \land a \land x$ and $[e]$ is a section complemented lattice with zero. Then $e \leq x \land w \leq w$, so there exists $y \in L$ such that
\[ y \lor (x \land w) = w \quad \text{and} \quad y \land (x \land w) = e. \]

Thus \[ y \land x = (y \land w) \land x = e \leq a \land x \quad \text{and} \quad y \lor x \leq w \lor x \leq a \lor x \] which shows that \( y \) is subperspective to \( a \).

Since \( J \) is a projective ideal and \( a \in J \), we have \( y \in J \). Hence, \( w = y \lor (x \land w) \) with \( y \in J \) and \( x \land w \in I \). Thus condition (ii) of Theorem 1.15 is satisfied, so \( J \) is standard.

We close this chapter with a list of examples of gSSC lattices.

1. Any section complemented lattice is a gSSC lattice. The following figure is an example of a section complemented lattice.

![Diagram of a section complemented lattice]

2. Any relatively complemented lattice is a section complemented lattice and hence a gSSC lattice.

3. Any SSC lattice is a gSSC lattice.

4. Any atomistic lattice is an SSC lattice. In particular, if \( V \) is an infinite dimensional vector space, the lattice of all finite dimensional subspaces,
the lattice of all finite and cofinite dimensional subspaces, and the lattice of all subspaces of $V$ are all atomistic and hence gSSC lattices.

5. Let $L$ be an SSC lattice with $1$ and $\Gamma$ be an infinite indexing set. Let $L_\Gamma$ be the set of all functions $f: \Gamma \rightarrow L$ such that $f(\gamma) = 1$ for all but finitely many $\gamma$ of $\Gamma$. Then $L_\Gamma$, partially ordered by $f \leq g$ if and only if $f(\gamma) \leq g(\gamma)$ for all $\gamma \in \Gamma$, is a gSSC lattice. Moreover, $L_\Gamma$ has no least element, so $L$ is not an SSC lattice.
II. COMPLETIONS OF gSSC LATTICES

The Conditional Completion by Cuts

In [8], M. F. Janowitz shows that the completion by cuts $\overline{L}$ of a lattice $L$ with 0 is SSC if and only if $L$ is SSC ([8], Theorem 2.4). We proceed here to generalize this theorem.

If $L$ is a gSSC lattice which has no least element, then $\overline{L}$ has $\emptyset$ for its least element. Since the intersection of two ideals is another ideal and is thus nonempty, a nonempty element $X \in \overline{L}$ cannot have a semicomplement in any interval. Thus $\overline{L}$ is not gSSC unless $0 \in L$. We shall show that the conditional completion by cuts is gSSC.

The conditional completion by cuts $\overline{\mathcal{L}}$ of a lattice $L$ is the set of all ideals $X$ of $L$ such that $X$ has an upper bound in $L$ and $X = X^{ut}$. The set $\overline{\mathcal{L}}$, partially ordered by inclusion, is a sublattice of $\overline{L}$ and is conditionally complete. It has a least element $J_0$ (resp. greatest element $J_1$) if and only if $L$ has 0 (resp. 1). Moreover, $\overline{L}$ may be constructed from $\mathcal{L}$ by adjoining a least element and a greatest element when such elements do not exist in $\mathcal{L}$. Clearly, $\overline{L}$ contains all principal ideals of $L$. 

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and the mapping \( x \mapsto J_x \) embeds \( L \) into \( \widetilde{L} \) so as to preserve all existing meets and joins.

We now show that the conditional completion by cuts \( \widetilde{L} \) of a gSSC lattice \( L \) is gSSC. We actually show more. The lattice \( \widetilde{L} \) is in fact principally gSSC with respect to \( L \).

**Definition 2.1.** Let \( \mathcal{J} \) be a lattice of ideals of a lattice \( L \). Then \( \mathcal{J} \) is said to be **principally gSSC with respect to** \( L \) if for each \( X \in \mathcal{J} \), there exists \( e \in L \) such that the principal ideal \( J_e \in \mathcal{J} \), \( J_e \leq X \) in \( \mathcal{J} \) and \([J_e]\) is an SSC lattice.

We define **principally gDSSC with respect to** \( L \) dually.

**Theorem 2.2.** A lattice \( L \) is gSSC if and only if its conditional completion by cuts \( \widetilde{L} \) is principally gSSC with respect to \( L \).

**Proof:** Let \( L \) be gSSC and \( X \in L \). Choose any \( w \in X \). Since \( L \) is gSSC, there exists \( e \in L \) such that \( e \leq w \) and \([e]\) is an SSC lattice. Then \( J_e \leq X \).

We show that \([J_e]\) is SSC.

Let \( J_e \leq I < J \) in \( \widetilde{L} \). There exists \( b_1 \in J \) such that \( b_1 \not\in I \). Then \( b = b_1 \vee e \in J, b \not\in I \), and \( b \geq e \). Now \( b \not\in I = \mathcal{I}^u \) so there is a \( y \in \mathcal{I}^u \) such
that \( b \wedge y < b \). Thus \( b \wedge y < b \). Note that \( e \in I \) and \( y \in I^u \) so \( e \leq y \). Since \( e \leq b \) also, \( e \leq b \wedge y < b \).

Now \([e]\) is SSC, so we can find \( x \in L \) such that \( e < x \leq b \) and \( x \wedge (b \vee y) = e \). But then \( x \wedge y = (x \wedge b) \wedge y = e \). It follows that \( J_e < J_x \leq J_b \leq J \) and \( J_e \leq J_x \wedge I \leq J_x \wedge J_y = J_{x \wedge y} = J_e \), so \( J_x \) is the required semicomplement of \( I \) in \([J_e, J]\).

Conversely suppose \( L' \) is principally gSSC with respect to \( L \). Let \( w \in L \). Then \( J_w \in L' \). By Definition 2.1, there exists \( J_e \leq J_w \) such that \([J_e]\) is SSC. Then \( e \leq w \). To show \([e]\) is SSC, let \( e \leq b < a \).

Then \( J_e \leq J_b < J_a \) so there exists \( C \in L' \) such that \( J_e < C \leq J_a \) and \( J_e = C \wedge J_b \). Let \( c_1 \in C \) and \( c_1 \notin J_e \). If \( c = c_1 \vee e \), then \( J_e < J_c \leq C \leq J_a \) and \( J_e \leq J_{c \wedge J_b} \leq C \wedge J_b = J_e \). Thus \( e < c \leq a \) and \( e = c \wedge b \), \([e]\) is SSC, and hence \( L \) is gSSC.

The following question remains open. Does there exist a lattice \( L \) such that its conditional completion by cuts \( L' \) is gSSC but is not principally gSSC with respect to \( L \)?

We now obtain Theorem 2.4 of [8] as a corollary of Theorem 2.2.

Corollary 2.3. A lattice \( L \) with 0 is SSC if and only if its completion by cuts \( L' \) is SSC.
Proof: If \( L \) is SSC, a greatest element may be adjoined to \( L \) (providing \( L \) has no greatest element) to form another SSC lattice \( L_0 \). By Theorem 2.2 \( L_0 \) is SSC. But \( L = \tilde{L} = L_0 \), so \( L \) is an SSC lattice. Conversely if \( L \) is an SSC lattice, so is \( \tilde{L} \). Since \( 0 \in L \) \( \tilde{L} \) is principally gSSC with respect to \( L \). By Theorem 2.2 \( L \) is SSC.

The dual of Theorem 2.2 also holds. It should be observed that the proof is not simply a dual proof.

Theorem 2.4. A lattice \( L \) is gDSSC if and only if its conditional completion by cuts \( \tilde{L} \) is principally gDSSC with respect to \( L \).

Proof: First let \( L \) be gDSSC and let \( X \) be an element of \( \tilde{L} \). Now \( X \) is bounded above so we can choose \( w \in X^u \). Since \( L \) is gDSSC, there exist \( e \in L \) such that \( e \geq w \) and \( (e] \) is DSSC. We claim \( (J_e] \) is DSSC in \( \tilde{L} \).

Let \( I < J < J_e \) in \( \tilde{L} \). There exists \( b_1 \in I^u \) such that \( b_1 \notin J^u \). If \( b = b_1 \wedge e \), then \( b \in I^u \), \( b \notin J^u \), and \( b \leq e \). Now \( b \notin J^u \) implies the existence of \( y \in J \) such that \( y \not\leq b \). Moreover \( e \in J^u \), so \( y \leq e \). Thus \( b < b \lor y \leq e \). Since \( (e] \) is DSSC, there is an \( x \in L \) satisfying \( b \leq x < e \) and \( x \lor (y \lor b) = e \). But then \( x \lor y = (x \lor b) \lor y = e \). It follows that
\[ J_e > J_x \geq J_b \geq I \text{ and } J_e \geq J_x \lor J \geq J_x \lor J_y = J_{x \lor y} = J_e. \text{ Thus } (J_e) \text{ is DSSC.} \]

Now let \( \tilde{L} \) be principally \( g \text{DSSC} \) with respect to \( L \) and let \( w \in L \). There exists \( J_e \in \tilde{L} \) such that \( J_w \leq J_e \) and \( (J_e) \) is DSSC. If \( e \geq a > b \), then \( J_e \geq J_a > b \), so we can find a \( C \in \tilde{L} \) satisfying \( J_e > C \geq J_b \) and \( J_e = J_a \lor C \). There exists \( c_1 \in C \) with \( c_1 \not\in J_e^u \), so if \( c = c_1 \land e \) then \( c \not\in J_e^u \) and \( c < e \). Thus \( J_e > J_c \geq C \geq J_b \) and \( J_e \geq J_c \lor J_a \geq C \lor J_a = J_e \) and hence \( e > c \geq b \) and \( e = c \lor a \). This shows that \( (e) \) is DSSC and it follows from the definition that \( L \) is \( g \text{DSSC} \).

Observing that every relatively complemented lattice is both \( g \text{SSC} \) and \( g \text{DSSC} \), we have the following corollary to Theorem 2.2 and Theorem 2.4.

**Corollary 2.5.** The conditional completion by cuts of a relatively complemented lattice is both \( g \text{SSC} \) and \( g \text{DSSC} \).

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**Lattices of Complete Ideals**

In this section we investigate two completions, \( K(L) \) and \( B(L) \), of a lattice \( L \) and list several examples. The following discussion of \( K(L) \) is from [2].
Definition 2.6. An ideal $X$ of a lattice $L$ is a complete ideal if $X$ contains $\forall S$ whenever $S \subseteq X$ and $\forall S$ exists in $L$.

If $L$ has a least element, let $K(L)$ denote the set of complete ideals; otherwise let $K(L)$ be the complete ideals together with $\phi$.

Clearly, the intersection of any family of elements of $K(L)$ is again in $K(L)$, so $K(L)$, partially ordered by inclusion, is a complete lattice. The greatest element of $K(L)$ is $L$ and the least element is either $\phi$ or the principal ideal $J_0$. The meet operation in $K(L)$ is set intersection. The join of any subset $S$ of $K(L)$ is the smallest complete ideal containing the members of $S$. If $X$ and $Y$ are complete ideals, we shall denote their join in $K(L)$ by $X \lor_K Y$ and their join in the lattice of all ideals $I(L)$ by $X \lor Y$.

Since the meet of $X$ and $Y$ is set intersection in $I(L)$, as well as $K(L)$, we shall denote this meet by $X \land Y$ in both lattices.

Now every principal ideal $J_a$ is complete, so $J_a \lor_K J_b = J_{a \lor b}$ and $J_a \land J_b = J_{a \land b}$. Further, if $a = \lor(a_\delta | \delta \in \Delta)$ in $L$, then $J_a$ is a complete ideal containing each $J_{a_\delta}$. Let $X$ be a complete ideal which contains each $J_{a_\delta}$. Then $a_\delta \in X$ for all $\delta$, and $\lor(a_\delta | \delta \in \Delta)$ exists. Since $X$ is complete, $a = \lor(a_\delta | \delta \in \Delta) \in X$ and hence, $J_a \leq X$. Therefore,
$J_a = \bigvee_k (J_{a_\delta} \mid \delta \in \Delta)$. Clearly, $b = \bigwedge (b_\delta \mid \delta \in \Delta)$ in $L$ implies $J_b = \bigwedge (J_{b_\delta} \mid \delta \in \Delta)$, so the mapping $x \mapsto J_x$ is a regular embedding of $L$ into $K(L)$.

Finally, we observe that if $L$ is complete, an ideal of $L$ is complete if and only if it is principal, and then $x \mapsto J_x$ is an isomorphism of $L$ onto $K(L)$.

Now let $X$ be a subset of $L$ such that $a \in L$, $S \subseteq X$, and $\forall S$ exists implies $a \land \forall S \in X$. If $X \neq \emptyset$, let $c \leq b$ with $b \in X$. Then $c = c \land \forall [b] \in X$. If $\emptyset \neq S \subseteq X$ such that $\forall S$ exists, then $\forall S \land \forall S \in X$ and hence $X$ is a complete ideal. Conversely, any complete ideal satisfies the condition. We also note that $\emptyset$ satisfies the condition if and only if $\forall \emptyset$ does not exist (i.e., if and only if $0 \notin L$), and hence if and only if $\emptyset \notin K(L)$. Summarizing the above statements, we have

**Theorem 2.7.** (i) A subset $X$ of $L$ is an element of $K(L)$ if and only if $a \land \forall S \in X$ whenever $a \in L$, $S \subseteq X$ and $\forall S$ exists in $L$.

(ii) The set $K(L)$, partially ordered by set inclusion, is a complete lattice. The infimum of any subset of $K(L)$ is set intersection.
(iii) If \( x \in L \), then the principal ideal \( J_x \in K(L) \), and the mapping \( x \mapsto J_x \) is a regular embedding of \( L \) into \( K(L) \).

(iv) If \( L \) is complete, then \( K(L) \) is isomorphic to \( L \).

As indicated in the discussion above, the join \( X \lor_Y Y \) of two complete ideals \( X \) and \( Y \) is, in general, the smallest complete ideal containing both \( X \) and \( Y \). This join is difficult to use in calculations. We now investigate a condition on a lattice \( L \) which allows the join in \( K(L) \) to be expressed explicitly. First, we need the following definition.

**Definition 2.8.** A nonempty subset \( S \) of a lattice \( L \) is called an *increasing directed set* if for every pair \( x, y \in X \), there exists an element \( z \) in \( S \) such that \( x \lor y \leq z \). The notation \( x \uparrow_I x \) shall denote that \( \{ x_\delta \} \) is an increasing directed set with \( x = \lor x_\delta \).

A lattice \( L \) is said to be *conditionally upper continuous* if \( x_\delta \uparrow_I x \) implies \( x_\delta \land y \uparrow_I x \land y \) for each \( y \in L \). If \( L \) is also complete, it is called *upper continuous*.

Crawley [2] shows that conditional upper continuity is a sufficient condition on \( L \) in order to describe the join in \( K(L) \) of two elements \( A \) and \( B \).
Theorem 2.9. Let $L$ be a conditionally upper continuous lattice. Then the following statements hold:

(i) The join in $K(L)$ is given by $A \vee_k B = \{x \mid x = VS \text{ for some } S \subseteq A \vee B\}$.

(ii) $K(L)$ is upper continuous.

Proof: (i) Let $Q$ denote $\{x \mid x = VS \text{ for some } S \subseteq A \vee B\}$. Since $A \vee B$ is the smallest ideal containing $A$ and $B$, and since $A \vee_k B$ is the smallest complete ideal containing $A$ and $B$, we have $Q \subseteq A \vee_k B$. Suppose $X$ is a subset of $Q$ such that $\forall X$ exists in $L$. For each $x \in X$, there is a subset $S_x \subseteq A \vee B$ with $x = VS_x$. Let $C$ be the ideal generated by $\bigcup \{S_x \mid x \in X\}$. Then $C \subseteq A \vee B$ and $\forall C = \forall (VS_x \mid x \in X) = \forall X$ exists in $L$. If $a \in L$, then $a \wedge \forall X = a \wedge \forall C = \forall (a \wedge c \mid c \in C)$ since $C$ may be considered as an increasing directed set. Noting that $[a \wedge c \mid c \in C] \subseteq C \subseteq A \vee B$, we have $a \wedge \forall X \notin Q$. It follows from Theorem 2.7 (i) that $Q$ is a complete ideal so $Q = A \vee_k B$.

(ii) Let $X_\delta \uparrow X (\delta \in \Delta)$ in $K(L)$ and $Y \in K(L)$. With a slight modification of the proof of (i), it is easily seen that $\forall_k (X_\delta \mid \delta \in \Delta) = \{x \mid x = VS \text{ for some } S \subseteq \forall (X_\delta \mid \delta \in \Delta)\}$. If $x \in X \wedge Y = \forall_k (X_\delta \mid \delta \in \Delta) \wedge Y$, then there is a subset $S \subseteq \forall_\delta X_\delta \wedge Y$ such that $\forall S = x$. 

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If \( s \in S \), then \( s \leq \bigvee_{i=1}^{n} x_i \) (\( x_i \in X_{S} \)). Since \([X_{S}]\) is directed, we can find \( \delta_0 \) such that \( s \leq \bigvee_{i=1}^{n} x_i \in \bigvee_{\delta \in \Delta} X_{\delta} \). Now \( s \in Y \), so \( s \in X_{\delta}, \forall Y \leq \bigvee_{\delta \in \Delta} X_{\delta} \). Since \( \bigvee_{\delta} (X_{\delta} \wedge Y) \) is complete, \( x = \bigvee S \in \bigvee_{\delta} (X_{\delta} \wedge Y) \), and hence \( X \wedge Y \leq \bigvee_{\delta} (X_{\delta} \wedge Y) \). The reverse inclusion always holds so \( X_{\delta} \wedge Y \neq X \wedge Y \).

Crawley also shows that if \( L \) is conditionally upper continuous, every identity which holds in \( L \) also holds in \( K(L) \). In particular, with this condition, \( L \) modular (or distributive) implies \( K(L) \) is modular (or distributive). The proof as given by Crawley is rather long, and as we shall not need the theorem in that generality, we present a shorter version.

**Theorem 2.10.** If \( L \) is a conditionally upper continuous modular (resp. distributive) lattice, then \( K(L) \) is modular (resp. distributive).

**Proof:** Suppose \( L \) is modular and let \( X, A, B \in K(L) \) with \( X \leq B \). Since \( X \bigvee_{\Delta} (A \wedge B) \leq (X \bigvee_{\Delta} A) \wedge B \) always holds, let \( y \in (X \bigvee_{\Delta} A) \wedge B \). By Theorem 2.9 (i) \( y = \bigvee S \) for some \( S \subseteq X \bigvee A \). Now \( y \in B \) so that \( S \subseteq B \) also. Recalling that \( L \) modular implies \( I(L) \) modular, we have \( S \subseteq (X \bigvee A) \wedge B = X \bigvee (A \wedge B) \leq X \bigvee_{\Delta} (A \wedge B) \in K(L) \). Therefore, \( y = \bigvee S \in X \bigvee_{\Delta} (A \wedge B) \) and \( K(L) \) is modular.
The distributive case is similar.

It is worth noting in the proof above that the crucial aspect is knowing the relationship between $A \lor K B$ and $A \lor B$.

Except for the proof of Theorem 2.10 all of the above discussion of $K(L)$ is essentially due to Crawley [2]. We have included proofs since we shall make extensive use of this material in the remainder of this paper.

We now define a new completion $B(L)$ of a lattice $L$. Let $B(L)$ denote the set of all complete ideals of $L$ which have upper bounds. Let $B(L)$ be partially ordered by inclusion. If elements $X$ and $Y$ of $B(L)$ are bounded above by $a$ and $b$ respectively, then $X \lor K Y$ is bounded above by $a \lor b$ since $J_{a \lor b}$ is a complete ideal containing both $X$ and $Y$. Thus the join of any two elements of $B(L)$ exists and is the join in $K(L)$. Similarly, the meet of $X$ and $Y$ in $B(L)$ is the same as that of $K(L)$. Thus $B(L)$ is a sublattice of $K(L)$. Moreover, if $S$ is a nonempty subset of $B(L)$ such that $S$ has an upper bound $X$ in $B(L)$, then in $K(L)$, $\lor K S \leq X$ since $X$ is complete. Now $X$ is bounded above, so $\lor K S$ is bounded above and thus $\lor K S \in B(L)$. It follows that $B(L)$ is conditionally complete and each join which exists in
B(L) is the join in K(L). A similar result holds for meets.

In view of the above remarks and the fact that each principal ideal is bounded above and complete, we have from Theorem 2.7 that the mapping \( x \mapsto J_x \) is an embedding of \( L \) into \( B(L) \) which preserves all existing meets and joins. Finally observing that \( B(L) \) is conditionally upper continuous (resp. modular, distributive) whenever \( K(L) \) is upper continuous (resp. modular, distributive), we have the following results for \( B(L) \) which correspond to Theorem 2.7, Theorem 2.9, and Theorem 2.10.

**Theorem 2.11.** The set \( B(L) \) partially ordered by set inclusion is a sublattice of \( K(L) \) and arbitrary meets and joins are the same in both \( B(L) \) and \( K(L) \) whenever they exist in \( B(L) \). Moreover, \( B(L) \) is conditionally complete.

**Theorem 2.12.** The mapping \( x \mapsto J_x \) is a regular embedding of \( L \) into \( B(L) \).

**Theorem 2.13.** If \( L \) is conditionally upper continuous, then \( B(L) \) is also conditionally upper continuous. If \( L \) is also modular (resp. distributive), then \( B(L) \) is modular (resp. distributive).
Remark 2.14. (i) If $L$ is conditionally complete, then a complete ideal has an upper bound if and only if it is principal. Thus the mapping $x \mapsto J_x$ is an isomorphism of $L$ onto $B(L)$.

(ii) As with the conditional completion by cuts $\bar{L}$, $B(L)$ has a least element (resp. greatest element) if and only if $L$ has $0$ (resp. $1$), and in this case the element is $J_0$ (resp. $J_1$).

We now investigate conditions for $K(L)$ and $B(L)$ to have atoms. Important to later developments is the fact that the atoms and finite elements (i.e., those elements which are finite joins of atoms) of $K(L)$ are the principal ideals generated by atoms and finite elements of $L$, and that $K(L)$ is atomistic whenever $L$ is. Note that similar results hold for the completion by cuts also (see [10], Lemma (12.6)).

Theorem 2.15. If $L$ is a lattice with $0$, then the following statements hold:

(i) An element $X \in K(L)$ is an atom if and only if $X = J_p$ for some atom $p \in L$.

(ii) $L$ is atomic if and only if $K(L)$ is atomic.

(iii) $L$ is atomistic if and only if $K(L)$ is atomistic.
(iv) An element $X$ is a finite element of $K(L)$ if and only if $X = J_a$ for some finite element $a \in L$.

(v) The above four statements remain true if $K(L)$ is replaced by $B(L)$.

Proof: (i) If $p$ is an atom of $L$, then $J_p = \{0, p\}$ so that $J_p$ covers $J_0$ in $K(L)$. Thus $J_p$ is an atom.

Conversely, suppose $X$ is an atom of $K(L)$. Since $J_0 < X$, there is an $a \in X$ such that $0 < a$. If $a$ is not an atom of $L$, there exists $b \in L$ satisfying $0 < b < a$ and hence $J_0 < J_b < J_a \leq X$ contrary to the statement that $X$ is an atom. Thus $a$ is an atom of $L$. Finally $0 < J_a \leq X$ implies $J_a = X$.

(ii) Suppose $L$ is atomic and $J_0 < I$ in $K(L)$. There exists $x \in I$ with $0 < x$ and hence an atom $p \in L$ such that $0 < p \leq x$. But then $J_0 < J_p \leq I$ so $K(L)$ is atomic.

If $K(L)$ is atomic and $0 < a$ in $L$, then there exists an atom $p$ with $J_0 < J_p \leq J_a$ in $K(L)$. Hence $0 < p \leq a$.

(iii) Let $L$ be atomistic and $I < J$ in $K(L)$. There exists $x \in J$ such that $x \notin I$. Let $S = \{p | p \leq x, p \text{ an atom}\}$. If $S \subseteq I$, then $x = \vee S \in I$ since $I$ is a complete ideal. Therefore, there exists
q ∈ S with q /∈ I. It follows that J_q is atom, J_q ≤ J_x ≤ J and J_q /∈ I. Thus K(L) is atomistic.

Conversely, assume 0 < a and a is not the join of the set S of its atoms. Then there is b < a such that b is an upper bound of S. Thus \( \bigvee_k (J_p | p ∈ S) \leq J_b < J_a \). Since \( J_p \) is an atom contained in \( J_a \) if and only if p is an atom contained in a, K(L) is not atomistic.

(iv) If \( X ∈ K(L) \) is finite, then \( X = J_{p_1} \lor J_{p_2} \lor \cdots \lor J_{p_n} \) for some atoms \( p_i \) (i = 1, ..., n). Conversely, for a finite element \( a = J_{p_1} \lor J_{p_2} \lor \cdots \lor J_{p_n} \), \( J_a = J_{p_1} \lor J_{p_2} \lor \cdots \lor J_{p_n} \) is finite in K(L).

(v) Clearly the atoms of B(L) are exactly those of K(L). The statements (i) through (iv) then hold for B(L).

Before proceeding we pause to look at various completions of certain lattices.

1. Let \( L \) be any chain. If \( X \) is an element of the completion by cuts \( \overline{L} \) and \( S ⊆ X \) such that \( \forall S \) exists, then \( \forall S \leq y \) for all \( y ∈ X^u \). Thus \( \forall S ∈ X^{ul} = X \), so \( X ∈ K(L) \). On the other hand, if \( X ∈ K(L) \), let \( y ∈ X^{ul} \). If \( y ≤ x \) for some \( x ∈ X \), then \( y ∈ X \).
If \( y \not\leq x \) for all \( x \in X \), then \( x \leq y \) for all \( x \in X \) since \( L \) is a chain. Since \( y \leq z \) for all \( z \in X^u \), 
\( y = \bigvee X \). But then \( X \) complete implies \( y \in X \), so 
\( X = X^u \) and \( X \in \overline{L} \). Thus \( K(L) = \overline{L} \). Similarly, 
\( B(L) = \overline{L} \). In particular, if \( Q \) is the lattice of rational numbers, \( B(L) \) is the lattice of real numbers.

2. Let \( V \) be an infinite dimensional vector space and \( L \) be the lattice of all finite dimensional subspaces of \( V \). Then \( L \) is conditionally complete so both \( \overline{L} \) and \( B(L) \) are isomorphic to \( L \). Furthermore, 
\( \overline{L} \) is isomorphic to \( L \cup \{ V \} \). If \( I \) is any ideal of \( L \) and \( S \subseteq I \) such that \( \forall S \) exists, then \( \forall S \) is finite dimensional and hence there exists a finite subset of \( I \) whose join is \( \forall S \). Thus \( \forall S \in I \), showing that \( I \) is complete. Therefore, \( K(L) = I(L) \). Moreover, if \( \langle I \rangle \) denotes the subspace of \( V \) generated by \( I \), then the mapping \( I \mapsto \langle I \rangle \) of \( K(L) \) into \( L(V) \), the lattice of all subspaces of \( V \), is an isomorphism onto \( L(V) \).

3. Let \( V \) be an infinite dimensional vector space and \( L \) be the lattice of all finite dimensional and cofinite dimensional subspaces of \( V \). Then 
\( B(L) = K(L) \) and \( \overline{L} = \overline{L} \). An ideal of \( L \) is complete if and only if it is a principal ideal generated by a finite or cofinite dimensional subspace, or if it
is an ideal of all finite subspaces of an infinite dimensional subspace which is not cofinite dimensional. As in the previous example, the mapping \(I \mapsto \langle I \rangle\) is an isomorphism of \(K(L)\) onto the lattice \(L(V)\) of all subspaces of \(V\).

Now \(L(V)\) is a complete modular DAC-lattice (see [10], Definition (27.1)), so from [10], Theorem (28.3), it follows that \(L(V)\) is isomorphic to the completion by cuts of the sublattice \(L\) of finite and cofinite elements of \(L(V)\). Thus \(\overline{L}\) is isomorphic to \(L(V)\) and \(\overline{L} = K(L)\).

4. Let \(H\) be an infinite dimensional Hilbert space and \(L\) be the lattice of all finite dimensional subspaces and their orthogonal complements. By [10], Theorem (28.3), the lattice \(L_C(H)\) of all closed subspaces of \(H\) is isomorphic to \(\overline{L}\). To determine the elements of \(K(L)\), let \(I\) be a complete ideal of \(L\). Then \(I\) is either a principal ideal generated by a finite subspace or the orthogonal complement of a finite subspace, or an ideal of all finite subspaces contained in an infinite dimensional subspace whose closure is not an element of \(L\). The mapping \(I \mapsto \langle I \rangle\) is an isomorphism of \(K(L)\) onto the lattice of all subspaces of \(H\) which are finite dimensional or orthogonal complements of finite dimensional subspaces, or whose closures are
not orthogonal complements of finite dimensional subspaces.

5. Let $V$ be an infinite dimensional vector space and let $A$ and $B$ be complementary infinite dimensional subspaces. Let $M_0$ denote the set of subspaces $\{(0), A, V, B' | B' \text{ is a subspace of } B \text{ and } B' \text{ is cofinite in } B\}$. If $L$ is the set $\{X + F | X \in M_0 \text{ and } F \text{ is a finite dimensional subspace of } V\}$ partially ordered by inclusion, then $K(L)$ is isomorphic to the partially ordered set of subspaces $\{X + F | X \in M \text{ and } F \text{ finite dimensional}\}$ where $M = \{(0), A, V, B' | B' \text{ is a subspace of } B\}$. This lattice $K(L)$ was first constructed by M. F. Janowitz as an example of a finite modular AC-lattice which is not $M$-symmetric (see Definitions 2.16, 2.22, and 2.27). The lattice $L$ also has this property.

The Supremum in $K(L)$

We now resume our investigation of lattices whose completion by complete ideals has a join expressable by a formula. Unlike the conditionally upper continuous case where the join of any two elements in $K(L)$ has such a representation, we shall be able to describe the join of only special, but useful elements. All the lattices considered here have the covering property.
Definition 2.16. In a lattice with \( 0 \), the following condition is called the **covering property**:

If \( p \) is an atom and \( a \land p = 0 \), then \( a \lor p \) covers \( a \).

An atomistic lattice with the covering property is called an **AC-lattice**.

All of the examples 2 through 5, pp. 34–36 are AC-lattices.

Theorem 2.17. In a lattice \( L \) with \( 0 \), the following statements are equivalent.

(i) \( L \) has the covering property.

(ii) If \( p \) is an atom, then \( (p,x)M \) for every \( x \in L \).

(iii) If \( p \) is an atom, then \( (p,x)M^* \) for every \( x \in L \).

Proof: Maeda [10], Theorem (7.6), p. 31.

Clearly, any modular (and hence, distributive) lattice with \( 0 \) has the covering property.

Lemma 2.18. Let a lattice \( L \) with \( 0 \) have the covering property. If \( p \) is an atom and \( \lor(x \lor p \mid x \in X) \) exists for a subset \( X \) of \( L \), then \( \lor X \) exists.
Proof: Let \( z = \lor (x \lor p \mid x \in X) \). Now \( z \) is an upper bound of \( X \). If \( z \) is not the supremum of \( X \), there exists an upper bound \( w_1 \) of \( X \) such that \( w_1 \neq z \). Set \( w = w_1 \land z \). Then \( w \) is an upper bound of \( X \) and \( w < z \).

Now \( w \land p = 0 \) for if \( w \land p \neq 0 \), then \( p \leq w \) and since \( w \) is an upper bound of \( X \), \( z = \lor (p \lor x \mid x \in X) \leq w \) contradicting \( w < z \). If \( y \) is any upper bound of \( X \), then \( y \land w \) is an upper bound of \( X \). Since \( (y \land w) \land p = y \land 0 = 0 \), by the covering property \( (y \land w) \lor p \) covers \( y \land w \). Now \( x \lor p \leq (y \land w) \lor p \) for every \( x \in X \) so \( z = \lor (x \lor p \mid x \in X) \leq (y \land w) \lor p \leq w \lor p \leq z \lor p = z \). Thus \( z \) covers \( y \land w \).

Since \( y \land w \leq w < z \) we have \( y \land w = w \) and \( w \leq y \).

Thus \( w = \lor X \).

If \( a \) is a finite element, then \( a = p_1 \lor p_2 \lor \ldots \lor p_n \) for atoms \( p_i \) \((i = 1, \ldots, n)\).

By induction we have

Corollary 2.19. Let a lattice with \( 0 \) have the covering property. If \( X \) is a subset of \( L \) and \( a \) is a finite element such that \( \lor (x \lor a \mid x \in X) \) exists, then \( \lor X \) exists.

Theorem 2.20. Let \( a \) be a finite element of a lattice \( L \) with the covering property and \( X \in K(L) \). If one of
X and $J_a$ is a standard ideal of $L$, then

$$X \vee_{\mathcal{K}} J_a = X \vee J_a.$$  

Moreover, if $a$ is a standard element of $L$, then $J_a$ is a standard element of $K(L)$.

**Proof:** Let $S \subseteq X \vee J_a$ such that $\forall S$ exists in $L$. Since $X$ or $J_a$ is standard, each of the elements of $X \vee J_a$ is of the form $x \vee a_1$ for some $x \in X$ and $a_1 \leq a$. Let $T = \{ x \mid x \in X, x \vee a_1 \in S \}$. Then $
(t \vee a_1 \in S) = \forall S \forall a$. By Corollary 2.19 $\forall T$ exists, and since $X$ is complete, $\forall T \subseteq X$. Thus $\forall S \subseteq \forall T \forall a \in X \vee J_a$, showing that $X \vee J_a$ is complete. Therefore, $X \vee_{\mathcal{K}} J_a = X \vee J_a$.

Now let $a$ be a standard element of $L$. Then $J_a$ is a standard ideal. If $X,Y \in K(L)$, then

$$(X \wedge J_a) \vee_{\mathcal{K}} (X \wedge Y) \leq X \wedge (J_a \vee Y) = X \wedge (J_a \vee Y) = (X \wedge J_a) \vee (X \wedge Y) \leq (X \wedge J_a) \vee_{\mathcal{K}} (X \wedge Y),$$

so $J_a$ is a standard element in $K(L)$.

If $L$ is a distributive lattice with 0, then every ideal is standard and $L$ has the covering property, so $X \vee_{\mathcal{K}} J_a = X \vee J_a$ for every finite element $a \in L$ and every $X \in K(L)$.

**Lemma 2.21.** Let $S \subseteq I \vee J_a$ in $I(L)$. If $\forall S$ exists in $L$ and $(x,a) \in S$ for all $x \in L$, then there is a
subset \( T \subseteq I \lor J_a \) such that

(i) \( t \in T \) implies \( t = x \lor a \) for some \( x \in I \),

(ii) \( \forall T \) exists in \( L \), and

(iii) \( \forall T \geq \forall S \).

**Proof:** Let \( z = \forall S \) with \( S \subseteq I \lor J_a \). Then \( z \lor a = \lor(s \lor a | s \in S) \). Let \( w = z \lor a \).

Now for each \( s \in S \), \( s \lor a \in I \lor J_a \) so we can find an \( s' \in I \) depending on \( s \) such that \( s \lor a \leq s' \lor a \).

Since \( w \geq s \lor a \geq a \) and \( (x, a)M^* \) for every \( x \in L \), \( s \lor a = w \land (s \lor a) \leq w \land (s' \lor a) = (w \land s') \lor a \leq w \land a = w \). Now let \( T = \{ (w \land s') \lor a | s \in S \} \). Note that \( w \land s' \in I \) so \( T \) satisfies (i). We have shown that \( w \) is an upper bound of \( T \). If \( y \) is any upper bound of \( T \), then \( y \geq (w \land s') \lor a = w \land (s' \lor a) \geq w \land (s \lor a) \geq s \lor a \) for every \( s \in S \). Thus \( y \geq \lor(s \lor a | s \in S) = w \) and hence \( w = \forall T \). Since \( w \geq z = \forall S \), (ii) and (iii) are satisfied.

Before proceeding further, we shall need the following definitions.

**Definition 2.22.** A lattice \( L \) is called \( M \)-symmetric (resp. \( M^* \)-symmetric) if \( (a, b)M \) implies \( (b, a)M \) (resp. \( (a, b)M^* \) implies \( (b, a)M^* \)).

Whenever \( (a, b)M \) implies \( (b, a)M^* \), \( L \) is said to be cross-symmetric.
Certainly, every modular lattice is $M$-symmetric, $M^*$-symmetric and cross-symmetric. Moreover,

**Lemma 2.23.** Every cross-symmetric lattice is $M$-symmetric. Every $M$-symmetric lattice with $0$ has the covering property.

**Proof:** The first statement is Maeda [10], Theorem (1.9) p. 3. The second follows from [10], Corollary (7.7), p. 32.

Now we are in a position to describe the join in $K(L)$ of any element and any atom when $L$ is cross-symmetric and has a least element.

**Theorem 2.24.** If a lattice $L$ with $0$ is cross-symmetric then $I \lor_{K} J_{p} = I \lor J_{p}$ for every atom $p$ and every $I \in K(L)$. In particular, if $L$ is a modular lattice with $0$, then $I \lor_{K} J_{p} = I \lor J_{p}$ where $p$ is an atom and $I \in K(L)$.

**Proof:** Let $L$ be a cross-symmetric lattice with $0$. We must show that $I \lor J_{p}$ is complete if $p$ is an atom and $I \in K(L)$. Let $S \subseteq I \lor J_{p}$ such that $\lor S$ exists in $L$. By Lemma 2.23 and Theorem 2.17, $(p,x)M$ for every $x \in L$ and since $L$ is cross-symmetric, $(x,p)M^*$ for every $x \in L$. Applying Lemma 2.21, we can find a subset $T \subseteq I \lor J_{p}$ such that $T = \{x \lor p \mid x \in X\}$ for some $X \subseteq I$ and $\lor T \geq \lor S$. From
Lemma 2.18 it follows that \( \forall X \) exists and since 
\( I \in K(L), \forall X \in I. \) Thus \( \forall S \leq \forall T = \forall X \lor p \in I \lor J_p, \)
so \( I \lor J_p \) is complete.

Since every modular lattice is cross-symmetric, the second statement follows directly.

In the proof above cross-symmetry is used only to show that the condition \((p,x)M^* \) and \((x,p)M^*\) for all \( x \in L \) and all atoms \( p \) holds in \( L \). The hypothesis that \( L \) is an \( M^* \)-symmetric lattice with the covering property is also sufficient for this condition and hence the conclusion of Theorem 2.24 to hold.

We shall need the following lemma whose proof is an obvious modification of the proof of Lemma 2.21.

**Lemma 2.25.** Let \( S \subseteq I \lor J_a \) in \( I(L) \) and \( \forall S \) exist in \( L \). If there exists \( b \in I \) such that \((x,a \lor b)M^* \) for all \( x \geq b \), then there is a subset \( T \subseteq I \lor J_a \) such that

(i) \( t \in T \) implies \( t = x \lor a \) for some \( x \in I, \)

(ii) \( \forall T \) exists in \( L, \) and

(iii) \( \forall T \geq \forall S. \)

**Lemma 2.26.** Let \( a \) be a finite element of a lattice \( L \) with the covering property. If the ideal \( I \in K(L) \)
has an element \( b \) such that \((x, b \lor a)M^*\) for all \( x \geq b \), then \( I \lor^\_J_a = I \lor J_a^\_\!

**Proof:** Let \( S \subseteq I \lor J_a \) with \( \forall S \in L \). By Lemma 2.25, there exists a subset \( T \subseteq I \lor J_a \) such that \( \forall T \geq \forall S \) and \( T = \{x \lor a | x \in X\} \) for some \( X \subseteq I \). It follows from Lemma 2.18 that \( \forall X \) exists, and since \( I \) is complete, \( \forall X \in I \). Thus \( \forall S \leq \forall T = \forall X \lor a \in I \lor J_a \) and hence \( I \lor J_a = I \lor^\_J_a \).

We pause briefly to describe certain AC-lattices which satisfy the conditions of Lemma 2.26.

**Definition 2.27.** An element \( a \) of a lattice \( L \) is a **modular element** if \((x, a)M\) for every \( x \in L \).

A lattice \( L \) is **finite modular** when every finite element is modular.

All of the examples 2 through 5, pp. 34-36 are finite modular AC-lattices. The following theorem characterizes finite modular AC-lattices.

**Theorem 2.28.** In an AC-lattice \( L \), the following statements are equivalent.

(i) \( L \) is finite modular.

(ii) If \( p \) and \( q \) are atoms and \( p \leq q \lor a \) \((a \neq 0)\), then there exists an atom \( r \in L \) such that \( p \leq q \lor r \) and \( r \leq a \).
(iii) If \( p \) is an atom, then \((x,p)_{\mathcal{M}^*}\) for every \( x \in L \).

(iv) \( L \) is \( \mathcal{M}^* \)-symmetric.

**Proof:** See Maeda [10], Lemma (9.2), p. 41 and Theorem (9.5), p. 42.

**Definition 2.29.** A lattice \( L \) with \( 0 \) is strongly planar if \( L \) satisfies the condition:

If \( p, q, \) and \( r \) are atoms, \( p \leq q \lor a \), and \( r \leq a \), then there exists an atom \( s \) such that \( p \leq q \lor r \lor s \) and \( s \leq a \).

**Remark 2.30.** By Theorem 2.28 (ii), every finite modular AC-lattice is strongly planar.

The following lemma shows that an AC-lattice is relatively AC and that the atoms of an interval are easily determined.

**Lemma 2.31.** Let \( a < b \) in an AC-lattice \( L \). Then \( [a,b] \) is an AC-lattice and an element \( c \in L \) is an atom of \( [a,b] \) if and only if there exists an atom \( p \) of \( L \) such that \( c = a \lor p \), \( a \land p = 0 \), and \( p \leq b \).

A similar statement holds for the sublattice \([a)\).

**Proof:** Maeda [10], Lemma (8.18), p. 39.
Lemma 2.32. An AC-lattice is strongly planar if and only if the sublattice \([p]\) is a finite modular AC-lattice for every atom \(p\) of \(L\).


We are now able to show that for a strongly planar AC-lattice \(L\), the join of a complete ideal and an atom of \(K(L)\) is the same in both \(K(L)\) and \(I(L)\).

Theorem 2.33. Let \(L\) be a strongly planar AC-lattice. If \(p\) is an atom and \(I \in K(L)\), then \(I \vee_k J_p = I \vee J_p\).

Proof: If \(I = J_0\) or \(p \notin I\) the proof is trivial so assume \(I \neq J_0\) and \(p \notin I\). Choose any atom \(q \in I\). Then \(q \neq p\) so by Lemma 2.31, \(p \vee q\) is an atom of \([q]\). It follows from Lemma 2.32 and Theorem 2.28 that \([q]\) is a finite modular AC-lattice and \((x, p \vee q)^M\) for every \(x \geq q\). By Lemma 2.26, \(I \vee_k J_p = I \vee J_p\).

Since every finite modular AC-lattice is strongly planar, we have

Corollary 2.34. If \(L\) is a finite modular AC-lattice, then \(I \vee_k J_p = I \vee J_p\) for every atom \(p\) and every \(I \in K(L)\).
Finally, we note that Theorem 2.33 and Corollary 2.34 can be extended to include all finite elements by the following theorem.

**Theorem 2.35.** Let $L$ be a lattice with $0$. If $I \vee \bigvee_{p \in L} J_p = I \vee J_p$ for every $I \in K(L)$ and every atom $p \in L$, then $I \vee \bigvee_{a \in L} J_a = I \vee J_a$ for every $I \in K(L)$ and every finite element $a \in L$.

**Proof:** Let $a = p_1 \vee p_2 \vee \cdots \vee p_n$. If $n = 1$, there is nothing to prove. By induction, $I \vee \bigvee_{a \in L} J_a = (I \vee \bigvee_{a \in L} J_a) \vee J_{p_n} = (I \vee J_{p_1} \vee \cdots \vee J_{p_{n-1}} \vee J_{p_n}) \vee J_{p_n} = I \vee (J_{p_1} \vee \cdots \vee J_{p_{n-1}} \vee J_{p_n}) = I \vee J_a$. 

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Our first goal in this chapter is to establish that the $B(L)$ conditional completion of a conditionally upper continuous modular gSSC lattice is a relatively complemental modular lattice. We shall have need of the following three lemmas, the first of which is a generalization of [11], Satz 1.4.

Lemma 3.1. Let $L$ be an upper continuous lattice. Then $[x,1]$ is semicomplemented for every $x \in L$ if and only if $[x,1]$ is complemented for every $x \in L$.

Proof: Recall that an upper continuous lattice is complete and hence $1 \in L$. Suppose that $[x,1]$ is semicomplemented for every $x \in L$ and $b < 1$ in $L$. If $b < c < 1$, let $C$ be the family of all semicomplements of $c$ in $[b,1]$. By hypothesis $C \neq \emptyset$. Let $X$ be any chain in $C$. Since $L$ is complete, $\bigvee X$ exists. Applying upper continuity we have $\bigvee X \wedge c = \bigvee (x \wedge c | x \in X) = b$. Of course $\bigvee X > b$ since $x > b$ for each $x \in X$. Thus $\bigvee X$ is a semicomplement of $c$ in $[b,1]$ showing $\bigvee X \in C$. By Zorn's Lemma there exists a maximal element $w \in C$. Suppose $w$ is not a complement of $c$ in $[b,1]$. Then $c \vee w < 1$ since we have $c \wedge w = b$. There exists a semicomplement $y$ of $c \vee w$ in $[w,1]$. Thus $b < w < y$ and
c \land y = c \land (c \lor w) \land y = c \land w = b \text{ showing that } y \in C.

But this contradicts the maximality of \( w \). We must conclude that \( c \lor w = 1 \) and hence \( w \) is a complement of \( c \) in \( [b, 1] \).

The converse is direct.

**Corollary 3.2.** Let \( L \) be a conditionally complete and conditionally upper continuous lattice. Then \( L \) is relatively semicomplemented if and only if it is relatively complemented.

**Proof:** Let \( b < a \) in \( L \). Then \( [b,a] \) is complete and upper continuous. If \( L \) is relatively semicomplemented, then \( [x,a] \) is semicomplemented for every \( x \in [b,a] \). It follows from Lemma 3.1 that \( [b,a] \) is complemented.

The converse is direct.

Among the properties preserved by the injection of \( L \) into \( B(L) \) is semicomplementation, at least in the following sense.

**Lemma 3.3.** In a lattice \( L \), \( [b,a] \) is semicomplemented if and only if \( [J_b,J_a] \) is semicomplemented in \( B(L) \).

**Proof:** Suppose first that \( [J_b,J_a] \) is semicomplemented. If \( b \leq c < a \), then \( J_b \leq J_c < J_a \) in \( B(L) \).

There exists \( X \in B(L) \) such that \( J_b < X \leq J_a \) and \( J_b = J_c \land X \). Let \( d \in X \) such that \( d \notin J_b \). Then
b < b ∨ d and b ∨ d ∈ X so that \( J_b \leq J_{b ∨ d} \land J_c \leq X \land J_c = J_b \) and \( J_b < J_{b ∨ d} \leq J_a \). It follows that 
\[ b < b ∨ d \leq a \quad \text{and} \quad b = (b ∨ d) \land c. \]

Now let \([b, a]\) be semicomplemented and \( J_b \leq X < J_a \). Then \( a \) is not the least upper bound of \( X \) for if \( a = \vee X \), then \( a \in X \) since \( X \) is complete. Therefore there exists an upper bound \( d \) of \( X \) such that \( d \not< a \). But then \( c = d \land a \) is an upper bound of \( X \) and \( c < a \). We have \( J_b \leq X \leq J_c < J_a \) and thus \( b \leq c < a \). Since \([b, a]\) is semicomplemented, there exists \( y \in L \) such that \( b = c \land y \) and \( b < y \leq a \). Then 
\[ J_b \leq X \land J_y \leq J_c \land J_y = J_{c \land y} = J_b \quad \text{and} \quad J_b < J_y \leq J_a \] 
so \([J_b, J_a]\) is semicomplemented.

Recall that complemented modular lattices are relatively complemented. In the case of semicomplemented modular lattices the following is true.

**Lemma 3.4.** Let \( L \) be a modular lattice. If \([b, a]\) is semicomplemented, then \([x, a]\) is semicomplemented for every \( x \in [b, a] \).

**Proof:** If \( b < x \leq c < a \), there exists \( d \in L \) such that \( b < d \leq a \) and \( c \land d = b \). Then \( (x \lor d) \land c = x \lor (d \land c) = x \lor b = x \) since \( L \) is modular, and \( x \leq x \lor d \leq a \). If \( x = x \lor d \), then \( d \leq x \) so that \( b < d \leq d \land x \leq d \land c = b \). Thus \( x < x \lor d \) and \( x \lor d \) is a
semicomplement of \( c \) in \([x,a]\).

**Corollary 3.5.** Every modular gSSC lattice \( L \) is relatively semicomplemented.

**Proof:** Let \( b < a \) in \( L \). Since \( L \) is gSSC, there exists \( e \leq b \) such that \([e]\) is SSC. But then \([e,a]\) is semicomplemented and by Lemma 3.4 \([b,a]\) is semicomplemented.

We now have established the tools needed to prove the following major theorem.

**Theorem 3.6.** Let \( L \) be a conditionally upper continuous modular gSSC lattice. Then \( B(L) \) is a conditionally complete, conditionally upper continuous relatively complemented modular lattice.

If \( L \) moreover has 0 and 1, then \( K(L) \) is an upper continuous complemented modular lattice.

**Proof:** The lattice \( B(L) \) is conditionally complete, conditionally upper continuous, and modular by Theorem 2.11 and Theorem 2.13, so we need only show that \( B(L) \) is relatively complemented.

Let \( X < Y \) in \( B(L) \). Since \( X \neq \emptyset \) and \( Y \) is bounded above, there exist \( a,b \in L \) such that \( J_b \leq X < Y \leq J_a \). By Corollary 3.5, \( L \) is relatively semicomplemented.

Thus \([b,a]\) is semicomplemented and by Lemma 3.3,
[J_{b}, J_{a}] is semicomplemented in B(L). Moreover, [J_{b}, J_{a}] is a complete upper continuous modular lattice. By Lemma 3.4 \([Z, J_{a}]\) is semicomplemented for every \(Z \in [J_{b}, J_{a}]\), and by Lemma 3.1 \([J_{b}, J_{a}]\) is complemented. Since a complemented modular lattice is relatively complemented \([X, Y]\) is complemented and hence B(L) is relatively complemented.

If L has 0 and 1, then B(L) = K(L) so K(L) is an upper continuous complemented modular lattice.

Note that the lattice L of Example 3, p. 34, of all finite and cofinite dimensional subspaces of an infinite dimensional vector space V is a conditionally upper continuous complemented modular (and hence, gSSC) lattice. Its completion by complete ideals is the lattice of all subspaces L(V) which is an upper continuous complemented modular lattice as required by Theorem 3.6.

Since the completion B(L) of a conditionally upper continuous lattice also preserves distributivity, we have the following corollary.

**Corollary 3.7.** If L is a conditionally upper continuous distributive gSSC lattice, then B(L) is a relatively complemented distributive lattice.

If L also has 0 and 1, then K(L) is a Boolean algebra.

The properties assumed on a lattice L in Theorem 3.6 are sufficiently strong to enable us to say something.
about the dual lattice. Namely, a lattice satisfying the conditions of Theorem 3.6 is also a \( g \text{DSSC} \) lattice.

**Lemma 3.8.** If \( \{ J_a \} \) is DSSC in \( B(L) \), then \( \{ a \} \) is DSSC in \( L \).

**Proof:** Let \( b < c < a \). Then \( J_b < J_c < J_a \) in \( B(L) \). Since \( \{ J_a \} \) is DSSC, there exists \( X \in B(L) \) such that \( J_b \leq X < J_a \) and \( J_c \lor_X X = J_a \). Now \( a \) is not the least upper bound of \( X \), for if \( a = \lor X \), then \( a \in X \) since \( X \) is complete and hence \( X < J_a \leq X \). Therefore there exists an upper bound \( d_1 \) of \( X \) such that \( d_1 \neq a \). If \( d = d_1 \wedge a \), then \( X \leq J_d < J_a \). It follows that \( J_b \leq X \leq J_d < J_a \) and \( J_a = J_c \lor_X X \leq J_c \lor J_d \leq J_a \). Thus \( b \leq d < a \) and \( c \lor d = a \) proving that \( \{ a \} \) is DSSC.

**Theorem 3.9.** Every conditionally upper continuous modular \( g \text{SSC} \) lattice \( L \) is also \( g \text{DSSC} \).

**Proof:** By Theorem 3.6 \( B(L) \) is relatively complemented and hence every interval is DSSC. For each \( a \in L \), \( \{ J_a \} \) is DSSC and by Lemma 3.8 \( \{ a \} \) is DSSC. Thus \( L \) is \( g \text{DSSC} \).

If the lattice satisfies the stronger property of distributivity a stronger connection exists between conditional upper continuity and the \( g \text{DSSC} \) property. To show
this we need the following lemma which generalizes [8], Lemma 3.8, p. 10.

**Lemma 3.10.** Every distributive gDSSC lattice $L$ is conditionally upper continuous.

**Proof:** Let $a_\delta \uparrow a$ and $b \in L$. Then $a \land b \geq a_\delta \land b$ for all $\delta$. Suppose $x \geq a_\delta \land b$ for all $\delta$, and assume $x < x \lor (a \land b)$. Since $L$ is gDSSC, we can find an $e \in L$ such that $e \geq a \lor b \lor x$ and $(e)$ is DSSC. Then there exists $y \in L$ satisfying $x \leq y < e$ and $y \lor (x \lor (a \land b)) = e$. Now $y \lor (a \land b) = y \lor x \lor (a \land b) = e$ and since $a \lor b \leq e$, we must have $y \lor a = y \lor b = e$.

From $y \geq x \geq a_\delta \land b$, it follows that for all $\delta$

$$y = y \lor (a_\delta \land b) = (y \lor a_\delta) \land (y \lor b) = (y \lor a_\delta) \land e = y \lor a_\delta \geq a_\delta.$$  Thus $y \geq a$ and $y = y \lor a = e$. But this is a contradiction since $y < e$. Therefore $x = x \lor (a \land b)$ so $a \land b \leq x$. Since $x$ was an arbitrary upper bound of $[a_\delta \land b]$, $a_\delta \land b \uparrow a \land b$.

**Theorem 3.11.** Let $L$ be a distributive gSSC lattice. Then $L$ is gDSSC if and only if $L$ is conditionally upper continuous.

**Proof:** If $L$ is conditionally upper continuous, then $L$ is a gDSSC lattice by Theorem 3.9. The converse holds by Lemma 3.10.
We now turn to the investigation of the completion by cuts of conditionally upper continuous modular gSSC lattices. The following lemma establishes a connection between $B(L)$ and the conditional completion by cuts $\sim L$.

**Lemma 3.12.** If $B(L)$ is principally gDSSC with respect to $L$, then $B(L)$ is the conditional completion by cuts $\sim L$ of $L$.

**Proof:** Let $X \in \sim L$ and $S \subseteq X$ such that $\forall S \in L$. If $y$ is any upper bound of $X$, then $y$ is an upper bound of $S$. Thus $\forall S \leq y$ for all $y \in X^u$ so $\forall S \in X^{ul} = X$. Therefore $X \in B(L)$.

If $X \in B(L)$ such that $X \notin \sim L$, then there exists $y \in X^{ul}$ such that $y \notin X$. Then $X \lor_{\sim} J_y > X$ and $X \lor_{\sim} J_y \in B(L)$. Since $B(L)$ is principally gDSSC with respect to $L$ there exists $a \in L$ such that $J_a \geq J_y \lor_{\sim} X > X$ and $(J_a)$ is DSSC. Therefore we can find a $Y \in B(L)$ satisfying $X \leq Y < J_a$ and $Y \lor_{\sim} (J_y \lor_{\sim} X) = J_a$. As in the proof of Lemma 3.8, since $Y$ is complete, there exists $c \in L$ such that $Y \leq J_c < J_a$. Thus $X \leq J_c < J_a$ and $J_a = Y \lor_{\sim} (J_y \lor_{\sim} X) \leq J_c \lor_{\sim} (J_y \lor_{\sim} X) \leq J_a$. It follows that $J_c \neq J_y \lor_{\sim} X$ and hence $c \notin (J_y \lor_{\sim} X)^u$. But $c \in X^u$ and $y \in X^{ul}$ so $y \leq c$ and hence $J_y \leq J_c$. Thus $J_y \lor_{\sim} X \leq J_c$ and $c \in (J_y \lor_{\sim} X)^u$ which is a
contradiction. Therefore \( X \notin \tilde{L} \).

It is worth noting that when \( B(L) \) is relatively complemented, it is principally \( gDSSC \) with respect to \( L \), so by Lemma 3.12 \( B(L) = \tilde{L} \).

Now if \( L \) has a greatest element \( 1 \) and \( K(L) \) is \( DSSC \), \( B(L) \) is \( DSSC \) and hence principally \( gDSSC \) with respect to \( L \). Thus \( B(L) = \tilde{L} \). Adjoining a least element to \( B(L) \) and \( \tilde{L} \) if such a least element does not exist, we have \( K(L) = \bar{L} \) the completion by cuts. In summary,

**Corollary 3.13.** If \( L \) has \( 1 \) and \( K(L) \) is \( DSSC \), then \( K(L) \) is the completion by cuts \( \bar{L} \) of \( L \).

**Theorem 3.14.** A lattice \( L \) is a conditionally upper continuous modular \( gSSC \) lattice if and only if the conditional completion by cuts \( \tilde{L} \) of \( L \) is a conditionally upper continuous relatively complemented modular lattice.

If \( L \) has \( 0 \) and \( 1 \), then \( L \) is a conditionally upper continuous modular \( SSC \) lattice if and only if its completion by cuts \( \bar{L} \) is an upper continuous complemented modular lattice.

**Proof:** If \( L \) is conditionally upper continuous, modular and \( gSSC \), then by Theorem 3.6 \( B(L) \) is a conditionally upper continuous relatively complemented modular lattice, and by Lemma 3.12 \( B(L) = \tilde{L} \).
Conversely, if \( \tilde{L} \) is relatively complemented, then it is principally gSSC with respect to \( L \) and by Theorem 2.2 \( L \) is gSSC. Since \( L \) is embedded into \( \tilde{L} \), the modularity of \( \tilde{L} \) implies the modularity of \( L \). To show that \( L \) is conditionally upper continuous, let \( \stackrel{\delta}{a} \uparrow a \) and \( b \in L \). If \( a \land b \) is not the join of \( \{ \stackrel{\delta}{a} \land b \} \), there exists an upper bound \( x \) of \( \{ \stackrel{\delta}{a} \land b \} \) such that \( x < a \land b \). Then \( \stackrel{\delta}{J}_a \land \stackrel{\delta}{J}_b = \stackrel{\delta}{J}_a \land \stackrel{\delta}{J}_b \leq \stackrel{\delta}{J}_x \land \stackrel{\delta}{J}_x \) for all \( \delta \). Since the embedding \( x \mapsto \stackrel{\delta}{J}_x \) preserves all joins, \( \stackrel{\delta}{J}_a \uparrow \stackrel{\delta}{J}_a \). Now \( \tilde{L} \) is conditionally upper continuous, so \( \stackrel{\delta}{J}_a \land \stackrel{\delta}{J}_b \uparrow \stackrel{\delta}{J}_a \land \stackrel{\delta}{J}_b \). Thus \( \stackrel{\delta}{J}_a \land \stackrel{\delta}{J}_b = \land \stackrel{\delta}{J}_a \land \stackrel{\delta}{J}_b = \lor (\stackrel{\delta}{J}_a \land \stackrel{\delta}{J}_b) \leq \stackrel{\delta}{J}_x \land \stackrel{\delta}{J}_x \) which is a contradiction. Therefore \( \stackrel{\delta}{a} \land \stackrel{\delta}{b} \uparrow a \land b \).

If \( L \) is a conditionally upper continuous distributive gSSC lattice, by Corollary 3.7 \( \mathcal{B}(L) \) is distributive. Clearly the statement of Theorem 3.14 still holds if the word "modular" is replaced by "distributive". Applying Theorem 3.11 we have the following corollary.

**Corollary 3.15.** A distributive lattice \( L \) is gSSC and gDSSC if and only if its conditional completion by cuts \( \tilde{L} \) is a relatively complemented distributive lattice.

If a distributive lattice \( L \) has \( 0 \) and \( 1 \), then it is SSC and DSSC if and only if its completion by cuts \( \tilde{L} \) is a Boolean algebra.
In a lattice with 0 and 1, \((x,1), (x,0), (x,y,1),\) and \((x,y,0)\) form modular pairs and distributive triples for all \(x, y \in L\) and all permutations of the elements. If the conditional completion by cuts \(\bar{L}\) of a lattice is modular or distributive, then adjoining a greatest element and a least element (when such elements do not exist) to obtain \(\bar{L}\) does not destroy that property. Thus the following statement follows from Theorem 3.14.

**Corollary 3.16.** Let \(L\) be a conditionally upper continuous gSSC lattice. If \(L\) is modular, then the completion by cuts \(\bar{L}\) is modular; if \(L\) is distributive, then \(\bar{L}\) is distributive.

**Remark 3.17.** Let \(L\) be an orthocomplemented (see [10], Definition (29.1)) modular lattice. By [10], Lemma (29.11), p. 131, the completion by cuts \(\bar{L}\) of \(L\) is also orthocomplemented. If \(L\) is also conditionally upper continuous (and by the dual automorphism is also conditionally continuous), by Theorem 3.14, \(\bar{L}\) is modular. Applying the celebrated theorem of Kaplansky (see [1], p. 273) we have that \(\bar{L}\) is a continuous geometry. Hence we may pose the following open question. Since \(\bar{L}\) is a continuous geometry, perspectivity is transitive in \(\bar{L}\). Is perspectivity transitive in \(L\)?
IV. AC-LATTICES

The Completion by Complete Ideals

We established in Chapter II that certain conditions, including the covering property, are sufficient to have $X \vee \bigvee_k J_a = X \vee J_a$ where $a$ is a finite element. In this section we shall make use of this equality to determine lattice properties which are preserved by the completion $K(L)$.

**Lemma 4.1.** If $(x,a)M^*$ (resp. $(a,x)M^*$) for all $x \in L$, then in $I(L)$, $(X,J_a)M^*$ (resp. $(J_a,X)M^*$) for all ideals $X$.

**Proof:** Let $(x,a)M^*$ for all $x \in L$ and $Y \supseteq J_a$. If $y \in Y \wedge (X \vee J_a)$, then $y \in Y$ and $y \leq x \vee a$ for some $x \in X$. Thus $y \vee a \leq x \vee a$ and $y \vee a \in Y \vee J_a = Y$. Then by $(x,a)M^*$, $y \leq y \vee a = (y \vee a) \wedge (x \vee a) = [(y \vee a) \wedge x] \vee a \in (Y \wedge X) \vee J_a$ so $Y \wedge (X \vee J_a) \subseteq (Y \wedge X) \vee J_a$.

Since the reverse inequality always holds we have $(X,J_a)M^*$.

The proof is similar for the statement that $(a,x)M^*$ for all $x$ implies $(J_a,X)M^*$ for all $X$. 

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Lemma 4.2. Let \( I, J \in K(L) \). If \( I \vee K J = \{ x | x = vS \text{ where } S \subseteq I \vee J \} \), then \((I,J)M^*\) in \( I(L) \) implies \((I,J)M^*\) in \( K(L) \).

Proof: Let \( Y \supseteq J \) in \( K(L) \). Note that \((Y \wedge I) \vee K J \subseteq Y \wedge (I \vee K J)\), so let \( w \in Y \wedge (I \vee K J) \). Then \( w \in I \vee K J \) and hence \( w = vS \) for some \( S \subseteq I \vee J \). But also \( w \in Y \) so that \( S \subseteq Y \wedge (I \vee J) \). Since \((I,J)M^*\) in \( I(L) \), we have \( S \subseteq Y \wedge (I \vee J) = (Y \wedge I) \vee J \subseteq (Y \wedge I) \vee K J \). Since \((Y \wedge I) \vee K J\) is complete, \( w = vS \in (Y \wedge I) \vee K J \).

Theorem 4.3. Let \( L \) be a lattice with 0. If \( L \) is cross-symmetric, then \( K(L) \) has the covering property.

Proof: By Lemma 2.23 \( L \) has the covering property, which by Theorem 2.17 is equivalent to \((p,x)M^*\) for every \( x \in L \) and every atom \( p \). If \( I \in K(L) \) and \( J_p \) is an atom of \( K(L) \), then \((J_p,I)M^*\) in \( I(L) \) by Lemma 4.1. From Theorem 2.24 we have \( I \vee K J_p = I \vee J_p \). Certainly then \( I \vee K J_p = \{ x | x = vS \text{ where } S \subseteq I \vee J_p \} \). By Lemma 4.2 \((J_p,I)M^*\) holds in \( K(L) \). Applying Theorem 2.17 again, we see that \( K(L) \) has the covering property.

By a slight modification of the above proof it can be shown that if \( L \) is an \( M^* \)-symmetric lattice with the covering property, then \( K(L) \) also has the covering property.
Corollary 4.4. If \( L \) is a modular lattice with 0, then \( K(L) \) has the covering property.

Theorem 4.5. If \( L \) is a strongly planar AC-lattice, then \( K(L) \) is a strongly planar AC-lattice.

Proof: We first show that \( K(L) \) is an AC-lattice. By Theorem 2.33 \( I \lor J_p = I \lor J_p \) for every atom \( p \in L \) and every \( I \in K(L) \). Since \( L \) has the covering property, \( (p,x)^* \) for all \( x \in X \). As in the proof of Theorem 4.3, \( (J_p,I)^* \) for all \( I \in K(L) \) follows from Lemma 4.1 and Lemma 4.2. Thus \( K(L) \) has the covering property. Finally, by Theorem 2.15 \( K(L) \) is atomistic.

To show that \( K(L) \) is strongly planar, let \( J_p \) be an atom of \( K(L) \). Then \( p \) is an atom of the strongly planar lattice \( L \), so by Lemma 2.32 \( (p) \) is a finite modular AC-lattice. If \( q \) is an atom of \( L \) with \( q \land p = 0 \), then from Lemma 2.31 and Lemma 2.28 \( (x,q \lor p)^* \) holds for all \( x \geq p \). By a proof similar to that of Lemma 4.1, \( (X,J_q \lor J_p)^* \) holds in \( I(L) \) for every \( X \). It follows from Theorem 2.33, Theorem 2.35, and Lemma 4.2 that \( (X,J_q \lor J_p)^* \) in \( K(L) \) for every atom \( J_q \) such that \( J_q \land J_p = 0 \) and every \( X \in K(L) \). Applying Lemma 2.28 once more, we have that \( [J_p] \) is finite modular for every atom \( J_p \in K(L) \). By Lemma 2.32 \( K(L) \) is strongly planar.
Theorem 4.6. If \( L \) is a finite modular AC-lattice, then \( K(L) \) is a finite modular AC-lattice.

Proof: Since every finite modular AC-lattice is strongly planar, \( K(L) \) is an AC-lattice. By Lemma 2.28 \((x,p)M^*\) holds for all \( x \in L \) and atoms \( p \in L \). By Lemma 4.1 and Lemma 4.2, we have \((X,J_p^p)M^*\) in \( K(L) \) for all atoms \( J_p \) and \( X \in K(L) \). Thus \( K(L) \) is finite modular.

Note that alternate proofs for Theorem 4.5 and Theorem 4.6 can be constructed using the remark following the proof of Theorem 4.3.

Corollary 4.7. If \( L \) is a modular atomistic lattice, then \( K(L) \) is a finite modular AC-lattice.

It is worth noting that the completion by cuts \( \overline{L} \) of a finite modular (or strongly planar) AC-lattice \( L \) is also a finite modular (strongly planar) AC-lattice (see [7], Theorem 4.5 and Remark 4.9). Moreover, the completion by cuts of an AC-lattice is an AC-lattice. This presents the following open questions. Is the completion \( K(L) \) of an arbitrary AC-lattice also an AC-lattice? If not, what other conditions are sufficient for \( K(L) \) to be an AC-lattice?

We have frequently used the strong connection between dual modularity in \( L \) and the nature of the
join $X \lor_K J_a$ (with a finite). We now establish a relationship between the join in $K(L)$ of arbitrary complete ideals and dual modularity in $K(L)$.

**Lemma 4.8.** Let $a$ and $b$ be nonzero elements of a finite modular AC-lattice. Then $(a,b)^{M^*}$ holds if and only if the following condition holds:

If $p$ is an atom such that $p \leq a \lor b$, then there exist two atoms $q$ and $r$ such that $p \leq q \lor r$, $q \leq a$, and $r \leq b$.

**Proof:** Maeda [10], Theorem (9.3), p. 42.

**Theorem 4.9.** Let $a$ and $b$ be nonzero elements of a strongly planar AC-lattice $L$. Then $(a,b)^{M^*}$ implies the following condition:

If $p$ is an atom such that $p \leq a \lor b$, then there exist atoms $q, r$ and $s$ such that $p \leq q \lor r \lor s$, $q \leq a \lor s$, and $r \lor s \leq b$.

**Proof:** Choose any atom $s$ such that $s \leq b$. We will divide the proof into three cases.

(Case 1) $s = b$. Since $p \leq a \lor b$, $p \leq a \lor s = (a \lor s) \lor s$. Since $L$ is strongly planar, by Definition 2.29 there exists an atom $q \leq a \lor s$ such that $p \leq q \lor s \lor s$. 

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(Case 2) \( s = a \). Then \( p \leq s \lor b \) and \( s \leq b \). By the definition of strongly planar there exists an atom \( r \leq b \) such that \( p \leq s \lor r \lor s \).

(Case 3) \( s \neq a \) and \( s \neq b \). Then \( s \lor a > s \) and \( b = s \lor b > s \). If \( p = s \), then \( p \leq q \lor s \lor s \) for any atom \( q \leq a \). Thus assume \( p \neq s \). Then \( p \lor s \) is an atom of \([s]\). Since \((a, b)M^*\), if \( y \geq b \), then 
\[
[y \land (a \lor s)] \lor b \leq y \land (a \lor s \lor b) = y \land (a \lor b) = (y \land a) \lor b \leq [y \land (a \lor s)] \lor b.
\]
Thus \((a \lor s, b)M^*\) in \([s]\). By Lemma 2.32 \([s]\) is a finite modular AC-lattice. We have \( a \lor s \) and \( b \) are nonzero elements of \([s]\), \((a \lor s, b)M^*\) in \([s]\) and \( p \lor s \) is an atom such that \( p \lor s \leq (a \lor s) \lor b \). By Lemma 4.8 there exist atoms \( q \) and \( r \) such that \( p \lor s \leq (q \lor s) \lor (r \lor s) \), \( q \lor s \leq a \lor s \), and \( r \lor s \leq b \).

Thus \( p \leq q \lor r \lor s \), \( q \leq a \lor s \), and \( r \lor s \leq b \).

**Theorem 4.10.** Let \( L \) be a strongly planar AC-lattice. If \((X,Y)M^*\) in \( K(L) \), then \( X \lor K Y = \{x| x = \lor S \text{ where } S \subseteq X \lor Y\} \).

**Proof:** By Theorem 4.5 \( K(L) \) is a strongly planar AC-lattice. Let \( X, Y \in K(L) \). Then \( \{x| x = \lor S \text{ where } S \subseteq X \lor Y\} \subseteq X \lor K Y \). If \( w \) is a nonzero element of \( X \lor K Y \), let \( p \) be an atom such that \( p \leq w \). Then \( p \in X \lor K Y \) and \( J_p \subseteq X \lor K Y \). By Theorem 4.9 there
exist atoms \( q, r \) and \( s \) such that \( p \leq q \lor r \lor s \), and
\( q \lor r \lor s \). Thus \( q \in X \lor J_s, r \lor s \in Y \) and \( p \leq q \lor r \lor s \), so \( p \in X \lor Y \). Letting \( S = \{p \mid p \text{ is an atom and } p \leq w \} \), we have \( S \subset X \lor Y \) and \( \forall S = w \).

Thus \( w \in \{x \mid x = \forall S \text{ where } S \subset X \lor Y \} \). Since the case where \( w = 0 \) is trivial, the theorem is proven.

If \( L \) is also modular the converse of Theorem 4.10 also holds.

**Theorem 4.11.** Let \( L \) be a modular atomistic lattice and \( X, Y \in K(L) \). Then \( X \lor Y = \{x \mid x = \forall S \text{ where } S \subset X \lor Y \} \) if and only if \((X,Y)M^* \) in \( K(L) \).

**Proof:** If \((X,Y)M^* \) in \( K(L) \), then from Theorem 4.10 we have \( X \lor Y = \{x \mid x = \forall S \text{ where } S \subset X \lor Y \} \).

Now suppose \( X \lor Y = \{x \mid x = \forall S \text{ where } S \subset X \lor Y \} \).

Since \( L \) is modular, \( I(L) \) is modular and hence \((X,Y)M^* \) holds in \( I(L) \). By Lemma 4.2 \((X,Y)M^* \) holds in \( K(L) \).

Observe the similarity between Theorem 4.11 and the theorem of Mackey which states that in the lattice \( L_C(V) \) of all closed subspaces of a linear system \( B \), the join of two closed subspaces \( W_1 \) and \( W_2 \) is \( W_1 + W_2 \) if and only if \((W_1,W_2)M^* \) holds in \( L_C(V) \).
Corollary 4.12. Let $L$ be a modular atomistic lattice. Then $K(L)$ is modular if and only if $X \vee_k Y = \{x \mid x = \vee S \text{ where } S \subseteq X \vee Y\}$ for all $X, Y \in K(L)$.

We close this section with an open question. Are there other conditions for AC-lattices (or for lattices in general) under which a formula for the joins in $K(L)$ can be explicitly determined?

Conditionally Upper Continuous AC-Lattices

In this section we investigate the $K(L)$ completion of conditionally upper continuous AC-lattices. The results are reminiscent of those in Chapter III.

Theorem 4.13. Let $L$ be a conditionally upper continuous lattice with 0. If $L$ has the covering property, then both $B(L)$ and $K(L)$ have the covering property.

Proof: Let $J_p$ be an atom of $K(L)$. Then $p$ is an atom of $L$ and since $L$ has the covering property, $(p, x) \ast$ for all $x \in L$. By Theorem 2.9 $X \vee_k J_p = \{x \mid x = \vee S \text{ where } S \subseteq X \vee J_p\}$. Applying Lemma 4.1 and Lemma 4.2, we have $(J_p, X) \ast$ for all
\( X \in K(L) \). Thus \( K(L) \) has the covering property.

A similar proof shows that \( B(L) \) has the covering property.

**Definition 4.14.** A complete lattice \( L \) is said to be **compactly atomistic** if it is atomistic and upper continuous.

A compactly atomistic \( M \)-symmetric lattice is called a **matroid lattice**.

**Theorem 4.15.** Let \( L \) be a compactly atomistic lattice. Then \( L \) is \( M \)-symmetric if and only if it has the covering property.

**Proof:** Maeda [10], Theorem (7.15), p. 34.

**Theorem 4.16.** Let \( L \) be a conditionally upper continuous lattice. If \( L \) is atomistic, then \( K(L) \) is compactly atomistic. If \( L \) is an AC-lattice, then \( K(L) \) is a matroid lattice.

**Proof:** The lattice \( K(L) \) is upper continuous by Theorem 2.9 and is atomistic by Theorem 2.15, so it is compactly atomistic. If \( L \) also has the covering property, then by Theorem 4.13 \( K(L) \) has the covering property. It follows from Theorem 4.15 that \( K(L) \) is a matroid lattice.
Lemma 4.17. If $L$ is a conditionally upper continuous AC-lattice, then $(a,b)M$ in $L$ if and only if $(J_a^*, J_b^*)M$ in $K(L)$.

Proof: Since $L$ is embedded into $K(L)$ by the mapping $x \mapsto J_x$, $(J_a^*, J_b^*)M$ implies $(a,b)M$.

If $(a,b)M$ holds in $L$, let $X \leq J_b^*$ in $K(L)$. Suppose $w \in (X \lor J_a^*) \land J_b^*$. Then $w \in X \lor J_a^*$ and $w \in J_b^*$. By Theorem 2.9 $w = \lor S$ for some $S \subseteq X \lor J_a^*$. Certainly $S \subseteq J_b^*$ also. If $s \in S$, then $s \leq x \lor a$ for some $x \in X$. Now $X \leq J_b^*$ so $x \leq b$ and by $(a,b)M$ we have $s = s \land b \leq (x \lor a) \land b = x \lor (a \land b) \in X \lor (J_a^* \land J_b^*) \leq X \lor K (J_a^* \land J_b^*)$. Thus $S \subseteq X \lor K (J_a^* \land J_b^*)$. Since $w = \lor S$ and since $X \lor K (J_a^* \land J_b^*)$ is complete, $w \in X \lor K (J_a^* \land J_b^*)$. Therefore $(X \lor K J_a^*) \land J_b^* \leq X \lor K (J_a^* \land J_b^*)$. Since the reverse inclusion is always true, $(J_a^*, J_b^*)M$ holds in $K(L)$.

Corollary 4.18. Every conditionally upper continuous AC-lattice $L$ in $M$-symmetric.

Proof: By Theorem 4.16 $K(L)$ is a matroid lattice and hence is $M$-symmetric. If $(a,b)M$ in $L$, by Lemma 4.17 $(J_a^*, J_b^*)M$ holds in $K(L)$. But then $(J_b^*, J_a^*)M$ in $K(L)$ and using Lemma 4.17 once more, we have $(b,a)M$. 

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Corollary 4.18 also follows from [7], Theorem 4.12.

**Theorem 4.19.** If $L$ is a conditionally upper continuous AC-lattice, then $B(L)$ is conditionally upper continuous relatively complemented AC-lattice.

**Proof:** $B(L)$ is a conditionally upper continuous AC-lattice by Theorems 2.13, 2.15, and 4.13. To show that $B(L)$ is relatively complemented, let $X < Y$ in $B(L)$. By Lemma 2.31, $[Z,Y]$ is an AC-lattice for every $Z \in [X,Y]$ and is thus semicomplemented. Since $B(L)$ is always conditionally complete, $[X,Y]$ is complete. Observing that $[X,Y]$ is also upper continuous, we can apply Lemma 3.1. It follows that $[X,Y]$ is complemented and hence $B(L)$ is relatively complemented.

**Remark 4.20.** An alternate proof of Theorem 4.19 can be constructed in the following manner. By Theorem 4.16 $K(L)$ is a matroid lattice. From Maeda [10], Theorem (7.15), p. 34, Theorem (3.9), p. 12, and Corollary (2.10), p. 8, we have that $K(L)$ is relatively complemented. It is easily seen that $B(L)$ is an ideal of $K(L)$ and is, therefore, also relatively complemented. Using this method, we can show that $B(L)$ is, in fact, left complemented (see [10], Definition (3.7), p. 11).
Corollary 4.21. Every conditionally upper continuous AC-lattice $L$ is gDSSC.

Proof: $B(L)$ is principally gDSSC with respect to $L$. By Lemma 3.12 $B(L) = \overset{\sim}{L}$. It follows from Theorem 2.4 that $L$ is gDSSC.

Our goal now is to establish that for a lattice $L$ with $1$ the conditions of Theorem 4.19 are sufficient for $L$ to be dual atomistic.

Lemma 4.22. Let $L$ be a lattice with $1$. Then

(i) $X$ is a dual atom of $K(L)$ if and only if $X = J_a$ for some dual atom $a \in L$.

(ii) $L$ is dual atomic if and only if $K(L)$ is dual atomic.

(iii) If $K(L)$ is dual atomistic, then $L$ is dual atomistic.

(iv) $X$ is a dual finite element of $K(L)$ if and only if $X = J_a$ for some dual finite element $a \in L$.

(v) The above statements remain true if $K(L)$ is replaced by $B(L)$.

Proof: (i) Let $a$ be a dual atom of $L$. If $J_a \leq X < J_1$, then there exists an upper bound $b$ of $X$ such that $b < 1$ (otherwise $1 = \vee X$ and $1 \in X$.
since $X$ is complete). Then $J_a \leq X \leq J_b < 1$ so that $a \leq b < 1$. Since $a$ is a dual atom, $a = b$ and hence $J_a = X$. Thus $J_a$ is a dual atom of $K(L)$.

Now let $X$ be a dual atom of $K(L)$. Since $X < J_1$ there exists an upper bound $a$ of $X$ such that $a < 1$. Then $X \leq J_a < 1$. But $X$ is a dual atom so $X = J_a$. If $a$ is not a dual atom of $L$, there exists $b \in L$ such that $a < b < 1$. But then $X = J_a < J_b < 1$ and $X$ is not a dual atom of $K(L)$. Thus $a$ is a dual atom of $L$.

(ii) Let $L$ be dual atomic and $X < J_1$ in $K(L)$. There exists an upper bound $b$ of $X$ such that $b < 1$, and a dual atom $a \in L$ such that $b \leq a < 1$. Then $X \leq J_b \leq J_a < 1$ and $J_a$ is an atom. Thus $K(L)$ is atomic.

If $K(L)$ is atomic, then $L$ is atomic since $x \mapsto J_x$ is an embedding of $L$ into $K(L)$.

(iii) If $L$ is not dual atomistic, there exists $d \in L$ such that $d$ is not the infimum of its dual atoms. Therefore we can find $c \in L$ such that $d < c \leq a$ for every dual atom $a$ such that $a \geq d$. Then $J_d < J_c \leq J_a$ for every dual atom $J_a$ of $J_d$, and so $K(L)$ is not dual atomistic.

(iv) This follows from (i) and the embedding $x \mapsto J_x$. 

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(v) The proofs are identical when \( B(L) \) replaces \( K(L) \).

The converse of Lemma 4.22 (iii) is not true in general. Let \( L \) be the lattice of all finite subspaces of an infinite dimensional Hilbert space \( H \) together with their orthogonal complements and let \( L_1 \) be the lattice of subspaces of \( H \) which is isomorphic to \( K(L) \) (see example 4, p. 35). The lattice \( L \) is dual atomistic. If \( X \) is a nonclosed subspace in \( L_1 \), then its closure \( \overline{X} \) is also in \( L_1 \). The dual atoms of \( L_1 \) which contain \( X \) are the same as those which contain \( \overline{X} \) so \( L_1 \) (and hence \( K(L) \)) is not dual atomistic.

Lemma 4.23. An AC-lattice with 1 is dual atomistic if and only if it is DSSC.


Theorem 4.24. Let \( L \) be a conditionally upper continuous AC-lattice with 1. Then \( L \) is dual atomistic, \( K(L) = \overline{L} \) and \( \overline{L} \) is a matroid lattice.

Proof: Since \( 0,1 \in L \), \( B(L) = K(L) \) and \( \overline{L} = \overline{L} \).

By Theorem 4.19, \( K(L) \) is relatively complemented and
is therefore DSSC. It follows from Corollary 3.13 that \( K(L) = \overline{L} \) and from Lemma 4.23 that \( K(L) \) is dual atomistic. Finally \( L \) is dual atomistic by Lemma 4.22 (iii).

**Corollary 4.25.** If \( L \) is a conditionally upper continuous modular atomistic lattice with \( 1 \), then \( \overline{L} \) is a modular matroid lattice and \( L \) is dual atomistic.
V. IDEALS AND CONGRUENCE RELATIONS

In this chapter we investigate properties of ideals which are kernels of congruence relations. Most of the results obtained here are extensions to lattices without 0 of theorems for atomistic and SSC lattices (see Janowitz [6] and [8]).

Theorem 5.1. For a principal ideal \([z]\) of a gSSC lattice \(L\), the following are equivalent:

(i) \([z]\) is a standard ideal.

(ii) \([z]\) is a distributive ideal.

(iii) \([z]\) is a homomorphism kernel.

(iv) \([z]\) is a projective ideal.

Proof: (iv) implies (i). Suppose \([z]\) is not a standard ideal. Then there must exist \(a, b \in L\) such that \((a \lor z) \land b > (a \land b) \lor (z \land b)\). Since \(L\) is gSSC, we can find an element \(e \in L\) such that \([e]\) is SSC and \(e \leq a \land b \land z\). Then \(e \leq (a \land b) \lor (z \land b) < (a \lor z) \land b\), so there is \(x \in L\) satisfying the inequalities \(e < x \leq (a \lor z) \land b\) and \(x \land [(a \land b) \lor (z \land b)] = e\). Now \(x \leq b\), so \(x \land a = x \land (a \land b) \leq x \land [(a \land b) \lor (z \land b)] = e \leq z \land a\). Moreover,
\( x \leq (a \lor z) \land b \leq a \lor z, \) so \( x \lor a \leq z \lor a. \) Thus \( x \) is subperspective to \( z \) and since \( (z) \) is projective, \( x \leq z. \) Recalling that \( x \leq b, \) we have \( x \leq z \land b \leq (a \land b) \lor (z \land b) \) so that \( e \leq x = x \land [(a \land b) \lor (z \land b)] = e \) which is impossible. Therefore, we must have \( (a \lor z) \land b = (a \land b) \lor (z \land b) \) for all \( a, b \in L, \) and thus \( (z) \) is standard.

Since (i) implies (ii), (ii) implies (iii), and (iii) implies (iv) always hold, the proof is complete.

**Corollary 5.2.** In a gSSC lattice, every distributive element is standard.

Following Grätzer and Schmidt ([4], Definition 4, p. 162) we define the notion of weakly modular as follows.

**Definition 5.3.** The lattice \( L \) is weakly modular if \( a > b, c > d \) and \( a, b \rightarrow c, d \) imply the existence of \( a_1, b_1 \in L \) such that \( a \geq a_1 > b_1 \geq b \) and \( c, d \rightarrow a_1, b_1. \)

Grätzer and Schmidt have shown that all modular lattices and all relatively complemented lattices are weakly modular (see [4], Lemma 16, p. 162).

The next three results are from [5]. For convenience we include their proofs.
Lemma 5.4. An element $d$ of a lattice $L$ is distributive if and only if $a \leq b \leq d \leq c \leq e$ and $\overline{a,b} \rightarrow \overline{c,d}$ imply $c = d$.

Proof: Let $d$ be a distributive element. Then $x \equiv y(\Theta(d))$ if and only if $[(x \wedge y) \vee d] \wedge (x \vee y) = x \vee y$ and, moreover, $[d]$ is the kernel of $\Theta(d)$. If $a \leq b \leq d \leq c \leq e$, then since $a \equiv b(\Theta(d))$, $\overline{a,b} \rightarrow \overline{c,d}$ implies $c \equiv d(\Theta(d))$. Therefore, $c = [(c \wedge e) \vee d] \wedge (c \vee e) = c \vee e = e$.

Conversely, if $d$ is not distributive, there exist $x,y \in L$ with $d \vee (x \wedge y) < (d \vee x) \wedge (d \vee y)$. Now $d \vee (x \wedge y) \equiv (d \vee x) \wedge (d \vee y) (\Theta(d))$, so there exist $a,b \in L$ such that $a \leq b \leq d$ and $d \vee (x \wedge y) \equiv (d \vee x) \wedge (d \vee y) (\Theta(a,b))$. By Theorem 1.12, there exists $y_i \in L$ such that $(d \vee x) \wedge (d \vee y) = y_0 \geq y_1 \geq \ldots \geq y_n = d \vee (x \wedge y)$ and $\overline{a,b} \rightarrow \overline{y_i,y_i'}$ for each $i$. Since $d \vee (x \wedge y) < (d \vee x) \wedge (d \vee y)$, $y_{i-1} > y_i$ for one of the pairs $y_{i-1}, y_i$. Denoting these elements by $e$ and $c$, respectively, we have $a \leq b \leq d \leq c \leq e$, $\overline{a,b} \rightarrow \overline{c,d}$, and $c \neq d$, so the condition fails.

Theorem 5.5. A standard element $s$ of a lattice $L$ is neutral if and only if $a \geq b \geq s \geq c \geq e$ and $\overline{a,b} \rightarrow \overline{c,e}$ imply $c = e$. 

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Proof: If \( s \) is neutral, then it is dual distributive and the condition holds by the dual of Lemma 5.4.

If \( s \) satisfies the condition, then by the dual of Lemma 5.4, it is dual distributive. Since it is also standard, \( s \) is neutral.

Corollary 5.6. If the lattice \( L \) is weakly modular, then every standard element in \( L \) is neutral.

Proof: Suppose \( s \) is a standard element which is not neutral. Then by Theorem 5.5, there exists elements \( a > b \geq s \geq c > d \) with \( \overline{a, b} \rightarrow \overline{c, d} \). Since \( L \) is weakly modular, there exist \( a_1, b_1 \in L \) such that \( a \geq a_1 > b_1 \geq b \) and \( \overline{c, d} \rightarrow \overline{a_1, b_1} \). But \( c \equiv d(\overline{s}) \), so \( a_1 \neq b_1(\overline{s}) \). Since \( s \) is standard, \( a_1 = b_1 \vee s_1 \) for some \( s_1 \leq s \). Then \( a_1 = a_1 \vee s = b_1 \vee s_1 \vee s = b_1 \) which is impossible. Therefore \( s \) must be neutral.

Our next goal is to find conditions which imply weak modularity. Then in view of Theorem 5.1 and Corollary 5.6 these same conditions are sufficient for principal projective ideals (and hence principal homomorphism kernels) of gSSC lattices to be neutral.

Theorem 5.7. If a lattice \( L \) is both gSSC and gDSSC, then \( L \) is weakly modular.
Proof: Let \( a > b \) and \( a \land x \geq a' > b' \geq b \land x \).

Since \( L \) is gSSC, there exists \( e \in L \) such that \( e \leq b \land x \) and \( (e) \) is SSC. Since \( e \leq b' < a' \), we can find \( y \in L \) such that \( e < y \leq a' \) and \( y \land b' = 0 \).

Then since \( y \leq a' \leq x \), \( e \leq y \land b = y \land b \land x = y \land b \land x \land b' = e \), so \( b < b \lor y \leq a \). Observe that \( (a' \land y) \lor b = y \lor b \) and \( (b' \land y) \lor b = e \lor b = b \), so that \( a',b' \rightarrow b \lor y,b \). A similar proof using the fact that \( L \) is gDSSC shows that if \( a \lor x \geq a'' > b'' \geq b \), then \( a'',b'' \rightarrow a_1,b_1 \) for some \( a_1,b_1 \in L \) such that \( a \geq a_1 > b_1 \geq b \).

Now let \( a > b \), \( c > d \), and \( a,b \rightarrow c,d \). Then there exist \( x_1, x_2, \ldots, x_n \in L \) such that
\[
[\ldots((a \lor x_1) \land x_2) \lor x_3) \land \ldots] \lor x_n = c \quad \text{and} \\
[\ldots((b \lor x_1) \land x_2) \lor x_3) \land \ldots] \lor x_n = d.
\]
By the above discussion and a simple induction proof, \( c,d \rightarrow a_1,b_1 \) where \( a \geq a_1 > b_1 \geq b \). Thus \( L \) is weakly modular.

Corollary 5.8. If a lattice \( L \) is both gSSC and gDSSC, then a principal ideal \( (a] \) is projective if and only if it is neutral.

Theorem 5.9. Let \( L \) be a gSSC lattice. Then \( L \) is weakly modular if and only if it satisfies the following condition:
\((*)\) \( a < b \leq c < d \) and \( \overline{a,b} \rightarrow \overline{c,d} \) together imply 
\( \overline{c,d} \rightarrow \overline{a_1,b_1} \) for some \( a_1 \) and \( b_1 \) such that 
\( a \leq a_1 < b_1 \leq b \).

**Proof:** First assume that \( L \) satisfies the condition. Let \( a > b, c > d, \) and \( \overline{a,b} \rightarrow \overline{c,d} \). Then there exist elements \( x_1, \ldots, x_n, y_1, \ldots, y_n \in L \) such that 
\( (\ldots[(a \lor x_1) \land y_1] \lor \ldots) \land x_n) \land y_n = c \) and
\( (\ldots[(b \lor x_1) \land y_1] \lor \ldots) \land x_n) \land y_n = d. \) We proceed by induction on \( n \) to find \( a_1, b_1 \in L \) such that 
\( b \leq b_1 < a_1 \leq a \) and \( \overline{c,d} \rightarrow \overline{a_1,b_1} \).

\((n = 1). \) Since \( L \) is gSSC, there exist \( e \in L \) such that \( e \leq b \land x_1 \land y_1 \) and \( [e] \) is SSC. Since \( c > d, \) \( (a \lor x_1) \land y_1 > (b \lor x_1) \land y_1. \) Then there exists \( z \in L \) such that \( e < z \leq (a \lor x_1) \land y_1 \) and \( z \land (b \lor x_1) \land y_1 = e. \) We consider two cases.

(Case 1) \( z \not< a. \) Now \( [(a \lor x_1) \land y_1 \land z] \lor a = z \lor a \) and \( [(b \lor x_1) \land y_1 \land z] \lor a = e \lor a = a, \) so \( \overline{a,b} \rightarrow z \lor a, a. \) Since \( z \not< a, \) \( b < a < z \lor a. \) Applying the condition, we have \( \overline{z \lor a, a} \rightarrow \overline{a_1,b_1} \) where 
\( b \leq b_1 < a_1 \leq a. \) But \( (a \lor x_1) \land y_1, (b \lor x_1) \land y_1 \rightarrow z \lor a, a, \) so \( (a \lor x_1) \land y_1, (b \lor x_1) \land y_1 \rightarrow \overline{a_1,b_1} \) and \( b \leq b_1 < a_1 \leq a. \)
(Case 2) $z \leq a$. Suppose first that $z \leq b$.

Since $z \leq (a \lor x_1) \land y_1 \leq y_1$, $e < z = (b \lor x_1) \land z = (b \lor x_1) \land y_1 \land z = e$ which is impossible. Thus $z \not< b$ and $z \lor b > b$. Now $[(a \lor x_1) \land y_1 \land z] \lor b = z \lor b$ and $[(b \lor x_1) \land y_1 \land z] \lor b = e \lor b = b$, so

$$\frac{(a \lor x_1) \land y_1, (b \lor x_1) \land y_1}{z \lor b, b}$$

with $b < z \lor b \leq a$.

(Induction step). Note that

$$\frac{(a \lor x_1) \land y_1, (b \lor x_1) \land y_1}{c,d}.$$ By induction $c,d \rightarrow h,k$ where $(b \lor x_1) \land y_1 \leq k < h \leq (a \lor x_1) \land y_1$.

As before, let $e \leq (b \land x_1) \land y_1$ with $[e]$ SSC and let $z$ satisfy $z \land k = e$ and $e < z \leq h$.

(Case 1) $z \not< a$. Now $(h \land z) \lor a = z \lor a$ and $(k \land z) \lor a = e \lor a$, so $a,b \rightarrow c,d \rightarrow h,k \rightarrow z \lor a,a$ with $b < a < z \lor a$. Applying the condition, we have $\frac{z \lor a,a}{a_1,b_1}$ for some $a_1,b_1 \in L$ such that $b \leq b_1 < a_1 \leq a$. It follows that $c,d \rightarrow h,k \rightarrow z \lor a,a \rightarrow a_1,b_1$.

(Case 2) $z \leq a$. Once again we have $z \not< b$ since $z \leq y_1$. Then $(h \land z) \lor b = z \lor b$ and $(k \land z) \lor b = e \lor b = b$, so $c,d \rightarrow h,k \rightarrow z \lor b,b$ and $b < z \lor b \leq a$.

Thus $L$ is weakly modular.

The converse is direct.
Corollary 5.10. If a gSSC lattice satisfies the condition (*) of Theorem 5.9, then a principal ideal \( \langle a \rangle \) is projective if and only if it is neutral.

We have been able to characterize only the principal homomorphism kernels in an arbitrary gSSC lattice \( L \). If the lattice is also conditionally complete we can say more.

Theorem 5.11. Let \( L \) be a conditionally complete gSSC lattice. A complete ideal \( J \) is the kernel of a congruence relation if and only if it is a standard ideal.

Proof: Let \( J \) be the kernel of the congruence relation \( \Theta \). By Theorem 1.16, it suffices to show that \( a \equiv b (\Theta) \) implies \( a \lor b = (a \land b) \lor s \) for some \( s \in J \). If \( a \equiv b (\Theta) \), then \( a \lor b = a \land b (\Theta) \). Choose \( s_1 \in J \). Since \( L \) is gSSC, there exists \( e \in L \) such that \( e \leq a \land b \land s_1 \) and \( \langle e \rangle \) is SSC. Note that \( e \in J \).

If \( a \lor b = a \land b \), then \( (a \lor b) = (a \land b) \lor e \). If \( a \lor b > a \land b \geq e \), there exists \( x \in L \) such that \( e < x \leq a \lor b \) and \( x \land a \land b = e \). Let \( S \) be the set of all semicomplements of \( a \land b \) in \( [e, a \lor b] \) and let \( s = \lor S \). Then \( (a \land b) \lor s \leq a \lor b \). If \( (a \land b) \lor s < a \lor b \), we can find \( z \in L \) such that \( e < z \leq a \lor b \) and \( z \land [(a \land b) \lor s] = e \). Thus \( z \land (a \land b) = e \) and \( z \in S \). Therefore
e < z = z ∧ s ≤ z ∧ [(a ∧ b) ∨ s] = e which is impossible. Hence (a ∧ b) ∨ s = a ∨ b.

Now if x ∈ S, then since a ∨ b ≡ a ∧ b (Θ), x = (a ∨ b) ∧ x = x ∧ (a ∧ b) = e (Θ). But e ∈ J and J is the kernel of Θ so x ∈ J. Thus S ⊆ J. Since J is complete and ∀s = s, we have s ∈ J, so a ∨ b = (a ∧ b) ∨ s with s ∈ J.

The converse is direct.

We have previously established the equivalence of principal projective ideals and homomorphism kernels in gSSC lattices. The following questions remain open. Are complete projective ideals in conditionally complete gSSC lattices homomorphism kernels and hence standard ideals? What can be said about arbitrary homomorphism kernels in gSSC lattices?

Definition 5.12. The element b of a lattice L is subprojective to a if there is a finite sequence b = x₀, x₁, ..., xₙ = a with xᵢ subperspective to xᵢ₊₁ (i = 0, 1, ..., n-1).

We shall let F(a) denote the set of all elements which are finite joins of elements each of which is subprojective to a.

Our goal now is to prove that in a section complemented lattice L, F(a) is the kernel of the smallest congruence relation Θ[a] which identifies the elements
of (a]. This is similar to a result for section complemented lattices with 0 which appears in [8]. There projectivity is defined using the following definition of subperspectivity: \( b \) is subperspective to \( a \) if there exists an \( x \) such that \( b \leq a \lor x \) and \( b \land x = 0 \).

Note that this implies subperspectivity in our sense of the word.

For the remainder of this chapter, \( L \) will denote a section complemented lattice.

Lemma 5.13. Let \( b \) be subprojective to \( a \) and \( c \leq b \lor x \). Then there exists \( y \in L \) such that \( c = y \lor (x \land c) \) and \( y \) is subprojective to \( a \).

Proof: Since \( L \) is section complemented, there exists \( e \in L \) such that \( e \leq x \land c \land b \) and \( [e] \) is a section complemented lattice with zero. Let \( y \) be a complement of \( x \land c \) in \( [e,c] \). Then \( y \land x = (y \land c) \land x \leq e \leq b \land x \) and \( y \lor x \leq c \lor x \leq b \lor x \). Thus \( y \) is subperspective to \( b \). Since \( b \) is subprojective to \( a \), \( y \) is subprojective to \( a \).

Finally, note that \( c = y \lor (x \land c) \) since \( y \) is a complement of \( (x \land c) \) in \( [e,c] \).

Lemma 5.14. Let \( b \in \mathcal{F}(a) \) and \( c \leq b \lor x \). Then there exists \( y \in \mathcal{F}(a) \) such that \( c = y \lor (x \land c) \).
Proof: Since \( b \in F(a) \), \( b = b_1 \lor \cdots \lor b_n \) with each \( b_i \) subprojective to \( a \). We proceed by induction on \( n \).

The case when \( n = 1 \) is Lemma 5.13, so assume \( b = b_1 \lor \cdots \lor b_k \lor b_{k+1} \). Then \( c \leq (b_1 \lor \cdots \lor b_{k+1}) \lor x = (b_1 \lor \cdots \lor b_k) \lor (b_{k+1} \lor x) \). By induction \( c = y \lor [(b_{k+1} \lor x) \land x] \) with \( y \in F(a) \). Now \( (b_{k+1} \lor x) \land c \leq b_{k+1} \lor x \), so by Lemma 5.13, \( (b_{k+1} \lor x) \land c = y_1 \lor (x \land c) \) with \( y_1 \) subprojective to \( a \). Thus \( c = y \lor [y_1 \lor (x \land c)] = (y \lor y_1) \lor (x \land c) \) and \( y \lor y_1 \in F(a) \).

Theorem 5.15. If \( L \) is a section complemented lattice, then \( F(a) \) is an ideal of \( L \).

Proof: It is clear that \( b, c \in F(a) \) imply that \( b \lor c \in F(a) \). If \( c \leq b \) and \( b \in F(a) \), then \( b = b_1 \lor b_2 \lor \cdots \lor b_n \) with \( b_i \) subprojective to \( a \). Choose \( e \in L \) such that \( e \leq c \land b_1 \land b_2 \land \cdots \land b_n \) and \( (e) \) is section complemented. Then \( c \leq b = (b_1 \lor b_2 \lor \cdots \lor b_{n-1}) \lor b_n \). By Lemma 5.14, there exists \( y \in F(a) \) such that \( c = y \lor (b_n \land c) \). Now \( b_n \land c \leq b_n = b_n \land e \), so by Lemma 5.13, \( b_n \land c = y_1 \lor [e \land (b_n \land c)] = y_1 \lor e \) with \( y_1 \) subprojective to \( a \). It is clear from the proof of Lemma 5.13 that \( y_1 \geq e \). Thus \( b_n \land c = y_1 \), so \( c = y \lor (b_n \land c) = y \lor y_1 \in F(a) \). Therefore \( F(a) \) is an ideal.
Theorem 5.16. Let $L$ be a section complemented lattice. If $\Theta(a)$ is the smallest congruence relation identifying $(a]$, then $F(a)$ is the kernel of $\Theta(a)$.

**Proof:** Since the kernel of $\Theta(a)$ is a projective ideal, $b$ subprojective to $a$ implies $b \in \ker \Theta(a)$. Thus $(a] \subseteq F(a) \subseteq \ker \Theta(a)$. Therefore it suffices to show that $F(a)$ is a homomorphism kernel.

Let $b \in F(a)$, $c \vee x \leq b \vee x$, and $c \wedge x \leq b \wedge x$. Since $c \leq b \vee x$, by Lemma 5.14, there exists $y \in F(a)$ such that $c = y \vee (c \wedge x)$. Since $c \wedge x \leq b \wedge x \leq b$, $c \wedge x \in F(a)$. Therefore $c = y \vee (c \wedge x) \in F(a)$ and hence $F(a)$ is a projective ideal. By Theorem 1.22, $F(a)$ is a homomorphism kernel.

Corollary 5.17. If $L$ is a section complemented lattice, then $x \equiv y (\Theta(a))$ if and only if $x \vee b_1 \vee \ldots \vee b_n = y \vee b_1 \vee \ldots \vee b_n$ for some $b_i$ ($i = 1, \ldots, n$) such that $b_i$ is subprojective to $a$ for each $i$.

**Proof:** $x \equiv y (\Theta(a))$ if and only if $x \equiv y (\Theta[F(a)])$ if and only if $x \vee y = (x \wedge y) \vee b$ for some $b \in F(a)$. Then $(x \wedge y) \vee b = x \vee y \vee b$, so $x \vee b = y \vee b$. Writing $b = b_1 \vee \ldots \vee b_n$ with $b_i$ subprojective to $a$, we have $x \equiv y (\Theta(a))$ implies $x \vee b_1 \vee \ldots \vee b_n = y \vee b_1 \vee \ldots \vee b_n$.

The converse is direct since $b_i \in F(a)$.

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