Convergence of Bounds in Optimization

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CONVERGENCE OF BOUNDS
IN OPTIMIZATION

by

Pascal D. Mubenga

A Dissertation
Submitted to the
Faculty of The Graduate College
in partial fulfillment
of the
Degree of Doctor of Philosophy

Western Michigan University
Kalamazoo, Michigan
April, 1972

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Pascal D. Mubenga
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I. INTRODUCTION

In this paper we consider the approximation of the extremal value of a real-valued function within the framework of Moore-Smith convergence. Specifically: given a real-valued function $I$ defined on a subset $D$ of an $L$-space $X$, together with a net of real-valued functions $I_b$ each defined on a subset $D_b$ of the space $X$, we seek conditions under which the infimum $i_b$ (respectively, supremum $i^*_b$) of $I_b$ over $D_b$ will converge to the infimum $i$ (resp., supremum $i^*$) of $I$ over $D$. The approximation of an optimal point of $I$ by means of the optimal points of $I_b$ is also taken into account, but only in a secondary way.

Perhaps the best known method of obtaining lower bounds for the number $i$ consists in extending the functional $I$ over a wider domain, under the practical provision that the infimum of the extension over this larger domain is relatively easier to compute. This basic principle has been used in the classical calculus of variations, and it has been developed extensively to obtain lower bounds for eigenvalues by A. Weinstein [26]. Books by S. H. Gould [15], and more recently by G. Fichera [12], give extensive references.
We point out that our general treatment does not require the functional $I_b$ to be an extension of the functional $I$. Furthermore, the assumed structure of the space $X$ is in general far weaker than that of the closure spaces studied by E. Čech [7, pp. 237-249], and therefore far weaker than a topology on $X$, thus allowing for still more generality.

We begin in Chapter II with a study of nets of extended real numbers. For such nets we define the notions of "limit superior" and "limit inferior" in a manner that actually generalizes similar notions for numerical sequences. The behavior of these limits is developed and then used to characterize lower and upper semi-continuous functions on a topological space. This characterization gives us a basis for defining lower and upper semi-continuous functions on a space weaker than a topological space, namely an $L$-space (see Definition 2.13). As defined here, our $L$-spaces are more general than "les espaces du type $L$" of J. Kisynski [18] and our $L^*$-spaces are weaker than those defined by F. Chimenti [8]. The basic results and terminology on Moore-Smith convergence assumed throughout can be found in J. L. Kelley [17] and in S. Mrowka [21].
Chapter III presents, mainly, sufficient conditions for the approximation of \( i \) (respectively, \( i^* \)) in terms of the net \( i_b \) (respectively, \( i_b^* \)). The functionals under consideration are defined on subsets of an L-space, and we rely heavily on the material of the preceding Chapter II. The main results of the chapter, in a general setting, are embodied in Theorem 3.4 and Theorem 3.8.

A restriction is made in Chapter IV to the case where \( X \) is a topological space and the bounds \( i_b \) form a sequence. We embark here upon a close examination of one of the assumptions made in the sufficient conditions of Chapter III. Among other things, we establish that the assumption (i.e., property \( (D+) \)) is implied in an appropriate setting by any one of the well-known notions of convergence of subsets, namely, Hausdorff convergence, topological convergence, Vietoris convergence.

Another assumption made in the sufficient conditions of Chapter III is examined individually in Chapter V. A possible method for improving upon given bounds of \( i \) (respectively, \( i^* \)) is indicated in Theorem 5.2 and Theorem 5.3. Under certain conditions we are able to make some comparison with a part in the work of B. M. Budak and E. M. Berkovich [5].

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The remaining chapters are devoted to applications. We first look at a natural extension of the general problem of linear programming (Chapter VI) and show how such a problem can be treated with the theory of Chapter III. In Chapter VII we use the theory of Chapters III and IV to establish the convergence of Weinstein's approximations of the least or the greatest spectral point of certain operators in a Hilbert space. The computation of the lower bounds and upper bounds involved in this construction was elegantly formulated by G. Fichera [12, lecture 15 and following]. Finally, in Chapter VIII we indicate how our theory may be applied to the method of penalty functions, and we close the chapter with an application to a problem in optimal control theory.

Basic Notations and Conventions

(1) \( \mathbb{N} \) will always denote the set of positive integers with the natural ordering.

(2) \( \mathbb{R}^p \), with \( p \in \mathbb{N} \), will stand for the \( p \)-dimensional Euclidean space. \( \mathbb{R}^1 \) will simply be written as \( \mathbb{R} \) and called the real line.
(3) $\mathbb{R}^+$ stands for the extended real line $\mathbb{R} \cup \{+\infty\}$ with its usual topology and its usual ordering.

(4) Addition in $\mathbb{R}^+$ is defined as follows:

a) the addition in $\mathbb{R}$ is retained;

b) if $x \in \mathbb{R}$ then

\[
(+\infty) + x = (+\infty) - x = +\infty,
\]

and

\[
(-\infty) + x = (-\infty) - x = -\infty;
\]

c) $(-\infty) + (-\infty) = -\infty$ and $(+\infty) + (+\infty) = +\infty$;

d) the expressions $(-\infty) + (+\infty)$ and $(+\infty) + (-\infty)$ remain undefined.

(5) Given $x \in \mathbb{R}^+$ and $y \in \mathbb{R}^+$, we shall say that "$x + y$ is defined in $\mathbb{R}^+$" if $x$ and $y$ fall in one of the patterns of (4).

(6) Given $a$ and $b$ in $\mathbb{R}^+$ with $a \leq b$ we let
a) \([a, b] = \{x \in \mathbb{R}^+: a \leq x \leq b\}\),

b) \((a, b] = \{x \in \mathbb{R}^+: a < x \leq b\}\),

c) \([a, b) = \{x \in \mathbb{R}^+: a \leq x < b\}\),

d) \((a, b) = \{x \in \mathbb{R}^+: a < x < b\}\).

(7) In a topological space, the closure of a subset \(A\) will be denoted by \(\bar{A}\).
II. SEMI-CONTINUITY AND NUMERICAL NETS

The basic terminology and results on Moore-Smith convergence, as used in the following pages, will be found in J.L. Kelley [17] and S. Mrowka [21].

**Definition 2.1**

Let $X$ be a non-empty set and $B$ a directed set (with direction $\geq$).

(a) A net $S : B \to X$ will be referred to as the net $(S_b, b \in B)$ in $X$, where $S_b = S(b)$.

(b) If $G$ is a cofinal subset of $B$ and $(S_b, b \in B)$ a net in $X$, the net $(S|G)_b, b \in G)$ will be simply written $(S_b, b \in G)$.

(c) If $X$ is a topological space, then $E(S_b, b \in B)$ will denote the set of cluster points of the net $(S_b, b \in B)$ in $X$.

(d) A net $(S_b, b \in B)$ in $\mathbb{R}^+$ will be called a numerical net and we shall define:

$$\liminf(S_b, b \in B) = \text{greatest lower bound of } E(S_b, b \in B),$$

$$\limsup(S_b, b \in B) = \text{least upper bound of } E(S_b, b \in B).$$
(e) A numerical net \((S^b, b \in B)\) will be called monotone if it is either increasing \((a \geq b \text{ in } B \Rightarrow S_a \geq S_b \text{ in } \mathbb{R}^+)\) or decreasing \((a \geq b \text{ in } B \Rightarrow S_a \leq S_b \text{ in } \mathbb{R}^+)\).

(f) If the net \((S^b, b \in B)\) in a topological space \(X\) converges to a point \(p \in X\), we shall write \((S^b, b \in B) \to p\), or \(\lim(S^b, b \in B) = p\) in case \(X\) is Hausdorff.

(g) A net \(S\) on a directed set \(B\) is said to have a property \(P\) for almost all \(b \in B\) if \(\exists b_o \in B\) such that for those \(b \in B\) with \(b \geq b_o\), \(S^b\) has property \(P\).

**Lemma 2.2**

(a) A subset \(M\) of \(\mathbb{N}\) is cofinal in \(\mathbb{N}\) if and only if \(M\) is infinite.

(b) If \(M\) is a cofinal subset of \(\mathbb{N}\) and \(K \subset M\), then \(K\) is cofinal in \(M\) if and only if \(K\) is infinite.

**Proof:**

(a) If \(M\) is cofinal in \(\mathbb{N}\), we may construct an infinite subset of \(M\) inductively as follows: Put \(k_1 = 1\). Then \(k_1 \in \mathbb{N}\) and \(M\) is cofinal in \(\mathbb{N}\), so there is \(m_1 \in M\) with \(m_1 \geq k_1\). Now put \(k_2 = 1 + m_1\).
Then \( k_2 \in \mathbb{N} \) and \( M \) is cofinal in \( \mathbb{N} \), so again there is \( m_2 \in M \) with \( m_2 \geq k_2 \). Proceed inductively with \( k_{t+1} = 1 + m_t \). The set \( \{m_t : t \in \mathbb{N}\} \) is an infinite subset of \( M \) since \( m_t \in M \) and \( m_{t+1} \geq k_{t+1} = 1 + m_t > m_t \) for every \( t \in \mathbb{N} \), and so \( M \) is infinite. Obviously an infinite subset of \( \mathbb{N} \) is cofinal in \( \mathbb{N} \).

(b) If \( K \) is an infinite subset of \( M \subseteq \mathbb{N} \), then \( K \) is cofinal in \( \mathbb{N} \) by (a), and so \( K \) is cofinal in \( M \). Conversely suppose \( K \) is cofinal in \( M \). Since \( M \) was assumed cofinal in \( \mathbb{N} \), it follows that \( K \) is cofinal in \( \mathbb{N} \). Therefore \( K \) is infinite by (a).

Q.E.D.

The notation of Definition 2.1 will be followed for sequences also. By a sequence \( (x_n, n \in L) \) in a set \( X \), we shall mean a function \( x: L \to X \), where \( x_n = x(n) \) and \( L \) is an infinite subset of \( \mathbb{N} \) with the natural ordering. Lemma 2.2 assures us that a sequence \( (x_n, n \in L) \) in \( X \) is a net in \( X \). Given a sequence \( (x_n, n \in L) \) in \( X \), every subsequence of \( (x_n, n \in L) \) is obtained by restricting \( x \) to some infinite subset \( M \) of \( L \), and the resulting
subsequence is then denoted \((x_n, n \in M)\) in accordance
with Definition 2.1.

**Theorem 2.3**

For every numerical net \((S_b, b \in B)\), each of the following statements holds:

(a) \(\lim \inf(S_b, b \in B) \leq \lim \sup(S_b, b \in B)\).

(b) Each cofinal subset \(G\) of \(B\) satisfies

\[E(S_b, b \in G) \subseteq E(S_b, b \in B),\]

\[\lim \sup(S_b, b \in G) \leq \lim \sup(S_b, b \in B)\]

and

\[\lim \inf(S_b, b \in G) \geq \lim \inf(S_b, b \in B).\]

(c) If there is \(x \in \mathbb{R}^+\) with \(S_b \leq x\) for almost all \(b \in B\), then \(E(S_b, b \in B) \subseteq [-\infty, x]\)
and \(\lim \sup(S_b, b \in B) \leq x\).

(d) If there is \(x \in \mathbb{R}^+\) with \(S_b \geq x\) for almost all \(b \in B\), then \(E(S_b, b \in B) \subseteq [x, +\infty]\)
and \(\lim \inf(S_b, b \in B) \geq x\).

(e) If there is \(x \in \mathbb{R}^+\) with \(\lim \sup(S_b, b \in B) > x\),
then there is a cofinal subset \(G\) of \(B\) with
\(S[G] \subseteq ]x, +\infty]\).
(f) If there is \( x \in \mathbb{R}^+ \) with \( \lim \inf (S_b, b \in B) < x \), then there is a cofinal subset \( G \) of \( B \) with \( S[G] \subset [-\infty, x] \).

**Proof:**

(a) The space \( \mathbb{R}^+ \) is compact, so \( \phi \notin E(S_b, b \in B) \). It follows that \( \inf E(S_b, b \in B) \leq \sup E(S_b, b \in B) \), i.e., \( \lim \inf (S_b, b \in B) \leq \lim \sup (S_b, b \in B) \).

(b) Assume \( G \) to be a cofinal subset of \( B \). Take \( x \in E(S_b, b \in G) \). Consider any nbhd \( V \) of \( x \) in \( \mathbb{R}^+ \) and \( b_o \in B \). Since \( G \) is cofinal in \( B \), we may find \( a_o \in G \) with \( a_o \geq b_o \). Now \( V \) is a nbhd of \( x \), \( a_o \in G \) and \( x \in E(S_b, b \in G) \) imply that there is \( c \in G \) with \( c \geq a_o \) and \( S_c \in V \). Therefore \( c \in B \), \( c \geq b_o \) and \( S_c \in V \). Consequently, \( (S_b, b \in B) \) is frequently in each nbhd of \( x \). It follows that \( x \in E(S_b, b \in B) \), so that \( E(S_b, b \in G) \subset E(S_b, b \in B) \). On the other hand we know from (a) that \( \phi \notin E(S_b, b \in G) \). Thus \( \sup E(S_b, b \in G) \leq \sup E(S_b, b \in B) \) and \( \inf E(S_b, b \in G) \geq \inf E(S_b, b \in B) \), proving (b).
(c) If $y \in \mathbb{R}^+$ and $x < y$, find a real number $t$ with $x < t < y$. Then $V = [t, +\infty]$ is a nbd of $y$ in $\mathbb{R}^+$, but the net $(S_b, b \in B)$ is eventually in the complement of $V$. Hence $y \notin E(S_b, b \in B)$. It follows that no element of $E(S_b, b \in B)$ exceeds $x$, i.e., $E(S_b, b \in B) \subset [-\infty, x]$. But then also $\limsup(S_b, b \in B) = \sup E(S_b, b \in B) \leq x$.

(d) Consider $z \in \mathbb{R}^+$ with $z < x$. Then a real number $t$ exists with $z < t < x$. This implies that $V = [-\infty, t]$ is a nbd of $z$ in $\mathbb{R}^+$, but the net $(S_b, b \in B)$ is eventually in the complement of $V$. Thus $z \notin E(S_b, b \in B)$. It follows that $E(S_b, b \in B) \subset [x, +\infty]$ and so $\liminf(S_b, b \in B) = \inf E(S_b, b \in B) \geq x$.

(e) Find a real number $\alpha$ with $\limsup(S_b, b \in B) > \alpha > x$. Hence $\sup E(S_b, b \in B) = \limsup(S_b, b \in B) > \alpha$, so $\exists z \in E(S_b, b \in B)$ with $z > \alpha$. Therefore $V = [\alpha, +\infty]$ is a nbd of $z$ in $\mathbb{R}^+$, with $z \in E(S_b, b \in B)$. It follows that for every $b \in B$ we may find $b_0 \in B$ with $b_0 \geq b$ and $S_{b_0} \in V$. If we select one such $b_0$ for every $b \in B$, the set $G = \{b_0 : b \in B\}$ is then cofinal in $B$ and $S[G] \subset V = [\alpha, +\infty] \subset ]x, +\infty]$ since $x < \alpha$. 

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(f) Choose a real number \( \beta \) with \( \liminf(S_b, b \in B) < \beta < x \), so that \( \inf E(S_b, b \in B) < \beta \).

Hence there is some \( y \in E(S_b, b \in B) \) with \( y < \beta \).

Thus \( U = [-\infty, \beta[ \) is a nbd of \( y \) in \( \mathbb{R}^+ \), with \( y \in E(S_b, b \in B) \). It follows that for every \( b \in B \) we may find some \( b_v \in B \) with \( b_v > b \) and \( S_{b_v} \in U \).

Selecting one such \( b_v \) for every \( b \) in \( B \), we obtain that the set \( G = \{ b_v : b \in B \} \) is cofinal in \( B \) and \( S[G] \subset U = [-\infty, \beta[ \), with \( [-\infty, \beta[ \subset [-\infty, x[ \) since \( \beta < x \).

Q.E.D.

**Lemma 2.4**

Let \( (S_b, b \in B) \) be a numerical net and \( x \in \mathbb{R}^+ \).

(a) If \( \limsup(S_b, b \in B) < x \), then the net \( (S_b, b \in B) \) is eventually in \( [-\infty, x[ \).

(b) If \( \liminf(S_b, b \in B) > x \), then the net \( (S_b, b \in B) \) is eventually in \( ]x, +\infty] \).

**Proof:**

(a) Assume that the conclusion fails. Given \( b \in B \) we may thus find some \( b' \in B \) with \( b' > b \) and \( S_{b'} > x \). We can select one such \( b' \) in \( B \) for every \( b \in B \) to get the cofinal subset.
G = \{b': b \in B\} of B. Since S_c \geq x for every c \in G, it follows by Theorem 2.3(a),(d) that \(\limsup(S_c, c \in G) \geq x\). This violates Theorem 2.3(b) since \(x > \limsup(S_b, b \in B)\).

(b) Let us suppose that the conclusion fails, that is, for every \(b \in B\) there is \(b' \in B\) with \(b' \geq b\) and \(S_{b'} \leq x\). Selecting one such \(b'\) for every \(b \in B\), we obtain a cofinal subset \(G = \{b': b \in B\}\) for which \(S_c \leq x\) for each \(c \in G\). Hence Theorem 2.3(a),(c) implies \(\liminf(S_c, c \in G) \leq x\), whence \(\liminf(S_b, b \in B) \leq x\) in view of Theorem 2.3(b).

Q.E.D.

Lemma 2.5

Consider two numerical nets \((S_b, b \in B)\) and \((T_b, b \in B)\).

(a) If \(S_b \leq T_b\) for almost all \(b \in B\), then

\[
\limsup(S_b, b \in B) \leq \limsup(T_b, b \in B)
\]

and

\[
\liminf(S_b, b \in B) \leq \liminf(T_b, b \in B).
\]
(b) If $S_b + T_b$ is defined in $\mathbb{R}^+$ for every $b \in B$, and if also $\limsup(S_b, b \in B) + \limsup(T_b, b \in B)$ is defined in $\mathbb{R}^+$, then

$$\limsup(S_b + T_b, b \in B) \leq [\limsup(S_b, b \in B)] + [\limsup(T_b, b \in B)].$$

(c) If $S_b + T_b$ is defined in $\mathbb{R}^+$ for every $b \in B$ and if also $\liminf(S_b, b \in B) + \liminf(T_b, b \in B)$ is defined in $\mathbb{R}^+$, then

$$\liminf(S_b + T_b, b \in B) \geq [\liminf(S_b, b \in B)] + [\liminf(T_b, b \in B)].$$

Proof:

(a) Put $\alpha = \limsup(S_b, b \in B), \beta = \limsup(T_b, b \in B)$ and consider $\alpha > \beta$. Take a real number $t$ for which $\alpha > t > \beta$. It follows by Lemma 2.4(a) that there is $b_0 \in B$ with $T_{b_0} < t$ for $b \geq b_0$ in $B$. If now $S_b \leq T_b$ for almost all $b \in B$, then we would have $S_b \leq T_b < t$ for almost all $b \in B$, so that $\limsup(S_b, b \in B) \leq t$ by Theorem 2.3(a), (c). This is impossible since $t < \limsup(S_b, b \in B)$. Consider the case $p = \liminf(S_b, b \in B) > q = \liminf(T_b, b \in B)$ and take a $t$ such that $p > t > q$. Lemma 2.4(b)
implies that $S_b > t$ for almost all $b \in B$. Thus, if it is true that $S_b \leq T_b$ for almost all $b \in B$, we would have $T_b > t$ for almost all $b \in B$, so that $q = \lim \inf (T_b, b \in B) \geq t$ by Theorem 2.3(d). This is of course impossible because we had $t > q$.

(b) Put

\[ \alpha = \lim \sup (S_b, b \in B), \]
\[ \beta = \lim \sup (T_b, b \in B), \]
\[ \gamma = \lim \sup (S_b + T_b, b \in B). \]

If either $\alpha = +\infty$ or $\beta = +\infty$, we have then $\alpha + \beta = +\infty$ and the conclusion is obvious. Thus we may assume $\alpha < +\infty$ and $\beta < +\infty$. In this case, the sum $u + v$ is defined in $\mathbb{R}^+$ whenever $u > \alpha$ and $u > \beta$, and the set $K = \{u + v : u > \alpha, u > \beta\}$ satisfies:

$\inf K = \alpha + \beta$. Let us consider $u + v \in K$. Therefore $u > \alpha$ and $v > \beta$, so that, by Lemma 2.4(a) there is $b_0 \in B$ with $S_b < u$ for $b \in B, b \geq b_0$, and there is $b_1 \in B$ with $T_b < v$ for $b \in B, b \geq b_1$. Find $c_0 \in B$ with $c_0 \geq b_0$ and $c_0 \geq b_1$ (such an element exists because $B$ is directed). Then $S_b < u$ and $T_b < v$ for $b \geq c_0, b \in B$. It follows that $S_b + T_b < u + v$ for $b \geq c_0, b \in B$. Thus $\gamma \leq u + v$ by Theorem 2.3(c). Since $u + v$ was an arbitrary element.
of $K$, we conclude that $\gamma \leq \inf K = \alpha + \beta$, which proves (b).

(c) Define

$$p = \lim \inf (S_b, b \in B)$$

$$q = \lim \inf (T_b, b \in B)$$

$$r = \lim \inf (S_b + T_b, b \in B).$$

In case $p = -\infty$ or $q = -\infty$, then $p + q = -\infty$ and $p + q \leq r$ is then true. We can therefore assume $p > -\infty$ and $q > -\infty$. The subset $K = \{m + n: m < p$ and $n < q\}$ of $\mathbb{R}^+$ satisfies: $\sup K = p + q$. Now given $m + n \in K$, we have $m < p$ and $n < q$, so that by Lemma 2.4(b), there is $b_0 \in B$ with $[S_b > m$ for $b \in B, b \geq b_0]$ and there is $b_1 \in B$ with $[T_b > n$ for $b \in B, b \geq b_1]$. Because $B$ is directed, we can find $c_0 \in B$ with $c_0 \geq b_0, c_0 \geq b_1$. Then $S_b > m$ and $T_b > n$ for $b \in B, b \geq c_0$. It follows that $S_b + T_b > m + n$ for $b \in B, b \geq c_0$. Theorem 2.3(d) now implies that $r \geq m + n$. Because $m + n$ was arbitrary in $K$, we conclude that $r \geq \sup K = p + q$.

Q.E.D.
The assumptions of Theorem 2.5(b),(c) may look clumsy, but they are necessary. If we let \( S_b = -b \) and \( T_b = b \), where \( B = \mathbb{N} \), then \( \lim \sup(S_b, b \in B) + \lim \sup(T_b, b \in B) \) and \( \lim \inf(S_b, b \in B) + \lim \inf(T_b, b \in B) \) both reduce to \( -\infty + \infty \), which is meaningless in \( \mathbb{R}^+ \). On the other hand taking \( B = \mathbb{N} \) again with

\[
S_b = \begin{cases} 
+\infty & \text{for } b \text{ even} \\
-\infty & \text{for } b \text{ odd}
\end{cases}
\]

and

\[
T_b = \begin{cases} 
-\infty & \text{for } b \text{ even} \\
+\infty & \text{for } b \text{ odd}
\end{cases}
\]

we see that \( S_b + T_b \) is not defined in \( \mathbb{R}^+ \), and so it would be meaningless to write \( \lim \sup(S_b + T_b, b \in B) \) or \( \lim \inf(S_b + T_b, b \in B) \).

**Theorem 2.6**

Let \( (S_b, b \in B) \) be a numerical net and \( x \in \mathbb{R}^+ \).

The following statements are equivalent:

(a) \( (S_b, b \in B) \) converges to \( x \) (in \( \mathbb{R}^+ \))

(b) \( E(S_b, b \in B) = \{x\} \).
(c) \( \lim \sup (S_b, \ b \in B) = \lim \inf (S_b, \ b \in B) = x. \)

Proof:

(a) implies (b): This implication is well known for any Hausdorff space, and \( \mathbb{R}^+ \) is Hausdorff.

(b) implies (c): This is obvious, since

\[
\lim \sup (S_b, \ b \in B) = \sup \{S_b, \ b \in B\} = x
\]

and

\[
\lim \inf (S_b, \ b \in B) = \inf \{S_b, \ b \in B\} = x
\]

by definition.

(c) implies (a): Because \( \mathbb{R}^+ \) is compact, it follows that \( E(S_b, \ b \in B) \neq \emptyset \), hence (c) implies that \( E(S_b, \ b \in B) = \{x\}, \) i.e., \( (c) = (b) \). Let \( V \) be a nbd of \( x \) in \( \mathbb{R}^+ \) such that \( (S_b, \ b \in B) \) is not eventually in \( V \). Then there is a cofinal subset \( G \) of \( B \) such that \( S_b \notin V \) for every \( b \in G \). It follows that \( x \in E(S_b, \ b \in G) \). But by Theorem 2.3(b) and the fact that \( \mathbb{R}^+ \) is compact we have \( \emptyset \neq E(S_b, \ b \in G) \subseteq E(S_b, \ b \in B) = \{x\} \). This is a contradiction.

Q.E.D.

Corollary 2.7

An increasing numerical net \( (S_b, \ b \in B) \) converges to \( \sup \{S_b; \ b \in B\} \). A decreasing numerical
net \((S_b, b \in B)\) converges to \(\inf\{S_b, b \in B\}\).

**Proof:**

Consider an increasing numerical net \((S_b, b \in B)\) and put \(x = \sup\{S_b: b \in B\}\). If \(y < x\) in \(R^+\), so that \(y < t < x\) for some \(t \in R\), then \(U = [-\infty, t]\) is a nbd of \(y\) in \(R^+\) which is disjoint from \(V = [t, x]\). From the definition of \(x\) and the fact that \(t < x\), we may find \(b_o \in B\) with \(t < S_{b_o} \leq x\).

If now \(b \in B\) and \(b \geq b_o\), we get \(S_{b_o} \leq S_b\ (\leq x)\) because the net is increasing, so that \(S_b \in V\). It follows that \((S_b, b \in B)\) is eventually in \(V\), and thus \((S_b, b \in B)\) is not frequently in \(U\), so \(y \notin E(S_b, b \in B)\). On the other hand if \(y \in R^+\) with \(x < y\), then \(S_b \leq x\) for every \(b \in B\), hence the net \((S_b, b \in B)\) is not at all in the nbd \([x, +\infty]\) of \(y\) in \(R^+\), and thus \(y \notin E(S_b, b \in B)\). We conclude: 
\(\{y \neq x \text{ in } R^+\} = [y \notin E(S_b, b \in B)\]. But \(R^+\) is compact and so \(E(S_b, b \in B) \neq \emptyset\). It follows that \(E(S_b, b \in B) = \{x\}\), and so Theorem 2.6 applies. The case of a decreasing net is treated similarly.

Q.E.D.
Corollary 2.8

If \((S_b, b \in B)\) is a numerical net, then

(a) \(\limsup(S_b, b \in B)\)

\[= \inf \{\sup \{S_b: b \in B \text{ and } b \geq c\}\} \text{,} \]

(b) \(\liminf(S_b, b \in B)\)

\[= \sup \{\inf \{S_b: b \in B \text{ and } b \geq c\}\} \text{.} \]

Proof:

(a) For \(c \in B\) define \(T_c = \sup \{S_b: b \in B \text{ and } b \geq c\}\). Then \((T_c, c \in B)\) is a decreasing numerical net and therefore it converges, by Corollary 2.7, to \(\inf \{T_c: c \in B\} = \inf \{\sup \{S_b: b \in B \text{ and } b \geq c\}\} = x\). Now \(S_b \leq T_b\) for every \(b \in B\). Therefore

\[\limsup(S_b, b \in B) \leq \limsup(T_c, c \in B)\]

(2.1)

\[= \lim(T_c, c \in B)\]

\[= x\]

by Lemma 2.5(a) and Theorem 2.6. Consider an open nbd \(V\) of \(x\) in \(\mathbb{R}^+\) and \(b_0 \in B\). There is \(c_0 \in B\) such that \([c \in B \text{ and } c \geq c_0 = T_c \in V\]. Find \(a_0 \in B\).
with \( a_0 \geq b_0, a_0 \geq c_0 \). Thus \( c \in B \) and \( c \geq a_0 \Rightarrow T_c \in V \). In particular, \( T_{a_0} = \sup\{S_b : b \in B \text{ and } b \geq a_0 \} \in V \). Since \( V \) is open, it follows that there is \( b \in B \) with \( b \geq a_0 \) and \( S_b \in V \). Thus \( b \in B, b \geq b_0 \) and \( S_b \in V \). The net \( (S_b, b \in B) \) is thus frequently in \( V \), and so \( x \in E(S_b, b \in B) \), whence

\[
(2.2) \quad x \leq \sup E(S_b, b \in B) = \lim \sup (S_b, b \in B).
\]

Combining (2.1) and (2.2), we obtain (a).

(b) The proof is analogous to part (a).

Q.E.D.

Definition 2.9

Let \( X \) be a topological space and \( p \in X \).

(a) A functional \( f: X \to \mathbb{R} \) is said to be lower semi-continuous (l.s.c.) at \( p \) if, whenever

\[-\infty < t < f(p), \]

there is a nbd \( V \) of \( p \) in \( X \) with \( f[V] \subseteq ]t, +\infty[. \)

(b) \( f: X \to \mathbb{R} \) is said to be upper semi-continuous (u.s.c.) at \( p \) if, whenever \( f(p) < t < +\infty \), there is a nbd \( V \) of \( p \) in \( X \) with \( f[V] \subseteq ]-\infty, t[. \)
Theorem 2.10

Let \( X \) be a topological space, \( p \in X \) and \( f: X \rightarrow \mathbb{R} \).

(a) \( f \) is l.s.c. at \( p \) if and only if each net \( (S_b, b \in B) \) in \( X \) converging to \( p \) in \( X \) satisfies:
\[
f(p) \leq \lim \inf(f(S_b), b \in B).
\]

(b) \( f \) is u.s.c. at \( p \) if and only if each net \( (S_b, b \in B) \) in \( X \) converging to \( p \) in \( X \) satisfies:
\[
f(p) \geq \lim \sup(f(S_b), b \in B).
\]

(c) In case \( p \) has a countable local base in \( X \), then nets may be replaced by sequences in (a) and (b) above.

Proof:

(a) Necessity: Suppose a net \( (S_b, b \in B) \) in \( X \) converges to \( p \) in \( X \). Consider a real number \( \varepsilon > 0 \) and put \( t = f(p) - \varepsilon \), so that \( -\varepsilon < t < f(p) \).

There exists a nbd \( V \) of \( p \) in \( X \) with \( t < f(x) \) for each \( x \in V \). Since \( V \) is a nbd of \( p \) in \( X \) and \( (S_b, b \in B) \rightarrow p \) in \( X \), \( \exists b_0 \in B \) so that \( S_b \in V \) for \( b \in B, b > b_0 \). Hence we have that if \( b \in B \) and \( b \geq b_0 \), then \( S_b \in V \), so that \( (f \circ S)_b = f(S_b) > t \). This means that the numerical
net \((f(S_b), b \in B)\) satisfies: \(f(S_b) > t\) for almost all \(b \in B\). Hence Theorem 2.3(d) implies that
\[
\liminf(f(S_b), b \in B) \geq t = f(p) - \varepsilon.
\]
Since this is valid for an arbitrary real \(\varepsilon > 0\), it follows that
\[
\liminf(f(S_b), b \in B) \geq f(p).
\]

Sufficiency: Define \(B = \{(x, V): x \in V \text{ and } V\text{ is a nbd of } p \text{ in } X\}\). The set \(B\) is directed by declaring \((x, V) \geq (y, U)\) to mean \(V \subset U\). Let us define \(S: B \rightarrow X\) by \(S(x, V) = x\). Then the net 
\((S(x, V), (x, V) \in B)\) converges to \(p\). Hence, by hypothesis, \(f(p) \leq \liminf((f \circ S)(x, V), (x, V) \in B) = \liminf(f(x), (x, V) \in B)\). Thus, if \(-\infty < t < f(p)\), it follows that \(t < \liminf((f \circ S)(x, V), (x, V) \in B)\), so that we can use Lemma 2.4(b) to conclude that 
\(\exists (x_0, V_0) \in B\) satisfying: \((x, V) \in B\) and \((x, V) \geq (x_0, V_0) = (f \circ S)(x, V) > t\), i.e., \(f(x) > t\).
Since \((x, V) \geq (x_0, V_0)\) for each \(x \in V_0\) and since \(V_0\) is a nbd of \(p\), we conclude that \(f[V_0] \subset ]t, +\infty[\). Therefore \(f\) is l.s.c. at \(p\), by definition.

(b) The arguments used in proving (a) may be used in a dual way to establish (b).
(c) In both (a) and (b), the proof of the necessity remains valid for sequences because a sequence is a net. Let us now assume that \( p \) has a countable local base \( \{ U_n : n \in \mathbb{N} \} \) in \( X \) and that the sufficiency in (a) holds for sequences. Suppose that some \( t \) exists with \(-\infty < t < f(p)\) but no matter what neighborhood \( V \) of \( p \) we take in \( X \), we may find \( x \in V \) with \( f(x) < t \).

Therefore for every \( n \in \mathbb{N} \) we may find \( S_n \in U_n \) with \( f(S_n) < t \). From the fact that \( S_n \in U_n \) for each \( n \in \mathbb{N} \) and that \( \{ U_n : n \in \mathbb{N} \} \) is a local base at \( p \), it follows that \( (S_n, n \in \mathbb{N}) \to p \). Since \( f(S_n) < t \) for each \( n \in \mathbb{N} \), Theorem 2.3(a), (c) implies that \( \lim \inf f(S_n), n \in \mathbb{N} \leq t \), with \( t < f(p) \). This contradicts the hypothesis. The sufficiency in (b) under similar hypotheses is handled in a dual way.

Q.E.D.

**Definition 2.11**

Let \( X \) be a topological space, \( p \in X \) and \( f: X \to \mathbb{R} \).

(a) \( f \) is said to be sequentially l.s.c. at \( p \) if and only if every sequence \((x_n, n \in \mathbb{N})\) in \( X \) which converges to \( p \) satisfies: \( f(p) \leq \lim \inf f(x_n), n \in \mathbb{N} \).
(b) \( f \) is sequentially u.s.c. at \( p \) if, and only if, whenever a sequence \( (x_n, n \in \mathbb{N}) \) in \( X \) converges to \( p \), it is true that \( f(p) \geq \lim \sup(f(x_n), n \in \mathbb{N}) \).

Theorem 2.10 allows us to define the notions of "lower semi-continuous functional", "upper semi-continuous functional" on a set \( X \neq \emptyset \) with no topology, as long as we know what nets in \( X \) "converge" to what elements in \( X \).

Definition 2.12

Let \( X \) be a non-void set and \( A \) the class of all nets in \( X \). A non-void subclass \( C \) of \( A \times X \) is called "a convergence structure" for \( X \) if:

(L1) When \( p \in X \) and \( B \) is a directed set and \( S: B \rightarrow X \) is defined by \( S_b = p \) for every \( b \in B \), then \( ((S_b, b \in B), p) \in C \).

(L2) If \( ((S_b, b \in B), p) \in C \) and \( (T_a, a \in M) \) is a subnet of \( (S_b, b \in B) \), then \( ((T_a, a \in M), p) \in C \).

Definition 2.13

(a) By "an L-space \( X \)" we shall mean a non-void set \( X \) for which a convergence structure has been defined.
(b) An L-space $X$ with a convergence structure $C$ will be said to be Hausdorff if it is true that

$((S_b, b \in B), p) \in C$ and $((S_b, b \in B), q) \in C \Rightarrow p = q$.

(c) In an L-space $X$ with a convergence structure $C$ we shall write $(S_b, b \in B) \rightarrow_c p$ (or simply $(S_b, b \in B) \rightarrow p$), and we shall say that $(S_b, b \in B)$ converges to $p$ with respect to $C$ (or simply $(S_b, b \in B)$ converges to $p$), if it is true that $((S_b, b \in B), p) \in C$.

**Definition 2.14**

Let $X$ be a non-empty set and $B$ the class of sequences in $X$. (Recall that a sequence in $X$ is a function $S: L \rightarrow X$ where $L$ is an infinite subset of $\mathbb{N}$). A non-void subclass $C$ of $B \times X$ is called a "sequential convergence structure" for $X$ if

(L'1) If $p \in X$ and $M$ is an infinite subset of $\mathbb{N}$ and $S: M \rightarrow X$ is defined by $S_n = p$ for each $n \in M$, then $((S_n, n \in M), p) \in C$.

(L'2) If $((S_n, n \in M), p) \in C$ and $(S_n, n \in L)$ is a subsequence of $(S_n, n \in M)$, then $((S_n, n \in L), p) \in C$.

We then call $X$ "a sequential L-space".
**Definition 2.15**

(a) An L-space $X$ is called an L*-space if the following holds:

(L3) If a net $(S_b, b \in B)$ in $X$ does not converge to a point $p \in X$, then there is a subnet of $(S_b, b \in B)$, no subnet of which converges to $p$.

(b) A sequential L-space $X$ is called a sequential L*-space if the following is true:

(L'3) If a sequence $(S_n, n \in M)$ in $X$ does not converge to a point $p \in X$, then there is a subsequence of $(S_n, n \in M)$, no subsequence of which converges to $p$.

**Remark 2.16**

(a) Every topological space is an L*-space and a sequential L*-space with respect to the convergence generated by the topology. The converse is generally false. A discussion of these matters appears in J.L. Kelley [17] and E. Čech [7].

(b) Certain notions in a topological space $X$ are defined, or can be characterized, in terms of the convergence generated by the topology. Among them let us single out:
(1) cluster points of a net \((S_b, b \in B)\) in \(X\): \(p \in X\) is a cluster point of \((S_b, b \in B)\) if and only if a subnet of \((S_b, b \in B)\) converges to \(p\);

(2) limit points of a sequence \((S_n, n \in M)\) in \(X\): \(p \in X\) is a limit point of \((S_n, n \in M)\) if and only if a subsequence of \((S_n, n \in M)\) converges to \(p\);

(3) sequential compactness: a subset \(Y\) of \(X\) is sequentially compact if each sequence in \(Y\) has a limit point \(p \in Y\);

(4) compactness: a subset \(Y\) of \(X\) is compact if and only if each net in \(Y\) has a cluster point \(p \in Y\);

(5) lower semi-continuity: see Theorem 2.10;

(6) upper semi-continuity: see Theorem 2.10;

(7) sequential lower semi-continuity: see Definition 2.11;

(8) sequential upper semi-continuity: see Definition 2.11.

In an \(L\)-space \(X\), we will use this convergence characterization as the definition of the above-listed
notions, as long as "converges to p" is interpreted according to Definition 2.13. This does not lead to confusion because, when a convergence structure C for the L-space X is generated by a topology τ for X, the expressions "converges to p with respect to τ" and "converges to p with respect to C" coincide.
III. SUFFICIENT CONDITIONS FOR APPROXIMATION

In this chapter we extend the results of [24] to nets with some relaxation of the assumptions.

Definition 3.1

Let $X$ be an $L$-space, $D$ a non-empty subset and $I: D \to \mathbb{R}$. Consider a directed set $B$ with a net $(D_b, b \in B)$ of non-empty subsets of $X$, and a net $(I_b, b \in B)$ of real-valued functionals, each $I_b$ defined on $D_b$. We will let $i = \inf I[D]$, $i^* = \sup I[D]$; $i_b = \inf I_b[D_b]$, $i_b^* = \sup I_b[D_b]$ for $b \in B$. We will also use the following:

**Property (m):** $i_b \leq i$ for every $b \in B$.

**Property (m*):** $i_b^* \geq i^*$ for every $b \in B$.

**Property (I):** If $G$ is a cofinal subset of $B$ and a net $(x_b, b \in G)$ in $X$ is such that $x_b \in D_b$ for every $b \in G$ and $(x_b, b \in G) \to y \in D$, then $I(y) \leq \lim \inf (I_b(x_b), b \in G)$.

**Property (I*):** As in Property (I), with the conclusion replaced by $I(y) \geq \lim \sup (I_b(x_b), b \in G)$.
Property (D): If a net \((x_b, b \in B)\) in \(X\) is such that \(x_b \in D_b\) for every \(b \in B\), then a cofinal subset \(G\) of \(B\) exists with a point \(y \in D\) such that \((x_b, b \in G) \rightarrow y\).

Property (D+): If \(H\) is any cofinal subset of \(B\) and a net \((x_b, b \in H)\) in \(X\) is such that \(x_b \in D_b\) for every \(b \in H\), then a cofinal subset \(G\) of \(H\) exists with a point \(y \in D\) such that \((x_b, b \in G) \rightarrow y\).

Remark 3.2

In case the set \(B\) of Definition 3.1 is \(\mathbb{N}\), we can use Lemma 2.2 to reformulate some of the properties of Definition 3.1 as follows:

Property (I): If a sequence \((x_n, n \in L)\) in \(X\) is such that \(x_n \in D_n\) for every \(n \in L\) and \((x_n, n \in L) \rightarrow y \in D\), then \(I(y) \leq \lim \inf(I_n(x_n), n \in L)\).

Property (I*): As in Property (I), with the conclusion replaced by \(I(y) \geq \lim \sup(I_n(x_n), n \in L)\).

Property (D): Each sequence \((x_n, n \in \mathbb{N})\) in \(X\) with \(x_n \in D_n\) for every \(n \in \mathbb{N}\) has a subsequence \((x'_n, n \in L)\) convergent to a point \(y\) of \(D\).
Property \((D^+)\): For each infinite set \(L \subseteq \mathbb{N}\), each sequence \((x_n, n \in L)\) in \(X\) with \(x_n \not\in D_n\) for every \(n \in L\) has a subsequence \((x_{n'}, n' \in M)\) convergent to a point \(y\) of \(D\).

**Lemma 3.3**

If \(I_b\) is an extension of \(I\) for every \(b \in B\), (i.e., \(D \subseteq D_b\) and \(I_b|D = I\)), then Property \((m)\) and Property \((m^*)\) are satisfied.

**Proof:**

Given \(b \in B\) we know that \(\emptyset \not\in I[D] \subseteq I_b[D_b]\), whence

\[
\inf I[D] \geq \inf I_b[D_b]
\]

and

\[
\sup I[D] \leq \sup I_b[D_b],
\]

that is, \(i \geq i_b\) and \(i^* \leq i_b^*\). It follows that Property \((m)\) and Property \((m^*)\) hold.

Q.E.D.
Theorem 3.4

(a) Property (m), Property (I) and Property (D+)
are sufficient for the numerical net \((i_b, b \in B)\) to approximate \(i\).

(b) Suppose Property (m), Property (I) and Property
(D+) are satisfied, and for each \(b \in B\) let \(x^*_b \in D_b\)
minimize \(I_b\). Then a cluster point \(x^*\) of \((x^*_b, b \in B)\)
lies in \(D\) and minimizes \(I\).

Proof:

(a) Property (m) together with Theorem 2.3(c)
gives us

\[(3.1) \quad i \geq \lim \sup (i_b, b \in B).\]

Let us now assume that \(i > \lim \inf (i_b, b \in B)\), so
that we may find a \(t \in \mathbb{R}^+\) with \(i > t > \lim \inf (i_b, b \in B)\). It follows by Theorem 2.3(f) that a cofinal
subset \(G\) of \(B\) exists with \(i_b < t\) for every
\(b \in G\). Hence for each \(b \in G\) we can pick \(x_b \in D_b\)
such that \(I_b(x_b) < t\), by the definition of \(i_b\).

Property (D+) now assures us that there exists a co-
final subset \(H\) of \(G\) (hence \(H\) is cofinal in \(B\))
and a point \(y \in D\) with \((x_b, b \in H) \to y\). Consequently,
\[ i \leq I(y) \leq \lim \inf (I_b(x_b), b \in H) \leq t < i \quad \text{by Theorem 2.3(a), (c) and our choice of} \quad t. \]

This is a contradiction. Therefore it is true that

\[ (3.2) \quad i \leq \lim \inf (i_b, b \in B). \]

We can now combine (3.1), (3.2), Theorem 2.3(a) and Theorem 2.6 to obtain \( i = \lim (i_b, b \in B) \), which proves part (a).

(b) Because \( x^*_b \in D_b \) for every \( b \in B \), Property (D+) implies, since \( B \) is a cofinal subset of itself, that a cofinal subset \( G \) of \( B \) exists with a point \( x^* \in D \) such that \( (x^*_b, b \in G) \to x^* \). Then

\[ i \leq I(x^*) \leq \lim \inf (I_b(x^*_b), b \in G) = \lim \inf (i_b, b \in G) = \lim (i_b, b \in B) = i. \]

(The next-to-last equality in (3.3) is obtained as follows:
(\(i_b, b \in G\)) is a subnet of \((i_b, b \in B)\) since \(G\) is a cofinal subset of \(B\), and \((i_b, b \in G)\) converges (uniquely since we are in the Hausdorff space \(R^+\)) to \(i\) by part (a), whence \((i_b, b \in G)\) converges uniquely to \(i\); now apply Theorem 2.6 to the numerical net \((i_b, b \in G)\). It follows by (3.3) that \(i = I(x^*)\).

Of course \(x^*\) is a cluster point of \((x^*_b, b \in B)\) since \(x^*\) was a limit of the subnet \((x^*_b, b \in G)\) of \((x^*_b, b \in B)\).

Q.E.D.

Corollary 3.5

Suppose \(I_b\) extends \(I\) for every \(b \in B\). Then Property (I) and Property (D+) are sufficient for the numerical net \((i_b, b \in B)\) to approximate \(i\), in which case every net \((x^*_b, b \in B)\) with \(x^*_b \in D_b\) and \(i_b = I_b(x^*_b)\) for every \(b \in B\) has a cluster point \(x^*\) in \(D\) which minimizes \(I\) over \(D\).

Proof:

Property (m) holds because of Lemma 3.3. Therefore we may apply Theorem 3.4.

Q.E.D.
Corollary 3.6

Suppose \( I_b \) extends \( I \) for each \( b \in B \), and suppose that \( I_b \mid (D_b \cap D_c) = I_c \mid (D_b \cap D_c) \) for \( b, c \in B \). Then Property (D+) and the lower semi-continuity on \( D \) of the functional \( f = \bigcup \{ I_b : b \in B \} \) are sufficient for the numerical net \((i_b, b \in B)\) to approximate \( i \), in which case every net \((x_b^*, b \in B)\) with \( x_b^* \in D_b \) and \( i_b = I_b(x_b^*) \) for every \( b \in B \) has a cluster point \( x^* \in D \) which minimizes \( I \).

Proof:

We only need to show that Property (I) holds, because then we can apply Corollary 3.5. (Observe that the assumption "\( I_b \mid (D_b \cap D_c) = I_c \mid (D_b \cap D_c) \) for every \( b, c \in B \)" guarantees that \( f = \bigcup \{ I_b : b \in B \} \) is a real-valued functional defined on \( \bigcup (D_b : b \in B) \) in the usual way: \( f(x) = I_b(x), \) for some \( b \in B \).) Consider then a cofinal subset \( G \) of \( B \) and a net \((x_b, b \in G)\) convergent to some \( y \in D \) with \( x_b \in D_b \) for every \( b \in G \). It follows from the lower semi-continuity of \( f \) on \( D \) that \( f(x) \leq \lim \inf(f(x_b), b \in G) \). However \( \lim \inf(f(x_b), b \in G) = \lim \inf(I_b(x_b), b \in G) \) since \( x_b \in D_b \) and \( f \) extends \( I_b \) for every \( b \in G \). Consequently, \( f(x) \leq \lim \inf(I_b(x_b), b \in G) \). Thus Property (I) is satisfied.

Q.E.D.
Corollary 3.7

Suppose that $X$ is a Hausdorff sequential $L$-space and Property (m), Property (I) and Property (D+) are satisfied with $B = \mathbb{N}$. Assume that $(t_n, n \in \mathbb{N})$ is a null sequence of positive real numbers. For every $n \in \mathbb{N}$, let $x_n \in D_n$ be such that $I_n(x_n) < i_n + t_n$. Then

(a) the numerical sequence $(i_n, n \in \mathbb{N})$ approximates $i$,

(b) the sequence $(x_n, n \in \mathbb{N})$ has at least one limit point in $D$, and

(c) every limit point of $(x_n, n \in \mathbb{N})$ lies in $D$ and minimizes $I$ over $D$. (In particular, if $x_n^* \in D_n$ minimizes $I_n$ over $D_n$ for every $n \in \mathbb{N}$, then $(x_n^*, n \in \mathbb{N})$ has a limit point in $D$ and every limit point of $(x_n^*, n \in \mathbb{N})$ lies in $D$ and minimizes $I$ over $D$.)

Proof:

(a) Immediate from Theorem 3.4.
(b) Since $x_n \in D_n$ for every $n \in \mathbb{N}$, it follows by Property (D+), Lemma 2.2 and Remark 3.2 that a subsequence $(x_{n_k}, n \in L)$ of $(x_n, n \in \mathbb{N})$ converges to a point $y \in D$. Hence $y$ is a limit point of $(x_n, n \in \mathbb{N})$ in $D$.

(c) Consider any limit point $y$ of $(x_n, n \in \mathbb{N})$. Then there is a subsequence $(x_{n_k}, n \in L)$ of $(x_n, n \in \mathbb{N})$ convergent to $y$. Since $x_n \in D_n$ for every $n \in L$, we know by Property (D+) and Remark 3.2 that a subsequence $(x_{n_k}, n \in M)$ of $(x_{n_k}, n \in L)$ converges to a point $z$ of $D$. Thus $z = y \in D$ because $X$ was assumed Hausdorff. Finally,

$$i \leq I(y) = i(z) \leq \lim \inf (I_{n_k}(x_{n_k}), n \in M)$$

(3.4)  $$\leq \lim \inf (i_{n_k} + t_{n_k}, n \in M) \leq \lim (i_{n_k} + t_{n_k}, n \in M) = i.$$

Thus $i = I(y)$, i.e., $y$ minimizes $I$ over $D$.

(The last equality in (3.4) is obtained as follows: $(i_{n_k}, n \in \mathbb{N}) \to i$ from part (a) and $(t_{n_k}, n \in \mathbb{N}) \to 0$.

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by hypothesis with $t_n$ real for every $n \in \mathbb{N}$, so

$i_n + t_n$ is defined in $\mathbb{R}^+$ for every $n \in \mathbb{N}$ and

$(i_n + t_n, n \in \mathbb{N}) \to i$; thus the subsequence

$(i_n + t_n, n \in M)$ converges to $i$, and therefore

$i = \lim \inf (i_n + t_n, n \in M)$ by Theorem 2.6.) For

the special case in which $x^*_n \in D_n$ with $i_n = I_n(x^*_n)$,

take $t_n = \frac{1}{n}$.

Q.E.D.

Theorem 3.8

(a) Property $(m^*)$, Property $(I^*)$ and Property
(D+) are sufficient for the convergence of $(i_b^*, b \in B)$
to $i^*$.

(b) Suppose Property $(m^*)$, Property $(I^*)$ and
Property $(D+)$ hold, and assume $x^*_b \in D_b$ with

$i_b^* = I_b(x^*_b)$ for every $b \in B$. Then a cluster point

$x^*$ of $(x_b^*, b \in B)$ lies in $D$ and maximizes $I$

over $D$.

Proof:

Analogous to the proof of Theorem 3.4.
Corollary 3.9

Suppose \( I_b \) extends \( I \) for every \( b \in B \). Then Property \((I^*)\) and Property \((D+)\) are sufficient for the numerical net \((i^*_b, b \in B)\) to approximate \( i^* \), in which case every net \((x^*_b, b \in B)\) with \( x^*_b \in D_b \) and \( i^*_b = I_b(x^*_b) \) for every \( b \in B \) has a cluster point \( x^* \) in \( D \) and \( x^* \) maximizes \( I \) over \( D \).

Proof:

It follows by Lemma 3.3 that Property \((m^*)\) holds. Therefore Theorem 3.8 applies.

Q.E.D.

Corollary 3.10

Let \( I_b \) extend \( I \) for every \( b \in B \), and suppose that \( I_b|_{(D_b \cap D_c)} = I_c|_{(D_b \cap D_c)} \) for every \( b, c \in B \). Then Property \((D+)\) and the upper semi-continuity on \( D \) of the functional \( g = \cup (I_b: b \in B) \) are sufficient for the net \((i^*_b, b \in B)\) to approximate \( i^* \) and, in this case, every net \((x^*_b, b \in B)\) with \( x^*_b \in D_b \) and \( i^*_b = I_b(x^*_b) \) for every \( b \in B \) has a cluster point \( x^* \in D \) maximizing \( I \) over \( D \).

Proof:

Analogous to the proof of Corollary 3.6.
Corollary 3.11

Let $X$ be a Hausdorff $L$-space, $B = \mathbb{N}$, and suppose Property (m*), Property (D+) and Property (I*) hold. Assume that $(t_n, n \in \mathbb{N})$ is a null sequence of positive real numbers and, for every $n \in \mathbb{N}$, let $x_n \in D_n$ be such that $i_n^* < I_n(x_n) + t_n$. Then

(a) the numerical sequence $(i_n^*, n \in \mathbb{N})$ approximates $i^*$,

(b) the sequence $(x_n, n \in \mathbb{N})$ has a limit point in $D$,

(c) every limit point of $(x_n, n \in \mathbb{N})$ lies in $D$ and maximizes $I$ over $D$. (In particular if $x_n^* \in D_n$ maximizes $I_n$ over $D_n$ for every $n \in \mathbb{N}$, then $(x_n^*, n \in \mathbb{N})$ has a limit point in $D$ and every limit point of $(x_n^*, n \in \mathbb{N})$ lies in $D$ and maximizes $I$ over $D$.)

Proof:

Analogous to the proof of Corollary 3.7.
Theorem 3.12

Let $I_b$ extend $I$ for every $b \in B$, and suppose $D_b \subset D_a$ and $I_a|D_b = I_b$ whenever $a \leq b$ in $B$. If Property (D) holds and if there is $b_o \in B$ such that $I_{b_o}$ is l.s.c. on $D$, then the net $(i_b, b \in B)$ approximates $i$ and each net $(x^*_b, b \in B)$ with $x^*_b \in D_b$, $i_b = I_b(x^*_b)$ for every $b \in B$ has a cluster point $x^*$ in $D$ and $x^*$ minimizes $I$ over $D$.

Proof:

If $a \leq b$ in $B$, it follows by the hypothesis that $I_a[D_a] \supset I_b[D_b] \supset I[D]$, so that $i \geq i_b \geq i_a$ in $\mathbb{R}^+$. Thus $(i_b, b \in B)$ is an increasing numerical net, and therefore $\lim(i_b, b \in B) = \sup\{i_b: b \in B\} \leq i$ exists by Corollary 2.7. Let us assume that $\lim(i_b, b \in B) < i$, and consider some $t \in \mathbb{R}^+$ with $\lim(i_b, b \in B) < t < i$. Then it is true that $\lim(i_b, b \in B) = \sup\{i_b: b \in B\} < t$, so $i_b < t$ for every $b \in B$, so we can find $x_b \in D_b$ with $I_b(x_b) < t$ for every $b \in B$. Applying Property (D), we obtain a cofinal subset $G$ of $B$ and a point $y \in D$ with $(x_b, b \in G) \to y$. Since $b_o \in B$ and $G$ is cofinal in $B$, $\exists a_o \in G$ such that $a_o \geq b_o$. Then $H = \{b \in G: b \geq a_o\}$ is a cofinal subset of $G$, $(x_b, b \in H) \to y$ and $x_b \in D_{b_o}$ for every $b \in H$. 

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so that \( I_{b_o}(x_b) = I_{a_o}(x_b) = I_b(x_b) < t \) for every \( b \in H \), by using the hypothesis. Thus

\[
(3.5) \quad \liminf(I_{b_o}(x_b), b \in H) \leq t
\]

by Theorem 2.3(a), (c). However since \( I_{b_o} \) is l.s.c. on \( D \) and \( y \in D \) and \( (x_b, b \in H) \to y \) with \( x_b \in D_{b_o} = \) domain of \( I_{b_o} \), it follows that

\[
(3.6) \quad I_{b_o}(y) \leq \liminf(I_{b_o}(x_b), b \in H).
\]

Combining (3.5), (3.6), the fact that \( t < i \), and \( I_{b_o}|D = I \) and \( y \in D \), we get \( I(y) < i \), which contradicts the definition of \( i \). Thus \( \lim(i_b, b \in B) \geq i \). Since we already had \( \lim(i_b, b \in B) \leq i \), we conclude that

\[
(3.7) \quad \lim(i_b, b \in B) = i.
\]

Finally let us consider a net \((x^*_b, b \in B)\) such that \( x^*_b \in D_{b} \) and \( i_b = I_{b}(x^*_b) \) for every \( b \in B \). We can apply Property (D) to obtain a point \( x^* \in D \) and a cofinal subset \( G \) of \( B \) with \((x^*_b, b \in G) \to x^*\).

Therefore \( H = \{b \in G: b > b_o\} \) is cofinal in \( G \) (and in \( B \)), and \( (x^*_b, b \in H) \to x^* \) with \( x^*_b \in D_{b_o} \) for each \( b \in H \), and \( I_{b_o}(x^*_b) = I_{b}(x^*_b) = i_b \).
for each \( b \in H \). It follows from (3.7) that
\[
\lim \inf (I_{b_0}(x^*), b \in H) = \lim \inf (i_b, b \in H) = i
\]
by an application of Theorem 2.6. Consequently
\[
i \leq I(x^*), \text{ since } x^* \in D \text{ and } i = \inf I[D];
\]
\[
= I_{b_0}(x^*), \text{ since } I_{b_0} \text{ extends } I;
\]
\[
\leq \lim \inf (I_{b_0}(x^*), b \in H), \text{ since } I_{b_0} \text{ is l.s.c. on } D;
\]
\[
= i, \text{ as shown above; so } i = I(x^*).
\]
Obviously \( x^* \) is a cluster point of \( (x^*_b, b \in B) \) since
\( x^* \) is a limit of the subnet \( (x^*_b, b \in G) \) of \( (x^*_b, b \in B) \).
Q.E.D.

**Theorem 3.13**

Let \( I_b \) extend \( I \) for every \( b \in B \), and suppose that \( D_b \subset D_a \) with \( I_a|_{D_b} = I_b \) whenever \( a \leq b \) in \( B \). If Property (D) holds and \( \exists b_0 \in B \) such that \( I_{b_0} \) is u.s.c. on \( D \), then the net \( (i^*_b, b \in B) \) approximates \( i^* \) and every net \( (x^*_b, b \in B) \) with \( x^*_b \in D_b \) and
\[
i^*_b = I_b(x^*_b)
\]
for each \( b \in B \) has a cluster point \( x^* \) in \( D \) which maximizes \( I \) over \( D \).

**Proof:**

The proof is essentially dual to the proof of
**Theorem 3.12.**
IV. A STUDY OF PROPERTY (D+)

This chapter is devoted to a comparison between Property (D+) and several notions of convergence in the power set exp(X). We mainly intend to provide sufficient or necessary conditions for Property (D+) in terms of the well-established forms of convergence in exp(X). We assume that X is a topological space, and we will deal exclusively with sequences, rather than nets, in exp(X).

A Convergence Structure in exp(X)

Lemma 4.1

Let D be a subset of a topological space X and \( (D_n, n \in \mathbb{N}) \) a sequence of subsets of X. Then

(i) Property (D+) implies Property (D); the converse is generally false.

(ii) If \( D_{n+1} \subseteq D_n \) for almost all \( n \in \mathbb{N} \), then Property (D+) and Property (D) are equivalent.

(iii) If \( D \subseteq D_n \) for almost all \( n \in \mathbb{N} \), then Property (D) implies that the set D is sequentially compact.
(iv) If $X$ is Hausdorff, then Property (D) implies that $D \supset \bigcap_{n=1}^{\infty} D_n$.

(v) If $X$ is Hausdorff and $D \subset D_n$ for each $n \in \mathbb{N}$, then Property (D) implies that $D = \bigcap_{n=1}^{\infty} D_n$.

Proof:

(i) That Property (D+) implies Property (D) is clear. In order to see that the converse need not hold, take $X$ = the real line with $D = [1, 2]$, $D_{2n} = [1, 2 + \frac{1}{2n}]$ and $D_{2n-1} = [-\frac{1}{2n-1}, 2]$. Let $E$ denote the set of even natural numbers and $O$ the set of odd natural numbers. If $x_n \in D_n$ for every $n \in \mathbb{N}$, then the subsequence $(x_n, n \in E)$ is bounded, so has a convergent subsequence $(x_n, n \in M)$. Since $1 \leq x_n \leq 2 + \frac{1}{n}$ for every $n \in M$, it follows that $1 \leq \lim (x_n, n \in M) \leq \lim (2 + \frac{1}{n}, n \in M) = 2$, so that $\lim (x_n, n \in M) \in [1, 2] = D$. Thus Property (D) is satisfied. On the other hand for every $n \in O$, $y_n = -\frac{1}{n} \in D_n$ and $\lim (y_n, n \in O) = 0$. Therefore each subsequence of $(y_n, n \in O)$ converges only to 0 with 0 $\notin D$. It follows that Property (D+) fails.
(ii) Let \( N \) be a positive integer such that
\[
D_{n+1} \subseteq D_n \quad \text{for each } n \geq N,
\]
and let us suppose that Property (D) is satisfied. Consider a subsequence
\((D_n, n \in M_1)\) with a sequence of points \((x_n, n \in M_1)\)
for which \(x_n \in D_n\) for every \(n \in M_1\). Define
\[M = \{n \in M_1: n \geq N\}.\]
We shall show that a subsequence of \((x_n, n \in M)\) converges to a point of \(D\). We begin
by ordering the set \(M\) linearly in the usual manner,
\[M = \{m_t: t \in N\}\]
in such a way that \(m_t < m_{t+1}\) for every \(t \in N\). Fix an element \(p \in X\) and define \(y_t = p\)
for \(t < m_1, t \in N\). Put \(y_t = x_t\) for \(t \in M\). If
\(n \in N \setminus M\) and \(n > m_1\), there is a unique \(t \in N\) with
\(m_t < n < m_{t+1}\), so that \(D_{m_t} \subseteq D_n \subseteq D_{m_{t+1}}\); define
\[y_n = x_{m_t} \in D_{m_t} \subseteq D_n.\]
We now have the sequence
\[(y_n, n \in N),\]
of which \((x_n, n \in M)\) is a subsequence, and for which \(y_n \in D_n\) for \(n \geq N\). It follows from
Property (D) that a subsequence \((y_n, n \in L)\) of
\[(y_n, n \in N),\]
converges to a point \(y \in D\). If \(M \cap L\)
is infinite, then the subsequence \((x_n, n \in M \cap L)\)
of \((x_n, n \in M)\) converges to \(y\). Should \(M \cap L\) be
finite, pick an integer \(b\) greater than \(m_1\) and
greater than each element of \(M \cap L\). The set
\[K = \{t \in L: t > b\}\]
is then an infinite subset of \(L \cap (N \setminus M)\), and \((y_n, n \in K)\) converges to \(y\). The
function \( f: K \to M \) defined by \( f(n) = m_{t+1} \), where \( t \in N \) is the unique integer satisfying the inequality \( m_t < n < m_{t+1} \), has the property: \( f(n) < f(t) \) if and only if \( n < t \). Thus \( f[K] \) is an infinite subset of \( M \) and \( (x_n, n \in f[K]) \) is a subsequence of \( (x_n, n \in M) \). We now show that \( (x_n, n \in f[K]) \to y \) as follows. Let a neighborhood \( V \) of \( y \) be given. Since \( (y_n, n \in K) \to y \), there is a positive integer \( T \in K \) such that \( n \in K \) together with \( n > T \) imply \( y_n \in V \). If \( s \in f[K] \) and \( s > f(T) \), there is a unique \( n \in K \) with \( s = f(n) > f(T) \), so that \( n > T \) in \( K \) and \( s = f(n) = m_{t+1} \) according to the above definition of \( f \). But then, from the construction of the sequence \( (y_n, n \in N) \), we obtain \( x_s = x_{m_{t+1}} = y_n \), and \( y_n \in V \) since \( n \in K \), \( n > T \). Hence \( x_s \in V \) for \( s \in f[K], s > f(T) \). The claim that \( (x_n, n \in f[K]) \) converges to \( y \) is therefore proved, and Property \((D+)\) holds. From part (i) of this lemma we know that Property \((D+)\) implies Property \((D)\). Hence the proof of (ii) is completed.

(iii) Assume that Property \((D)\) holds and that \( D \subset D_n \) for almost all \( n \in N \). If \( (x_n, n \in N) \) is a sequence in \( D \), then \( x_n \in D_n \) for almost all \( n \in N \); so, by Property \((D)\), a subsequence of \( (x_n, n \in N) \) converges to a point of \( D \).
(iv) Let us suppose that Property (D) holds and the space $X$ is Hausdorff. If $x \in \bigcap_{n=1}^{\infty} D_n$, define $x_n = x$ for every $n \in \mathbb{N}$, so that $(x_n, n \in \mathbb{N}) \to x$. By Property (D) a subsequence $(x_{n'}, n' \in M)$ of $(x_n, n \in \mathbb{N})$ converges to a point $y$ of $D$. Since the space is Hausdorff, we must have $x = y$; so $x \in D$ because $y \in D$.

(v) Let Property (D) be satisfied with $X$ Hausdorff and $D \subseteq D_n$ for every $n \in \mathbb{N}$. The condition $D \subseteq D_n$ for every $n \in \mathbb{N}$ implies $D \subseteq \bigcap_{n=1}^{\infty} D_n$, and from (iv) above we know that $D \supseteq \bigcap_{n=1}^{\infty} D_n$.

Q.E.D.

**Definition 4.2**

Let $X$ be a topological space.

(a) $\exp(X)$ will denote the class of all subsets of $X$, $\exp_0(X)$ will denote the class of all non-empty subsets of $X$, $\mathcal{S}_c(X)$ will denote the class of all non-empty, sequentially compact subsets of $X$.

(b) If $D \in \exp(X)$ and $(D_n, n \in \mathbb{N})$ is a sequence in $\exp_0(X)$, we shall write $(D_n, n \in \mathbb{N}) \to D$ to mean that Property (D+) is satisfied.
Lemma 4.3
Under the "convergence" ♦ of Definition 4.2, $\exp(X)$ satisfies axioms (L'2) and (L'3) but not necessarily axiom (L'1) (see Definitions 2.14, 2.15).

Proof:

(L'2): Let $(D_n, n \in \mathbb{N})$ ♦ D in $\exp(X)$, and suppose that $(D_n, n \in M)$ is a subsequence of $(D_n, n \in \mathbb{N})$. Given any subsequence $(D'_n, n \in L)$ of $(D_n, n \in M)$ and any sequence of points $(x^*_n, n \in L)$ for which $x^*_n \in D_n$ for every $n \in L$, it is true that $(D'_n, n \in L)$ is a subsequence of $(D'_n, n \in \mathbb{N})$ and $x^*_n \in D_n$ for each $n \in L$; therefore Property (D+), which holds because $(D_n, n \in \mathbb{N})$ ♦ D, implies that a subsequence $(x^*_n, n \in T)$ of $(x^*_n, n \in L)$ converges in $X$ to a point $y$ of $D$. Therefore $(D'_n, n \in M)$ ♦ D.

(L'3): Consider a sequence $(D_n, n \in \mathbb{N})$ in $\exp(X)$ and a set $D \in \exp(X)$ such that the condition "$(D_n, n \in \mathbb{N})$ ♦ D" fails. Therefore there exists an infinite set $M$ of natural numbers and a sequence $(x^*_n, n \in M)$ with $x^*_n \in D_n$ for every $n \in M$, but each subsequence of $(x^*_n, n \in M)$ fails to converge to a point of $D$. We shall show that the condition "$(D'_n, n \in L)$ ♦ D" fails for every subsequence $(D'_n, n \in L)$ of $(D_n, n \in M)$. Indeed let $(D'_n, n \in L)$
be a subsequence of \((D_n, n \in M)\). Then \(L\) is an infinite subset of \(M\), and so \((x_n', n \in L)\) is a subsequence of \((x_n', n \in M)\). Since no subsequence of \((x_n', n \in M)\) converges to a point of \(D\), and since every subsequence of \((x_n', n \in M)\) is a subsequence of \((x_n', n \in L)\), it follows that no subsequence of \((x_n', n \in L)\) converges to a point of \(D\). Thus Property \((D+)\) fails for the set \(D\) and the sequence \((D_n, n \in L)\). Hence the condition "\((D_n, n \in L) \not\rightarrow D\)" is not satisfied. In order to see that axiom (L') will generally fail here, it is sufficient to find a space \(X\) with a non-empty subset \(D\) which is not sequentially compact, so that there is a sequence \((x_n, n \in N)\) in \(D\) no subsequence of which converges to a point of \(D\).

If we define \(D_n = D\) for every \(n \in N\), the existence of the above sequence \((x_n, n \in N)\) means that condition "\((D_n, n \in N) \not\rightarrow D\)" fails.

Q.E.D.

Apart from trivial cases, every convergence form defined in \(\exp(X)\) using the topology for \(X\) behaves rather wildly unless restricted to an appropriate subclass of \(\exp(X)\), as attested by the works of F. Hausdorff [16], E. Michael [20], S. Mrowka [21], Z. Frolik [14], and F.A. Chimenti [8]. In fact, for deeper studies of
such convergence forms in \( \exp(X) \), further restrictions are usually placed on the space \( X \) itself.

**Theorem 4.4**

Let \( X \) be a topological space. Under the convergence of Definition 4.2, \( \text{Sc}(X) \) is a sequential \( L^* \)-space with the following properties:

(i) If a sequence \( (x_n, \ n \in \mathbb{N}) \) converges to \( x \) in \( X \), then \( ([x], \ n \in \mathbb{N}) \overset{*}{\rightarrow} [x] \) in \( \text{Sc}(X) \).

(ii) The function \( f: X \to \text{Sc}(X) \) defined by \( f(x) = [x] \) is continuous.

(iii) If \( \text{card} \ X \geq 2 \), the space \( \text{Sc}(X) \) is not Hausdorff.

**Proof:**

From Lemma 4.3 we know that the axioms \( (L'1) \) and \( (L'3) \) are satisfied. To see that also the axiom \( (L'1) \) is satisfied in \( \text{Sc}(X) \), consider a constant sequence \( (D_n, \ n \in \mathbb{N}) \) in \( \text{Sc}(X) \), \( D_n = D \in \text{Sc}(X) \) for each \( n \in \mathbb{N} \). If \( M \) is an infinite set of natural numbers and \( (x_n, \ n \in M) \) a sequence of points with \( x_n \in D_n \) for every \( n \in M \), then \( (x_n, \ n \in M) \) is a sequence in the sequentially compact set \( D \), so it has a subsequence converging to a point of \( D \). Thus Property \( (D+) \) holds, so \( (D_n, \ n \in \mathbb{N}) \overset{*}{\rightarrow} D \).
(i) Since \((x_n, n \in \mathbb{N})\) converges to \(x\) in \(X\), then every subsequence of \((x_n, n \in \mathbb{N})\) converges to \(x\) in \(X\).

(ii) This is an immediate consequence of (i) and the definition of continuity for a function between two \(L^\ast\)-spaces.

(iii) Find \(x \in X\) and \(y \in X\) with \(x \neq y\). Put \(D = \{x\}\) and \(F = \{x, y\}\). Then \(D \in \text{Sc}(X),\ F \in \text{Sc}(X)\) and \(D \neq F\). In \(\text{Sc}(X)\), the constant sequence \(D_n = D\) converges to both \(D\) and \(F\).

Q.E.D.

We have not been able to determine whether axiom (L'4) holds in \(\text{Sc}(X)\), except for the trivial case where \(X\) is the indiscrete space.

Comparison with Hausdorff Convergence

Definition 4.5

Consider a metric space \((X, d)\). For \(A \in \exp_o(X)\) and \(G \in \exp_o(X)\) put
\[ h(A, G) = \max \{\sup\{d(x, G): x \in A\}, \sup\{d(y, A): y \in G\}\}. \]

The function \(h\) of Definition 3.5 was first introduced, with a restricted domain, by F. Hausdorff [16]. This function has been also studied by several people,
among them C. Berge [3]. When the domain of \( h \) is enlarged as we have done in Definition 3.5, some of the metric properties of \( h \) are lost, but enough of them are left for our purposes.

**Theorem 4.6**

The function \( h \) of Definition 4.5 has the following properties:

1. **(HD0)**: If \( A, G \) are bounded, then \( 0 \leq h(A, G) < +\infty \).

2. **(HD1)**: If \( A, G \in \exp_{o}(X) \), then \( h(A, G) = 0 \) if and only if \( A \subseteq \overline{G} \) and \( G \subseteq \overline{A} \). In particular, \( h(A, A) = h(A, \overline{A}) = h(\overline{A}, A) = 0 \) for each \( A \in \exp_{o}(X) \).

3. **(HD2)**: \( h \) is a generalized pseudo-metric for \( \exp_{o}(X) \).

4. **(HD3)**: \( h(A, G) = h(\overline{A}, \overline{G}) \).

5. **(HD4)**: If \( \emptyset \neq G \subseteq \overline{A} \subseteq X \), then \( h(A, G) = \sup \{ d(x, G) : x \in A \} \).

6. **(HD5)**: If \( A, G \in \exp_{o}(X) \) are both closed, then \( h(A, G) = 0 \) if and only if \( A = G \).

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Proof:

(HD0): It is obvious from the definition of \( h \) that 
\[ 0 \leq h(A, G) \leq +\infty \] for any \( A, G \in \exp_0(X) \). If \( A \) and \( G \) are also bounded, then \( \{d(x, G): x \in A\} \) and \( \{d(y, A): y \in G\} \) are bounded, non-void subsets of real numbers, and so \( \sup\{d(x, G): x \in A\} < +\infty \) and \( \sup\{d(y, A): y \in G\} < +\infty \), whence \( h(A, G) < +\infty \).

(HD1): \( h(A, G) = 0 \)

\[ \Rightarrow \max\{\sup\{d(x, G): x \in A\}, \sup\{d(y, A): y \in G\}\} = 0, \]

\[ \Rightarrow 0 = \sup\{d(x, G): x \in A\} = \sup\{d(y, A): y \in G\}, \]

\[ \Rightarrow d(x, G) = 0 \quad \text{for each} \quad x \in A \quad \text{and} \quad d(y, A) = 0 \quad \text{for each} \quad y \in G, \]

\[ \Rightarrow x \in \overline{G} \quad \text{for each} \quad x \in A \quad \text{and} \quad y \in \overline{A} \quad \text{for each} \quad y \in G, \]

\[ \Rightarrow A \subseteq \overline{G} \quad \text{and} \quad G \subseteq \overline{A}. \]

(HD2): As we already indicated under (HD0), we know that \( 0 \leq h(A, G) \leq +\infty \) for \( A, G \in \exp_0(X) \). From (HD1) we obtain \( h(A, A) = 0 \) for any \( A \in \exp_0(X) \). Now

\[ h(A, G) = \max\{\sup\{d(x, G): x \in A\}, \sup\{d(y, A): y \in G\}\} = \max\{\sup\{d(y, A): y \in G\}, \sup\{d(x, G): x \in A\}\} = h(G, A). \]

To complete the proof of (HD2), it only remains to
establish the triangle inequality. So let \( A, G, E \in \exp_\circ(X) \). If \( h(A, G) = +\infty \) or \( h(G, E) = +\infty \), the inequality \( h(A, E) \leq h(A, G) + h(G, E) \) is clear. We shall thus assume \( h(A, G) < +\infty \) and \( h(G, E) < +\infty \).

Fix \( x \in A \). Take any real \( \varepsilon > 0 \). Because \( d(x, G) = \inf\{d(x, y) : y \in G\} \), we may find \( u \in G \) with

\[
(4.1) \quad d(x, G) > d(x, u) - \frac{\varepsilon}{2}.
\]

Similarly, there is \( v \in E \) with

\[
d(u, E) > d(u, v) - \frac{\varepsilon}{2}.
\]

Thus \( d(u, v) - \frac{\varepsilon}{2} < d(u, E) \)

\[
\leq \sup\{d(y, E) : y \in G\}
\leq h(G, E),
\]

i.e.,

\[
(4.2) \quad d(u, v) - \frac{\varepsilon}{2} < h(G, E).
\]

By (4.1) and (4.2),

\[
d(x, G) + h(G, E) > d(x, u) - \frac{\varepsilon}{2} + d(u, v) - \frac{\varepsilon}{2}
\]

\[
= [d(x, u) + d(u, v)] - \varepsilon
\]

\[
\geq d(x, v) - \varepsilon
\]

\[
\geq d(x, E) - \varepsilon \quad \text{since} \quad v \in E; \quad \text{so that}
\]

\[
d(x, G) + h(G, E) > d(x, E) - \varepsilon.
\]
Since this inequality holds for each real \( \varepsilon > 0 \), it follows that

\[(4.3) \quad d(x, G) + h(G, E) \geq d(x, E).\]

Because \( x \) was an arbitrary element of \( A \), we may take the supremum over \( A \) in (4.3) to obtain

\[(4.4) \quad \sup\{d(x, G) : x \in A\} + h(G, E) \geq \sup\{d(x, E) : x \in A\}.\]

The definition of \( h \) is such that \( h(A, G) \leq \sup\{d(x, G) : x \in A\} \), so we can combine this inequality with (4.4) to get

\[(4.5) \quad h(A, G) + h(G, E) \leq \sup\{d(x, E) : x \in A\}.\]

We observe that relation (4.5) holds for any three sets \( A, G, E \), taken in that order. Therefore, taking these three sets in the order \( E, G, A \) and using (4.5), we obtain

\[(4.6) \quad h(A, G) + h(G, E) \leq \sup\{d(y, A) : y \in E\},\]

or, since we showed that \( h \) is symmetric,

\[(4.6) \quad h(A, G) + h(G, E) \leq \sup\{d(y, A) : y \in E\}.\]

It follows at once from (4.5), (4.6) and the definition of \( h \) that \( h(A, G) + h(G, E) \geq h(A, E) \), which completes the proof of (HD2).
(HD3): \( h(A, G) \leq h(A, \overline{A}) + h(\overline{A}, G) + h(G, G) = h(\overline{A}, G) \)
by (HD1), so \( h(A, G) \leq h(\overline{A}, G) \). Similarly we have
\( h(\overline{A}, G) \leq h(\overline{A}, G) + h(G, A) + h(A, \overline{A}) = h(G, A) \). It
follows that \( h(A, G) \leq h(\overline{A}, G) \) and \( h(\overline{A}, G) \leq h(A, G) \),
so that \( h(A, G) = h(\overline{A}, G) \) as desired.

(HD4): If \( y \in G \), then \( y \in \overline{A} \), so \( 0 = d(y, A) \).
Therefore \( \sup\{d(y, A) : y \in G\} = 0 \). Because
\( \sup\{d(x, G) : x \in A\} \geq 0 \), it now follows from the defi-
nition of \( h \) that \( h(A, G) = \sup\{d(x, G) : x \in A\} \).

(HD5): \( h(A, G) = 0 \)
\( \Rightarrow A \subseteq \overline{G} \) and \( G \subseteq \overline{A} \) by (HD1),
\( \Rightarrow A \subseteq G \) and \( G \subseteq A \) because \( A \) and \( G \) are both
closed,
\( \Rightarrow A = G \).

Q.E.D.

Since \( h \) is nothing more than the extension of
the Hausdorff metric, we shall call \( h \) the Hausdorff
distance. As a generalized pseudo-metric on \( \exp_\circ(X) \)
according to (HD2), \( h \) can be used to generate a topology
for \( \exp_\circ(X) \), and hence a convergence structure for
\( \exp_\circ(X) \) is obtained. We proceed directly to the defini-
tion of this convergence, as it is what we need, rather
than the topology it generates.
Definition 4.7

Let \((X, d)\) be a metric space, \(D \in \exp_0(X)\) and \((D_n, n \in \mathbb{N})\) a sequence in \(\exp_0(X)\). We shall say that \((D_n, n \in \mathbb{N})\) converges to \(D\) under the Hausdorff distance if the sequence of extended real numbers \((h(D_n, D), n \in \mathbb{N})\) converges to zero.

Of course, when this convergence does occur, it follows that \(h(D_n, D) < +\infty\) for \(n\) sufficiently large. This, however, does not necessarily mean that \(D\) is bounded or that \(D_n\) is bounded for \(n\) sufficiently large, as can be seen by taking \(X\) as the set of real numbers, \(d\) as the usual metric, \(D = \mathbb{R}^+ \cup \{0\}\) and \(D_n = \mathbb{R}^+, \frac{1}{n}\}.\) Here, \(h(D_n, D) = \frac{1}{n} \to 0\), but all the sets are unbounded. Now should this convergence take place and should \(D\) be bounded, the following lemma shows that \(D_n\) is then bounded for \(n\) sufficiently large.

Lemma 4.8

Let \((X, d)\) be a metric space, \(D\) a bounded member of \(\exp_0(X)\) and \((D_n, n \in \mathbb{N})\) a sequence in \(\exp_0(X)\). If the sequence \((\sup\{d(x, D) : x \in D_n\}, n \in \mathbb{N})\) is bounded for almost all \(n \in \mathbb{N}\) (in particular, if \((D_n, n \in \mathbb{N})\) converges to \(D\) under the Hausdorff distance), then there is an \(N\) such that \(\bigcup_{n \geq N} D_n\) is bounded.
Proof:

There is a real $b > 0$ and an $N$ for which
\[ \sup \{d(x, D) : x \in D_n \} < b \text{ for } n \geq N. \]
Also there is a real $c > 0$ with $d(x, y) \leq c$ if $x, y \in D$.

Put $a = 2b + c$. If $x, z \in \bigcup_{n \geq N} D_n$, then $x \in D_m$
and $z \in D_l$ for some $m, l \geq N$. Now
\[ d(x, D) \leq \sup \{d(y, D) : y \in D_m \} \]
\[ < b \text{ since } m \geq N; \]
and similarly $d(z, D) < b$. Thus there exists $y \in D$
with $d(x, y) < b$ and there exists $u \in D$ with
\[ d(z, u) < b. \]
It follows that
\[ d(x, z) \leq d(x, y) + d(y, u) + d(z, u) \]
\[ < b + c + b \]
\[ = a. \]

Thus \( \bigcup_{n \geq N} D_n \) is bounded by the real number $a$.

Q.E.D.

Theorem 4.9

Let $X$ be a metric space, $D$ a compact member
of $\exp_0(X)$ and $(D_n, n \in \mathbb{N})$ a sequence in $\exp_0(X)$.
The following are equivalent:

(i) Property $(D+)$

(ii) $(\sup \{d(x, D) : x \in D_n \}, n \in \mathbb{N}) \to 0$. 
Proof:

(ii) implies (i): Consider an infinite set $M$ of natural numbers and a sequence $(x_n, n \in M)$ with $x_n \in D_n$ for each $n \in M$. For each $n \in M$, the compactness of $D$ assures us that there exists $y_n \in D$ with $d(x_n, y_n) = d(x_n, D)$. The sequence $(y_n, n \in M)$, being in the compact set $D$, must have a subsequence $(y_n, n \in L)$ converging to some point $z \in D$. For $n \in L$ we have

\[
0 \leq d(x_n, z) \leq d(x_n, y_n) + d(y_n, z) = d(x_n, D) + d(y_n, z) \leq \sup\{d(x, D) : x \in D_n}\right) + d(y_n, z),
\]

since $x_n \in D_n$.

By (ii) we know that $(\sup\{d(x, D) : x \in D_n\}, n \in L) \to 0$. Since $(y_n, n \in L) \to z$, it follows from the inequalities in (4.7) that $(x_n, n \in L) \to z$. Thus we have extracted from $(x_n, n \in M)$ a subsequence converging to a point of $D$. Therefore Property (D+) holds.

(i) implies (ii): Assume (i) and suppose (ii) fails. Take a real $t > 0$ and an infinite set $M$ of natural numbers such that $\sup\{d(x, D) : x \in D_n\} > t$ for every $n \in M$. Therefore given any $n \in M$ we may
find \( x_n \in D_n \) satisfying \( d(x_n, D) > t \). If \( (x_n, n \in L) \) is any subsequence of \( (x_n, n \in M) \) and \( z \in D \), then we have \( d(x_n, z) \geq d(x_n, D) > t \) for any \( n \in L \), because \( L \subset M \);

whence \( (d(x_n, z), n \in L) \) is bounded below by the positive constant \( t \), so cannot converge to zero. Therefore \( (x_n, n \in L) \) does not converge to \( z \). In other words, no subsequence of \( (x_n, n \in M) \) converges to a point of \( D \). This contradicts (i) since \( x_n \in D_n \) for each \( n \in M \).

Q.E.D.

Corollary 4.10

Let \((X, d)\) be a metric space, \(D\) a compact member of \(\exp_0(X)\) and \( (D_n, n \in N) \) a sequence in \(\exp_0(X)\).

If the sequence \( (D_n, n \in N) \) converges to \(D\) under the Hausdorff distance, then Property \((D^+)\) is satisfied.

Proof:

We have \( 0 \leq \sup\{d(x, D): x \in D_n\} \leq h(D_n, D) \) for each \( n \in N \), and \( (h(D_n, D), n \in N) \to 0 \) by assumption.

Therefore Theorem 4.8 applies.

Q.E.D.
Corollary 4.11

Let \((X, d)\) be a metric space, \(D\) a compact member of \(\exp_0(X)\) and \((D_n, n \in \mathbb{N})\) a sequence in \(\exp_0(X)\), \(D \subseteq D_n\) for almost all \(n \in \mathbb{N}\). Then Property \((D^+)\) is satisfied if and only if the sequence \((D_n, n \in \mathbb{N})\) converges to \(D\) under the Hausdorff distance.

Proof:

The sufficiency was proved in Corollary 4.10. To prove the necessity we may first apply Theorem 4.9 to get \(\sup\{d(x, D) : x \in D_n\} \to 0\). Next, because \(D \subseteq D_n\) for almost all \(n \in \mathbb{N}\), we use Theorem 3.6, part (\text{HD4}), to conclude that \(h(D_n, D) = \sup\{d(x, D) : x \in D_n\}\). It thus follows that \((D_n, n \in \mathbb{N})\) converges to \(D\) under the Hausdorff distance, as claimed.

Q.E.D.

Corollary 4.12

Given a metric space \((X, d)\), a compact member \(D\) of \(\exp_0(X)\) and a sequence \((D_n, n \in \mathbb{N})\) in \(\exp_0(X)\) such that \(D \subseteq D_{n+1} \subseteq D_n\) for almost all \(n \in \mathbb{N}\), a necessary and sufficient condition for Property \((D)\) is that the sequence \((D_n, n \in \mathbb{N})\) converges to \(D\) under the Hausdorff distance.
**Proof:**

Because \( D_{n+1} \subset D_n \) for almost all \( n \in \mathbb{N} \), Property (D) is equivalent to Property (D+) by part (ii) of Lemma 4.1. Since the assumption "\( D \subset D_n \) for almost all \( n \in \mathbb{N} \)" is also made, we may apply Corollary 4.11 above.

Q.E.D.

**Remark 4.13**

(i) In Corollary 4.10, the assumption that \( D \) be compact may not be dropped in general, as is shown by taking \( D = \]0, 1[ \) and \( D_n = [0, 1 + \frac{1}{n}] \) on the real line: here \( h(D_n, D) = \frac{1}{n} \), so that \( (D_n, n \in \mathbb{N}) \) converges to \( D \) under the Hausdorff distance; but Property (D+) fails since \( x_n = 1 + \frac{1}{n} \notin D_n \) for every \( n \in \mathbb{N} \), while \( (x_n, n \in \mathbb{N}) \) has no subsequence converging to a point of \( D \). This remark applies to Theorem 4.9 as well.

(ii) If the requirement that "\( D \subset D_n \) for almost all \( n \in \mathbb{N} \)" is dropped in Corollary 4.11, the conclusion may fail, as can be seen in the trivial example \( D = [0, 2], D_{2n} = [0, 2], D_{2n-1} = [0, 1] \) on the real line. Property (D+) certainly holds, for if \( M \) is an infinite set of natural numbers and \( x_n \in D_n \) for every \( n \in M \), then \( (x_n, n \in M) \) is a sequence in
the compact set \([0, 2] = D\), hence \((x_n, n \in M)\) must have a subsequence converging to a point of \([0, 2] = D\).

On the other hand we have \(D \subset D_n\) for each odd natural number \(n\), and so "\(D \subset D_n\) for almost all \(n \in N\)" is not met. We observe that \(h(D_{2n}, D) = 0\) while \(h(D_{2n-1}, D) = 1\), whence \(h(D_n, D)\) does not converge to 0 and thus \((D_n, n \in N)\) does not converge to \(D\) under the Hausdorff distance.

Comparison with Topological Convergence

**Definition 4.14**

Consider a topological space \(X\) and a sequence \((D_n, n \in N)\) in \(\exp(X)\). We define

(i) \(L X(D_n, n \in N) = \{x \in X: \text{if } V \text{ is a nbd of } x \text{ in } X, \text{ there is an } N \text{ such that } n > N \Rightarrow V \cap D_n \neq \emptyset\}\) and

(ii) \(L s X(D_n, n \in N) = \{x \in X: \text{if } V \text{ is a nbd of } x \text{ in } X, \text{ there is an infinite subset } M \text{ of } N \text{ with } V \cap D_n \neq \emptyset \text{ for each } n \in M\}\).

We shall say that \((D_n, n \in N)\) converges topologically to \(D \in \exp(X)\) and write \(D = \lim X(D_n, n \in N)\),

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if it is true that \( D = \liminf_X(D_n, n \in \mathbb{N}) = \limsup_X(D_n, n \in \mathbb{N}). \)

\( \liminf_X(D_n, n \in \mathbb{N}) \) is called the topological limit inferior of \( (D_n, n \in \mathbb{N}) \);

\( \limsup_X(D_n, n \in \mathbb{N}) \) is called the topological limit superior of \( (D_n, n \in \mathbb{N}) \);

\( \lim_X(D_n, n \in \mathbb{N}) \) is called the topological limit of \( (D_n, n \in \mathbb{N}). \)

Topological limits of sequences in \( \exp(X) \) were introduced as early as 1905. A generalization to nets was made in 1937 by Garrett Birkhoff [4]. For a historical survey, see the dissertation of F. Chimenti [8].

Proofs of the following six well-known properties of topological limits may be found in Z. Frolik [14], C. Berge [3], and S. Mrowka [21].

(i) \( \liminf_X(D_n, n \in \mathbb{N}) \subseteq \limsup_X(D_n, n \in \mathbb{N}); \)

(ii) \( \liminf_X(D_n, n \in \mathbb{N}) \) and \( \limsup_X(D_n, n \in \mathbb{N}) \) are closed subsets of \( X; \)

(iii) If \( D_n \subseteq A_n \) for almost all \( n \in \mathbb{N}, \) then \( \liminf_X(D_n, n \in \mathbb{N}) \subseteq \liminf_X(A_n, n \in \mathbb{N}) \) and \( \limsup_X(D_n, n \in \mathbb{N}) \subseteq \limsup_X(A_n, n \in \mathbb{N}); \)
(iv) If $D_n = D \in \exp(X)$ for almost all $n \in \mathbb{N}$, then $D = \lim_{n \to \infty} D_n$.

(v) $\text{Li}_X(D_n, n \in \mathbb{N}) = \text{Li}_X(D_n, n \in \mathbb{N})$ and $\text{Ls}_X(D_n, n \in \mathbb{N}) = \text{Ls}_X(D_n, n \in \mathbb{N})$;

(vi) If $(D_n, n \in \mathbb{N})$ is contracting, then $\bigcap_{n=1}^{\infty} D_n = \lim_{n \to \infty} D_n$.

**Lemma 4.15**

Let $X$ be a first countable topological space and $(D_n, n \in \mathbb{N})$ a sequence in $\exp(X)$. Then

(a) $x \in \text{Li}_X(D_n, n \in \mathbb{N})$ if and only if there is a sequence $(x_n, n \in \mathbb{N})$ converging to $x$ and such that $x_n \in D_n$ for almost all $n \in \mathbb{N}$;

(b) $x \in \text{Ls}_X(D_n, n \in \mathbb{N})$ if and only if there is an infinite set $M$ of natural numbers and a sequence $(x_n, n \in M)$ converging to $x$ such that $x_n \in D_n$ for every $n \in M$.

**Proof:**

(a) Sufficiency: Given a neighborhood $V$ of $x$, find a natural number $N_1$ such that $n \geq N_1$ implies $x_n \in V$. Also, since $x_n \in D_n$ for almost all $n \in \mathbb{N}$, find a natural number $N_2$ such that $n \geq N_2$ implies...
Putting $N = \max\{N_1, N_2\}$, we see that $n \geq N = x_n \in V \cap D_n \neq \emptyset$. It follows that $x \in \text{Li}_X(D_n^+, n \in \mathbb{N})$.

Necessity: Let $x \in \text{Li}_X(D_n^+, n \in \mathbb{N})$. Take a countable local base $\{V_n, n \in \mathbb{N}\}$ at $x$ with $V_{n+1} \subseteq V_n$ for every $n \in \mathbb{N}$.

$V_1$ is a nbd of $x$ and $x \in \text{Li}_X(D_n^+, n \in \mathbb{N})$, so we may find a natural number $N_1$ such that $n \geq N_1$ implies $V_1 \cap D_n \neq \emptyset$. $V_2$ is a nbd of $x$ and $x \in \text{Li}_X(D_n^+, n \in \mathbb{N})$, therefore there exists a natural number $N'$ such that $n \geq N'$ implies $V_2 \cap D_n \neq \emptyset$. Put $N_2 = 1 + \max\{N_1, N'\}$. So $n \geq N_2 = V_2 \cap D_n \neq \emptyset$. Proceed inductively to obtain infinitely many natural numbers $N_1 < N_2 < N_3 < \cdots < N_k < N_{k+1} < \cdots$ for which $n \geq N_k$ implies $V_k \cap D_n \neq \emptyset$, for every $k \in \mathbb{N}$. We construct the sequence $(x_t, t \in \mathbb{N})$ as follows: for $t < N_1$, choose $x_t \in X$ arbitrarily; if $t = N_k$ for some $k \in \mathbb{N}$, then $V_k \cap D_t \neq \emptyset$ by construction, so pick $x_t \in V_k \cap D_t$; if $t \neq N_k$ for every $k \in \mathbb{N}$ and $t > N_1$, find $k \in \mathbb{N}$ with $N_k < t < N_{k+1}$ (such a $k$ is obviously unique), so that $V_k \cap D_t \neq \emptyset$, and therefore pick $x_t \in V_k \cap D_t$.
Our construction is such that \( x_t \in D_t \) for \( t \geq N_1 \), and it is clear that \( (x_t, t \in \mathbb{N}) \to x \) since \( \{V_k, k \in \mathbb{N}\} \) was a local base for \( x \).

(b) Necessity: If \( x \in Ls_x(D_n, n \in \mathbb{N}) \), let \( \{V_n : n \in \mathbb{N}\} \) be a countable local base at \( x \) with \( V_{n+1} \subseteq V_n \) for every \( n \in \mathbb{N} \). \( V_1 \) is a nbd of \( x \) and \( x \in Ls_x(D_n, n \in \mathbb{N}) \), so there is a natural number \( n_1 \) such that \( V_1 \cap D_{n_1} \neq \emptyset \). \( V_2 \) is a nbd of \( x \) and \( x \in Ls_x(D_n, n \in \mathbb{N}) \), so there is a natural number \( n_2 > n_1 \) such that \( V_2 \cap D_{n_2} \neq \emptyset \). Proceed inductively to obtain infinitely many natural numbers \( \{n_k : k \in \mathbb{N}\} \) with \( n_k < n_{k+1} \) and \( V_k \cap D_{n_k} \neq \emptyset \) for every \( k \in \mathbb{N} \). Pick \( x_{n_k} \in V_k \cap D_{n_k} \) for every \( k \in \mathbb{N} \), and set \( M = \{n_k : k \in \mathbb{N}\} \). Because \( \{V_k : k \in \mathbb{N}\} \) is a local base for \( x \) and \( x_{n_k} \in V_k \) for every \( k \in \mathbb{N} \), it follows that \( (x_{n_k}, k \in \mathbb{N}) \to x \).

Sufficiency: For any nbd \( V \) of \( x \), the convergence of \( (x_n, n \in M) \) to \( x \) implies that there is a natural number \( N \) such that \( n \in M, n \geq N \) imply \( x_n \in V \). Since \( x_n \in D_n \) for every \( n \in M \), this implies that \( x_n \in V \cap D_n \neq \emptyset \) for \( n \in M, n \geq N \). Hence \( V \cap D_n \neq \emptyset \) for each \( n \) in the infinite set \( \{n \in M : n \geq N\} \). It follows that \( x \in Ls_x(D_n, n \in \mathbb{N}) \). Q.E.D.
Theorem 4.16

If \( X \) is a Hausdorff, first countable topological space, \((D_n, n \in \mathbb{N})\) a sequence in \( \text{exp}(X) \) and \( D \in \text{exp}(X) \), then Property \((D^+)\) implies \( D \supset \text{Ls}\_X(D_n, n \in \mathbb{N}) \). (Equality need not hold in this containment.) The converse is generally false even if we assume \( D = \text{Lim}\_X(D_n, n \in \mathbb{N}) \).

Proof:

Let \( x \in \text{Ls}\_X(D_n, n \in \mathbb{N}) \). Lemma 4.15 assures us that there is an infinite set \( M \) of natural numbers and a sequence \((x_n, n \in M)\) converging to \( x \) with \( x_n \in D_n \) for each \( n \in M \). Now Property \((D^+)\) guarantees that a subsequence \((x'_n, n \in K)\) of \((x_n, n \in M)\) converges to a point \( y \) of \( D \). Since the space is Hausdorff, we have \( x = y \in D \). Hence \( D \supset \text{Ls}\_X(D_n, n \in \mathbb{N}) \). To see that equality need not hold, take the real line with \( D = [-1, 3] \), \( D_{2n} = [1, 2 + \frac{1}{2n}] \) and \( D_{2n-1} = [-\frac{1}{2n-1}, 2] \): all the assumptions of the proposition are met, but \( \text{Ls}\_X(D_n, n \in \mathbb{N}) = [0, 2] \neq D \). (This same example shows that the assumptions of the theorem do not guarantee the existence of \( \text{Lim}\_X(D_n, n \in \mathbb{N}) \) since \( \text{L}i\_X(D_n, n \in \mathbb{N}) = [1, 2] \). To verify that the converse of the proposition may fail even if we assume \( D = \text{Lim}\_X(D_n, n \in \mathbb{N}) \), take the real line again with \( D = [0, 1] \) and \( D_n = \{n\} \cup [0, 1 + \frac{1}{n}] \).
The assumption that the space be Hausdorff cannot be omitted in Theorem 4.16. To see this, take $X$ as the set of real numbers with the topology generated by the pseudo-metric $d(x, y) = |x^2 - y^2|$, and define $D = [-3, -1]$, $D_n = [1, 2 + \frac{1}{n}]$. The space $(X, d)$ has the following properties:

(a) if $(x_n, n \in \mathbb{N})$ converges to $x$ in $R$, then $(x_n, n \in \mathbb{N})$ converges to $x$ in $(X, d)$;

(b) if $(x_n, n \in \mathbb{N})$ converges to $x$ in $(X, d)$, then $(x_n, n \in \mathbb{N})$ converges to $-x$ in $(X, d)$.

Now observe that if $M$ is an infinite set of natural numbers and $(x_n, n \in M)$ a sequence with $x_n \in D_n$ for each $n \in M$, then $(x_n, n \in M)$ is a sequence in the compact subset $[1, 3]$ of $R$, so has a subsequence $(x_n, n \in K)$ converging in $R$ to $x \in [1, 3]$, hence $(x_n, n \in K)$ converges in $(X, d)$ to $-x \in [-3, -1] = D$. Therefore Property (D+) is satisfied. It is easily verified that $D = [-3, -1] \not\subseteq \text{Ls}_X(D_n, n \in \mathbb{N}) = D \cup [1, 2]$. Q.E.D.

**Theorem 4.17**

Let $D$ be a subset of a Hausdorff first countable space $X$ and suppose $(D_n, n \in \mathbb{N})$ is a sequence in $\text{exp}(X)$ such that $D \subseteq D_n$ for almost all $n \in \mathbb{N}$. Then
Property (D+) implies \( D = \lim_x(D_n, n \in \mathbb{N}) \).

**Proof:**

Theorem 4.16 gives us at once \( D \supset L_x(D_n, n \in \mathbb{N}) \).

Since \( D \subset D_n \) for almost all \( n \in \mathbb{N} \) we have
\[ D \subset \lim_x(D_n, n \in \mathbb{N}) \]. Consequently \( D = \lim_x(D_n, n \in \mathbb{N}) \).

Q.E.D.

Taking \( D = [0, 1] \) and \( D_n = [0, 1 + \frac{1}{n}] \cup \{n\} \) in \( \mathbb{R} \) we see that the converse of Theorem 4.17 fails.

**Definition 4.18**

A subset \( B \) of a topological space \( X \) will be called almost sequentially compact if each sequence in \( B \) has a subsequence converging to a point of \( X \).

**Lemma 4.19**

(i) In a topological space, a sequentially compact subset is almost sequentially compact.

(ii) In a topological space, if \( D \subset B \) and \( B \) is almost sequentially compact, then so is \( D \).

(iii) In a metric space, an almost sequentially compact subset is bounded.

(iv) In a Euclidean space, a subset is almost sequentially compact if and only if it is bounded.
Proof:

(i) and (ii) are obvious from the definition.

(iii): Take an unbounded subset $B \neq \emptyset$ in a metric space $(X,d)$ and fix an element $x_0 \in X$. Find $x_1 \in B$ with $\alpha_1 = d(x_0, x_1) > 1$. Find $x_2 \in B$ with $\alpha_2 = d(x_0, x_2) > \max\{2, \alpha_1\}$. Proceed inductively to obtain a sequence $(x_n, n \in \mathbb{N})$ in $B$ with $d(x_0, x_{n+1}) > \max\{n+1, d(x_0, x_n)\}$. Then every subsequence of $(x_n, n \in \mathbb{N})$ is unbounded and therefore cannot converge.

(iv) Necessity: See (iii).

Sufficiency: If $B$ is bounded, then $\overline{B}$ is sequentially compact. Since $B \subseteq \overline{B}$, we may use (i) and (ii).

Q.E.D.

Theorem 4.20

For every sequence $(D_n, n \in \mathbb{N})$ of subsets of a Euclidean space $X$ and an arbitrary subset $D \neq \emptyset$ of $X$, Property (D+) implies that there exists a natural number $N$ such that $\bigcup_{n \geq N} D_n$ is almost sequentially compact.
Assume the conclusion of the theorem fails. It follows by (iv) of Lemma 4.19 that $\bigcup_{n \geq N} D_n$ is unbounded for every natural number $N$. Put $k_1 = 1$. Then $\bigcup_{n \geq k_1} D_n$ is unbounded, so we may find $y \in \bigcup_{n \geq k_1} D_n$ with $|y| > 1$. Thus $y \in D_t$ for some natural number $t \geq k_1$ and $|y| > 1$. Define $n_1 = t$ and $x_{n_1} = y$. Note $x_{n_1} \in D_{n_1}$ and $|x_{n_1}| > 1$. Now put $k_2 = 1 + n_1$. Again $\bigcup_{n \geq k_2} D_n$ is unbounded, so $z \in \bigcup_{n \geq k_2} D_n$ with $|z| > 2$, so that $z \in D_t$ for some natural number $t \geq k_2$. Define $n_2 = t$ and $x_{n_2} = z$. This gives $x_{n_2} \in D_{n_2}$ and $|x_{n_2}| > 2$.

We may thus proceed inductively to obtain the natural numbers $n_{t+1} \geq k_{t+1} = 1 + n_t > n_t$ and the sequence $(x_{n_t}, t \in \mathbb{N})$ with $x_{n_t} \in D_{n_t}$ and $|x_{n_t}| > t$. Every subsequence of $(x_{n_t}, t \in \mathbb{N})$ is thus unbounded, and therefore, cannot converge. Thus Property (D+) fails.

Q.E.D.

The validity of Theorem 4.20 in an arbitrary topological space $X$ and, generally, in an arbitrary $L$-space is an open question, even when the sequence is contracting.
Theorem 4.21

Let $D$ be a subset of an arbitrary topological space $X$ and $(D_n, n \in \mathbb{N})$ a sequence in $\exp(X)$ with $D \supseteq \text{Ls}_X(D_n, n \in \mathbb{N})$. If there is a natural number $N$ such that $\bigcup_{n \in \mathbb{N}} D_n$ is almost sequentially compact (in particular if $X$ is sequentially compact), then Property $(D^+)$ is satisfied.

Proof:

Given an infinite set $M$ of natural numbers and a sequence $(x_n, n \in M)$ with $x_n \in D_n$ for each $n \in M$, put $K = \{n \in M: n \geq N\}$. Then $(x_n, n \in K)$ is a sequence in the almost sequentially compact set $\bigcup_{n \in \mathbb{N}} D_n$, and must therefore have a subsequence $(x_n, n \in L)$ converging to $x \in X$. The theorem will be proved by showing that $x \in D$. Now take an arbitrary nbd $V$ of $X$. Since $(x_n, n \in L) \to x$, there is a natural number $T$ such that $n \in L$ and $n \geq T$ imply $x_n \in V$. The set $B = \{n \in L: n \geq T\}$ is infinite and $x_n \in V \cap D_n \neq \emptyset$ for every $n \in B$. Thus $x \in \text{Ls}_X(D_n, n \in \mathbb{N})$. It follows by the hypothesis that $x \in D$.

Q.E.D.
Corollary 4.22

For a Hausdorff, first countable space $X$, a subset $D$ of $X$ and a sequence $(D_n, n \in \mathbb{N})$ in $\exp(X)$ such that there is a natural number $N$ with $\bigcup_{n \geq N} D_n$ almost sequentially compact, Property (D+) is equivalent to $D \supset Ls_{\kappa}(D_n, n \in \mathbb{N})$.

Proof:

That Property(D+) implies $D \supset Ls_{\kappa}(D_n, n \in \mathbb{N})$ was proved in Theorem 4.16. The reverse implication is Theorem 4.21.

Q.E.D.

Corollary 4.23

Let $X$ be a Hausdorff, first countable and sequentially compact space. For every subset $D$ of $X$ and every sequence $(D_n, n \in \mathbb{N})$ in $\exp(X)$, Property (D+) is equivalent to the relation $D \supset Ls_{\kappa}(D_n, n \in \mathbb{N})$.

Proof:

Here, the set $\bigcup_{n=1}^{\infty} D_n$ is almost sequentially compact, and we may apply Corollary 4.22.

Q.E.D.
**Corollary 4.24**

Assume $D$ to be a subset of a Hausdorff, first countable topological space $X$ and $(D_n, n \in \mathbb{N})$ a sequence in $\exp(X)$ such that $D \subseteq D_n$ for almost all $n \in \mathbb{N}$, and suppose that $\bigcup_{n \geq N} D_n$ is almost sequentially compact for some natural number $N$. Then Property $(D^+)$ is equivalent to $D = \lim_{\mathcal{X}} (D_n, n \in \mathbb{N})$.

**Proof:**

Property $(D^+)$ implies $D = \lim_{\mathcal{X}} (D_n, n \in \mathbb{N})$ as was shown in Theorem 4.17. If now $D = \lim_{\mathcal{X}} (D_n, n \in \mathbb{N})$, then $D \supseteq \lim_{\mathcal{X}} (D_n, n \in \mathbb{N})$. We may therefore apply Theorem 4.21.

Q.E.D.

**Corollary 4.25**

Let $X$ be a Hausdorff, first countable and sequentially compact topological space. For every subset $D$ of $X$ and every sequence $(D_n, n \in \mathbb{N})$ in $\exp(X)$ such that $D \subseteq D_n$ for almost all $n \in \mathbb{N}$, Property $(D^+)$ is equivalent to the relation $D = \lim_{\mathcal{X}} (D_n, n \in \mathbb{N})$.

**Proof:**

Since $X$ is sequentially compact and $(D_n, n \in \mathbb{N})$ a sequence in $\exp(X)$, the set $\bigcup_{n=1}^{\infty} D_n$ is almost sequentially compact. Therefore Corollary 4.24 applies.

Q.E.D.
Comparison with Vietoris Convergence

**Definition 4.26**

If $X \neq \emptyset$ is a topological space and $U_1, \ldots, U_n$ finitely many open non-empty subsets of $X$, put

$$[U_1, \ldots, U_n] = \{E \in \exp_0(X) : E \subseteq \bigcup_{k=1}^{n} U_k\}.$$  Let $T_v$ denote the family of all subclasses $S$ of $\exp_0(X)$ for which $E \in S$ implies that there exist finitely many non-empty open subsets $U_1, \ldots, U_n$ of $X$ with $E \in [U_1, \ldots, U_n] \subseteq S$.

**Theorem 4.27**

If $X \neq \emptyset$ is a topological space, then $T_v$ is a topology for $\exp_0(X)$.

**Proof:**

The empty subclass of $\exp_0(X)$ belongs to $T_v$ (vacuously); and $\exp_0(X) = [X] \in T_v$.

If $[S_\alpha : \alpha \in A]$ is a collection of members of $T_v$ and $E \in S = \bigcup_{\alpha \in A} S_\alpha$, then $\exists \alpha_0 \in A$ with $E \in S_{\alpha_0}$.

Since $S_{\alpha_0} \in T_v$, we can find non-void open subsets $U_1, U_2, \ldots, U_n$ of $X$, $n$ finite, such that $E \in [U_1, \ldots, U_n] \subseteq S_{\alpha_0}$; whence $E \in [U_1, \ldots, U_n] \subseteq S$.

It follows that $S \in T_v$. 

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Let us now consider $S_1, S_2, \in T_v$ with $E \in S_1 \cap S_2$. We may find finitely many non-void open subsets $U_1, \ldots, U_n$ of $X$ and finitely many non-void open subsets $V_1, \ldots, V_m$ of $X$ satisfying

$$E \in [U_1, \ldots, U_n] \subset S_1,$$

(4.8)

$$E \in [V_1, \ldots, V_m] \subset S_2.$$

Put $W = \bigcup (U_i \cap V_j : 1 \leq i \leq n, 1 \leq j \leq m)$. Then $W$ is open and from (4.8), $\emptyset \neq E \subset W$, so that $W \neq \emptyset$.

As the definition of $W$ indicates, we have $W \subset \bigcup_{i=1}^{m} U_i$ and $W \subset \bigcup_{j=1}^{n} V_j$. It follows that $E \in [W] \subset [U_1, \ldots, U_n] \cap [V_1, \ldots, V_n] \subset S_1 \cap S_2$, and thus $S_1 \cap S_2 \in T_v$.

One obtains the topology $T_v$ for $\exp_0(X)$ from a relaxation of the requirements used by L. Vietoris [25] (see also E. Michael [20]). The present approach gives us more nbds for $E \in \exp_0(X)$. We shall call $T_v$ "the Vietoris topology for $\exp_0(X)$".

**Theorem 4.28**

Let $D \neq \emptyset$ be a subset of a topological space $X$ and $(D_n : n \in \mathbb{N})$ a sequence in $\exp_0(X)$. Then Property (D+) implies that $(D_n : n \in \mathbb{N})$ converges to $D$ under the Vietoris topology $T_v$. 

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Proof:

Suppose \( S \) is a nbd of \( D \) in \( \exp_\phi(X) \) under \( T_v \). We can thus find finitely many non-empty open subsets \( V_1, \ldots, V_k \) of \( X \) with \( D \in [V_1, \ldots, V_k] \subset S \). Suppose it is false that \( D_n \subset \bigcup_{i=1}^{k} V_i \) for almost all \( n \in \mathbb{N} \). Therefore there is an infinite subset \( M \) of natural numbers such that \( D_n \not\subset \bigcup_{i=1}^{k} V_i \) for every \( n \in M \). Consequently, if \( n \in M \), we may find \( x_n \in D_n \setminus \bigcup_{i=1}^{k} V_i \). It follows from Property (D+) that a subsequence \( (x_n', n \in L) \) of \( (x_n, n \in M) \) converges to some \( x \in D \). But then \( x \in D \subset \bigcup_{i=1}^{k} V_i \) and thus \( \bigcup_{i=1}^{k} V_i \) is a nbd of \( x \) in \( X \), and yet \( x_n \not\in \bigcup_{i=1}^{k} V_i \) for every \( n \in L \), which is impossible. We conclude that \( D_n \subset \bigcup_{i=1}^{k} V_i \) for almost all \( n \in \mathbb{N} \), so that \( D_n \in [V_1, \ldots, V_k] \subset S \) for almost all \( n \in \mathbb{N} \).

Q.E.D.

Theorem 4.29

Let \( D \neq \emptyset \) be a compact and closed subset of a space \( X \). Assume that \( X \) is regular (not necessarily Hausdorff), first countable and locally compact. If a sequence \( (D_n, n \in \mathbb{N}) \) in \( \exp_\phi(X) \) converges to \( D \) under the Vietoris topology, then Property (D+) is satisfied.
Proof:

Since \( D \) is compact and \( X \) is locally compact, there is an open subset \( V \) of \( X \) with \( V \supset D \) and \( \overline{V} \) compact, whence \( \overline{V} \) is sequentially compact (because \( X \) is first countable). Now \( V \supset D \neq \emptyset \) and \( V \) open imply \( D \in [V] \in T_\nu \). It follows from the Vietoris convergence of \( (D_n, n \in \mathbb{N}) \) that we can find a natural number \( N_1 \) with \( D_n \in [V] \) for \( n \geq N_1 \), i.e., \( D_n \subset V \subset \overline{V} \) for \( n \geq N_1 \). Consider now an infinite set \( M \) of natural numbers and a sequence \( (x_n, n \in M) \) such that \( x_n \in D_n \) for each \( n \in M \). Put \( L = \{n \in M: n \geq N_1 \} \). The sub-sequence \( (x_n, n \in L) \) then lies in the set \( \overline{V} \), hence has a subsequence \( (x_n', n \in K) \) converging to some \( z \in \overline{V} \). If \( z \notin D \), then, since \( X \) is regular and \( D \) is closed, we may find two open subsets \( G \) and \( H \) of \( X \) with \( D \subset G \), \( z \in H \), \( G \cap H = \emptyset \). Thus \( [G] \) is a nbd of \( D \) in \( \exp_\nu(X) \) under \( T_\nu \), so that \( D_n \in [G] \) for almost all \( n \in \mathbb{N} \); but \( x_n \in D_n \) for each \( n \in K \), and so \( x_n \in G \) for almost all \( n \in K \). Because \( G \cap H = \emptyset \), it follows that \( x_n \in H \) only for finitely many \( n \in K \), contradicting the fact that \( (x_n', n \in K) \to z \) and \( H \) is a nbd of \( z \). This shows that \( z \in D \). Thus Property \( (D+) \) holds as claimed.

Q.E.D.
It appears possible that the assumption of local compactness on $X$ may be removed in Theorem 4.29. This may be verified in case $X$ is a linear topological space, but we are unable to provide a general proof for other spaces.

**Remark 4.30**

Let $D$ be a non-empty subset of a topological space $X$, $(D_n, n \in \mathbb{N})$ a sequence in $\exp_0(X)$, $I$ a real-valued function defined on $D$ and for each $n \in \mathbb{N}$, $I_n$ a real-valued function defined on $D_n$. Suppose that

(a) There exists a real-valued, lower semi-continuous function $J$ defined on $X$ such that $I = J|D$ and $I_n = J|D_n$ for every $n \in \mathbb{N}$;

(b) Whenever $G$ is an open subset of $X$ containing $D$, it is true that $D_n \subseteq G$ for almost all $n \in \mathbb{N}$.

Then $\lim \inf(i_n, n \in \mathbb{N}) \geq i$, where $i = \inf I[D]$ and $i_n = \inf I_n[D_n]$. (In particular if $i_n \leq i$ for every $n \in \mathbb{N}$, then $\lim(i_n, n \in \mathbb{N}) = i$.)

**Proof:**

Assume that $\lim \inf(i_n, n \in \mathbb{N}) < i$, so that we can find a real number $t$ with $\lim \inf(i_n, n \in \mathbb{N}) < t < i$. Put $G = J^{-1}[t, +\infty[$. If $y \in X \setminus G$, then there exists a net $(S_b, b \in B)$ in $X \setminus G$ such that $(S_b, b \in B) \to y$. 

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It follows from the lower semi-continuity of $J$ (see Theorem 2.10(a)) that $J(y) \leq \lim \inf(J(S_b), b \in B)$. But $S_b \notin X \setminus G$ for every $b \in B$, and so $J(S_b) \leq t$ for every $b \in B$. Applying Theorem 2.3(a), (c), we see that $\lim \inf(J(S_b), b \in B) \leq t$. Hence $J(y) \leq t$, i.e., $y \notin X \setminus G$. We conclude that $X \setminus G$ is a closed subset of $X$, whence $G$ is an open subset of $X$.

We further observe that if $x \in D$, then $J(x) = I(x) > t$, so that $x \in G$. Thus $G$ is an open subset of $X$ containing $D$. It follows by the hypothesis that there exists a natural number $N$ for which $n \in N$ and $n \geq N$ imply $D_n \subset G$. Now, then, for $n \in N$ with $n \geq N$ we have $D_n \subset G$, so $I_n[D_n] = J[D_n] \subset J[G]$, from which it follows that $i_n > t$. We may thus apply Theorem 2.3(d) to conclude that $\lim \inf(i_n, n \in N) \geq t$, which is a contradiction. This proves the assertion made in the remark.

The assumption made in part (b) of the foregoing remark is certainly satisfied if the sequence $(D_n, n \in N)$ converges to $D$ under the topology $T_v$ constructed in Definition 4.26 (and Theorem 4.27). The latter convergence was shown to be implied by Property $(D^+)$ (see Theorem 4.28).
This indicates that when dealing with a topological space (rather than an L-space), Property (D+) might possibly be replaced with some weaker requirement, as long as the functionals $I$ and $I_n$ are the restrictions of one appropriate functional $J$ defined on the whole space.
V. A STUDY OF PROPERTIES (I) AND (I*)

We continue to use the notation of Definition 3.1. The purpose of the present chapter is to discuss the semi-continuity Properties (I) and (I*) used in the sufficient conditions of Chapter III.

Theorem 5.1

Suppose that $I_e$ extends $I$ for every $e \in B$ and that $I_b|_{{(D_b \cap D_c)}} = I_c|_{{(D_b \cap D_c)}}$ for every $b \in B$ and every $c \in B$. Let $f = \cup(I_b : b \in B)$.

(a) If $f$ is lower semi-continuous on $D$, then Property (I) holds.

(b) If $f$ is upper semi-continuous on $D$, then Property (I*) holds.

(c) If $f$ is continuous on $D$, then both Property (I) and Property (I*) hold.

Proof:

We first observe that $f: \cup[D_b : b \in B] \to R$ is indeed a function defined by $f(x) = I_b(x)$ for some $b \in B$. 

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(a) Consider a cofinal subset $G$ of $B$ and a net $(x_b, b \in G)$ in $X$ with $x_b \in D_b$ for each $b \in G$ and $(x_b, b \in G) \to y \in D$. It follows from the lower semi-continuity of $f$ on $D$ that $f(y) \leq \lim \inf(f(x_b), b \in G)$. Since $f(y) = I(y)$ and $f(x_b) = I_b(x_b)$, we have $I(y) \leq \lim \inf(I_b(x_b), b \in G)$. Hence Property (I) is satisfied.

(b) The proof is analogous to that of (a).

(c) If $f$ is continuous on $D$, it follows by Theorem 2.6 that $f$ is both upper and lower semi-continuous on $D$, so that (a) & (b) apply.

Q.E.D.

Theorem 5.2

Let $X, D, I, B, I_b$ and $D_b$ be as in Definition 3.1. For each $b \in B$, let $J_b : D_b \to R$ be non-negative. Define $I'_b = I_b + J_b$, $b \in B$.

(a) If Property (I) holds for $I$ and the functionals $I_b$, then also Property (I) holds for $I$ and the functionals $I'_b$.

(b) $i'_b = \inf I'_b[D_b] \geq i_b$ for each $b \in B$. 

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Proof:

(b) Fix \( b \in B \). If \( x \in D_b \), then

\[
I'_b(x) = I_b(x) + J_b(x) \geq I_b(x)
\]

\[
\geq \inf I_b[D_b] = i_b \quad \text{since } J_b(x) \geq 0.
\]

Thus \( I'_b(x) \geq i_b \) for each \( x \in D_b \). Hence \( i'_b \geq i_b \).

(a) Suppose \( G \) is a cofinal subset of \( B \) and the net \( (x_b^g, b \in G) \) converges to \( y \in D \) with \( x_b \in D_b \) for each \( b \in G \). Then \( I(y) \leq \lim \inf(I_b(x_b^g), b \in G) \). Since \( I'_b(x_b^g) \leq I'_b(x_b^g) \) for each \( b \in G \), it follows by Lemma 2.5(a) that \( \lim \inf(I'_b(x_b^g), b \in G) \leq \lim \inf(I'_b(x_b^g), b \in G) \). Thus \( I(y) \leq \lim \inf(I'_b(x_b^g), b \in G) \).

Q.E.D.

Theorem 5.3

Let \( X, D, I, B, I_b \) and \( D_b \) be as in Definition 3.1. For every \( b \in B \) let \( J_b: D_b \to \mathbb{R} \) be non-negative and define \( I'_b = I_b - J_b, b \in B \).

(a) If Property \((I^*)\) holds for \( I \) and the functionals \( I_b \), then Property \((I^*)\) also holds for \( I \) and the functionals \( I'_b \).

(b) \( i'_b^* = \sup I'_b[D_b] \leq i_b^* \) for each \( b \in B \).
Proof:
This is the dual of Theorem 5.2.
Q.E.D.

Remark 5.4

(a) In Theorem 5.2, let the functionals \( J_b \) satisfy the further property that \( i'_b \leq i \) for each \( b \in B \).
Thus \( i_b \leq i'_b \leq i \) for each \( b \in B \), so that Property (m) holds for \( I \) and \( I'_b \), \( b \in B \). If Property (D+) holds, and if \( I \) and \( \{I_b : b \in B\} \) satisfy Property (I), it then follows from Theorem 3.4 that \( (i_b, b \in B) \rightarrow i \).
But then, in view of Theorem 5.2, it is true that Property (I) holds for \( I \) and \( I'_b \), and thus also \( (i'_b, b \in B) \rightarrow i \). Since \( i_b \leq i'_b \leq i \) for each \( b \in B \), the net of bounds \( (i'_b, b \in B) \) has a rate of convergence at least as good as the net \( (i_b, b \in B) \), and possibly better.

(b) A similar observation can be made about Theorem 5.3.

(c) In case \( B = \mathbb{N} \), the expression "\( f \) is lower semi-continuous on \( D \)" in Theorem 5.2 (respectively, "\( f \) is upper semi-continuous on \( D \)" in Theorem 5.3) can be replaced by "\( f \) is sequentially lower semi-continuous on \( D \)" (respectively, "\( f \) is sequentially upper semi-continuous on \( D \)").
**Lemma 5.5**

Let $Y$ be a topological space, $A$ a non-empty subspace and $\emptyset \in A$, with $J: A \rightarrow \mathbb{R}^+$ and $\mathcal{B}$ a local base for $\emptyset$ in $Y$. Define

$$J^B = \sup_{V \in \mathcal{B}} \{\inf J[V \cap A]\}$$

and

$$J^B = \inf_{V \in \mathcal{B}} \{\sup J[V \cap A]\}.$$  

Then:

(a) The number $J^B$ is invariant with respect to local bases for $\emptyset$ in $Y$; i.e., if $\mathcal{B}$ and $\mathcal{A}$ are local bases for $\emptyset$ in $Y$, then $J^B = J^A$. We shall let $\liminf_{a \rightarrow \emptyset} J(a)$ denote $J^B$, where $\mathcal{B}$ is an arbitrary local base for $\emptyset$ in $Y$.

(b) The number $J^B$ is invariant with respect to local bases for $\emptyset$ in $Y$; i.e., if $\mathcal{B}$ and $\mathcal{A}$ are both local bases for $\emptyset$ in $Y$, then $J^B = J^A$. We shall let $\limsup_{a \rightarrow \emptyset} J(a)$ denote $J^B$, where $\mathcal{B}$ is an arbitrary local base for $\emptyset$ in $Y$.

**Proof:**

(a) Let $A$ be the class of all nbds of $\emptyset$ in $Y$. Let us now suppose that $\mathcal{B}$ is an arbitrary local base of $\emptyset$ in $Y$. It follows that $\mathcal{B} \subseteq A$, whence $\{\inf J[U \cap A]: U \in \mathcal{B}\} \subseteq \{\inf J[U \cap A]: U \in A\}$. Taking the supremum
throughout we obtain

(5.1) \[ J_B \leq J_A \]

On the other hand if \( U \in A \), then, since \( B \) is a local base for \( \theta \), there is \( V \in B \) such that \( V \subset U \); so that \( J[U \cap A] \supset J[V \cap A] \), whence

\[ \inf J[U \cap A] \leq \inf J[V \cap A], \]

\[ \leq \sup \{ \inf J[T \cap A] \} \text{ because } V \in B; \]

\[ = J_B. \]

Therefore \( \inf J[U \cap A] \leq J_B \) for each \( U \in A \), whence

\[ \sup \{ \inf J[H \cap A] \} \leq J_B; \text{ i.e., } \]

(5.2) \[ J_A \leq J_B. \]

We can combine (5.1) and (5.2) to obtain \( J_A = J_B \), which completes the proof of (a).

(b) Let \( A \) & \( B \) be as in (a) above. Thus again

\( B \subset A \), so \([ \sup J[V \cap A]; V \in B ] \subset [ \sup J[V \cap A]; V \in A ] \).

Taking the infimum throughout, we obtain

(5.3) \[ J_B \geq J_A. \]

Let us now suppose that \( U \in A \). Since \( B \) is a local base for \( \theta \) in \( X \), \( \exists V \in B \) with \( V \subset U \), so
\[
\sup J[U \cap A] \geq \sup J[V \cap A],
\]
\[
\geq \inf \{\sup J[T \cap A] \} \quad \text{since} \quad V \in B,
\]
\[
= J^B
\]

It follows that \( \sup J[U \cap A] \geq J^B \) for each \( U \in A \), whence \( \inf \{\sup J[U \cap A] \} \geq J^B \), i.e.,
\[
(5.4) \quad J^A \geq J^B.
\]

We may combine (5.3) and (5.4) to conclude that
\( J^A = J^B \), which proves (b).

Q.E.D.

Lemma 5.6

In Lemma 5.5 above, if \( A = \mathbb{N} \), \( Y = \mathbb{R}^+ \) and
\( \emptyset = +\infty \), then
\[
\liminf J(n) = \liminf (J(n), n \in \mathbb{N}), \quad \text{as} \quad n \to +\infty.
\]
\[
\limsup J(n) = \limsup (J(n), n \in \mathbb{N}). \quad \text{as} \quad n \to +\infty
\]

Proof:

For every \( n \in \mathbb{N} \), put \( V_n = [n, +\infty] \). Then
\( \{V_n : n \in \mathbb{N}\} \) is a local base for \( +\infty \) in \( \mathbb{R}^+ \), and
\[
(5.5) \quad V_n \cap \mathbb{N} = \{k \in \mathbb{N} : k \geq n\}, \quad \text{for each} \quad n \in \mathbb{N}.
\]
It follows by Lemma 5.5 that
\[
\liminf_{n \to +\infty} J(n) = \sup_{n \in \mathbb{N}} \{\inf_{f \in \mathcal{F}} J_{n}^f \}
\]
and
\[
\limsup_{n \to +\infty} J(n) = \inf_{n \in \mathbb{N}} \{\sup_{f \in \mathcal{F}} J_{n}^f \}.
\]

But
\[
\sup_{n \in \mathbb{N}} \{\inf_{f \in \mathcal{F}} J_{n}^f \}
\]

\[
= \sup_{n \in \mathbb{N}} \{\inf_{k \geq n \text{ and } k \in \mathbb{N}} J(k)\}
\]

\[
= \liminf_{n \in \mathbb{N}} (J(n), n \in \mathbb{N})
\]

by (5.5) and Corollary 2.8(b),

and
\[
\inf_{n \in \mathbb{N}} \{\sup_{f \in \mathcal{F}} J_{n}^f \}
\]

\[
= \inf_{n \in \mathbb{N}} \{\sup_{k \geq n \text{ and } k \in \mathbb{N}} J(k)\}
\]

\[
= \limsup_{n \in \mathbb{N}} (J(n), n \in \mathbb{N})
\]

by (5.5) and Corollary 2.8(a). This proves the lemma.

Q.E.D.

**Definition 5.7**

Let \( X \) be a set, \( D \) a non-empty subset and \( I : X \to \mathbb{R} \). Consider a topological space \( Y \), a non-empty subset \( A \) of \( Y \), a point \( \theta \in \overline{A} \) and a function \( G : A \to \exp_0(X) \). Put \( G_a = G(a), a \in A \).
(a) We shall say that the problem of finding \( \inf I[D] \) is lower stable with respect to the family \( \{G_a : a \in A\} \) if the function \( J: A \to \mathbb{R}^+ \) defined by
\[
J(a) = \inf I[G_a]
\]
satisfies \( \liminf_{a \to \theta} J(a) \geq \inf I[D] \).

(b) We shall say that the problem of finding \( \inf I[D] \) is upper stable with respect to the family \( \{G_a : a \in A\} \) if \( \limsup_{a \to \theta} J(a) \leq \inf I[D] \), where \( J: A \to \mathbb{R}^+ \) is defined by \( J(a) = \inf I[G_a] \).

**Theorem 5.8**

Let \( X \) be a topological space, \( D \) a non-empty subset and \( (D_n, n \in \mathbb{N}) \) a sequence in \( \exp_\circ(X) \). Consider a functional \( I: X \to \mathbb{R} \) and put \( I_n = I|D_n \). If the Property (D+) and Property (I) hold, then the problem of finding \( \inf I[D] \) is lower stable with respect to the family \( \{D_n : n \in \mathbb{N}\} \).

**Proof:**

Here \( Y = \mathbb{R}^+ \), \( A = \mathbb{N} \) and \( \theta = +\infty \). See Lemma 5.6.

Put \( i = \inf I[D] \) and assume that \( \liminf_{n \to +\infty} J(n) < i \),

so that we may find a real number \( t \) with \( \liminf_{n \to +\infty} J(n) < t < i \). It follows by Lemma 5.6 that \( \liminf_{n \in \mathbb{N}} J(n), n \in N < t \). Hence by Theorem 2.3(f) and Lemma 2.2, we
can find an infinite subset \( M \) of \( \mathbb{N} \) such that
\[ J(n) < t \text{ for each } n \in M; \text{ i.e., } \inf I[D_n] = \inf I_n[D_n] < t \text{ for each } n \in M. \]
Thus for every \( n \in M \) there is a point \( x_n \in D_n \) with \( I_n(x_n) = I(x_n) < t \). It follows by Property \((D^+)\) that the sequence \((x_n', n \in M)\) has a subsequence \((x_n', n \in L)\)
which converges to a point \( y \) of \( D \). But then
Property \((I)\) implies that \( I(y) \leq \lim \inf (I_n(x_n'), n \in L) \). Since \( I_n(x_n') < t < i \) for each \( n \in L \)
\((L \text{ is a subset of } M)\), it follows by Theorem 2.3
\((a),(c)\) that \( \lim \inf (I_n(x_n'), n \in L) \leq t < i \), so that
I(y) \leq t < i. This contradicts the definition of \( i \)
because \( y \in D \).

Q.E.D.

**Theorem 5.9**

Let \( D \) be a non-empty subset of a topological
space \( X \) and \((D_n, n \in \mathbb{N})\) a sequence in \( \exp_0(X) \).
Consider any functional \( I: X \to \mathbb{R} \) and let \( I_n = I|D_n \).
If Property \((D^+)\) and Property \((I^*)\) hold, then the prob-
lem of finding \( \inf I[D] \) is upper stable with respect
to the family \(\{D_n: n \in \mathbb{N}\}\). (Here, \( Y = \mathbb{R}^+, A = \mathbb{N}, \)
\( \emptyset = +\infty, G_n = D_n \)).
Proof:

The proof is essentially dual to that of Theorem 5.8.

Remark 5.10

The concept of "stability" introduced in Definition 5.7 is due to B. M. Budak and E. M. Berkovich [5].
VI. A PROBLEM IN LINEAR PROGRAMMING

The ordinary problem of linear programming consists of optimizing a linear functional $f: \mathbb{R}^n \to \mathbb{R}$ constrained by finitely many linear equations and inequalities. Usually, the feasible region is further restricted to vectors whose components are non-negative. Then, using well-known elementary operations, all the constraints in the general LP problem may be brought under the following canonical form:

$$
(6.1) \quad x \geq 0 \text{ and } a^k \cdot x \geq d_k \text{ for } 1 \leq k \leq n,
$$

where: $n$ is a positive integer; $x \in \mathbb{R}^n$; "$x \geq 0$" means that each component of $x$ is non-negative; $a^k \in \mathbb{R}^n$ for every positive integer $k \leq n$; $a^k \cdot x$ denotes the inner product of $a^k$ and $x$ in $\mathbb{R}^n$; and $d_k \in \mathbb{R}$ for every positive integer $k \leq n$. A thorough discussion of these operations and related matters appears in M. Simonnard [22] and G.B. Dantzig [11].

Let us now consider an extended version of the above problem, namely:
minimize the linear functional \( f: \mathbb{R}^p \to \mathbb{R} \)

subject to \( x \geq 0 \) and \( a^k \cdot x \geq d_k \) for each \( k \in \mathbb{N} \).

where the notation is that of line (6.1).

**Definition 6.1**

Let

(a) \( D' = \{ x \in \mathbb{R}^p : x \geq 0 \) and \( a^k \cdot x \geq d_k \) for each \( k \in \mathbb{N} \} \),

(b) \( D'_n = \{ x \in \mathbb{R}^p : x \geq 0 \) and \( a^k \cdot x \geq d_k \) for \( 1 \leq k \leq n \} \),

(c) \( J = f|D' \),

(d) \( J_n = f|D'_n \),

(e) \( j = \inf J[D'] \) and \( j_n = \inf J_n[D'_n] \).

**Lemma 6.2**

(a) \( D' = \bigcap_{n=1}^{\infty} D'_n \) and \( D' \subseteq D'_n+1 \subseteq D'_n \) for each \( n \in \mathbb{N} \).

(b) \( j_n \leq j_{n+1} \leq j \) for each \( n \in \mathbb{N} \) if \( D' \neq \emptyset \).
(c) $D'_n$ is convex and closed for each $n \in \mathbb{N}$.

**Proof:**

(a) The equality $D' = \bigcap_{n=1}^{\infty} D'_n$ is clear from the definition of the sets $D'$ and $D'_n$, and so is the containment $D' \subseteq D'_{n+1} \subseteq D'_n$ for each $n \in \mathbb{N}$.

(b) It follows from (a) that $\mathbf{j}[D'] \subseteq \mathbf{j}[D'_{n+1}] \subseteq \mathbf{j}[D'_n]$ for each $n \in \mathbb{N}$, whence $\phi \neq J[D'] \subseteq J_{n+1}[D'_{n+1}] \subseteq J_n[D'_n]$ for every $n \in \mathbb{N}$, so $\inf J[D'] \geq \inf J_{n+1}[D'_{n+1}] \geq \inf J_n[D'_n]$ for every $n \in \mathbb{N}$; i.e., $j_n \leq j_{n+1} \leq j$ for each $n \in \mathbb{N}$.

(c) The set $G = \{x \in \mathbb{R}^P : x \geq 0\}$ is closed and convex. Also, for every $k \in \mathbb{N}$, the half-space $H_k = \{x \in \mathbb{R}^P : a^k x \geq d_k\}$ is closed and convex. Since $D'_n$ is the intersection of $G, H_1, H_2, \ldots, H_n$, it follows that $D'_n$ is closed and convex.

Q.E.D.

**Definition 6.3**

A subset $E$ of $\mathbb{R}^P$ is said to be convex at the point $x \in \mathbb{R}^P$ if $tx + (1 - t)y \in E$ whenever $y \in E$ and $t \in [0, 1]$. 

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Lemma 6.4

Let $E$ be a subset of $\mathbb{R}^p$.

(a) If $E$ is convex at $x$, then $x \in E$ provided $E \neq \emptyset$.

(b) If $E$ is convex at $x$, then $\overline{E}$ is convex at $x$.

(c) If $E$ is a convex subset, then $E$ is convex at $x$ for every $x \in E$.

Proof:

(a) Take $t = 1$ and $y$ an arbitrary element of $E$.

(b) Let $y \in \overline{E}$ and $t \in [0, 1]$. There exists a sequence $(x_n', n \in \mathbb{N})$ in $E$ such that $y = \lim_{n \to \infty} x_n', n \in \mathbb{N})$. For each $n \in \mathbb{N}$, $tx + (1 - t)x_n \in E$ because $E$ is convex at $x$. Therefore $\lim_{n \to \infty} (tx + (1 - t)x_n', n \in \mathbb{N}) \in \overline{E}$. Since $tx + (1 - t)y = \lim_{n \to \infty} (tx + (1 - t)x_n', n \in \mathbb{N})$, we conclude that $tx + (1 - t)y \in \overline{E}$. Thus $\overline{E}$ is convex at $x$.

(c) Clear.

Q.E.D.

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In our attempt to make the problem of line (6.2) fit into the scheme of Chapter III, we will need the following result.

**Lemma 6.5**

Suppose $E$ is a closed and unbounded subset of $\mathbb{R}^P$ which is convex at the point $x$. Then there exists a unit vector $u \in E - x = \{y - x : y \in E\}$ such that $x + tu \in E$ for every non-negative number $t \in \mathbb{R}$.

**Proof:**

Put $B = E - x$. Then $B$ is closed, unbounded and convex at the origin $0$. Thus for every $n \in \mathbb{N}$ we can find $x_n \in B$ with $\|x_n\| > n$. But then $G_n = \{tx_n : t \in [0, 1]\} \subset B$ for each $n \in \mathbb{N}$. For a given $n \in \mathbb{N}$, it is true that $0 < t_n = \frac{1}{\|x_n\|} < 1$ (because $\|x_n\| > n$); thus

$$u_n = t_n x_n \in G_n \subset B \quad \text{and} \quad \|u_n\| = 1.$$ 

The bounded sequence $(u_n, n \in \mathbb{N})$ in $\mathbb{R}^P$ has a subsequence $(u_n', n \in M)$ converging to some $u \in \mathbb{R}^P$ where $M$ is an infinite subset of $\mathbb{N}$. Obviously, $\|u\| = 1$, and $u \in B$ since $B$ is closed. Fix a natural number $k$. The set $M$ is an infinite subset of $\mathbb{N}$, so $\exists N \in M$ with $k < N$. If now $n \in M$ and $n \geq N$, then $\|x_n\| > n > k \geq 1 > 0$, so that
0 < \frac{k}{\|x_n\|} < 1$, whence \( k_u = \frac{k}{\|x_n\|} x_n \in G_n \subseteq B \). It follows that \( k_u \in B \) for \( n \in M, n \geq N \). Therefore \( \lim(ku_n, n \in M) \in B \) because \( B \) is closed. Since \( \lim(ku_n, n \in M) = ku \), it is true that \( ku \in B \). Hence \( ku \in B \) for each \( k \in \mathbb{N} \). We can now use the convexity of \( B \) at the origin \( \theta \) to conclude that \( \{tku : t \in [0, 1]\} \subseteq B \) for every \( k \in \mathbb{N} \). However \( \{tk : k \in \mathbb{N} \) and \( t \in [0, 1]\} \) is the set of all non-negative real numbers. Consequently \( tu \in B = E - x \) for every non-negative real number \( t \), whence \( x + tu \in E \) for every non-negative real number \( t \).

Q.E.D.

**Theorem 6.6**

Let \( x \in \mathbb{R}^p \) and suppose that for every \( n \in \mathbb{N} \), \( E_n \) is a subset of \( \mathbb{R}^p \) which is convex at \( x \). If \( L_{\mathbb{R}^p}(E_n, n \in \mathbb{N}) \) is bounded, then at least one of the sets \( E_n \) is bounded.

**Proof:**

Assume that each \( E_n \) is unbounded. Then by Lemma 6.4 each \( E_n \) is closed, unbounded and convex at the point \( x \). Therefore given \( n \in \mathbb{N} \), Lemma 6.5 implies that \( \exists \) a unit vector \( v_n \in \mathbb{E}_n - x \) such that \( \{x + tv_n : t \in \mathbb{R} \) and \( t \geq 0\} \subseteq E_n \). The bounded
sequence \((v_n, n \in \mathbb{N})\) has a subsequence \((v_m, m \in \mathbb{M})\) converging to some \(u \in \mathbb{R}^p\). We note that \(\|u\| = 1\) since \(\|v_n\| = 1\) for each \(n \in \mathbb{N}\). Consider a real scalar \(t \geq 0\). Then \(x + tv_n \in E_n^p\) for each \(n \in \mathbb{M}\) and \(\mathbb{M}\) is an infinite subset of \(\mathbb{N}\); so \(\lim(x+tv_n, n \in \mathbb{M}) \in \text{Ls}_{R^p}(E_n, n \in \mathbb{N}) = \text{Ls}_{R^p}(E_n', n \in \mathbb{N})\) by Lemma 4.15(b); i.e., \(x + tu \in \text{Ls}_{R^p}(E_n', n \in \mathbb{N})\), for every real scalar \(t \geq 0\). Therefore \(\text{Ls}_{R^p}(E_n', n \in \mathbb{N})\) would contain the unbounded set \([x + tu; t \in \mathbb{R} \text{ and } t \geq 0]\).

Q.E.D.

The following lemma paves the way for showing the compactness Property \((D+)\) of the approximating sets \(D_n, n \in \mathbb{N}\), which are used in the intermediate problems.

**Lemma 6.7**

Fix \(\overline{x} \in D'\) and let \(D = \{x \in D': f(x) \leq f(\overline{x})\}\), \(D_n = \{x \in D_n': f(x) \leq f(\overline{x})\}\), \(I = J[D], I_n = J_n[D_n']\), \(i = \inf I[D]\) and \(i_n = \inf I_n[D_n]\). Then

(a) \(D = \bigcap_{n=1}^{\infty} D_n\) and \(D \subset D_{n+1} \subset D_n\) for each \(n \in \mathbb{N}\).
(b) \( D_n \) is closed and convex for each \( n \in \mathbb{N} \);
\( D \) is convex and closed.

(c) \( i = j \) and \( i_n = j_n \) for each \( n \in \mathbb{N} \).

(d) If \( y \in D'_n \) and \( j_n = J_n(y) \), then \( y \in D_n \) and \( j_n = i_n = I_n(y) \).

(e) If \( z \in D' \) and \( j = J(z) \), then \( z \in D \) and \( j = i = I(z) \).

**Proof:**

(a) \( x \in D \Leftrightarrow f(x) \leq f(x) \) and \( x \in D' \),
\[ \Leftrightarrow f(x) \leq f(x) \] and \( x \in \bigcap_{n=1}^{\infty} D'_n \) since
\[ D' = \bigcap_{n=1}^{\infty} D'_n, \]
\[ \Leftrightarrow x \in D_n \] for each \( n \in \mathbb{N} \) by the
definition of \( D_n \).

Thus \( D = \bigcap_{n=1}^{\infty} D_n \), so \( D \subseteq D_n \) for each \( n \in \mathbb{N} \). To
complete the proof of (a), let us suppose that
\( x \in D_{n+1} \). Then \( f(x) \leq f(x) \) and \( x \in D'_{n+1} \); therefore
\( f(x) \leq f(x) \) and \( x \in D'_n \) by Lemma 6.2(a); hence
\( x \in D_n \) by the definition of \( D_n \).
(b) The set \( \{ x \in \mathbb{R}^p : f(x) \leq f(\bar{x}) \} \) is closed and convex because \( f \) is linear. The set \( D' \) and each \( D_n' \) are closed and convex by Lemma 6.2(a), (c). Therefore \( D = D' \cap \{ x \in \mathbb{R}^p : f(x) \leq f(\bar{x}) \} \) is closed and convex, and \( D_n = D_n' \cap \{ x \in \mathbb{R}^p : f(x) \leq f(\bar{x}) \} \) is closed and convex.

(c) Since \( I \) is the restriction of \( J \) to \( D \) and \( I_n \) the restriction of \( J_n \) to \( D_n \), it is immediate that \( i \geq j \) and \( i_n \geq j_n \) for each \( n \in \mathbb{N} \).

Take now \( x \in D' \). Either \( f(x) \leq f(\bar{x}) \) and so \( J(x) = I(x) \geq i \), or else \( f(x) > f(\bar{x}) \) and therefore \( J(x) = f(x) > f(\bar{x}) = J(\bar{x}) = I(\bar{x}) \geq i \). Hence \( J(x) \geq i \) for each \( x \in D' \), whence \( j \geq i \). Similarly for a given \( n \in \mathbb{N} \) let \( x \in D_n' \). Then either \( f(x) \leq f(\bar{x}) \) and so \( J_n(x) = I_n(x) \geq i_n \), or else \( f(x) > f(\bar{x}) \) and so \( J_n(x) = f(x) > f(\bar{x}) = J_n(\bar{x}) = I_n(\bar{x}) \geq i_n \). Thus \( J_n(x) \geq i_n \) for each \( n \in \mathbb{N} \), so that \( j_n \geq i_n \). Combining \( i \geq j, j \geq i, i_n \geq j_n \) and \( j_n \geq i_n \) proves part (c).

(d) Consider \( y \in D_n' \) with \( j_n = J_n(y) \). Then \( f(y) = J_n(y) = j_n \leq J_n(\bar{x}) = f(\bar{x}) \), so \( y \in D \), and therefore \( i_n = j_n = J_n(y) = I_n(y) \) by part (c).
(e) Take $z \in D'$ with $J(z) = j$. Thus $f(z) = J(z) = j \leq J(\bar{x}) = f(\bar{x})$, so $z \in D$. Furthermore

$\bar{i} = j$

$= J(z) = I(z)$ by part (c),

Q.E.D.

Theorem 6.8

For the extended LP problem of line (6.2), let

$D' = \{x \in \mathbb{R}^p : x \geq \mathbb{R}^p : x \geq 0 \text{ and } a^k \cdot x \geq d_k \text{ for each } k \in \mathbb{N}\}$ and assume that $\exists \bar{x} \in D'$ with $\{x \in D' : f(x) \leq f(\bar{x})\}$ compact. Put:

\[
\begin{align*}
J &= f|D', \ j = \inf J[D'], \\
D'_n &= \{x \in \mathbb{R}^p : x \geq 0 \text{ and } a^k \cdot x \geq d_k \text{ for } x \leq k \leq n\}, \\
J_n &= f|D'_n \text{ and } j_n = \inf J_n[D'_n].
\end{align*}
\]

Then

(a) The sequence $(j_n, n \in \mathbb{N})$ converges to $j$ monotonically;

(b) If $x^*_n \in D'_n$ and $j_n = J_n(x^*_n)$ for each $n \in \mathbb{N}$, then a limit point $x^*$ of $(x^*_n, n \in \mathbb{N})$ exists in $D'$ with $j = J(x^*)$ and furthermore, every limit point $y$ of $(x^*_n, n \in \mathbb{N})$ lies in $D'$ and satisfies $j = J(y)$. 

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Proof:

Let \( \overline{x} \) be as in the theorem, so that \( D' \neq \emptyset \), and define \( D, I, D_n, I_n, i \) and \( i_n \) as in Lemma 6.7. We will show that:

1. Property \((D+)\) holds for \( D \) and \( (D_n', n \in \mathbb{N}) \), and
2. Property \((I)\) holds for \( I \) and \( (I_n', n \in \mathbb{N}) \).

Proof of (1):

Since \( D_{n+1} \subset D_n \) and \( D_n \) is closed for each \( n \in \mathbb{N} \) by Lemma 6.7 (a), (b), it follows that

\[
\bigcap_{n=1}^{\infty} D_n = \lim_{n \to \infty} D_n = \lim_{n \to \infty} (D_n', n \in \mathbb{N}) = \text{Ls}_p \left( D_n', n \in \mathbb{N} \right)
\]

by Lemma 6.7(a).

Now \( \overline{x} \in D_n \) for each \( n \in \mathbb{N} \), and \( D_n \) is a convex set for each \( n \in \mathbb{N} \) by Lemma 6.7(b); thus for every \( n \in \mathbb{N} \), the set \( D_n \) is convex at \( \overline{x} \) by Lemma 6.4(c). According to the hypothesis, the set \( D = \text{Ls}_p \left( D_n', n \in \mathbb{N} \right) \) is compact and hence, bounded. Therefore Theorem 6.6 applies: \( \exists \) a natural number \( N \) such that the set \( D_N \) is bounded. Because \( D_N \) is closed by Lemma 6.7(b), it follows that \( D_N \) is compact. Therefore

\[
\bigcup_{n \geq N} D_n = D_N \text{ is almost sequentially compact, and since } D = \lim_{n \to \infty} D_n = \text{Ls}_p \left( D_n', n \in \mathbb{N} \right), \text{ Corollary 4.24 implies that Property (D+) is satisfied, proving (1).}
\]
Proof of (2):

The functional $f$ is continuous on $\mathbb{R}^p$ and $I = I_n|_D$, $I_n|_{D_{n+1}} = I_{n+1}$, $I_n = f|_{D_n}$. Thus Theorem 5.1(c) applies, and both Property (I) and Property (I*) hold.

We can now prove Theorem 6.8:

(a) It follows by Corollary 3.5(a) that $(i_n', n \in \mathbb{N}) \rightarrow i$. However $i = j$ and $i_n = j_n$ and $j_n \leq j_{n+1} \leq j$ for each $n \in \mathbb{N}$ by Lemma 6.7(c) and Lemma 6.2(b). Therefore the sequence $(j_n, n \in \mathbb{N})$ converges to $j$ monotonically.

(b) Corollary 3.7 applies here because $\mathbb{R}^p$ is a Hausdorff space. Thus a limit point $x^*$ of $(x_n^*, n \in \mathbb{N})$ exists in $D$ with $i = I(x^*)$, and every limit point $y$ of $(x_n^*, n \in \mathbb{N})$ lies in $D$ with $i = I(y)$. Now since $D \subset D'$ and since $J$ extends $I$, both $x^*$ and $y$ referred to above lie in $D'$ and satisfy $i = J(x^*)$, $i = J(y)$, whence $j = J(x^*)$, $j = J(y)$ by Lemma 6.7(c).

Q.E.D.
Remark 6.9

It should be noticed that the problem in line (6.2) is precisely the problem of computing \( j \) and a point \( x^* \in D' \) with \( j = J(x^*) \). Thus in Theorem 6.8 we have provided approximate solutions of Problem (6.2) in the form of \( j_n \) and \( x_n^* \). This immediately raises the question about the computation of \( j_n \) and \( x_n^* \). But \( j_n \) and \( x_n^* \) are precisely the solutions of the ordinary LP problem:

\[
\begin{align*}
\text{minimize} & \quad \text{the linear functional } f: \mathbb{R}^p \to \mathbb{R} \\
\text{subject to} & \quad x \geq 0 \quad \text{and} \quad a^k \cdot x \geq d_k \quad \text{for} \quad 1 \leq k \leq n.
\end{align*}
\]

One may therefore compute \( j_n \) and \( x_n^* \), for an appropriate \( n \in \mathbb{N} \), by the well-established simplex method, whose exposition can be found in M. Simonnard [22] or F. A. Ficken [13].

Remark 6.10

In Theorem 6.8 we made the assumption: \( \exists \overline{x} \in D' \) such that \( \{ x \in D': f(x) \leq f(\overline{x}) \} \) is compact. This is certainly satisfied if \( D' \) is compact to begin with (take \( \overline{x} \) to be an arbitrary element of \( D' \)). However the converse is false: for \( p = 1 \), let \( f(x) = bx \) where \( b \) is a positive real constant, let \( a^k = k \) and \( d_k = 1 \) for each \( k \in \mathbb{N} \); then \( D' = [1, +\infty[ \), which is not compact, whereas for any \( \overline{x} \in D' \), the set
\{x \in D': f(x) \leq f(\bar{x}) \} \text{ is closed, bounded above by } \bar{x} \text{ and below by } 1, \text{ hence is compact. Finally let us point out that Theorem 6.8 does not require the computation of the point } \bar{x}, \text{ but only the knowledge that such an } \bar{x} \text{ exists.}

Remark 6.11

A procedure dual to the present one may be developed for the maximization problem.

Remark 6.12

The requirement "x \geq 0" in Problem (6.2) may be dropped without changing the conclusions of Theorem 6.8. However, the resulting ordinary LP problems (to find \( j_n \) and \( x^*_n \in D'_n \) such that \( j_n = J_n(x^*_n) \)) will no longer contain this constraint. It is known in linear programming that in such cases it is sometimes impossible to apply the simplex method.

Remark 6.13

Let us suppose that a linear functional \( f: \mathbb{R}^p \to \mathbb{R} \) is given together with a non-empty, convex, closed subset \( D' \) of \( \mathbb{R}^p \), and we wish to minimize \( f \) over \( D' \). It is known (see, for example, C. Berge [3]) that \( D' \) can be written as a countable intersection

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of closed half-spaces of the form \( \{x \in \mathbb{R}^p : a_k^\top x \geq d_k\} \).

If \( \exists \bar{x} \in D' \) such that \( \{x \in D' : f(x) \leq f(\bar{x})\} \) is compact, or if \( D' \) is compact, we can apply Theorem 6.8 to approximate \( j = \inf f[D'] \) and a point \( x^* \in D' \) with \( j = f(x^*) \). In practice, one hardly expects to be able to compute \( a_k \) and \( d_k \) for every \( k \in \mathbb{N} \).

But once \( a^n \) and \( d^n \) have been determined up to some natural number \( n \) (if any), then Theorem 6.8 insures that the number \( j_n \), computed from the corresponding ordinary LP problem, is a good lower estimate for \( j \).
VII. WEINSTEIN APPROXIMATION
OF EXTREME SPECTRAL POINTS

A method for exterior approximation of eigenvalues of certain differential operators was devised in 1937 by A. Weinstein [26] in his study of elastic thin plates. Since then, extensions of Weinstein's methods have been formulated by several mathematicians, among them N. Aronszajn, N. W. Bazley [2], H.F. Weinberger and A. Weinstein himself [27]. A comprehensive survey of these methods can be found in S.H. Gould [15] and G. Fichera [12]. The sufficient conditions of Chapter III can be applied in certain cases to obtain the convergence of these approximations to a desired eigenvalue or spectral point.

**Theorem 7.1**

Let \( H \) be a Hilbert space and \( T: H \rightarrow H \) a linear operator.

(a) If \( T \) is a positive operator, then the functional \( f: H \rightarrow \mathbb{R} \) defined by \( f(x) = x \cdot Tx \) is weakly sequentially lower semi-continuous on \( H \) (here, \( x \cdot Tx \) denotes the inner product in \( H \)).
(b) If $T$ is a compact self-adjoint operator, then the functional $f$ of part (a) is weakly sequentially continuous on $H$.

Proof:

(a) Let a sequence $(x_n, n \in \mathbb{N})$ in $H$ converge weakly to $y \in H$. Since $T$ is positive, and hence self-adjoint, we have $0 \leq (x_n - y) \cdot T(x_n - y) = x_n \cdot Tx_n - Ty_x + y \cdot Ty - x_n \cdot Ty$. From this inequality and the fact that $x_n \cdot Tx_n$ is real for every $n \in \mathbb{N}$, it follows that the quantity $- Ty_x + y \cdot Ty - x_n \cdot Ty$ is real and

$$(7.1) \quad Ty_x - y \cdot Ty + x_n \cdot Ty \leq x_n \cdot Tx_n \quad \text{for every} \quad n \in \mathbb{N}.$$ 

On the other hand, the weak convergence of $(x_n, n \in \mathbb{N})$ to $y$ implies that the sequences $(Ty_x, n \in \mathbb{N})$ and $(x_n \cdot Ty, n \in \mathbb{N})$ converge to $Ty \cdot y$ and $y \cdot Ty$, respectively. We note that $Ty \cdot y = y \cdot Ty$, because $T$ is a positive operator. Hence $\lim(Ty_x - y \cdot Ty + x_n \cdot Ty, n \in \mathbb{N}) = y \cdot Ty$. It follows by line (7.1) that

$$y \cdot Ty = \lim(Ty_x - y \cdot Ty + x_n \cdot Ty, n \in \mathbb{N}) \leq \lim \inf(x_n \cdot Tx_n, n \in \mathbb{N});$$

that is, $f(y) \leq \lim \inf(f(x_n), n \in \mathbb{N})$. 

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(b) Suppose a sequence \((x_n, n \in \mathbb{N})\) in \(H\) converges weakly to \(y \in H\). Then there exists a real number \(c\) with \(\|x_n\| < c\) for every \(n \in \mathbb{N}\), and, since \(T\) is a compact operator, the sequence of vectors \((Tx_n, n \in \mathbb{N})\) converges in norm to \(Ty\). However, we have

\[
|x_n \cdot Tx_n - y \cdot Ty| \leq |x_n \cdot Tx_n - x_n \cdot Ty| + |x_n \cdot Ty - y \cdot Ty|
\]

\[
\leq c|Tx_n - Ty| + |x_n \cdot Ty - y \cdot Ty|
\]

for every \(n \in \mathbb{N}\),

and the sequence \((x_n \cdot Ty, n \in \mathbb{N})\) converges to \(y \cdot Ty\) because of the weak convergence of \((x_n, n \in \mathbb{N})\) to \(y\). This shows that \((x_n \cdot Tx_n, n \in \mathbb{N}) \to y \cdot Ty\), so that \(f(y) = \lim(f(x_n), n \in \mathbb{N})\).

Q.E.D.

**Theorem 7.2**

Let \(V\) be a linear subspace of a Hilbert space \(H\) for which the orthogonal complement \(V^\perp\) has a complete independent sequence \((a_n, n \in \mathbb{N})\). Put \(C = \{x \in H: \|x\| \leq 1\}\), \(W_n = \text{subspace spanned by } \{a_1, a_2, \ldots, a_n\}\), \(V_n = W_n^\perp\), \(D = C \cap V\), \(D_n = C \cap V_n\). Then:
(a) \( V \subset V_{n+1} \subset V_n \) (so \( D \subset D_{n+1} \subset D_n \)) for every \( n \in \mathbb{N} \).

(b) Property (D+) holds for \( D \) and the sequence \( (D_n, n \in \mathbb{N}) \) with respect to the weak topology for \( H \).

**Proof:**

(a) If \( x \in V \), then \( x \) is orthogonal to \( V^l \), so \( x \) is orthogonal to \( a_k \) for each \( k \in \mathbb{N} \), whence \( x \) is orthogonal to \( W_n \) for every \( n \in \mathbb{N} \), i.e., \( x \in V_n \) for every \( n \in \mathbb{N} \). Thus \( V \subset V_n \) for every \( n \in \mathbb{N} \). It is obvious from the construction that \( W_n \subset W_{n+1} \) for every \( n \in \mathbb{N} \), and therefore \( W_n^l \supset W_{n+1}^l \) for every \( n \in \mathbb{N} \), so that \( V_n \supset V_{n+1} \) for each \( n \in \mathbb{N} \).

(b) Here, in view of Lemma 4.1(b), Property (D+) and Property (D) are equivalent, and we shall therefore only establish Property (D). Let then \( (x_n, n \in \mathbb{N}) \) be a sequence in \( H \) with \( x_n \in D_n \) for each \( n \in \mathbb{N} \). Thus \( x_n \in C \) for every \( n \in \mathbb{N} \), and so a subsequence \( (x_n', n \in M) \) converges weakly to some \( y \in H \). Because the norm is a weakly sequentially lower semi-continuous functional and \( \|x_n\| \leq 1 \) for every \( n \in M \), we obtain \( \|y\| \leq \lim \inf(\|x_n\|, n \in M) \leq 1 \), so that \( y \in C \). It now only remains to show that \( y \in V \) in order to conclude that \( y \in D \). Fix \( k \in \mathbb{N} \). If \( l \in M \) and \( l \geq k \),

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then, by (a), \( x^*_\ell \in V_\ell \subseteq V_k \) and \( a_k \in W_k \), so that 
\( x^*_\ell \cdot a_k = 0 \) by the definition of \( V_k \). Thus the sequence 
\((x_n \cdot a_k, n \in M)\) satisfies the property: \( x_n \cdot a_k = 0 \)
for almost all \( n \in M \). This implies that 
\( \lim(x_n \cdot a_k, n \in M) = 0 \). But since (\( x_n, n \in M \)) con­
verges weakly to \( y \) and \( a_k \) is fixed, it follows
that \( \lim(x_n \cdot a_k, n \in M) = y \cdot a_k \). We conclude that 
\( 0 = y \cdot a_k \). We observe that \( k \) was an arbitrary member 
of \( \mathbb{N} \); therefore \( 0 = y \cdot a_k \) for every \( k \in \mathbb{N} \), whence 
y is orthogonal to \( \mathcal{V} \), i.e., \( y \in \mathcal{V} \), as claimed.

**Theorem 7.3**

Let \( A \) be a self-adjoint operator defined on a
Hilbert space \( H \), and suppose \( V, V_n, D, D_n \) are as 
in Theorem 7.2. Let \( Q \) be the orthogonal projector 
of \( H \) onto \( V \) and \( Q_n \) the orthogonal projector of
\( H \) onto \( V_n \), \( f(x) = x \cdot QAQx \) for \( x \in H \), \( I = f|D, I_n = f|D_n \), \( i = \inf I[D], i_n = \inf I_n[D_n] \), \( i^* = \sup I[D] \)
and \( i_n^* = \sup I_n[D_n] \). Then:

(a) \( i \) is the least spectral point of the operator 
\( QAQ \), \( i^* \) is the largest spectral point of the operator 
\( QAQ \), \( i_n \) is the least spectral point of the operator 
\( Q_n A Q_n \) for every \( n \in \mathbb{N} \), and \( i_n^* \) is the
largest spectral point of the operator \( Q_n A Q_n \) for
every \( n \in \mathbb{N} \).
(b) \(i_n \leq i_{n+1} \leq i\) and \(i^* \leq i^*_{n+1} \leq i^*\) for every \(n \in \mathbb{N}\).

(c) If the operator \(A\) is positive, then
\[
\lim(i_n, n \in \mathbb{N}) = i.
\]

(d) If the operator \(A\) is compact, then
\[
\lim(i_n, n \in \mathbb{N}) = i \text{ and } \lim(i^*_n, n \in \mathbb{N}) = i^*.
\]

Proof:

(a) The operator \(QAQ\) is self-adjoint. Thus the least spectral point of \(QAQ\) is given by
\[
\inf\{x^*QAQx: ||x|| \leq 1\}
\]
and the largest spectral point of \(QAQ\) is given by \(\sup\{x^*QAQx: ||x|| \leq 1\}\). We shall show that \(\{x^*QAQx: ||x|| \leq 1\} = \{y^*QAQx: y \in D\}\). Since \(D = V \cap C \subseteq C = \{x \in H: ||x|| \leq 1\}\), we obtain at once \(\{x^*QAQx: x \in D\} \subseteq \{y^*QAQx: ||y|| \leq 1\}\). In order to get the reverse containment, let \(x \in H\) with \(||x|| \leq 1\). We can find two vectors \(u\) and \(v\) such that \(x = u + v\) and \(u \in V, v \in V^l\). Then \(||u|| \leq ||x|| \leq 1\), so \(u \in D\), and \(x^*QAQx = u^*QAQu \in \{y^*QAQy: y \in D\}\). The remaining assertions in part (a) are proved in a similar manner by showing that \(\{x^*Q_nA_nx: ||x|| \leq 1\} = \{y^*Q_nA_ny: y \in D_n\}\).

(b) It follows by Theorem 7.2(a) that \(D \subseteq D_{n+1} \subseteq D_n\) for every \(n \in \mathbb{N}\). Since \(I\) and \(I_n, n \in \mathbb{N}\),

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are restrictions of the functional \( f \), it follows that
\[
I[D] \subseteq I_{n+1}[D_{n+1}] \subseteq I_n[D_n]
\]
for every \( n \in \mathbb{N} \), so
\[
\inf I[D] \geq \inf I_{n+1}[D_{n+1}] \geq \inf I_n[D_n]
\]
and
\[
\sup I[D] \leq \sup I_{n+1}[D_{n+1}] \leq \sup I_n[D_n]
\]
for every \( n \in \mathbb{N} \), proving part (b).

(c) Since the operator \( A \) is positive, so is the
operator \( T = QAQ \). Hence \( f \) is weakly sequentially
lower semi-continuous on \( H \) by Theorem 7.1(a), so that
Property (I) holds in view of Theorem 5.1(a). On the
other hand Property (D+) holds because of Theorem 7.2(b).
Therefore Theorem 3.4(a) applies (Property (m) was
established in part (b) above).

(d) Given that the operator \( A \) is compact, it
follows that the operator \( QAQ = T \) is also compact,
so that \( f \) is weakly sequentially continuous on \( H \) by
Theorem 7.1(b). Thus, by Theorem 5.1(c), both Properties
(I) and (I*) are satisfied. Now Property (D+) holds
according to Theorem 7.2(b) and both Property (m)
and Property (m*) were established in part (b) above.
Thus Theorem 3.4(a) and Theorem 3.8(a) apply.

Remark 7.4

For the computation of the approximations \( i_n \) and
\( i_n^* \) of Theorem 7.3, see G. Fichera [12].
VIII. THE METHOD OF PENALTY FUNCTIONS

The theory of penalty functions originated from a conjecture of R. Courant in 1943 and was subsequently developed in [10] and [6]. Our treatment will be based on Chapter III of this paper.

General Theory

Problem 8.1

Given a set $X$ and $m+1$ real-valued functionals $J, g_1, \ldots, g_m$ defined on $X$ with $g_k \geq 0$ for $1 \leq k \leq m$, minimize $J$ over those members $x$ of $X$ satisfying $g_k(x) = 0$ for $1 \leq k \leq m$. Here, $m$ is a positive integer.

Let us define

$$E = \bigcap_{t=1}^{m} g_t^{-1}(0).$$

For obvious reasons we shall assume that the set $E$ is not void. Problem 8.1 is equivalent to the problem of finding $\inf J[E]$ together with a possible point $x^* \in E$ such that $J(x^*) = \inf J[E]$. We shall deal with Problem 8.1 in this form.
The standard penalty method involves the following machinery:

(8.2) For every integer \( t \) with \( 1 \leq t \leq m \),
\[
(K(n,t), n \in \mathbb{N}) \text{ is a strictly increasing sequence of positive real numbers not bounded above.}
\]

(8.3) For every \( n \in \mathbb{N} \), \( J_n = J + \sum_{t=1}^{m} K(n,t) \delta_t \).

The fundamental question is whether \( \inf J_n[X] \) converges to \( \inf J[E] \). Observe that all the functionals \( J_n \) have a fixed domain, \( X \), so that the theory of Chapter III does not apply immediately.

In order to pass from the above procedure to the setting of Chapter III, we make the following adjustments.

(8.4) \( w \) is a fixed point in the set \( E \).

\[ D = \{ x \in E : J(x) \leq J(w) \} ; \]

(8.5) \[ D_n = \{ x \in X : J_n(x) \leq J(w) \} . \]

(8.6) \( I = J[D] \), \( I_n = J_n[D_n] \), \( i = \inf I[D] \), \( i_n = \inf I_n[D_n] \).
Under these adjustments, we have a real-valued functional $I$ defined on a subset $D$ of a set $X$, a sequence $I_n$ of real-valued functionals each defined on a subset $D_n$ of the set $X$, and we may check the conditions of Chapter III under which the sequence $(i_n, n \in \mathbb{N})$ converges to $i$. The equivalence of this formulation to the standard penalty method is established in the following lemma (parts (b) and (c)).

Lemma 8.2

(a) $D \subseteq E$ and $D \subseteq D_{n+1} \subseteq D_n$ for every $n \in \mathbb{N}$.

(b) $i = \inf E$; given $x^* \in X$, $x^*$ minimizes $J$ over $E$ if and only if $x^*$ minimizes $I$.

(c) $i_n = \inf J_n[X]$; given $y \in X$, $y$ minimizes $J_n$ over $X$ if and only if $y$ minimizes $I_n$.

(d) $J_n \leq J_{n+1}$ for each $n \in \mathbb{N}$.

(e) $i_n \leq i_{n+1} \leq i$ for every $n \in \mathbb{N}$.

Proof:

(a) These containments are immediate from (8.5), (8.2) and the fact that $g_t \geq 0$ for $t = 1, 2, \ldots, m$. 

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(b) It follows from part (a) and the definition of I that \( I[D] \subset J[E] \), whence \( i = \inf I[D] \geq \inf J[E] \). Since the point \( u \) of (8.4) lies in E and \( E \supset D \) and \( J \) extends I over E, it is immediate that \( \inf J[E] \geq \inf I[D] = i \). Hence \( i = \inf J[E] \). The second assertion in part (b) follows from this equality.

(c) The proof is analogous to that of part (b) above.

(d) Let \( n \in \mathbb{N} \), and take \( x \in X \). For every integer \( t \) with \( 1 \leq t \leq m, 0 < K(n,t) < K(n+1, t) \) by (8.2). Since \( g_t(x) \geq 0 \), we have \( 0 \leq K(n,t)g_t(x) \leq K(n+1, t)g_t(x) \). Summation over \( t \) yields

\[
0 \leq \sum_{t=1}^{m} K(n,t)g_t(x) \leq \sum_{t=1}^{m} K(n+1, t)g_t(x).
\]

We may add \( J(x) \) throughout these inequalities and use (8.3) to get \( J(x) \leq J_n(x) \leq J_{n+1}(x) \), which proves the assertion in (d).

(e) The functional \( I_n \) clearly extends I, so that \( i_n \leq i \) for each \( n \in \mathbb{N} \). Now suppose that there exists \( n \in \mathbb{N} \) with \( i_{n+1} < i_n \). The definition of \( i_{n+1} \) then implies that we may find \( x \in D_{n+1} \) with
that is, \( j(x) + \sum_{t=1}^{m} K(n+1,t)g_t(x) \leq J(x) \) and \( I_{n+1}(x) = J_{n+1}(x) = J(x) + \sum_{t=1}^{m} K(n+1,t)g_t(x) < \epsilon \). It follows by part (d) that

\[ I_n(x) = J_n(x) \leq J_{n+1}(x) < \epsilon \] with \( x \in D_{n+1} \subset D_n \), contradicting the definition of \( \epsilon \).

**Remark 8.3**

We can combine Definition 4.18 and the remarks at the end of Chapter II to define "an almost sequentially compact" subset in an arbitrary (sequential) L-space \( X \). Given a real-valued sequentially l.s.c. functional \( f \) defined on an arbitrary (sequential) L-space \( X \), the following statements can be proved just as in the case of a topological space.

(a) If \( \emptyset \neq B \subset X \) and \( B \) is almost sequentially compact, then \( f \) is bounded below on \( B \).

(b) If \( \emptyset \neq B \subset X \) and \( B \) is sequentially compact, then \( f \) is bounded below on \( B \) and attains a minimum there.

**Remark 8.4**

By Lemma 2.5 and Theorem 2.10, a finite sum of (sequentially) l.s.c. real functions is (sequentially) l.s.c. Similarly for upper semi-continuity.
Theorem 8.5

Suppose there exists a sequential convergence structure $C$ on $X$ (see Definition 2.12) such that, with respect to $C$, the functionals $J, g_1, \ldots, g_m$ are l.s.c. on $X$ and for some $N \in \mathbb{N}$, the set $D_N$ is almost sequentially compact. Then $i = \lim(i_n, n \in \mathbb{N})$.

Proof:

Let $C$ and $N$ be as in the theorem. We shall show that:

(a) Property (D+) holds, with respect to $C$, for the sets $D$ and $D_n$ of line (8.5);

(b) Property (I) holds, with respect to $C$, for the functionals $I$ and $I_n$ of line (8.6);

(c) Property (m) is satisfied.

Once (a), (b), (c) are proved, the conclusion of Theorem 8.5 then follows from Theorem 3.4(a).

Proof of (c):

This is Lemma 8.3(e).
Proof of (b):

Suppose that for an infinite subset $M$ of $\mathbb{N}$, the sequence $(x_n, n \in M)$ converges to $y \in D$ with respect to $C$, where $x_n \in D_n$ for each $n \in M$. Then $I(y) = J(y)$ and $I_n(x_n) = J_n(x_n)$ for every $n \in M$. It follows by Lemma 8.2(d) that $J(x_n) \leq J_n(x_n)$ for every $n \in M$, so that $\lim \inf(J(x_n), n \in M) \leq \lim \inf(J_n(x_n), n \in M) = \lim \inf(I_n(x_n), n \in M)$. On the other hand, the lower semi-continuity of $J$ on $X$ with respect to $C$ implies that $J(y) \leq \lim \inf(J(x_n), n \in M)$. Therefore we have $I(y) = J(y) \leq \lim \inf(J(x_n), n \in M) \leq \lim \inf(J_n(x_n), n \in M) = \lim \inf(I_n(x_n), n \in M)$.

Proof of (a):

Given an infinite subset $M$ of $\mathbb{N}$, let the sequence $(x_n, n \in M)$ be such that $x_n \in D_n$ for every $n \in M$. Put $P = \{n \in M: n \geq N\}$, where $N$ is as in the theorem. Then $P$ is an infinite subset of $M$ and by Lemma 8.2(a), $(x_n, n \in P)$ is a sequence in the almost sequentially compact set $D_N$. It follows that for some infinite subset $L$ of $P$, the sequence $(x_n, n \in L)$ converges, with respect to $C$, to some $y \in X$. Note that $(x_n, n \in L)$ is a subsequence of $(x_n, n \in M)$. The proof will be completed by showing that $y \in D$, that is, $y \in E$ and $J(y) \leq J(\omega)$.
Since \( J \) is l.s.c. on \( X \) and \( D_N \) is an almost sequentially compact subset of \( X \) and \( x_n \in D_N \) for each \( n \in L \), we may apply Remark 8.3(a) to obtain a real number \( b \) such that \( b \leq J(x_n) \) for every \( n \in L \). This together with the fact that \( x_n \in D_N \) for each \( n \in L \) imply that

\[
(8.7) \quad b + \sum_{t=1}^{m} K(n,t)g_t(x_n) \leq J(x_n) + \sum_{t=1}^{m} K(n,t)g_t(x_n)
\]

\[
= J_n(x_n) \leq J(w), \quad n \in L.
\]

If now \( \ell \) is a natural number with \( 1 \leq \ell \leq m \), we have \( K(n,\ell)g_{\ell}(x_n) \geq 0 \), so \( b + K(n,\ell)g_{\ell}(x_n) \leq b + \sum_{t=1}^{m} K(n,t)g_t(x_n) \). Application of line (8.7) and the fact that \( k(n,\ell) > 0 \) then yields

\[
(8.8) \quad g_{\ell}(x_n) \leq \frac{J(w) - b}{K(n,\ell)} \quad \text{for every} \quad n \in L.
\]

Since \( J(w) - b \) is a constant real number, and since \( (K(n,\ell), \quad n \in L) \) is a strictly increasing sequence of positive real numbers not bounded above, it is true
that \( (\frac{J(w) - b}{K(n,t)}, n \in L) \to 0 \). Thus line (8.8) implies that \( \lim \inf (g_t(x_n), n \in L) \leq 0 \). But the subsequence \( (x_n, n \in L) \) of \( (x_n, n \in M) \) still converges to \( y \) with respect to \( C \). It follows from the lower semi-continuity of \( g_t \) and the above inequality that \( g_t(y) \leq \lim \inf (g_t(x_n), n \in L) \leq 0 \). Because \( g_t \) is non-negative, we conclude that \( g_t(y) = 0 \). Since \( t \) was arbitrary, we have

\[(8.9) \quad g_t(y) = 0 \quad \text{for} \quad t = 1, 2, \ldots, m.\]

Finally according to line (8.7) \( J_n(x_n) \leq J(w) \) for each \( n \in L \). Therefore Lemma 8.2(d) implies that the inequality \( J(x_n) \leq J(w) \) also holds for every \( n \in L \), whence \( \lim \inf (J(x_n), n \in L) \leq J(w) \). Since \( J \) is l.s.c. and \( (x_n, n \in L) \subset C \) \( y \), it follows that \( J(y) \leq \lim \inf (J(x_n), n \in L) \leq J(w) \). This and line (8.9) imply that \( y \in D \), by the definition of \( D \).

Q.E.D.

**Remark 8.6**

If the assumptions of Theorem 8.5 hold, then for every integer \( k \geq N \) (\( N \) as in Theorem 8.5), the set \( D_k \) is sequentially compact with respect to \( C \) and \( J_k \) attains a minimum over \( X \).
Proof:

It follows by Remark 8.4 and the hypothesis that $J_k$ is sequentially l.s.c. on $X$. Therefore if $D_k$ is sequentially compact, we may use Remark 8.3(b) to conclude that $J_k$ attains a minimum over $D_k$; then Lemma 8.2(c) guarantees that the minimum of $J_k$ over $D_k$ is the same as the minimum of $J_k$ over the whole set $X$. We thus only need to establish that $D_k$ is sequentially compact. Since $k \geq N$, a sequence $(x_n, n \in \mathbb{N})$ in $D_k$ is a sequence in the almost sequentially compact set $D_N$; hence for an infinite subset $M$ of $\mathbb{N}$, the subsequence $(x_n, n \in M)$ converges to some $y \in X$. The assumption that $x_n \in D_k$ implies that $J_k(x_n) \leq J(w)$ for each $n \in M$, so the lower semi-continuity of $J_k$ implies that $J_k(y) \leq \lim \inf(J_k(x_n), n \in M) \leq J(w)$, whence $y \in D_k$ by (8.5).

An Illustration in Euclidean Space

As an illustration of the above theory, we shall consider an elementary problem of non-linear programming. This problem can be solved by other methods, such as elimination or the method of Lagrange multipliers, and it will be included here only as an illustration. A similar example is given in [10]. Our problem will
be: to minimize the functional $J: \mathbb{R}^2 \to \mathbb{R}$, $J(x,y) = x^2 + y^2$, subject to $3x^2 + 5y = 2$.

With reference to the notation of the previous section, let $X = \mathbb{R}^2$, $m = 1$, $g_1(x,y) = (3x^2 + 5y - 2)^2$.

It is easily verified that the above problem is equivalent to: minimize $J(x,y)$ subject to $g_1(x,y) = 0$. Since the equation $3x^2 + 5y = 2$ determines a parabola in $\mathbb{R}^2$, the set $E$ of (8.1) is non-empty, so we may fix some point $\omega = (\omega_1, \omega_2) \in E$ as in (8.4). It is unnecessary to specify the point $\omega$. Put $K(n,1) = n$.

With respect to the Euclidean topology, both $J$ and $g_1$ are obviously lower semi-continuous, and $J_n$, as in (8.3), has continuous partial derivatives.

Let us observe that the set $D_1$ of (8.5) here satisfies

$$D_1 = \{(x,y) \in \mathbb{R}^2: x^2 + y^2 + (3x^2 + 5y - 2)^2 \leq \omega_1^2 + \omega_2^2\}$$

$$\subseteq \{(x,y) \in \mathbb{R}^2: x^2 + y^2 \leq \omega_1^2 + \omega_2^2\}$$

$$= \{z \in \mathbb{R}^2: \|z\| \leq \|\omega\|\}, \text{ where } \|\| \text{ is Euclidean norm.}$$

Hence $D_1$ is bounded and, by Lemma 4.19(d), is almost sequentially compact. Thus all the assumptions in Theorem 8.5 hold with $N = 1$. We conclude from Remark 8.6 that for each integer $n \geq 1$, $D_n$ is sequentially
compact and $J_n$ attains a minimum over $\mathbb{R}^2$, which is therefore a global minimum. Consequently
\[
\inf J_n[\mathbb{R}^2] \text{ is attained at a stationary point of } J_n.
\]

Given $n \in \mathbb{N}$, the equations $\frac{\partial J_n}{\partial x}(u,v) = 0$ and $\frac{\partial J_n}{\partial y}(u,v) = 0$ give the easily computed solutions $x_n^* = 0$, $y_n^* = \frac{10n}{1+25n} = \frac{10}{25+n}$, so that the minimum of $J_n$ over $\mathbb{R}^2$ is given by
\[
J_n(x_n^*, y_n^*) = \frac{10}{625+50n^{-1}+n^{-2}} + \frac{4n^{-1}}{625+50n^{-1}+n^{-2}}
= \frac{100+4n^{-1}}{625+50n^{-1}+n^{-2}}.
\]

It follows from Lemma 8.2(c) that $i_n = \frac{100+4n^{-1}}{625+50n^{-1}+n^{-2}}$ and, by Theorem 8.5, this sequence converges to the numerical solution of the problem. Obviously we have $\lim(i_n, n \in \mathbb{N}) = \frac{100}{625} = \frac{4}{25} = 0.16$, so the infimum (minimum) of $J(x,y)$ subject to $3x^2 + 5y = 2$ is $0.16$.

An Initial Value Problem with Control

Let us now apply the method of penalty functions, as developed in the first section of this chapter, to a control problem with fixed initial value. Often the
difficulty involved in solving such problems arises from the differential equations which constrain the state coordinates. An appropriate use of penalty functions can avoid the actual solving of the given differential equations by replacing the original dynamic problem with a sequence of intermediate nondynamic problems whose solutions approximate the solution of the original problem.

**Problem 8.7**

Consider a class of control functions \( u: [a,b] \rightarrow \mathbb{R}^l \) and corresponding absolutely continuous functions \( x: [a,b] \rightarrow \mathbb{R}^k \) satisfying

\[
(8.10) \quad x'(t) = f(t,x(t),u(t)), \quad x' \in L^2, \quad x(a) = x_0.
\]

Minimize the integral \( \int_a^b g(t,x(t),u(t))dt \) over the corresponding class of pairs \( (x,u) \).

In a form appropriate to a penalty method, the differential equation in (8.10) can be expressed as

\[
\int_a^b |x'(t) - f(t,x(t),u(t))|^2 dt = 0.
\]

We shall impose certain conditions on \( g,f \) and the class of control functions \( u \). For convenience we list most of these restrictions at once, even though
some of them will be used only later on in the discussion.

The numbers $a$ and $b$ are to be real and fixed with $a < b$. Also $k$ and $l$ are fixed positive integers and $\mu$ will denote Lebesgue measure. The vector $x_0$ is fixed in $\mathbb{R}^k$. The norm $|y|$ of a vector $y \in \mathbb{R}^n$ will be $|y| = \sum_{i=1}^{n} |y_i|$. $\mathbb{R}^{l \times k}$ will be the set of $l \times k$ real matrices with norm $|w| = \sum_{j=1}^{l} \sum_{i=1}^{k} |w_{ij}|$. The class of control functions will be denoted $U$, and each $u \in U$ is an $L^2$ function defined on $[a,b]$ with values in $\mathbb{R}^l$. We assume that

(8.11) $U$ is $L^2$ bounded and weakly closed.

The class $U$ will be weakly closed if, for example, $U$ is convex and $L^2$ closed. We assume that

(8.12) The domain of $f$ contains the set

$$\Omega = \{(t,y,u(t)): t \in [a,b], y \in \mathbb{R}^k, u \in U\}.$$

We shall assume the following two conditions on $f$.

(8.13) $|f(t,y,u(t))| \leq \varphi(t) |y| + \Upsilon(t) |u(t)| + \Theta(t)$

for each $(t,y,u(t)) \in \Omega$. 

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where \( \varphi, \gamma \) and \( \theta \) are fixed non-negative \( L^2 \) functions on \([a,b]\).

\( f(t,y,v) = p(t,y) + v \cdot q(t,y) \) on \( \Omega \),

where \( p \) and \( q \) are continuous functions on \([a,b] \times R^k\) such that \( p \) has range in \( R^k \) and \( q \) has range in \( R^k \times R^k \).

Observe that if \( p \) and \( q \) are assumed bounded on the set \([a,b] \times R^k\), then \( (8.14) \) implies \( (8.13) \) for \( \varphi = 0, \theta \) and \( \gamma \) then taken as any positive constants bounding \( p \) and \( q \), respectively.

To apply the theory developed at the beginning of this chapter, we now define an appropriate space \( X \).

**Definition 8.8**

Let \( X = \{(x,u) : u \in U, x : [a,b] \to \mathbb{R}^k, x(a) = x_o, x \) absolutely continuous with \( x' \in L^2, \|x' - f(t,x,u)\|_2 \leq \alpha \}. \) Here \( \alpha \) is any fixed positive real constant.

We define convergence in \( X \) as follows:

**Definition 8.9**

Given \((x,u) \in X \) and a sequence \((x_n,u_n), n \in \mathbb{N}\) in \( X \), we shall say that "\((x_n,u_n), n \in \mathbb{N}\) converges to \((x,u)\) in \( X\)" if \((x_n, n \in \mathbb{N}\) converges uniformly to \( x \) on \([a,b]\) and \((u_n, n \in \mathbb{N})\) converges weakly to \( u \) in \( L^2 \).
It is easily verified that, under this convergence, $X$ is a sequential L-space in the sense of Definition 2.14. We shall prove that $X$ is sequentially compact in this convergence as follows:

(1) The class $X_1 = \{x: [a,b] \to \mathbb{R}^k, \exists u \in U \text{ with } (x,u) \in X\}$ shall be proved uniformly bounded and equi-continuous (Lemmas 8.10, 8.11).

(2) The class $U$ is weakly compact. Hence for any sequence $((x_n,u_n), n \in \mathbb{N})$ in $X$, there exist $u \in U$, a continuous function $x: [a,b] \to \mathbb{R}^k$ and an infinite subset $M$ of $\mathbb{N}$ such that $(x_n, n \in M) \to x$ uniformly on $[a,b]$ and $(u_n, n \in M) \to u$ weakly in $L^2$. We shall show (proof of Theorem 8.12) that $(x,u) \in X$. The sequential compactness of $X$ will then follow. The almost sequential compactness of $D_N$, used in Theorem 8.5, will then be immediate for $N$ arbitrary.

**Lemma 8.10**

The class $X_1 = \{x: [a,b] \to \mathbb{R}^k, \exists u \in U \text{ with } (x,u) \in X\}$ is uniformly bounded.

**Proof:**

If $(x,u) \in X$, define $z(x,u)(t) = x'(t) - f(t, x(t), u(t))$. By line (8.11), $\exists$ a real number $\beta > 0$ such that $\|u\|_2 \leq \beta$ for every $u \in U$. Put
the right hand side, we see that:

\[
\int_{\mathbb{R}^n} \left( |f(x)|^p + |g(x)|^p \right)^{\frac{q}{p}} dx = \left( \int_{\mathbb{R}^n} |f(x)|^q dx \right)^{\frac{p}{q}}
\]

... and we apply Hölder's inequality.

\[
|\int f(x) g(x) dx| \leq \left( \int |f(x)|^p dx \right)^{\frac{1}{p}} \left( \int |g(x)|^q dx \right)^{\frac{1}{q}}
\]

\[
|\int f(x) g(x) dx| \leq \left( \int |f(x)|^p dx \right)^{\frac{1}{p}} \left( \int |g(x)|^q dx \right)^{\frac{1}{q}}
\]

... above inequality, over the integrals of each term occurring on the right hand side. If we integrate both sides of the inequalities, we get:

\[
\int \left( |f(x)|^p + |g(x)|^p \right)^{\frac{q}{p}} dx = \left( \int |f(x)|^q dx \right)^{\frac{p}{q}}
\]

... and we apply Hölder's inequality.

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\]

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|\int f(x) g(x) dx| \leq \left( \int |f(x)|^p dx \right)^{\frac{1}{p}} \left( \int |g(x)|^q dx \right)^{\frac{1}{q}}
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\]

\[
|\int f(x) g(x) dx| \leq \left( \int |f(x)|^p dx \right)^{\frac{1}{p}} \left( \int |g(x)|^q dx \right)^{\frac{1}{q}}
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... above inequality, over the integrals of each term occurring on the right hand side. If we integrate both sides of the inequalities, we get:

\[
\int \left( |f(x)|^p + |g(x)|^p \right)^{\frac{q}{p}} dx = \left( \int |f(x)|^q dx \right)^{\frac{p}{q}}
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|\int f(x) g(x) dx| \leq \left( \int |f(x)|^p dx \right)^{\frac{1}{p}} \left( \int |g(x)|^q dx \right)^{\frac{1}{q}}
\]

\[
|\int f(x) g(x) dx| \leq \left( \int |f(x)|^p dx \right)^{\frac{1}{p}} \left( \int |g(x)|^q dx \right)^{\frac{1}{q}}
\]

... above inequality, over the integrals of each term occurring on the right hand side. If we integrate both sides of the inequalities, we get:

\[
\int \left( |f(x)|^p + |g(x)|^p \right)^{\frac{q}{p}} dx = \left( \int |f(x)|^q dx \right)^{\frac{p}{q}}
\]

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\[
|\int f(x) g(x) dx| \leq \left( \int |f(x)|^p dx \right)^{\frac{1}{p}} \left( \int |g(x)|^q dx \right)^{\frac{1}{q}}
\]

\[
|\int f(x) g(x) dx| \leq \left( \int |f(x)|^p dx \right)^{\frac{1}{p}} \left( \int |g(x)|^q dx \right)^{\frac{1}{q}}
\]

... above inequality, over the integrals of each term occurring on the right hand side. If we integrate both sides of the inequalities, we get:

\[
\int \left( |f(x)|^p + |g(x)|^p \right)^{\frac{q}{p}} dx = \left( \int |f(x)|^q dx \right)^{\frac{p}{q}}
\]

... and we apply Hölder's inequality.

\[
|\int f(x) g(x) dx| \leq \left( \int |f(x)|^p dx \right)^{\frac{1}{p}} \left( \int |g(x)|^q dx \right)^{\frac{1}{q}}
\]

\[
|\int f(x) g(x) dx| \leq \left( \int |f(x)|^p dx \right)^{\frac{1}{p}} \left( \int |g(x)|^q dx \right)^{\frac{1}{q}}
\]
\[ V(t) \exp\left(-\int_a^t \varphi(s) \, ds\right) \leq |x_0| \left[ \int_a^t \varphi^2(s) \, ds \right]^\frac{1}{2} \left[ \int_a^t 1 \, ds \right]^\frac{1}{2} \\
+ \left[ \int_a^t \varphi^2(s) \, ds \right]^\frac{1}{2} \left[ \int_a^t |u(s)|^2 \, ds \right]^\frac{1}{2} \\
+ \left[ \int_a^t \theta^2(s) \, ds \right]^\frac{1}{2} \left[ \int_a^t 1 \, ds \right]^\frac{1}{2} \\
+ \left[ \int_a^t |z(x,u)(s)|^2 \, ds \right]^\frac{1}{2} \left[ \int_a^t 1 \, ds \right]^\frac{1}{2} \\
\leq |x_0| \|\varphi\|_2 (b-a)^\frac{1}{2} + \|\varphi\|_2 \|u\|_2 \\
+ \|\theta\|_2 (b-a)^\frac{1}{2} + \|z(x,u)\|_2 (b-a)^\frac{1}{2} \\
\leq |x_0| \|\varphi\|_2 (b-a)^\frac{1}{2} + \|\varphi\|_2 \beta \\
+ \|\theta\|_2 (b-a)^\frac{1}{2} + \alpha (b-a)^\frac{1}{2}. \]

where we have used the fact that \( \|u\|_2 \leq \beta \) and \( \|z(x,u)\|_2 \leq \alpha \). Thus \( V(t) \leq \exp\left(\int_a^b \varphi(s) \, ds\right) \left[ |x_0| \|\varphi\|_2 (b-a)^\frac{1}{2} \\
+ \beta \|\varphi\|_2 \|\theta\|_2 (b-a)^\frac{1}{2} + \alpha (b-a)^\frac{1}{2} \right] \). If we combine this inequality with line (8.16), we obtain \( |x(t)| \leq c \), where \( c \) is the constant of line (8.15).

**Lemma 8.11**

The class \( X_1 = \{x: [a,b] \to \mathbb{R}^k, \exists u \in U \text{ with } (x,u) \in X\} \) is equicontinuous.
Proof:

By Lemma 8.10, there is a constant real number $c > 0$ such that $|y(r)| \leq c$ for every $y \in X_1$ and every $r \in [a,b]$. Put $d = \max\{1,c\}$. By an argument similar to the one used in the proof of Lemma 8.10 it can be shown that

$$|x(t) - x(s)| \leq d \left[ \int_s^t \left[ \varphi(r) + \psi(r)|u(r)| + \Theta(r) + |z(x,u)(r)| \right] dr. \right.$$  

However, if $a \leq s \leq t \leq b$ and $(x,u) \in X$ we may use Hölder's inequality to obtain

$$\int_s^t \left[ \varphi(r) + \psi(r)|u(r)| + \Theta(r) + |z(x,u)(r)| \right] dr \leq \|\varphi\|_2 (t-s)^{\frac{1}{2}} + \|u\|_2 \left[ \int_s^t \psi^2(r) dr \right]^{\frac{1}{2}} + \|\Theta\|_2 (t-s)^{\frac{1}{2}} + \|z(x,u)\|_2 (t-s)^{\frac{1}{2}} + \beta \left[ \int_s^t \psi^2(r) dr \right]^{\frac{1}{2}} + \|z(x,u)\|_2 (t-s)^{\frac{1}{2}} + \phi (t-s)^{\frac{1}{2}},$$

where $\beta > 0$ is a real number with $\|v\|_2 \leq \beta$ for each $v \in U$. The equicontinuity of $X_1$ now follows from line (8.17).
**Theorem 8.12**

Let \( ((x_n, u_n), n \in \mathbb{N}) \) be a sequence in \( X \), let \( x: [a,b] \to \mathbb{R}^k \) be continuous, and let \( u \in U \) be such that \( (x_n, n \in \mathbb{N}) \to x \) uniformly on \([a,b]\) and \( (u_n, n \in \mathbb{N}) \to u \) weakly in \( L^2 \). Then \( (x,u) \in X \).

**Proof:**

The sequence \( (z(x_n, u_n), n \in \mathbb{N}) \), where \( z(x_n, u_n)(t) = x'_n(t) - f(t, x_n(t), u_n(t)) \) is \( L^2 \) bounded by \( \alpha \). Hence there exists an infinite subset \( M \) of \( \mathbb{N} \) such that \( (z(x_n, u_n), n \in M) \) converges weakly in \( L^2 \). The limit \( y \) satisfies \( \|y\|_2 \leq \alpha \) by weak lower semi-continuity of the norm.

By absolute continuity of \( x_n' \), we have

\[
x_n(t) = x_n(a) + \int_a^t z(x_n, u_n)(s) \, ds + \int_a^t [p(s, x_n(s)) + u_n(s) \cdot q(s, x_n(s))] \, ds,
\]

where we have used line (8.14). Since \( X_1 \) is uniformly bounded (Lemma 8.10), we can work with bounded restrictions of \( p \) and \( q \). A well known theorem states that the integral of a linear function of \( u \) with continuous bounded coefficients is continuous in the above convergence. Thus

\[
x(t) = x(a) + \int_a^t y(s) \, ds + \int_a^t [p(s, x(s)) + u(s) \cdot q(s, x(s))] \, ds.
\]
Hence $x$ is absolutely continuous and
\[ x'(t) = y(t) + p(t,x(t)) + u(t) \cdot q(t,x(t)) \quad \text{a.e.} \]

Consequently $x' \in L^2$, and $z(x,u) = y$ so $\|z(x,u)\| \leq q$.

Q.E.D.

Finally, in order to apply Theorem 8.5, we shall define $J$. We need the following conditions on $g$.

\[
\begin{cases}
  g: T \times S \to \mathbb{R} \text{ with } g \geq 0, \ g \text{ continuous and} \\
  \text{convex in } u, \ T \text{ a closed subset of } \mathbb{R}^{k+1} \\
  \text{containing } \{(t,x(t)): t \in [a,b], x \in X_1\}, \\
  S \text{ a closed convex subset of } \mathbb{R}^t \text{ containing} \\
  \{u(t): t \in [a,b], u \in U\} \text{ and} \\
  \int_a^b g(t,x(t),u(t))dt < +\infty \text{ for } (x,u) \in X.
\end{cases}
\]

**Definition 8.13**

For $(x,u) \in X$, let $J(x,u) = \int_a^b g(t,x(t),u(t))dt$.

**Theorem 8.14**

$J$ is sequentially lower semi-continuous on $X$ with respect to the convergence of Definition 8.9.

**Proof:**

This is a special case of [23, Theorem 4], since $U$ is $L^2$ bounded (and hence $L^1$ bounded).
**Definition 8.15**

For \((x,u) \in X\), let \(g_1(x,u) = \int_a^b \left| x'(t) - f(t,x(t),u(t)) \right|^2 \, dt\).

**Theorem 8.16**

Under the convergence of Definition 8.9, \(g_1\) is sequentially lower semi-continuous on \(X\).

**Proof:**

Let \(z(x,u) = x' - f(t,x,u)\). If \(((x_n,u_n), \ n \in \mathbb{N})\) converges to \((x,u)\) in \(X\), let \(M\) be an infinite subset of \(\mathbb{N}\) such that \(\|z(x_n,u_n)\|^2, \ n \in M\) → \(\lim \inf \|z(x_n,u_n)\|^2, \ n \in \mathbb{N}\). Since the sequence \((z(x_n,u_n), \ n \in M)\) is \(L^2\) bounded by \(\alpha\), there is an infinite subset \(P\) of \(M\) such that the sequence \((z(x_n,u_n), \ n \in P)\) converges weakly to \(z(x,u)\) by the same reasoning as in Theorem 8.12. The square of the norm is weakly sequentially lower semi-continuous, so

\[
\|z(x,u)\|^2 \leq \lim \inf \|z(x_n,u_n)\|^2, \ n \in P
\]

\[
= \lim(\|z(x_n,u_n)\|^2, \ n \in M)
\]

\[
= \lim \inf(\|z(x_n,u_n)\|^2, \ n \in \mathbb{N}).
\]

Q.E.D.
Summary

We transformed Problem 8.7 into the equivalent problem of minimizing the functional \( J(x,u) \) (Definition 8.13) over the set \( X \) (Definition 8.8) subject to \( g_1(x,u) = 0, \) \( g_1 \) given in Definition 8.15. Then we made \( X \) into a sequential L-space which is sequentially compact and on which \( J \) and \( g_1 \) are sequentially lower semi-continuous. We may take \( m = 1 \) and \( K(n,1) = n \) in Problem 8.1 and line (8.2), respectively. Thus each set \( D_n \) of line (8.5) is an almost sequentially compact subset of \( X \). Therefore all the hypotheses of Theorem 8.5 are satisfied. Consequently the (non-dynamic) intermediate problems (minimize \( J_n \) over \( X \), \( J_n \) as in line (8.3)) approximate Problem 8.7 numerically. We refer to [1] for indications concerning the computation of \( \min J_n [X] \).

A Boundary Value Problem with Control

By use of the penalty method, a boundary value problem can be replaced with intermediate problems which do not contain the terminal conditions, and which approximate the original problem. See C. M. Kashmar and E. L. Peterson [19].

Problem 8.17

Consider a class of control functions \( u; [a,b] \rightarrow \mathbb{R}^t \)
and corresponding absolutely continuous functions
\[ x: [a, b] \to \mathbb{R}^k \] satisfying:
\[
\begin{cases}
x'(t) = f(t, x(t), u(t)), & x' \in L^2, \\
x(a) = x_0, & x(b) = y_0.
\end{cases}
\] (8.19)

Minimize the integral \( \int_a^b g(t, x(t), u(t)) \, dt \) over the corresponding class of pairs \((x, u)\).

We shall impose the same restrictions on \( f \) and \( g \) as in the preceding section. Also the quantities \( a, b, k, \ell, U, x_0 \) will be as in the preceding section. The vector \( y_0 \) will be fixed in \( \mathbb{R}^k \). Our procedure will be parallel to that of the preceding section.

**Definition 8.18**

Let \( \tilde{X} = \{(x, u): u \in U, x: [a, b] \to \mathbb{R}^k, x(a) = x_0, x \text{ is absolutely continuous on } [a, b], x' \in L^2, x'(t) = f(t, x(t), u(t))\} \).

**Theorem 8.19**

\( \tilde{X} \) is a sequentially compact subset of \( X \), where \( X \) is as in Definition 8.8.

**Proof:**

That \( \tilde{X} \subset X \) is obvious. If \( \{(x_n, u_n), n \in \mathbb{N}\} \) is a sequence in \( \tilde{X} \subset X \), it follows from the
sequential compactness of $X$ that, for an infinite subset $M$ of $\mathbb{N}$, the subsequence $((x^n, u^n), n \in M)$ converges to some $(x, u) \in X$. By an argument similar to that used in the proof of Theorem 8.12, we obtain $x'(t) = f(t, x(t), u(t))$ because $x_n'(t) = f(t, x_n(t), u_n(t))$ for each $n \in M$.

Throughout, we shall consider $\tilde{X}$ with the convergence inherited from $X$. Theorem 8.19 states that under this convergence $\tilde{X}$ is a sequentially compact space.

**Definition 8.20**

Let $\tilde{J} = J|\tilde{X}$. For $(x, u) \in \tilde{X}$, let $\tilde{g}_1(x, u) = |x(b) - y_0|^2$.

**Theorem 8.21**

$\tilde{J}$ and $\tilde{g}_1$ are sequentially lower semi-continuous on the space $\tilde{X}$.

**Proof:**

The lower semi-continuity of $\tilde{J}$ follows from that of $J$. The lower semi-continuity of $\tilde{g}_1$ is obvious from the definition of $\tilde{g}_1$.

Q.E.D.

Problem 8.17 is now equivalent to the problem of minimizing $\tilde{J}(x, u)$ over $\tilde{X}$ subject to $\tilde{g}_1(x, u) = 0$, which is in the form of the general Problem 8.1, where

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\( \tilde{J} \) plays the role of \( J \), \( \tilde{g}_l \) plays the role of \( g_l \), \( m = 1 \) and \( \tilde{X} \) plays the role of \( X \). The sequence 

\[ K(n,1) \text{ may be defined by } K(n,1) = n, \text{ and } \tilde{J}_n = \tilde{J} + n\tilde{g}_l. \]

All the assumptions of Theorem 8.5 are satisfied: \( \tilde{J} \) and \( \tilde{g}_l \) are l.s.c. on \( \tilde{X} \) by Theorem 8.21, \( D_1(N = 1) \) is almost sequentially compact as a subset of the sequentially compact space \( \tilde{X} \). Hence the intermediate problems (minimize \( \tilde{J}_n \) over \( \tilde{X} \)) approximate Problem 8.17. Observe from Definition 8.18 that these intermediate problems do not contain the terminal condition \( x(b) = y_0 \).


