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Discrete Approximations to Continuous Optimal Control Problems

Maurice L. Eggen
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DISCRETE APPROXIMATIONS
TO
CONTINUOUS OPTIMAL CONTROL PROBLEMS

by

Maurice L. Eggen

A Dissertation
Submitted to the
Faculty of The Graduate College
in partial fulfillment
of the
Degree of Doctor of Philosophy

Western Michigan University
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Maurice Eggen
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I. INTRODUCTION

Many times when one considers an "Optimal Control Problem", one seeks to minimize or maximize a given integral, the "cost" function. Generally this integral is defined on the solutions of a system of differential equations, the solution of the system of equations depending on a particular "control" variable. Thus one seeks the particular control function, determining the solution of the system of differential equations, which will make the value of the cost function a maximum or a minimum, hence the "optimal" control.

According to various authors, an optimal control problem which is representable by a system of ordinary differential equations, an integral cost function, and control and state variable constraints is a "Continuous Optimal Control Problem."

In most cases when one seeks the optimal control for such a problem, one must resort to numerical methods. In such cases, the original differential problem is replaced by a difference problem, that is, the integral is replaced by an "approximating" sum, and the differential equation is replaced by a difference equation.
One then seeks the optimal control for the "discrete" problem, and hopes that the minimum so obtained is a reasonable representative of the actual minimum. Thus, for a sequence of such approximating problems, there arises the question of the convergence of the optimal value of the difference problem to the optimal value of the differential problem. In this paper, this is studied for two classes of problems.

In the first class, the infimum of a given integral is sought. This integral is defined on the solutions of a first order initial value system of differential equations. This problem is studied in chapter three, and sufficient conditions for the desired convergence are presented.

In the second class, again the infimum of an integral is sought. In this case, the integral is defined on the solutions of a second order boundary value differential equation. This problem is studied in chapter four and again sufficient conditions for the desired convergence are presented.

Both of the questions of chapter three and chapter four are expressed as special cases of a more general optimization problem. In chapter two of this paper, the more general problem is considered and a general theorem...
is obtained which gives sufficient conditions for the convergence of infima.

With the convergence theorems at hand, the continuous optimal control problem is essentially reduced to a sequence of discrete optimal control problems. The discrete problems may also present some difficulty in obtaining their optimal controls. With this in mind, chapters five and six present an adaptation of a method of V.F. Krotov [13] for finding upper and lower bounds for the infima of the discrete problems which arise in chapters three and four.

We conclude the introduction with some basic notations that will be used throughout the paper.

**Note 1.1:** Throughout, we have used $x_n$ to represent a sequence. It will be clear from context whether we are talking about the entire sequence or its $n$th entry.

**Note 1.2:** We denote the closure of the subset $A$ of a topological space $X$ by $\overline{A}$. The interior of $A$ is $A^\circ$. The boundary of $A$ is $\partial A$. 

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Note 1.3: By the uniform topology on a space of continuous functions, we mean the topology derived from the supremum norm,

\[ \|x\| = \sup |x(t)|. \]

Note 1.4: Wherever possible, unnecessary symbols are omitted, for example

\[ \inf_{x \in D} J(x) = \inf_{D} J. \]

Note 1.5: In this paper when we say "nhd" (neighborhood), we mean "open nhd."

II. GENERAL CONVERGENCE THEOREM

The material of this chapter will present sufficient conditions for the infimum of an approximating functional to converge to the infimum of a given functional. The major result of this chapter is essentially an extension, simplification, and clarification of work presented by B. M. Budak and E. M. Berkovic in [1]. The result presented here will be the foundation for the major results of the next two chapters. Before this material can be presented, it is necessary to state some definitions and prove some
fundamental theorems.

§2.1 Topological Limits Inferior and Superior

Let $E$ be a topological space with topology $\tau$ and let $D_n$ be a sequence of subsets of $E$.

**Definition 2.1:** The topological limit inferior of the sequence $D_n$, denoted $\liminf_{\tau} D_n$, is defined by

$$\liminf_{\tau} D_n = \{x \in E: \text{ For each nhd } V \text{ of } x, \text{ there is an integer } N(V) \text{ such that if } n > N, \text{ then } V \cap D_n \neq \emptyset \}.$$

**Definition 2.2:** The topological limit superior of the sequence $D_n$, denoted $\limsup_{\tau} D_n$, is defined by

$$\limsup_{\tau} D_n = \{x \in E: \text{ For each nhd } V \text{ of } x \text{ and each integer } N > 0, \text{ there is } n > N \text{ such that } V \cap D_n \neq \emptyset \}.$$

**Definition 2.3:** The limit inferior of the sequence $D_n$, denoted $\liminf D_n$, is defined by

$$\liminf D_n = \bigcup_{N} \bigcap_{n > N} D_n.$$
**Definition 2.4:** The limit superior of the sequence \( D_n \), denoted \( \limsup D_n \), is defined by

\[
\limsup D_n = \bigcap_{n>N} \bigcup_{N=n} D_n.
\]

The following properties state the relationships between these definitions.

**Proposition 2.5:** \( \liminf \tau D_n = \liminf \tau \text{cl}(D_n) \).

**Proof:** \( x \in \liminf \tau D_n \) iff for every nhd \( V \) of \( x \), there exists \( N \) such that if \( n > N \) then \( V \cap D_n \neq \emptyset \) iff for every nhd \( V \) of \( x \) there exists \( N \) such that if \( n > N \) then \( V \cap \text{cl}(D_n) \neq \emptyset \) iff \( x \in \liminf \tau \text{cl}(D_n) \).

**Proposition 2.6:** \( \limsup \tau D_n = \limsup \tau \text{cl}(D_n) \).

**Proof:** Similar to that of Proposition 2.5.

**Proposition 2.7:** \( \liminf \tau D_n \) is a closed set.

**Proof:** Let \( x \) be an element of \( \text{cl}(\liminf \tau D_n) \). Then for any nhd of \( V \) of \( x \), \( V \cap \liminf \tau D_n \neq \emptyset \). Therefore, \( V \) contains a point \( x' \) of \( \liminf \tau D_n \). By definition, there is \( N \) such that if \( n > N \), then \( V \cap D_n \neq \emptyset \). Since this is true for each nhd of \( x \), \( x \in \liminf \tau D_n \).
Proposition 2.8: \( \limsup_{\tau} D_n \) is a closed set.

Proof: Similar to that above.

The next proposition relates definitions 2.1 and 2.3.

Proposition 2.9: \( \liminf_{\tau} \overline{D}_n \subseteq \liminf_{\tau} D_n \), with equality holding if \( \tau \) is the discrete topology.

Proof: Let \( x \) be an element of \( \liminf_{\tau} \overline{D}_n \). Then for some \( N \), \( x \in \bigcap \{ \overline{D}_n : n > N \} \). Therefore, for this \( N \), \( x \in \overline{D}_n \) for all \( n > N \). Let \( V \) be any nhd of \( x \). Therefore for \( n > N \), \( x \in V \cap \overline{D}_n \neq \emptyset \), and since \( V \) is open, \( V \cap D_n \neq \emptyset \) for \( n > N \). Hence \( x \in \liminf_{\tau} D_n \).

Let \( \tau \) be the discrete topology. We require the opposite containment, so let \( x \) be an element of \( \liminf_{\tau} D_n \). Since \( \tau \) is discrete, \( \{x\} \) is a nhd of \( x \), so by definition, there is an integer \( N \) such that if \( n > N \), then \( \{x\} \cap D_n \neq \emptyset \), i.e. \( x \in D_n \). This implies \( x \in \bigcup_{N} \bigcap_{n > N} \overline{D}_n = \liminf_{\tau} \overline{D}_n \).

The following two propositions determine what happens when additional hypotheses are added to the sequence \( D_n \).
Proposition 2.10: If $D_n$ is a contracting sequence, then $\liminf_{\tau_n} D_n = \cap \text{cl}(D_n)$.

Proof: Let $x$ be an element of $\cap \text{cl}(D_n)$. Let $V$ be any nhd of $x$. Then for every $n$, $x \in V \cap \text{cl}(D_n) \neq \emptyset$. This implies that for each $n$, $V \cap D_n \neq \emptyset$, which means that $x \in \liminf_{\tau_n} D_n$. Note that this did not use the contracting condition, so that $\cap \text{cl}(D_n) \subseteq \liminf_{\tau_n} D_n$ in any case.

For the opposite containment, let $x \in \liminf_{\tau_n} D_n$, and let $V$ be any nhd of $x$. Then there is $N$ such that for $n > N$, $V \cap D_n \neq \emptyset$. But the sequence $D_n$ is contracting, so $V \cap D_n \neq \emptyset$ for every $n$. Therefore $x \in \text{cl}(D_n)$ for every $n$, so $x \in \cap \text{cl}(D_n)$.

Proposition 2.11: If the sequence $D_n$ is expanding, then

$$\liminf_{\tau_n} D_n = \text{cl}(U D_n).$$

Proof: Let $x$ be an element of $\text{cl}(U D_n)$, and let $V$ be any nhd of $x$. Then $V \cap (U D_n) \neq \emptyset$. This implies that $V \cap D_n \neq \emptyset$ for some $N$. But the sequence $D_n$ is expanding, so $V \cap D_n \neq \emptyset$ for $n > N$. Therefore $x \in \liminf_{\tau_n} D_n$. 

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For the opposite containment, let \( x \in \lim \inf_{\tau} D_n \), and let \( V \) be any nhd of \( x \). Then there is an integer \( N \) such that if \( n > N \), then \( V \cap D_n \neq \emptyset \). This implies that \( V \cap (\cup D_n) \neq \emptyset \). Therefore \( x \in \text{cl}(\cup D_n) \). Note that, in proving this last containment, we did not use the hypothesis that \( D_n \) be expanding. Therefore, we conclude \( \lim \inf_{\tau} D_n \subseteq \text{cl}(\cup D_n) \) in any case.

§ 2.2. Sufficient Conditions for Convergence of Infima.

With the basic machinery at hand, we shall now explain the setting for the major result of this section.

Consider a topological space \( E \) with two (possibly different) topologies \( \tau \) and \( \xi \), a subset \( D \) of \( E \), a sequence of subsets \( D_n \), and a sequence of subsets \( E_n \). Assume \( J \) is a functional defined on \( E \). Consider also a sequence of functionals \( J_n \) defined on a sequence of sets \( \Delta_n \), and mappings \( P_n: \Delta_n \rightarrow E_n \), and \( Q_n: D_n \rightarrow \Delta_n \), as in the following diagram.

![Diagram](image-url)
Theorem 2.12: If the following conditions are satisfied,

then  \( \lim \inf_{n \to \infty} J_n = \inf_{D_n} D \)

(2.1) \( J \) is \( \tau \)-upper semi-continuous on \( E \).

(2.2) \( \lim \inf_{D_n} D \).

(2.3) \( \limsup_{n \to \infty} (J_n(z_n) - J(z_n)) \leq 0 \) for any sequence \( z_n \) with \( z_n \in D_n \) for each \( n \).

(2.4) \( \liminf_{n \to \infty} (J_n(y_n) - J(y_n)) \geq 0 \) for any sequence \( y_n \) with \( y_n \in D_n \) for each \( n \).

(2.5) \( J \) is \( \xi \)-lower semi-continuous on \( E \).

(2.6) For any sequence of points \( x_j \) with \( x_j \in E_{n_j} \), \( E_{n_j} \) is a subsequence of \( E_n \), there exists a subsequence \( x_{j_k} \) converging to a point \( x \in D \) in the \( \xi \)-topology.

The proof of Theorem 2.12 will be done in the following four lemmas.

Lemma 2.13: Conditions (2.1) and (2.2) imply that

\[
\limsup_{n \to \infty} \inf_{D_n} D \leq \inf_{D} D
\]
Lemma 2.14: Condition (2.3) implies that
\[
\limsup_{n \to \infty} \inf_{D_n} J \leq \limsup_{n \to \infty} \inf_{D_n} J.
\]

Lemma 2.15: Condition (2.4) implies that
\[
\liminf_{n \to \infty} \inf_{E_n} J \leq \liminf_{n \to \infty} \inf_{A_n} J_n.
\]

Lemma 2.16: Conditions (2.5) and (2.6) imply that
\[
\liminf_{n \to \infty} \inf_{E_n} J = \inf_{D_n} J.
\]

Combining the results of Lemmas (2.13) to (2.16),
\[
\inf_{D_n} J \leq \liminf_{n \to \infty} \inf_{E_n} J \leq \liminf_{n \to \infty} \inf_{A_n} J_n \leq \limsup_{n \to \infty} \inf_{A_n} J_n \leq \limsup_{n \to \infty} \inf_{D_n} J \leq \inf_{D_n} J.
\]

Therefore \( \liminf_{n \to \infty} J_n = \inf_{D_n} J \).

Proof of Lemma 2.13: Let \( \varepsilon > 0 \) be given. There exists \( x \in D \) such that \( J(x) < \inf_{D_n} J + \varepsilon \). Since \( J \) is upper semi-continuous, there is a nhδ \( V \) of \( x \) such that if \( y \in V \) then \( J(y) - \varepsilon \leq J(x) \). Also since \( \liminf_{n \to \infty} D_n = D \), there is an \( N \) such that if \( n > N \), \( V \cap D_n \neq \emptyset \). Therefore, for \( n > N \), \( \exists y_n \in D_n \) such that \( \inf_{D_n} J - \varepsilon \leq J(y_n) - \varepsilon \leq J(x) < \inf_{D_n} J + \varepsilon \).

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Therefore, $\inf J < \inf J + 2\varepsilon$. Hence

$$\limsup_{n \to \infty} \inf_{D_n} J \leq \inf J.$$

**Proof of Lemma 2.14:** Assume $\limsup_{n \to \infty} \inf_{D_n} J_n > \limsup_{n \to \infty} \Delta_n$.

Choose $c$ so that $\limsup_{n \to \infty} \inf_{D_n} J > c > \limsup_{n \to \infty} \inf_{D_n} J$. By definition, there is $N$ such that for $n > N$, $\inf_{D_n} J < c$. Therefore, there is $z_n \in D_n$ so that $J(z_n) < c$, for $n > N$. For $n \leq N$, let $z_n \in D_n$ be arbitrary. Then, for the sequence $z_n$,

$$J_n(z_n) - J(z_n) > J_n(Q_n z_n) - c \quad \text{for} \quad n > N.$$

Hence

$$\limsup (J_n(Q_n z_n) - J(z_n)) \geq \limsup (J_n(Q_n z_n) - c) \geq \limsup (J_n(Q_n z_n)) - c \geq \limsup J_n(Q_n z_n) - \limsup \inf_{\Delta_n} J_n \geq 0.$$
But this contradicts the hypothesis.

**Proof of Lemma 2.15:** Let $\varepsilon > 0$ be given. By definition of infimum, there is $y_n \in \Delta_n$ such that

$$\inf J_n + \varepsilon > J_n(y_n).$$

By hypothesis, $\lim \inf(J_n(y_n) - J(p_\infty y_n)) \geq 0$. Therefore, there is $N$ such that

$$J_n(y_n) - J(p_\infty y_n) > -\varepsilon \quad \text{for} \quad n > N.$$ Hence, for $n > N$,

$$\inf \left. \Delta_n \right| J_n + \varepsilon > J_n(y_n) > J(p_\infty y_n) - \varepsilon \geq \inf \left. E_n \right| J - \varepsilon.$$ Thus

$$\lim \inf \inf \left. \Delta_n \right| J_n \geq \lim \inf \inf \left. E_n \right| J - 2\varepsilon.$$ Since $\varepsilon$ is arbitrary, we have the desired result.

**Proof of Lemma 2.16:** Assume $\lim \inf \inf \left. E_n \right| J < \inf \left. D \right| J$. Choose a real number $c$ such that $\lim \inf \inf \left. E_n \right| J < c < \inf \left. D \right| J$. There is a subsequence $E_{nk}$ of $E_n$ which satisfies $\inf \left. D \right| J < c < \inf \left. D \right| J$, for each $k$. Therefore, there exists $x_{nk} \in E_{nk}$ so that $J(x_{nk}) < c < \inf \left. D \right| J$. Hence, by (2.6), there is a subsequence $x_{nk_j}$ of $x_{nk}$.
which converges to a point $x \in D$. Hence
\[
J(x) \leq \liminf_{k,j} J(x_{n_k,j}) \leq c < \inf J.
\]

But this contradicts the definition of infimum.

**Remark 2.17:** If $E_n \subseteq D$, then the result of Lemma 2.16 is immediate even without the lower semicontinuity of $J$ since in that case, $\inf E_n \geq \inf J$, and this implies $\liminf_{n \to \infty} \inf E_n \geq \inf J$. Similarly, if $D \subseteq D_n$, then the result of Lemma 2.13 is immediate without using upper semicontinuity of $J$.

**Remark 2.18:** If the topology $\tau$ of conditions (2.1) and (2.2) of Theorem 2.12 is the discrete topology, then the condition $\liminf_{n} D_n \supseteq D$ reduces to the condition $\liminf_{n} D_n \supseteq D$. This result follows from Proposition 2.9. Also, the semicontinuity of $J$ is immediate since in the discrete topology, every function is continuous.

**Remark 2.19:** Another case worthy of consideration is the case where $Q_n = P_n^{-1}$ and $D_n = E_n$ for each $n$. Then conditions (2.3) and (2.4) are equivalent to the single condition
Using this condition we in this case may obtain the results of Lemmas 2.14 and 2.15.

III. DIFFERENCE APPROXIMATIONS FOR INITIAL VALUE PROBLEMS

The material of this chapter will examine the general theory presented in chapter two in the setting of a particular problem. The functional $J$ of chapter two will be an integral, $J_n$ will be an approximating sum; the sets $D_n$ and $A_n$ are classes of functions defined by differential equations (initial value problems) and corresponding difference equations. The primary results of this chapter are extensions and clarifications of the work done by B. M. Budak, E. M. Berkovic, and E. N. Soloveva in [2].

§ 3.1. Statement of the Problem

Consider the minimization of the functional

\[ J(x,u) = \int_{t_0}^{T} g(x(\tau),u(\tau),\tau) d\tau \]

which is defined on the solutions of the initial value problem
\( (3.2) \ \frac{dx}{dt} = f(x, u(t), t), \ \ t_o \leq t \leq T, \ \ x(t_0) = x_o \)

corresponding to all admissible controls \( u \) in a certain class \( U \). Here \( x = (x^1, \ldots, x^N) \) and \( f = (f^1, \ldots, f^N) \) are \( N \)-vector valued functions, \( u = (u^1, \ldots, u^r) \) is an \( r \)-vector valued function, and \( g \) is a scalar valued function.

Divide the interval \([t_0, T]\) into \( n \) parts by the partition \( \Sigma_{in}: t_0 = t_{no} < t_{n1} < \ldots < t_{nn} = T \) and consider the difference problem of minimizing

\( (3.3) \ J_n(x_n, u_n) = \sum_{i=0}^{n-1} g(x_{ni}, u_{ni}, t_{ni}) \tau_{ni} \)

where the functional \( J_n \) is defined on the solutions of the difference equation

\( (3.4) \ x_{ni+1} = x_{ni} + f(x_{ni}, u_{ni}, t_{ni}) \tau_{ni}, \) \n
\[ i = 0, 1, 2, \ldots, n-1, \ x_{no} = x_o, \]

corresponding to all admissible controls \( u_n \) in a certain class \( U_n \). Here, \( \tau_{ni} = t_{ni+1} - t_{ni} \) is the length of the \( i \)-th subinterval of the partition \( \Sigma_{in} \). Also, \( u_n \) is to be a vector valued function of the integral argument.
i, where $u_n$ takes the values $u_{n0}, u_{n1}, \ldots, u_{nn-1}$ as $i$ takes the values 0, 1, 2, ..., $n-1$.

The problem is to show that, under appropriate conditions, the infima of the functionals $J_n$ converge to the infimum of $J$ as $n$ tends to infinity.

§ 3.2. Basic Assumptions, Preliminary Results.

In the following $|x|$ represents Euclidean norm of $x$.

The control set $U$ will be assumed to satisfy either

(3.5) $U \subseteq \{u: \|u\|_s < K < +\infty\}$, where $U = \text{cl}(U^0)$ in the $L_s$ metric topology, for some $s$, $s < +\infty$, or, to cover the case $s = +\infty$,

(3.6) $U = \{u: u(t) \in H$ for almost every $t\}$ where $H$ is a homeomorphic image of $I^x$, the Euclidean unit cube.

In addition, assume that the function $f$ satisfies

(3.7) $|f(x,u,t)| \leq A_1|x| + A_2|u|^p + A_3$, for $A_1, A_2$ and $A_3$ constants and some $p \leq s$. 

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\[(3.8) \quad |f(x_1,u_1,t) - f(x_2,u_2,t)| \leq A_4 |x_1 - x_2| + B(u_1,u_2),\]

where \( B \) is continuous, \( B(u,u) = 0 \), and

\[\int_{t_0}^{T} B(u_1,u_2)dt \to 0 \quad \text{as} \quad u_1 \to u_2 \quad \text{in the} \quad L_p \quad \text{norm.}\]

\( A_4 \) is constant.

**Remark:** Some functions \( B \) which satisfy the requirement of (3.8) are the following:

In Reference [2], the function \( B \) with \( p = 2 \) is given by

\[B(u_1,u_2) = A_5 (|u_1|^2 + |u_2|^2) + A_6 |u_1 - u_2|^2, \quad A_5,A_6 \text{ constants.}\]

Clearly, \( B(u_1,u_2) = 0 \), and using Hölder's Inequality, we can see that the second requirement is also satisfied if \( s \geq 2 \).

A slightly more general situation is given by

\[B(u_1,u_2) = A_8 (1 + |u_1| + |u_2|) (|u_1-u_2|) + A_6 |u_1-u_2|^p.\]

Again it is clear that \( B(u,u) = 0 \), and if \( 2 \leq p \) with \( U \subseteq \{u: \|u\|_s \leq K\} \), then an application of Hölder's Inequality will give the third requirement on \( B \) in (3.8) also satisfied.
The function $g$ will be assumed to satisfy

\[(3.9) \quad |g(x,u,t)| \leq C_1(x) + C_2(x)|u|^p, \text{ where } C_1 \text{ and } C_2 \text{ are any functions which are bounded on bounded sets.}\]

\[(3.10) \quad \text{Moreover, } g \text{ is continuous in } (x,u,t) \text{ and continuous in } (x,t) \text{ uniformly in } u. \text{ } f \text{ is continuous in } t, \text{ uniformly in } x \text{ and } u. \]

The following results study various possible control sets.

Theorem 3.1: In the $L_s$ metric topology, the sets

\[
A = \{x: \|x-a\|_s \leq K_1, \ a \in L_s\},
\]

\[
B = \{x: \|x\|_s \leq K_2\},
\]

\[
C = \{x: \|x-a_1\|_s \leq r_1\} \cap \{x: \|x-a_2\|_s \leq r_2\}
\]

all satisfy $\text{cl}(X^o) = X$, provided $C^0 \neq \emptyset$.

Proof: To establish this result, recall the following theorem: If $X$ is a topological vector space and if $A$ is a closed convex subset of $X$ with nonempty interior, then $\text{cl}(A^o) = A$. (For a proof of this, see [9, p.110]). Certainly $L_s$ is a topological vector space, and certainly each of $A$, $B$, and $C$ are closed and convex. $A^o \neq \emptyset$ since $a \in A^o$.
**Theorem 3.2**: Each of the sets $A$, $B$, $C$ of the previous theorem, their arbitrary intersections (with non-empty interior), and their finite unions are possible control sets according to Condition (3.5).

**Proof**: In order to prove this theorem, all that must be done is to show that if $S_1$ and $S_2$ are two subsets of a metric space $X$ satisfying $\text{cl}(S_1^o) = S_1$, $\text{cl}(S_2^o) = S_2$ then $\text{cl}[(S_1 \cup S_2)^o] = S_1 \cup S_2$. To show this, let $x \in \text{cl}[(S_1 \cup S_2)^o]$. Then there is a sequence $x_n$ in $(S_1 \cup S_2)^o$ such that $x_n$ converges to $x$. But $(S_1 \cup S_2)^o \subseteq S_1 \cup S_2$ and $S_1 \cup S_2$ is closed, so $x \in S_1 \cup S_2$.

Now, let $x \in S_1 \cup S_2$. If $x \in (S_1 \cup S_2)^o$, then $x \in \text{cl}[(S_1 \cup S_2)^o]$, so we may assume $x \in \partial(S_1 \cup S_2)$. Since $\partial(S_1 \cup S_2) \subseteq \partial S_1 \cup \partial S_2$, $x \in \partial S_1$ or $x \in \partial S_2$.

If $x \in \partial S_1$, there is a sequence $x_n$ in $S_1^o \subseteq (S_1 \cup S_2)^o$ such that $x_n$ converges to $x$. So $x \in \text{cl}[(S_1 \cup S_2)^o]$.

**Theorem 3.3**: Each of the following sets

$$G_1 = \{u: u \in L_{\infty} \text{ and } \|u\|_\infty \leq K < +\infty\}$$

and

$$G_2 = \{u: \sup_{t_0 \leq t \leq T} |u(t)| \leq K < +\infty\}$$

are possible control sets according to Condition (3.6).
Proof: Let $H = \{x: x \in \mathbb{R}^r \text{ and } |x| \leq K\}$. Then $H$ is a homeomorphic image of the unit cube $I^r$, and if $u \in G_1$ then $|u(t)| \leq \|u\|_\infty \leq K$ for almost every $t$, so $u(t) \in H$ for almost every $t$ and therefore $G_1$ is a possible control set. Also, for any $u$ in $G_2$ and any $t$, $|u(t)| \leq K$. So $u(t) \in H$ for every $t$ and therefore $G_2$ is also a possible control set according to Condition (3.6).

§3.3. Existence and Uniqueness of Solutions

The following lemmas are used for the proofs of the existence theorems.

Lemma 3.4: If $\|x'\| \leq A \|x\| + B$, where $x$ is absolutely continuous and $A$ and $B$ are integrable on $[t_0, T]$, then

$$\|x(t)\| \leq (\|x(t_0)\| + \int_{t_0}^{t} B(s)ds) \cdot \exp \int_{t_0}^{t} A(s)ds.$$

Proof: Let $x(t_0)$ be denoted by $x_0$. We may take $A$ and $B$ nonnegative. We have $x(t) - x_0 = \int_{t_0}^{t} x'(s)ds$. Therefore, $\|x(t) - x_0\| \leq \int_{t_0}^{t} \|x'(s)\|ds$. This and the assumptions imply that
\[(3.11) \quad ||x(t)|| \leq ||x_0|| + \int_{t_0}^{t} (A(s)||x(s)|| + B(s)) ds.\]

Let \(R(t) = \int_{t_0}^{t} (A(s)||x(s)|| + B(s)) ds.\) Then \(R'(t) = A(t)||x(t)|| + B(t) \) a.e. But then \(R'(t) \leq A(t)(||x_0|| + R(t)) + B(t).\) This implies

\[(3.12) \quad R'(t) - A(t)R(t) \leq A(t) ||x_0|| + B(t).\]

We shall multiply both sides of (3.12) by \(\exp(-\int_{t_0}^{t} A(s) ds).\)

We get

\[
\left( R(t) \exp\left(-\int_{t_0}^{t} A(s) ds \right) \right)'
\leq (A(t)||x_0|| + B(t)) \exp\left(-\int_{t_0}^{t} A(s) ds \right).
\leq (A(t)||x_0|| + B(t)) \exp\left(-\int_{t_0}^{t} A(s) ds \right).
\]

Therefore

\[
R(t) \leq \left( \exp \int_{t_0}^{t} A(s) ds \right) \int_{t_0}^{t} (A(s)||x_0|| + B(s)) \exp\left(-\int_{t_0}^{s} A(r) dr \right) ds.
\]
Using the definition of $R(t)$ and line (3.11), it is seen that

\[ (3.13) \quad \| x(t) \| \leq \| x_0 \| + \left( \exp \int_{t_0}^{t} A(s) \, ds \right) \cdot \int_{t_0}^{t} (A(s) \| x_0 \| ) \\
+ B(s) \cdot \exp \left( -\int_{t_0}^{s} A(r) \, dr \right) \, ds \\
= \exp \left( \int_{t_0}^{t} A(s) \, ds \right) \cdot \left( \| x_0 \| \exp \left( -\int_{t_0}^{t} A(s) \, ds \right) \right) \\
+ \| x_0 \| \int_{t_0}^{t} A(s) \exp \left( -\int_{t_0}^{t} A(s) \, ds \right) \, ds \\
+ \int_{t_0}^{t} B(s) \exp \left( -\int_{t_0}^{t} A(s) \, ds \right) \, ds \\
= \exp \left( \int_{t_0}^{t} A(s) \, ds \right) \left( \| x_0 \| \right) \\
+ \int_{t_0}^{t} B(s) \exp \left( -\int_{t_0}^{s} A(r) \, dr \right) \, ds \]

Hence the result.
Remark 3.5: The preceding lemma is valid for any norm; the only thing necessary is that norm of integral is less than or equal to integral of norm.

Remark 3.6: In view of Lemma 3.4 and assumption (3.7), the solutions of equation (3.2) are bounded uniformly over $u \in U$. Here the norm is the Euclidean norm, the function $A$ is constant, and the role of the function $B$ is played by $A_2 |u|^p + A_3$.

We shall now take up the question of the existence and uniqueness of the solutions of (3.2) under the stated assumptions.

Proposition 3.7: For each $x$, the function $f(x,u,t)$ is continuous in $(u,t)$.

Proof: By assumption (3.10), $f$ is continuous in $t$. Condition (3.8) gives $f$ continuous in $u$ uniformly in $t$. Continuity in one variable uniformly with respect to the other, and continuity in the other implies continuity in both together.

Proposition 3.8: The function $f(x,u(t),t)$ of (3.2) is measurable in $t$ for fixed $x$ and continuous in $x$ for fixed $u$ and $t$. 

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Proof: The second assertion is immediate. The first part is a special case of: if $F(u,v)$ is real valued and continuous, and $u(t), v(t)$ are measurable, then $F^*(t) = F(u(t), v(t))$ is measurable. Let $G$ be an open set.

$F^*$ is measurable if and only if $F^{-1}(G)$ is measurable.

$$F^{-1}(G) = \{ t: (u(t), v(t)) \in F^{-1}(G) \}$$

$F^{-1}(G)$ is open, and consequently it is a union of a sequence of hypercubes. Therefore

$$F^{-1}(G) = \bigcup_n \{ t: (u(t), v(t)) \in I_n \times J_n \}$$

$$= \bigcup_n \{ t: u(t) \in I_n \} \cap \{ t: v(t) \in J_n \}$$

$$= \bigcup_n \{ t: t \in u^{-1}(I_n) \} \cap \{ t: t \in v^{-1}(J_n) \}.$$ 

Since $u$ and $v$ are measurable, we have the result.

Theorem 3.9: Under the conditions (3.7)-(3.10), the differential equation (3.2) has a solution in a nhd of $(t_o, x_o)$ satisfying $x(t_o) = x_o$. 

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Proof: To prove this theorem, we shall quote the following result which is due to Carathéodory: Let $R$ denote the rectangle $|t-\tau| \leq a$, $|x-\xi| \leq b$, where $(\tau, \xi)$ is some fixed point in the $(t,x)$ plane; $a$ and $b$ are positive real numbers. Let $f$ be defined on $R$ and suppose it is measurable in $t$ for each fixed $x$ and continuous in $x$ for each fixed $t$. If there exists a Lebesgue integrable function $m$ on the interval $|t-\tau| \leq a$ such that $|f(x,t)| \leq m(t)$, $(t,x) \in R$, then there exists a solution $\varphi$ of the differential equation $x' = f(x,t)$ on some interval $|t-\tau| \leq \beta$, $\beta > 0$ satisfying $\varphi(\tau) = \xi$.

For a proof of this, see [3, Chapter 2]. In order to establish our Theorem 3.9, we shall show that the conditions of the Carathéodory theorem are satisfied.

Let $a = T$, let $b$ be the bound guaranteed by Lemma 3.4. For a given $u$, the measurability of $f$ in $t$ is guaranteed by Proposition 3.8 as is the continuity of $f$ in $x$.

Now, according to (3.7), $f$ satisfies

\begin{equation}
|f(x,u(t),t)| \leq A_1|x| + A_2|u|^p + A_3.
\end{equation}

Lemma 3.4 assures $x$ bounded, and using $B$ for this bound we may write
\[|f(x,u(t),t)| \leq A_2 |u|^P + (A_1 B + A_3).\]

Therefore, with \( m(t) = A_2 |u|^P + (A_1 B + A_3) \), the conditions of the Carathéodory theorem are satisfied and we have a solution of (3.2) for a given \( u \).

**Theorem 3.10:** The solutions of (3.2) guaranteed by the previous theorem may be continued so that each solution is defined on the entire interval \([t_0, T]\).

**Proof:** To establish this result, we will again quote a general theorem, whose proof may be found in [3, Chapter 2].

In a domain \( D \) of the \((t,x)\) plane, let the function \( f \) be defined, measurable in \( t \) for fixed \( x \), and continuous in \( x \) for fixed \( t \). Let there exist a Lebesgue integrable function \( m \) such that

\[|f(x,t)| \leq m(t) \text{ for } (t,x) \in D.\]

Then given a solution \( \varphi \) of \( x' = f(t,x) \) for \( t \) in an interval \((a,b)\), it is the case that \( \varphi(b-0) \) exists, and if \((b, \varphi(b-0)) \in D\), then \( \varphi \) can be continued over \((a,b + \delta)\) for some \( \delta > 0 \). Thus the solution can be continued up to the boundary of \( D \).

If we let
\[ D = \{ x : |x| \leq B \} \times \{ t : t_0 \leq t \leq T \} \]

then the conditions of the above result are satisfied and therefore there is a solution on the whole interval \([t_0, T]\).

**Theorem 3.11**: The solutions of (3.2) guaranteed by the preceding theorems are unique.

**Proof**: Assume that \( x_1 \) and \( x_2 \) are two solutions of (3.2) corresponding to a fixed \( u \in U \) and satisfying \( x(t_0) = x_0 \). According to line (3.8), \( f \) satisfies a Lipschitz condition given by \[ |f(x_1, u(t), t) - f(x_2, u(t), t)| \leq A_4 |x_1 - x_2|. \] Therefore, \[ |(x_1 - x_2)'| \leq A_4 |x_1 - x_2|, \] and the bounding lemma, Lemma 3.4, with \( A = A_4 \) and \( B = 0 \) gives \[ |x_1(t) - x_2(t)| \leq \exp \left( \int_{t_0}^{T} A_4 ds \right) |x_1(t_0) - x_2(t_0)| = 0. \] Thus \( x_1 = x_2 \).

§3.4. Setting in the Language of Chapter Two.

We are now in a position to describe the sets and functionals of Chapter two.

**Definition**: The control \( u_n(t) \) is said to be a piece-wise constant extension of the discrete control \( u_n \) if it is defined by
\[ u_n(t) = u_{n_i} \text{ when } t_{n_i} \leq t < t_{n_i+1} \]

\[ i = 0, 1, \ldots, n-1. \]

**Notation:** We shall denote the solution of the differential equation (3.2) corresponding to the piecewise constant control \( u_n(t) \) by \( \tilde{x}_n(t) \). The solution of the difference equation (3.4) corresponding to the discrete control \( u_n \) will be denoted by \( x_n \).

Let \( U \) be defined according to (3.5) or (3.6). Define

(3.14) \( U_n = \{ u_n : \text{the piecewise constant extension of } u_n \text{ is in } U \} \)

(3.15) \( \tilde{U}_n = \{ \text{piecewise constant extensions of } u_n \in U_n \} \)

(3.16) \( D = \{ (x,u) : x' = f(x,u,t), x(t_0) = x_0, u \in U \} \)

(3.17) \( D_n = \{ (\tilde{x}_n,u_n) : \tilde{x}_n' = f(\tilde{x}_n,u_n,t), \tilde{x}_n(t_0) = x_0, u_n \in \tilde{U}_n \} \)

(3.18) \( \Delta_n = \{ (x_n,u_n) : x_{n_i+1} - x_{n_i} = f(x_{n_i},u_{n_i},t_{n_i}) \tau_{n_i}, \]

\[ i = 0, 1, \ldots, n-1, \quad x_{n_0} = x_0, \quad u_n \in U_n \} \)
Recall that the functional $J$ is defined by (3.1), and the functional $J_n$ is defined by (3.3).

In this special case of Chapter two, $P_n = Q_n^{-1}$ and $P_n: \Delta_n \to D_n$ is defined by

$$P_n(x_n,u_n) = (\tilde{x}_n,u_n)$$

where in the left side, $x_n$ is the solution of the difference equation corresponding to the discrete control $u_n$, and in the right side, $\tilde{x}_n$ is the solution of the differential equation corresponding to the piecewise constantly extended control $u_n$. Note that the mapping $D_n$ is clearly 1:1 and onto.

We also note that $E_n = D_n \subseteq D$, since $U_n \subseteq U$. Because $E_n \subseteq D$, Remark 2.17 applies.

Note: In the following material, and also in the previous material, $U_n$ and $\tilde{U}_n$ will be used interchangeably. It will be clear from the context which is being used, the discrete control or the piecewise constantly extended discrete control.

§ 3.5. The Uppersemicontinuity of $J$.

**Theorem 3.12:** If $G \geq 0$, $G$ continuous, then

$$J(y) = \int G(y(t),t)dt$$

is $L_p$ lower semicontinuous.
Proof: Let $y_n$ be a sequence such that $y_n$ converges to $y$ in the $L_p$ norm. By definition of inferior limit, we may extract a subsequence $y_{n_k}$ of the sequence $y_n$ such that

$$\lim_{k \to \infty} J(y_{n_k}) = \lim_{n \to \infty} \inf J(y_n).$$

Furthermore, it must be the case that $y_{n_k}$ converges to $y$ in the $L_p$ norm.

The convergence of the sequence $y_{n_k}$ to $y$ in the $L_p$ norm implies the existence of a subsequence $y_{n_{k_j}}$ of $y_{n_k}$ such that $y_{n_{k_j}}$ converges to $y$ pointwise almost everywhere. Therefore

$$J(y) = \int G(y(t),t) dt$$

$$= \int \lim_{j} G(y_{n_{k_j}}(t),t) dt$$

$$\leq \liminf_{j} \int G(y_{n_{k_j}}(t),t) dt \quad \text{(Fatou's Lemma)}$$

$$= \lim_{k} \int G(y_{n_k}(t),t) dt$$

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\[= \lim \inf \int G(y_n(t), t) \, dt\]

\[= \lim \inf J(y_n).\]

Hence \(J\) is \(L_p\) lower semicontinuous.

In the above theorem, \(G \geq 0\) was an essential hypothesis. We shall partially remove this restriction in the following corollary.

**Corollary 3.13:** If \(G\) is continuous, if there is a function \(h\) which satisfies \(G \geq h\) and \(\int h(y, t) \, dt\) \(L_p\) lower semicontinuous, then \(J(y) = \int G(y, t) \, dt\) is \(L_p\) lower semicontinuous.

**Proof:** Let \(y_n\) be a sequence such that \(y_n\) converges to \(y\) in the \(L_p\) norm. By Theorem 3.12,

\[\int (G(y, t) - h(y, t)) \, dt\]

\[\leq \lim \inf \int G(y_n, t) - h(y_n, t) \, dt\]

\[\leq \lim \inf \int G(y_n, t) \, dt - \lim \inf \int h(y_n, t) \, dt.\]

Therefore,
\[ \int G(y,t)\,dt \leq \liminf \int G(y_n,t)\,dt - \left( \liminf \int h(y_n,t)\,dt \right) - \int h(y,t)\,dt. \]

Hence \[ \int G(y,t)\,dt \leq \liminf \int G(y_n,t)\,dt \] provided
\[ \liminf \int h(y_n,t)\,dt \geq \int h(y,t)\,dt. \] But this is the case since by assumption \( \int h(y,t)\,dt \) is lower semicontinuous.

The next result relates the control functions \( u \in U \) to the phase functions \( x \).

**Proposition 3.14:** If \( u_n \) is a sequence of control functions such that \( u_n \) converges to \( u \) in the \( L_p \) norm, then the corresponding sequence \( x_n \) converges to \( x \) uniformly.

**Proof:** Let \( |z_n| = |x_n - x| \). Then
\[ |z'_n| = |x'_n - x'| = |f(x_n,u_n,t) - f(x,u,t)| \leq A_4 |x_n - x| + B(u_n,u). \]

Therefore \[ |z'_n| \leq A_4 |z_n| + B(u_n,u). \] An application of Lemma 3.4 gives
\[ |z_n(t)| = |x_n(t) - x(t)| \leq \exp(A_4(t-t_0)) \int_{t_0}^{t} B(u_n, u) dt. \]

Let \( M = \max \exp(A_4(t-t_0)) \). Then

\[ |x_n(t) - x(t)| \leq M \int_{t_0}^{t} B(u_n, u) dt. \]

So

\[
\sup_{t_0 \leq t \leq T} |x_n(t) - x(t)| \leq M \int_{t_0}^{T} B(u_n, u) dt.
\]

The result now follows from properties of \( B(u_1, u_2) \) in line (3.8).

**Theorem 3.15**: Under the conditions (3.7)-(3.10) of Section 3.2, the functional \( J(x,u) \int_{t_0}^{t} g(x,u,t) dt \) of (3.1) is upper semicontinuous on \( D \) with respect to \( L^p \) convergence of \( u \).

**Proof**: First note that if \( u_n \to u \) in the \( L^p \) norm, then \( x_n \to x \) uniformly from Proposition 3.14. Consequently \( x_n \to x \) in the \( L^p \) norm also. Hence

\((x_n, u_n) \to (x, u)\) in the \( L^p \) norm, where here we are using Euclidean absolute value.

We shall state now a dual of the result in Corollary 3.13: If \( g \) is continuous, if there is a function \( h \) which satisfies \( g \leq h \) and \( \int h(y,t) dt \) is \( L^p \) upper
semicontinuous, then \( J(y) = \int g(y, t) dt \) is \( L_p \) upper semicontinuous.

Now, with the pair \((x, u)\) playing the role of \(y\), we shall show that the hypothesis of the above result is satisfied.

First, \(g\) is continuous from (3.10). Second, it is assumed in (3.9) that

\[
|g(x, u, t)| \leq C_1(x) + C_2(x)|u|^p
\]

where \(C_1\) and \(C_2\) are functions which are bounded on bounded sets. Therefore, Lemma 3.4, the bounding lemma, allows us to write

\[
|g(x, u, t)| \leq A + B|u|^p,
\]

where \(A\) is a bound for \(C_1\) and \(B\) is a bound for \(C_2\).

The function \(\int_{t_0}^{T} h(x, u, t) dt = \int_{t_0}^{T} (A + B|u|^p) dt\) is continuous with respect to \(L_p\) convergence. Hence we have the result.

Remark 3.16: The function \(h\) that we have used above is not the only possibility. Note that the function

\[
h(x, u, t) = A|x| + B|u|^p + C
\]

also satisfies the indicated conditions. With \(p = 2\), this is the function used by Budak, Berkovich and Solov'eva in [2].
§ 3.6. Proof that $\liminf_{\tau_n} D_n \supseteq D$.

Recall in the previous section we defined $D$ and $D_n$ to be

$$D = \{(x,u): x' = f(x,u,t), x(t_0) = x_0, u \in U\}$$

$$D_n = \{(\tilde{x}_n, u_n): \tilde{x}_n = f(\tilde{x}_n, u_n, t), \tilde{x}_n(t_0) = x_0, u_n \in \tilde{U}_n\}.$$  

The result of this section is to show that $\liminf_{\tau_n} D_n \supseteq D$. We shall first show that $\liminf_{\tau_n} \tilde{U}_n = U$, for the two possible types of control set $U$.

**Proposition 3.17:** If $U$ is defined according to (3.5) and if the partition $\Sigma_{in}$ satisfies $\tau_n = \max \tau_{ni} \to 0$ as $n \to \infty$, then $\liminf_{\tau_n} \tilde{U}_n = U$. In this case, $\tau$ is the $L_p$ metric topology, $p \leq s$.

**Proof:** It is clear that $\liminf_{\tau_n} \tilde{U}_n \subseteq U$, since $U$ is closed. We must show the opposite containment. Let $u \in U$ and let $\varepsilon > 0$. We must show that there is an integer $N$ such that if $n > N$ then there exists $u_n \in \tilde{U}_n$ satisfying $\|u - u_n\|_s < \varepsilon$.  

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Since $\text{cl}(U^0) = U$, there is $u'$ such that $u' \in U^0$ and $\|u' - u\|_s < \varepsilon/4$. Let $\varepsilon' = \min\{\varepsilon/4, K - \|u'\|_s\}$. Choose a continuous function $v$ such that $\|v - u'\|_s < \varepsilon'$. Then

$$\|v\|_s = \|v - u' + u'\|_s$$

$$\leq \|v - u'\|_s + \|u'\|_s$$

$$< K - \|u'\|_s + \|u'\|_s = K.$$

Moreover,

$$\|v - u\|_s = \|v - u' + u' - u\|_s$$

$$\leq \|v - u'\|_s + \|u' - u\|_s$$

$$< \varepsilon/4 + \varepsilon/4 = \varepsilon/2.$$

Hence $v$ is a continuous function satisfying $v \in U^0$ and $\|v - u\|_s < \varepsilon/2$.

Now, $v$ is continuous on $[t_0, T]$, hence it is uniformly continuous there. So for arbitrary $\alpha > 0$, there exists $\beta > 0$ such that if $|t - t'| < \beta$ then $|v(t) - v(t')| < \alpha$. 

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Consider the positive number \( \frac{\epsilon}{2(T-t_0)^{1/s}} \).

There is a \( \delta > 0 \) such that if \( |t - t'| < \delta \) then
\[
|v(t) - v(t')| < \frac{\epsilon}{2(T-t_0)^{1/s}}.
\]
By assumption on \( \Sigma_{in} \), there is \( N \) such that if \( n > N \) then \( \max_i \tau_{ni} < \delta \).

We must then have, for \( n > N \),
\[
\left| \max_{t \in [t_{ni}, t_{ni+1}]} v(t) - \min_{t \in [t_{ni}, t_{ni+1}]} v(t) \right| < \frac{\epsilon}{2(T-t_0)^{1/s}}.
\]

Let \( \hat{t} \) be the point in \( [t_{ni}, t_{ni+1}] \) which makes \( |v(t)| \) a minimum. That is, \( |v(\hat{t})| \leq |v(t)| \) for every \( t \in [t_{ni}, t_{ni+1}] \). Define the piecewise constant function \( v_n(t) \) by
\[
v_n(t) = v(\hat{t}) \text{ for } t \in [t_{ni}, t_{ni+1}].
\]
Then
\[
\int_{t_0}^{T} |v_n(t) - v(t)|^s \, dt = \sum_{i=0}^{n-1} \int_{t_{ni}}^{t_{ni+1}} |v(\hat{t}) - v(t)|^s \, dt < \sum_{i=0}^{n-1} \int_{t_{ni}}^{t_{ni+1}} \left( \frac{\epsilon}{2(T-t_0)^{1/s}} \right)^s \, dt.
\]

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\[
\left[ \frac{\varepsilon}{2(T-t_o)^{1/s}} \right]^{s}\left(T-t_o\right) = (\varepsilon/2)^{s}.
\]

Therefore, \( \|v_n(t) - v(t)\| < \varepsilon/2 \). Hence,

\[
\|u - v_n\|_s \leq \|u - v\|_s + \|v - v_n\|_s < \varepsilon/2 + \varepsilon/2 = \varepsilon.
\]

Moreover,

\[
\int_{t_o}^{T} |v_n|_sdt = \sum_{i=0}^{n-1} \int_{t_{ni}}^{t_{ni+1}} |v(t)|_sdt
\]

\[
\leq \sum_{i=0}^{n-1} \int_{t_{ni}}^{t_{ni+1}} |v(t)|_sdt
\]

\[
= \int_{t_o}^{T} |v(t)|_sdt \leq K^s
\]

Hence \( \|v_n\|_s \leq K \), i.e., \( v_n \in \tilde{U}_n \).

Therefore,

\[
\lim \inf_{L_s} \tilde{U}_n = U.
\]
So we have $\liminf_{L^s_n} \tilde{U}_n = U$. We would like to have $\liminf_{L^p_n} \tilde{U}_n = U$, where $p \leq s$. We recall the following result from measure theory:

If $X$ is a finite measure space, and if $u \in L^s(X)$, then $u \in L^p(X)$ for $1 \leq p \leq s$. Moreover, the inequality $\|u\|_p \leq \|u\|_s \mu(X)^{(s-p)/sp}$ also holds.

Hence for given $u \in U$, if we can find a $v_n$ which is close to $u$ in the $L^s$, then $v_n$ must also be close to $u$ in $L^p$, since the following inequality must hold:

$$\|v_n - u\|_p \leq \|v_n - u\|_s \mu(X)^{(s-p)/sp}. $$

This shows that if $\liminf_{L^s_n} \tilde{U}_n = U$, then $\liminf_{L^p_n} \tilde{U}_n = U$.

**Proposition 3.18:** For the control set $U$ defined according to (3.6), $\liminf_{\tau_n} \tilde{U}_n = U$, provided $\tau_n = \max_i \tau_{ni} \rightarrow 0$ as $n \rightarrow \infty$. $\tau$ in this case is the $L^p$ metric topology.
Proof: For convenience, the restatement of line (3.6) is

\[(3.6) \quad U = \{u: u(t) \in H \text{ for almost every } t \in [t_0, T]\}\]

where $H$ is a homeomorphic image of $I^r$, the Euclidean unit cube.

Let $u \in U$ and let $\epsilon > 0$ be given. Then $u(t) \in H$ for almost every $t$. Let $h: X[0,1] \to H$ be the homeomorphism. Then $h^{-1} \circ u: [t_0, T] \to X[0,1]$ for almost every $t$. From Lusin's Theorem, [8, p. 110], there is a closed set $F$ contained in $[t_0, T]$ such that $\mu([t_0, T] \setminus F) < \epsilon'$ and $h^{-1} \circ u|_F$ is continuous. From Tietze's extension theorem, $h^{-1} \circ u|_F$ may be extended to a continuous function, say $g'$ which is defined on all of $[t_0, T]$. Moreover, $g'(t) \in X[0,1]$ for every $t \in [t_0, T]$. Consider the continuous function $h \circ g': [t_0, T] \to H$. We have

$$\int_{t_0}^{T} |u - h \circ g'|^p dt = (F) \int_{t_0}^{T} |u - h \circ g'|^p dt$$

$$+ ([t_0, T] - F) \int_{t_0}^{T} |u - h \circ g'|^p dt$$

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\[ = \left( [t_0, T] - F \right) \int |u - h \circ g'|^p dt, \]

since \( h^{-1} \circ u \) and \( g' \) agree on \( F \).

Now, \( u \) and \( h \circ g' \) are functions with values in \( H \) for almost every \( t \), consequently they are bounded for almost every \( t \), say by \( B \). Therefore,

\[ \int_{t_0}^T |u - h \circ g'|^p dt \leq \sup_{t \in [t_0, T]} |u - h \circ g'|^p \cdot \mu([t_0, T] \cap F) < 2B^p \epsilon' \cdot \]

Hence we have a continuous function \( v = h \circ g' \) such that \( v \in U \) and \( \|v - u\|^p_p < \epsilon \) if \( \epsilon' = \epsilon^p / 2B^p \).

Now, \( v \) is continuous on \([t_0, T] \), hence it is uniformly continuous there. There is an \( N \) such that if \( n > N \), \( \tau_n \) is so small that \( t_{n+1} - t_n < \delta \) implies that

\[
\left| \max_{t \in [t_n, t_{n+1}]} v(t) - \min_{t \in [t_n, t_{n+1}]} v(t) \right| < \epsilon .
\]

So, for \( n > N \), define \( v_n(t) = \hat{v}(\hat{t}) \), where \( \hat{t} \) is the point in \([t_n, t_{n+1}]\) which makes \( |v(t)| \) a minimum. Then \( v_n \) is a piecewise constant function such that range \( v_n \subset H \). Also
\[
\int_{t_0}^{T} |v - v_n|^p \, dt = \sum_{i=0}^{n-1} \int_{t_{ni}}^{t_{ni+1}} |v(t) - v(t_{ni+1})|^p \, dt
\]

\[
< \sum_{i=0}^{n-1} \int_{t_{ni}}^{t_{ni+1}} \varepsilon^p \, dt
\]

\[
= \varepsilon^p (T - t_0) \text{ if } n > N.
\]

Therefore, we have the result.

**Theorem 3.19**: For \( U \) defined by (3.5) or (3.6) and for \( D \) defined by (3.15), \( \liminf_{\tau_n} D_n \supseteq D \), where \( \tau \) is the product of the uniform topology on \( x \) and the \( L_p \) topology on \( u \).

**Proof**: Let \((x,u) \in D\) and let \( V \) be any neighborhood of \((x,u)\). We may take \( V = V_1 \times V_2 \) where \( x \in V_1 \), \( u \in V_2 \), \( V_1 \) open in the uniform topology, \( V_2 \) open in the \( L_p \) topology. From Proposition 3.14 choose \( V_3 \) such that \( u \in V_3 \subseteq V_2 \) and \( V_3 \) is so small that if \( u_n \in V_3 \) then \( x_n \in V_1 \). Since \( \liminf_{L_p} U_n = U \), choose \( N(V_3) \) such that if \( n > N(V_3) \), then \( U_n \cap V_3 \neq \emptyset \). This implies that
\[ D_n \cap V_1 \times V_3 \subset D_n \cap V_1 \times V_2 = D_n \cap V \neq \phi. \]

Hence \( \lim \inf_{\tau_n} D_n \supset D. \)

§ 3.7. Proof that \( J_n(y_n) - J(p_n y_n) \to 0 \)

Referring to § 3.4 and noting that \( P_n = Q_n^{-1} \), we may take care of the proofs of (2.3) and (2.4) by simply proving the following theorem:

Theorem 3.20: For any sequence \( y_n \) with \( y_n \in A_n \) for each \( n \),

\[ \lim_{n \to \infty} (J_n(y_n) - J(p_n y_n)) = 0 \tag{3.19} \]

If \( y_n = (x_n(u_n), u_n) \) where \( x_n(u_n) \) is the solution of line (3.4) corresponding to the discrete control \( u_n \), then \( p_n y_n = (x(u_n), u_n) \) where \( x(u_n) \) is the solution of (3.2) corresponding to the piecewise constantly extended discrete control \( u_n \). Theorem 3.20 may then be written as

Theorem 3.20: If \( (x_n, u_n) \) is any sequence with \( (x_n, u_n) \in A_n \) for each \( n \), then

\[ \lim_{n \to \infty} (J_n(x_n(u_n), u_n) - J(x(u_n), u_n)) = 0. \tag{3.20} \]
In order to prove this theorem, we shall need the following lemma. We will use the notation $\tilde{x}_n(t) = x(u_n)$.

Lemma 3.21: $z_{ni} = |\tilde{x}(t_{ni}) - x_{ni}| \to 0$ uniformly in $\mathcal{E} \text{ as } n \to \infty$ provided $\tau_n = \max_{i} \tau_{ni} = O(n^{-1})$ as $n \to \infty$.

Here $t_{ni}$ is the $i$th point of the partition $\Sigma_{in}$.

Proof: From (3.2) we may write:

$$\tilde{x}_n(t_{ni+1}) - \tilde{x}(t_{ni}) = \int_{t_{ni}}^{t_{ni+1}} f(\tilde{x}_n(\tau), u_{ni}, \tau) d\tau$$

and from (3.4):

$$x_{ni+1} - x_{ni} = \int_{t_{ni}}^{t_{ni+1}} f(x_{ni}, u_{ni}, t_{ni}) d\tau$$

Therefore,

$$z_{ni+1} \leq z_{ni}$$

$$\leq \int_{t_{ni}}^{t_{ni+1}} f(\tilde{x}_n(\tau), u_{ni}, \tau) - f(x_{ni}, u_{ni}, t_{ni}) d\tau$$

$$\leq z_{ni} + \int_{t_{ni}}^{t_{ni+1}} f(\tilde{x}_n(\tau), u_{ni}, \tau) - f(x_n(t_{ni}), u_{ni}, \tau) d\tau$$

(3.21)

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\[
+ \int_{t_{ni}}^{t_{ni+1}} |f(\tilde{x}_n(t_{ni}), u_{ni}, \tau) - f(x_{ni}, u_{ni}, \tau)|\,d\tau
\]

\[
+ \int_{t_{ni}}^{t_{ni+1}} |f(x_{ni}, u_{ni}, \tau) - f(x_{ni}, u_{ni}, t_{ni})|\,d\tau.
\]

From (3.8),

\[
(3.22) \quad z_{ni+1} \leq z_{ni} + A_4 \int_{t_{ni}}^{t_{ni+1}} |\tilde{x}_n(\tau) - \tilde{x}_n(t_{ni})|\,d\tau
\]

\[
+ A_4 \int_{t_{ni}}^{t_{ni+1}} |\tilde{x}_n(t_{ni}) - x_{ni}|\,d\tau
\]

\[
+ \int_{t_{ni}}^{t_{ni+1}} |f(x_{ni}, u_{ni}, \tau)
\]

\[
- f(x_{ni}, u_{ni}, t_{ni})|\,d\tau.
\]

Since the integrand of the third term of the right hand side of (3.22) is just \( z_{ni} \), we may write

\[
(3.23) \quad z_{ni+1} \leq (1 + A_4 \tau_n) z_{ni}
\]

\[
+ A_4 \int_{t_{ni}}^{t_{ni+1}} |\tilde{x}_n(\tau) - \tilde{x}_n(t_{ni})|\,d\tau
\]
\[ + \int_{t_{ni}}^{t_{ni+1}} |f(x_{ni}, u_{ni}, \tau) - f(x_{ni}, u_{ni}, t_{ni})| d\tau. \]

Consider the second term of the right hand side of (3.23).

\[ \tilde{x}_n(\tau) = \tilde{x}_n(t_{ni}) + \int_{t_{ni}}^{\tau} f(\tilde{x}_n(t), u_n(t), t) dt. \]

Therefore, from (3.7),

\[ |\tilde{x}_n(\tau) - \tilde{x}_n(t_{ni})| \leq A_1 \int_{t_{ni}}^{t_{ni+1}} |\tilde{x}_n| d\tau \]

(3.24)

\[ + A_2 \int_{t_{ni}}^{t_{ni+1}} |u_n(t)|^P dt + A_3 \tau_{ni}. \]

Now, \( x_n \) is a solution of Equation (3.2), hence it is bounded uniformly, say by \( B \).

Thus

(3.25) \[ |\tilde{x}_n(\tau) - \tilde{x}(t_{ni})| \leq (A_1 B + A_3) \tau_{ni} + A_2 \int_{t_{ni}}^{t_{ni+1}} |u_n(t)|^P dt. \]
Consequently

\[ A_4 \int_{t_{ni}}^{t_{ni+1}} |x_n(\tau) - x_n(t_{ni})| d\tau \]

\[ \leq A_4 (A_1B + A_3) \tau_n \tau_n \]

\[ + A_4 A_2^{\tau} \int_{t_{ni}}^{t_{ni+1}} |u_n(t)|^p dt. \]

(3.26)

Now, consider the third term on the right hand side of (3.23). From (3.10), \( f \) is uniformly continuous in \( t \), uniformly in \( x \) and \( u \). Hence if \( \tau_n \) is sufficiently small, that is if \( n \) is sufficiently large,

\[ |f(x_{ni}, u_{ni}, \tau) - f(x_{ni}, u_{ni}, t_{ni})| \leq \varepsilon_n, \]

where \( \varepsilon_n \to 0 \).

Therefore,

\[ \int_{t_{ni}}^{t_{ni+1}} |f(x_{ni}, u_{ni}, \tau) - f(x_{ni}, u_{ni}, t_{ni})| d\tau \leq \varepsilon_n \tau_n. \]

(3.27)

(3.28)

Now, using (3.26) and (3.28) in (3.23), with \( \tau_n = A_4 (A_1B + A_3) \tau_n + \varepsilon_n \), we have
\[(3.29) \quad z_{n+1} \leq (1 + A_4 T_n)z_n + r_n \tau_n + \tau_n A_4 A_2 \int_{t_n}^{t_{n+1}} |u_n(t)|^p dt\]

where \(r_n \to 0\) as \(n \to \infty\).

We must consider the difference inequality (3.29). First, \(z_{n_0} = |\tilde{x}_n(t_{n_0}) - x_{n_0}| = |x_o - x_o| = 0\). Then

\[z_{n_1} \leq (1 + A_4 T_n)z_{n_0} + r_n \tau_n + \tau_n A_4 A_2 \int_{t_{n_0}}^{t_{n_1}} |u_n(t)|^p dt,\]

\[z_{n_1} \leq (r_n + A_4 A_2 \int_{t_{n_0}}^{t_{n_1}} |u_n(t)|^p dt) \tau_n\]

\[z_{n_2} \leq (1 + A_4 T_n)z_{n_1} + (r_n + A_4 A_2 \int_{t_{n_1}}^{t_{n_2}} |u_n(t)|^p dt) \tau_n\]

\[\leq \left\{ (1 + A_4 T_n) (r_n + A_4 A_2 \int_{t_{n_0}}^{t_{n_1}} |u_n(t)|^p dt) \right\} \tau_n + (r_n + A_4 A_2 \int_{t_{n_1}}^{t_{n_2}} |u_n(t)|^p dt) \tau_n\]

Proceeding inductively, we get
\[ 0 \leq z_{ni} \leq \{(1+A_4 \tau_n)^{i-1} + (1+A_4 \tau_n)^{i-2} + \ldots \]
\[ + (1+A_4 \tau_n) + 1\} \tau_n \]
\[ + \left\{(1+A_4 \tau_n)^{i-1} A_4 A_2 \int_{t_{no}}^{t_{ni}} |u_n(t)| P dt \right\} \tau_n \]
\[ + (1+A_4 \tau_n)^{i-2} A_4 A_2 \int_{t_{ni}}^{t_{n2}} |u_n(t)| P dt + \ldots \]
\[ + A_4 A_2 \int_{t_{ni}}^{t_{ni-1}} |u_n(t)| P dt \right\} \tau_n \]
\[ \leq \left\{ r_n \sum_{k=0}^{i-1} (1+A_4 \tau_n)^k + (1+A_4 \tau_n)^n A_4 A_2 \int_{t_o}^{T} |u_n(t)| P dt \right\} \tau_n \]
\[ \leq r_n \tau_n \left(1 - (1+A_4 \tau_n)^i\right) + \tau_n \left(\frac{A_4 \tau_n}{1-(1+A_4 \tau_n)}\right)^n A_4 A_2 \int_{t_o}^{T} |u_n(t)| P dt \]
\[ = \frac{r_n}{A_4} ((1+A_4 \tau_n)^i - 1) + \tau_n e^{n A_4 \tau_n} A_4 A_2 \int_{t_o}^{T} |u_n(t)| P dt \]
\[ \leq \frac{r_n}{A_4} (e^{n A_4 \tau_n - 1}) + \tau_n e^{n A_4 \tau_n} A_4 A_2 \int_{t_o}^{T} |u_n(t)| P dt.\]
Now, by hypothesis, \( \tau_n = o(n^{-1}) \), so there is a constant \( C \) and an integer \( M \) such that when \( n > M \),
\[
\tau_n \leq \frac{C}{n}.
\]
Then
\[
0 \leq z_{ni} \leq \frac{r_n}{A_4} (e^{-A_4} - 1) + \tau_n A_4^2 A_2 \int_{t_0}^{T} |u_n(t)|^2 dt.
\]

For the control set specified by either (3.5) or (3.6), \( \int_{t_0}^{T} |u_n(t)|^2 dt \leq K \), where \( K \) is a constant independent of \( n \). Therefore, \( z_{ni} \to 0 \) uniformly in \( i \) as \( n \to \infty \).

A **Counterexample**: The following example shows that
\[
\int_{t_{ni}}^{t_{ni+1}} |\tilde{x}_n(\tau) - \tilde{x}_n(t_{ni})| d\tau
\]
need not be bounded by \( r_n \tau_n \) where \( r_n \to 0 \), and \( \tau_n = o(n^{-1}) \). This example is given with reference to [2], Lemma 2.

Let \( t_0 = 0 \), \( T = 1 \). Consider a uniform subdivision of the interval \([0, 1]\):
\[
\sum_{in} : 0 = t_{n0} < t_{n1} = \frac{1}{n} < t_{n2} = \frac{2}{n} < ... < t_{nn} = 1.
\]

Let the piecewise constant extended discrete control
$u_n$ be defined by

$$u_n(t) = \begin{cases} \sqrt{n} & \text{if } t \in [0, \frac{1}{n}) \\ 0 & \text{if } t \in \left[\frac{1}{n}, 1 \right] \end{cases}$$

Then $\int_0^1 |u_n(t)|^2 \, dt = 1 < +\infty$, so that $u_n \in L_2[0,1]$.

Let $f(x,u,t)$ be defined by

$$f(x,u,t) = x + u^2.$$

This $f$ clearly satisfies the conditions set forth in this paper as well as those in [2].

We would like to find the solution of $x' = f(x,u,t)$ where $u$ is the piecewise constantly extended discrete control given above. We have

$$\tilde{x}_n' = \tilde{x}_n + u^2 = \tilde{x}_n + n.$$

Thus $\tilde{x}_n(t) = n(e^t - 1)$, and

$$\tilde{x}_n(t_{\frac{1}{n-1}}) = \tilde{x}_n(\frac{1}{n}) = n(e^{1/n} - 1) \to 1 \quad \text{as} \quad n \to \infty.$$

We also will calculate the solution of the difference equation $\Delta^+ x_{ni} = f(x_{ni}, u_{ni}, t_{ni})$, $x_{no} = 0$, $i = 0, 1, \ldots, n-1$, corresponding to this discrete control. We
have

\[ x_{n1} = x_{no} + f(x_{no}, u_{no}, t_{no}) \frac{1}{n} \]

so

\[ x_{n1} = 0 + (x_{no} + n) \frac{1}{n} = 1. \]

In [2], the quantity

\[ z_{ni} = |\tilde{x}_{n}(t_{ni}) - x_{ni}| \]

is considered. Here \( z_{n1} = |\tilde{x}_{n}(t_{n1}) - x_{n1}| = |n(e^{1/n} - 1) - 1| \). So \( \lim_{n \to \infty} z_{n1} = 0 \).

However,

\[
\int_{t_{no}}^{t_{n1}} |x_{n}(\tau) - x_{n}(t_{no})| \, d\tau = \int_{t_{no}}^{t_{n1}} |n(e^\tau - 1)| \, d\tau
\]

\[ = \int_{0}^{1/n} n(e^\tau - 1) \, d\tau
\]

\[ = ne^\tau - n\tau \bigg|_{0}^{1/n}
\]

\[ = ne^{1/n} - 1 - n
\]

\[ = n(e^{1/n} - \frac{1}{n} - 1). \]
Suppose \( n(e^{1/n} - \frac{1}{n} - 1) \leq r_n \tau_n \). For large \( n \), \( \tau_n \leq \frac{C}{n} \), consider

\[
n^2(e^{1/n} - \frac{1}{n} - 1) \leq C\tau_n.
\]

But

\[
\lim_{n \to \infty} n^2(e^{1/n} - \frac{1}{n} - 1) = \lim_{x \to 0} \frac{e^x - x - 1}{x^2}
\]

\[
= \lim_{x \to 0} \frac{e^x - 1}{2x}
\]

\[
= \lim_{x \to 0} \frac{e^x}{2}
\]

\[
= \frac{1}{2}.
\]

Thus it is impossible that \( \lim_{n \to \infty} r_n = 0 \).

**Proof of Theorem 3.20:** We must show that

\[
(3.20) \quad \lim_{n \to \infty} (J_n(x_n(u_n),u_n) - J(x(u_n),u_n)) = 0.
\]

\[
|J(x(u_n),u_n) - J_n(x_n(u_n),u_n)|
\]

\[
= \left| \int_0^T g(\tilde{x}_n(t),u_n(t),t)dt - \sum_{i=0}^{n-1} g(x_{n_i},u_{n_i},t_{n_i}) \tau_{n_i} \right|
\]
\[ \leq \sum_{i=0}^{n-1} \int_{t_{ni}}^{t_{ni+1}} |g(x_{ni}(\tau), u_{ni}, \tau) - g(x_{ni}, u_{ni}, t_{ni})| \, d\tau. \]

We have

\[ |\tilde{x}_{n}(\tau) - x_{ni}| \leq |\tilde{x}_{n}(\tau) - \tilde{x}_{n}(t_{ni})| + |\tilde{x}_{n}(t_{ni}) - x_{ni}|. \]

So \( \lim |\tilde{x}_{n}(\tau) - x_{ni}| = 0 \) for \( \tau \in [t_{ni}, t_{ni+1}] \) uniformly in \( i \), from (3.27) or (3.29), and Lemma 3.21. Therefore,

\[ |g(\tilde{x}_{n}(\tau), u_{ni}, \tau) - g(x_{ni}, u_{ni}, t_{ni})| \to 0 \text{ as } n \to \infty \text{ uniformly in } i \text{ from (3.10).} \]

Hence we have the result of Theorem 3.20.

Summary: If \( u^* \in U \) is the optimal control for the system described by (3.1) and (3.2), if \( u^*_n \in U_n \) is the optimal control for the system described by (3.3) and (3.4), if \( f, g \), and the control set \( U \) satisfy the conditions as set forth in (3.5)-(3.10), and finally if the partition \( \Sigma_{in} \) satisfies \( \tau_n = \max_{i} \tau_{in} = O(n^{-1}) \) as \( n \to \infty \), then the results of this chapter show that the hypothesis of the general convergence theorem of Chapter two, Theorem 2.12, are satisfied. Therefore

\[ \lim_{n \to \infty} J_n(x_n(u^*_n), u^*_n) = J(x(u^*), u^*). \]
Thus it has been proved theoretically that certain continuous optimal control problems can be replaced by sequences of discrete optimal control problems.

IV. DIFFERENCE APPROXIMATIONS FOR BOUNDARY VALUE PROBLEMS

In this chapter we shall consider another application of the general theory presented in Chapter 2, this time to boundary value problems. Again the functional $J$ will be an integral, $J_n$ will be an approximating sum, the set $D$ will be defined by a boundary value problem, and $\Delta_n$ will be a corresponding difference problem.

§ 4.1 Statement of the Problem.

Consider minimizing the integral

$$ (4.1) \quad J(x,u) = \int_0^1 g(x,u,t)\,dt $$

which is defined on the solutions of the boundary value problem

$$ (4.2) \quad (px')' + qx = u, \quad x(0) = x(1) = 0; \quad 0 \leq t \leq 1, $$
where the control variable $u$ belongs to some particular control set $U$.

As before, let $\Sigma_{in}$ be an arbitrary subdivision of the interval $[0,1]$:

$$\sum_{\Sigma_{in}}: 0 = t_{n0} < t_{n1} < \ldots < t_{nn} = 1.$$ \hspace{1cm} (4.3)

The partial intervals of $\Sigma_{in}$ have length

$$\tau_{ni} = t_{ni+1} - t_{ni}.$$ \hspace{1cm}

An approximating functional is then

$$J_n(x_n, u_n) = \sum_{i=1}^{n-1} g(x_{ni}, u_{ni}, t_{ni}) \tau_{ni}$$ \hspace{1cm} (4.4)

and is defined on the solutions of the discrete boundary value problem

$$\Delta^{-}(p_{ni} \Delta^{+} x_{ni})$$ \hspace{1cm} (4.5)

$$+ q_{ni} x_{ni} = u_{ni}, \ i = 1, \ldots, n-1,$$

$$x_{n0} = x_{nn} = 0.$$
\( p_{ni} \) is the value of the function \( p \) at the point \( t_{ni} \) of the partition \( \Sigma_{ni} \), similarly for \( q_{ni} \).

The difference operators \( \Delta^+ \) and \( \Delta^- \) are defined by

\[
\Delta^+ x_{ni} = \frac{x_{ni+1} - x_{ni}}{\tau_{ni}}
\]

\[
\Delta^- x_{ni} = \frac{x_{ni} - x_{ni-1}}{\tau_{ni}}.
\]

The control variable \( u_n \) will come from a certain class \( U_n \).

It will be required to show that the infima of the functionals \( J_n \) converge to the infimum of the functional \( J \) as \( n \) tends to infinity.

§4.2 Basic Assumptions

Assume that the functions \( p \) and \( q \) satisfy

\( p(t) \) is continuous on \([0,1]\), \( p'(t) \) exists on \([0,1]\) and that \( p(t) > 0 \) on \([0,1]\).

\( q(t) \) is continuous on \([0,1]\).

Furthermore, assume that the only solution of

\( Lx = (px')' + qx = 0 \) which is zero at both \( t = 0 \) and \( t = 1 \) is the solution \( x \equiv 0 \).
In addition to the above, we shall assume that the function $g$ satisfies the same conditions as it did in Chapter 3, namely conditions (3.9) and (3.10). Moreover, assume that the control set $U$ is defined as it was in Chapter 3, that is, it satisfies (3.5) or (3.6).

§4.3 Calculation of the Continuous Green's Function

For simplicity, in this section and those that follow, we shall express equation (4.2) as

\[ L_x (px')' + qx = u, \]
\[ x(0) = x(1) = 0, \quad 0 \leq t \leq 1 \]

and equation (4.5) as

\[ L_n x_n = \Delta^{-1} (p_n \Delta^+ x_n) + q_n x_n = u_n', \]
\[ x_{n0} = x_{nn} = 0, \quad i = 1, 2, \ldots, n-1. \]

The next results calculate a Green's function for the equation (4.11).
Let \( y(t) \) be the unique solution of \( Ly = 0 \) satisfying \( y(0) = 0, \ y'(0) = 1 \). Also, let \( z \) be the unique solution of \( Lz = 0 \) satisfying \( z(1) = 0, \ z'(1) = 1 \).

**Proposition 4.1:** The Wronskian \( W(y,z) = yz' - zy' \) never vanishes for \( t \in I = [0,1] \).

**Proof:**

\[
(pW)' = pW' + p'W
\]

\[
= p(yz'' - zy'') + p'(yz' - y'z)
\]

\[
= y(pz')' - z(py')'
\]

\[
= y(-qz) - z(-qy) = 0.
\]

Hence \( pW \) is constant for \( t \in I \). Since from (4.8) \( p > 0 \) in \( I \), either \( W = 0 \) in \( I \) or \( W \) is never zero for \( t \in I \). Assume that \( W = 0 \) in \( I \). Then \( yz' = zy' \) for every \( t \in I \). At \( t = 0 \), \( y(0)z'(0) = z(0)y'(0) \), which implies that \( z(0) = 0 \). Therefore \( z \) is a nontrivial solution of \( Lz = 0 \) which is zero at both ends, contrary to assumption (4.10). Therefore \( W \) never vanishes in \( I \).
Proposition 4.2: Given any point $\tau$ in $I$, there exist uniquely determined quantities $C_1(\tau), C_2(\tau)$ such that

$$C_1(\tau)y(\tau) - C_2(\tau)y(\tau) = 0$$

and

$$C_1(\tau)y'(\tau) - C_2(\tau)z'(\tau) = -\frac{1}{p(\tau)}.$$

Proof: From Proposition 4.1, $pW$ is constant on $I$ and not zero. Let $k = (pW)^{-1}$ and define

$$C_1(\tau) = kz(\tau), \quad C_2(\tau) = ky(\tau).$$

Then $C_1(\tau)y(\tau) - C_2(\tau)z(\tau) = ky(\tau)z(\tau) - ky(\tau)a(\tau) = 0$ and $kz(\tau)y'(\tau) - ky(\tau)z'(\tau) = k(-W) = -\frac{1}{p(\tau)}$. The quantities $C_1$ and $C_2$ so defined are unique since

$$\begin{vmatrix} y(\tau) - z(\tau) \\ y'(\tau) - z'(\tau) \end{vmatrix} = -W \neq 0$$

by Proposition 4.1.

Proposition 4.3: The function $G(t, \tau)$ defined by

$$G(t, \tau) = \begin{cases} 
  kz(t)y(\tau) & \text{if } 0 \leq \tau \leq t \\
  ky(t)z(\tau) & \text{if } t \leq \tau \leq 1
\end{cases}$$

(4.13)
is continuous on the closed square $0 \leq t, \tau \leq 1$,
and $G(t, \tau) = G(\tau, t)$.

**Proof:** It is apparent that, on the triangle $0 \leq \tau \leq t$, $G(t, \tau) = k z(t)y(\tau)$ is a continuous function, since it is a product of continuous functions. Also, $G(t, \tau) = k y(t)z(\tau)$ is a product of continuous functions on the triangle $t \leq \tau \leq 1$. Furthermore, when $t = \tau$, the two definitions of $G(t, \tau)$ agree. Hence $G(t, \tau)$ is continuous on the closed square.

The second assertion of the proposition is immediate from the definition of $G$.

**Proposition 4.4:** Using the function $G$ defined in (4.13), the solutions of the differential equation

$$Lx = u, \quad u \in U, \quad x(0) = x(1) = 0, \quad 0 \leq t \leq 1$$

may be represented by

$$x(t) = \int_0^1 G(t, \tau)u(\tau)d\tau.$$
Proof: Let $v(t)$ and $w(t)$ be any twice differentiable functions defined for $t \in [\tau_1, \tau_2]$ where $0 \leq \tau_1 < \tau_2 \leq 1$. Then

$$\int_{\tau_1}^{\tau_2} [vLw - wLv] d\tau$$

$$= \int_{\tau_1}^{\tau_2} \left[ v(pw')' + vqw - w(pv')' - wqv \right]$$

$$= \left. (vpq' - wpv') \right|_{\tau_1}^{\tau_2}.$$  

Let $t \in I$. We shall apply (4.15) to each of the intervals $0 \leq \tau \leq t$ and $t \leq \tau \leq 1$ using $w(t) = G(t, \tau)$. We get

$$\int_{0}^{t} (vLG(t, \tau) - G(t, \tau)Lv) d\tau$$

$$= \left[ vpG_{\tau}(t, \tau) - G(t, \tau)pv' \right]_0^t.$$
Since \( LG(t, \tau) = 0 \), the above gives

\[-\int_0^t G(t, \tau) L v d \tau = [v p G_\tau(t, \tau) - G(t, \tau) p v']_0^t.\]

Similarly,

\[-\int_1^t G(t, \tau) L v d \tau = [v p G_\tau(t, \tau) - G(t, \tau) p v']_1^t.\]

If these are added, we get

\[-\int_0^t G(t, \tau) L v d \tau - \int_t^1 G(t, \tau) L v d \tau = v p G_\tau(t, \tau)]_0^t - G(t, \tau) p v']_0^t -

\[+ v p G_\tau(t, \tau)]_1^t - G(t, \tau) p v']_1^t.\]

Manipulating these with the aid of the definition of \( G \) and Proposition 4.2, we ultimately arrive at

\[\int_0^1 G(t, \tau) L v d \tau = v p G_\tau(t, \tau)]_0^1 - v(t).\]

Therefore, if \( x \) is a solution to the initial value problem \( Lx = u, x(0) = x(1) = 0, \) we simply substitute
\[ x(t) = \int_{0}^{1} G(t, \tau)u(\tau)d\tau. \]

**Theorem 4.5**: For a given \( u \in U \), the function \( x \) defined by

\[ x(t) = \int_{0}^{1} G(t, \tau)u(\tau)d\tau \]

is a solution of \( Lx = 0, \ x(0) = x(1) = 0. \)

**Proof**: It is clear that this \( x \) satisfies \( x(0) = x(1) = 0. \)

\[ x(t) = \int_{0}^{1} G(t, \tau)u(\tau)d\tau \]

\[ = \int_{0}^{t} k z(t)y(\tau)u(\tau)d\tau + \int_{t}^{1} k y(t)z(\tau)u(\tau)d\tau. \]

Therefore,

\[ p(t)x'(t) = kp(t)z'(t)\int_{0}^{t} y(\tau)u(\tau)d\tau \]

\[ + kp(t)y'(t)\int_{t}^{1} z(\tau)u(\tau)d\tau \]
So

\[(p(t)x'(t))' + q(t)x(t)\]

\[= k(p(t)z'(t))' \int_0^t y(\tau)u(\tau) \, d\tau\]

\[+ k(p(t)y'(t))' \int_t^1 z(\tau)u(\tau) \, d\tau\]

\[+ k(q(t)z(t))^t_0 y(\tau)u(\tau) \, d\tau\]

\[+ kq(t)y(t) \int_t^1 z(\tau)u(\tau) \, d\tau\]

\[+ kp(t)z'(t)y(t)u(t)\]

\[= u(t)\]

from Proposition 4.2 and from the fact that \(y\) and \(z\) satisfy the homogeneous equation.
§ 4.4 Calculation of the Discrete Green's Function

The next results calculate a Green's function for equation (4.12).

Remark: There is a nonzero solution of the equation

\[ L_n(x_n) = \Delta^{-1}(p_{ni} \Delta^+ x_{ni}) + q_{ni}x_{ni} = 0, \quad i = 1, \ldots, n-1 \]

satisfying \( x_{no} = 0 \).

Proof: If we write out the equation \( L_n(x_n) = 0 \) in detail, we get

\[ L_n(x_n) = \Delta^{-1}(p_{ni} \Delta^+ x_{ni}) + q_{ni}x_{ni} \]

\[ = \frac{p_{ni}}{\tau_{ni}} x_{ni+1} + \left( q_{ni} - \frac{p_{ni}}{\tau_{ni}} - \frac{p_{ni-1}}{\tau_{ni} \tau_{ni-1}} \right) x_{ni} \]

\[ + \frac{p_{ni-1}}{\tau_{ni} \tau_{ni-1}} x_{ni-1} = 0. \]

\[ = A_{ni}x_{ni+1} + B_{ni}x_{ni} + C_{ni}x_{ni-1} = 0. \]

At \( i = 1 \), \( A_{ni}x_{n2} + B_{ni}x_{n1} + C_{ni}x_{no} = 0. \) By assumption, \( x_{no} = 0 \), and furthermore, the coefficient \( A_{ni} \neq 0 \) for any \( i \). Therefore we may solve for \( x_{n2} \) in terms of \( x_{n1} \):
\[ x_{n2} = \frac{-B_{nl}}{A_{nl}} x_{nl}. \]

Therefore, assigning \( x_{nl} \) any nonzero value, and solving inductively for the remaining \( x_{ni} \)'s, we have the result.

Take any nonzero solution \( y_n \) of the equation \( L_n y_n = 0 \) satisfying \( y_{n0} = 0 \). Also, take any nonzero solution \( z_n \) of \( L_n z_n = 0 \) satisfying \( z_{nn} = 0 \). In the following we shall assume in addition to the previous assumptions that any solution of the equation \( L_n x_n = 0 \) which is zero at both \( x_{no} \) and \( x_{nn} \) is the identically zero solution. This assumption gives uniqueness of solutions of the equation \( L_n x_n = u_n', x_{no} = x_{nn} = 0 \), corresponding to one given \( u_n \) in the following way:

Suppose \( r_n \) and \( s_n \) are two solutions of \( L_n x_n = u_n', x_{no} = x_{nn} = 0 \), corresponding to a given \( u_n \). Then \( t_n = r_n - s_n \) is a solution of \( L_n t_n = 0 \), and \( t_n \) is zero both at \( t_{no} \) and \( t_{nn} \). Thus \( t_n = 0 \) and \( r_n = s_n \).

**Proposition 4.6:** Any solution of the equation (4.12) can be expressed in the form

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\[(4.17) \quad x_{ni} = \sum_{k=1}^{n-1} G_n(i,k)u_{nk}^\tau nk\]

where

\[(4.18) \quad G_n(i,k) = \begin{cases} 
edependentonlyonn. 
\end{cases}\]

**Proof:** This shall be demonstrated by direct substitution of (4.17) into (4.12).

\[(4.19) \quad \Delta^{-}(p_{ni} x_{ni}^\tau_{ni}) + q_{ni} x_{ni}^\tau_{ni} = \Delta^{-}(p_{ni} x_{ni}^\tau_{ni}) + q_{ni} x_{ni}^\tau_{ni} \]

\[
= \Delta^{-}\left(\frac{p_{ni}}{2} \sum_{k=1}^{n-1} G_n(i+1,k)u_{nk}^\tau nk\right) + q_{ni} \sum_{k=1}^{n-1} G_n(i,k)u_{nk}^\tau nk
\]

\[
= \frac{p_{ni}}{2} \sum_{k=1}^{n-1} G_n(i+1,k)u_{nk}^\tau nk
\]

\[
- \frac{p_{ni-1}}{\tau_{ni-1}} \sum_{k=1}^{n-1} G_n(i,k)u_{nk}^\tau nk
\]

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\[ -\frac{p_{ni}}{\tau_{ni}} \sum_{k=1}^{n-1} G_n(i,k)u_{nk}\tau_{nk} + \frac{p_{ni-1}}{\tau_{ni}\tau_{ni-1}} \sum_{k=1}^{n-1} G_n(i-1,k)u_{nk}\tau_{nk} \]

\[ + q_{ni} \sum_{k=1}^{n-1} G_n(i,k)u_{nk}\tau_{nk} \]

\[ = -\frac{p_{ni}}{\tau_{ni}} \left( \sum_{k=1}^{i} K_n z_{ni} y_{nk} u_{nk}\tau_{nk} + \sum_{k=i+1}^{n-1} K_n y_{ni} z_{nk} u_{nk}\tau_{nk} \right) \]

\[ + \frac{p_{ni-1}}{\tau_{ni}\tau_{ni-1}} \left( \sum_{k=1}^{i} K_n z_{ni} y_{nk} u_{nk}\tau_{nk} + \sum_{k=i+1}^{n-1} K_n y_{ni} z_{nk} u_{nk}\tau_{nk} \right) \]

\[ + \frac{p_{ni}}{\tau_{ni}} \left( \sum_{k=1}^{i} K_n z_{ni} y_{nk} u_{nk}\tau_{nk} + \sum_{k=i+1}^{n-1} K_n y_{ni} z_{nk} u_{nk}\tau_{nk} \right) \]

\[ + q_{ni} \left( \sum_{k=1}^{i} K_n z_{ni} y_{nk} u_{nk}\tau_{nk} + \sum_{k=i+1}^{n-1} K_n y_{ni} z_{nk} u_{nk}\tau_{nk} \right) . \]
We shall now consider the five terms in the above expression (4.19) which involve $z_{ni}$. We will add and subtract terms so that in each case the upper limit of summation is $i$. We get:

\[
\begin{align*}
\frac{p_{ni}}{\tau_{ni}} z_{ni+1} & \sum_{k=1}^{i} K_{n}^{\gamma nk} u_{nk} \tau_{nk} + \frac{p_{ni}}{\tau_{ni}} K_{n} z_{ni+1} y_{ni+1} u_{ni+1} \tau_{ni+1} \\
- \frac{p_{ni-1}}{\tau_{ni} \tau_{ni-1}} z_{ni-1} & \sum_{k=1}^{i} K_{n}^{\gamma nk} u_{nk} \tau_{nk} \\
- \frac{p_{ni}}{2} z_{ni} & \sum_{k=1}^{i} K_{n}^{\gamma nk} u_{nk} \tau_{nk} \\
+ \frac{p_{ni-1}}{\tau_{ni} \tau_{ni-1}} z_{ni-1} & \sum_{k=1}^{i} K_{n}^{\gamma nk} u_{nk} \tau_{nk} \\
- \frac{p_{ni-1}}{\tau_{ni} \tau_{ni-1}} K_{n} z_{ni-1} & y_{ni} u_{ni} \tau_{ni} \\
+ q_{ni} z_{ni} & \sum_{k=1}^{i} K_{n}^{\gamma nk} u_{nk} \tau_{nk} \\
= \left( \frac{p_{ni}}{\tau_{ni}} (z_{ni+1} - z_{ni}) - \frac{p_{ni-1}}{\tau_{ni} \tau_{ni-1}} (z_{ni} - z_{ni-1}) \right) \\
+ q_{ni} z_{ni} & \sum_{k=1}^{i} K_{n}^{\gamma nk} u_{nk} \tau_{nk}
\end{align*}
\]
Now, the sum of the terms in the parenthesis is zero since \( z_n \) satisfies the equation \( L(z_n) = 0 \).

In like manner, the sum of the 5 terms in (4.19) which involve \( y_{ni} \) is

\[
- \frac{P_{ni-l}}{\tau_{ni}^{\tau_{ni-l}}} K_n z_{ni-l}^{y_{ni-l}} y_{ni-l}^{u_{ni-l}} z_{ni-l}^{\tau_{ni-l}}
\]

Therefore,

\[
\Delta (p_{ni} \Delta x_{ni}) + q_{ni} x_{ni} = \frac{P_{ni-l}}{\tau_{ni}^{\tau_{ni-l}}} K_n y_{ni-l}^{u_{ni-l}} \tau_{ni-l}
\]

\[
- \frac{P_{ni-l}}{\tau_{ni}^{\tau_{ni-l}}} K_n z_{ni-l}^{y_{ni-l}} y_{ni-l}^{u_{ni-l}} \tau_{ni-l}
\]

\[
= \frac{P_{ni-l}}{\tau_{ni-l}} n_{ni}^{u_{ni-l}} (y_{ni-l}^{z_{ni-l}} z_{ni-l}^{y_{ni-l}} - y_{ni-l}^{z_{ni-l}} z_{ni-l}^{y_{ni-l}}).
\]
Hence, with \( K_n = \left( \prod_{i=1}^{n} (c_i) \right)^{-1} \),

\[ \Delta^{-}(p_n \Delta^+ x_{ni}) + q_{ni} x_{ni} = u_{ni}. \]

It remains to show that \( K_n \) is indeed constant. To prove this, consider the "discrete Wronskian" defined by

\[
\Delta^+ y_{ni} \quad \Delta^+ z_{ni} \\
\Delta^+ y_{ni} \quad \Delta^+ z_{ni}
\]

\[ W_{ni} = \det \left| \begin{array}{cc} y_{ni} & z_{ni} \\ \Delta^+ y_{ni} & \Delta^+ z_{ni} \end{array} \right| = y_{ni} \Delta^+ z_{ni} - z_{ni} \Delta^+ y_{ni} \]

We shall show that \( p_n W_{ni} \) is constant by showing that \( \Delta^{-}(p_n W_{ni}) = 0. \)

\[
\Delta^{-}(p_n W_{ni}) = \Delta^{-}(p_n y_{ni} \Delta^+ z_{ni} - p_n z_{ni} \Delta^+ y_{ni})
\]

\[ = y_{ni} \Delta^{-}(p_n \Delta^+ z_{ni}) + p_n \Delta^+ z_{ni} \Delta^{-} y_{ni} \]

\[ - z_{ni} \Delta^{-}(p_n \Delta^+ z_{ni}) - p_n \Delta^+ y_{ni} \Delta^{-} z_{ni} \]

\[ = y_{ni} (-q_{ni} z_{ni}) - z_{ni} (q_{ni} y_{ni}) \]

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\[ + p_{ni-1} (\frac{z_{ni}-z_{ni-1}}{\tau_{ni-1}}) (\frac{y_{ni}-y_{ni-1}}{\tau_{ni}}) \]

\[ - p_{ni} (\frac{y_{ni}-y_{ni-1}}{\tau_{ni-1}}) (\frac{z_{ni}-z_{ni-1}}{\tau_{ni}}) \]

\[ = 0. \]

Therefore, \( p_{ni} W_{ni} \) is constant.

\[ p_{ni} W_{ni} = p_{ni} (y_{ni} \Delta y_{ni} - z_{ni} \Delta y_{ni}) \]

\[ = p_{ni} (\frac{y_{ni}}{\tau_{ni}} (z_{ni+1} - z_{ni}) - \frac{z_{ni}}{\tau_{ni}} (y_{ni+1} - y_{ni})) \]

\[ = \frac{p_{ni}}{\tau_{ni}} (y_{ni} z_{ni+1} - z_{ni} y_{ni+1}). \]

Now, under the assumption that \( p > 0 \), and the assumption that the only solution of the homogeneous equation which is zero at both ends is the identically zero solution, then the constant \( p_{ni} W_{ni} \) is nonzero.

To see this, assume that \( p_{ni} W_{ni} = 0 \) for every \( i \). This implies that \( W_{ni} = 0 \) for every \( i \). From the definition of \( W_{ni} \), we get that \( y_{ni} z_{ni+1} = y_{ni+1} z_{ni} \), for every \( i \). At \( i = 0 \),
\[ z_{n1}y_{no} = z_{no}y_{n1}. \]

\[ z_{nn} = 0, \ y_{no} = 0, \] so \( z_{no} \neq 0. \) Therefore \( y_{n1} = 0. \) From the fact that \( L_n(y_n) = 0, \) at \( i = 1, \) we get

\[
\frac{p_{n1}}{\tau_{n1}}(y_{n2} - y_{n1}) - \frac{p_{no}}{\tau_{n1} \tau_{no}}(y_{n1} - y_{no}) + q_{n1}y_{n1} = 0.
\]

This implies \( \frac{p_{n1}}{\tau_{n1}}y_{n2} = 0, \) i.e., \( y_{n2} = 0. \) By induction, we get \( y_{ni} = 0 \) for every \( i, \) but this contradicts the assumption that \( y_n \) was a nonzero solution of \( L_n(y_n) = 0. \)

§ 4.5 Setting in Chapter 2, Relation to Chapter 3

We shall make the following definitions

(4.21) \( D = \{(x,u): (px')'+ qx = u, \ t \in [0,1], \ x(0) = x(1) = 0, \ u \in U \} \)

(4.22) \( D_n = \{(x,u): (px')'+ qx = u, \ t \in [0,1], \ x(0) = x(1) = 0, \ u \in U_n \} \)

(4.23) \( \Delta_n = \{(x,u): \Delta^-(p_{ni}\Delta^+x_{ni}) + q_{ni}x_{ni} = u_{ni}, \ i = 1, \ldots, n-1, \ x_{no} = x_{nn} = 0, \ u_n \in U_n \}. \)
Recall that the functionals $J$ and $J_n$ are defined by (4.1) and (4.4) respectively.

The next results will verify that the hypothesis of Theorem 2.12 is satisfied. The first result is used to obtain the upper semicontinuity of $J$.

**Theorem 4.7:** If $u_n \in U$ and $u_n \to u$ in $L^p$, $1 \leq p \leq \infty$, then $x_n \to x$ uniformly, i.e. $x$ is continuous with respect to $u$.

**Proof:** It must be shown that, for given $\varepsilon > 0$, there is a $\delta > 0$ such that if $\|u_n - u\|_p < \delta$, then $\sup |x_n - x| < \varepsilon$. From relation (4.14), we get

$$|x_n(t) - x(t)| \leq \int_0^1 |G(t, \tau)| |u_n(\tau) - u(\tau)| d\tau$$

$$\leq \|u_n - u\|_p \left\{ \int_0^1 |G(t, \tau)| \right\}^{1/q} .$$

From Proposition 4.3, $G$ is continuous on the closed square $0 \leq t, \tau \leq 1$, and so it is bounded there, say by $B$. Therefore

$$|x_n(t) - x(t)| \leq B \|u_n - u\|_p .$$
Thus \( \sup |x_n(t) - x(t)| \leq B \|u_n - u\|_p \).

With this result the upper semicontinuity of \( J \) follows as in § 3.5 of Chapter 3.

**Corollary 4.8:** It is equally easy to show that if \( u \in L_p, 1 \leq p < \infty \), and \( x \in L_r, 1 \leq r < \infty \), then \( x \) is continuous with respect to \( u \).

We must verify condition (2.2) of Theorem 2.12, that \( \liminf \tau_n \geq D \). We do this in the following result:

**Proposition 4.9:** For \( U \) defined according to (3.5) or (3.6), and for \( D \) and \( D_n \) defined by (4.21) and (4.22),

\[
\liminf_{\tau_n} D_n \geq D,
\]

where \( \tau \) is the product of the \( L_p \) metric topology on the \( u \)'s and the uniform topology on the \( x \)'s.

**Proof:** First note that \( U, U_n \) are defined in the same way as in Chapter 3, so for either definition of \( U \), we have \( \liminf_{L_p} U_n = U \).

Let \( (x,u) \in D \). We must show that \( (x,u) \in \liminf_{\tau_n} D_n \). Recall that \( (x,u) \in \liminf_{\tau_n} D_n \)

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if and only if for every nhd $V$ of $(x,u)$, there is an $N$ such that if $n > N$ then $D_n \cap V \neq \emptyset$. We may take $V = V_1 \times V_2$, where $u \in V_2$ is open in $L_p$, $x \in V_1$ is open in the uniform topology. Since $x$ is continuous with respect to $u$ (Theorem 4.7), choose $V_3$ such that $u \in V_3 \subset V_2$ and $V_3$ is so small that if $u_n \in V_3$, then $x_n \in V_1$. Since $\liminf_{L_p} U_n = U$, choose $N(V_3)$ such that if $n > N(V_3)$, then $U_n \cap V_3 \neq \emptyset$. This implies that, for $n > N(V_3)$,

$$D_n \cap (V_1 \times V_3) \subset D_n \cap (V_1 \times V_2) = D_n \cap \emptyset \neq \emptyset.$$ 

Hence $\liminf_{\tau} D_n \supset D$.

Remark: Note that $D_n = E_n \subset D$ in this setting, consequently Remark 2.17 applies, and so conditions (2.5) and (2.6) of Theorem 2.12 are satisfied.

All that remains to be shown of the hypothesis of Theorem 2.12 are conditions (2.3) and (2.4). We may verify these conditions by proving the following theorem.

Theorem 4.10: If $u_n$ is any sequence in $U_n$, then

$$\lim_{n \to \infty} \{J_n(x_n(u_n),u_n) - J(x(u_n),u_n)\} = 0$$
where \( x_n(u_n) \) is the solution of equation (4.5) and \( x(u_n) \) is the solution of (4.2) corresponding to the same discrete control \( u_n \), piecewise constantly extended in the second case.

As it did in Chapter 3, the proof relies heavily on the following lemma:

**Lemma 4.11:** \( |x'(t_i) - x_n| \to 0 \) uniformly in \( i \) as \( n \to \infty \) provided \( \Sigma_{ni} \) is the uniform subdivision of \([0,1]\).

**Proof:** Define

(4.24) \( u_n(\tau) = u_{nk} \) if \( \frac{k}{n} \leq \tau < \frac{k+1}{n} \).

(4.25) \( G_{ni}(\tau) = G_n(i,k) \) if \( \frac{k}{n} \leq \tau < \frac{k+1}{n} \).

Here again we are using the notation \( \tilde{x}_n(t) = x(u_n) \).

With the above definitions, we have

\[
\tilde{x}_n(t) = \int_0^1 G_n(\frac{i}{n}, \tau) u_n(\tau) \, d\tau \quad \text{and} \quad \tilde{x}_{ni} = \sum_{k=1}^{n-1} G_n(i,k) u_{nk} \cdot \frac{1}{n} \]

\[
= \int_0^1 G_{ni}(\tau) u_n(\tau) \, d\tau .
\]
Then \[ |x_n(t) - x_{ni}| \leq \int_0^1 |G(x_n(\tau), \tau) - G_{ni}(\tau)| |u_n(\tau)| d\tau. \]

(4.26) \[ \leq \|u_n\|_p \left\{ \int_0^1 |G(x_n(\tau), \tau) - G_{ni}(\tau)| q \right\}^{1/q}. \]

We shall show that \[ |G(x_n(\tau), \tau) - G_{ni}(\tau)| \to 0 \] uniformly in \( i \) and in \( \tau \) as \( n \to \infty \). To do this, we shall repeat some of the considerations involved in obtaining the Green's functions.

Recall, \( y \) is the unique solution of \( Lx = 0 \) satisfying \( y(0) = 0, \ y'(0) = 1, \) and \( z \) is the unique solution of \( Lx = 0 \) satisfying \( z(1) = 0, \ z'(1) = 1. \)

If the subdivision \( \Sigma_n \) is the uniform subdivision, then \( \Delta^+ u_{ni} = \frac{x_{ni+1} - x_{ni}}{1/n} \) and \( \Delta^- u_{ni} = \frac{x_{ni} - x_{ni-1}}{1/n}. \)

We shall take \( y_n \) as the unique solution of \( L_n(y_n) = 0 \) which satisfies \( y_{no} = 0, \Delta^+ y_{no} = 1, \) and also we shall take \( z_n \) as the unique solution of \( L_n(z_n) = 0 \) which satisfies \( z_{nn} = 0, \Delta^- z_{nn} = 1. \) Then

\[ G(t, \tau) = \begin{cases} Kz(t) y(\tau) & \text{if } 0 \leq \tau \leq t \\ Ky(t) z(\tau) & \text{if } t \leq \tau \leq 1 \end{cases} \]
and $K = [p(yz' - zy')]^{-1} = \text{constant. Also,}$

$$G_n(i, k) = \begin{cases} 
K_n z_{ni} y_{ni} & 0 \leq k \leq i \\
K_n y_{ni} z_{nk} & i < k \leq n 
\end{cases}$$

and $K_n = \{nP_n(z_{ni+1} y_{ni} - y_{ni+1} z_{ni})\}^{-1}$.

We shall show first that $|y(\frac{i}{n}) - y_{ni}| \to 0$ uniformly in $i$ as $n \to \infty$.

Consider the equation $(py')' + qy = 0$. If we let $v = py'$, then $v' = -qy$, and the second order differential equation becomes the following first order differential system

$$y' = v/p$$

(4.27)

$$v' = -qy$$

with initial conditions given by those on $y$, that is, $y(0) = 0$, $v(0) = p(0)$.

Consider also the equation $L_n y_n = \Delta^{-}(p_{ni} \Delta^{+} y_{ni}) + q_{ni} y_{ni} = 0$. As above, let $v_{ni} = p_{ni} \Delta^{+} y_{ni}$, then $\Delta^{-} v_{ni} = q_{ni} y_{ni}$. Then the second order difference equation becomes the following first order difference system:
\[ \Delta^+ y_{ni} = \frac{v_{ni}}{p_{ni}} \]

(4.28)
\[ \Delta^+ v_{ni} = -q_{ni} y_{ni} + \left( q_{ni} y_{ni} - q_{ni+1} y_{ni+1} \right) \]

with initial conditions \( y_{no} = 0, \ v_{no} = p_{no} \).

We have the following relationships:

(4.29) (a) \[ y\left(\frac{i+1}{n}\right) = y\left(\frac{i}{n}\right) + \int_{i/n}^{(i+1)/n} v(t)/p(t) \, dt \]

(4.29) (b) \[ y_{ni+1} = y_{ni} + \int_{i/n}^{(i+1)/n} v_{ni}/p_{ni} \, dt \]

(4.29) (c) \[ v\left(\frac{i+1}{n}\right) = v\left(\frac{i}{n}\right) + \int_{i/n}^{(i+1)/n} -q(t)y(t) \, dt \]

(4.29) (d) \[ v_{ni+1} = v_{ni} + \int_{i/n}^{(i+1)/n} -q_{ni} y_{ni} \, dt + \frac{1}{n}(q_{ni} y_{ni} - q_{ni+1} y_{ni+1}) \]

If we write \( w = (y, v)' \) and \( A = \begin{pmatrix} 0 & 1/p \\ -q & 0 \end{pmatrix} \), then the differential system (4.27) becomes

(4.30) \[ w' = Aw, \ w(0) = \begin{pmatrix} 0 \\ p(0) \end{pmatrix} \]
If we define \( w_{ni} = (Y_{ni}) \), \( i = 0, 1, \ldots, n \)
and \( A_{ni} = \begin{pmatrix} 0 & 1/F_{ni} \\ -q_{ni} & 0 \end{pmatrix} \), \( i = 0, 1, \ldots, n \), we may write, upon subtracting in (4.29),

\[
(4.31) \quad w^{(i+1)/n} - w_{ni+1} = w^{(i)/n} - w_{ni} + \int_{i/n}^{(i+1)/n} (A w - A_{ni} w_{ni}) dt + b_{ni}
\]

where

\[
b_{ni} = \begin{pmatrix} 0 \\ \frac{1}{n} (q_{ni} Y_{ni} - q_{ni+1} Y_{ni+1}) \end{pmatrix}
\]

The equations written above are first order initial value problems, which suggests that methods like those of Chapter 3, in particular Lemma 3.21, may be applied. To apply this machinery, it is necessary to see that the conditions of Chapter 3 are satisfied by the system (4.30).
We must verify conditions (3.7) and (3.8) for the function $Aw = f(w,u,t)$.

$$|f(w,u,t)| = |Aw|$$

$$= \left| \begin{pmatrix} v(t) \\ p(t) \\ -q(t)y(t) \end{pmatrix} \right|$$

$$= \left( \frac{v^2}{p} + q^2y^2 \right)^{1/2}.$$

Let $A^* = \max\left\{ \max_{0 \leq t \leq 1} \left( \frac{1}{p(t)} \right), \max_{0 \leq t \leq 1} q(t) \right\}$. Then $|f(w,u,t)| \leq A^*(v^2 + y^2)^{1/2} = A^*|w|$. Hence condition (3.7) is satisfied. Since the function $f(w,u,t)$ is independent of $u$ and linear in $w$, an examination of the above shows that condition (3.8) is satisfied with $A^*_4 = A^*$ and $B \equiv 0$.

Hence, from Lemma 3.21, we would have $|w_{n,i}^{(i)} - w_{ni}| \to 0$ uniformly in $i$ as $n \to \infty$ if it were not for the additional term $b_{ni}$ that we have in (4.31). However, upon examining the proof of Lemma 3.21, if we can show that...
\[ |q_{ni+1}y_{ni+1} - q_{ni}y_{ni}| = r_n \rightarrow 0 \text{ as } n \rightarrow \infty \]

then this term may be included in the right hand side of inequality (3.29). To show this, consider

\[ |q_{ni+1}y_{ni+1} - q_{ni}y_{ni}| \]

(4.32) \[ = |q_{ni+1}y_{ni+1} - q_{ni}y_{ni} + q_{ni}y_{ni+1} - q_{ni}y_{ni}| \]

\[ \leq |q_{ni+1} - q_{ni}| |y_{ni+1}| + |y_{ni+1} - y_{ni}| |q_{ni}|. \]

Now, \( q \) is continuous on a closed interval, so it is bounded there. We need to show that \( y_n \), the solution of the difference equation, is also bounded, and that this bound is independent of \( n \).

To show this, we shall rewrite the representation for the vector \( w_{ni} \) slightly. We have

\[ \Delta^+ w_{ni} = v_{ni}/p_{ni}, \text{ and} \]

\[ \Delta^- v_{ni} = -q_{ni}y_{ni}. \]

Therefore,
\[
\frac{v_{ni+1} - v_{ni}}{1/n} = -q_{ni+1} v_{ni+1}
\]

\[
= -q_{ni+1} \frac{1}{n} \left( \frac{v_{ni+1} - v_{ni}}{1/n} \right) - q_{ni+1} v_{ni}
\]

\[
= \frac{-q_{ni+1}}{np_{ni}} v_{ni} - q_{ni+1} v_{ni}.
\]

Therefore, with \( w_{ni} = \begin{pmatrix} v_{ni} \\ v_{ni} \end{pmatrix} \), we have

\[
\Delta^+ w_{ni} = \begin{pmatrix} 0 & 1/p_{ni} \\ -q_{ni+1} & -q_{ni+1}/(np_{ni}) \end{pmatrix} \cdot w_{ni}
\]

Consider \( |w_{ni}| \) (Euclidean absolute value). Since each entry of the matrix is bounded, the matrix is bounded say by \( K \). Therefore

\[
\Delta^+ |w_{ni}| \leq |\Delta^+ w_{ni}| \leq K |w_{ni}|.
\]

At \( i = 0 \), with \( |w_{no}| = \left| \begin{pmatrix} 0 \\ p_{no} \end{pmatrix} \right| \), \( |w_{n1}| - |w_{no}| \leq \frac{K}{n} |w_{no}| \), so \( |w_{n1}| \leq (1 + \frac{K}{n}) |w_{no}| \). At \( i = 1 \),

\[
|w_{n2}| - |w_{n1}| \leq \frac{K}{n} |w_{n1}|, \quad \text{so} \quad |w_{n2}| \leq (1 + \frac{K}{n})^2 |w_{no}|.
\]
Proceeding inductively,

\[ |w_{ni}| \leq (1 + \frac{K}{n})^i |w_{no}| \]

\[ \leq (1 + \frac{K}{n})^n |w_{no}| \]

\[ \leq (e^{K/n})^n |w_{no}| \]

\[ = e^K |w_{no}| . \]

Therefore \( |w_{ni}| \), and consequently \( |y_{ni}| \) and \( |v_{ni}| \) are bounded independent of \( n \).

Let \( B \) be the maximum of the bounds for \( |y_{ni}| \), \( |v_{ni}| \), and \( |q_{ni}| \). Therefore, from inequality (4.32),

\[ |q_{ni+1}y_{ni+1} - q_{ni+1}y_{ni}| \leq B |q_{ni+1} - q_{ni}| + B |v_{ni+1} - v_{ni}| \]

\[ \leq B |q_{ni+1} - q_{ni}| + B \frac{|v_{ni}|}{|p_{ni}|} \frac{1}{n} \]

\[ \leq B |q_{ni+1} - q_{ni}| + B \frac{1}{|p_{ni}|} \frac{1}{n} \to 0 \]

as \( n \to \infty \), independent of \( i \).
Hence \( |w_{n_i} - w\left(\frac{i}{n}\right)| \to 0 \) uniformly in \( i \) as \( n \to \infty \),

and consequently, \( |y_{n_i} - y\left(\frac{i}{n}\right)| \to 0 \) uniformly in \( i \) as \( n \to \infty \).

In like manner, by applying the material of Chapter 3, it may be shown that \( |z\left(\frac{i}{n}\right) - z_{n_i}| \to 0 \) uniformly in \( i \) as \( n \to \infty \).

Also, we will need the result that \( |K_n - K| \to 0 \) as \( n \to \infty \). If we can show that \( 1/K_n \to 1/K \) as \( n \to \infty \), then this will be sufficient.

\[
l/K_n = nP_{n_i}(z_{n_i+l} - y_{n_i+l}z_{n_i})
\]

\[
= P_{n_i}(y_{n_i}z_{n_i} - z_{n_i}y_{n_i}).
\]

Upon evaluating \( 1/K_n \) at \( i = 0 \), we see that

\[
l/K_n = P_{n_0}(-z_{n_0}).
\]

On the other hand, evaluating \( 1/K \) at \( t = 0 \), we get \( 1/K = P(0)(-z(0)) \). Therefore, using the result that \( z\left(\frac{i}{n}\right) \to z_{n_i} \) uniformly in \( i \), we get

\[
\lim_{n \to \infty} P_{n_0}(-z_{n_0}) = -P(0) \lim_{n \to \infty} z_{n_0}
\]

\[
= -P(0)z(0)
\]

\[
= 1/K.
\]
Having these facts at hand, we shall complete the argument that \( |G_{\frac{i}{n}}, \tau) - G_{\frac{i}{n}}(\tau) | \to 0 \) uniformly in \( i \) as \( n \to \infty \).

\[
|G_{\frac{i}{n}}, \tau) - G_{\frac{i}{n}}(\tau) | = \begin{cases} 
|Kz_{\frac{i}{n}}y(\tau) - K_n z_{ni} y_{nk}| & \text{if } \tau \leq \frac{i}{n} \\
|Kz_{\frac{i}{n}}y(\tau) - K_n z_{ni} y_{nk}| & \text{if } \tau \geq \frac{i}{n} \\
|Kz_{\frac{i}{n}}y(\tau) - K_n z_{ni} y_{nk}| & \text{if } \tau \leq \frac{k}{n} \leq \frac{(k+1)}{n}
\end{cases}
\]

We note that \( \tau \leq \frac{i}{n} \) and \( \frac{k}{n} \leq \tau \leq \frac{(k+1)}{n} \) imply that \( k \leq i \), and that \( \tau \geq \frac{i}{n} \) and \( \frac{k}{n} \leq \tau \leq \frac{(k+1)}{n} \) imply that \( k \geq i \).

Now, for arbitrary \( i, 0 \leq i \leq n \) and for \( k \leq i \), \( \frac{k}{n} \leq \tau \leq \frac{k+1}{n} \), \( |Kz_{\frac{i}{n}}y(\tau) - K_n z_{ni} y_{nk}| \leq |K-K_n| |z_{\frac{i}{n}}y(\tau)| + |z_{\frac{i}{n}} - z_{ni}| |K_n y(\tau)| + |y(\tau)-y_{nk}| |K_n z_{ni}| \). The bounding lemma, Lemma 3.4, assures that \( w \), and hence \( z \) and \( y \) are bounded. Therefore, we are guaranteed that for this case, \( |G_{\frac{i}{n}}, \tau) - G_{\frac{i}{n}}(\tau) | \to 0 \) uniformly. The considerations are similar for the second case. Therefore, from inequality (4.26), \( |x_{\frac{i}{n}} - x_{ni}| \to 0 \) uniformly in \( i \) as \( n \to \infty \). This concludes the proof of Lemma 4.11.
With Lemma 4.11 at hand, the proof of Theorem 4.10 follows as in Chapter 3.

**Summary:** To approximate the minimum value for the system defined by (4.1) and (4.2), subdivide the interval $[0,1]$ into $n$ equal parts by the subdivision points $\xi_{in}: 0 < \frac{1}{n} < \frac{2}{n} < \ldots < \frac{n-1}{n} < 1$. Assume that an optimal control $\bar{u}_n$ can be found for each system described by (4.4) and (4.5). The theorems of this chapter show that the hypothesis of Theorem 2.12 as presented in Chapter two are satisfied under the assumptions (4.8)-(4.10), and (3.9), (3.10). Consequently, the minimum value

$$\inf_{u \in U} J_n(x_n(u), u) = J_n(\bar{x}_n(\bar{u}_n), \bar{u}_n)$$

is such that

$$\lim_{n \to \infty} J_n(\bar{x}_n(\bar{u}_n), \bar{u}_n) = \inf_{u \in U} J(x(u), u).$$
§ 4.6 An Example

In this section, we shall consider a special example of the material presented previously in this chapter. The reason for doing this is that the Green's functions can be calculated and the convergence can be shown directly.

We are required to minimize

\[(4.33) \quad J(x,u) = \int_0^1 g(x,u,t)dt \]

which is defined on the solutions of the boundary value problem

\[(4.34) \quad x'' = u, \quad 0 \leq t \leq 1, \quad x(0) = x(1) = 0. \]

Here \( u \) belongs to a control set \( U \) which satisfies either (3.5) or (3.6). Also, \( g \) is a scalar valued function which satisfies (3.9) and (3.10).

Consider also an arbitrary subdivision \( \Sigma_{in} \) of \([0,1]\) into intervals of length \( t_{ni} = t_{ni+1} - t_{ni} \) by the subdivision points \( 0 = t_{n0} < t_{n1} < t_{n2} < \ldots < t_{nn} = 1 \). In accordance with the previously presented material, consider the discrete problem defined by
(4.35) \[ J_n(x_n, u_n) = \sum_{k=1}^{n-1} g(x_{ni}, u_{ni}, t_{ni}) \tau_{ni} \]

where \( J_n \) is defined on the solutions of the difference equation

(4.36) \[ \Delta^- (\Delta^+ x_{ni}) = u_{ni}, \quad x_{no} = x_{nn} = 0, \quad i = 1, 2, \ldots, n-1 \]

for control functions \( u_n \) in the class \( U_n \) as previously described.

We shall now calculate the Green's functions for both the differential equation and its corresponding difference equation.

Remark: The Green's functions for the problem \( x'' = u, \quad x(0) = x(1) = 0 \) may be calculated directly using simple integration by parts.

Consider the equation \( x'' = 0 \). We seek the solution \( y \) which satisfies \( y(0) = 0, y'(0) = 1 \). This solution is clearly \( y = t \). We also want the solution \( z \) which satisfies \( z(1) = 0, z'(1) = 1 \). This solution is clearly \( z(t) = t - 1 \). Thus, the Green's function for this differential equation is
\[
G(t, \tau) = \begin{cases} 
K(t-\tau) & 0 \leq \tau \leq t \\
Kt(\tau-l) & t \leq \tau \leq l
\end{cases}
\]

where \( K = \frac{1}{p(t)(yz' - z'y')} = 1 \). So

\[
(4.37) \quad G(t, \tau) = \begin{cases} 
(\tau(t-l)) & 0 \leq \tau \leq t \\
t(\tau-l) & t \leq \tau \leq 1
\end{cases}
\]

Thus, the solution of the equation

\[ x'' = u, \quad x(0) = x(1) = 0, \quad 0 \leq t \leq 1 \]

may be represented by

\[ x(t) = \int_0^1 G(t, \tau)u(\tau)d\tau. \]

To calculate the Green's function for the difference equation, we shall use a uniform subdivision of \([0,1]\). Then the difference equation

\[
(4.38) \quad \Delta^- (\Delta^+ x_{ni}) = u_{ni}, \quad i = 1, 2, \ldots, n-1, \quad x_{n0} = x_{nn} = 0
\]

becomes

\[
(4.39) \quad x_{ni+1} - 2x_{ni} + x_{ni-1} = \frac{1}{n^2} u_{ni}.
\]
\( i = 1, 2, \ldots, n-1, x_{no} = x_{nn} = 0. \)

Let \( y_n \) be the solution of \( \Delta^- \Delta^+ y_{ni} = 0 \) satisfying \( y_{no} = 0, \Delta^+ y_{no} = 1 \). This solution is

\[
(4.40) \quad y_{ni} = \frac{i}{n}, \quad i = 0, 1, \ldots, n.
\]

Let \( z_n \) be the solution of \( \Delta^- \Delta^+ z_{ni} = 0 \) satisfying \( z_{nn} = 0, \Delta^- z_{nn} = 1 \). This solution is

\[
(4.41) \quad z_{ni} = \frac{i-n}{n}, \quad i = 0, 1, \ldots, n.
\]

Thus the Green's function has the form

\[
G_n(i,k) = \begin{cases} 
K_n \left( \frac{i-n}{n} \right) \left( \frac{k}{n} \right) & k \leq i \\
K_n \left( \frac{k-n}{n} \right) \left( \frac{i}{n} \right) & k \geq i 
\end{cases}
\]

where \( K_n = \left[ n(y_{ni}z_{ni+1} - z_{ni}y_{ni+1}) \right]^{-1} = 1 \). Therefore

\[
(4.42) \quad G_n(i,k) = \begin{cases} 
\left( \frac{i}{n} - 1 \right) \frac{k}{n} & k \leq i \\
\left( \frac{k}{n} - 1 \right) \frac{i}{n} & i \leq k 
\end{cases}
\]
Therefore, the solution of the difference equation (4.36) may be represented by

\[ x_{ni} = \frac{1}{n} \sum_{k=1}^{n-1} G_n(i,k)u_{nk}. \]

As before, in order to complete the discussion of the convergence of infima by an application of Theorem 2.12, the following lemma must be proved:

**Lemma 4.16:** If \( \Sigma_{in} \) is the uniform subdivision of \([0,1]\), and if \( \tilde{x}_n \) is the solution of (4.34) corresponding to the piecewise constantly extended discrete control \( u_n \), then \( |\tilde{x}_n(1/n) - x_{ni}| \to 0 \) uniformly in \( i \) as \( n \to \infty \).

**Proof:** Define \( G_{ni}(\tau) = G_n(i,k) \) if \( k/n \leq \tau < (k+1)/n \).

Then

\[
|\tilde{x}_n(i/n) - x_{ni}| = \int_0^1 G(i/n, \tau)u_n(\tau)d\tau - \frac{1}{n} \sum_{k=0}^{n-1} G_n(i,k)u_{nk} \mid
\]

\[ = \int_0^1 G(i/n, \tau)u_n(\tau)d\tau \]
\[-\int_0^1 G_{ni}(\tau)u_n(\tau)\,d\tau\]

\[\leq \int_0^1 |G(i/n, \tau) - G_{ni}(\tau)| |u_n(\tau)|\,d\tau\]

\[\leq \|u_n\|_p \left\{ \sum_{k=0}^{n-1} \int_{k/n}^{(k+1)/n} |G(i/n, \tau) - G_{ni}(\tau)|^{q} d\tau \right\}^{1/q} \]

\[= \|u_n\|_p \left\{ \sum_{k=0}^{n-1} \int_{k/n}^{(k+1)/n} |G(i/n, \tau) - G_{ni}(\tau)|^{q} d\tau \right\}^{1/q} \]

\[= \|u_n\|_p \left\{ \sum_{k=0}^{n-1} \int_{k/n}^{(k+1)/n} |G(i/n, \tau) - \frac{i}{n} - \frac{k}{n}|^{q} d\tau \right\}^{1/q} \]

\[= \|u_n\|_p \left\{ \sum_{k=0}^{n-1} \int_{k/n}^{(k+1)/n} \left| \frac{i}{n} - \frac{k}{n} \right|^{q} d\tau \right\}^{1/q} \]

\[+ \sum_{k=i}^{n-1} \int_{k/n}^{(k+1)/n} \frac{i}{n} \left( \tau - 1 \right) \left( \frac{k}{n} - 1 \right) \left| \frac{i}{n} \right|^{q} d\tau \]

\[\leq \|u_n\|_p \left\{ \sum_{k=0}^{n-1} \int_{k/n}^{(k+1)/n} \left| \frac{i}{n} - \frac{k}{n} \right|^{q} d\tau \right\}^{1/q} \]
\[
\|u_n\|_p \left\{ \sum_{k=0}^{i-1} \int_{k/n}^{(k+1)/n} (\tau - \frac{k}{n})^q (1 - \frac{i}{n})^q \, d\tau \right. \\
+ \sum_{k=i}^{n-1} \left. \int_{k/n}^{(k+1)/n} \left( \frac{i}{n} \right)^q (\tau - \frac{k}{n})^q \, d\tau \right\}^{1/q}
\]

\[
= \|u_n\|_p \left\{ \sum_{k=0}^{i-1} \frac{(\tau - \frac{k}{n})^q+1}{q+1} \left( 1 - \frac{i}{n} \right)^q \right|_{k/n}^{(k+1)/n} \right\}^{1/q}
\]

\[
= \|u_n\|_p \left\{ \sum_{k=0}^{i-1} \left( \frac{1}{n} \right)^q+1 \frac{1}{q+1} \left( 1 - \frac{i}{n} \right)^q \right. \\
+ \sum_{k=i}^{n-1} \left. \left( \frac{1}{n} \right)^q+1 \frac{1}{q+1} \left( \frac{i}{n} \right)^q \right\}^{1/q}
\]

\[
= \|u_n\|_p \left\{ \frac{1}{(q+1)n^{q+1}} \left( 1 - \frac{i}{n} \right)^q \cdot i \right. \\
+ \frac{1}{(q+1)n^{q+1}} \cdot \left( \frac{i}{n} \right)^q \cdot (n-i) \right\}^{1/q}
\]
\[ = \|u_n\|_p \left\{ \frac{(n-i)^q + i^q}{(q+1)n^q + 1} \right\}^{1/q} \]

\[ \leq \|u_n\|_p \left\{ \frac{n(n^q) + n^2}{(q+1)n^q + 1} \right\}^{1/q} \]

\[ = \frac{\|u_n\|_p 2^{1/q}}{(q+1)^{1/q}} \cdot \frac{1}{n}. \]

Therefore \( 0 \leq \lim_{n \to \infty} |x_n(i/n) - x_{ni}| \leq \lim_{n \to \infty} \frac{\|u_n\|_p}{(q+1)^{1/q}} \cdot \frac{1}{n} = 0. \)

Note that the above may be done for any \( p, 1 < p < \infty. \)

V. KROTOV'S METHOD FOR INITIAL VALUE DIFFERENCE PROBLEMS

In this chapter we shall present a method of approximation for initial value difference problems. The method given here may possibly be used to approximate the infima of those difference problems which arise in Chapter 3.

§ 5.1 Description of the Method.

Consider an arbitrary subdivision of \([0, 1]\) by the points \( \Sigma_{in}: 0 = t_{no} < t_{n1} < ... < t_{nn} = 1 \) and consider the discrete functional
which is defined on the solutions of the difference equation

\[ \Delta^+ x_{ni} = f(x_{ni}, u_{ni}, t_{ni}), \]

\[ i = 0, 1, \ldots, n-1, \quad x_{no} = A, \] a constant.

We are required to minimize the given functional for \( u_n \) in a certain class \( U_n \). It will be assumed that the functions \( f, g \), and the control set \( U_n \) satisfy conditions (3.5)-(3.10).

An adaptation of the method of V. F. Krotov for approximating the infimum of such functionals will be given. For a complete exposition, see reference [10]. We give here a method designed to suit our own purposes.

Let \( D_n = \{(x_n, u_n): \Delta^+ x_{ni} = f(x_{ni}, u_{ni}, t_{ni}), x_{no} = A, \quad u_n \in U_n \} \) and let \( D_n^+ = \{(x_n, u_n); x_{no} = A, u_n \in U_n \} \).

Clearly \( D_n^+ \supset D_n \). Let \( \theta_n \) be an arbitrary discrete function and note that
\[
\sum_{i=0}^{n-1} (\theta_{ni+1}x_{ni+1} - \theta_{ni}x_{ni}) = \theta_{nn}x_{nn} - \theta_{no}x_{no}
\]

(5.3) 

\[
= \sum_{i=0}^{n-1} \{\theta_{ni+1}(x_{ni+1} - x_{ni}) + x_{ni}(\theta_{ni+1} - \theta_{ni})\}.
\]

Therefore, the functional \( J^n \) defined by

\[
J^n = \sum_{i=0}^{n-1} \{g(x_{ni}^+, u_{ni}^+, t_{ni}) \tau_{ni} + \theta_{ni+1} f(x_{ni}, u_{ni}, t_{ni}) \tau_{ni} + x_{ni}(\theta_{ni+1} - \theta_{ni})\} - (x_{nn} \theta_{nn} - x_{no} \theta_{no})
\]

is such that \( J^n|_{D^n} = J^n \). Also, since \( D^n \supset D_n \)

\[
i^n = \inf_{D^n} J^n \leq \inf_{D_n} \leq i_n.
\]

The objective here of course is to make a good choice of the functions \( \theta_n \) so that \( i_n \) and \( i^n \) are close.

Define the function
\[ R_n(x_{ni}, u_{ni}, t_{ni}, \theta_{ni}) = g(x_{ni}, u_{ni}, t_{ni}) \tau_{ni} \]

\[ + \theta_{ni+1} f(x_{ni}, u_{ni}, t_{ni}) \tau_{ni} \]

\[ + x_{ni} (\theta_{ni+1} - \theta_{ni}) \]

and

\[ P_n(x_{ni}, t_{ni}, \theta_{ni}) = \inf_{u_n \in U_n} R_n(x_{ni}, u_{ni}, t_{ni}, \theta_{ni}) \]

It is suggested in [10] that a good method for determining the functions \( \theta_n \) is the following:

Choose a network of graphs in phase space distributed more less uniformly, say \( \{x^n_{\beta}; \beta \in B\} \). It is shown in [10] that the problem of finding a minimum for the continuous analogue of this problem could be completely solved if the function \( \theta_n \) could be chosen so that the function \( P_n(x_{ni}, t_{ni}, \theta_{ni}) \) is independent of \( x_n \). With this in mind, we will determine the function \( \theta_n \) so that \( P_n \) is approximately independent of \( x_n \) by choosing, for \( \beta_o \) fixed,

\[ P_n(x_{ni}, t_{ni}, \theta_{ni}) = P_n(x^\beta_{ni}, t_{ni}, \theta_{ni}); \beta \in B. \]
This will determine, upon solving the equations, a particular function $\overline{\theta}_n$, which we may substitute into the function $P_n$. Then, the lower bound $i^+_n$ will be given by

$$i^+_n = \inf_{x_n} \sum_{i=0}^{n-1} P_n(x_n, t_n, \overline{\theta}_n).$$

The information above may be used to find an upper bound for the sought for infimum $i_n$. After finding $\overline{\theta}_n$, and by using the relation obtained when finding $\inf R_n$, we may determine $\overline{u}_n = \overline{u}_n(\overline{\theta}_n)$, and then through the use of (5.2), find $\overline{x}_n = \overline{x}_n(\overline{u}_n)$. Then $(\overline{x}_n, \overline{u}_n) \in D_n$ and so $J_n(\overline{x}_n, \overline{u}_n) \geq \inf_{D_n} J_n(\overline{x}_n, \overline{u}_n)$, i.e. an upper bound for the infimum $i_n$. We may then determine the closeness of our approximations to the actual infimum by considering the difference $J_n(\overline{x}_n, \overline{u}_n) - i^+_n$.

Remark: If better approximations to the infima are desired, one might try a sum of the following form in place of the sum in (5.3):

$$\sum_{i=0}^{n-1} (\theta_{ni+1} x_{ni+1} - \theta_{ni} x_{ni}) = \theta_{nn} x_{nn}^2 - \theta_{no} x_{no}^2$$

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\[
J = \sum_{i=0}^{n-1} \left\{ \theta_{ni+1} (x_{ni+1} + x_1) (x_{ni+1} - x_{ni}) + x_{ni}^2 (\theta_{ni+1} - \theta_{ni}) \right\}.
\]

To put this idea another way, one might extend the functional \( J_n \) to the functional \( J_n^+ \) by means of the function \( \theta_n x_n^2 \) rather than using \( \theta_n x_n \). In general one might extend using the function

\[
\theta_n^{(1)} x_n + \theta_n^{(2)} x_n^2 + \theta_n^{(3)} x_n^3 + \ldots + \theta_n^{(k)} x_n^k,
\]

where the functions \( \theta_n^{(1)}, \ldots, \theta_n^{(k)} \) are to be unknown functions determined through the described procedure.

§ 5.2 Example 1. Problem Without Control

We are required to minimize the functional

\[
(5.7) \quad J = \int_0^1 (u^2 + tx^2) dt
\]

which is defined on the solutions of the initial value problem

\[
(5.8) \quad x' = u, \quad x(0) = 1.
\]

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Consider a uniform subdivision of $[0,1]$ and the approximating functional

\begin{equation}
J_n = \sum_{i=0}^{n-1} \left( u_{ni}^2 + \frac{1}{n} x_{ni}^2 \right) \frac{i}{n}
\end{equation}

which is defined on the solutions of the difference equation

\begin{equation}
x_{ni+1} = x_{ni} + \frac{1}{n} u_{ni}, \quad x_{no} = 1, \quad i = 0, 1, \ldots, n-1.
\end{equation}

Now,

\begin{equation}
J_n^+ = \sum_{i=0}^{n-1} \left( \frac{i}{n} u_{ni}^2 + \frac{1}{n} x_{ni}^2 + \frac{1}{n} \theta_{ni+1} u_{ni} \right)
+ x_{ni} (\theta_{ni+1} - \theta_{ni}) - (\theta_{nn} x_{nn} - \theta_{no} x_{no})
\end{equation}

In addition to the restrictions indicated above, we shall choose $\theta_n$ so that $\theta_{nn} = 0$. With this, and with $x_{no} = 1$, the last term in (5.11) becomes simply $\theta_{no}$.
When calculating $\inf \mathcal{R}$, we find

\begin{equation}
(5.12) \quad u_{ni} = -\frac{1}{2} \theta_{ni+1} \quad i = 0, 1, \ldots, n-1.
\end{equation}

Substituting this into (5.10), we get

\begin{equation}
(5.13) \quad j^+ \geq \sum_{i=0}^{n-1} (-\frac{1}{4n} \theta_{ni+1}^2 + \frac{i}{n} x_{ni}^2 + x_{ni} (\theta_{ni+1} - \theta_{ni})) + \theta_{no}.
\end{equation}

Therefore, $P = \left( -\frac{1}{4n} \theta_{ni+1}^2 + \frac{i}{n} x_{ni}^2 + x_{ni} (\theta_{ni+1} - \theta_{ni}) \right)$. 

To make $P$ approximately independent of $x_n$, we shall choose

\begin{equation}
P(x_n = 1) = P(x_n = 0).
\end{equation}

This gives the following difference equation for $\theta_n$:

\begin{equation}
(5.14) \quad -\frac{i}{n^2} = \theta_{ni+1} - \theta_{ni}, \quad \theta_{nn} = 0.
\end{equation}

This difference equation has solution

\begin{equation}
(5.15) \quad \theta_{ni} = \frac{n(n-1)-i(i-1)}{2n^2}, \quad i = 0, 1, 2, \ldots, n.
\end{equation}
We now minimize (5.13) with respect to $x_n$ and obtain
\begin{equation}
(5.16) \quad J_n^+ \geq \sum_{i=0}^{n-1} \left( - \frac{1}{4n^2} \theta_{ni+1}^2 - \frac{n^2}{4i} (\theta_{ni+1} - \theta_{ni})^2 \right) + \theta_{n0}.
\end{equation}

If we substitute (5.14) and (5.15) into (5.16), we obtain an expression for a lower bound for the functional (5.9).
\begin{equation}
(5.17) \quad i_n^+ = \sum_{i=0}^{n-1} \left( - \frac{1}{4n} \left[ \frac{n(n-1)-i(i+1)}{2n^2} \right]^2 - \frac{i}{4n} \right) + \frac{n-1}{2n}.
\end{equation}

To obtain an expression for an upper bound, (5.12) and (5.15) can be used to give
\begin{equation}
(5.18) \quad \overline{u}_{ni} = \frac{i(i+1)-n(n-1)}{-4n^2} \quad i = 0, 1, ..., n.
\end{equation}

Then, using the difference equation (5.10), $x_{ni}$ may be found to be
\begin{equation}
(5.19) \quad x_{ni} = 1 + \frac{i(i^2-1)}{12n^3} - \frac{i(n-1)}{4n^2}, \quad i = 0, 1, ..., n.
\end{equation}

If we substitute (5.18) and (5.19) into (5.9), we get

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\begin{equation}
J(x_n, u_n) = \frac{1}{n} \sum_{i=0}^{n-1} \left[ \frac{i(i+1) - n(n-1)}{4n^2} \right]^2 + \frac{1}{n^2} \sum_{i=0}^{n-1} i\left(1 + \frac{i^2 - 1}{12n^3} - \frac{i(n-1)}{4n^2}\right)^2.
\end{equation}

This is an expression for an upper bound for the infimum in, since $(x_n, u_n) \in D$.

Conclusion: It is apparent that the functions $g(x, u, t) = u^2 + tx^2$ and $f(x, u, t) = u$ satisfy conditions (3.7)–(3.10) as set forth in section (3.2) of Chapter 3. Consequently, we must have

$$\lim_{n \to \infty} \inf J_n = \inf J.$$  

However, for each $n$, $i_n^+$ and $J_n(x_n, u_n)$ give upper and lower bounds for $\inf J_n$. With some work, it may be shown that

$$\lim_{n \to \infty} i_n^+ = 0.3416$$

and

$$\lim_{n \to \infty} J_n(x_n, u_n) = 0.43038.$$
Therefore, these must be upper and lower bounds for the infimum of the functional $J$ given by (5.7).

§ 5.3 Example 2. Problem with Control.

We are required to minimize the integral

\[ J(x,u) = \int_0^1 (4x^2 + u^2) \, dt \]

which is defined on the solutions of the differential equation

\[ x' = u, \quad x(1) = 1, \quad 0 \leq t \leq 1. \]

We shall assume that the control variable $u$ is to lie in the set

\[ \{ u: \ |u(t)| \leq 1, \ 0 \leq t \leq 1 \}. \]

Again consider a uniform subdivision of $[0,1]$ and the approximating sum

\[ J_n(x_n,u_n) = \sum_{i=0}^{n-1} (4x_{ni}^2 + u_{ni}^2) \frac{1}{n} \]

which is defined on the solutions of the corresponding difference equation.
(5.25) \[ x_{ni+1} = x_{ni} + \frac{1}{n} u_{ni}, \quad x_{nn} = 1, \quad i = 0, 1, 2, \ldots, n-1. \]

Here assume that the control functions \( u_n \) satisfy the restriction

\[ u_n \in \{ u_n : |u_{ni}| \leq 1, \quad i = 0, 1, \ldots, n \}. \]

The function \( J^+ \) is defined by

\[
J^+_n = \sum_{i=0}^{n-1} \left[ \frac{4}{n} x_{ni}^2 + \frac{1}{n} u_{ni}^2 + \frac{1}{n} \theta_{ni+1} u_{ni} \right. \\
\left. + x_{ni} (\theta_{ni+1} - \theta_{ni}) \right] \\
- (\theta_{nn} x_{nn} - \theta_{no} x_{no}).
\]

In addition, choose \( \theta_n \) so that \( \theta_{no} = 0 \), so that the last term on the right hand side of (5.26) becomes simply \(-\theta_{nn}\). We would now like to calculate the function

\[ p_n = \inf_{u_n} P_n. \]
We do this by ordinary calculus methods, and obtain the following relationships:

(5.27)

(a) If \( \left| -\frac{1}{2} \theta_{n_i+1} \right| \leq 1 \), then \( u_{n_i} = -\frac{1}{2} \theta_{n_i+1} \).

(b) If \( -\frac{1}{2} \theta_{n_i+1} > 1 \), then \( u_{n_i} = 1 \) and

(c) if \( -\frac{1}{2} \theta_{n_i+1} < -1 \), then \( u_{n_i} = -1 \).

If (5.27) (a) holds, then the function \( P_n \) is

(5.28)

(a) \( P_{in} = \frac{4}{n} x_{ni}^2 + x_{ni} (\theta_{ni+1} - \theta_{ni}) - \frac{1}{4n} \theta_{ni+1}^2 \).

In case (5.27) (b),

(5.28)

(b) \( P_{in} = \frac{4}{n} x_{ni}^2 + x_{ni} (\theta_{ni+1} - \theta_{ni}) + \left( \frac{1}{n} + \frac{1}{n} \theta_{ni+1} \right) \).

In case (5.27) (c),

(5.28)

(c) \( P_{in} = \frac{4}{n} x_{ni}^2 + x_{ni} (\theta_{ni+1} - \theta_{ni}) + \left( \frac{1}{n} - \frac{1}{n} \theta_{ni+1} \right) \).
Thus we shall write

\[ J_n^+ \cong \sum_{i=0}^{n-1} (p_{in}) - \theta_{nn} \]  

with the three possible choices of \( p_{in} \) in mind.

To choose \( p_n \) approximately independent of \( x_n \), we shall choose \( p_n(x_n = 0) = p_n(x_n = 1) \). This gives the following difference equation for \( \theta_n \):

\[ \theta_{ni+1} - \theta_{ni} = -\frac{4}{n}, \quad \theta_{n0} = 0, \quad i = 0, 1, \ldots, n-1. \]

This difference equation has solution

\[ \theta_{ni} = -\frac{4i}{n} \]

which gives

\[ \theta_{nn} = -4. \]

According to equations (5.27), we must determine for which values of \( i \) we have \( |-\frac{1}{2}\theta_{ni+1}| \leq 1 \).

\[ |-\frac{1}{2}\theta_{n+1}| = \left| \frac{2(i+1)}{n} \right| \] is less than or equal to one if \( i \) satisfies \(-\frac{n}{2} - 1 \leq i \leq \frac{n}{2} - 1 \). Since the
side of the inequality is always negative, \( \left| -\frac{1}{2} n_i + 1 \right| \leq 1 \) when \( 0 < i < \frac{n}{2} - 1 \). Thus we must use the function \( P_{ni} \) given by (5.28)(a) for \( 0 < i < \frac{n}{2} - 1 \), and we must use the function \( P_{ni} \) determined by (5.28)(b) for \( \frac{n}{2} - 1 < i < n-1 \). For notational convenience, let \( j = \lceil \frac{n}{2} - 1 \rfloor \), where the square brackets indicate the greatest integer function. Then, from (5.29),

\[
(5.33) \quad J_n^+ \geq \sum_{i=0}^{j} \left( \frac{4}{n} x_i^2 + x_i (\theta_{ni+1} - \theta_{ni}) - \frac{1}{4 n} \theta_{ni+1}^2 \right) 
+ \sum_{i=j+1}^{n-1} \left( \frac{4}{n} x_i^2 + x_i (\theta_{ni+1} - \theta_{ni}) \right) 
+ \frac{1}{n} \frac{1}{n} x_{ni+l} \right) - \theta_{nn} 

= \sum_{i=0}^{n-1} \frac{4}{n} x_i^2 + x_i (\theta_{ni+1} - \theta_{ni}) + \sum_{i=0}^{1} \frac{1}{4 n} \theta_{ni+1}^2 
+ \sum_{i=j+1}^{n} \left( \frac{1}{n} + \frac{1}{n} \theta_{ni+l} \right) - \theta_{nn}
\]
Now, again using ordinary calculus, we shall minimize the above expression with respect to the variable $x_n$, and we obtain

$$x_n = -\frac{n}{8}(\theta_{ni+1} - \theta_{ni}).$$

If we substitute this into (5.33), we get

$$J_n^+ \geq \sum_{i=0}^{n-1} -\frac{n}{16}(\theta_{ni+1} - \theta_{ni})^2 + \sum_{i=0}^{j} -\frac{1}{4n}\theta^2_{ni+1}$$

$$+ \sum_{i=j+1}^{n-1} \left(\frac{1}{n} + \frac{1}{n}\theta_{ni+1}\right) + 4. \tag{5.34}$$

The substitution of (5.31) into (5.34) gives the following expression for $i_n^+$:

$$i_n^+ = 3 - \frac{2}{3n^2}(j+1)(j+2)(2j+3) + \frac{1}{n}(n-j-1)$$

$$- \frac{2}{n^4}[n(n+1) - (j+1)(j+2)].$$

To obtain an upper bound, we shall again use the results of the equations (5.27). We must calculate the following:
(5.36) For $0 \leq i \leq j$, \( x_{ni+1} - x_{ni} = \frac{1}{n^2} \frac{2(i+1)}{n} \), and for \( j < i \leq n - 1 \), \( x_{ni+1} - x_{ni} = \frac{1}{n} \cdot 1 \), \( x_{nn} = 1 \).

The solution of this difference equation is:

\[
\begin{align*}
x_{ni} &= \frac{i(i+1)+(n-2-j)(j+1)}{n^2} \quad 0 \leq i \leq j \\
x_{ni} &= \frac{i}{n} \quad j + 1 \leq i \leq n - 1.
\end{align*}
\]

(5.37) Using the particular expressions for \( \bar{u}_n \) obtained by substituting (5.31) into (5.27), and using the expressions of (5.37) for our particular \( \bar{x}_n \), we obtain the following expression for an upper bound:

\[
J_n(\bar{x}_n, \bar{u}_n) = \sum_{i=0}^{n-1} \frac{4}{n} x_{ni}^2 + \frac{1}{n} u_{ni}^2
\]

\[
= \sum_{i=0}^{j} \left\{ \frac{4}{n} \left( \frac{i(i+1)+(n-2-j)(j+1)}{n^2} \right)^2 + \frac{1}{n} \left( \frac{2(i+1)}{n} \right)^2 \right\} + \sum_{i=j+1}^{n-1} \left\{ \frac{4}{n} \left( \frac{i}{n} \right)^2 + \frac{1}{n} \right\}.
\]
Conclusion: As in the previous example, it can be shown with some work that

$$\lim_{n \to \infty} i_n^+ = 1.833$$

and that

$$\lim_{n \to \infty} J_n(x_n, u_n) = 2.233.$$  

We conclude that these numbers are in fact upper and lower bounds for the infimum of the functional $J$ given by equation (5.21).

VI. KROTOV'S METHOD FOR BOUNDARY VALUE DIFFERENCE PROBLEMS

In this chapter we shall present an example to show how the method of Krorov may be modified to give upper and lower bounds for the infimum of a discrete functional defined on the solutions of a second order difference equation. We note that the method may be used to approximate the infima of those problems which arise in Chapter 4.

Another problem is considered in this chapter. It is to minimize a functional defined on the solutions of a first order boundary value problem. This example is
included to show the diversity of Krorov's method.

§ 6.1 Second Order Problem

Suppose we are required to minimize the functional

$$J(x, u) = \int_0^1 \left( r_1 x + g x^2 \right) dt$$

defined on the solutions of the differential equation

$$\frac{d^2 x}{dt^2} = u, \quad x(0) = x(1) = 0, \quad 0 \leq t \leq 1$$

where the control variable \( u \) is to belong to the set

$$U = \{ u : |u(t)| \leq 1, \quad 0 \leq t \leq 1 \}.$$

Consider a uniform subdivision of \([0,1]\) and the approximating functional

$$J_n(x_n, u_n) = \sum_{i=1}^{n-1} (r x_{ni} + \beta x_{ni}^2) \tau_{ni}, \quad r, \beta \text{ constants}$$

which is defined on the solutions of the difference equation

$$\Delta^-(\Delta^+ x_{ni}) = u_{ni}, \quad x_{nn} = x_{no} = 0, \quad i = 1, 2, \ldots, n-1.$$
\[ U_n = \{ u_n : |u_{ni}| \leq 1, i = 1, 2, \ldots, n-1 \} \]

Consider the sum

\[
(6.6) \sum_{i=1}^{n-1} \Delta^{-}(x_{ni} \Delta^{+} \theta_{ni}) = x_{n-1} \Delta^{+} \theta_{n-1} - x_{n} \Delta^{+} \theta_{n} - \ldots.
\]

Also,

\[
(6.7) \sum_{i=1}^{n-1} \Delta^{-}(x_{ni} \Delta^{+} \theta_{ni})
\]

\[
= \sum_{i=1}^{n-1} (x_{ni} \Delta^{-} \Delta^{+} \theta_{ni} + \Delta^{+} \theta_{ni-1} \Delta^{-} x_{ni}).
\]

Consider now the sum

\[
(6.8) \sum_{i=1}^{n-1} \Delta^{-}(\theta_{ni} \Delta^{+} x_{ni}) = \theta_{n-1} \Delta^{+} x_{n-1} - \theta_{n} \Delta^{+} x_{n} - \theta_{n} \Delta^{+} x_{n}.
\]

Also,

\[
(6.9) \sum_{i=1}^{n-1} \Delta^{-}(\theta_{ni} \Delta^{+} x_{ni})
\]

\[
= \sum_{i=1}^{n-1} (\theta_{ni} \Delta^{-} \Delta^{+} x_{ni} + \Delta^{+} x_{ni-1} \Delta^{-} \theta_{ni}).
\]
If we subtract (6.7) from (6.9), we get

\[ \sum_{i=1}^{n-1} (\theta_{ni} - x_{ni} + x_{ni} - \theta_{ni}) = n(-x_{n1} \theta_{no} - x_{nn-1} \theta_{nn}) , \]

recalling that \( x_{nn} = x_{no} = 0 \). We shall restrict consideration to those \( \theta \)'s which satisfy \( \theta_{nn} = \theta_{no} = 0 \). We may now define the extension of \( J \) to be

\[ J_n^+ = \sum_{i=1}^{n-1} [(x_{ni} + \beta x_{ni}^2) \frac{1}{n} + \theta_{ni} u_{ni} - x_{ni} - \theta_{ni}]. \]

It is clear from line (6.10) that \( J_n^+ = J_n \) if the difference equation (6.5) is satisfied.

Now, define the function \( R_n = R_n(x_n, u_n, t_n, \theta_n) \) to be the summand of (6.11). We would like to calculate \( \inf R_n \), and in so doing, we find

\[ u_{ni} = -\text{sign} (\theta_{ni}) \]

must hold for \( i = 1, 2, \ldots, n-1 \). Thus the function \( P = P(x_n, t_n, \theta_n) \) becomes

\[ P_{ni} = \frac{1}{n} (x_{ni} + \beta x_{ni}^2) + \theta_{ni} (-\text{sign} \theta_{ni}) - x_{ni} - \theta_{ni} \].
To make $P_n$ approximately independent of $x_n$, we shall choose $P_n(x_n = 1) = P_n(x_n = 0)$. This gives the following difference equation for $\theta_n$:

$$\Delta^{-\Delta^+} \theta_{ni} = \frac{1}{n}(r+\beta), \quad i = 1, 2, \ldots, n-1, \quad \theta_{no} = \theta_{nn} = 0.$$  

This difference equation has solution

$$\theta_{ni} = \frac{(r+\beta)i(i-n)}{2n^3} \quad i = 0, 1, 2, \ldots, n.$$

We now minimize (6.11) with respect to $x_n$, and take into account (6.12) to obtain

$$J_n^+ \geq \sum_{i=1}^{n-1} \left\{ \frac{\Delta^{-\Delta^+} \theta_{ni} - \frac{r}{n}}{4n^3} - \theta_{ni} \text{sign } \theta_{ni} \right\}.$$

With $\theta_{ni} = \frac{(r+\beta)i(i-n)}{2n^3}$, and with $\Delta^{-\Delta^+} \theta_{ni} = \frac{r+\beta}{n}$, and also assuming that $(r+\beta) > 0$ so that $\text{sign } \theta_{ni} = -1$, we obtain the following expression for a lower bound for the functional (6.4):

$$i_n^+ = \sum_{i=1}^{n-1} \left[ -\frac{\beta}{4n^3} + \frac{1}{2n^3}(r+\beta)i(i-n) \right].$$
If the constant $r + \beta < 0$, then the expression for $\iota_n^+$ is the same except for a negative sign in the second term of the summand.

To obtain an upper bound for the functional (6.4), recall that $u_n$ must satisfy $u_{ni} = -\text{sign } \theta_{ni}$, $i = 1, \ldots, n-1$.

Case 1: Assume $r + \beta > 0$. Then for $i = 1, \ldots, n-1$, $\theta_{ni}$ is negative and hence $u_{ni} = 1$. From (6.5), we obtain

\begin{equation}
\Delta - \Delta^+ x_{ni} = 1, \quad x_{no} = x_{nn} = 0.
\end{equation}

This difference equation has solution

\begin{equation}
x_{ni} = \frac{i(i-n)}{2n^2} \quad i = 0, 1, 2, \ldots, n.
\end{equation}

The substitution of (6.18) into (6.4) yields the following expression for an upper bound for (6.4)

\begin{equation}
J_n(x_n, u_n) = \frac{1}{n} \sum_{i=1}^{n-1} \left( \frac{i(i-n)}{2n^2} \right) + \frac{2}{n} \sum_{i=1}^{n-1} \left( \frac{i(i-n)}{2n^2} \right)^2.
\end{equation}
Case 2: Assume $r + \beta < 0$. Then $\bar{u}_{ni} = -1$, and

$$\bar{x}_{ni} = \frac{i(n-i)}{2n^2}.$$ Thus the expression for the upper bound is the same except for a change of sign inside each summand. With some work, it may be shown that

$$\lim_{n \to \infty} \frac{i^+}{n} = \frac{-r - 4\beta}{12}$$

and that

$$\lim_{n \to \infty} J_n(\bar{x}_n, \bar{u}_n) = \frac{-10r + \beta}{120}.$$ Both of these calculations were made with the assumption that $r + \beta > 0$, and that $\beta > 0$. Similar calculations can be made in case $r + \beta < 0$, $\beta > 0$. The condition $\beta > 0$ is necessary in order to insure that (6.11) has a minimum with respect to $x_n$.

Remark: Note that the approximations are best for small values of $\beta$. When $\beta = 0$, we have actual equality.

Summary: It is apparent that the functional $J$ and the approximating functionals $J_n$ satisfy the conditions as set forth in Chapter 4 of this paper. Consequently, we must have
\[ \lim \inf_{n \to \infty} J_n = \inf J. \]

However, for each \( n \), (6.16) and (6.19) represent upper and lower bounds for the functional \( J_n \). Therefore, we conclude that the expressions (6.20) and (6.21) represent lower and upper bounds respectively for the infimum of the functional \( J \) given by (6.1).

§ 6.2 First Order Problem

We would like to minimize the sum

\[ J_n(x_n, u_n) = \sum_{i=0}^{n-1} (u_{ni}^2 + \frac{i}{n} x_{ni}^2) \frac{1}{n} \]

where \( J_n \) is defined on the solutions of

\[ \Delta^+ x_{ni} = u_{ni}, \quad i = 0, 1, \ldots, n-1 \]

which satisfy \( x_{n0} = 0, \ x_{nn} = 1 \). Here \( x_n \) and \( u_n \) are taken to be discrete functions defined on the points of a uniform subdivision of \([0,1]\) into \( n \) parts; \( x_{ni} \) is the value of \( x_n \) at the subdivision point \( \frac{i}{n} \).

Extend the functional \( J_n \) to \( J_n^+ \) by means of the function \( \theta_n x_n \), as in Chapter 5. We then get
\[
J_n^+ = \sum_{i=0}^{n-1} \left\{ \left( u_{ni}^2 + \frac{i}{n} x_{ni}^2 \right) \frac{1}{n} \right\} + \frac{1}{n} u_{ni} \theta_{ni+1} + x_{ni} (\theta_{ni+1} - \theta_{ni}) \} - \theta_{nn}.
\]

(6.24)

Define the function \( R_n = R_n(x_n, u_n, t_n, \theta_n) \) to be the summand of the expression in (6.24).

When calculating \( \inf R_n \), we find that

\[
u_n = -\frac{1}{2} \theta_{ni+1} \quad i = 0, 1, ..., n-1.
\]

(6.25)

If we substitute this into (6.24), we get

\[
J_n^+ \geq \sum_{i=0}^{n-1} \left\{ -\frac{1}{4n} \theta_{ni+1}^2 + \frac{i}{2} x_{ni}^2 + x_{ni} (\theta_{ni+1} - \theta_{ni}) \right\} - \theta_{nn}.
\]

(6.26)

We define the function \( P_n(x_n, t_n, \theta_n) \) by

\[
P_{ni} = \left\{ -\frac{1}{4n} \theta_{ni+1}^2 + \frac{i}{2} x_{ni}^2 + x_{ni} (\theta_{ni+1} - \theta_{ni}) \right\}, \quad i = 1, 2, ..., n-1.
\]

To make \( P_n \) approximately independent of \( x_n \), we shall choose \( P_n(x_n = 0) = P_n(x_n = 1) \). This gives the following difference equation for \( \theta_n \):
This difference equation has solution

\begin{equation}
\theta_{ni+1} - \theta_{ni} = -\frac{i}{n^2} \quad i = 0, 1, \ldots, n-1.
\end{equation}

where \( \theta_{no} \) is a constant to be determined. In order to determine this constant, we shall use (6.23) and (6.25). The combination of these gives

\begin{equation}
x_{ni+1} - x_{ni} = -\frac{1}{2n} \theta_{ni+1}.
\end{equation}

Therefore, \( x_{ni+1} - x_{ni} = -\frac{1}{2n} \left( \theta_{no} - \frac{i(i+1)}{2n^2} \right) \).

If we sum both sides and take into account the boundary conditions on \( x_n \), we obtain

\begin{equation}
1 = -\frac{1}{2n} \sum_{i=0}^{n-1} \left( \theta_{no} - \frac{i(i+1)}{2n^2} \right).
\end{equation}

Therefore,

\[ \theta_{no} = -\frac{1}{n} \left( -2n + \sum_{i=0}^{n-1} \frac{i(i+1)}{2n^2} \right). \]

This may be simplified to give
\[(6.31) \quad \Theta_{n_0} = \frac{-ln^2-1}{6n^2}.\]

Therefore, from (6.28),

\[(6.32) \quad \Theta_{ni} = -\frac{ln^2+1}{6n^2} - \frac{i(i-1)}{2n^2} \quad i = 0, 1, 2, \ldots, n.\]

We shall now minimize the right hand side of (6.26) with respect to \(x_n\), and obtain

\[(6.33) \quad J_n^+ \geq \sum_{i=0}^{n-1} \left( -\frac{1}{4n} \Theta_{ni+1}^2 - \frac{n^2}{4i} (\Theta_{ni+1} - \Theta_{ni})^2 \right) - \Theta_{nn}.\]

The substitution of (6.32) into the above yields the following expression for a lower bound for the given functional:

\[(6.34) \quad j_n^+ = -\frac{1}{4n} \sum_{i=0}^{n-1} \left( -\frac{i(i+1)}{2n^2} - \frac{ln^2+1}{6n^2} \right)^2\]

\[+ \frac{3(n-1)}{8n} + \frac{ln^2+1}{6n^2}.\]

In order to obtain an upper bound for the given functional, we shall use (6.23), (6.25), and (6.32)
The combination of these gives the following solution for \( x_n \):

\[
(6.35) \quad x_{ni} = \frac{i(i^2-1)}{12n^3} - \frac{i}{2n} \theta_{no} \quad i = 0, 1, 2, \ldots, n.
\]

This may be simplified to give

\[
(6.36) \quad x_{ni} = \frac{i(i^2+11n^2)}{12n^3} \quad i = 0, 1, 2, \ldots, n.
\]

From (6.25) and (6.28), we have the following expression for \( u_n \):

\[
(6.37) \quad u_{ni} = -\frac{1}{2n} \left( -\frac{i(i+1)}{2n^2} + \theta_{no} \right).
\]

We now substitute (6.35) and (6.37) directly into (6.22) to obtain the following expression for an upper bound for the functional \( J_n \)

\[
(6.38) \quad J_n (x_n, u_n) = \frac{1}{4n} \sum_{i=0}^{n-1} \left( -\frac{i(i+1)}{2n^2} + \theta_{no} \right)^2
\]

\[
+ \frac{1}{144n^8} \sum_{i=0}^{n-1} i^3 (i^2 + 11n^2)^2
\]

where \( \theta_{no} = -\frac{11n^2-1}{6n^2} \).
We would now like to give some numerical data for the given problem.

For $n = 2$:

The actual minimum by direct calculation is found to be $1.05882...$, compared with an upper bound from (6.38) of $1.05884...$ and a lower bound from (6.34) of $1.05858...$.

For $n = 3$:

The actual minimum by direct calculation is $1.10517223...$ compared with an upper bound of $1.1052177...$ and a lower bound of $1.09705...$.

The formulas (6.38) and (6.34) were programmed on a model PDP 10 data processing system in FORTRAN IV using the following program to give upper and lower bounds for some selected values of $n$. The program may easily be adapted to give more precise results, and also to give upper and lower bounds for other values of $n$ than those listed.
DOUBLE PRECISION SUM1,SUM2,XU,XL,XI,XK
DIMENSION NV(31)
DATA NV/2,3,4,5,6,7,8,9,10,11,12,13,14,
     15,16,17,18,19,20,21,22,23,24,25,50,100,
     2150,200,1000,5000,10000/
DO 1 I = 1,31
XN = NV(I)
XK = (-11.*XN*XN-1.)/(6.*XN*XN)
SUM 1 = 0
SUM 2 = 0
DO 2 J = 1,N
XI = J - 1
SUM1 = SUM1 + (-XI*(XI+1.)/(2.*XN*XN)+XK)**2
SUM2 = SUM2 + (XI*XI*XI)*(XI*XI+11.*XN*XN)**2
XU = SUM1/(4.*XN)+SUM2/(144.*XN**8)
XL = -SUM1/(4.*XN)+3.*(XN-1.)/(8.*XN)-XK
WRITE (5,3) N,XU,XL
END

The program was executed for the values of \( n \) indicated in the first column. Respective upper and lower bounds are contained in the second and third columns of this print-out.
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